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Semi E^h-b-preinvexity and its applications to optimization problems

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Abstract In this paper, the class of semi E^h-b-preinvex and pseudo E^h-b-preinvex functions are defined as an extension of E-B-preinvex and h-preinvex functions. In this extension the functions $E: \mathbb{R}^n \to \mathbb{R}^n$, $h: [0,1] \to \mathbb{R}$, and $b: \mathbb{R}^n \times \mathbb{R}^n \times [0,1] \to \mathbb{R}^+$ are taken into consideration. Various properties for the new functions as well as some properties and characterizations of quasi semi E-preinvex functions are discussed. Some optimality properties for semi E^h-b-preinvex nonlinear optimization problems are proved.

AMS Subject Classification: 46N10, 47N10, 90C48, 90C90, 49K27

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1. Introduction

The research on convexity considered as one of the most important criteria in optimization problems and applied mathematics (see e.g., [1-4]). In particular convex sets and convex functions are widely studied in the literature. Many attempts have been made to generalize convex sets and convex functions by relaxing the convexity assumptions to deal with many practical problems (see [5-26] and the references therein). One of the important generalizations of convex sets and convex functions is the so-called invex sets [5,6] and preinvex functions [7,8]. Another important generalization is E-convex sets and E-convex functions introduced first by Youness [9] and widely studied and applied to optimization problems by many researchers (see [10-16]). By combining invex set and E-convex sets (resp., preinvex and E-convex) functions, Fulga and Perda [17] defined E-invex sets (resp., E-preinvex and E-prequasiinvex) functions. Syau et. al. in [18] introduced E-B-preinvex functions as a combination of B-preinvex function [19] and E-convex function. Recently, Luo and Jian [20] extended the class of E-convex, semi E-convex [16], and E-preinvex maps to semi E-preinvex and quasi semi E-preinvex maps in Banach space. Very recently, Enad and Majeed [21] introduced the class of E^h-b-preinvex functions as an extension of h-preinvex functions [22] and E-B-preinvex functions. They also discussed some properties of local E^h-b-preinvex optimization problems. Motivated by [18, 20-22], we introduce in this paper new concepts of generalized convexity which are semi E^h-b-preinvex and pseudo semi E^h-b-preinvex functions. In section 2, we recall some preliminary definitions studied in the literature that motivate us to present the new generalized convex functions and to study their properties. In section 3, some basic properties of semi E^h-b-preinvex and pseudo semi E^h-b-preinvex functions are presented and the relationship between semi E^h-b-preinvex and pseudo semi E^h-b-preinvex is studied (see Propositions 3.1-3.3 and Propositions 3.5 and 3.6). A new relationship between E-prequasiinvex and quasi semi E-preinvex functions is proved (see Proposition 3.7). Some necessary conditions for a function f to be semi E^h-b-preinvex using the level sets M_γ and M_γ^E and the epigraph of f are given (see Propositions 3.8-3.10, 3.13). Also, a new

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characterization of quasi semi E-preinvex functions using E level sets M_γ^E and $E - M_\gamma$ are given (see Propositions 3.11 and 3.12). Finally, a sufficient condition for a function to be f to be quasi semi E-preinvex in terms of $epif$ is shown (see Proposition 3.14). Section 4 is devoted to study the optimality properties of a non-linear optimization problems (denoted by (P_E)) in which the objective function $f \circ E$ is semi E^h-b-preinvex (see Propositions 4.4 and 4.5) and when the function f is E^h-b-preinvex function (see Proposition 4.6).

2. Preliminaries

In this paper, $\mathbb{R}^n$ denotes the n-dimensional Euclidean space and $\mathbb{R}^+$ be a set of non-negative real numbers. For brevity in writing the statements, the following assumption is needed.

Assumption (I) Let $\emptyset \neq M \subseteq \mathbb{R}^n$ and $f: \mathbb{R}^n \to \mathbb{R}$, $h: [0,1] \to \mathbb{R}$ be two real valued functions. Assume also that $E: \mathbb{R}^n \to \mathbb{R}^n \psi: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, and $b: \mathbb{R}^n \times \mathbb{R}^n \times [0,1] \to \mathbb{R}^+$ are given mappings where $\lambda b(x, y, \lambda) \in [0,1]$ for all $x, y \in \mathbb{R}^n$ and $\lambda \in [0,1]$.

Next, we recall the necessary definitions and related concepts that is needed throughout the paper. Unless otherwise stated, M, f, E, ψ and b are defined as in assumption (I).

Definition 2.1 Let $M, E,$ and ψ are defined as in Assumption (I) then, $\forall m_1, m_2 \in M$ and $\forall \lambda \in [0,1]$, M is called

1. E-convex if $\lambda E(m_1) + (1 - \lambda)E(m_2) \in M$. [9]
2. An invex set with respect to ψ (for short, M is an invex w.r.t. ψ) if $m_2 + \lambda\psi(m_1, m_2) \in M$. [6]
3. An E-invex set w.r.t. ψ if $E(m_2) + \lambda\psi(E(m_1), E(m_2)) \in M$. [17]

Definition 2.2 [17] Let M_1 and M_2 be two subsets of $\mathbb{R}^n$ and ψ defined be as in Assumption (I). Then M_1 is said to be slack-E-invex w.r.t. M_2 if, for every $m, m^* \in M_1 \cap M_2$ and every $0 \leq \lambda \leq 1$ such that $E(m^*) + \lambda\psi\big(E(m), E(m^*)\big) \in M_2$ we get $E(m^*) + \lambda\psi\big(E(m), E(m^*)\big) \in M_1$.

