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Pole's decomposition for Burgers' hierarchy with dynamical systems of poles for solutions of its first and second level

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Abstract. Singularities (poles in particular) often occur in solutions of partial differential equations. In this paper we consider the Burgers hierarchy family of equations. We prove that this family enjoys poles decomposition property. This property may provide rational solutions with movable poles. We find the motion of these poles as dynamical systems for the first and second level of the hierarchy.

1. Introduction

A lot of nonlinear differential equations have local solutions that develop singularities and there is often a physical interpretation for occurrence of singularities in mathematical models (e.g., ignition in combustion, focusing in optics, cusps in free-surface flows, etc.) [30]. (Here all singularities are assumed to be poles). It is often a mathematical challenge to prove that all time regularity or occurring of singularity for solution of a nonlinear partial differential equation [29]. There exist a lot of tools that can be used to find domains of analyticity of solutions of nonlinear PDEs, such as: Pade approximants, Painleve expansions, pole dynamics, spectral methods, the abstract Cauchy-Kowalewski theorem, and, more generally, methods that involve analytic norms [26]. It is important to determine the singularities' location, nature, and their dependence on the initial data [30]. Kruskal's work [20] originated the method of pole dynamics then Calogero [6] and the Choodnovskys [8] developed a very powerful method to solve nonlinear PDEs. They showed that a very large class of nonlinear evolution equations has an associated N-body problem or equivalent N-body formulation. Case in 1979 [7] found the dynamic of poles for Bebjamin-Ono related equations. In [12], Fournier and Frisch considered complex spatial singularities for the inviscid Burgers equation from a deterministic and a statistical viewpoint. Thual, Frisch, and Henon [15] used pole dynamics to find the stationary pole distribution and stationary solution of a Sivashinsky-type flame-front propagation pseudo-differential equation. Bessis and Fournier [2, 3] analyzed the spatial analytic properties of Burgers equation for the inviscid and viscous deterministic cases using generic initial data. Kimura [18] described complex space and time pole positions for the Burgers equation with periodic initial data by solving for the roots of the Cole-Hopf variable. Senouf in 1997 [26] used the Mittag-Leffler expansion for pole where the complex spatial poles are time dependent, in his work, compatibility conditions are found as a

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dynamical system for these poles called the Calogero dynamical system. Deconinck, Kimura and Segur in 2007 [10], gives a special form of rational solution for Burgers' equation, they showed that the dynamical system which governed poles' motion of the rational solution of Burgers' equation is completely integrable. For Burgers hierarchy of level more than one, we do not know if there is any similar study This paper studies Burgers hierarchy (specially for the first and second levels corresponding to Burgers and Sharma-Tasso-Olver equations respectively). Our strategy is using poles' decomposition to get the rational solution, by which we can find the dynamical system for the poles of the rational solutions analytically. In section two, the rational solutions are derived for the Burgers hierarchy by pole decomposition; and then in section three the dynamics of poles for the rational solutions for the first and second level of Burgers hierarchy are derived.

2. decomposition of poles property for Burgers' hierarchy

2.1 Burgers' Hierarchy of Nonlinear Evolution Equations

The Burgers hierarchy (BH) is a family of nonlinear evolution equations, which can be written as: [21]

$$u_t + \alpha \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} + u\right)^n u = 0, \ \ \alpha \in \mathbb{R}, \ \ and \ \ \ n = 1,2,3, \dots \ . \tag{2.1}$$

When $n = 1$ the Eq. (2.1) is the Burgers equation (BE) [4].

$$u_t + 2\alpha u u_x + \alpha u_{xx} = 0 \, . \tag{2.2}$$

This equation can be linearized using the Cole-Hopf transformation [9, 17]. And it has Exact solutions that were discussed in e.g. [11,25].

