IOP Conference Series: Materials Science and Engineering

Essential-small submodules relative to an arbitrary submodule

IOP Conf. Series: Materials Science and Engineering **571** (2019) 012030 doi:10.1088/1757-899X/571/1/012030

Essential-small sub modules relative to an arbitrary submodule

M. S. ABBAS

Department of Mathematics, College of Science, Mustansiriyah University, Baghdad, Iraq.

mhdsabass@gmail.com

Abstract. Let R be an arbitrary ring and T a submodule of an R-module M. A submodule N is said to be Te-small in M, if for each essential submodule X of M, T ⊆ N + X implies that T ⊆ X. In this work we study this mentioned notion which is a generalization of the essential-small submodules as well as the T-small submodule. We use this notion to investigate T-essential radical of module, also to introduce generalized T-essential Hopfian modules.

1. Introduction

Throughout this work, all rings are associative with nonzero identity and all modules are unitary left R-modules. We use the notation " ⊆ " and " ≤ "to denote inclusion and submodule, respectively. Let R be a ring and M an R-module. Recall that a submodule N of M is small, denoted by $N \ll M$, if for any submodule X of M, M = N + X implies that X = M. Dually, a submodule N is essential in an R-module M, if for any submodule K of M, N ∩ K = 0 implies that K = 0. In this case we denote $K \trianglelefteq M$. For more details about small and essential submodule see [1]. The notion of small submodules plays an important role in ring and module theory. D. X. Zhou and X.R.Zhang [4] generaliz the concept of small submodules to that of essential-small by considering the class of all essential submodules in place of all submodules. Let N be a submodule of an R-module M. N is called essential-small in M denoted by $N \ll_e M$ if N + L = M then L = M for all essential submodule L of M. Also R. Beyranvand and F. Moradi [2] generalize the notion of small submodules by replacing an arbitrary submodule T (say) instead M. Let T be an arbitrary submodule of an R-module M. A submodule N of M is called T-small in M if for each submodule X of M, T ⊆ N + X implies T ⊆ X. The notion of smallness and T-smallness are coincide if T = M.

The concept of essential-smallness and T-smallness are investing to investigate some radicals of modules. In [2] the authors define the essential radical of an R-module M, denoted by $\mathrm{Rad}_e(M)$ as $\mathrm{Rad}_e(M) = \cap\{N \leq M \mid N \text{ is essential and maximal in } M\}$, and they proved that radical is equivalent to the sum of all essential-small submodules of M. While in [4], they proved the following. Let T be a nontrival finitely generated submodule of an R-module M. Then $\bigcap_{k\in B} K = \sum_{L\in A} L$ where B = { K ≤ M | K is a T-maximal submodule of M} and A = { L ≤ M | L is a T-small in M and L + K ⊆ T + K, for all T-maximal submodule K of M}. $\bigcap_{k\in B}$ Kis called the T-radical of M and we denoted by $J_T(M)$.
This motivates us to define a new generalization of T-small submodules as well as of essential-small submodules. Let T be an arbitrary submodule of an R-module M. We say that a submodule N of M is an T-essential-small of M provided that T ⊆ N+X implies that T ⊆ X for all essential submodule X of M, Note that, every T-small submodule is T-essential-small and every essential-small submodule is M-essential-small.

In the first section, we investigate the basic properties of T-essential-small submodules. In section two, we use the notion of T-maximal submodule [2], and introduce the T-essential radical of modules, T-essential maximal submodules, we used this new class of submodules to investigate another radical of modules. Also introduce T-essential cosemisimple module and give some of their properties and characterizations. Finally in section three, we introduce the notion of generalized T-essential (T-

IOP Conf. Series: Materials Science and Engineering **571** (2019) 012030 doi:10.1088/1757-899X/571/1/012030

essential-closed) Hopfian modules and give their characterizations in terms of T-essential-small submodules.

§1 .T-essential-small submodules.

In this section, as a generalization of essential-small submodules and T-small submodules, T-essential-small submodules isintroduced, and their various properties are given.

Definition. (1.1). Let Tbe an arbitrary submodule of an R-module M. A submodule N of M is called T-essential-small in M (simply Te-small), denote by $N \ll_{Te} M$, if for each essential submodule X of M, $T \subseteq N + X$ implies $T \subseteq X$.

According to the definition, if T = 0, then every submodule of M is Te-small in M. Furthermore if T = M, then $N \ll_{Te} M$ if and only if $N \ll_e M$.It is clear that $0 \ll_{Te} M$ and$M \not\ll_{Te} M$ for any submodule T of M.

