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Natural frequencies of a Timoshenko beam subjected to axial forces by the differential transform method

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Abstract. Rotating beams are extensively used in different mechanical and aeronautical installations. In this paper, a systematic approach is presented in order to solve the eigenvalues problem through the Timoshenko beam theory. The equations of motion are deduced by using the Hamiltonian approach. These equations are then solved by the differential transform method (DTM). The obtained numerical results using DTM are compared with the exact solution. Natural frequencies are determined, and the effects of the rotational speed and axial force on the natural frequencies are investigated. Results show high accuracy and efficiency of the differential transform method.

1. Introduction

Rotating structures can be found in turning machinery systems such as motors, engines, and turbines. The bending vibrations analysis of the beams aroused considerable interest for the engineers. The natural frequencies and mode shapes of such systems are indispensable in the design of structures. Zu and Han [1] analytically solved the free flexural vibration of a spinning Timoshenko beam with classical boundary conditions. Zhang [2] studied the free vibration of axially loaded shear beam column and obtained a very simple frequency equation to utilize for axially loaded beam as well as to obtain the buckling loads by setting the natural frequencies to disappear.

Farchaly and Shebl [3] determined two sets of exact general frequency and mode shape equations to study the vibration and stability of a Timoshenko beam that carries an end masses of finite length. Lee [4] enclosed the constant axial force and found that it had a considerable effect on the magnitude of the dynamic response. Ouyang [5] established a dynamic model for a rotating Timoshenko beam subjected to three force components acting on the surface. The deflection of the beam examined and found it had proportional increases with respect to the deflection and the frequency components when the axial force component is included.

The effect of such parameters such as moving velocity, the skew force angle, and the rotating speed on the system dynamic response is investigated by utilizing the global assumes mode method by considering boundary conditions [6]. The dynamic green function is used to introduce the free vibration of elastically supported Timoshenko beam on a partly Winkler foundation [7]. The finite element method is used to investigate the behaviour of the natural frequencies and to determine the influence of the rotating speed profile on the vibration of the cantilevered beam based on the dynamic modelling method by using the stretch deformation [8].



Among different types of numerical techniques, finite element method, the finite difference approach, the Galerkin method, etc. In the present study, the differential transform method is used. The proposition of the concept of the differential transform method was presented for the first time by Zhou [9] in 1986 when the problems of linearity and no-linearity that involved the electrical circuit problems had been solved.

2. Governing equation of motion

Figure 1 shows a uniform beam of the circular cross-section A , in an inertial coordinate system $oxyz$ subjected to axial force P along the x -axis

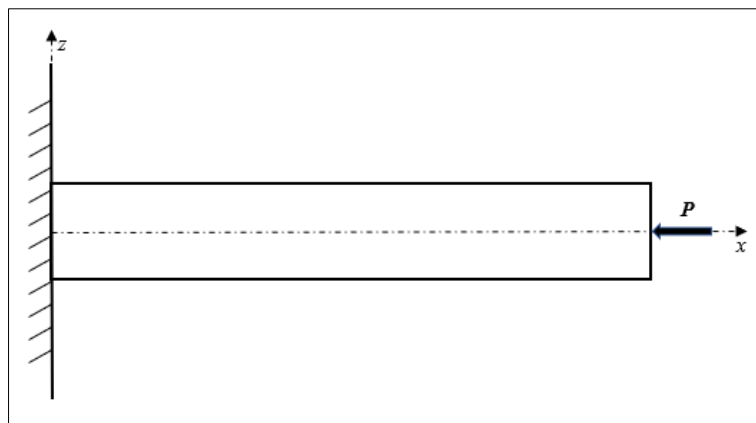


Figure 1. Cantilevered beam geometry.