When $n = 2$, the Eq. (2.1) give us the Sharma - Tasso - Olver (STO) equation [24,28]

$$u_t + \alpha u_{xxx} + 3\alpha u_x^2 + 3\alpha u u_{xx} + 3\alpha u^2 u_x = 0 \tag{2.3}$$

In [16,22] Some exact solutions of this equation were obtained.

when $n = 3$ and $n = 4$, we get the following PDEs:

$$u_t + \alpha u_{xxxx} + 10\alpha u_x u_{xx} + 4\alpha u u_{xxx} + 12\alpha u u_x^2 + 6\alpha u^2 u_{xx} + 4\alpha u^3 u_x = 0 \, . \tag{2.4}$$

$$u_t + \alpha u_{xxxxx} + 10\alpha u_{xx}^2 + 15\alpha u_x u_{xxx} + 5\alpha u u_{xxx} + 15\alpha u_x^3 + 50\alpha u u_x u_{xx} + 10\alpha u^2 u_{xxx} + 30\alpha u^2 u_x^2 + 10\alpha u^3 u_{xx} + 5\alpha u^4 u_x = 0 \tag{2.5}$$

2.2 Poles Decomposition Technique

The technique of allowing solution of certain PDE, or partial integro equation to be reduced to a finite set of ODEs for the position of poles in complex plane, is called poles decomposition technique.

Most of the completely integrable models of partial differential equations have finite dimensional solutions whose degrees of freedom are movable singularities (poles) in the complex plane [1,14]. Some of the most notable examples of pole decomposition in integrable systems can be found in [5,8,19, 23].

2.3 Poles Decomposition for Burgers' Hierarchy

Suppose that $u(x,t)$ be a solution for a given PDE, such that $u(x,t)$ is meromorphic function in prescribed region D, hence $u(x,t)$ has finite number of poles say (N). Assuming $u(x,t) = f(P(x,t))$, where f is a function and P is explicitly known polynomial (algebraic or trigonometric) in space variable x, with polynomial dependence on the time t. Frisch [14] proved that such assumed solution has polar singularities on complex algebraic varieties, and called *P pole decomposition* for the given equation.

Consider BE (2.2), (the simplest case of BH), which is linearized using Cole-Hopf transformation [17].

$$u(x,t) = -2\alpha \frac{P_x(x,t)}{P(x,t)} \quad , \qquad or \quad u(x,t) = -2\alpha \frac{\partial}{\partial x} \log P(x,t) \tag{2.6}$$

that maps BE into the heat equation

$$P_t(x,t) = \alpha P_{xx}(x,t) \tag{2.7}$$

Therefore, BE has the pole decomposition [8], [13].
using this transformation for BE we have,

$$u_t + \alpha \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} + u\right)^n u = \frac{\partial}{\partial x}\left(\frac{P_t + \alpha P_{n+1,x}}{p}\right) \tag{2.8}$$

here, $P_{k,x}$ we mean k^{th} partial derivative of P for x [21].

Kudryashov [21], presented A generalization of the Cole-Hopf transformation, which he used to find different types of exact solutions: the rational, the periodic, and the solitary wave solutions.

Note 2.1

$$\frac{\partial}{\partial x}\, log \prod_{i=1}^{N} F_i(x) = \frac{\partial}{\partial x} \sum_{i=1}^{N} logF_i(x) = \sum_{i=1}^{N} \frac{\frac{\partial}{\partial x} F_i(x)}{F_i(x)}, \qquad n = 1,2,\dots . \tag{2.9}$$

Proof: (straightforward from the property of log function) □

Proposition 2.1

If that the PDE,

$$F(x,t,u,u_x,u_t,\dots) = 0 \tag{2.10}$$

is transformed into the PDE,

$$G(x,t,p,p_x,p_t,\dots) = 0 \tag{2.11}$$

By using the Cole-Hopf transformation

$$u(x,t) = \frac{\partial}{\partial x} \log[P(x,t)]. \tag{2.12}$$

Then Eq. (2.10) has the pole decomposition property, when Eq. (2.11) has a polynomial solution of degree N.

Proof:

Suppose $P$is an algebraic polynomial of degree N, as follows:

$$P(x,t) = \prod_{i=1}^{N} \big(x - z_i(t)\big) \quad . \tag{2.13}$$

hence;

$$\log P(x,t) = \sum_{i=1}^{N} \log\big(x - z_i(t)\big) \quad . \tag{2.14}$$

Therefore, using lemma 1

$$\frac{\partial}{\partial x}\, log \prod_{i=1}^{N} \big(x - z_i(t)\big) = \sum_{i=1}^{N} \frac{1}{x - z_i(t)} . \tag{2.15}$$

Or.

$$u(x,t) = \sum_{i=1}^{N} \frac{1}{x - z_i(t)} . \tag{2.16}$$

In the ansatz (2.16), any rational solutions is called pole representation or a polar form of the solution (2.12), where $z_i(t)$, $i = 1, \dots, N$ are N simple poles of (2.16).