Examples and Remarks (1.2).
1. It is clear that every T-small submodule of an R-module M is Te-small. The converse is true on uniform modules.
2. For any positive integer m, the zero submodule is the only $(m\mathbb{Z})$e-small in the $\mathbb{Z}$-module$\mathbb{Z}$.
3. Let $\mathbb{Z}_{p^n} = < 1 / p^n + \mathbb{Z} >$ and $\mathbb{Z}_{p^m} = < 1 / p^m + \mathbb{Z} >$ be submodules of the $\mathbb{Z}$-module $\mathbb{Z}_{p^\infty}$. Then m > n if and only if $\mathbb{Z}_{p^n} \ll_{(\square_{p^m})e} \mathbb{Z}_{p^\infty}$
4. Let M be a semisimple R-module. Since M is the only essential submodule of M, then every proper submodule of M is Te-small.
5. Consider the $\mathbb{Z}$-module $\mathbb{Z}_{24}$and for T = $2\mathbb{Z}_{24}$, $8\mathbb{Z}_{24} \ll_{Te} \mathbb{Z}_{24}$, but $8\mathbb{Z}_{24}$is not T-small in$\mathbb{Z}_{24}$

Proposition. (1.3). Let L, T, and K be submodules of an R-module M with $L \subseteq T$. Then
1. $K \ll_{Te} M$ implies that $K \cap T \ll_e M$.
2. $L \ll_{Te} M$ if and only if $L \ll_e T$.

Proof.1. Let X be an essential submodule of M and $(K \cap T) + X = M$. Then $T \subseteq (K \cap T) + X \subseteq K + X$ and since $K \ll_{Te} M$, then $T \subseteq X$. Thus $K \cap T \subseteq X$ and hence $X = (K \cap T) + X = M$.
2. Suppose $L \ll_{Te} M$ and $L + X = T$ for essential submodule X of T. Then $T \subseteq L + X$ implies that $T \subseteq X$. Thus T = X. Conversely, suppose $L \ll_e T$and $T \subseteq L + X$ for essential submodule X of M. Then $T = (L+X) \cap T = L + (X \cap T)$ and hence $X \cap T = T$, so $T \subseteq X$. □

Proposition. (1.4). Let M be an R-module with submodule N, K,T and T,N ⊆ K. If $N \ll_{Te} K$, then $N \ll_{Te} M$.

Proof. Assume $T \subseteq N + X$ for some essential submodule X of M. Then $T \subseteq (N + X) \cap K = N + (X \cap K)$ and hence $T \subseteq X \cap X$ □

Proposition. (1.5). Let N, K and T be submodules of an R-module M. Then $N \ll_{Te} M$ and $K \ll_{Te} M$ if and only if $N + K \ll_{Te} M$

Proof. Let X be an essential submodule of M with $T \subseteq (N + K) + X$. Then $T \subseteq K + X$ and hence $T \subseteq X$. Conversely, if $T \subseteq N + X$ and $T \subseteq K + X$, then $T \subseteq (N+K) + X$ which implies that $T \subseteq X$. □

Proposition. (1.6). Let M be an R-module with submodules N, K and T such that $K \subseteq N$ and $K \subseteq T$. If $N \ll_{Te} M$, then $K \ll_{Te} M$ and $N/K \ll_{(T/K)e} M/K$.

Proof. Assume that $N \ll_{Te} M$. For each essential submodule X of M, if $T \subseteq K + X$, then $T \subseteq N + X$ and hence $T \subseteq X$, so $K \ll_{Te} M$. Suppose $T/K \subseteq N/K + X/K$ for some essential submodule X/K of M/K. Then $T \subseteq N + X$ and so $T \subseteq X$ which implies that $T/K \subseteq X/K$. This shows that $N/K \ll_{(T/K)e}$ M/K □

Proposition. (1.7). Let M be an R-module with $N_i \leq M_i \leq M$ (i = 1,2) such that $T \subseteq M_1 \cap M_2$. Then $N_i \ll_{Te} M_i$ (i = 1, 2) if and only if $N_1 + N_2 \ll_{Te} M_1 + M_2$.

Proof. Let $N_i \ll_{Te} M_i$(i = 1, 2). By proposition.(1.4), $N_i \ll_{Te} M_1 + M_2$.By the help of proposition.(1.5), $N_1 + N_2 \ll_{Te} M_1 + M_2$. The other direction is clear. □

Proposition. (1.8). Let M and N be R-modules and $\alpha : M \rightarrow N$ an R-homomorphism. If K and T are submodule of M with $K \ll_{Te} M$, Then $\alpha(K) \ll_{\alpha(T)e} N$. In particular, if $K \ll_{Te} M \leq N$, then $K \ll_{Te} N$.

Proof. Let X be an essential of N such that $\alpha(T) \subseteq \alpha(K) + X$. If $t \in T$, then $\alpha(t) = x + \alpha(k)$ for some $x \in X$ and $k \in K$. Thus $\alpha(t\text{-}k) \in X$ and so $t - k \in \alpha^{-1}(X)$ and hence $T \subseteq K + \alpha^{-1}(X)$. By Te-smallness of K in M, we have $T \subseteq \alpha^{-1}(X)$ and $\alpha(T) \subseteq X$. □

Theorem. (1.9). Let M be an R-module with submodules N, T and α a surjective endomorphism of M whose kernel is closed. Then $N \ll_{Te} M$ if and only if $\alpha(N) \ll_{\alpha(T)e} M$.

