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Mathematical Analysis of Fourier Expansion Using Gauss Partial Sum

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Abstract

Fourier series is one of major specific and standard series of periodic functions that can be found in a wide range of theoretical and practical applications. Despite the Fourier series is old but still brand new in their practical applications especially in the field of communications. The present paper is a new version of the mathematical analysis for such a series using the partial Gauss sum. The functions that will be discussed herein are uniformly continuous and periodic. It is important also that by studying extension of the Fourier series sum for periodic functions, this will be helpful for one to be able to extend Gauss partial sum. The theoretical implementation is proved by solving an example and the results seem to be acceptable.

Key Words: Fourier series, Fourier coefficients, Gauss Partial Sum, continuous functions, piecewise continuous function

1. Introduction

Fourier series, presented by the French scientist Joseph Fourier, is one of the most important mathematical tools used by communications engineers in their applications [1-4]. The series can write any mathematical function through a set of Sine, Cosine. Because many physical phenomena are important phenomena such as the flow of fluids [5-7], electrical and magnetic solutions for such phenomena require the expression of some functions periodically and can be through chains called chains Fourier so we will study one of the ways by which the expression of certain functions in terms of this pattern of The strings are called strings Fourier and we apply these strings to find solutions to many of the problems. Under certain conditions, the function can be expressed as an infinite series containing the corner pockets and the pockets of the whole angle or multiples of these angles [8-11]. In mathematics, Fourier series is a method that allows the writing of any periodic mathematical function in a sequential form or sum of the functions of the sinus and cosine multiplied by a given parameter [12-13]. Fourier transform is a mathematical process used to convert mathematical functions from time domain to frequency domain. It is useful for analyzing signals and identifying the frequencies they contain, and has an application in solving differential equations. The name of the process is derived from the name of the French scientist Fourier [13].

2. Mathematical Formulation

Let the analysis starts by writing the general formula for Fourier series, defined on a general interval.

$$\frac{a_0}{2} + \frac{a_o}{2} \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad (1)$$

This series is also known as Fourier trigonometric series, the parameters $a_o, a_n, b_n, n \in N$ are the coefficients of trigonometric series. If the series is convergent, then its sum is periodic function as $f(x)$ with period 2π , and as the functions and $\cos nx \sin nx$ are convergent with period 2π therefore:

$$f(x) = f(x + 2\pi) \quad (2)$$

Now, a question will appears directly, that is, 'what conditions should be satisfied that make the Fourier series converges into its sum equal with the considered function at the corresponding points?.

To distinguish the satisfying condition showing the function $f(x)$ by its Fourier series, we will present the following theorem:



3. Mathematical Analysis

Theorem

Consider

$$S(x) = \sum_{m \leq x} \left(\frac{m+a_1}{p_1} \right) \left(\frac{m+a_2}{p_2} \right) \dots \left(\frac{m+a_k}{p_k} \right) \quad (3)$$

Where

$$k \in N \quad (4-1)$$

$$p_1, p_2, \dots, p_r \quad (4-2)$$

$$Q = p_1 p_2 \dots p_r \quad (4-3)$$

$$1 \leq x \leq Q, Q \in R \quad (4-4)$$

$$a_1, a_2, \dots, a_r \in Z \quad (4-5)$$

Assume that the function:

$$S_o(x) = \frac{s(x+0) + s(x-0)}{2} \quad (5)$$

Equation (5) is a periodic function with period Q , therefore, the extension of $S_o(x)$ may be appeared as follows:

$$S_o(x) = \sum_{n=-\infty}^{n=+\infty} c_n e^{\frac{2\pi i n x}{Q}} \quad (6)$$

The series in equation (6) is convergent Fourier series; furthermore, the Fourier coefficients can be calculated as following form:

$$C_n = \begin{cases} \frac{1}{2\pi i n} \prod_{s=1}^k \left(e^{2\pi i n \frac{Q'_s a'_s}{P_s}} \left(-n \frac{Q'_s}{p_s} \right) \Gamma_p \right) & \text{For } n \neq 0 \\ \sum_{n=1}^Q \left(1 - \frac{n}{Q} \right) \left(\frac{n+a_1}{p_1} \right) \dots \left(\frac{n+a_k}{p_k} \right) & \text{For } n = 0 \end{cases} \quad (7)$$

By using the features of Gauss sum and Vinograd, the coefficients C_n may be calculated.