Definition 2.3 Let M, f, E, b and ψ are defined as in Assumption (I) then, $\forall m_1, m_2 \in M$ and for every $\lambda \in [0,1]$, f is called

1. E-preinvex function w.r.t. ψ on the E-invex set M if,
   $f(E(m_2) + \lambda\psi(E(m_1), E(m_2))) \leq \lambda f(E(m_1)) + (1 - \lambda)f(E(m_2))$. [17]
2. E-prequasiinvex w.r.t. ψ on the E-invex set M if,
   $f(E(m_2) + \lambda\psi\big(E(m_1), E(m_2)\big)) \leq \max\{f(E(m_1)), f(E(m_2))\}$. [17]
3. E-B-preinvex function w.r.t. ψ on the E-invex set M w.r.t. ψ if,
   $f(E(m_2) + \lambda\psi(E(m_1), E(m_2))) \leq \lambda b(m_1, m_2, \lambda)\ f(E(m_1)) + (1 - \lambda b(m_1, m_2, \lambda)\ )f(E(m_2))$. [18]
4. semi E-preinvex function w.r.t. ψ on the E-invex set M if,
   $f(E(m_2) + \lambda\psi(E(m_1), E(m_2))) \leq \lambda f(m_1) + (1 - \lambda)f(m_2)$. [20]
5. quasi semi E-preinvex function w.r.t. ψ on the E-invex set M if
   $f\Big(E(m_2) + \lambda\psi(E(m_1), E(m_2))\Big) \leq \max\{f(m_1), f(m_2)\}$. [20]

Remark 2.4 Wherever it seems convenient

1. We omit the parentheses from $E(x)$, and writing it instead as Ex.

2. E-invex set w.r.t. ψ and (E-preinvex, E-prequasiinvex, E-B-preinvex, semi E-preinvex, quasi semi E-preinvex) functions w.r.t. ψ will be called E-invex set, and (E-preinvex, E-prequasiinvex, E-B-preinvex, semi E-preinvex, quasi semi E-preinvex) functions.
3. We omit the argument of the mapping b and express $b(m_1, m_2, \lambda)$ as b.

Definition 2.5 A function $f: \mathbb{R}^n \longrightarrow \mathbb{R}$ is called

1. sublinear if $f(\alpha_1 x_1 + \alpha_1 x_2) \leq \alpha_1 f(x_1) + \alpha_2 f(x_2)$ $\forall x_1, x_2 \in \mathbb{R}^n$ and $\alpha_1, \alpha_2 \in \mathbb{R}$. [1]
2. idempotent if $f^2(x) = f(x)$ $\forall x \in \mathbb{R}^n$. [11]
3. non-decreasing if whenever $x, y \in \mathbb{R}^n$ such that $x \leq y$ (i.e., $x_i \leq y_i$, $\forall i = 1, \ldots, n$) we get $f(x) \leq f(y)$. [23]

Different types of γ-level sets associated with f and E are introduced in the literature. Some of these sets are listed below.

Definition 2.6 Let $\gamma \in \mathbb{R}$. Then,

1. $M_\gamma = \{m \in M: f(m) \leq \gamma\}$. [1]
2. E-$M_\gamma = \{m \in M: f(Em) \leq \gamma\}$. [13]
3. $M_\gamma^E = \{Em \in E(M): f(m) \leq \gamma\}$. [13]

The epigraph of a function f is defined as $epif = \{(x, \gamma) \in M \times \mathbb{R}: f(x) \leq \gamma\}$[1]. Next, the definition of h-convex introduced in [24] is recalled. Noting that, in [25,26], other versions of h-convex functions are defined.

Definition 2.7 [24] Let $h: [0,1] \rightarrow \mathbb{R}$ be a function. Then $f: \mathbb{R}^n \rightarrow \mathbb{R}$ is said to be h-convex function if for each $m_1, m_2 \in M$, and each $\lambda \in [0,1]$ we have $f(\lambda m_1 + (1-\lambda) m_2) \leq h(\lambda) f(m_1) + h(1-\lambda) f(m_2)$.

Matloka in 2014 introduced the class of h-preinvex function.

Definition 2.8 [22] Let $h: [0,1] \rightarrow \mathbb{R}$ be a positive function. Then a positive function $f: \mathbb{R}^n \rightarrow \mathbb{R}$ is said to be h-preinvex on the invex set M if for each $m_1, m_2 \in M$, and each $0 \leq \lambda \leq 1$ we have $f(m_2 + \lambda \psi(m_1, m_2)) \leq h(\lambda) f(m_1) + h(1-\lambda) f(m_2)$.

Very recently, Enad and Majeed in [21] defined the class of E^h-b-preinvex functions using the definitions of h-preinvex and E-B-preinvex functions as follows.

Definition 2.9 Let M, f, E, b, ψ and h are expressed as in Assumption (I) such that M is an E-invex set. Then f is said to be E^h-b-preinvex function on M if for each $m_1, m_2 \in M$, and each $0 \leq \lambda \leq 1$

$$f(E\mathrm{m}_2 + \lambda \psi(Em_1, Em_2)) \leq h(\lambda b) f(Em_1) + h(1 - \lambda b) f(Em_2).$$

Motivated by semi E-preinvex, h-convex, and E-B-preinvex functions, we introduce the classes of semi E^h-b-preinvex and pseudo semi E^h-b-preinvex functions as follows.

Definition 2.10 Let M, f, E, b, ψ and h are expressed as in Assumption (I) such that M is an E-invex set. Then, f is said to be

1. semi E^h-b-preinvex function on M if for each $m_1, m_2 \in M$, and each $0 \leq \lambda \leq 1$

$$f(E\mathrm{m}_2 + \lambda \psi(Em_1, Em_2)) \leq h(\lambda b) f(m_1) + h(1 - \lambda b) f(m_2).$$

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2. strict semi E^h-b-preinvex function on M if for each $m_1, m_2 \in M$, $m_1 \neq m_2$ and each $0 < \lambda < 1$

$$f(E\mathrm{m}_2 + \lambda\psi(Em_1, Em_2)) < h(\lambda b)f(m_1) + h(1-\lambda b)f(m_2).$$

3. pseudo semi E^h-b-preinvex on M if there exists a strictly positive function $g: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ such that whenever $f(m_1) < f(m_2)$ then
$$f\left(Em_2 + \lambda\psi(Em_1, Em_2)\right) \leq f(m_2) + h(\lambda b)h(\lambda b - 1)g(m_1, m_2),$$
for each $m_1, m_2 \in M$, and each $0 \leq \lambda \leq 1$.

An example of semi E^h-b-preinvex function is given next.