Proof. The " only if " part follows from proposition. (1.8). Conversely, for essential submodule X of M suppose that $T \subseteq N + X$. Then $\alpha(T) \subseteq \alpha(N) + \alpha(X)$. There is an isomorphism $\overline{\alpha} : M / \ker(\alpha) \rightarrow N$such that $\overline{\alpha} \circ \pi = \alpha$ where $\pi : M \rightarrow M / \ker(\alpha)$ is the natural epimorphism. Now,$\alpha(X) = \overline{\alpha}(X/ \ker(\alpha))$. Since ker(α) is closed, then by ([3], proposition (1-4)) we have α(X) is essential in M. By α(T)e-maximality of α(N) in M we get $\alpha(T) \subseteq \alpha(X)$ and hence $T \subseteq X$. This completes the proof. □

Proposition. (1.10). Let $\{T_\alpha\}_{\alpha \in \Lambda}$ be an indexed family of submodules of an R-module M and N a submodule of M. If $N \ll_{T\alpha e} M$ for each $\alpha \in \Lambda$, then $N \ll_{(\sum_{\alpha \in \Lambda} T_\alpha)e} M$.

Proof. For an essential submodule X of M, assume that $\sum_{\alpha \in \Lambda} T_\alpha \subseteq N + X$. Then for each $\alpha \in \Lambda$, $T_\alpha \subseteq N + X$. As $N \ll_{(T_\alpha)e} M$, then $T\alpha \subseteq X$ for each $\alpha \in \Lambda$ and hence $\sum_{\alpha \in \Lambda} T_\alpha \subseteq X$ □

Corollary. (1.11). Let N_1 and N_2be two submodules of an R-module M. If N_1 and N_2 are matually essential-small in M, then $N_1 \cap N_2 \ll_{(N1+N2)e} M$.

Proof. Assume that $N1 \ll_{N_2e} M$ and $N2 \ll_{N_1e} M$. By proposition.(1.6). $N_1 \cap N_2 \ll_{N_1e} M$ and $N_1 \cap N_2 \ll_{N_2e} M$.So proposition. (1.10) implies that $N_1 \cap N_2 \ll_{(N1+N2)e} M$ □

Proposition. (1.12). Let M be an R-module with submodules N and T(≠ 0). Then the following are equivalent

1. $N \ll_{Te} M$

2. For any R-module L and essential R-homomorphism $\alpha : L \rightarrow M$, $T \subseteq N + \alpha(L)$ implies that $T \subseteq \alpha(L)$.

Proof.(1) → (2). It is clearbe the definition. (2) → (1). Suppose $T \subseteq N + X$ for essential submodule X of M. Let $i : X \rightarrow M$ be the inclusion mapping. Then by (2) $T \subseteq N + X = N + i(X)$ implies that $T \subseteq X$.

Let M be an R-module with submodules N and T. A submodule N' of M is called Te-supplement of N in M, if N' is minimal essential submodule with the property $T \subseteq N + N'$.

Proposition. (1.13). Let M be an R-module with submodules N, N', T and N' is Te-supplement of N in M. If $N \ll_{Te} M$, then $T \subseteq N'$. If in additional, T is essential, then T = N'.

Proof. Since $T \subseteq N + N'$ and $N \ll_{Te} M$, then $T \subseteq N'$. Furthermore, If T is essential in M, then minimality of N' and $T \subseteq N + T$ implies that T = N' □

Theorem. (1.14). Let M be an R-module with submodules K,T and K' an T-supplement of K in M. Then $K \ll_{K'e} M$ if and only if for each essential submodule N of M, $T \subseteq K + N$ implies that $K' \subseteq N$.

Proof. The " only if " part is clear from the definition. For the " if " part, let X be an essential submodule of M with $K' \subseteq X + K$. Since $T \subseteq K + K' \subseteq X + K$, by the hypothesis, $K' \subseteq X$. □

§2 . T-essential radicals of module

Let M be an R-module and T a submodule of M. Recall that a submodule K of M is T-maximal if $T \nsubseteq K$ and there exists no proper submodule W of K + T which contain K properly [2]. This is equivalent to saying that (K + T)/K is a simple R-module. It is clear that a submodule N is maximal in M if and only if N is M-maximal.

For an R-module M, and a submodule T of M, consider the following two families of submodules

$$V = \{K \leq M \mid K \text{ essential and T-maximal in } M\}$$
$$\text{and } W = \{L \ll_{Te} M \mid L + K \subseteq T + K \text{ for all T-maximal submodule K in M}\}$$

Theorem. (2.1). Let Mbe an R-module and T a nontrivial finitely generated submodule of M. Then $\bigcap_{k\in V} K = \sum_{L\in W} L$

