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A subclass of spiral–like functions defined by generalized komatu operator with (r-k)integral operator

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Abstract : In this paper , we introduced a subclass of spiral – like functions defined by generalized Komotu operator with (R – K)integral operator with positive coefficients in the unit disk U . We obtain some new result of this class.

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Keywords : spiral – like functions, distortion bounded, growth bounded, Komotu operator,

(R – K)integral operator, Bernoulli polynomials .

1.Introduction :

Let KA denoted the class of functions defined by the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (a_n \geq 0, z \in U), \tag{1}$$

Which are analytic and univalent function . For $|J| \leq \dfrac{\pi}{2}$, a function f in the (1) is said in the class $S_{p,k}$ if and only if

$$\operatorname{Re}\left\{ e^{iJ} \frac{zf'(z)}{f(z)} \right\} > 0, \quad z \in U , \tag{2}$$

where U is unit disk defined by $U = \{z \in C : |z| < 1\}$. The class $S_{p,k}$ is a class of all J - spiral – like functions.

Let the function $g(z) = z + \sum_{n=2}^{\infty} b_n z^n, \quad b_n \geq 0, z \in U$ .The Hadamard product or convolution defined as following:

$$f(z) * g(z) = (f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n , \tag{3}$$

For $J = 0, S_{p,k}(0) = S$ is well – Known the class of functions starlike with respect to the origin . For $J \neq 0$, it is known that $S_{p,k}(J)$ is not contained in .

there are many of others studied of this object as Robertson [9] showed that the radius of starlikeness of $S_{p,k}(J)$ is $\left(\cos J + |\sin(J)|\right)^{-1}$, Spacek [10] and Zamerski [11].

Definition1. 1: The generalized integral operator of $f \in S_{p,k}$ is defined as following :

$$P_{\lambda,\alpha,\theta,k}^{\mu,\beta,\ell}\left(f(z)\right) = \frac{\theta k(\lambda - \beta + 2)^{\mu-\alpha+1}}{\ell^{\mu-\alpha+1}\Gamma(\lambda - \alpha + 1)} \int_0^1 \left[\log \frac{1}{\tau^{\ell}}\right]^{\mu-\alpha} f(\frac{z\tau}{\theta k}) d\tau \ , \tag{4}$$

where $\lambda - \alpha < 1, \ell > 0, \tau > 0, \theta > 0, k > 0$,

so from (4) , we have the following

$$P_{\lambda,\alpha,\theta,k}^{\mu,\beta,\ell}\left(f(z)\right) = z + \sum_{n=2}^{\infty} \left[\frac{\lambda - \beta + 2}{\lambda - \beta + n + 1}\right]^{\mu-\alpha+1} a_n z^n . \tag{5}$$

If $\beta = \alpha = \ell = \theta = k = 1$, we have the operator defined by Komatu operator [5] .
this type of function(spiral – like functions) is studied by many researchers such as M. Hussain [2] , K. I. Noor, Nazar Khan, M. A. Noor[8] ,K. I. Noor , Nazar Khan and Q. Z. Ahamd [7], K. I. Noor and S. Z. Bukhari[6] And gave good results used in other research.
In the following we introduced integral operator studied by R.H Buti and K.A. Jassim denoted by (R –K) integral operator .

Definition 1.2: The integral operator (R –K) involving Euler formula $e^{iA} = (cocA + i\sin A)$ is defined by :

$$[R - K]_q^{A,r,c}\left(g(z)\right) = \frac{r}{q} e^{-iA} \int_0^1 \tau^{r+1+c} g\left(\frac{qze^{iA}\tau^c}{\tau^c}\right) d\tau$$

$$= z + \sum_{n=2}^{\infty} \Psi(A,n,c,r) b_n z^n \ , \tag{6}$$

And $\Psi(A,n,c,r) = \dfrac{q^{n-1} e^{iA(n-1)}}{r(r - c(n-1))}$, $r > c(n-1), c > 0$. (7)

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So, from(5) and (6) we have the hadamard product or convolution defined by the following :

$$[RKA]^{\mu,\beta,\ell,A,r}_{\lambda,\alpha,\theta,k,c}\left(f(z)*g(z)\right)=P^{\mu,\beta,\ell}_{\lambda,\alpha,\theta,k}\left(f(z)\right)*[R-K]^{A,r,c}_{q}\left(g(z)\right)$$

$$=z+\sum_{n=2}^{\infty} D(A,n,c,r,\lambda,\beta,\mu,\alpha)a_n b_n z^n, \tag{8}$$

where $D(A,n,c,r,\lambda,\beta,\mu,\alpha)=\left[\dfrac{\lambda-\beta+2}{\lambda-\beta+n+1}\right]^{\mu-\alpha+1}*\Psi(A,n,c,r).$ (9)