Example 2.11 Let $M = [-2,2] \subseteq \mathbb{R}$. Let $f, E: \mathbb{R} \to \mathbb{R}$, $b: \mathbb{R} \times \mathbb{R} \times [0,1] \to \mathbb{R}^+$, and $\psi: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are defined as follows.

$$f(x) = \begin{cases} 1 & -2 \leq x \leq 0 \\ \frac{1}{2} & otherwise \end{cases}, \quad b(x, y, \lambda) = \begin{cases} \frac{1}{\lambda} & \lambda \in (0,1] \\ 0 & \lambda = 0 \end{cases}$$

$$\psi(x, y) = \begin{cases} x - y & x, y \in [-2,0] \ or\ x, y \in (0,2] \\ y - x & otherwise \end{cases}$$

Let $E(x) = |x|$ $\forall x \in \mathbb{R}$ and $h: [0,1] \to \mathbb{R}$ be defined as $h(\lambda) = 2\lambda$ $\forall \lambda \in [0,1]$. We show that f is a semi E^h-b-preinvex function on M. Direct calculations yields $[-2,2]$ is an E-invex set. Now, considering $m_1, m_2 \in [-2,2]$ and $\lambda \in [0,1]$, we have four possible cases

Case (1): If $m_1, m_2 > 0$,i.e., $m_1, m_2 \in (0,2]$; then

$$f\left(Em_2 + \lambda\psi(Em_1, Em_2)\right) = f\left(m_2 + \lambda\psi(m_1, m_2)\right)$$
$$= f\big(m_2 + \lambda(m_1 - m_2)\big) = f(\lambda m_1 + (1-\lambda)m_2) = \frac{1}{2},$$

and $h(\lambda b)f(m_1) + h(1-\lambda b)f(m_2) = 2\lambda b\left(\frac{1}{2}\right) + 2(1-\lambda b)\left(\frac{1}{2}\right) = 1.$

Case (2): If $m_1, m_2 \leq 0$,i.e., $m_1, m_2 \in [-2,0]$; then

$$f\left(Em_2 + \lambda\psi(Em_1, Em_2)\right) = f(\lambda|\mathrm{m}_1| + (1-\lambda)|\mathrm{m}_2|) = \frac{1}{2},$$

and $h(\lambda b)f(m_1) + h(1-\lambda b)f(m_2) = 2\lambda b(1) + 2(1-\lambda b)(1) = 2.$

Case (3): If $m_1 > 0$, $m_2 \leq 0$; i.e., $m_1 \in (0,2]$ and $m_2 \in [-2,0]$; then

$$f\left(Em_2 + \lambda\psi(Em_1, Em_2)\right) = f\left(|\mathrm{m}_2| + \lambda\psi(m_1, |\mathrm{m}_2|)\right)$$
$$= f\big(|\mathrm{m}_2| + \lambda(|\mathrm{m}_2| - \mathrm{m}_1)\big) = 1 \text{ or } \frac{1}{2},$$

and $h(\lambda b)f(m_1) + h(1-\lambda b)f(m_2) = 2\lambda b\left(\frac{1}{2}\right) + 2(1-\lambda b)(1)$

$$= 2 - \lambda b = \begin{cases} 1 & \lambda \in (0,1] \\ 2 & \lambda = 0 \end{cases}.$$

Case (4): If $m_1 \leq 0, m_2 > 0$; i.e., $m_1 \in [-2,0]$ and $m_2 \in (0,2]$ then

$$f\left(Em_2 + \lambda\psi(Em_1, Em_2)\right) = f\left(m_2 + \lambda\psi(|\mathrm{m}_1|, m_2)\right) = f(-\lambda|\mathrm{m}_1| + (1+\lambda)m_2) = 1 \text{ or } \frac{1}{2}$$

and $h(\lambda b)f(m_1) + h(1-\lambda b)f(m_2) = 2\lambda b(1) + 2(1-\lambda b)\left(\frac{1}{2}\right)$

$$= \lambda b + 1 = \begin{cases} 2 & \lambda \in (0,1] \\ 1 & \lambda = 0 \end{cases}.$$

In all cases, we have $f\left(Em_2 + \lambda\psi(Em_1, Em_2)\right) \leq h(\lambda b)f(m_1) + h(1-\lambda b)f(m_2)$ as it is required to show. ■

Note that the class of semi E^h-b-preinvex is a generalization of the class of E-B-preinvex functions when $E = h = I$. The following proposition confirms this observation.

Proposition 2.12 Every E-B-preinvex function is a semi E^h-b-preinvex (respectively, pseudo semi E^h-b-preinvex) function when $E = h = I$.

Proof. It is easy to prove that every E-B-preinvex function is a semi E^h-b-preinvex by choosing $E = h = I$. To show every E-B-preinvex function is a pseudo semi E^h-b-preinvex, let f is an E-B-preinvex function defined on an E-invex set M and such that $E = h = I$. Since $f(m_1) < f(m_2)$ and f is an E-B-preinvex on M, then for every $m_1, m_2 \in M$ and $\lambda \in (0,1)$, we have

$$\begin{aligned} f\left(Em_2 + \lambda\psi(Em_1, Em_2)\right) &\leq \lambda b f(Em_1) + (1-\lambda b)f(Em_2) \\ &= \lambda b f(m_1) + (1-\lambda b)f(m_2) \\ &= f(m_2) + \lambda b\big(f(m_1) - f(m_2)\big) \\ &\leq f(m_2) + \lambda b(1-\lambda b)\big(f(m_1) - f(m_2)\big) \\ &= f(m_2) + \lambda b(\lambda b - 1)\big(f(m_2) - f(m_1)\big) \\ &= f(m_2) + \lambda b(\lambda b - 1)g(m_1, m_2), \end{aligned}$$

where $g(m_1, m_2) = f(m_2) - f(m_1) > 0$. This mean f is pseudo semi E^h-b-preinvex. ■

Remark 2.13 The converse of the proceeding proposition may not be true. In other words, A semi E^h-b-preinvex function is not necessary E-B-preinvex function. Consider Example 2.11 in which we have proved f is semi E^h-b-preinvex function. However, f is not E-B-preinvex. To show this, take $m_1 = 2, m_2 = 0$, and $\lambda = 1$. Then

$$\begin{aligned} f\left(Em_2 + \lambda\psi(Em_1, Em_2)\right) = f\left(0 + \psi(2,0)\right) = f(-2) = 1 \\ > \lambda b f(Em_1) + (1-\lambda b)f(Em_2) \\ = f(2) = \frac{1}{2}. \end{aligned}$$

From Definition 2.3(3), f is not E-B-preinvex function.