Proof. Let $L \in W$. We show that $L \subseteq K$ for each $K \in V$. If not then $K \subsetneqq L + K \leq T + K$. Since K is T-maximal we have L + K = T +K and hence $T \subseteq L + K$. But $L \ll_{Te} M$, then $T \subseteq K$ and hence (T+K)/K = 0 with a contradiction. Thus $\sum_{L\in W} L \subseteq \bigcap_{k\in V} K$ Conversely, let $x \in \bigcap_{k\in V} K$. we show $Rx \in V$, for each essential submodule X of M, suppose $T \subseteq Rx + X$ and $T \nsubseteq X$. Consider the following family $\mathbb{C} = \{K \leq M \mid K \text{ is essential in M}, T \nsubseteq K \text{ and } X \subseteq K\}$. It is clear that $\mathbb{C}$ is nonempty family and we can order $\mathbb{C}$ by inclusion. Let $T = \sum_{i=1}^{n} Rx_i$, where $x_1, x_2, \ldots, x_n \in M$. Let $\mathbb{C}'$ be a chain in $\mathbb{C}$. It is clear that $N \subseteq \bigcup_{k\in \mathbb{C}'} K \leq M$. if $T \subseteq \bigcup_{k\in \mathbb{C}'} K$, then there exists $\{K_1, K_2, \ldots, K_n\} \subseteq \mathbb{C}'$ such that for any $1 \leq i \leq n$, $x_i \in K_i$, we may assume $K_i \leq K_n$ for all $1 \leq i \leq n$. Thus $T \subseteq K_n$which is a contradiction. Thus $T \nsubseteq \bigcup_{k\in \mathbb{C}'} K$ and hence $\bigcup_{k\in \mathbb{C}'} K \in \mathbb{C}$ and an upper bound of $\mathbb{C}'$. By Zorn's lemma $\mathbb{C}$ has a maximal element K_0 (say). We claim that K_0 is T-maximal. First we note that K_0 is an essential submodule of M and $(T+K_0)/K_0 \neq 0$. Assume that $K_0 \subsetneqq U \leq T + K_0$. By maximality of K_0, we get $T \subseteq U$ and hence $U = T + K_0$. Thus K_0 is T-maximal and $x \in K_0$ which is a contradiction, because $x \in \bigcap_{k\in W} K$. This shows that $Rx \ll_{Te} M$. on the other hand, for any T-maximal submodule K of M, K = Rx+ K and so $Rx \in V$. Therefore $\bigcap_{k\in V} K \subseteq \sum_{L\in W} L$ □

Let M be an R-module with a submodule T. we denote the intersection of all essential T-maximal submodules of M by $J_{Te}(M)$, and call it the T-essential radical of M. By the proof of theorem. (2.1), we get the following evident result

Corollary. (2.2). Let Mbe an R-module with nontrivial finitely generated submodule T. Then for any x ∊ M and all T-maximal submodule K of M,x∊J_{Te}(M) if and only if Rx $\ll_{Te}$ M and Rx + T ⊆ T + K. Let M be an R-module. Then we have the following inclusion relation J_T(M) ⊆J_{Te}(M) for any nontrivial finitely generated submodule T of M.

Definition. (2.3). Let Mbe an R-module and T a submodule of M. A submodule K of M is called Te-maximal, if T ⊈ K and there is no essential submodule W of M with the property K ⪇ W ⪇ K + T. This equivalent to saying that (K + T)/K is nozero and (K + T)/K is the only essential submodule of (K + T)/K.

Let M be an R-module and T a submodule of M. Consider the following two families of submodules of M.

V' = {K ≤ M | K is Te-maximal submodule of M}

W' = { L⊴ M | L $\ll_T$ M, L + K ⊆ T + K for all Te-maximal submodule K of M}

Theorem. (2.4). Let Mbe an R-module and T a nontrivial finitely generated submodule of M. Then $\sum_{L\in W'} L \subseteq \bigcap_{k\in v'} K \subseteq \sum_{L\in W''} L$ where

W'' = {L $\ll_{Te}$ M | L + K ⊆ T + K for all Te-maximal submodule K of M}.

Proof. Let L ∊ W' and there is a submodule K ∊ V' such that L ⊈ K. Then K ⪇ L + K ≤ T + K. Since K is Te-maximal in M, we have L + K = T + K and hence T⊆ L + K. Since L $\ll_T$ M, then T ⊆ K wich is a contradiction. Thus L ⊆ K for each Te-maximal submodule K of M and hence $\sum_{L\in W'} L \subseteq \bigcap_{K\in V'} K$. Let x $\in \bigcap_{K\in V'} K$. We show that Rx ∊ W''. Suppose that there is an essential submodule X of M with T ⊆ Rx + X and T ⊈ X. Consider the following family ℂ = {K ⊴ M | X ⊆ K and T ⊈ K}. Then ℂ is nonempty and we can ordered ℂ by inclusion. Let ℂ' = {K_α∊ ℂ | α∊ ʌ} be a chain in ℂ. It is clear that X $\subseteq \bigcup_{\alpha\in\Lambda} K_\alpha$ ⊴ M. T = $\sum_{i=1}^{n} Rx_i$ where x_1, x_2, …,x_n∊ T. If T $\subseteq \bigcup_{\alpha\in\Lambda} K_\alpha$, then there exists {$K_{\alpha 1}$, $K_{\alpha 2}$, …, $K_{\alpha n}$} ⊆ ℂ' . For any 1 ≤ i ≤ n , x_i∊$K_{\alpha i}$, we may assume that $K_{\alpha i}$ ≤ $K_{\alpha n}$for all 1 ≤ i ≤ n. Thus T ⊆$K_{\alpha n}$which is a contradiction. Thus T $\nsubseteq \bigcup_{\alpha\in\Lambda} K_\alpha$ and hence $\bigcup_{\alpha\in\Lambda} K_\alpha$ ∊ ℂ and an upper bound of ℂ'. By Zorn's Lemma ℂ has a maximal element K_0 (say). We claim that K_0 is Te-maximal in M. it is clear T ⊈ K_0. Assume that K_0 ⪇ U ≤ K_0 + T, where U is an essential submodule of M. By maximalty of K_0, we have T ⊆ U and hence U = K_0 + T. Thus K_0 is Te-maximal in M and x ∊ K_0 which is a contradiction, because x $\in \bigcap_{K\in V'} K$. This shows that Rx $\ll_{Te}$ M. On other hand, for any Te-maximal submodule K of M, K = Rx + K ⊆ T + K and so Rx ∊ W'' . □

In the following proposition we see the behavior of Te-maximal submodules under homomorphisms.

