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Comparison of four methods to estimate the scale parameter for Extended Poisson Exponential distribution

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Abstract. In this paper, it was obtained estimators of the scale parameter of Extended Poisson Exponential distribution by four methods using complete data, these methods are the Maximum Likelihood Estimator (MLE), the Least Squares Estimator (LSE), Plotting Position Estimator (PPE) and White Estimator (WE), these four methods have been compared by the criteria of Mean Square Error (MSE) using Monte Carlo simulation, and from the results on the samples in the simulation, the comparison in this study showed that Maximum Likelihood Estimator method is best from other methods followed by the White Estimator method then the Least Squares Estimator method.

Keywords. Extended Poisson Exponential distribution, Maximum Likelihood Estimator, Least Squares Estimator, plotting position Estimator, White Estimator.

1.Introduction

The Extended Poisson Exponential distribution was developed by Anum Fatima et al [1] in 2015 by taking the maximum random variable of Modified Exponential distribution when the sample size follows to the zero truncated Poisson, this distribution has increasing and decreasing in failure rates. the probability density function of the Extended Poisson Exponential distribution is given by

$$f\,(x;\alpha,\beta,\lambda)=\frac{\alpha\beta\lambda e^{-\lambda-\beta x}}{(1-e^{-\lambda})(1-\bar{\alpha}e^{-\beta x})^2}e^{\lambda(1-e^{-\beta x})/(1-\bar{\alpha}e^{-\beta x})}\quad x>0 \text{ and } \alpha,\beta,\lambda>0\ ,\ \bar{\alpha}=1-\alpha \qquad (1)$$

If $\alpha=1$, resulting distribution Poisson-Exponential distribution, when λ approaches to zero then resulting distribution converges to be modified exponential distribution. Also when both $\alpha=1$ and λ approaches to zero then the resulting distribution converges to exponential distribution.

Hilary et al. [2] use The Extended Poisson Exponential distribution to obtained classes of ordinary differential equations obtained for the probability functions,

The cumulative distribution function and survival function respectively for the Extended Poisson Exponential distribution are

$$F(x;\alpha,\beta,\lambda)=\frac{e^{-\alpha\lambda e^{-\beta x}/\left(1-\bar{\alpha}e^{-\beta x}\right)}-e^{-\lambda}}{1-e^{-\lambda}} \qquad (2)$$

$$S\,(x;\alpha,\beta,\lambda)=\frac{1-e^{-\alpha\lambda e^{-\beta x}/\left(1-\bar{\alpha}e^{-\beta x}\right)}}{1-e^{-\lambda}} \qquad (3)$$

We review in this paper four methods using complete data, which are the Maximum Likelihood Estimator (MLE), Least Squares Estimator (LSE), Plotting Position Estimator (PPE) and White Estimator (WE). These methods are compared in Section (6) by using the criteria the mean square error (MSE) in the Simulation, all these four methods in this paper which are estimate the scale parameter β for the Extended Poisson Exponential distribution.

2. Maximum Likelihood Estimator Method (MLE)

From the probability density function of the Extended Poisson Exponential distribution (1), then the likelihood function is:

$$L=\left[\alpha\beta\lambda(1-e^{-\lambda})^{-1}\right]^{n} e^{\sum_{i=1}^{n}(-\lambda-\beta x_i)}\left[\prod_{i=1}^{n}(1-\bar{\alpha}e^{-\beta x_i})^{-2}\right]e^{\sum_{i=1}^{n}\lambda(1-e^{-\beta x_i})/(1-\bar{\alpha}e^{-\beta x_i})} \tag{4}$$

so the logarithm of this likelihood function is

$$\ln L = n\ln(\alpha\beta\lambda)-n\lambda-\beta\sum_{i=1}^{n}x_i-n\ln(1-e^{-\lambda})-2\sum_{i=1}^{n}\ln(1-\bar{\alpha}e^{-\beta x_i})+\sum_{i=1}^{n}\frac{\lambda(1-e^{-\beta x_i})}{1-\bar{\alpha}e^{-\beta x_i}} \tag{5}$$

The partial derivative of the logarithm of likelihood function with respect to β is

$$\frac{\partial \ln L}{\partial\beta}=\frac{n}{\beta}-\sum_{i=1}^{n}x_i-2\bar{\alpha}\sum_{i=1}^{n}\frac{x_i e^{-\beta x_i}}{1-\bar{\alpha}e^{-\beta x_i}}+\alpha\lambda\sum_{i=1}^{n}\frac{x_i e^{-\beta x_i}}{(1-\bar{\alpha}e^{-\beta x_i})^2} \tag{6}$$

