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Moduli Space of the Category of Fuzzy Topographic Topological Mappings

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Abstract. The moduli space has concept emerged to classify objects of certain categories by carrying out their parameterization. In fact, the theory of moduli space has been widely appreciated from a distant time in diverse fields not only of mathematics, but also of physics. This paper is devoted to establish the moduli space of the category of the mathematical model fuzzy topographic topological mappings, which is adopted in solving neuromagnetic inverse problem during a seizure. In this study, the moduli functor of fuzzy topographic topological mapping was constructed. The applicability of the theory of moduli space is uniquely revealed with specifying its sort.

1. Introduction

Fuzzy Topographic Topological Mapping (FTTM) is a mathematical model which was introduced in 1999 to solve the neuromagnetic inverse problem for determining the location of epileptic foci in epilepsy disorder patient. Currently, there are three versions of FTTM which are FTTM1 [1], fuzzy topographic topological mapping version 2 (FTTM2) [2], and fuzzy topographic topological mapping digital (FTTMdig) [3]. All these versions are structured based on mathematical concepts of topology and fuzzy set. Furthermore, FTTM1 and FTTM2 were developed to present a 3-dimensional view of unbounded single current source and bounded multi current sources, respectively. In addition, the sequence of n versions of FTTM is defined by [4].

The concept of “moduli” for mathematical objects has various meanings. It arises as solutions to classification problems. The moduli problem refers to the all complex of problems related with existing moduli spaces for certain families of objects having something in common with the given one, in addition to the study of the algebraic-geometric properties and the methods for compactification for moduli space. Generally, moduli space is used to describe a set of parameters determining the isomorphism class of mathematical objects. The moduli space is an old subject and it has two types, which are moduli space of Riemann surfaces and moduli space of curves. The principal inventor of the moduli space was Bernard Riemann in 1857, when he justified the formula for the dimension of the space of isomorphism classes of a compact Riemann surface [5].

The study of the moduli space of Riemann surfaces has a long history. It has been at the centre of pure mathematics for more than 100 years now. Despite its age, this field still attracts a lot of attention. This space lies within the intersection of the many fields of mathematics and might,



therefore, be studied from various points of view. Many in-depth results have been obtained throughout history by many famous mathematicians. In addition, understanding the geometry and topology of the moduli space of Riemann surfaces and the corresponding mapping class groups has been a goal of central importance in mathematics for many years. In the last 20 years there have been several new perspectives on moduli spaces that have not only increased our understanding of these important objects but have fundamentally affected major research directions of several areas of topology and geometry.

Later, the precise formulation of Riemann's results were declared in 1939 by Teichmüller, by establishing some of these results by using his theory of extremal quasi-conformal mappings. Indeed, he characterized the Teichmüller theory through using the tools of geometry, topology, and analysis to study moduli spaces of Riemann surfaces. Moreover, Rauch adopted a series of investigations on the moduli space of Riemann surfaces, some 1951 on numerical moduli space, while the other investigations were in 1955 on the algebraic Riemann surfaces [6].

According to [5] the proof of other Teichmüller formulations of the statements of Riemann were verified by Bers in 1960. Meanwhile, Bers and Ahlfors independently devised the Teichmüller theory, which were consequently influenced to appear in the school of Ahlfors-Bers of the complex analytic approach to the Teichmüller theory for studying moduli spaces of Riemann surfaces. Since then, this school has had a strong presence in the USA.

On the other hand, many studies have been conducted on the moduli space of Riemann surfaces in algebraic geometry. In fact, Mumford in 1960 introduced algebro-geometric approach to construct the moduli space of Riemann surfaces via geometric invariant theory and he gave precise definitions of moduli spaces and methods for constructing them. In addition, Mumford stated his approach in construction of the moduli space of Riemann surfaces in his book in 1965, which is titled "Geometric Invariant Theory". Currently, this approach is being developed in India [7].

On another hand, the moduli space of curves has proven itself a central object in geometry due to its importance in many subjects such as geometry, algebraic topology and topology differential, physics, and combinatorics. During the past few decades, moduli spaces of curves have achieved notoriety amongst mathematicians for their incredible structure. However, the study of this space was developed with great difficulty. It is worth noting that the moduli space of curves means algebraic isomorphism classes of curves of algebraic genus [8].

In 1960, Grothendieck formulated the moduli problems in terms of the moduli functors by using Teichmüller spaces in complex analytical geometry as an example. In general, Grothendieck established the category approach to find a moduli space. In addition, he presented the representable functor for the fundamental problem of moduli in order to determine the type of the moduli space. In 1962, Baily verified that the moduli space of curves is a quasi-projective. Then, Mumford in 1965 used the method of Grothendieck in the construction the moduli space of curves, which represented the first purely algebraic construction of the moduli space of curves. Further, he used the geometric invariant theory to find the moduli space of curves. Up to date, the categorical approach is still vigorously pursued due to it is applicable and accurate. Interestingly, it is influenced in extension several inventions as shown in this paper [7].

The aim of this paper is to employ the category approach of moduli space in construction the moduli space of the category of FTTM. The space determined by the parameters is called moduli space as given in the following definition [9].

Definition 1. Given a category C . A moduli problem is comprised the following things:

- Specifying a class of objects with the notion of a family of these objects over an object B which is called a base of a family.
- Choosing an equivalent relation on the set $S(B)$ (family over B) of all such families over each base B .
- Determining the set of all isomorphism classes between the families, which is the moduli space $\mathcal{M}_C$ for the category C .

2. Fuzzy Topographic Topological Mapping (FTTM)

Fuzzy Topographic Topological Mapping Version 1 (FTTM1) is used to solve the inverse problem for determining single current source. It consists of four components with three algorithms that link between them. These components of FTTM1 are magnetic contour plane (M_1), base magnetic plane (B_1), fuzzy magnetic field (F_1), and topographic magnetic field (T_1) (refer to Figure 1) [2].

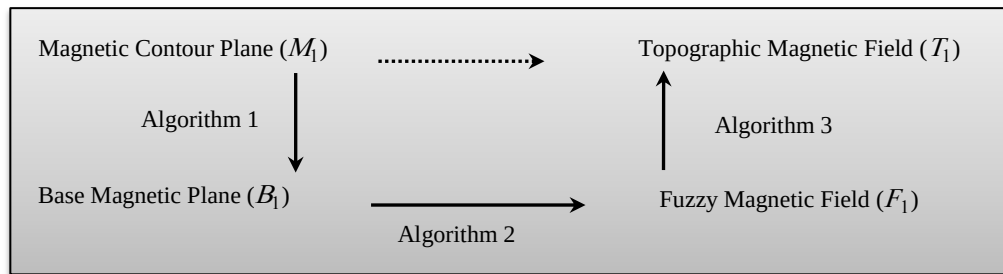


Figure 1. FTTM version 1.