Proposition. (2.5). Let M and N be R-modules and α : M → N an R-homomorphism. If T is a submodule of M and K is a Te-maximal submodule of M with ker(α) ⊆ K, then α(K) is α(T)e-maximal in N.

Proof. If α(T) ⊆α(K), then for each t ∊ T, α(t) = α(k) for some k ∊ K and hence t - k ∊ker(α) ⊆ K. Thus t ∊ K which is a contradiction. Now, let W be an essential submodule of N with α(K) ⪇ W ≤ α(K + T). Then K = α^{-1}(α(K)) ≤ α^{-1}(W) ≤ K + T. On the other hand K ⪇α^{-1}(W), if not, that is K = α^{-1}(W) and since α(K) ⪇ W, there exists w ∊ W\\α^{-1}(K). But W ⊆α(K + T), this implies that w = α(k + t) for some k ∊ K and t ∊T. Thus k + t ∊α^{-1}(W), and hence t ∊ K. It follows that α(t) ∊α(K) which is a contradiction. α^{-1}(W) is an essential submodule of M and K is Te-maximal in M, that α^{-1}(W) = K + T and hence W = α(K) + α(T). □

An R-epimorphismα : M → N is called closed if ker(α) is closed in M.

Proposition. (2.6). Let M and N be R-modules and $\alpha : M \to N$ a closed R-epimorphism, if T is a submodule of M and K is α(T)e-maximal submodule in N, then $\alpha^{-1}(K)$ is Te-maximal submodule of M.

Proof. First we show that $T \nsubseteq \alpha^{-1}(K)$, if not, then $T \subseteq \alpha^{-1}(K)$ and hence $\alpha(T) \subseteq K$ which is a contradiction. Suppose that W is an essential submodule of M with $\alpha^{-1}(K) \subseteq W \subseteq \alpha^{-1}(K) + T$. Then $K = \alpha(\alpha^{-1}(K)) \subseteq \alpha(W) \subseteq K + \alpha(T)$. There is an isomorphism $\overline{\alpha} : M / \ker(\alpha) \to N$ such that $\overline{\alpha} \circ \pi = \alpha$ where $\pi : M \to M / \ker(\alpha)$ is the natural epimorphism. Thus $\alpha(W) = \overline{\alpha}(W/\ker(\alpha))$. Since $\ker(\alpha)$ is closed, then $W/\ker(\alpha)$ is essential in M and hence $\alpha(W)$ is essential in N ([3], proposition(1-4)). By α(T)e-maximalty of K in N, then $\alpha(W) = K$ or $\alpha(W) = K + \alpha(T)$. If $\alpha(W) = K$, then $W \subseteq \alpha^{-1}(\alpha(W)) = \alpha^{-1}(K)$ with the other case, if $\alpha(W) = K + \alpha(T)$, let $w \in \alpha^{-1}(K) + T$ then $w = a + t$ for some $a \in \alpha^{-1}(K)$ and $t \in T$ and so $\alpha(w) = \alpha(a) + \alpha(t)$, hence $a + t - w \in \ker(\alpha) \subseteq \alpha^{-1}(K) \subseteq W$ and $a + t \in W$, this implies that $\alpha^{-1}(K) + T \subseteq W$. Thus $\alpha^{-1}(K) + T = W$. This implies that $\alpha^{-1}(K)$ is Te-maximal submodule in M □

Let M be R-module and T a submodule of M. We denote $J'_{Te}(M)$ the intersection of all Te-maximal submodules of M. Then we have the following inclusion relation $J'_{Te}(M) \subseteq J_T(M) \subseteq J_{Te}(M)$.

Example. (2.7). Let $R = \mathbb{Z}$, $M = \mathbb{Z}_{24}$ and $N \leq M$. Then all submodules of M have the following properties

Example. (2.7). Let $R = \mathbb{Z}$, $M = \mathbb{Z}_{24}$ and $N \leq M$. Then all submodules of M have the following properties

$N \leq M$	$N \trianglelefteq M$	$N \ll_M M$	$N \ll_{2\mathbb{Z}_{24}} M$	$N \ll_{3\mathbb{Z}_{24}} M$	$N \ll_{4\mathbb{Z}_{24}} M$	$N \ll_{6\mathbb{Z}_{24}} M$	$N \ll_{8\mathbb{Z}_{24}} M$	$N \ll_{12\mathbb{Z}_{24}} M$	$N \ll_0 M$	$N \ll_{(M)e} M$	$N \ll_{(2\mathbb{Z}_{24})e} M$	$N \ll_{(3\mathbb{Z}_{24})e} M$	$N \ll_{(4\mathbb{Z}_{24})e} M$	$N \ll_{(6\mathbb{Z}_{24})e} M$	$N \ll_{(8\mathbb{Z}_{24})e} M$	$N \ll_{(12\mathbb{Z}_{24})e} M$	$N \ll_{0e} M$
M	√	x	x	x	x	x	x	x	√	x	x	x	x	x	x	x	√
$2\mathbb{Z}_{24}$	√	x	x	√	x	x	x	x	√	√	x	√	x	√	x	√	√
$3\mathbb{Z}_{24}$	x	x	x	x	x	x	√	x	√	x	x	x	√	√	√	√	√
$4\mathbb{Z}_{24}$	√	x	x	x	x	√	x	x	√	√	x	√	x	√	x	√	√
$6\mathbb{Z}_{24}$	x	√	x	√	x	x	√	x	√	√	√	√	√	√	√	√	√
$8\mathbb{Z}_{24}$	x	x	x	√	x	√	x	x	√	√	x	√	x	√	x	√	√
$12\mathbb{Z}_{24}$	x	√	√	√	x	√	√	x	√	√	√	√	√	√	√	√	√
0	x	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√