Now we make this partial derivative equal to zero

$$\frac{\partial \ln L}{\partial\hat{\beta}}=0$$

Then

$$\frac{n}{\beta}-\sum_{i=1}^{n}x_i-2\bar{\alpha}\sum_{i=1}^{n}\frac{x_i e^{-\beta x_i}}{1-\bar{\alpha}e^{-\beta x_i}}+\alpha\lambda\sum_{i=1}^{n}\frac{x_i e^{-\beta x_i}}{(1-\bar{\alpha}e^{-\beta x_i})^2}=0 \tag{7}$$

$$\hat{\beta}_{MLE}=\frac{n}{\sum_{i=1}^{n}x_i+2\bar{\alpha}\sum_{i=1}^{n}\frac{x_i e^{-\beta x_i}}{1-\bar{\alpha}e^{-\beta x_i}}-\alpha\lambda\sum_{i=1}^{n}\frac{x_i e^{-\beta x_i}}{(1-\bar{\alpha}e^{-\beta x_i})^2}} \tag{8}$$

Where this symbol $\hat{\beta}_{MLE}$ is indicating to estimator the parameter β by using Maximum Likelihood estimator method,

3. Least Squares Estimator Method (LSE)

The idea of Least Squares Estimator method is to minimize the sum of squared differences between the random sample values and the estimate values by linear approximation [6].

$$\varepsilon_i = y_i - b_0 - b_1 x \tag{9}$$

$$\sum_{i=1}^{n}\varepsilon_i^2=\sum_{i=1}^{n}[y_i-\hat{y}_i]^2 \tag{10}$$

$$\sum_{i=1}^{n}\varepsilon_i^2=\sum_{i=1}^{n}[y_i-b_0-b_1x]^2 \tag{11}$$

by using the cumulative distribution function of Extended Poisson Exponential distribution (2) with respect to random sample values as follows

$$F(x_i)=\frac{e^{-\alpha\lambda e^{-\hat{\beta}x_i}/\left(1-\bar{\alpha}e^{-\hat{\beta}x_i}\right)}-e^{-\lambda}}{1-e^{-\lambda}} \tag{12}$$

$$(1-e^{-\lambda})F(x_i)+e^{-\lambda}=e^{-\alpha\lambda e^{-\hat{\beta}x_i}/\left(1-\bar{\alpha}e^{-\hat{\beta}x_i}\right)} \tag{13}$$

By taking the logarithm of the equation (13) getting

$$\ln\left((1-e^{-\lambda})F(x_i)+e^{-\lambda}\right)=-\alpha\lambda e^{-\hat{\beta}x_i}\Big/\left(1-\bar{\alpha}e^{-\hat{\beta}x_i}\right) \tag{14}$$

$$\ln\left((1-e^{-\lambda})F(x_i)+e^{-\lambda}\right)=-\alpha\lambda e^{-\hat{\beta}x_i}+\bar{\alpha}e^{-\hat{\beta}x_i}\ln\left((1-e^{-\lambda})F(x_i)+e^{-\lambda}\right) \quad (15)$$

$$\ln\left((1-e^{-\lambda})F(x_i)+e^{-\lambda}\right)=e^{-\hat{\beta}x_i}\left[-\alpha\lambda+\bar{\alpha}\ln\left((1-e^{-\lambda})F(x_i)+e^{-\lambda}\right)\right] \quad (16)$$

$$\ln\left[\ln\left((1-e^{-\lambda})F(x_i)+e^{-\lambda}\right)\right]=-\hat{\beta}x_i+\ln\left[-\alpha\lambda+\bar{\alpha}\ln\left((1-e^{-\lambda})F(x_i)+e^{-\lambda}\right)\right] \quad (17)$$

Comparing the equation (17) with the following simple linear model

$$Z=b_0+b_1Y+\varepsilon \quad (18)$$

We get

$$Z=\ln\left[\ln\left((1-e^{-\lambda})F(x_i)+e^{-\lambda}\right)\right] \quad (19)$$

$$b_0=\ln\left[-\alpha\lambda+\bar{\alpha}\ln\left((1-e^{-\lambda})F(x_i)+e^{-\lambda}\right)\right] \quad (20)$$

$$b_1=-\beta \quad (21)$$

$$Y=x_i \quad (22)$$

$$\varepsilon=\ln\left[\ln\left((1-e^{-\lambda})F(x_i)+e^{-\lambda}\right)\right]+\hat{\beta}x_i-\ln\left[-\alpha\lambda+\bar{\alpha}\ln\left((1-e^{-\lambda})F(x_i)+e^{-\lambda}\right)\right] \quad (23)$$