The governing differential equations of motion as well as the boundary conditions can be derived through Hamilton's principle which can be stated as follows:

$$\delta I = \delta \int_{t_1}^{t_2} L dt = 0 \quad (1)$$

$$L = T - U_1 - U_2 \quad (2)$$

Where δ is the variational operator and L is the Lagrangian of the model. T and U_1 are the kinetic energy and the potential energy respectively of the Timoshenko beam adapted from reference [10].

$$T = \frac{\rho}{2} \int_0^l \left\{ A \left[\left(\frac{dv}{dt} \right)^2 + \left(\frac{dw}{dt} \right)^2 \right] + I \left[\dot{\phi}^2 + \dot{\psi}^2 - 2\Omega(\dot{\phi}\psi - \phi\dot{\psi}) + 2\Omega^2 \right] \right\} dx \quad (3)$$

$$U_1 = \frac{1}{2} \int_0^l \left\{ EI \left[\left(\frac{d\phi}{dt} \right)^2 + \left(\frac{d\psi}{dt} \right)^2 \right] + \kappa GA (\gamma_{xz}^2 + \gamma_{yz}^2) \right\} dx \quad (4)$$

U_2 is the potential energy caused by the work of the axial load P according to the Engesser approach [11] given by the following equation:

$$U_2 = -\frac{1}{2} \int_0^l \left[\left(\frac{dv}{dx} \right)^2 + \left(\frac{dw}{dx} \right)^2 \right] dx \quad (5)$$

Where v, w are the deflections of the rotating beam in the y, z directions respectively, and the angular rotations around y and w axes by ϕ and ψ . A is the cross-section area of the beam, and I is the moment of the area.

$$I = \frac{\pi r^4}{4}$$

Where r is the radius of the beam ρ is the mass density of the beam material. E and G are Young's modulus and shear modulus respectively, and κ is the shear coefficient. γ_{xy} and γ_{xz} denote the shear strain expressed in (6) and (7).

$$\gamma_{xy} = \frac{\partial v}{\partial x} - \phi \quad (6)$$

$$\gamma_{xz} = \frac{\partial w}{\partial x} - \psi \quad (7)$$

By introducing the following non-dimensional variables and parameters:

$$\mu^{*2} = \frac{\rho AL^4}{EI} \omega \Omega, \quad \mu^2 = \frac{\rho AL^4}{EI} \omega^2$$

$$r^2 = \frac{I}{AL^2}, \quad s^2 = \frac{EI}{\kappa GAL^2}, \quad \bar{P} = \frac{P(x)L^2}{EI} \quad (8)$$

And presuming that the beam incurs harmonic motion:

$$\bar{Z}(x,t) = \Gamma(x)e^{-j\omega t}, \quad \alpha(x,t) = \Phi(x)e^{-j\omega t} \quad (9)$$

The differential equations of motion are obtained after integration, and its dimensionless form are as follows:

$$\left(\frac{1}{s^2} - \bar{P} \right) \frac{\partial^2 \Gamma}{\partial x^2} + \mu^2 \Gamma + \frac{1}{s^2} \frac{\partial \Phi}{\partial x} = 0 \quad (10)$$

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{1}{s^2} \left(\Phi - \frac{\partial \Gamma}{\partial x} \right) - 2\omega \mu^2 r^2 \Phi = 0 \quad (11)$$

As well as the associated boundary conditions:

$$\left[\left(\frac{1}{s^2} + \bar{P} \right) \frac{\partial \Gamma}{\partial x} - \frac{1}{s^2} \Phi \right] \delta \Gamma \Big|_0^l = 0 \quad (12)$$

$$\left(\frac{\partial \Phi}{\partial x} \right) \delta \Phi \Big|_0^l = 0 \quad (13)$$

3. Differential transform method

The differential transform method (DTM) is a semi analytical-numerical method that is suitable for solving initial and/or boundary value problems and can provide highly accurate results with small computational effort. [12, 13] The DTM has gained the attention of many researchers recently.