3. Some properties of semi E^h-b-preinvex and quasi semi E-preinvex functions

In this section, we discuss some basic properties of semi E^h-b-preinvex and pseudo semi E^h-b-preinvex functions. Some necessary conditions for a function f to be semi E^h-b-preinvex using level sets M_γ and M_γ^E and the epigraph of the semi E^h-b-preinvex functions are given. Also, new characterizations of quasi semi E-preinvex functions using E level sets $E - M_\gamma$ M_γ^E are proved.

Proposition 3.1 Let M, f, E, ψ, b and h are expressed as in Assumption (I). Let f be a semi E^h-b-preinvex on the E-invex set M. Let $\phi: \mathbb{R} \to \mathbb{R}$ be a non-decreasing sublinear function. Then the composite function $\phi \circ f$ is a semi E^h-b-preinvex.

Proof. For all $m_1, m_2 \in M$ then

$$(\phi \circ f)\big(Em_2 + \lambda\psi(Em_1, Em_2)\big) = \phi\left(f\big(Em_2 + \lambda\psi(Em_1, Em_2)\big)\right).$$

From the assumptions on ϕ and f, then the right-hand side yields,

$$\leq \phi(h(\lambda b)f(m_1) + h(1-\lambda b)f(m_2))$$

$$\leq h(\lambda b)\phi(f(m_1)) + h(1-\lambda b)\phi(f(m_2))$$

$$= h(\lambda b)(\phi \circ f)(m_1) + h(1-\lambda b)(\phi \circ f)(m_2))$$

Then, $\phi \circ f$ is a semi E^h-b-preinvex. ∎

Proposition 3.2 Let M, f, E, ψ, b and h are expressed as in Assumption (I) such that f is a pseudo semi E^h-b-preinvex on the E-invex set M. Let $\phi: \mathbb{R} \to \mathbb{R}$ be a non-decreasing strictly positive and sublinear function. Then, $\phi \circ f$ is a pseudo semi E^h-b-preinvex.

Proof. Let $m_1, m_2 \in M$, $\lambda \in [0,1]$. From the definition of f, we have

If $f(m_1) < f(m_2)$ then

$$f\left(Em_2 + \lambda\psi(Em_1, Em_2)\right) \leq f(m_2) + h(\lambda b)h(\lambda b - 1)g(m_1, m_2)$$

Since ϕ is a non-decreasing and strictly positive. Then, if $(\phi \circ f)(m_1) < (\phi \circ f)(m_2)$ we get

$$(\phi \circ f)(Em_2 + \lambda\psi(Em_1, Em_2)) = \phi(f\left(Em_2 + \lambda\psi(Em_1, Em_2)\right))$$

$$\leq \phi(f(m_2) + h(\lambda b)h(\lambda b - 1)g(m_1, m_2))$$

From the sublinearity of ϕ, the right-hand side of the last inequality yields,

$$(\phi \circ f)\left(Em_2 + \lambda\psi(Em_1, Em_2)\right) \leq (\phi \circ f)(m_2) + h(\lambda b)h(\lambda b - 1)\,(\phi \circ g)(m_1, m_2)$$

$$= (\phi \circ f)(m_2) + h(\lambda b)h(\lambda b - 1)k(m_1, m_2),$$

where $k(m_1, m_2) = (\phi \circ g)(m_1, m_2)$. Since ϕ and g are strictly positive functions then $k(m_1, m_2)$ is a strictly positive. Hence, $\phi \circ f$ is a pseudo semi E^h-b-preinvex. ∎

Proposition 3.3 Let M, f, E, ψ, b and h are expressed as in Assumption (I) such that f is a semi E^h-b-preinvex on the E-invex set M.Assume also that $h(1) = 1$ and $h(0) = 0$.Then $f(Em) \leq f(m)$ for each $m \in M$.

Proof. Since f is a semi E^h-b-preinvex on the E-invex set M. Then, for any $m_1, m_2 \in M$, we have

$$f\left(Em_2 + \lambda\psi(Em_1, Em_2)\right) \leq h(\lambda b)f(m_1) + h(1 - \lambda b)f(m_2)$$

Thus, for $\lambda = 0$, we get $f(Em_2) \leq h(1)f(m_2) = f(m_2)$. Then, $f(Em) \leq f(m)$ $\forall m \in M$. ■

Next we provide a non-semi E^h-b-preinvex function by employing Proposition 3.3. This example is inspired by Example 1 in [20].

Example 3.4 $M = [-2, -1] \cup [1,2]$. Let $f, E: \mathbb{R} \to \mathbb{R}$, $b: \mathbb{R} \times \mathbb{R} \times [0,1] \to \mathbb{R}^+$, and $\psi: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are defined as follows. $f(x) = x^2$,

$$\psi(x, y) = \begin{cases} x - y, & x, y \geq 0, \quad or \quad x, y \leq 0; \\ -1 - y, & x > 0,\ y \leq 0 \quad or \quad x \geq 0,\ y < 0; \\ 1 - y, & x < 0,\ y \geq 0 \quad or \quad x \leq 0, y > 0, \end{cases}$$

$$E(x) = \begin{cases} x^2 & if\ |x| \leq \sqrt{2}; \\ -1 & if\ |x| > \sqrt{2}. \end{cases} \quad , \quad b(x, y, \lambda) = \begin{cases} \frac{1}{2\lambda}, & \lambda \neq 0 \\ 0, & \lambda = 0 \end{cases}$$

Define $h(\lambda) = \lambda^n$ $\forall \lambda \in [0,1]$ and $n \in N$. From [20, Example 1], M is an E-invex set. Note that $h(0) = 0$, $h(1) = 1$ and $f\left(E(\sqrt{2})\right) = 4 > f(\sqrt{2}) = 2$. Then from proposition 3.3, it yields f is not a semi E^h-b-preinvex.