Example 2.1
The equation $p_t = p_{xx}$, (heat equation), has the polynomial solution

$$p(x,t) = 2c_1 t + 6c_2 t\, x + c_1 x^2 + c_2 x^3 = \prod_{i=1}^{3}(x - z_i(t)) \ .$$

where c_1 , $c_2 \in \mathbb{C}$ and $z_i(t) = z_i(c_1, c_2, t)$.
Therefore BE. has the pole decomposition property, hence it has a rational solution of the ansatz (2.16) with $n = 3$,

$$u(x,t) = \sum_{i=1}^{3} \frac{1}{x - z_i(t)}$$

Corollary 2.1
For any n The Burgers hierarchy equation, has pole decomposition property.
Proof:
as eq. (2.1) can be linearized by using Cole-Hopf transformation [21], the transformation (2.12), gives the rational solution (2.16).
in the above corollary, transformation (2.6), gives the rational solution:

$$u(x,t) = -2\alpha \sum_{i}^{N} \frac{1}{x - z_i(t)} \quad . \qquad (2.17)$$

3. Dynamical system of poles for rational Solution of the BE and the STO equation

Here, we find the dynamics of poles of rational solutions analytically, especially for the first and second levels of Burgers hierarchy, which corresponds to BE and STO equation, respectively. using poles decomposition which lead to the rational solutions.
In addition, we find the dynamics of poles of new rational solutions resulting from the linear combination of the (2.16) and its conjugate.

3.1 Poles Motion for Rational Solution in the Complex Plane

Here We proved that: the pole expansion of the rational solution, leads to a dynamical system with constraints describing the time evolution of the poles.
The ansatz of the rational solution (2.17), can be used for BE, STO equation or viscous Burgers equation,

$$u_t + uu_x - \alpha u_{xx} = 0 \ , \ \ x \in \mathbb{C}, \ \ and\ \alpha \in \mathbb{R}, \qquad (3.1)$$

and then the behaviour near each pole is examined. for the pole z_l, put $\epsilon = x - z_l$, where x is in a sufficiently small neighbourhood $N_\epsilon(z_l)$ of z_l . This gives a singular term as $\epsilon \longrightarrow 0$, for each non-positive power of ϵ. The dynamical system of the pole z_l is then found by equating the coefficients of these ϵ with non-positive powers to zero. this leads to the motion of poles.

Lemma 3.1
For every complex numbers $x\ and\ z_j$ where $j = 1, \dots, N$, we have the following:
(a)

$$\sum_{i}^{N}\sum_{i\neq j}^{N}\frac{1}{x-z_i}\ \frac{1}{(x-z_j)^2}=\sum_{i}^{N}\sum_{i\neq j}^{N}\frac{1}{z_i-z_j}\ \frac{1}{(x-z_j)^2}$$
$$=\sum_{i}^{N}\sum_{i\neq j}^{N}\frac{1}{z_j-z_i}\ \frac{1}{(x-z_i)^2} \tag{3.2}$$

(b)

$$\sum_{i}^{N}\sum_{i\neq j}^{N}\frac{1}{x-z_j}\ \frac{1}{(x-z_i)^3}$$
$$=\sum_{i}^{N}\sum_{i\neq j}^{N}\frac{-2}{(z_i-z_j)^3}\ \frac{1}{(x-z_i)}-\frac{1}{(z_i-z_j)^2}\ \frac{1}{(x-z_i)^2}$$
$$-\frac{1}{(z_i-z_j)}\ \frac{1}{(x-z_i)^3} \tag{3.3}$$