Let $f(x)$ be an analytic function in a domain D and let $x = x_0$ represent any point in domain D ; therefore, the differential transform of $f(x)$ is given by [14]:

$$F(k) = \frac{1}{k!} \left[\frac{d^k f(x)}{dx^k} \right]_{x=x_0} \quad (14)$$

Where $F(k)$ is the transformed function in the transformation domain, $f(x)$ is the original function and k is the transformation parameter. The inverse transformation is defined as:

$$f(x) = \sum_{k=0}^{\infty} (x-x_0)^k F(k) \quad (15)$$

Combining equations (13) and (14), we obtain:

$$f(x) = \sum_{k=0}^{\infty} \frac{(x-x_0)^k}{k!} \left[\frac{d^k f(x)}{dx^k} \right]_{x=x_0} \quad (16)$$

In the real case, the number of series is limited to N and equation (16) can be written as follows:

$$f(x) = \sum_{k=0}^N \frac{(x-x_0)^k}{k!} \left[\frac{d^k f(x)}{dx^k} \right]_{x=x_0} \quad (17)$$

Where the number N is defined according to the convergence criterion. The basic theorems of the differential transform are listed in Table 1 [14].

Table 1. Formatting sections, subsections and subsubsections.

Original function	Transformed function
$f(x) = h(x) \pm g(x)$	$F(k) = H(k) \pm G(k)$
$f(x) = \beta g(x)$	$F(k) = \beta G(k)$
$f(x) = h(x).g(x)$	$F(k) = H(k) \sum_{l=0}^k (k-l)G(l)$
$f(x) = \frac{dg(x)^n}{d(x)^n}$	$F(k) = [(k+n)!]G(k+n)$
$f(x) = x^n$	$F(k) = \delta(k-n) = \begin{cases} 0 & \text{if } k \neq n \\ 1 & \text{if } k = n \end{cases}$

4. Application of the DTM

The following recursive expressions are determined by applying DTM to equations (10) and (11) using theorems listed in Table 2.

$$F(k+2) = \frac{-s^2}{P} H(k+2) - \frac{s^2 \mu^2 \omega^2}{P(k+1)(k+2)} F(k) \quad (18)$$

$$H(k+2) = \frac{1}{s^2(k+1)(k+2)} H(k) + \frac{1}{s^2(k+2)} F(k+1) + \frac{r^2 \omega (2j\mu^{*2} + \omega \mu^2)}{(k+1)(k+2)} H(k) \quad (19)$$

Applying DTM to equations (12) and (13), the boundary conditions are as follows:

At $x=0$:

$$\begin{cases} F(0) = 0 \\ H(0) = 0 \end{cases} \quad (20)$$

At $x=l$:

$$\sum_{k=0}^N kH(k) = 0 \quad (21)$$

$$\sum_{k=0}^N H(k) = \sum_{k=0}^N kF(k) \quad (22)$$

$$(22) \Rightarrow \sum_{k=0}^N [kF(k) - H(k)] = 0 \quad (23)$$

For any $F(k)$ and $H(k)$ with $k \geq 2$ are expressed regarding the first two terms in function of ω, c_1 , and c_2 . By inserting the terms of $F(k)$ and $H(k)$ into the boundary conditions in (21) and (23) the system of equations obtained will be as follows:

$$A_{11}^N(\omega)c_1 + A_{12}^N(\omega)c_2 = 0, \quad i=1,2,3...N \quad (24)$$

Where $A_{11}^N(\omega)c_1, A_{12}^N(\omega)c_2$ are polynomials of ω corresponding to n^{th} term. The equation (24) can be rewritten in the matrix form as follows:

$$\begin{bmatrix} A_{11}^N(\omega) & A_{12}^N(\omega) \\ A_{21}^N(\omega) & A_{22}^N(\omega) \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (25)$$

The natural frequencies are obtained by equating the characteristic equation of the determinant of the equation (25).

Table 2. DTM theorems for boundary conditions

$x = 0$		$x = l$	
Original BC	Transformed BC	Original BC	Transformed BC
$f(x) = 0$	$F(0) = 0$	$f(1) = 0$	$\sum_{k=0}^N F(k) = 0$
$\frac{\partial f}{\partial x}(0) = 0$	$F(1) = 0$	$\frac{\partial f}{\partial x}(1) = 0$	$\sum_{k=0}^N kF(k) = 0$
$\frac{\partial^2 f}{\partial x^2}(0) = 0$	$F(2) = 0$	$\frac{\partial^2 f}{\partial x^2}(1) = 0$	$\sum_{k=0}^N k(k-1)F(k) = 0$

5. Numerical results and discussion

For the determination of the natural frequencies for a cantilevered beam, a computer package Matlab is used. In order to validate the computed results, an associate illustrative example taken from [3] is solved and also the results are compared with those within the same reference paper.