$J'_{Me}(M) = 6\mathbb{Z}_{24}$, $J_M(M) = 6\mathbb{Z}_{24}$, $J_{Me}(M) = 2\mathbb{Z}_{24}$
$J'_{(2\mathbb{Z}_{24})e}(M) = 12\mathbb{Z}_{24}$, $J_{2\mathbb{Z}_{24}}(M) = 12\mathbb{Z}_{24}$, $J_{(2\mathbb{Z}_{24})e}(M) = 4\mathbb{Z}_{24}$
$J'_{(3\mathbb{Z}_{24})e}(M) = 0$, $J_{3\mathbb{Z}_{24}}(M) = 6\mathbb{Z}_{24}$, $J_{(3\mathbb{Z}_{24})e}(M) = 2\mathbb{Z}_{24}$
$J'_{(4\mathbb{Z}_{24})e}(M) = 0$, $J_{4\mathbb{Z}_{24}}(M) = 0$, $J_{(4\mathbb{Z}_{24})e}(M) = M$
$J'_{(6\mathbb{Z}_{24})e}(M) = 0$, $J_{6\mathbb{Z}_{24}}(M) = 12\mathbb{Z}_{24}$, $J_{(6\mathbb{Z}_{24})e}(M) = 4\mathbb{Z}_{24}$
$J'_{(8\mathbb{Z}_{24})e}(M) = 0$, $J_{8\mathbb{Z}_{24}}(M) = 0$, $J_{(8\mathbb{Z}_{24})e}(M) = M$
$J'_{(12\mathbb{Z}_{24})e}(M) = 0$, $J_{12\mathbb{Z}_{24}}(M) = 0$, $J_{(12\mathbb{Z}_{24})e}(M) = M$
$J'_{0e}(M) = M$, $J_0(M) = M$, $J_{0e}(M) = M$

Theorem. (2.8). Let M and N be R-modules and $\alpha : M \to N$ a closed R-epimorphism such that $\ker(\alpha) \subseteq J'_{Te}(M)$. Then $\alpha(J'_{Te}(M)) = J'_{\alpha(T)e}(N)$.

Proof. Consider the following two families,

$$A = \{K \leq M \mid K \text{ is Te-maximal submodule of } M\}$$
$$\text{and } B = \{\alpha(K) \leq N \mid \alpha(K) \text{ is } \alpha(T)\text{e-maximal submodule of } N\}.$$

Then by proposition (2.5) and (2.6) we have $\alpha(J'_{Te}(M)) = \alpha(\bigcap_{K\in A} K) = \bigcap_{\alpha(K)\in B} \alpha(K) = J'_{\alpha(T)e}(N)$.

Proposition. (2.9). Let M and N be an R-modules and $\alpha : M \to N$ a closed R-epimorphism if T is a submodule of M and K is an essential T-maximal submodule of M with $\ker(\alpha) \subseteq K$, then $\alpha(K)$ is essential and $\alpha(T)$-maximal in N.

Proof. First we show $\alpha(K)$ in N. Since α is an epimorphism, then there is an isomorphism $\overline{\alpha} : M / \ker(\alpha) \to N$ such that $\overline{\alpha}\circ\pi = \alpha$ where $\pi : M \to M / \ker(\alpha)$ is the natural epimorphism. Since $\ker(\alpha)$ is closed and K is essential in M, then $K/\ker(\alpha)$ is essential in $M/\ker(\alpha)$, and hence $\alpha(K) = \overline{\alpha}(K/\ker(\alpha))$ is essential in N, ([3], 1.4). The rest of the proof as in ([2], lemma 3.4). □

The proof of the proposition is a similar to that of lemma 3.5 [2].

Proposition. (2.10). Let M and N be R-modules and $\alpha : M \to N$ an R-epimorphism. If T is a submodule of M and K is essential $\alpha(T)$-maximal submodule of N, then $\alpha^{-1}(K)$ is essential T-maximal in M. □

Theorem. (2.11). Let M and N be R-modules and $\alpha : M \to N$ a closed R-epimorphism such that $\ker(\alpha) \subseteq J_{Te}(M)$. Then $\alpha(J_{Te}(M)) = J_{\alpha(T)e}(M)$.