(c)

$$\sum_{i}^{N}\sum_{i\neq j}^{N}\ \frac{1}{(x-z_i)^2(x-z_j)^2}=2\sum_{i}^{N}\sum_{i\neq j}^{N}\frac{1}{(z_i-z_j)^2}\ \frac{1}{(x-z_i)^2}$$
$$-4\sum_{i}^{N}\sum_{i\neq j}^{N}\frac{1}{(z_i-z_j)^3}\ \frac{1}{(x-z_i)} \tag{3.4}$$

(d)

$$\sum_{i}^{N}\sum_{i\neq j}^{N}\sum_{i\neq k,j\neq k}^{N}\frac{1}{(x-z_i)^2(x-z_j)(x-z_k)}$$
$$=\sum_{i}^{N}\sum_{i\neq j}^{N}\sum_{i\neq k,j\neq k}^{N}\frac{1}{(z_i-z_j)(z_i-z_k)(x-z_i)^2} \tag{3.5}$$

Proof: (straightforward)

3.2 Dynamics of Poles for Rational Solution of BE

Theorem 3.1
If the solution of BE(3.1) is of the form of ansatz (2.17) then the motion of poles of this solution is described by the dynamical system:

$$\dot{z}_l=-2\alpha\sum_{l\neq j}^{N}\frac{1}{z_j-z_l}\quad l=1,2,\dots,N$$

Proof:
substitute (2.17) in (3.1), we get:

$$-2\alpha\sum_{l}^{N}\dot{z}_l\,(x-z_l)^{-2}-4\alpha^2\left(\sum_{l}^{N}(x-z_l)^{-1}\right)\left(\sum_{l}^{N}(x-z_l)^{-2}\right)+4\alpha^2\sum_{l}^{N}(x-z_l)^{-3}=0\ ,$$
$$\dot{z}_l=\frac{dz_l(t)}{dt} \tag{3.6}$$

$$\sum_{l}^{N} \dot{z}_l \,(x-z_l)^{-2} + 2\alpha \sum_{\boldsymbol{l}}^{\boldsymbol{N}} \sum_{\boldsymbol{j}}^{\boldsymbol{N}} (x-z_l)^{-2} \left(x-z_j\right)^{-1} - 2\alpha \sum_{l}^{N} (x-z_l)^{-3} = 0 \qquad (3.7)$$

$$\sum_{l}^{N} \dot{z}_l \,(x-z_l)^{-2} + 2\alpha \sum_{\boldsymbol{l}}^{\boldsymbol{N}} \sum_{\boldsymbol{l\neq j}}^{\boldsymbol{N}} (x-z_l)^{-2} \left(x-z_j\right)^{-1} + 2\alpha \sum_{\boldsymbol{l}}^{\boldsymbol{N}} (x-z_l)^{-3} - 2\alpha \sum_{l}^{N} (x-z_l)^{-3} = 0 \qquad (3.8)$$

By eq. (3.2), (3.8) becomes

$$\sum_{l}^{N} \dot{z}_l \frac{1}{(x-z_l)^2} + 2\alpha \sum_{\boldsymbol{l}}^{\boldsymbol{N}} \sum_{\boldsymbol{l\neq j}}^{\boldsymbol{N}} \frac{1}{z_l - z_j} \; \frac{1}{(x-z_l)^2} = 0 \qquad (3.9)$$

if x is near z_l, then the lift hand side (LHS) of (3.9) has the double pole z_l. Thus (3.9) becomes

$$\left[\dot{z}_l + 2\alpha \sum_{\boldsymbol{l\neq j}}^{\boldsymbol{N}} \frac{1}{z_l - z_j} \right] (x-z_l)^{-2} = 0 \quad , \qquad l = 1,2 \ldots, N \qquad (3.10)$$

since $\epsilon = (x - z_l)$ is non-zero, we have

$$\dot{z}_l + 2\alpha \sum_{\boldsymbol{l\neq j}}^{\boldsymbol{N}} \frac{1}{z_l - z_j} = 0 \qquad l = 1,2 \ldots, N \qquad (3.11)$$

Or,

$$\dot{z}_l = -2\alpha \sum_{\boldsymbol{l\neq j}}^{\boldsymbol{N}} \frac{1}{z_l - z_j} \qquad l = 1,2 \ldots, N \qquad (3.12)$$

The last equation (3.12) is given in [18], [23]. They give this result without any proof, or the method to get it. While here, we state the proof.