The results for cantilevered beam are tabulated in Table 3 for different values of P . The results demonstrate the effect of the parameter of the axial force on the natural frequency. A good agreement up to the third digit in the first mode and up to the first digit in the second mode are shown between the present study and that mentioned in reference [3]. The found results show a decrease of the natural non-dimensional frequencies parameter by the increasing of the non-dimensional of the axial force parameter P .

Table 3. Variation of the natural non-dimensional frequencies for a cantilever beam for various values of P concerning ($r^2=0.01$ and $s^2=3r^2$).

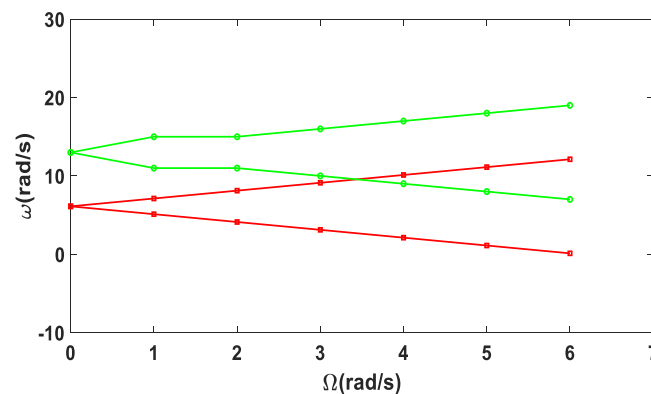
P	First mode		Second mode	
	Present	Ref.[3]	Present	Ref.[3]
0	1.799	1.799	3.819	3.820
1	1.579	1.579	3.708	3.715
10	-	-	2.333	2.199

The results presented in Table 4 show the effect of the rotational speed $\Omega(\text{rad/s})$ on the fundamental natural frequencies of the beam concerning the material and geometric properties of the beam are: $\rho = 8480 \text{ kg/m}^3$, $L = 0.4 \text{ m}$, $I = 1.886 \times 10^{-5} \text{ m}^4$, $E = 2 \times 10^7 \text{ pa}$. By raising the rotational speed gives rise to all natural frequencies ω which occurs as a result of the increase of the centrifugal tension force.

Table 4. Variation of the fundamental natural frequency of a rotating Timoshenko beam for various of rotational speed Ω .

P	1 st mode	2 nd mode	3 rd mode	4 th mode
0	6.121	12.998	15.89	17.012
50	6.159	13.073	15.898	17.173
100	6.197	13.147	15.91	17.330
150	6.235	13.221	15.925	17.486
200	6.272	13.295	15.942	17.639
250	6.309	13.369	15.894	17.792
300	6.346	13.443	15.979	17.944

Figure 2 gives the variation of the natural frequencies as a function of the variation of the speed of rotation. The influence of the speed of rotation on the natural frequencies of the beam is illustrated as can be seen from the Figure 2. We notice that for each speed of rotation, there are two frequencies, there are increasing values according to the speed of rotation called modes in direct precession, and other decreasing ones called retrograde modes. The gyroscopic effect causes this phenomenon. Indeed, we also notice that there is a proportional variation between the speed of rotation and the natural frequencies caused by the centrifugal stiffening

**Figure 2.** Whirl speed map of the rotating beam.

6. Conclusion.

In this paper, the natural frequencies of a non-rotating and rotating cantilevered Timoshenko beam are studied by using the DTM. The effects of the axial force and rotational speed on the natural frequencies are investigated. A good agreement is observed through a comparison of the DTM results and exact solutions from the reference [3] mentioned above.

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