By taking the sum square for the two sides of the equation (23) then

$$\sum_{i=1}^{n}\varepsilon_i^2=\sum_{i=1}^{n}\left(\ln\left[\ln\left((1-e^{-\lambda})F(x_i)+e^{-\lambda}\right)\right]+\hat{\beta}x_i-\ln\left[-\alpha\lambda+\bar{\alpha}\ln\left((1-e^{-\lambda})F(x_i)+e^{-\lambda}\right)\right]\right)^2 \quad (24)$$

$$\frac{\partial(\sum_{i=1}^{n}\varepsilon_i^2)}{\partial\beta}=2\sum_{i=1}^{n}x_i\left(\ln\left[\ln\left((1-e^{-\lambda})F(x_i)+e^{-\lambda}\right)\right]+\hat{\beta}x_i-\ln\left[-\alpha\lambda+\bar{\alpha}\ln\left((1-e^{-\lambda})F(x_i)+e^{-\lambda}\right)\right]\right) \quad (25)$$

we make this partial derivative equal to zero as follows

$$\frac{\partial(\sum_{i=1}^{n}\varepsilon_i^2)}{\partial\beta}=0 \quad (26)$$

then

$$2\sum_{i=1}^{n}x_i\ln\left[\ln\left((1-e^{-\lambda})\tilde{F}(x_i)+e^{-\lambda}\right)\right]+2\hat{\beta}\sum_{i=1}^{n}x_i^2-2\sum_{i=1}^{n}x_i\ln\left[-\alpha\lambda+\bar{\alpha}\ln\left((1-e^{-\lambda})\tilde{F}(x_i)+e^{-\lambda}\right)\right]=0 \quad (27)$$

$$\hat{\beta}_{LSE}=\frac{\sum_{i=1}^{n}x_i\ln\left[-\alpha\lambda+\bar{\alpha}\ln\left((1-e^{-\lambda})\tilde{F}(x_i)+e^{-\lambda}\right)\right]-\sum_{i=1}^{n}x_i\ln\left[\ln\left((1-e^{-\lambda})\tilde{F}(x_i)+e^{-\lambda}\right)\right]}{\sum_{i=1}^{n}x_i^2} \quad (28)$$

Where this symbol $\hat{\beta}_{LSE}$ is indicating to estimator the parameter β by using least squares estimator method, and $\tilde{F}(x_i)$ her is indicating to Empirical Distribution Functions as following [4].

$$\tilde{F}(x_i)=\frac{i-0.5}{n} \quad (29)$$

4. Plotting Position Estimator Method (PPE):

This method proposed in (1982) by Whitten and Cohen, as new modification about moment method [5], by following equation

$$E\left(\hat{F}(x_{(i)})\right)=F(x_{(i)}) \quad (30)$$

Where $x_{(i)}$ is first order random variable, and $\hat{F}(x_{(i)})$ is estimated unbiased for distribution function $F(x_{(i)})$ and replacement $F(x_{(i)})$ by the following plotting position formula

$$P_i=\frac{i}{n+1}, i=1,2,...,n \quad (31)$$

$$\hat{F}(x_{(i)})=\frac{i}{n+1}, i=1,2,...,n \quad (32)$$

So that

$$\hat{F}(x_{(1)}) = \frac{1}{n+1} \qquad (33)$$

From equations (2) and (33), we get:

$$\frac{e^{-\alpha\lambda e^{-\hat{\beta}x(1)} \big/ \left(1-\bar{\alpha}e^{-\hat{\beta}x(1)}\right)} - e^{-\lambda}}{1-e^{-\lambda}} = \frac{1}{n+1} \qquad (34)$$

$$e^{-\alpha\lambda e^{-\hat{\beta}x(1)} \big/ \left(1-\bar{\alpha}e^{-\hat{\beta}x(1)}\right)} = \frac{1-e^{-\lambda}}{n+1} + e^{-\lambda} \qquad (35)$$