Proof. Consider the following two families

$$A = \{K \trianglelefteq M \mid K \text{ is T-maximal submodule of } M\}$$
$$\text{and } B = \{\alpha(K) \trianglelefteq N \mid \alpha(K) \text{ is } \alpha(T)\text{-maximal submodule of } N\}.$$

Then by proposition(2.9) and (2.10), we have $\alpha(J_{Te}(M)) = \alpha(\bigcap_{K\in A} K) = \bigcap_{\alpha(K)\in B} \alpha(K) = J_{\alpha(T)e}(N)$ □

Proposition. (2.12). Let Mbe an R-module and T a submodule of M. If every proper essential submodule X of M with $T \nsubseteq X$ is contained in a T-maximal submodule of M, then $J_{Te}(M)$ (resp.$J_T(M)$, $J'_T(M)$)$\ll_{Te}$ M.

Proof. Assume X is a proper essential submodule of M with $T \subseteq J_{Te}(M) + X$. $T \nsubseteq X$, then by the hypothesis, $X \subseteq K$ for some T-maximal submodule K of M. Then K is essential in M. Thus $J_{Te}(M) \subseteq K$, so $T \subseteq K + X = K$ which is a contradiction. Thus $T \subseteq X$ and hence $J_{Te}(M) \ll_{Te} M$. The relation $J'_{Te}(M) \subseteq J_T(M) \subseteq J_{Te}(M)$ and proposition(1.6) imply that $J_T(M)$ ($J'_T(M)$) $\ll_{Te}$ M. □

As application of Zorn's lemma, we have the following corollary

Corollary. (2.13). Let Mbe a finitely generated R-module. Then $J_{Te}(M)$ (resp. $J_T(M)$, $J'_T(M)$)$\ll_{Te}$ M □

Let M be an R-module and T a nonzero submodule of M. We say that M is Te-cosemisimple, if each submodule of M is intersection of Te-maximal submodule of M.

Theorem. (2.14). Let Mbe an R-module and T a nonzero submodule of M.Then M is Te-cosemisimpleif and only if $J'_{(K+T/K)e}$(M/K) = 0 for each closed submodule K of M.

Proof. Suppose that M is Te-cosemisimple and K a closed submodule of M. Then K = $\bigcap_{S\in B} S$, where B is the family of Te-maximal submodules of M. Consider the following two families

A = {S/K ≤ M/K | S/K is (T+K/K)e-maximal submodule of M/K}

and A' = {S ≤ M | K ≤ S and S is Te-maximal submodule of M}.

We note that by proposition(2.5) and (2.6), B ⊆ A' and S/K ∈ A if and only if S ∈ A' . Thus $J'_{(K+T/K)e}$ (M/K) = $\bigcap_A S/K$ = ($\bigcap_{A'} S$)/K= ($\bigcap_{A'} S$)/($\bigcap_B S$) = 0. Conversely, suppose that $J'_{(K+T/K)e}$ (M/K) = 0 for all closed submodule K of M. Then 0 = $J'_{(K+T/K)e}$ (M/K) = $\bigcap_A S/K$ = ($\bigcap_{A'} S$)/K. This mean that K = $\bigcap_{A'} S$ □

Proposition. (2.15). Let Mbe an R-module and T a nonzero submodule of M. If M is Te-cosemisimple, then

1. Every submodule of M containing T is Te-cosemisimple.
2.M/N is ((T+N)/N)e-cosemisimple for each submodule N of M.

Proof. 1. Suppose T ≤ N ≤ M and M is Te-cosemisimple. If L is a submodule of N, then L = L ∩ N = ($\bigcup_A S$) ∩ N = $\bigcap_A (S \cap N)$where A is the set of all Te-maximal submodule of M. Since ((S∩N)+T)/(S∩N) ≃ T/((S∩N)∩T) = T/(S∩T) ≃ S+T/T is a simple R-module, then ST/T is a simple R-module, then S∩N is Te-maximal submodule of N and hence N is Te-cosemisimple.
2. Let L/N be a submodule of M/N. Since M is Te-cosemisimple, then L/N = ($\bigcap_A S$)/N = $\bigcap_A (S/N)$, where A is the set of all Te-maximal submodules of M. By proposition(2.5) S/N is ((T+N)/N)e-maximal in M/N. Thus M/N is ((T+N)/N)e-cosemisimple. □

§3. Generalized T-essential(-closed) Hopfian modules

We start by the following proposition.

Proposition. (3.1). Let Mbe an R-module with submodule K ⊆ N, K ⊆ T and K is closed in M. If K $\ll_{Te}$ M and N/K $\ll_{(T/K)e}$ M/K, then N $\ll_{Te}$ M.

Proof. For an essential submodule X of M, assume that T ⊆ N+X. Then T/K ⊆ (N+X)/K ⊆ N/K + (K+X)/K. Since X + K is essential in M and K is closed in M, then (X+K)/K is essential in M/K. The hypothesis implies that T/K ⊆ (K+X)/K and hence T ⊆ K + X, but K $\ll_{Te}$ M. Thus T ⊆ X. □

The following corollaries follow from proposition(3.1) and (1.6).