The component M_1 is a magnetic field on a plane above a current source with $z = 0$, whereas the plane is lowered down to B_1 and it is generated by the single current source, such that

$$M_1 = \left\{ \left((x,y)_0, B_Z(x,y) \right) : x,y \in \mathbb{R}, \right. \\ \left. B_Z(x,y) \in [B_{Z\min}, B_{Z\max}] \right\} \quad (1)$$

Where

$$B_Z(x,y) = \frac{\mu_o I}{2\pi} \left[\frac{(y-y_p)}{(y-y_p)^2 + [h + |x-x_p| \tan(\varphi - 90^\circ)]^2} \right] \quad (2)$$

$B_{Z\min}$ is the smallest magnetic fields reading, $B_{Z\max}$ is the largest magnetic field reading, μ_o is the permeability of free space and its value is $4\pi \times 10^{-7}$ (meter. Tesla/ ampere), I is the magnitude of current in ampere, $((x_p, y_p)_0, B_Z(x_p, y_p))$ is the element of M_1 which is exactly above the current source, φ is the angle between current source and z -axis, and h is the distance between M_1 and the current source in meter [2].

In addition, Faisal in 2011 proposed that every single second of elliptic seizure is stored in a square matrix, which includes the position of electrode on M_1 [10]. Subsequently, M_1 became a set of $(n \times n)$ square matrices (EEG signals) defined as:

$$M_{1n}(\mathbb{R}) = \left\{ \begin{array}{l} [\beta_{ij}(z)_t]_{n \times n} : i, j \in \mathbb{Z}^+, \\ \beta_{ij}(z)_t \in \mathbb{R} \end{array} \right\} \quad (3)$$

where β_{ij} is a potential difference reading of EEG signals from a particular at time t . Base magnetic plane (B_1) is a plane of the current source with $z = -h$. Then the entire B_1 is fuzzified into a fuzzy environment (F_1) where all the magnetic field readings are fuzzified. Finally, a three-dimensional presentation of F_1 is plotted on B_1 . The final process is defuzzification of the fuzzified data to obtain a 3-dimensional view of the current source (T_1). The components of FTTM are defined as follows:

$$B_1 = \left\{ \begin{array}{l} ((x, y)_{-h}, B_{Z(x,y)}): x, y \in \mathbb{R}, \\ B_{Z(x,y)} \in [B_{Z \min}, B_{Z \max}] \end{array} \right\} \quad (4)$$

$$F_1 = \left\{ \begin{array}{l} ((x, y)_{-h}, \mu_{B_{Z(x,y)}}): x, y, -h \in \mathbb{R}, \\ \mu_{B_{Z(x,y)}} \in [0, 1] \end{array} \right\} \quad (5)$$

such that:

$$\mu_{B_{Z(x,y)}} = \frac{|B_{Z(x,y)}| - MB_{Z \min}}{B_{Z \max} - B_{Z \min}} \quad (6)$$

and $B_{Z(x,y)}$ as in Equation (2).

As well,

$$T_1 = \left\{ \begin{array}{l} (x, y, z_{B_{Z(x,y)}}): x, y \in \mathbb{R}, \\ z_{B_{Z(x,y)}} \in [-h, 0] \end{array} \right\} \quad (7)$$

where

$$z_{B_{Z(x,y)}} = h(\mu_{B_{Z(x,y)}} - 1) \quad (8)$$

The components of FTTM1 were proven spaces in 2001 by [1] and homeomorphics in 2005 by Ahmad et al. [11] as given in the following:

Theorem 1. In FTTM1, $M_1 \cong B_1 \cong F_1 \cong T_1$ by the following homeomorphisms:

a) $b_1: M_1 \rightarrow B_1$ such that:

$$b_1((x, y)_0, B_{Z(x,y)}) = ((x, y)_{-h}, B_{Z(x,y)}), \forall ((x, y)_0, B_{Z(x,y)}) \in M_1 \quad (9)$$

b) $f_1: B_1 \rightarrow F_1$ such that:

$$f_1((x, y)_{-h}, B_{Z(x,y)}) = ((x, y)_{-h}, \mu_{B_{Z(x,y)}}), \forall ((x, y)_{-h}, B_{Z(x,y)}) \in B_1 \quad (10)$$

c) $t_1: F_1 \rightarrow T_1$ such that:

$$t_1((x, y)_{-h}, \mu_{B_{Z(x,y)}}) = (x, y, z), \forall ((x, y)_{-h}, \mu_{B_{Z(x,y)}}) \in F_1 \quad (11)$$

with $-h < 0$ is a constant, and $B_z \in B \subseteq \mathbb{R}$.

On the other hand, FTTM2 is the extended version of FTTM1 which is specifically designed to solve the inverse problem of multi-current source. Similar to FTTM1, the model is comprised of four components. They are Magnetic Image Plane (M_2), Base Magnetic Image Plane (B_2), Fuzzy Magnetic Image Field (F_2) and Topographic Magnetic Image Field (T_2) [2] as shown in Figure 2.

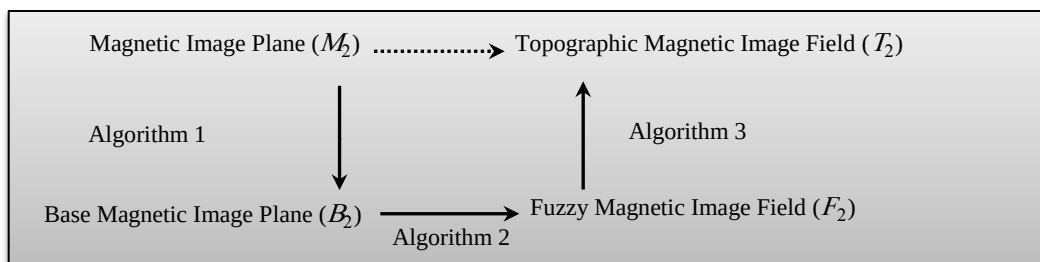


Figure 2. FTTM version 2.

As mentioned in [2] that the magnetic fields data which is laid on the component M_1 of FTTM1 are at once analyzed and transformed into image processing data in order to carry out the process in FTTM2. Therefore, the first component in FTTM2 is the component M_2 which has the grey scale image for readings [0, 255] of magnetic field. It lies on a plane above a current source with $z = 0$. When the plane is lowered down to the current source with $z = -h$, the component B_2 is obtained. The field of the component F_2 is the fuzzified B_2 plane, i.e. all the grey scale readings are fuzzified into a fuzzy environment. Finally, a three-dimensional presentation of F_2 field is plotted on T_2 field. This

final process is the defuzzification of the fuzzified data to obtain a 3-dimensional view of the current source.

FTTM2 is an improvement of FTTM1 because it presents the three-dimensional view of current source in four angles of observation (upper, left, right and back part of a head model). On the contrary, FTTM1 only presents the three-dimensional view of current source in one angle of observation (upper part of a head model). Furthermore, FTTM2 can be applied on single and multiple, bounded and unbounded current source in contrast to FTTM1 which can only be applied to single and unbounded current source.