Definition 1.3: A function f defined by (1)be in the $S_{p,k}(J)$ is said in the class $KAS_p(A,n,c,r,\lambda,\beta,\mu,\alpha,\theta,k,\gamma,b,\eta,J,q)=MR_p$ if satisfies the condition

$$\left|\frac{(1-b)\left[\dfrac{z\left[[RKA]^{\mu,\beta,\ell,A,r}_{\lambda,\alpha,\theta,k,c}\left(f(z)*g(z)\right)\right]'}{[RKA]^{\mu,\beta,\ell,A,r}_{\lambda,\alpha,\theta,k,c}\left(f(z)*g(z)\right)}-1\right]}{2\eta\left[\dfrac{z\left[[RKA]^{\mu,\beta,\ell,A,r}_{\lambda,\alpha,\theta,k,c}\left(f(z)*g(z)\right)\right]'}{[RKA]^{\mu,\beta,\ell,A,r}_{\lambda,\alpha,\theta,k,c}\left(f(z)*g(z)\right)}-\left(1-(1-\gamma)e^{-iJ}\dfrac{\cos J}{\cos b}\right)\right]}\right.$$

$$\left.\times\frac{1}{-\left[\dfrac{z\left[[RKA]^{\mu,\beta,\ell,A,r}_{\lambda,\alpha,\theta,k,c}\left(f(z)*g(z)\right)\right]'}{[RKA]^{\mu,\beta,\ell,A,r}_{\lambda,\alpha,\theta,k,c}\left(f(z)*g(z)\right)}-1\right]-bz^2\left[[RKA]^{\mu,\beta,\ell,A,r}_{\lambda,\alpha,\theta,k,c}\left(f(z)*g(z)\right)\right]'')}\right|<1, \tag{10}$$

and $0\le\gamma<1\quad,0<\eta\le1\quad,|J|\le\dfrac{\pi}{2}\quad,\lambda>\beta+n+1\quad 0\le b<1$.

The class $KAS_p(0,n,0,1,\lambda,1,\mu,1,1,1,\gamma,0,\eta,J,1)$ is studied by [1] .

In the following theorem , we obtain a necessarily and sufficient condition for the class belongs to the class MR_p.

Theorem 1.4 : A function defined by (1) be in the class MR_p if and only if

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$$\sum_{n=2}^{\infty}\left[(n-1)\left[(1-\eta)+\frac{b}{2}(n-1)\right]+\eta(1-\gamma)\left|e^{-iJ}\frac{\cos J}{\cos b}\right|\right]D(A,n,c,r,\lambda,\beta,\mu,\alpha)a_n b_n \le \eta(1-\gamma)\left|e^{-iJ}\frac{\cos J}{\cos b}\right| \quad (11)$$

Proof: let $|z|=1$. Then

$$\left|(1-b)\left[z\left[[RKA]_{\lambda,\alpha,\theta,k,c}^{\mu,\beta,\ell,A,r}(f(z)*g(z))\right]' - [RKA]_{\lambda,\alpha,\theta,k,c}^{\mu,\beta,\ell,A,r}(f(z)*g(z))\right]\right| -$$

$$\left|2\eta\left[z\left[[RKA]_{\lambda,\alpha,\theta,k,c}^{\mu,\beta,\ell,A,r}(f(z)*g(z))\right]' - \left(1-(1-\gamma)e^{-iJ}\frac{\cos J}{\cos b}\right)[RKA]_{\lambda,\alpha,\theta,k,c}^{\mu,\beta,\ell,A,r}(f(z)*g(z))\right] - \right.$$

$$\left. z\left[[RKA]_{\lambda,\alpha,\theta,k,c}^{\mu,\beta,\ell,A,r}(f(z)*g(z))\right]' - [RKA]_{\lambda,\alpha,\theta,k,c}^{\mu,\beta,\ell,A,r}(f(z)*g(z)) - bz^2\left[[RKA]_{\lambda,\alpha,\theta,k,c}^{\mu,\beta,\ell,A,r}(f(z)*g(z))\right]''\right|$$