Proof

Consider the function $S(x)$ which is as sum of Legendre symbol such that:

$$S(x) = \sum_{m \leq x} \left(\frac{m+a_1}{p_1} \right) \left(\frac{m+a_1}{p_1} \right) \dots \left(\frac{m+a_k}{p_k} \right) \quad (8)$$

Define $S_o(x)$ in which converges into $S(x)$ such that:

$$S_o(x) = \frac{s(x+0) + s(x-0)}{2} \quad (9)$$

is periodic function with periodic Q and the function $S_o(x)$ is piecewise uniformly continuously on their period. Thus, the sum of obtained sum equal the value of function at the continuously points and of the des continuously point equal white the average right & left limit of function.

Therefore, the obtained function series is converged at every point:

$$S_o(x) = \sum_{n=-\infty}^{n=+\infty} C_n \exp\left(\frac{2\pi i n x}{Q}\right) \quad (10)$$

Now, we can calculate the Fourier coefficients as following form:

$$C_0 = \frac{1}{Q} \int_0^Q S_0(x) dx = \left(\frac{1}{Q} \right) \sum_{m=1}^Q \sum_{n \leq m} \left(\frac{n+a_1}{p_1} \right) \left(\frac{n+a_2}{p_2} \right) \dots \left(\frac{n+a_k}{p_k} \right) \quad (11)$$

$$= \sum_{n=1}^Q \left(1 - \frac{n}{Q} \right) \left(\frac{n+a_1}{p_1} \right) \dots \left(\frac{n+a_k}{p_k} \right)$$

$$c_n = \frac{1}{Q} \int_0^Q S_0(x) e^{-2\pi i \frac{nx}{Q}} dx = - \left(\frac{1}{2\pi i n} \right) \int_0^Q S_0(x) d e^{-\frac{2\pi i nx}{Q}}$$

$$= - \left(\frac{1}{2\pi i n} \right) \left\{ S_0(x) e^{-\frac{2\pi i nx}{Q}} \right\}_0^Q + \left(\frac{1}{2\pi i n} \right) \int_0^Q e^{-\frac{2\pi i nx}{Q}} d S_0(x) \quad (12)$$

$$= \frac{1}{2\pi i n} \sum_{m=1}^Q \left(\frac{m+a_1}{p_1} \right) \dots \left(\frac{m+a_k}{p_k} \right) e^{-\frac{2\pi i m n x}{Q}}$$

Assume that:

$$P_s Q_s = Q, \quad s = 1, 2, 3, \dots, k \quad (13)$$

Therefore, each reminisces m and module Q we find a unique series $m_1, m_2, m_3, \dots, m_k$ such that:

$$m = m_1 Q_1 + m_2 Q_2 + m_3 Q_3 + \dots + m_k Q_k \quad (14)$$

Where

$$0 < m_1 < p_1$$

$$0 < m_2 < p_2$$

$$0 < m_3 < p_3 \quad (15)$$

.....

$$0 < m_k < Q_k$$

$$c_n = \left(\frac{1}{2\pi i n} \right) \sum_{m=1}^Q \left(\frac{m_1 Q_1 + a_1}{p_1} \right) \left(\frac{m_2 Q_2 + a_2}{p_2} \right) \dots \left(\frac{m_k Q_k + a_k}{p_k} \right) \quad (16)$$

$$\times \exp \left((-2\pi i n) \left(\frac{m_1 Q_1 + \dots + m_k Q_k}{Q} \right) \right) = \frac{1}{2\pi i n} \prod_{s=1}^k A_s$$

$$A_s = \sum_{m_s=0}^{p_s-1} \left(\frac{m_s Q_s + a_s}{p_s} \right) \exp \left(\frac{-2\pi i n m_s}{p_s} \right) \quad (17)$$

Consider $S = 1, 2, 3, \dots, k$ and moreover there is the equivalent relates for reminisce Q_s with model P_s , then:

$$Q_s Q'_s = 1 \pmod{P_s}$$

$$\Gamma_P = \sum_{m_s=0}^{p_s-1} \left(\frac{m}{p_s} \right) \exp \left(\frac{-2\pi i n m_s}{p_s} \right) \quad (18)$$

We can consider the Gauss sum with model P :

$$\begin{aligned}
A_s &= \sum_{m_s=0}^{p_s-1} \left\{ \left(\frac{m_s Q_s + a_s}{p_s} \right) \exp \left\{ \left(\frac{-2\pi i n Q_s'}{p_s} \right) \left(\frac{m_s Q_s + a_s}{p_s} \right) \right\} \right. \\
&\quad \left. \times \exp \left\{ \left(\frac{-2\pi i n Q_s'}{p_s} \right) \left(\frac{a_s}{p_s} \right) \right\} \right\} \\
&= \exp \left(\frac{2\pi i n Q_s'}{p_s} \right) \sum_{m_s=0}^{p_s-1} \left\{ \left(\frac{m}{p_s} \right) \exp \left(\frac{-2\pi i n Q_s'}{p_s} \right) \right\} \\
&= \exp \left(\frac{2\pi i n Q_s'}{p_s} \right) \sum_{m_s=0}^{p_s-1} \left\{ \exp \left(\frac{2\pi i n Q_s' a_s}{p_s} \right) \left(\frac{-n Q_s'}{p_s} \right) \right\} \Gamma_{p_s} \\
&= \frac{1}{2\pi i n} \prod_{s=1}^k \exp \left(\frac{2\pi i n Q_s' a_s}{p_s} \right) \left(\frac{-n Q_s'}{p_s} \right) \Gamma_{p_s} \tag{19}
\end{aligned}$$