In addition, [2] verified that the components of FTTM2 are spaces and she confirmed that FTTM2 is also designed to have equivalent topological structures between its components, as follow:

Theorem 2. In FTTM2, $M_2 \cong B_2 \cong F_2 \cong T_2$ by following homeomorphisms:

a) $b_2: M_2 \rightarrow B_2$ such that :

$$b_2((x, y)_0, M_I(x, y)) = ((x, y)_{-h}, M_I(x, y)), \forall ((x, y)_0, M_I(x, y)) \in M_2 \quad (12)$$

b) $f_2: B_2 \rightarrow F_2$ such that:

$$f_2((x, y)_{-h}, M_I(x, y)) = ((x, y)_{-h}, \mu_{M_I(x, y)}), \forall ((x, y)_{-h}, M_I(x, y)) \in B_2 \quad (13)$$

c) $t_2: F_2 \rightarrow T_2$ such that :

$$t_2((x, y)_{-h}, \mu_{M_I(x, y)}) = (x, y, z_{M_I(x, y)}), \forall ((x, y)_{-h}, \mu_{M_I(x, y)}) \in F_2 \quad (14)$$

with $-h < 0$ is a constant, $\mu_{M_I(x, y)} \in [0, 1]$ and $z_{M_I(x, y)} \in [-h, 0]$.

In the following theorem, the homeomorphism between each component of FTTM1 and its corresponding components of FTTM2 are given [2].

Theorem 3. Between FTTM1 and FTTM2, $M_1 \cong M_2$, $B_1 \cong B_2$, $F_1 \cong F_2$ and $T_1 \cong T_2$ by the following homeomorphisms:

a) $m_{1,2}: M_1 \rightarrow M_2$ such that:

$$m_{1,2}((x, y)_0, B_Z(x, y)) = ((x, y)_0, M_I(x, y)), \forall ((x, y)_0, B_Z(x, y)) \in M_1 \quad (15)$$

b) $b_{1,2}: B_1 \rightarrow B_2$ such that:

$$b_{1,2}((x, y)_{-h}, B_Z(x, y)) = ((x, y)_{-h}, M_I(x, y)), \forall ((x, y)_{-h}, B_Z(x, y)) \in B_1 \quad (16)$$

c) $f_{1,2}: F_1 \rightarrow F_2$ such that:

$$f_{1,2}((x, y)_{-h}, \mu_{B_Z(x, y)}) = ((x, y)_{-h}, \mu_{M_I(x, y)}), \forall ((x, y)_{-h}, \mu_{B_Z(x, y)}) \in F_1 \quad (17)$$

d) $t_{1,2}: T_1 \rightarrow T_2$ such that:

$$t_{1,2}(x, y, z_{B_Z(x, y)}) = (x, y, z_{M_I(x, y)}), \forall (x, y, z_{B_Z(x, y)}) \in T_1 \quad (18)$$

In 2010, [4] Ahmad *et al.* introduced a sequence of n versions of FTTM which is defined as FTTM1, FTTM2, FTTM3, ..., FTTM n where $n \in \mathbb{Z}^+$. The relations between the components in that sequence and the properties of these components are given in the following theorem.

Theorem 4. In the sequence of n versions of FTTM that:

a) $M_v \cong B_v \cong F_v \cong T_v$, in each version $v=1, 2, \dots, n \in \mathbb{Z}^+$ (see Figure 3).

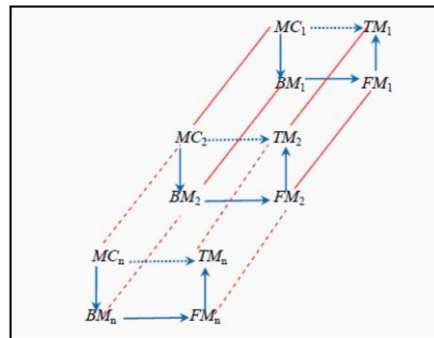


Figure 3. A sequence of n versions of FTTM.

- b) $M_v \cong M_{v+1}, B_v \cong B_{v+1}, F_v \cong F_{v+1}, T_v \cong T_{v+1}$, for any version $v=1, 2, \dots, (n-1) \in \mathbb{Z}^+$.
Furthermore, Faisal proved that FTTM is a large category [12].

3. Construction of Moduli Space of FTTM

The general goal in this section is to point out the moduli space of FTTM (MFTTM). For this purpose, a sample of seizure patients will be considered, which will be then divided into families based on the various properties and relations for the ingredients of FTTM. The following remarks contain the main abbreviations that will be used henceforth in this work.

Remark 1. Let m be the number of the patients, c_p be the time for patient p , and n be the number of versions of FTTM at each second s_p , such that $m, c_p, n \in \mathbb{Z}^+$ for each $p=1,2,3,\dots,m$, $s_p=1,2,3,\dots,c_p$, and $v=1,2,3,\dots,n$.

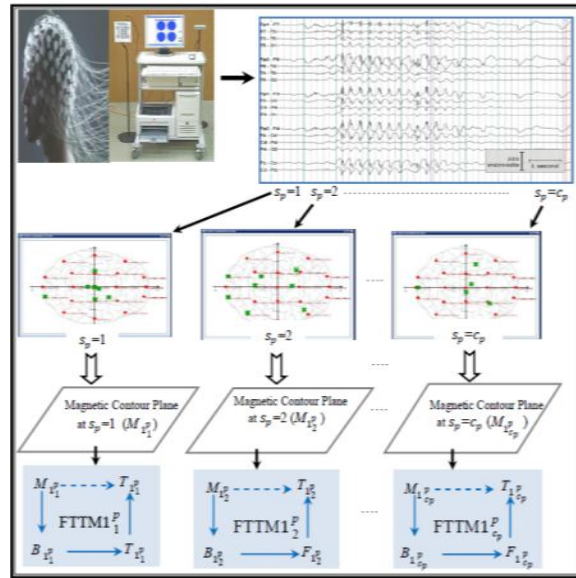
Then,

- a) Denote C_{FTTM} be a category of m patients in which there are n versions of FTTM at each second s_p , for any patient p . Then, FTTM is defined by:

$$\text{FTTM} = \{\text{FTTM}_v^{s_p} = (M_v^{s_p}, B_v^{s_p}, F_v^{s_p}, T_v^{s_p}) : M_v^{s_p} \cong B_v^{s_p} \cong F_v^{s_p} \cong T_v^{s_p},$$

$$\forall v=1,2,3,\dots,n, \forall s_p=1,2,3,\dots,c_p \text{ and } m, c_p, n \in \mathbb{Z}^+, \forall p=1,2,3,\dots,m\} \quad (19)$$

b) The component M_1 generates FTTM1 (see Figure 4), and then FTTM at each second for any patient.