$$=\left|\sum_{n=2}^{\infty}(1-b)(n-1)D(A,n,c,r,\lambda,\beta,\mu,\alpha)a_n b_n z^n\right| -$$

$$\left|2\eta z(1-\gamma)e^{-iJ}\frac{\cos J}{\cos u} - \sum_{n=2}^{\infty}[(n-1)(1-2\eta+nb) - 2\eta z(1-\gamma)e^{-iJ}\frac{\cos J}{\cos u}]D(A,n,c,r,\lambda,\beta,\mu,\alpha)a_n b_n z^n\right|$$

$$\le \sum_{n=2}^{\infty}(1-b)(n-1)D(A,n,c,r,\lambda,\beta,\mu,\alpha)a_n b_n - 2\eta(1-\gamma)\left|e^{-iJ}\frac{\cos J}{\cos u}\right| +$$

$$\sum_{n=2}^{\infty}[(n-1)(1-2\eta+nb) - 2\eta z(1-\gamma)\left|e^{-iJ}\frac{\cos J}{\cos u}\right|]D(A,n,c,r,\lambda,\beta,\mu,\alpha)a_n b_n$$

$$=\sum_{n=2}^{\infty}\left[(n-1)\left[(1-\eta)+\frac{b}{2}(n-1)\right]+\eta(1-\gamma)\left|e^{-iJ}\frac{\cos J}{\cos b}\right|\right]D(A,n,c,r,\lambda,\beta,\mu,\alpha)a_n b_n - \eta(1-\gamma)\left|e^{-iJ}\frac{\cos J}{\cos b}\right| \le 0.$$

Thus by maximum modulus theorem we have the function belong to the class.

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Conversely , let

$$\left| \frac{(1-b)\left[\dfrac{z\left[[RKA]^{\mu,\beta,\ell,A,r}_{\lambda,\alpha,\theta,k,c}(f(z)*g(z))\right]'}{[RKA]^{\mu,\beta,\ell,A,r}_{\lambda,\alpha,\theta,k,c}(f(z)*g(z))}-1\right]}{2\eta\left[\dfrac{z\left[[RKA]^{\mu,\beta,\ell,A,r}_{\lambda,\alpha,\theta,k,c}(f(z)*g(z))\right]'}{[RKA]^{\mu,\beta,\ell,A,r}_{\lambda,\alpha,\theta,k,c}(f(z)*g(z))}-\left(1-(1-\gamma)e^{-iJ}\dfrac{\cos J}{\cos b}\right)\right]}\right.$$

$$\left.\times\frac{1}{-\left[\dfrac{z\left[[RKA]^{\mu,\beta,\ell,A,r}_{\lambda,\alpha,\theta,k,c}(f(z)*g(z))\right]'}{[RKA]^{\mu,\beta,\ell,A,r}_{\lambda,\alpha,\theta,k,c}(f(z)*g(z))}-1\right]-bz^2\left[[RKA]^{\mu,\beta,\ell,A,r}_{\lambda,\alpha,\theta,k,c}(f(z)*g(z))\right]''\Big)}\right|<1,$$

$$=\left|\frac{\sum_{n=2}^{\infty}(1-b)(n-1)D(A,n,c,r,\lambda,\beta,\mu,\alpha)a_nb_nz^n}{2\eta z(1-\gamma)e^{-iJ}\dfrac{\cos J}{\cos u}-\sum_{n=2}^{\infty}[(n-1)(1-2\eta+nb)-2\eta z(1-\gamma)e^{-iJ}\dfrac{\cos J}{\cos u}]D(A,n,c,r,\lambda,\beta,\mu,\alpha)a_nb_nz^n}\right|$$

Since $|\mathrm{Re}(z)|\le|z|$ for all z , we have

$$\mathrm{Re}\left\{\frac{\sum_{n=2}^{\infty}(1-b)(n-1)D(A,n,c,r,\lambda,\beta,\mu,\alpha)a_nb_nz^n}{2\eta z(1-\gamma)e^{-iJ}\dfrac{\cos J}{\cos u}-\sum_{n=2}^{\infty}[(n-1)(1-2\eta+nb)-2\eta z(1-\gamma)e^{-iJ}\dfrac{\cos J}{\cos u}]D(A,n,c,r,\lambda,\beta,\mu,\alpha)a_nb_nz^n}\right\}<1$$