By the end of equation (19), the proof is completed. As regarding the above theorem one may present the following example.

Example

Assume The Gauss sum is given as:

$$G(x) = \sum_{m \leq x} \left\{ \exp \left(\frac{2\pi i m^2}{N} \right) \right\}$$

$$1 < x < N$$

$$x \in \mathbb{Z}$$

$$N : \text{odd}$$

Then there is a function as

$$G_0(x) = \frac{G(x+0) + G(x-0)}{2}$$

Such that we can extend it as Fourier convergent series, moreover, its coefficients may found as

$$G_0(x) = \sum_{n=-\infty}^{+\infty} g_n \exp\left(\frac{-2\pi i n x}{N}\right)$$

$$g_n = \frac{1}{N} \int_0^N G_0(x) \exp\left(\frac{-2\pi i n x}{N}\right) dx$$

$$g_0 = \sum_{m \leq N} \left(1 - \frac{m}{N}\right) \exp\left(\frac{-2\pi i m^2}{N}\right)$$

$$g_{2n} = \left[\frac{G}{4\pi i n}\right] \exp\left(\frac{-2\pi i n^2}{N}\right), n \neq 0$$

$$g_{2n+1} = \frac{H}{2\pi i (2n+1)} \exp\left[-2\pi i \left(\frac{2n+1}{4n}\right)^2\right]$$

$$G = \sum_{m=1}^N \exp\left(\frac{2\pi i m^2}{N}\right)$$

$$H = \sum_{m=1}^N \exp\left(\frac{2\pi i \left(m - \frac{1}{2}\right)^2}{N}\right)$$

Define $G_0(x)$ as:

$$G_0(x) = \frac{G(x+0) - G(x-0)}{2}, \quad 1 < x < N$$

Which is a periodic function with period N and it is piecewise continuously uniform such that it can be extended as Fourier series.

$$G_0(x) = \sum_{n=-\infty}^{+\infty} g_n \exp\left(\frac{2\pi i n x}{N}\right)$$

$$g_n = \frac{1}{N} \int_0^N G_0(x) \exp\left(\frac{-2\pi i n x}{N}\right) dx$$

$$\begin{aligned} g_0 &= \frac{1}{N} \int_0^N G_0(x) dx = \frac{1}{N} \sum_{K \leq N} \sum_{m \leq K} \exp\left(\frac{2\pi i m^2}{N}\right) \\ &= \sum_{m \leq N} \left(1 - \frac{m}{N}\right) \exp\left(\frac{2\pi i m^2}{N}\right) \end{aligned}$$

$$\begin{aligned}
g_n &= \left(-\frac{1}{2\pi in}\right) \int_0^N G_0(x) \exp\left(\frac{-2\pi inx}{N}\right) dx, n \neq 0 \\
&= \left(\frac{1}{2\pi in}\right) \int_0^N \exp\left(\frac{-2\pi inx}{N}\right) dG_0(x) \\
&= \left(\frac{1}{2\pi in}\right) \sum_{m=1}^N \exp\left(\frac{2\pi i(m^2 - mn)}{N}\right) \\
g_{2n} &= \left(\frac{G}{4\pi in}\right) \exp\left(\frac{-2\pi in^2}{N}\right)
\end{aligned}$$

$$g_{2n+1} = \frac{H}{2\pi i(2n+1)} \exp\left(\frac{-2\pi i(2n+1)^2}{4N}\right)$$

$$H = \sum_{m=1}^N \exp\left(\frac{2\pi i}{N} \left(m - \frac{n}{2}\right)^2\right)$$

4. Conclusion

As it has been seen throughout the text of the paper, and the mathematical manipulation of the Fourier series using Gauss partial sum, one can conclude that it is an easy in treatment in applications where using the Fourier series in its classical form makes confusion throughout the application treatment. Also, it has been noticed a little number of conditions should be satisfied to use the corresponding Gauss sum of the original Fourier series. The results due to the solved example showed an excellent agreement with the theoretical assumption and give a promise for more analysis.

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