**Figure 4.** Generation FTTM1 at each second for a patient.

- c) Without loss of generality, for any patient, FTTM1, FTTM2, ..., FTTMn will be indicated to n versions of FTTM at any second. For each version $v=1,2,3,\dots,n$, let $\text{FTTM}_v = (M_v, B_v, F_v, T_v)$ such that $M_v \cong B_v \cong F_v \cong T_v$, by the homeomorphisms $b_v: M_v \rightarrow B_v, f_v: B_v \rightarrow F_v$, and $t_v: F_v \rightarrow T_v$. In fact, there are other homeomorphisms between the components of FTTM_v , as the inclusion mappings that are $i_{M_v}: M_v \hookrightarrow M_v, i_{B_v}: B_v \hookrightarrow B_v, i_{F_v}: F_v \hookrightarrow F_v$, and $i_{T_v}: T_v \hookrightarrow T_v$. Moreover, the mappings $b_v^{-1}: B_v \rightarrow M_v, f_v^{-1}: F_v \rightarrow M_v, t_v^{-1}: T_v \rightarrow F_v, t_v \circ f_v \circ b_v: T_v \rightarrow M_v, b_v^{-1} \circ f_v^{-1} \circ t_v^{-1}: M_v \rightarrow T_v, f_v \circ b_v: M_v \rightarrow F_v, b_v^{-1} \circ f_v^{-1}: F_v \rightarrow M_v, t_v \circ f_v: B_v \rightarrow T_v, f_v^{-1} \circ t_v^{-1}: T_v \rightarrow B_v$.

Interestingly, the structure of FTTM differs from other structures in mathematics. Furthermore, FTTM has a distinguishing category [10].

In the current set up, FTTM is not a pullback. Since for $v=1, 2, 3, \dots, n \in \mathbb{Z}^+$, pick FTTM $_v$ as the version v of FTTM (see Figure 5).

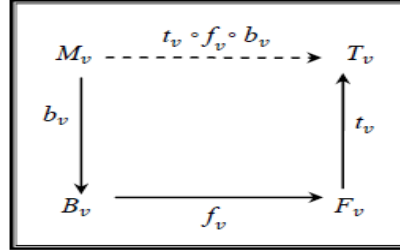


Figure 5. Version v of FTTM.

Notice that $M_v \cong B_v \cong F_v \cong T_v$ by the homeomorphisms $b_v : M_v \rightarrow B_v$, $f_v : B_v \rightarrow F_v$, and $t_v : F_v \rightarrow T_v$, respectively. By the definitions of the functions b_v, f_v , and t_v in Remark 1, none of them is a projection function as the components M_v, B_v, F_v and T_v are mutually disjoint. In fact, M_v is the first component in FTTM $_v$ in which are generated all the other components in FTTM $_v$. For example, lowered down M_v generates B_v , then B_v is fuzzified to generate F_v , and defuzzification of F_v produces T_v . Thus, none of these components has a product space structure for any two others. Therefore, FTTM $_v$ is not a pullback, for each $v=1, 2, 3, \dots, n \in \mathbb{Z}^+$.

To construct the moduli space of FTTM for m patients, we will discuss first the moduli space of FTTM $_1$ at each second, since it has a special category (refer to Figure 6) [10].

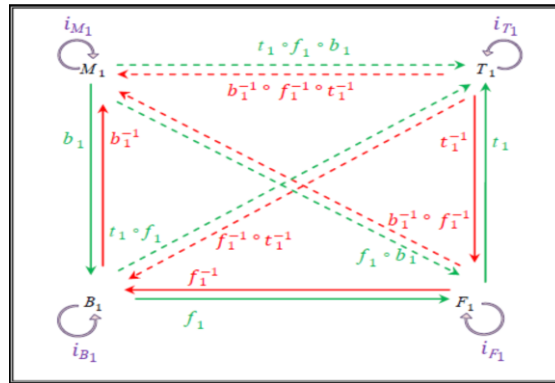


Figure 6. The Category of FTTM $_1$.

In the following, the moduli space of FTTM $_1$ ($\mathcal{M}_{\text{FTTM}_1}$ for short) is established at any second, for any patient.

Lemma 1. $\mathcal{M}_{\text{FTTM}_1} = \{[M_1]\}$.

Proof. Denote the sets of objects and morphisms in FTTM $_1$ by $C_{\text{FTTM}_1 O}$ and $C_{\text{FTTM}_1 M}$, respectively. Then, using the structures of FTTM $_1$, we conclude that:

$$C_{\text{FTTM}_1 O} = \{M_1, B_1, F_1, T_1\} \quad (20)$$

and

$$C_{\text{FTTM}_1 M} = \{i_{M_1}, i_{B_1}, i_{F_1}, i_{T_1}, b_1, b_1^{-1}, f_1, f_1^{-1}, t_1, t_1^{-1}, t_1 \circ f_1 \circ b_1, b_1^{-1} \circ f_1^{-1} \circ t_1^{-1}, f_1 \circ b_1, b_1^{-1} \circ f_1^{-1}, t_1 \circ f_1, f_1^{-1} \circ t_1^{-1}\} \quad (21)$$

Since b_1, f_1 , and t_1 are homeomorphisms as cited in Theorem 1, all the morphisms in $C_{\text{FTTM}_1 M}$ are homeomorphisms. Define a relation $\sim$ on $C_{\text{FTTM}_1 O}$ by:

$$\sim_{C_{\text{FTTM}_1 O}} = \{(A, B) \in C_{\text{FTTM}_1 O} \times C_{\text{FTTM}_1 O} : A \cong B\} \quad (22)$$

From $C_{\text{FTTM}_1 M}$ in Equation (33), $\sim_{C_{\text{FTTM}_1 O}}$ can be obtained by:

$$\sim_{C_{\text{FTTM}_1 O}} = \{(M_1, M_1), (B_1, B_1), (F_1, F_1), (T_1, T_1), (M_1, B_1), (B_1, M_1), (B_1, F_1), (F_1, B_1), (F_1, T_1), (T_1, F_1), (M_1, T_1), (T_1, M_1), (M_1, F_1), (F_1, M_1), (B_1, F_1), (F_1, B_1)\} \quad (23)$$

Clearly, $\sim_{C_{\text{FTTM}_1 O}}$ is an equivalent relation on $C_{\text{FTTM}_1 M}$. So, the quotient set for $C_{\text{FTTM}_1 O}$ on $\sim_{C_{\text{FTTM}_1 O}}$ is given by: $C_{\text{FTTM}_1 O} / \sim_{C_{\text{FTTM}_1 O}} := \{[M_1]\}$ (24)

By invoking Definition 1 part (c), $\mathcal{M}_{\text{FTTM}_1} := C_{\text{FTTM}_1 O} / \sim_{C_{\text{FTTM}_1 O}}$. Thereby, $\mathcal{M}_{\text{FTTM}_1} := \{[M_1]\}$ that is by Definition 1 part (a), M_1 is a base for a family of FTTM.