We can choose z on the real axis so that $[RKA]^{\mu,\beta,\ell,A,r}_{\lambda,\alpha,\theta,k,c}(f(z)*g(z))$ is real . Let $z\to 1$ through real value , so we get

$$\sum_{n=2}^{\infty}\left[(n-1)\left[(1-\eta)+\frac{b}{2}(n-1)\right]+\eta(1-\gamma)\left|e^{-iJ}\frac{\cos J}{\cos u}\right|\right]D(A,n,c,r,\lambda,\beta,\mu,\alpha)a_nb_n\le\eta(1-\gamma)\left|e^{-iJ}\frac{\cos J}{\cos u}\right| .$$

Corollary 1.5 : Let $f \in MR_p$, then

$$a_n \leq \frac{\eta(1-\gamma)\left|e^{-iJ}\frac{\cos J}{\cos u}\right|}{\left[(n-1)\left[(1-\eta)+\frac{b}{2}(n-1)\right]+\eta(1-\gamma)\left|e^{-iJ}\frac{\cos J}{\cos u}\right|\right]D(A,n,c,r,\lambda,\beta,\mu,\alpha)b_n} .$$

Now , we application the (H-R) fractional calculus introduced by [3].

Definition 1.6: The fractional integral of order $s(s=0,1,2,...)$ if defined by

$$_z^{-s}Df(z)=\frac{1}{\Gamma(2(s+1))}\int_0^z (z-u)^{2s+1}f(u)du \quad . \tag{12}$$

Where $f(z)$ is analytic function in simply connected region of z-plane containing the origin and the multiplicity of $(z-u)^{2s+1}$ is removed by required $\log(z-u)$ to be real when $(z-u)>0$.

Definition 1.7: The fractional derivative of order $s(s=2,3,...)$ if defined by

$$_z^{s}Df(z)=\frac{1}{\Gamma(s-1)}\frac{d}{dz}\int_0^z (z-u)^{s-1}f(u)du \quad . \tag{13}$$

Where $f(z)$ is analytic function in simply connected region of z-plane containing the origin and the multiplicity of $(z-u)^{2s+1}$ is removed by required $\log(z-u)$ to be real when $(z-u)>0$.

Definition 1.8: From above Definition , we have the following functions $G(z)$ and $H(z)$ defined as following :

$$G(z)=z^{2(s+1)}\Gamma(2+2(s+1))\,_z^{-s}Df(z)$$

$$=z+\sum_{n=2}^{\infty}\frac{n!\Gamma(2+2(s+1))}{\Gamma(n-1+2(s+1))}a_nz^n \quad . \tag{14}$$

And

$$H(z)=z^{2-z}\Gamma(s)\,_z^{s}Df(z)$$

$$=z+\sum_{n=2}^{\infty}\frac{n!\Gamma(s)}{\Gamma(n+s-1)}a_nz^n . \tag{15}$$

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In the next theorem we discuss the distortion theorem for the class MR_p .

Theorem 1.9: Let $f(z)$ defined by (1) be in the class MR_p, then

$$|G(z)| \leq |z| + \frac{\eta(1-\gamma)\left|e^{-iJ}\ \frac{\cos J}{\cos u}\right|}{\left[\left[(1-\eta)+\frac{b}{2}\right]+\eta(1-\gamma)\left|e^{-iJ}\ \frac{\cos J}{\cos u}\right|\right](2+s)D(A,2,c,r,\lambda,\beta,\mu,\alpha)b_2}$$

And

$$|G(z)| \geq |z| - \frac{\eta(1-\gamma)\left|e^{-iJ}\ \frac{\cos J}{\cos u}\right|}{\left[\left[(1-\eta)+\frac{b}{2}\right]+\eta(1-\gamma)\left|e^{-iJ}\ \frac{\cos J}{\cos u}\right|\right](2+s)D(A,2,c,r,\lambda,\beta,\mu,\alpha)b_2} .$$

Proof : From Theorem 1.4 , we have

$$\sum_{n=2}^{\infty} a_n \leq \frac{\eta(1-\gamma)\left|e^{-iJ}\ \frac{\cos J}{\cos u}\right|}{\left[\left[(1-\eta)+\frac{b}{2}\right]+\eta(1-\gamma)\left|e^{-iJ}\ \frac{\cos J}{\cos u}\right|\right]D(A,2,c,r,\lambda,\beta,\mu,\alpha)b_2} , \tag{16}$$

So , from Definition 1.8 , we have

$$G(z) = z + \sum_{n=2}^{\infty} \frac{n!\Gamma(2+2(s+1))}{\Gamma(n-1+2(s+1))} a_n z^n$$

$$= z + \sum_{n=2}^{\infty} \Phi(n.s) a_n z^n , \tag{17}$$

where $\Phi(n.s) = \dfrac{n!\Gamma(2+2(s+1))}{\Gamma(n-1+2(s+1))}$.