By taking the logarithm to equation (35), we get:

$$\frac{-\alpha\lambda e^{-\hat{\beta}x_{(1)}}}{\left(1-\bar{\alpha}e^{-\hat{\beta}x_{(1)}}\right)} = \ln\left(\frac{1-e^{-\lambda}}{n+1} + e^{-\lambda}\right) \qquad (36)$$

$$-\alpha\lambda e^{-\hat{\beta}x_{(1)}} + \bar{\alpha}e^{-\hat{\beta}x_{(1)}} \ln\left(\frac{1-e^{-\lambda}}{n+1} + e^{-\lambda}\right) = \ln\left(\frac{1-e^{-\lambda}}{n+1} + e^{-\lambda}\right) \qquad (37)$$

$$e^{-\hat{\beta}x_{(1)}} = \frac{\ln\left(\frac{1-e^{-\lambda}}{n+1} + e^{-\lambda}\right)}{-\alpha\lambda + \bar{\alpha}\ln\left(\frac{1-e^{-\lambda}}{n+1} + e^{-\lambda}\right)} \qquad (38)$$

$$\hat{\beta}_{PPE} = \frac{-1}{x_{(1)}} \ln\left(\frac{\ln\left(\frac{1-e^{-\lambda}}{n+1} + e^{-\lambda}\right)}{-\alpha\lambda + \bar{\alpha}\ln\left(\frac{1-e^{-\lambda}}{n+1} + e^{-\lambda}\right)}\right) \qquad (39)$$

Where this symbol $\hat{\beta}_{PPE}$ is indicating to estimator the parameter β by using plotting position Estimator method.

5.White Estimator Method (WE)

The White estimator method is based on the survival function of the distribution when the formula of the function converted to the formula of the linear regression equation, and characteristics of linear regression equation are to use its estimators as primary estimators for the parameters estimation methods.

To apply this method to the Extended Poisson Exponential distribution then from formula (3) we obtain the following

$$(1-e^{-\lambda})S(x) = 1-e^{-\alpha\lambda e^{-\beta x} \big/ \left(1-\bar{\alpha}e^{-\beta x}\right)} \qquad (40)$$

by taking the logarithm to both sides of equation (41) we get

$$\ln\left[1-(1-e^{-\lambda})S(x)\right] = \frac{-\alpha\lambda e^{-\beta x}}{1-\bar{\alpha}e^{-\beta x}} \qquad (41)$$

$$\ln\left[1-(1-e^{-\lambda})S(x)\right] = \bar{\alpha}e^{-\beta x} \ln\left[1-(1-e^{-\lambda})S(x)\right] - \alpha\lambda e^{-\beta x} \qquad (42)$$

$$\ln\left(\ln\left[1-(1-e^{-\lambda})S(x)\right]\right)=-\beta x+\ln\left(\bar{\alpha}\ln\left[1-(1-e^{-\lambda})S(x)\right]-\alpha\lambda\right) \quad (43)$$

Comparing the equation (43) with the following linear regression formula [7]

$$z=\theta y+\phi \quad (44)$$

$$z=\ln\left(\ln\left[1-(1-e^{-\lambda})\tilde{S}(x)\right]\right),\ \theta=\beta,\ y=-x,\ \phi=\ln\left(\bar{\alpha}\ln\left[1-(1-e^{-\lambda})S(x)\right]-\alpha\lambda\right) \quad (45)$$

and from properties of linear regression formula then

$$\hat{\theta}=\frac{\sum_{i=1}^{n}(y_i-\bar{y})(z_i-\bar{z})}{\sum_{i=1}^{n}(y_i-\bar{y})^2} \quad (46)$$

where

$$\bar{z}=\frac{\sum_{i=1}^{n}z_i}{n}=\frac{\sum_{i=1}^{n}\ln\left(\ln\left[1-(1-e^{-\lambda})\tilde{S}(x_i)\right]\right)}{n} \quad (47)$$

where

$$\tilde{S}(x_i)=1-\tilde{F}(x_i)=1-\frac{i-0.5}{n} \quad (48)$$

Set $\hat{\theta}$ form (46) in (45) we obtain :

$$\hat{\beta}_{WE}=\hat{\theta} \quad (49)$$

Where this symbol $\hat{\beta}_{WE}$ is indicating to estimator the parameter β by using White estimator method.

6.The Simulation

In this section, we used the Monte Carlo simulation to compare the MSE criterion for estimators of the scale parameter β for Extended Poisson Exponential distribution with respect to MLE, LSE, PPE and WE methods. We used the three models in this paper, the first model is $\alpha=0.4$, $\beta=2$, $\lambda=5$, the second model is $\alpha=2$, $\beta=0.1$, $\lambda=1$ and the third model is $\alpha=0.3$, $\beta=7$, $\lambda=0.02$.