Since FTTM2 is similar to FTTM1 in possessing a special category as mentioned in Remark 1, we will give a similar result to Lemma 1 on FTTM2 by using the same relation in Equation (22).

Lemma 2. $\mathcal{M}_{\text{FTTM}_2} = \{[M_2]\}$.

An immediate consequence of Lemmas 1 and 2 is the following:

Remark 2. For $m, n \in \mathbb{Z}^+$, let C_{FTTM} be a category of m patients with n versions of FTTM at each second s_p for patient p , then the moduli space for each version v at any second s for any patient p is given by:

$$\mathcal{M}_{\text{FTTM}_v s_p} = \{[M_v s_p]\} \quad (25)$$

where $s_p = 1, 2, 3, \dots, c_p$, $v = 1, 2, 3, \dots, n$, and $c_p \in \mathbb{Z}^+$ for each $p = 1, 2, 3, \dots, m$. This is because the components in $\text{FTTM}_v s_p$ are related.

In what follows, the moduli space for all the versions of FTTM at one second for any patient is discussed.

Theorem 5. Let $n \in \mathbb{Z}^+$ and let C_{FTTM} be a category for one patient containing n versions at one second. Then, $\mathcal{M}_{\text{FTTM}} := \mathcal{M}_{\text{FTTM}(1, 2, \dots, n)} := \{[M_1]\}$.

Proof. Denote the set of objects in C_{FTTM} for a sequence of n versions of FTTM by $C_{\text{FTTM}_O} := C_{\text{FTTM}(1, 2, \dots, n)_O}$. Through description of the sequence of n versions of FTTM in Remark 1 and illustrated in Figure 3, we get:

$$\begin{aligned} C_{\text{FTTM}_O} &:= C_{\text{FTTM}(1, 2, \dots, n)} = \{M_1, B_1, F_1, T_1, M_2, B_2, F_2, T_2, \dots, M_n, B_n, F_n, T_n\} \\ &= \{(M_v, B_v, F_v, T_v) : M_v \cong B_v \cong F_v \cong T_v, \forall v = 1, 2, 3, \dots, n \in \mathbb{Z}^+\} \\ &= \{C_{\text{FTTM}_v O} : \forall v = 1, 2, 3, \dots, n \in \mathbb{Z}^+\} \\ &= \{C_{\text{FTTM}_1 O}, C_{\text{FTTM}_2 O}, \dots, C_{\text{FTTM}_n O}\}. \end{aligned}$$

By defining the same relation $\sim$ in Equation (22) on $C_{\text{FTTM}(1, 2, \dots, n)}$, we have

$$\begin{aligned} \sim_{C_{\text{FTTM}_O}} &= \{(A, B) \in C_{\text{FTTM}_O} \times C_{\text{FTTM}_O} : A \cong B\} \\ \sim_{C_{\text{FTTM}_O}} &= \{(A, B) \in C_{\text{FTTM}(1, 2, \dots, n)} \times C_{\text{FTTM}(1, 2, \dots, n)} : A \cong B\} \end{aligned} \quad (26)$$

Incidentally, from Lemmas 1 and 2, and Remark 2 we find $\mathcal{M}_{\text{FTTM}_1} = \{[M_1]\}$, $\mathcal{M}_{\text{FTTM}_2} = \{[M_2]\}$, ..., $\mathcal{M}_{\text{FTTM}_n} = \{[M_n]\}$. This is because in a sequence of n FTTM, we have $M_v \cong M_{v+1}$, $B_v \cong B_{v+1}$, $F_v \cong F_{v+1}$, and $T_v \cong T_{v+1}$.

For each $v = 1, 2, \dots, (n-1)$, and $n \in \mathbb{Z}^+$ as indicated in Remark 1. Therefore, $M_1 \cong M_2$ implies $M_1 \in [M_2]$, and $M_2 \in [M_1]$. Thus, $[M_1] \subseteq [M_2]$, and $[M_2] \subseteq [M_1]$ means that $[M_1] = [M_2]$. Hence, $\mathcal{M}_{\text{FTTM}(1, 2)} = \{[M_1]\} = \{[M_2]\}$. Similarly, $M_2 \cong M_3 \cong \dots \cong M_n$ leads to $\mathcal{M}_{\text{FTTM}(1, 2, \dots, n)} = \{[M_1]\} = \{[M_2]\} = \dots = \{[M_n]\}$. Notice M_1 is the generation of all the n versions of FTTM per second as mentioned in Remark 1 part (b). So, $\mathcal{M}_{\text{FTTM}} := \mathcal{M}_{\text{FTTM}(1, 2, \dots, n)} = \{[M_1]\}$. From Definition 1 part (a), M_1 is a base for a family of FTTM. ■

The next theorem is relevant to our discussion of establishing the moduli space of FTTM, which will be this case for one patient at all seconds, and for all versions.

Theorem 6. Let $n, c_p, p \in \mathbb{Z}^+$ and let $C_{\text{FTTM } p}$ denote a category for one patient p containing n versions at each second s_p , for all $s_p = 1, 2, 3, \dots, c_p$. Then,

$$\mathcal{M}_{\text{FTTM } p} = \bigcup_{s_p=1}^{c_p} M_{1_{s_p}}^p$$

Proof. Suppose that $C_{\text{FTTM } p_o}$ is the set of objects in $C_{\text{FTTM } p}$, where:

$$\begin{aligned} C_{\text{FTTM } p_o} = & \{M_{1_1}^p, B_{1_1}^p, F_{1_1}^p, T_{1_1}^p, M_{2_1}^p, B_{2_1}^p, F_{2_1}^p, T_{2_1}^p, \dots, M_{n_1}^p, B_{n_1}^p, F_{n_1}^p, T_{n_1}^p, M_{1_2}^p, B_{1_2}^p, F_{1_2}^p, \\ & T_{1_2}^p, M_{2_2}^p, B_{2_2}^p, F_{2_2}^p, T_{2_2}^p, \dots, M_{n_2}^p, B_{n_2}^p, F_{n_2}^p, T_{n_2}^p, \dots, M_{1_{c_p}}^p, B_{1_{c_p}}^p, F_{1_{c_p}}^p, T_{1_{c_p}}^p, M_{2_{c_p}}^p, B_{2_{c_p}}^p, F_{2_{c_p}}^p, \\ & T_{2_{c_p}}^p, \dots, M_{n_{c_p}}^p, B_{n_{c_p}}^p, F_{n_{c_p}}^p, T_{n_{c_p}}^p\} \\ = & \{(M_{v_{s_p}}^p, B_{v_{s_p}}^p, F_{v_{s_p}}^p, T_{v_{s_p}}^p) : M_{v_{s_p}}^p \cong B_{v_{s_p}}^p \cong F_{v_{s_p}}^p \cong T_{v_{s_p}}^p, \forall v = 1, 2, 3, \dots, n, \forall s_p = \\ & 1, 2, 3, \dots, c_p, \text{ and } n, c_p, p \in \mathbb{Z}^+\} \\ = & \{C_{\text{FTTM}(v_{s_p}^p)_o} : \forall v = 1, 2, 3, \dots, n, \forall s_p = 1, 2, 3, \dots, c_p, \text{ and } n, c_p, p \in \mathbb{Z}^+\} \\ = & \{C_{\text{FTTM}(1_1^p)_o}, C_{\text{FTTM}(2_1^p)_o}, \dots, C_{\text{FTTM}(n_1^p)_o}, C_{\text{FTTM}(1_2^p)_o}, C_{\text{FTTM}(2_2^p)_o}, \dots, C_{\text{FTTM}(n_2^p)_o}, \\ & \dots, C_{\text{FTTM}(1_{c_p}^p)_o}, C_{\text{FTTM}(2_{c_p}^p)_o}, \dots, C_{\text{FTTM}(n_{c_p}^p)_o}\} \end{aligned}$$