We see that $\Phi(n.s)$ is a decreasing function of n and

$$0 < \Phi(n.s) < \Phi(2,.s) = \frac{1}{2+s} .$$

So by using (11) and (12) , we have

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$$|G(z)| \le |z| + \frac{\eta(1-\gamma)\left|e^{-iJ}\dfrac{\cos J}{\cos u}\right|}{\left[\left[(1-\eta)+\dfrac{b}{2}\right]+\eta(1-\gamma)\left|e^{-iJ}\dfrac{\cos J}{\cos u}\right|\right](2+s)D(A,2,c,r,\lambda,\beta,\mu,\alpha)b_2}$$

And

$$|G(z)| \ge |z| - \frac{\eta(1-\gamma)\left|e^{-iJ}\dfrac{\cos J}{\cos u}\right|}{\left[\left[(1-\eta)+\dfrac{b}{2}\right]+\eta(1-\gamma)\left|e^{-iJ}\dfrac{\cos J}{\cos u}\right|\right](2+s)D(A,2,c,r,\lambda,\beta,\mu,\alpha)b_2}.$$

Theorem 1.10: Let $f(z)$ defined by (1) be in the class MR_p. Then

$$|H(z)| \le |z| + \frac{2\eta(1-\gamma)\left|e^{-iJ}\dfrac{\cos J}{\cos u}\right|}{\left[\left[(1-\eta)+\dfrac{b}{2}\right]+\eta(1-\gamma)\left|e^{-iJ}\dfrac{\cos J}{\cos u}\right|\right]sD(A,2,c,r,\lambda,\beta,\mu,\alpha)b_2}$$

And

$$|H(z)| \ge |z| - \frac{2\eta(1-\gamma)\left|e^{-iJ}\dfrac{\cos J}{\cos u}\right|}{\left[\left[(1-\eta)+\dfrac{b}{2}\right]+\eta(1-\gamma)\left|e^{-iJ}\dfrac{\cos J}{\cos u}\right|\right]sD(A,2,c,r,\lambda,\beta,\mu,\alpha)b_2}.$$

Proof : From Theorem 1.4 , we have

$$\sum_{n=2}^{\infty} a_n \le \frac{\eta(1-\gamma)\left|e^{-iJ}\dfrac{\cos J}{\cos u}\right|}{\left[\left[(1-\eta)+\dfrac{b}{2}\right]+\eta(1-\gamma)\left|e^{-iJ}\dfrac{\cos J}{\cos u}\right|\right]D(A,2,c,r,\lambda,\beta,\mu,\alpha)b_2},$$

So , from Definition1.8 , we have

$$H(z) = z^{2-z}\Gamma(s)\,{}_z^s Df(z)$$

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$$= z + \sum_{n=2}^{\infty} \frac{n!\Gamma(s)}{\Gamma(n+s-1)} a_n z^n , \tag{18}$$

Where $R(n.s) = \dfrac{n!\Gamma(s)}{\Gamma(n+s-1)}$.

We see that $R(n.s)$ is a decreasing function of n and

$$0 < R(n.s) < R(n.s) = \frac{2}{s}.$$

So by using (11) and (13) , we have

$$|H(z)| \le |z| + \frac{2\eta(1-\gamma)\left|e^{-iJ}\dfrac{\cos J}{\cos u}\right|}{\left[\left[(1-\eta)+\dfrac{b}{2}\right]+\eta(1-\gamma)\left|e^{-iJ}\dfrac{\cos J}{\cos u}\right|\right] sD(A,2,c,r,\lambda,\beta,\mu,\alpha)b_2}$$

And

$$|H(z)| \ge |z| - \frac{2\eta(1-\gamma)\left|e^{-iJ}\dfrac{\cos J}{\cos u}\right|}{\left[\left[(1-\eta)+\dfrac{b}{2}\right]+\eta(1-\gamma)\left|e^{-iJ}\dfrac{\cos J}{\cos u}\right|\right] sD(A,2,c,r,\lambda,\beta,\mu,\alpha)b_2}.$$

The proof is complete .