Next, we will use new rational solutions (3.13a) and (3.13b), which are a linear combinations of the solutions (2.16) and (2.17) with their conjugate respectively, these solution is given by:

$$u(x,t) = \sum_{l}^{N} \frac{1}{x - z_l(t)} + \sum_{l}^{N} \frac{1}{x - \overline{z_l}(t)} \qquad (3.13\ a)$$

$$u(x,t) = -2\alpha \sum_{l}^{N} \frac{1}{x - z_l(t)} - 2\alpha \sum_{l}^{N} \frac{1}{x - \overline{z_l}(t)} \qquad (3.13\ b)$$

Theorem 3.2

If the solution of Burgers equation (3.1), is of the ansatz (3.13b) then the motion of poles of this solution is described by the dynamical system:

$$\dot{z}_l = -2\alpha \sum_{l\neq j}^{N} \frac{1}{z_l - z_j} - 2\alpha \sum_{j}^{N} \frac{1}{z_l - \overline{z_j}}$$

Proof:

Similar to the proof of theorem 3.1, we have

$$\dot{z}_l = -2\alpha \sum_{l\neq j}^{N} \frac{1}{z_l - z_j} - 2\alpha \sum_{j}^{N} \frac{1}{z_l - \overline{z_j}} \qquad (3.14)$$

$$\dot{\overline{z_l}} = -2\alpha \sum_{l\neq j}^{N} \frac{1}{\overline{z_l} - \overline{z_j}} - 2\alpha \sum_{j}^{N} \frac{1}{\overline{z_l} - z_j} \qquad (3.15)$$

Replacing the pole z_l by its complex conjugate $\overline{z_l}$ in (3.14) we get (3.15) so (3.14) is the required dynamical system.

□

3.3 Dynamics of Poles for Rational Solution of STO equation

Theorem 3.3
If the solution of STO equation (2.3) is of the form (2.16), then the motion of the poles of this solution is described by the dynamical system

$$\dot{z}_l = 3\alpha \sum_{l\neq j}^{N} \sum_{\substack{l\neq k,\\ j\neq k}}^{N} \frac{1}{(z_l - z_j)(z_l - z_k)}$$

With the two constraints:

$$0 = \sum_{l\neq j}^{N} \frac{1}{(z_l - z_j)^3}$$

$$0 = \sum_{l\neq j}^{N} \frac{1}{(z_l - z_j)}$$

Proof:
Substitute (2.16), in (2.3) we have

$$\sum_{l}^{N} \dot{z}_l (x - z_l)^{-2} - 6\alpha \sum_{l}^{N} (x - z_l)^{-4} + 3\alpha \left(\sum_{l}^{N} (x - z_l)^{-2} \right) . \left(\sum_{l}^{N} (x - z_l)^{-2} \right) + 6\alpha \left(\sum_{l}^{N} (x - z_l)^{-1} \right) . \left(\sum_{l}^{N} (x - z_l)^{-3} \right) - 3\alpha \left(\sum_{l}^{N} (x - z_l)^{-1} \right) . \left(\sum_{l}^{N} (x - z_l)^{-1} \right) . \left(\sum_{l}^{N} (x - z_l)^{-2} \right) = 0 \quad (3.16)$$

i.e,

$$\sum_{l}^{N} \dot{z}_l (x - z_l)^{-2} + 3\alpha \left(\sum_{l}^{N} \sum_{l\neq j}^{N} (x - z_j)^{-2} (x - z_l)^{-2} \right) + 6\alpha \left(\sum_{l}^{N} \sum_{\boldsymbol{l\neq j}}^{N} (x - z_j)^{-1} (x - z_l)^{-3} \right) - 3\alpha \sum_{\boldsymbol{l}}^{\boldsymbol{N}} \sum_{\boldsymbol{l\neq j}}^{\boldsymbol{N}} \sum_{\substack{\boldsymbol{l\neq k},\\ \boldsymbol{j\neq k}}}^{\boldsymbol{N}} (x - z_l)^{-2} (x - z_k)^{-1} (x - z_j)^{-1} = 0 \quad (3.17)$$