$$\text{So, } C_{\text{FTTM } p_o} = \{C_{\text{FTTM}((1,2,\dots,n)_1^p)_o}, C_{\text{FTTM}((1,2,\dots,n)_2^p)_o}, \dots, C_{\text{FTTM}((1,2,\dots,n)_{c_p}^p)_o}\} \quad (27)$$

By using the same relation $\sim$ defined in Equation (22) on $C_{\text{FTTM } o}$, we conclude $\sim_{C_{\text{FTTM } p_o}} = \{(A, B) \in C_{\text{FTTM } p_o} \times C_{\text{FTTM } p_o} : A \cong B\}$

Thus, by Theorem 5 we get $\mathcal{M}_{\text{FTTM } p} = \{[M_{1_1}^p], [M_{1_2}^p], \dots, [M_{1_{c_p}}^p]\}$

Notice as referred to in Remark 1 that for $v = 1, 2, 3, \dots, n$, $s_p = 1, 2, 3, \dots, c_p$ and $n, c_p, p \in \mathbb{Z}^+$, M_{1_s} is a set of $(n \times n)$ square matrix (EEG signals) defined as:

$$M_{1_{s_p}}^p = \left\{ \begin{array}{l} [\beta_{ij}(z)_{s_p}^p]_{n \times n} : i, j \in \mathbb{Z}^+, \\ \beta_{ij}(z)_{s_p}^p \in \mathbb{R} \end{array} \right\} \quad (28)$$

where $\beta_{ij}(z)$ is a potentially different reading of EEG signals from a particular time s . In addition, the objects of $\mathcal{M}_{\text{FTTM}}$ are equivalent classes. Therefore, by the properties of equivalent classes, we have either $[M_{1_k}^p] = [M_{1_u}^p]$ or $[M_{1_k}^p] \cap [M_{1_u}^p] = \emptyset$ at any two distinct seconds k and u such that $1 \leq k, u \leq c_p$ and $k, u, c_p \in \mathbb{Z}^+$. If $[M_{1_k}^p] = [M_{1_u}^p]$, in this case we conclude from Equation (28) that

$$\begin{aligned} \forall [\beta_{i_k j_k}(z)_k^p]_{n_k \times n_k} \in [M_{1_k}^p], \exists! [\beta_{i_u j_u}(z)_u^p]_{n_u \times n_u} \in [M_{1_k}^p] \ni n_k = n_u \quad \text{and} \\ [\beta_{i_k j_k}(z)_k^p]_{n_k \times n_k} = [\beta_{i_u j_u}(z)_u^p]_{n_u \times n_u}, \quad \text{for } i_k, j_k, i_u, j_u, n_k, n_u \in \mathbb{Z}^+ \quad \text{and} \\ \beta_{i_k j_k}(z)_k^p, \beta_{i_u j_u}(z)_u^p \in \mathbb{R}. \end{aligned}$$

Thus, we can omit one of them from $\mathcal{M}_{\text{FTTM}}$ and substitute it with another one. Otherwise, when $[M_{1_k}^p] \cap [M_{1_u}^p] = \emptyset$ as in Equation (28)

$$\begin{aligned} \exists [\beta_{i_k j_k}(z)_k^p]_{n_k \times n_k} \in [M_{1_k}^p] \ni [\beta_{i_k j_k}(z)_k^p]_{n_k \times n_k} \neq [\beta_{i_u j_u}(z)_u^p]_{n_u \times n_u} \quad \text{for some} \\ [\beta_{i_u j_u}(z)_u^p]_{n_u \times n_u} \in [M_{1_u}^p], \text{ such that } i_k, j_k, i_u, j_u, n_k, n_u \in \mathbb{Z}^+, \text{ and } \beta_{i_k j_k}(z)_k^p, \beta_{i_u j_u}(z)_u^p \in \mathbb{R}. \end{aligned}$$

As a result, both of them will represent their families. Accordingly,

$$\mathcal{M}_{\text{FTTM}}^p = \bigcup_{s_p=1}^{c_p} \left(\bigcup_{v=1}^n \mathcal{M}_{\text{FTTM}}^{p_{s_p}} \right)$$

$$\mathcal{M}_{\text{FTTM}}^p = \bigcup_{s_p=1}^{c_p} ([M_{1_{s_p}}^p]) \quad (29)$$

For short, $\mathcal{M}_{\text{FTTM}}^p = \bigcup_{s_p=1}^{c_p} M_{1_{s_p}}^p$. ■

In fact, Theorem 6 entails that the moduli space of FTTM for any patient will be the union of all the generations at any second of seizure provided use one of the equal generations. Then, we mainly discuss in the following theorem the final step in finding the moduli space of FTTM for a sample of finitely many patients.

Theorem 7. Let C_{FTTM} denote a category of m patients, each patient p has n versions at each of the c_p seconds, where $n, c_p, m \in \mathbb{Z}^+$, for all $p=1,2,3,\dots,m$. Then,

$$\mathcal{M}_{\text{FTTM}} = \bigcup_{p=1}^m \left(\bigcup_{s_p=1}^{c_p} M_{1_{s_p}}^p \right)$$