The next result provides necessary and sufficient condition for an E^h-b-preinvex function to be semiE^h -b-preinvex function.

Proposition 3.5 Let M, f, E, ψ,b and h are expressed as in Assumption (I). Suppose that f is an E^h-b-preinvex on the E-invex set M. Assume also that $h(1) = 1$ and $h(0) = 0$. Then f is a semi E^h-b-preinvex on M if and only if $f(Em) \leq f(m)$ for each $m \in M$.

Proof. The direct implication is true due from Proposition 3.3. Conversely suppose that f is E^h-b-preinvex on M and $f(Em) \leq f(m)$ for each $m \in M$. Then for any $m_1, m_2 \in M$ we have

$$\begin{aligned} f\left(Em_2 + \lambda\psi(Em_1, Em_2)\right) &\leq h(\lambda b)f(Em_1) + h(1 - \lambda b)f(Em_2) \\ &\leq h(\lambda b)f(m_1) + h(1 - \lambda b)f(m_2). \end{aligned}$$

Thus, f is a semi E^h-b-preinvex on M. ■

The next propositions show that every semi E^h-b-preinvex function is a pseudo semi E^h-b-preinvex under some conditions on h.

Proposition 3.6 Let M, f, E, ψ, b and h are expressed as in Assumption (I) such that h is a positive sublinear function and $h(1) = 1$. If f is semi E^h-b-preinvex function on the E-invex set M. Then f is a pseudo semi E^h-b-preinvex.

Proof. Assume $f(m_1) < f(m_2)$ and f is semi E^h-b-preinvex function on the E-invex set M, then for all m_1,$m_2 \in M$ we have

$$f\left(Em_2 + \lambda\psi(Em_1, Em_2)\right) \leq h(\lambda b)f(m_1) + h(1 - \lambda b)f(m_2).$$

Since h is a sublinear and $h(1) = 1$, the last inequality gives

$$\leq h(\lambda b)f(m_1) + h(1)f(m_2) - h(\lambda b)f(m_2)$$

$$= f(m_2) + h(\lambda b)(f(m_1) - f(m_2)),$$

since h is positive, the last expression yields,

$$< f(m_2) + h(\lambda b)h(\lambda b - 1)(f(m_2) - f(m_1)).$$

Set $g(m_1, m_2) = f(m_2) - f(m_1)$ which is strictly positive, then the last inequality becomes

$$f\big(Em_2 + \lambda\psi(Em_1, Em_2)\big) \leq f(m_2) + h(\lambda b)h(\lambda b - 1)g(m_1, m_2).$$

Then, f is a pseudo semi E^h-b-preinvex. ∎

Next proposition introduces a relationship between E-prequasiinvex and quasi semi E-preinvex functions.

Proposition 3.7 Let M, f, E and ψ are expressed as in Assumption (I). Suppose that f is E-prequasiinvex on the E-invex set M. Then f is a quasi-semi E- preinvex on M if and only if $f(Em) \leq f(m)$ for each $m \in M$.

Proof. First, we prove that $f(Em) \leq f(m)$ for each $m \in M$. Since f is a quasi-semi E- preinvex on M then $\forall$ m_1,$m_2 \in M$

$$f\left(Em_2 + \lambda\psi(Em_1, Em_2)\right) \leq \max\{f(m_1), f(m_2)\}.$$

Take $\lambda = 0$ and $m_1 = m_2$, we get $f(Em_1) \leq f(m_1)$. Now, we show that f is a quasi-semi E-preinvex on M. From the assumptions $f(Em) \leq f(m)$ for each $m \in M$ and f is an E-prequasiinvex function, then $f\left(Em_2 + \lambda\psi(Em_1, Em_2)\right) \leq \max\{f(Em_1), f(Em_2)\} \leq \max\{f(m_1), f(m_2)\}$, as required. ∎

Some properties related to the γ-level sets are given next. First, a necessary conditions for f to be semi E^h-b-preinvex using the E-invexity of the level set M_γ is given as follows.

Proposition 3.8 Let M, f, E, ψ , b and h are expressed as in Assumption (I) such that f is a semi E^h-b-preinvex on the E-invex set M. Assume $h(\alpha) \leq \alpha$ $\forall \alpha \in [0,1]$ and h is a non-negative function. Then, the level set M_γ is an E-invex set for any $\gamma \in \mathbb{R}$.

Proof. For any m_1,$m_2 \in M_\gamma$, $f(m_1) \leq \gamma$ and $f(m_2) \leq \gamma$. Since f is a semi E^h-b-preinvex on the E-invex set, then $Em_2 + \lambda\psi(Em_1, Em_2) \in M$, (1)

and $f\left(Em_2 + \lambda\psi(Em_1, Em_2)\right) \leq h(\lambda b)f(m_1) + h(1 - \lambda b)f(m_2)$

$$\leq h(\lambda b)\gamma + h(1-\lambda b)\gamma \leq \lambda b\gamma + (1-\lambda b)\gamma = \gamma. \qquad (2)$$

From (1) and (2), $Em_2 + \lambda\psi(Em_1, Em_2) \in M_\gamma$ as required. ■

Another necessary condition for f to be semi E^h-b-preinvex using the E-invexity of the level set M_γ^E is shown next.

Proposition 3.9 Let M, f, E, ψ, b and h are expressed as in Assumption (I). Let M be an E-invex set w.r.t. ψ and $E(M)$is an invex set w.r.t. $E \circ \psi$. If f is a semi E^h-b-preinvex on M and assume that E is a linear and idempotent and h is a sublinear such that $h(1) = 1$.Then, M_γ^E is an E-invex set w.r.t. $E \circ \psi$ $\forall \gamma \in \mathbb{R}$.