using eq. (3.3) - (3.5), (3.17) becomes

$$\sum_{l}^{N} \dot{z}_l \frac{1}{(x - z_l)^2} + 3\alpha \left(\sum_{l}^{N} \sum_{l\neq j}^{N} \frac{2}{(z_l - z_j)^2} \frac{1}{(x - z_l)^2} - \sum_{l}^{N} \sum_{l\neq j}^{N} \frac{4}{(z_l - z_j)^3} \frac{1}{(x - z_l)} \right) + 6\alpha \left(\sum_{l}^{N} \sum_{l\neq j}^{N} \frac{-2}{(z_l - z_j)^3} \frac{1}{(x - z_l)} - \frac{1}{(z_l - z_j)^2} \frac{1}{(x - z_l)^2} - \frac{1}{(z_l - z_j)} \frac{1}{(x - z_l)^3} \right) - 3\alpha \sum_{\boldsymbol{l}}^{\boldsymbol{N}} \sum_{\boldsymbol{l\neq j}}^{\boldsymbol{N}} \sum_{\substack{\boldsymbol{l\neq k},\\ \boldsymbol{j\neq k}}}^{\boldsymbol{N}} \frac{1}{(z_l - z_j)(z_l - z_k)(x - z_l)^2} = 0 \quad (3.18)$$

Or,

$$\sum_{l}^{N}\dot{z}_l\frac{1}{(x-z_l)^2}-24\alpha\sum_{l}^{N}\sum_{l\neq j}^{N}\frac{1}{(z_l-z_j)^3}\ \frac{1}{(x-z_l)}-6\alpha\sum_{l}^{N}\sum_{l\neq j}^{N}\frac{1}{(z_l-z_j)}\ \frac{1}{(x-z_l)^3}$$
$$-3\alpha\sum_{\boldsymbol{l}}^{\boldsymbol{N}}\sum_{\boldsymbol{l\neq j}}^{\boldsymbol{N}}\sum_{\substack{\boldsymbol{l\neq k,}\\ \boldsymbol{j\neq k}}}^{\boldsymbol{N}}\frac{1}{(z_l-z_j)(z_l-z_k)(x-z_l)^2}=0 \qquad (3.19)$$

if x is near z_l, for any l, then the LHS of (3.19) has a pole of order 3, z_l, assuming $\epsilon=(x-z_l)$, we get

$$\sum_{l}^{N}\dot{z}_l\,\epsilon^{-2}-24\alpha\sum_{l}^{N}\sum_{l\neq j}^{N}\frac{1}{(z_l-z_j)^3}\ \epsilon^{-1}-6\alpha\sum_{l}^{N}\sum_{l\neq j}^{N}\frac{1}{(z_l-z_j)}\ \epsilon^{-3}$$
$$-3\alpha\sum_{\boldsymbol{l}}^{\boldsymbol{N}}\sum_{\boldsymbol{l\neq j}}^{\boldsymbol{N}}\sum_{\substack{\boldsymbol{l\neq k,}\\ \boldsymbol{j\neq k}}}^{\boldsymbol{N}}\frac{1}{(z_l-z_j)(z_l-z_k)}\,\epsilon^{-2}=0. \qquad (3.20)$$

for any $l=1,2,\dots N$, we have

$$\left[\dot{z}_l-3\alpha\sum_{l\neq j}^{N}\sum_{\substack{l\neq k,\\ j\neq k}}^{N}\frac{1}{(z_l-z_j)(z_l-z_k)}\right]\epsilon^{-2}-24\alpha\sum_{l\neq j}^{N}\frac{1}{(z_l-z_j)^3}\ \epsilon^{-1}-6\alpha\sum_{l\neq j}^{N}\ \frac{1}{(z_l-z_j)}\epsilon^{-3}=0\,,$$
$$l=1,2,\dots N. \qquad (3.21)$$

since ϵ is non-zero, and l is arbitrary , we have the dynamical system with two constraints:

$$\dot{z}_l=3\alpha\sum_{l\neq j}^{N}\sum_{\substack{l\neq k,\\ j\neq k}}^{N}\frac{1}{(z_l-z_j)(z_l-z_k)} \qquad (3.22a)$$

$$0=\sum_{l\neq j}^{N}\frac{1}{(z_l-z_j)^3} \qquad (3.22b)$$

$$0=\sum_{l\neq j}^{N}\frac{1}{(z_l-z_j)} \qquad (3.22c)$$

The motion of poles z_l of rational solution for STO equation, described by The dynamical system (3.22a) and the constraints (3.22b)- (3.22c). □

Next, we use the rational solution (3.13a), for the STO equation.