Proof. Suppose that C_{FTTM_O} is the set of objects in C_{FTTM} , where:

$$\begin{aligned} C_{\text{FTTM}_O} = & \{M_{1_1}, B_{1_1}, F_{1_1}, T_{1_1}, M_{2_1}, B_{2_1}, F_{2_1}, T_{2_1}, \dots, M_{n_1}, B_{n_1}, F_{n_1}, T_{n_1}, M_{1_{\frac{1}{2}}}, B_{1_{\frac{1}{2}}}, F_{1_{\frac{1}{2}}}, T_{1_{\frac{1}{2}}}, \\ & M_{2_{\frac{1}{2}}}, B_{2_{\frac{1}{2}}}, F_{2_{\frac{1}{2}}}, T_{2_{\frac{1}{2}}}, \dots, M_{n_{\frac{1}{2}}}, B_{n_{\frac{1}{2}}}, F_{n_{\frac{1}{2}}}, T_{n_{\frac{1}{2}}}, \dots, M_{1_{\frac{1}{c_p}}}, B_{1_{\frac{1}{c_p}}}, F_{1_{\frac{1}{c_p}}}, T_{1_{\frac{1}{c_p}}}, M_{2_{\frac{1}{c_p}}}, B_{2_{\frac{1}{c_p}}}, F_{2_{\frac{1}{c_p}}}, T_{2_{\frac{1}{c_p}}}, \dots, \\ & M_{n_{\frac{1}{c_p}}}, B_{n_{\frac{1}{c_p}}}, F_{n_{\frac{1}{c_p}}}, T_{n_{\frac{1}{c_p}}}, M_{1_{\frac{2}{c_p}}}, B_{1_{\frac{2}{c_p}}}, F_{1_{\frac{2}{c_p}}}, T_{1_{\frac{2}{c_p}}}, M_{2_{\frac{2}{c_p}}}, B_{2_{\frac{2}{c_p}}}, F_{2_{\frac{2}{c_p}}}, T_{2_{\frac{2}{c_p}}}, \dots, M_{n_{\frac{2}{c_p}}}, B_{n_{\frac{2}{c_p}}}, F_{n_{\frac{2}{c_p}}}, T_{n_{\frac{2}{c_p}}}, M_{1_{\frac{2}{2}}}, B_{1_{\frac{2}{2}}}, \\ & F_{1_{\frac{2}{2}}}, T_{1_{\frac{2}{2}}}, M_{2_{\frac{2}{2}}}, B_{2_{\frac{2}{2}}}, F_{2_{\frac{2}{2}}}, T_{2_{\frac{2}{2}}}, \dots, M_{n_{\frac{2}{2}}}, B_{n_{\frac{2}{2}}}, F_{n_{\frac{2}{2}}}, T_{n_{\frac{2}{2}}}, \dots, M_{1_{\frac{2}{c_p}}}, B_{1_{\frac{2}{c_p}}}, F_{1_{\frac{2}{c_p}}}, T_{1_{\frac{2}{c_p}}}, M_{2_{\frac{2}{c_p}}}, B_{2_{\frac{2}{c_p}}}, F_{2_{\frac{2}{c_p}}}, \\ & T_{2_{\frac{2}{c_p}}}, \dots, M_{n_{\frac{2}{c_p}}}, B_{n_{\frac{2}{c_p}}}, F_{n_{\frac{2}{c_p}}}, T_{n_{\frac{2}{c_p}}}, \dots, M_{1_{\frac{m}{c_p}}}, B_{1_{\frac{m}{c_p}}}, F_{1_{\frac{m}{c_p}}}, T_{1_{\frac{m}{c_p}}}, M_{2_{\frac{m}{c_p}}}, B_{2_{\frac{m}{c_p}}}, F_{2_{\frac{m}{c_p}}}, T_{2_{\frac{m}{c_p}}}, \dots, M_{n_{\frac{m}{c_p}}}, B_{n_{\frac{m}{c_p}}}, F_{n_{\frac{m}{c_p}}}, T_{n_{\frac{m}{c_p}}}, \\ & M_{2_{\frac{m}{c_p}}}, B_{2_{\frac{m}{c_p}}}, F_{2_{\frac{m}{c_p}}}, T_{2_{\frac{m}{c_p}}}, \dots, M_{n_{\frac{m}{c_p}}}, B_{n_{\frac{m}{c_p}}}, F_{n_{\frac{m}{c_p}}}, T_{n_{\frac{m}{c_p}}}\} \\ = & \{(M_{v_{s_p}}^p, B_{v_{s_p}}^p, F_{v_{s_p}}^p, T_{v_{s_p}}^p) : M_{v_{s_p}}^p \cong B_{v_{s_p}}^p \cong F_{v_{s_p}}^p \cong T_{v_{s_p}}^p, \forall v = 1,2,3,\dots,n, s_p = 1,2,3,\dots,c_p, \text{ and } n, c_p, m \\ & \in \mathbb{Z}^+, \forall p = 1,2,3,\dots,m\} \\ = & \{C_{\text{FTTM}}(v_{s_p}^p)_O : \forall v = 1,2,3,\dots,n, \forall s_p = 1,2,3,\dots,c_p \text{ and } n, c_p, m \in \mathbb{Z}^+, \forall p = 1,2,3,\dots,m\} \\ = & \{C_{\text{FTTM}}((1,2,\dots,n)_1^1)_O, C_{\text{FTTM}}((1,2,\dots,n)_2^1)_O, \dots, C_{\text{FTTM}}((1,2,\dots,n)_{c_p}^1)_O, C_{\text{FTTM}}((1,2,\dots,n)_1^2)_O, \\ & C_{\text{FTTM}}((1,2,\dots,n)_2^2)_O, \dots, C_{\text{FTTM}}((1,2,\dots,n)_{c_p}^2)_O, \dots, C_{\text{FTTM}}((1,2,\dots,n)_1^m)_O, C_{\text{FTTM}}((1,2,\dots,n)_2^m)_O, \\ & \dots, C_{\text{FTTM}}((1,2,\dots,n)_{c_p}^m)_O\} \end{aligned}$$

Likewise, via the same relation $\sim$ defined in Equation (22) on C_{FTTM_O} , then

$$\sim_{C_{\text{FTTM}_O}} = \{(A, B) \in C_{\text{FTTM}_O} \times C_{\text{FTTM}_O} : A \cong B\}.$$

By Theorem 6, $\mathcal{M}_{\text{FTTM}} = \{\bigcup_{s_1=1}^{c_1} [M_{1_{s_1}}^1], [\bigcup_{s_2=1}^{c_2} [M_{1_{s_2}}^2], \dots, \bigcup_{s_m=1}^{c_m} [M_{1_{s_m}}^m]]\}.$

Notice that there is no fixed time for a seizure which happens during a special period that may be equal to a period for another patient. Thus, $c_1, c_2, \dots$, and c_m represent the times for patient 1, 2, ..., $m \in \mathbb{Z}^+$, respectively. In another aspect, there is no repetition of the generations of FTTM for any patient as discussed in Theorem 6. Therefore, the elements in each $\bigcup_{s_p=1}^{c_p} [M_{1_{s_p}}^p]$ are mutually disjoint for all $p = 1,2,3,\dots,m \in \mathbb{Z}^+$.