Proof. Let $\gamma \in \mathbb{R}$ and $Em_1, Em_2 \in M_\gamma^E$, i.e., $m_1, m_2 \in M$ and $f(m_1) \leq \gamma, f(m_2) \leq \gamma$. Since $Em_1, Em_2 \in E(M)$ and $E(M)$ is an E-invex set w.r.t. $E \circ \psi$. Then,

$$E^2m_2 + \lambda(E \circ \psi)(E^2m_1, E^2m_2) \in E(M)$$

From the assumption, $E$is linear and idempotent, thus

$$E\left(Em_2 + \lambda\psi(Em_1, Em_2)\right) \in E(M). \qquad (3)$$

Since f is a semi E^h-b-preinvex, then

$$f(Em_2 + \lambda\psi(Em_1, Em_2) \leq h(\lambda b)f(m_1) + h(1-\lambda b)f(m_2)$$

Since h is a sublinear function and $h(1) = 1$, we have

$$f(Em_2 + \lambda\psi(Em_1, Em_2)) \leq h(\lambda b)\gamma + \gamma - h(\lambda b)\gamma = \gamma. \qquad (4)$$

Using (3) and (4), we obtain M_γ^E is an E-invex set w.r.t. $E \circ \psi$. ■

Proposition 3.10 Let M, f, E, ψ, b and h are expressed as in Assumption (I). Let M be an E-invex set w.r.t. ψ and $E(M)$is an E-invex set w.r.t. $E \circ \psi$. If f is a semi E^h-b-preinvex on M and assume that E is a linear and idempotent and h is a sublinear function such that $h(1) = 1$. Then M_γ^E is slack-E-invex w.r.t. $E(M)$ for all $\in \mathbb{R}$.

Proof. Let $\gamma \in \mathbb{R}$ and $Em_1, Em_2 \in M_\gamma^E[f] \cap E(M)$ such that

$$E^2m_2 + \lambda(E \circ \psi)(E^2m_1, E^2m_2) \in E(M)$$

Using the same procedure of the proof of Proposition 3.9, we get

$$E^2m_2 + \lambda(E \circ \psi)(E^2m_1, E^2m_2) \in M_\gamma^E.$$

This shows M_γ^E is slack-E-invex w.r.t. $E(M)$. ■

A necessary and sufficient condition for f to be quasi semi E-preinvex using the E-invexity of M_γ^E w.r.t. $E \circ \psi$ is given below.

Proposition 3.11 Let M, f, ψ and E are expressed as in Assumption (I) such that M is an E-invex w.r.t. ψ and $E(M)$ is an invex w.r.t. $E \circ \psi$. Assume E is a linear and idempotent mapping. Then M_γ^E is an E-invex set w.r.t. $E \circ \psi$ for all $\gamma \in \mathbb{R}$ if and only if f is a quasi-semi E-preinvex on M.

Proof. Let us first prove that f is a quasi semi E-preinvex on the E-invex set M. Let $m_1, m_2 \in M$ and by setting $\gamma = \max\{f(m_1), f(m_2)\}$. Then $Em_1, Em_2 \in M_\gamma^E$ and $f(m_1) \leq \gamma, f(m_2) \leq \gamma$. Since M_γ^E is an E-invex set w.r.t.$E \circ \psi$, then

$$E^2 m_2 + \lambda E \circ \psi(E^2 m_1, E^2 m_2) \in M_\gamma^E \quad \text{for all } \gamma \in \mathbb{R}.$$

Since E is a linear and idempotent then $E(Em_2 + \lambda\psi(Em_1, Em_2)) \in M_\gamma^E$.

Applying now the definition of M_γ^E to get

$$f\left(Em_2 + \lambda\psi(Em_1, Em_2)\right) \leq \gamma = \max\{f(m_1), f(m_2)\}.$$

This shows f is a quasi semi E-preinvex. Next, we show that M_γ^E is an E-invex set w.r.t. $E \circ \psi$ for all $\gamma \in \mathbb{R}$. Let $\gamma \in \mathbb{R}$ and $Em_1, Em_2 \in M_\gamma^E$ then $Em_1, Em_2 \in E(M)$ and $f(m_1) \leq \gamma, f(m_2) \leq \gamma$. Since $E(M)$ is an invex w.r.t. $E \circ \psi$, then

$$E^2 m_2 + \lambda\left(E \circ \psi\right)(E^2 m_1, E^2 m_2) = E\left(Em_2 + \lambda\psi(Em_1, Em_2)\right) \in E(M), \qquad (5)$$

where in (5) we used the fact that E is linear and idempotent. Because f is a quasi semi E-preinvex and $Em_2 + \lambda\psi(Em_1, Em_2) \in M$, then

$$f(Em_2 + \lambda\, \psi(Em_1, Em_2) \leq \max\{f(m_1), f(m_2)\} \leq \gamma \qquad (6)$$

From (5) and (6), we get M_γ^E is an E-invex set w.r.t. $E \circ \psi$ for all $\gamma \in \mathbb{R}$. ∎

Next, we give characterization of the quasi semi E-preinvex of $f \circ E$.

Proposition 3.12 Let M, f, ψ and E are expressed as in Assumption (I) such that M be an E-invex set. Then $f \circ E$ is a quasi-semi E-preinvex on M iff E-M_γ is an E-invex set for each $\gamma \in \mathbb{R}$.

Proof. First, we prove that E-M_γ is an E-invex set. $\forall\, \gamma \in \mathbb{R}$ and $m_1, m_2 \in E$-M_γ then $f(Em_1) \leq \gamma$ and $f(Em_2) \leq \gamma$, i.e., $(f \circ E)(m_1) \leq \gamma$ and $(f \circ E)(m_2) \leq \gamma$. Since $m_1, m_2 \in M$ and M is an E-invex set, we have

$$Em_2 + \lambda\psi(Em_1, Em_2) \in M \qquad (7)$$

Since $f \circ E$ is quasi semi E-preinvex on M. Then

$$(f \circ E)(\, Em_2 + \lambda\psi(Em_1, Em_2)) \leq \max\{(f \circ E)(m_1), (f \circ E)(m_2)\} \leq \gamma \qquad (8)$$

Then by (7) and (8), we get $Em_2 + \lambda\psi(Em_1, Em_2) \in E$-$M_\gamma$. Therefore, E-M_γ is an E-invex w.r.t. ψ for each $\gamma \in \mathbb{R}$. Now, we prove that $f \circ E$ is quasi semi E-preinvex on M. Let $m_1, m_2 \in M$. Since M is an E-invex set, we have

$$Em_2 + \lambda\psi(Em_1, Em_2) \in M$$

Set $\gamma = \max\{(f \circ E)(m_1), (f \circ E)(m_2)\}$. Since E-M_γ is an E-invex w.r.t. ψ, we get

$$(f \circ E)(Em_2 + \lambda\psi(Em_1, Em_2)) \leq \gamma = \max\{(f \circ E)(m_1), (f \circ E)(m_2)\}$$

This shows $f \circ E$ is quasi semi E-preinvex on M. ■

A necessary condition for f to be semi E^h-b-preinvex using the $E \times I$ invexity of $epif$.