The MSE for these three models in this simulation is calculated by using 16,650 simulated samples. All computations are performed using MATLAB R2014a in this simulation. We have chosen sample sizes n=10, 25 and 50. We have generated the random numbers of in this simulation by using the inversion of the cumulative distribution function of the Extended Poisson Exponential distribution, the replicate the data for experiments are N times, (N=100,250,500,1000) with sample size n, finally, the results of this simulation presented in the following Tables.

Table-1. *The MSE for estimators of β for the first model*

$\alpha = 0.4, \beta = 2, \lambda = 5$

Cr.	MSE				N
Meth. / n	**MLE**	**LSE**	**PPE**	**WE**	
10	5.05722578	3.93320694	9.10016640	2.86776632	100
	51.0351620	2.22073188	100.710227	1.87701302	250
	13.5631492	3.54698655	168.424068	2.66079295	500
	29.0217696	2.73924136	187.962907	1.94794646	1000
25	0.90847354	0.76988574	146.008230	0.88660994	100
	0.46866467	0.83791225	226.717036	0.93909255	250
	0.55686371	0.95335735	157.126656	0.95963150	500
	0.65985539	0.84114734	245.092237	0.90665362	1000
50	0.12794017	0.45685368	149.469595	0.70077099	100
	0.21753848	0.50855325	143.638433	0.68247000	250
	0.17642587	0.47419249	84.3494201	0.67220023	500
	0.20042505	0.41626251	144.881126	0.64036557	1000

Table-2. *The MSE for estimators of β for the second model*

$\alpha = 2, \beta = 0.1, \lambda = 1$

Cr.	MSE				N
Meth. / n	**MLE**	**LSE**	**PPE**	**WE**	
10	0.00731123	32.2374937	10.2819239	0.11004120	100
	0.01650053	32.3534698	36.3749590	0.11112239	250
	0.01455291	32.1996015	14.5882560	0.10827133	500
	0.00967459	32.2637709	3.70105474	0.10731274	1000
25	0.00170161	34.9469741	9.85426441	0.10591912	100
	0.00132984	34.8635290	1.46266773	0.10626035	250
	0.00121820	35.3751739	16.6628251	0.10546927	500
	0.00153725	35.2454909	12.4958577	0.10574951	1000
50	0.00043728	36.4193879	1.21211616	0.10445325	100
	0.10445325	36.6110719	4.64319272	0.10450335	250
	0.00037247	36.9552395	3.60435816	0.10319193	500
	0.00045065	36.3927136	9.08742174	0.10406135	1000

Table-3. *The MSE for estimators of β for the third model*

$\alpha = 0.3, \beta = 7, \lambda = 0.02$

Cr.	MSE				N
Meth. / n	**MLE**	**LSE**	**PPE**	**WE**	
10	4.43437700	48.9092654	69.2456390	47.937308	100
	3.69142302	48.9063453	61.6323112	45.520512	250
	3.91629449	48.9104247	69.2648908	49.300202	500
	3.89564879	48.9076370	65.3759113	40.540275	1000
25	1.31814472	48.8997468	64.5364294	44.671430	100
	1.19785384	48.9050830	67.5422129	95.439691	250
	1.10533600	48.9005774	68.5356965	66.813775	500
	1.16809250	48.8990950	60.6954242	36.102083	1000
50	0.51217300	48.9010279	68.0145756	40.281347	100
	0.67876019	48.8971059	62.2632288	45.037464	250
	0.63081670	48.8981232	63.2391621	45.586029	500
	0.59750559	48.8968837	63.6201295	43.090108	1000

7. The conclusion

We can make the comments of results in the above Simulation tables as the following:

1. The results presented in Table-1 of the simulation is conclusions may be summarized as following: According to the MSE criterion, when n=10 then the WE is best method and comes after LSE with all replicates (N=100,250,500,1000), and when n=25, 50 thin the MLE is best method and comes after LSE with all replicates N.

2. The results presented in Table-2 of the simulation is conclusions may be summarized as following: According to the MSE criterion, then the MLE is best method and comes after WE with all sample sizes (n=10, 25,50) and all replicates (N=100,250,500,1000).

3. The results presented in Table-3 of the simulation is conclusions may be summarized as following: According to the MSE criterion, then the MLE is best method and comes after WE with all sample sizes (n=10, 25,50) and all replicates (N=100,250,500,1000).

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