For discussing the recursion between any two patients, we have two possible choices. In the first choice, we have for any two patients p_k and p_u in C_{FTTM} , either $[M_{1_{s_k}}^{p_k}] = [M_{1_{s_u}}^{p_u}]$, or $[M_{1_{s_k}}^{p_k}] \cap [M_{1_{s_u}}^{p_u}] = \emptyset$, at any time s_k and s_u for patients p_k and p_u , respectively. When

$[M_{1_{s_k}}^{p_k}] = [M_{1_{s_u}}^{p_u}]$, some generations are equal for patients p_k and p_u . Since $[M_{1_{s_k}}^{p_k}]$ and $[M_{1_{s_u}}^{p_u}]$ are two equivalent classes in $\mathcal{M}_{\text{FTTM}}$, one of them will be written in expression $\mathcal{M}_{\text{FTTM}}$ as discussed in Equation (29). In addition, if $\bigcup_{s_k=1}^{c_k} [M_{1_{s_k}}^{p_k}] = \bigcup_{s_u=1}^{c_u} [M_{1_{s_u}}^{p_u}]$, for some patients p_k and p_u . This means $c_k = c_u$ and $\forall [M_{1_{s_k}}^{p_k}] \in \mathcal{M}_{\text{FTTM}}^{p_k}, \exists! [M_{1_{s_u}}^{p_u}] \in \mathcal{M}_{\text{FTTM}}^{p_u} \exists [M_{1_{s_k}}^{p_k}] = [M_{1_{s_u}}^{p_u}]$, for $1 \leq s_k \leq c_k, 1 \leq s_u \leq c_u$, and $c_k, c_u \in \mathbb{Z}^+$ (30)

Thus, as discussed for one class, one of them will be in $\mathcal{M}_{\text{FTTM}}$. While in the second choice if $[M_{1_{s_k}}^{p_k}] \cap [M_{1_{s_u}}^{p_u}] = \emptyset$, both of them will be in $\mathcal{M}_{\text{FTTM}}$ which has been discussed in Equation (30). This implies that $\mathcal{M}_{\text{FTTM}} = \bigcup_{p=1}^m \left(\bigcup_{s_p=1}^{c_p} M_{1_{s_p}}^p \right)$.

In general, the moduli space for a sample of finitely many patients is the union of all the mutually disjoint bases of FTTM, namely $M_{1_{s_p}}^p$, where $s_p=1,2,3,\dots,c_p$ and $c_p, m \in \mathbb{Z}^+$ for each $p=1,2,3,\dots,m$.

4. Uniqueness of $\mathcal{M}_{\text{FTTM}}$

There is no other formal structure of the moduli space of FTTM. This result is presented formally as a theorem.

Theorem 8. Let C_{FTTM} be a category of m patients, each patient p has n versions at each of the c_p seconds, where $n, c_p, m \in \mathbb{Z}^+$, for all $p=1,2,3,\dots,m$. Then, $\mathcal{M}_{\text{FTTM}} = \bigcup_{p=1}^m \left(\bigcup_{s_p=1}^{c_p} M_{1_{s_p}}^p \right)$ is unique.

Proof. Based on Definition 1 part (c), the moduli space of FTTM must be established by the isomorphism relation between the components of FTTM. Thus, we may assume that $\mathcal{M}'_{\text{FTTM}}$ as another moduli space of FTTM under an isomorphism relation between the components of FTTM defined in Equation (22). Therefore,

$$\mathcal{M}'_{\text{FTTM}} = \left\{ [A]: B \in [A] \text{ if } A \cong B, \text{ for } A, B \in C_{\text{FTTM}_O} \right\}, \quad (31)$$

In order to find the relation between $\mathcal{M}_{\text{FTTM}}$ and $\mathcal{M}'_{\text{FTTM}}$, let $[X] \in \mathcal{M}'_{\text{FTTM}}$. Then, $X \in C_{\text{FTTM}_O}$. This entails X is a component in some version v_X of FTTM at fixed time s_X for patient p_X . Thereby, $X \in C_{\text{FTTM}(v_X \frac{p_X}{s_X})_O}$. That is, $X \in \text{FTTM} v_X \frac{p_X}{s_X}$. Hence, $X \in \{M_{v_X \frac{p_X}{s_X}}^{p_X}, B_{v_X \frac{p_X}{s_X}}^{p_X}, F_{v_X \frac{p_X}{s_X}}^{p_X}, T_{v_X \frac{p_X}{s_X}}^{p_X}\}$. We have $M_{v_X \frac{p_X}{s_X}}^{p_X} \cong B_{v_X \frac{p_X}{s_X}}^{p_X} \cong F_{v_X \frac{p_X}{s_X}}^{p_X} \cong T_{v_X \frac{p_X}{s_X}}^{p_X}$ in $\text{FTTM} v_X \frac{p_X}{s_X}$. Thus, $X \cong M_{v_X \frac{p_X}{s_X}}^{p_X} \cong B_{v_X \frac{p_X}{s_X}}^{p_X} \cong F_{v_X \frac{p_X}{s_X}}^{p_X} \cong T_{v_X \frac{p_X}{s_X}}^{p_X}$. By Remark 2, $\mathcal{M}_{\text{FTTM} v_X \frac{p_X}{s_X}} = \{[M_{v_X \frac{p_X}{s_X}}^{p_X}]\}$, and then from Theorem 5, we get $\mathcal{M}_{\text{FTTM}}^{p_X} = \{[M_{1_{s_X}}^{p_X}]\}$. Since $M_{v_X \frac{p_X}{s_X}}^{p_X}, B_{v_X \frac{p_X}{s_X}}^{p_X}, F_{v_X \frac{p_X}{s_X}}^{p_X}, T_{v_X \frac{p_X}{s_X}}^{p_X} \in [M_{1_{s_X}}^{p_X}]$. Thus, $X \in [M_{1_{s_X}}^{p_X}]$. From the properties of the equivalence relation, we conclude that $[X] = [M_{1_{s_X}}^{p_X}]$. But $[M_{1_{s_X}}^{p_X}] \in \mathcal{M}_{\text{FTTM}}$, which will lead to $[X] \in \mathcal{M}_{\text{FTTM}}$. Thereby,

$$\mathcal{M}'_{\text{FTTM}} \subseteq \mathcal{M}_{\text{FTTM}} \quad (32)$$

The same preceding argument in the first direction also shows that

$$\mathcal{M}_{\text{FTTM}} \subseteq \mathcal{M}'_{\text{FTTM}} \quad (33)$$

Overall, Equations (32) and (33) satisfy that $\mathcal{M}_{\text{FTTM}} = \mathcal{M}'_{\text{FTTM}}$. ■

As a result, for Theorem 8, the formal expression for the moduli space of FTTM in Theorem 7 is unique and it is established as the only collection for all the isomorphism classes represented by the generation components M_1 at any time and for any patient.

5. Conclusion

In a synopsis, this paper not only aimed at constructing a new moduli space, but also the uniqueness of this moduli space is proved. Specifically, FTTM has a moduli space, which is uniquely constructed by the first component at any time and for any patient.

Therefore, we conclude in epilepsy patients that different reaction and the similarity of the symptoms of the disease will lead to the same equivalence class in $\mathcal{MFTTM}$.

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