2.Some application on ((Bernoulli polynomial))

Definition 2.1 [4] : The Bernoulli numbers are defined by the relation

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!} , \tag{19}$$

Such that $B_n(x)(n = 0,1,...)$ is called Bernoulli polynomials and defined as following

$$B_n(x) = \sum_{n=0}^{\infty} C_n^k B_k x^{n-k} (B+x)^n , (B^k = B_k)$$

Such that $B_n(1) = \sum_{n=0}^{\infty} C_n^k B_k (B+1)^n , (B^k = B_k)$

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And so , we have

$$B_0(x) = 1$$

$$B_1(x) = x - \frac{1}{2}$$

$$B_2(x) = x^2 - x + \frac{1}{6}$$

3. Definition the Bernoulli polynomials by the generator function:

The Bernoulli polynomials defined by the generator function as following :

$$G(t,x) = \frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n(x)t^n}{n!} \quad , \tag{20}$$

Such that ,

$$G(t,0) = \frac{t}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n(0)t^n}{n!} \quad , \quad (B_n(0) = B_n) \tag{21}$$

Now, we defined $G(z,x)$ by the form :

$$G(z,x) = \frac{ze^{xz}}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n(x)z^n}{n!} . \tag{22}$$

So, from (16) , we get

$$G(z,x) = B_0(x) + B_1(x)\frac{z}{1!} + \sum_{n=2}^{\infty} \frac{B_n(x)z^n}{n!}$$

$$= 1 + z(x - \frac{1}{2}) + \sum_{n=2}^{\infty} \frac{B_n(x)z^n}{n!}$$

Now, let the function $\mathrm{K}(z,x)$ defined as following:

$$\mathrm{K}(z,x) = \frac{3}{2}z - (1 + zx) + G(z,x)$$

$$= z + \sum_{n=2}^{\infty} \frac{B_n(x)z^n}{n!} \quad , \tag{23}$$

So, the hadamard product of (18) and (1) is defined as following :

$$\Lambda(z,x) = f(z) * \mathrm{K}(z,x)$$

$$= z + \sum_{n=2}^{\infty} \frac{B_n(x)}{n!} a_n z^n \quad . \tag{24}$$

In the next theorem we show that the function defined by (23) be in the class MR_p

Theorem 3.1: The function defined by (19) be in the class MR_p, where $x = 0$.

Proof : If $x = 0$, we have from (19) the next function :

$$\Lambda(z,0) = z + \sum_{n=0}^{\infty} \frac{B_n(0)}{n!} a_n z^n . \tag{25}$$

Then , we must to show that

$$\sum_{n=2}^{\infty} \left[(n-1)\left[(1-\eta) + \frac{b}{2}(n-1)\right] + \eta(1-\gamma)\left| e^{-iJ} \frac{\cos J}{\cos u} \right| \right] D(A,n,c,r,\lambda,\beta,\mu,\alpha) \frac{B_n(0)}{n!} a_n b_n ,$$
$$\leq \eta(1-\gamma)\left| e^{-iJ} \frac{\cos J}{\cos u} \right| \quad .$$

So,

$$\sum_{n=2}^{\infty} \left[(n-1)\left[(1-\eta) + \frac{b}{2}(n-1)\right] + \eta(1-\gamma)\left| e^{-iJ} \frac{\cos J}{\cos u} \right| \right] D(A,n,c,r,\lambda,\beta,\mu,\alpha) \frac{B_n(0)}{n!} a_n b_n$$
$$\leq \eta(1-\gamma)\left| e^{-iJ} \frac{\cos J}{\cos u} \right| \quad .$$

since $\frac{B_n(0)}{n!} \leq 1$, for all $n \geq 2$, we have

$$\leq \sum_{n=2}^{\infty} \left[(n-1)\left[(1-\eta) + \frac{b}{2}(n-1)\right] + \eta(1-\gamma)\left| e^{-iJ} \frac{\cos J}{\cos u} \right| \right] D(A,n,c,r,\lambda,\beta,\mu,\alpha) a_n b_n \leq \eta(1-\gamma)\left| e^{-iJ} \frac{\cos J}{\cos u} \right| \quad .$$

Then the function $\Lambda(z,0)$ is in the class MR_p.

The proof is complete .

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