Theorem 3.4

If the ansatz (3.13a), is a solution of STO equation (2.3) then, the following dynamical system describe the motion of poles of this solution.

$$\dot{z}_l=3\alpha\sum_{l\neq j}^{N}\sum_{\substack{l\neq k,\\ j\neq k}}^{N}\left(\frac{1}{(z_l-z_j)(z_l-z_k)}+\frac{1}{(z_l-\overline{z_j})(z_l-z_k)}+\frac{1}{(z_l-z_j)(z_l-\overline{z_k})}+\frac{1}{(z_l-\overline{z_j})(z_l-\overline{z_k})}\right)$$

With two constraints:

$$0=\sum_{l\neq j}^{N}\left(\frac{1}{(z_l-z_j)^3}+\frac{1}{(z_l-\overline{z_j})^3}\right)$$

$$0=\sum_{l\neq j}^{N}\left(\frac{1}{(z_l-z_j)}+\frac{1}{(z_l-\overline{z_j})}\right)$$

Proof:
Similar to the proof of theorem 3.3, we have

$$\dot{z}_l = 3\alpha \sum_{l\neq j}^{N} \sum_{\substack{l\neq k,\\ j\neq k}}^{N} \left(\frac{1}{(z_l - z_j)(z_l - z_k)} + \frac{1}{(z_l - \overline{z_j})(z_l - z_k)} + \frac{1}{(z_l - z_j)(z_l - \overline{z_k})} + \frac{1}{(z_l - \overline{z_j})(z_l - \overline{z_k})}\right) \qquad (3.23a)$$

With two constraints:

$$0 = \sum_{l\neq j}^{N} \left(\frac{1}{(z_l - z_j)^3} + \frac{1}{(z_l - \overline{z_j})^3}\right) \qquad (3.23b)$$

$$0 = \sum_{l\neq j}^{N} \left(\frac{1}{(z_l - z_j)} + \frac{1}{(z_l - \overline{z_j})}\right) \qquad (3.23c)$$

And

$$\dot{\overline{z}}_l = 3\alpha \sum_{l\neq j}^{N} \sum_{\substack{l\neq k,\\ j\neq k}}^{N} \left(\frac{1}{(\overline{z_l} - z_j)(\overline{z_l} - z_k)} + \frac{1}{(\overline{z_l} - \overline{z_j})(\overline{z_l} - z_k)} + \frac{1}{(\overline{z_l} - z_j)(\overline{z_l} - \overline{z_k})} + \frac{1}{(\overline{z_l} - \overline{z_j})(\overline{z_l} - \overline{z_k})}\right) \qquad (3.24a)$$

with two constraints:

$$0 = \sum_{l\neq j}^{N} \left(\frac{1}{(\overline{z_l} - \overline{z_j})^3} + \frac{1}{(\overline{z_l} - z_j)^3}\right) \qquad (3.24b)$$

$$0 = \sum_{l\neq j}^{N} \left(\frac{1}{(\overline{z_l} - \overline{z_j})} + \frac{1}{(\overline{z_l} - z_j)}\right) . \qquad (3.24c)$$

Clearly replacing the pole z_l in (3.23) by its conjugate $\overline{z_l}$ we get (3.24); so (3.23) is required dynamical system.

4. Conclusion

We proved that for any n The general Burgers hierarchy, enjoys the pole decomposition property, by using the generalization Cole-Hopf transformation for general Burgers hierarchy family of equations, also we found the dynamical systems pols of rational solutions for Burgers equation and STO equation, that correspond to $n = 1$ and $n = 2$, respectively in Burgers hierarchy family of equations. To cover the cases for $n > 2$, we need to generalize the lemma 3.1.

5. References

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