Proposition 3.13 Let M, f, E, ψ, b and h are expressed as in Assumption (I) such that f is a semi E^h-b- preinvex function on the E-invex set M. Assume that $h(\alpha) \leq \alpha \ \forall \alpha \in [0,1]$ and h is a non-negative function .Then $epif$ is $E \times I$ invex on $M \times \mathbb{R}$ w.r.t. $\psi \times \psi_\circ$ where $\psi_\circ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that $\psi_\circ(x, y) = b(x, y) \ \forall x, y \in \mathbb{R}$.

Proof. Let (m_1, δ),$(m_2, \beta) \in epif$ such that $f(m_1) \leq \delta$,$f(m_2) \leq \beta$. From the assumption, f is semi E^h-b- preinvex function on the E-invex set M. Hence,

$$\begin{aligned} f\left(Em_2 + \lambda\psi(Em_1, Em_2)\right) &\leq h(\lambda b)f(m_1) + h(1 - \lambda b)f(m_2) \\ &\leq \lambda b\delta + (1 - \lambda b)\beta, \end{aligned}$$

where in the last inequality we used the assumptions on h. This implies,

$$(Em_2 + \lambda\psi(Em_1, Em_2), \lambda b\delta + (1 - \lambda b)\beta) \in epif \qquad (9)$$

Now, $epif$ is an $E \times I$ invex w.r.t. $\psi \times \psi_\circ$ where $\psi_\circ(\delta, \beta) = b(\delta - \beta)$. This means

$$\lambda b\delta + (1 - \lambda b)\beta = \beta + \lambda b(\delta - \beta) = I(\beta) + \lambda\psi_\circ\left(I(\delta), I(\beta)\right).$$

Therefore, (9) can be re-written as $(Em_2 + \lambda\psi(Em_1, Em_2), I(\beta) + \lambda\psi_\circ\left(I(\delta), I(\beta)\right) \in epi\, f$. Then $epi\, f$ is an $E \times I$ invex on $M \times \mathbb{R}$ w.r.t. $\psi \times \psi_\circ$. ■

The following proposition gives sufficient condition for f to be quasi semi E-preinvex function.

Proposition 3.14 Let M, f, E, ψ, b and h are expressed as in Assumption (I). Suppose $epif$ is $E \times I$ invex w.r.t. $\psi \times \psi_\circ$ where $\psi \times \psi_\circ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. Then f is a quasi-semi E-preinvex on M.

Proof. Assume that $(m_1, f(m_1))$,$(m_2, f(m_2)) \in epif$ which is $E \times I$-invex w.r.t. $\psi \times \psi_\circ$. Then $\left(Em_2 + \lambda\psi(Em_1, Em_2), f(m_2) + \lambda\psi_\circ\left(f(m_1), f(m_2)\right)\right) \in epi\, f$,

where $\psi_\circ\left(f(m_1), f(m_2)\right) = b\left(f(m_1) - f(m_2)\right)$. Then $\left(Em_2 + \lambda\psi(Em_1, EM_2), \lambda bf(m_1) + (1 - \lambda b)f(m_2)\right) \in epif$,

i.e., $f(Em_2 + \lambda\psi(Em_1, Em_2) \leq \lambda bf(m_1) + (1 - \lambda b)f(m_2)$. (10)

Define

$$b(m_1, m_2, \lambda) = \begin{cases} \dfrac{1}{\lambda} & if\ \lambda \in (0,1] \\ 0 & if\ \ \lambda = 0 \end{cases}$$

If $\lambda \in (0,1]$, then $\lambda b = 1$. Hence, (10) is written as

$$f(Em_2 + \lambda\psi(Em_1, Em_2) \leq f(m_1). \qquad (11)$$

If $\lambda = 0$, then $\lambda b = 0$, Hence, (10) is written as $f(Em_2 + \lambda\psi(Em_1, Em_2) \leq f(m_2)$ (12)

From (11) and (12) we get $f(Em_2 + \lambda\psi(Em_1, Em_2) \leq \max\{f(m_1), f(m_2)\}\}$. Thus, f is a quasi-semi E-preinvex on M. ■

4. Applications to non-linear optimization problems

Consider the following non-linear optimization problem which will be denoted as (P_E)

$$\min\ (f \circ E)(m)$$

$$\text{subject to } m \in M$$

where M, f and E are defined as in assumption (I). In this section, we study some optimality properties of Problem (P_E) for which the objective function $f \circ E$ is semi E^h-b-preinvex or the function f is E^h-b-preinvex function.

Remark 4.1 In Problem (P_E), if M is an E-invex set and $f \circ E$ is a semi E^h-b-preinvex functions on M then Problem (P_E) is called semi E^h-b-preinvex optimization problem.

Definition 4.2 The set of all global minima (or optimal solutions) of Problem (P_E) is denoted by $argmin_M f \circ E$ and is defined as

$$argmin_M f \circ E = \{m^* \in M: (f \circ E)(m^*) \leq (f \circ E)(m) \quad \forall m \in M\}.$$

Definition 4.3 A point $m^* \in \mathbb{R}^n$ is said to be local minimum if there exists $\varepsilon > 0$ such that $f(m^*) \leq f(m) \quad \forall m \in B(m^*, \varepsilon) \cap M$ where $(m^*, \varepsilon) = \{m \in \mathbb{R}^n : ||m - m^*|| \leq \varepsilon\}$ is the neighborhood of m^* with radius ε.