Corollary. (3.2). Let Mbe an R-module with submodules N, K and T such that K ⊆ N, K ⊆ T and K is closed. Then N $\ll_{Te}$ M if and only if K $\ll_{Te}$ M and N/K $\ll_{Te}$ M/K. □

Corollary. (3.3).Let M be an R-module with submodules K ⊆ N and K closed. Then N $\ll_{Te}$ M if and only if K $\ll_{Te}$ M and N/K $\ll_{Te}$ M/K. □

In proposition(1.9), we considered condition under which the image of Te-small submodules in a module being α(T)e-small in M for some endomorphism α. Now, we are interesting to consider the inverse imagesof Te-small submodules. Recall that an R-module M is generalized to Hopfian (gH), if every surjective endomorphism of M has small kernel.

Here, we introduce the following

Definition. (3.4). Let Mbe an R-module and T a submodule of M. We say that M is
1. generalized T-essential-Hopfian (GTe-H), ifker(α) $\ll_{Te}$ M for each surjective endomorphism α of M.
2. generalized T-essential-closed-Hopfian (GTec-H), if for each surjective endomorphism α of M, we have ker(α) is closed in M and ker(α)$\ll_{Te}$ M.

Theorem. (3.5). Let Mbe R-module and T a submodule of M. Then the following statements are equivalent
1. M is GTec-H
2. If N $\ll_{Te}$ M, then for every surjective R-endomorphism α of M, ker(α) is closed in M and α^{-1}(N) $\ll_{\alpha^{-1}(T)e}$ M.

Proof.(1) → (2). Let α be a surjective endomorphism of M. For each essential submodule L/K of M/K where K = ker(α), assume α^{-1}(T)/K $\subseteq\alpha^{-1}$(N)/K + L/K, then α^{-1}(T) $\subseteq\alpha^{-1}$(N) + L and hence T ⊆ N + α(L). There exists an isomorphism $\overline{\alpha}$: M / k → M such that $\overline{\alpha}\circ\pi$ = α where π : M → M / K is the natural epimorphism. Then α(L) = $\overline{\alpha}$(L/K) and hence α(L) is essential in M,([3] proposition 1.4). Thus T ⊆α(L) and hence α^{-1}(T)⊆ L. This implies that α^{-1}(T)/K ⊆ L/K and hence α^{-1}(N)/K $\ll_{\alpha^{-1}((T)/K)e}$ M/K. By (1). K $\ll_{Te}$ M and K is closed in M. By proposition(3.1) we have α^{-1}(N) $\ll_{\alpha^{-1}(T)e}$M.
(2) → (1). Let α be a surjective endomorphism of M. By(2) ker(α) is closed in M and ker(α)$\ll_{Te}$ M. Since 0 $\ll_{Te}$ M for any submodule T of M, in particular 0 $\ll_{\alpha(T)e}$ M. Then ker(α) = α^{-1}(0) $\ll_{\alpha^{-1}(T)e}$ M and hence ker(α)$\ll_{(T+ker(\alpha))e}$M. This implies that ker(α)$\ll_{Te}$ M. Thus M is GTec-H. □

Corollary. (3.6). Let Tbe a submodule of GTec-H R-module M and α a surjective endomorphism of M. Then the following are equibalent for a submodule N of M.
1. N $\ll_{Te}$ M.
2.α(N) $\ll_{\alpha(T)e}$M.
3.α^{-1}(N) $\ll_{\alpha^{-1}(T)e}$ M. □

Corollary. (3.7). The following statements are equivalent for an R-module M
1. M is GMec-H
2. If N $\ll_{e}$ M, then for every surjective endomorphism α of M, ker(α) is closed in M and ker(α)$\ll_{e}$ M.

Proposition. (3.8). Let Mbe an R-module and T a submodule of M. Then M is GTe-H if and only if there exists a closed fully invaniant submodule N $\ll_{Te}$ M such that M/N is G(T/N)e-H

Proof. The "if" part is trivial by taking N = 0. For only if" part, suppose that N is closed fully invariant submodule of M such that N $\ll_{Te}$ M and M/N is G(T/N)e-H. Let α : M → M be a surjective. Then define $\overline{\alpha}$: M/N → M/N by $\overline{\alpha}$(m+N) = α(m) + N for all m ∊ M. Clearly. $\overline{\alpha}$is well-defined surjective endomorphism of M/N. Then ker($\overline{\alpha}$) $\ll_{(T/N)e}$ M/N. Let ker($\overline{\alpha}$) = L/N for some submodule L of M. Then L/N $\ll_{(T/N)e}$ M/N and N $\ll_{Te}$ M, so by proposition (3.1), L $\ll_{Te}$ M, but ker(α) ≤ L, then ker(α)$\ll_{Te}$ M and hence M is GTe-H. □

Corollary. (3.9). Let Mbe an R-module. Then M is GMe-H if and only if there exists a closed fully invariant submodule N $\ll_{Te}$ M such that M/N is G(M/N)e-H. □

References

[1] F. W. Anderson and K. R. Fuller, Rings and categories of modules, New York, Springer-Verlag. 1992.
[2] R. Beyranvand and F. Moradi, Small submodules with respect to an arbitrary submodule, Journal of Algebra and Related Topics, Vol. 3, No. 2, (2015), 43-51.
[3] K. R. Goodearl, Ring Theory, New York and Basel, Marcell Dekker, INC. 1976.
[4] DX. Zhou and X. R. Zhang, Small-Essential submodules and Morita Duality, Southeast Asia Bulletion of Mathematics, 35, (2011), 1051-1062.