Proposition 4.4 Consider Problem (P_E) such that $f \circ E$ is a strictly semi E^h-b-preinvex on the E-invex set M. Assume that h is a sublinear function and $h(1) = 1$. Then the global optimal solutions of problem (P_E) is unique.

Proof. Let $m_1, m_2 \in M$ be two different global optimal solutions of problem (P_E). Then $(f \circ E)(m_1) = (f \circ E)(m_2)$. Since M is an E-invex and $f \circ E$ is strictly semi E^h-b- preinvex then for each $\lambda \in (0,1)$, we have $z = Em_2 + \lambda\psi(Em_1, Em_2) \in M$ where $z \neq m_1 (\neq m_2)$ and

$$(f \circ E)(z)) < h(\lambda b)(f \circ E)(m_1) + h(1 - \lambda b)(f \circ E)(m_2)$$

$$\leq h(\lambda b)(f \circ E)(m_1) + h(1)(f \circ E)(m_2) - h(\lambda b)(f \circ E)(m_2) = (f \circ E)(m_2),$$

The last expression contradicts the optimality of m_2 for problem (P_E). This shows the global optimal solution of problem (NLP_E) is unique. ■

Proposition 4.5 Consider semi E^h-b-preinvex optimization problem (P_E) defined as in Remark 4.1 such that h is a sublinear function and $h(1) = 1$. If m^* be a local minimum of Problem (P_E) such that $m^* = Em^*$. Then m^* is a global minimum.

Proof. Since $m^* = Em^*$ is a local minimum, then there exists $\varepsilon > 0$ such that

$$(f \circ E)(m^*) \leq (f \circ E)(m) \ \ \forall m \in M \cap B(m^*, \varepsilon) = H.$$

We must prove now that $(f \circ E)(m^*) \leq (f \circ E)(m) \quad \forall m \in M/H$. By contradiction, assume that there exists $\bar{m} \in M$ and $\bar{m} \notin B(m^*, \varepsilon)$ such that $(f \circ E)(m^*) > (f \circ E)(\bar{m})$. (13)

Since M is an E-invex and $f \circ E$ is a semi E^h-b-preinvex, then $Em^* + \lambda\psi(E\bar{m}.Em^*) \in M$, and $(f \circ E)\ (Em^* + \lambda\psi(E\bar{m}, Em^*)) \leq\ h(\lambda b)(f \circ E)(\bar{m}) + h(1 - \lambda b)(f \circ E)(m^*)$

Applying (13) and the assumption on h

$$(f \circ E)\big(Em^* + \lambda\psi(E\bar{m}, Em^*)\big) < h(\lambda b)(f \circ E)(m^*) + (f \circ E)(m^*) - h(\lambda b)(f \circ E)(m^*)$$

$$= (f \circ E)(Em^*) = (f \circ E)(m^*) \qquad (14)$$

If $\psi(E\bar{m}, Em^*) = 0$, then (14) becomes $(f \circ E)(Em^*) < (f \circ E)(m^*)$, which is a contradiction. If now $\psi(E\bar{m}, Em^*) \neq 0$, choose ε sufficiently small such that $\lambda = \frac{\varepsilon}{\|\psi(E\bar{m}, Em^*)\|} \leq 1$,

$$\|m^* - [Em^* + \lambda\psi(E\bar{m}, Em^*)]\| = \|Em^* - Em^* - \lambda\psi(E\bar{m}, Em^*)\|$$

$$= \lambda\|\psi(E\bar{m}, Em^*)\| = \frac{\varepsilon}{\|\psi(E\bar{m}, Em^*)\|}\|\psi(E\bar{m}, Em^*)\| = \varepsilon$$

This shows $Em^* + \lambda\psi(E\bar{m}, Em^*) \in B(m^*, \varepsilon)$, Again (14) contradicts the fact that m^* is a local minimum. Hence, $(f \circ E)(m^*) \leq (f \circ E)(m) \quad \forall m \in M\ /\ H$ as required. ■

The following optimality property holds when f in Problem (P_E) is E^h-b-preinvex functions.

Proposition 4.6 Consider Problem (P_E) such that f is an E^h-b-preinvex on the E-invex set M. Assume that $f(Em) \leq f(m)$ for all $m \in M$ and h is a sublinear and $h(1) = 1$. Then $argmax_M f \circ E$ occure on the boundary of M where $argmax_M f \circ E = \{m^* \in M: (f \circ E)(m^*) \geq (f \circ E)(m) \quad \forall m \in M\}$.

Proof. On contrary, assume that there exists m^* belongs to the interior of M such that $(f \circ E)(m^*) \geq (f \circ E)(m) \quad \forall m \in M$. (15)

Make a line through m^* and cutting the boundary of M at m_1, m_2. Since M is an E-invex set then $m^* = Em_2 + \lambda\psi(Em_1, Em_2) \in M$ for some $\lambda \in (0,1]$. Since f is an E^h-b-preinvex on M and $f(Em) \leq f(m)$ for all $m \in M$ then

$$(f \circ E)(m^*) \leq f(m^*) = f(Em_2 + \lambda\psi(Em_1, Em_2))$$

$$\leq h(\lambda b)f(Em_1) + h(1 - \lambda b)f(Em_2) \qquad (16)$$

If $f(Em_1) \leq f(Em_2)$, hence from (16) and the assumptions on h, the above expression yields

$$(f \circ E)(m^*) \leq h(\lambda b)f(Em_2) + h(1)f(Em_2) - h(\lambda b)f(Em_2) = (f \circ E)(m_2)$$

The last inequality contradicts (15). Similarly, if $f(Em_2) \leq f(Em_1)$, we get $(f \circ E)(m^*) \leq (f \circ E)(m_1)$ which also contradicts (15). This shows $argmax_M f \circ E$ occure on the boundary of M. ■

Conclusion In this paper, new types of generalized convex functions, namely, semi E^h-b-preinvex and pseudo E^h-b-preinvex functions are proposed. Some properties of these functions as well as quasi-semi E-preinvex functions are discussed. As an application of semi E^h-b-preinvex functions to optimization problems, some optimality properties are established.

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