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New Analytic Approach of Mild Solution for General Formula of Extensible Beam Model Equation

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Abstract.
In this paper. The generalized of extensible beam equation such that this equation is a special case where α is ordinary positive integer number and the nonlinear function is more general of the nonlinear function of beam equation and since the solvability of this equation is difficult some time. So, we turned to fractional order differential equation to use the nonlinear factional analyses to check the solvability exist.

1. Introduction.

There are many types of integrodifferential equations which are refer to physical, engineering models and another science. Many researchers are trying to develop methods to solve them, an example, in [2] and [12] the researchers have been discussed the existence of solutions of the following integrodifferential equation

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^4 u}{\partial x^4} - \left(\beta_1 + \beta_2 \int_0^L \left|\frac{\partial u(y,s)}{\partial y}\right|^2 dy\right)\frac{\partial^2 u}{\partial x^2} + g\left(\frac{\partial u}{\partial t}\right) = 0.$$

where $\beta_1, \beta_2, L > 0$, g is a nondecreasing numerical function. This equation can be converted to an abstract differential equation as

$$u'' + B^2 u + M\left(\left\|B^{1/2}u\right\|_H^2\right)Bu + g(u') = 0$$

In [1], the researchers have been discussed the existence of mild solution for integrodifferential equation that is taken the mathematical model in the form

$$\frac{\partial^2 u}{\partial t^2} - \lambda\frac{\partial^4 u}{\partial t^2 \partial x^2} + \frac{\partial^4 u}{\partial x^4} - \left(\beta_1 + \beta_2 \int_0^L \left|\frac{\partial u(y,s)}{\partial y}\right|^2 dy\right)\frac{\partial^2 u}{\partial x^2} = f\left(\frac{\partial u}{\partial t}\right).$$

This equation can be converted to an abstract differential equation as

$$Ku''(t) = Au(t) + f(t, u(t), u'(t))$$

$$u(0) = u_0 , \quad u'(0) = y_0 , \quad t \in [0, T].$$

Recently the development studies of the differential equation theory convert differential equation from differential equation with integer order to differential equation with fractional order (fractional differential equation) since the fractional calculus application in various branches of engineering and science.

In this article we discussed the existence of mild solution for type of more general form of integrodifferential equation as fractional-order partial differential equation when its mathematical model is the initial boundary value problem with Riemann-Liouville fractional derivative

$$ {}^{R}D_t^{\alpha} z - \beta\, {}^{R}D_t^{\alpha}\, {}^{R}D_x^{\alpha} z + {}^{R}D_x^{2\alpha} z - \left(\beta_1 + \beta_2 \int_0^L \left|\frac{\partial z(y,s)}{\partial y}\right|^2 dy\right) {}^{R}D_x^{\alpha} z = f\left(t, z, {}^{R}D_t^{\gamma} z\right) \quad (1.1)$$

$$D_t^{\alpha-1} z(t,x)|_{t=0} = \tilde{z}_0, \qquad D_t^{\alpha-2} z(t,x)|_{t=0} = \tilde{y}_0$$

where $\beta_1, \beta_2, L > 0$, the deflection of a point *x* at time *t* is *z*(*x*, *t*),

$1 < \alpha \le 2$, $0 < \gamma \le 1$ and $\beta \ge 0$.

Now, let the operator $A^{\alpha} = {}^{R}D_x^{\alpha} z(t,x)$ .Then we have

$$(I - \beta A^{\alpha})\, {}^{R}D_t^{\alpha} z(t,x) + A^{2\alpha} z(t,x) - \left(\beta_1 + \beta_2 \int_0^L \left|\frac{\partial z(y,s)}{\partial y}\right|^2 dy\right) A^{\alpha} z(t,x)$$

$$= f\left(t, z(t,x), {}^{R}D_t^{\gamma} z(t,x)\right). \quad (1.2)$$

$$(I - \beta A^{\alpha})\, {}^{R}D_t^{\alpha} z(t,x) + A^{2\alpha} z(t,x) - \left(\beta_1 + \beta_2 M\left[\left\|A^{\alpha/2} z\right\|^2\right]\right) A^{\alpha} z(t,x)$$

$$= f\left(t, z(t,x), {}^{R}D_t^{\gamma} z(t,x)\right). \quad (1.3)$$

M is real functions. Then the equation (1.3) can be written in the abstract form

$$\begin{cases} P\, {}^{R}D_t^{\alpha} z(t,x) = \tilde{A}^{\alpha} z(t,x) + f\left(t, z(t,x), {}^{R}D_t^{\gamma} z(t,x)\right), \\ {}^{R}D_t^{\alpha-1} z(t,x)\big|_{t=0} = \tilde{z}_0, \qquad {}^{R}D_t^{\alpha-1} z(t,x)\big|_{t=0} = \tilde{y}_0 \end{cases} \quad (1.4)$$

where, $1 < \alpha \le 2$, $0 < \gamma \le 1$, $t \in J = [0,T]$ and $x \in X$ (*X* is a Banach space)

$$\tilde{A}^{\alpha} = \left(\beta_1 + \beta_2 M\left[\left\|A^{\alpha/2} z\right\|^2\right]\right) A^{\alpha} - A^{2\alpha}, \quad P = (I - \beta A^{\alpha})$$

where P and $\tilde{A}^{\alpha}$ are linear operators in the Banach space

$$C_{\nu}^{RL}([0,T],X) = \left\{z \in C([0,T],X) \colon {}^{R}D_t^{\gamma} z \in C([0,T],X)\right\}$$

with the norm $\|z\|^* = \|z\|_C + \left\|{}^{R}D_t^{\gamma} z\right\|_C$ and $\|.\|_C$ is the sup norm in $C([0,T],X)$, $\nu = max\{\alpha, \gamma\}$ and the nonlinear function $f: J \times C_{\nu}^{RL}([0,T],X) \times C_{\nu}^{RL}([0,T],X) \to C_{\nu}^{RL}([0,T],X)$.

Our problem is more general and development than other "extensible Beam Equation" such as [1], [2] and [12], since it will be with fractional orders and our solvability of the problem is to use advance nonlinear functional analysis and give a necessary and sufficient condition later on to discuss the existence of the mild solution for the system (1.4) and the operator $\tilde{A}^{\alpha}$ which defined on some types of spaces as well as the semigroup $\tilde{S}_{\alpha}(t)$ that will be defined later by involving the Mainardi's function.

2. ***Preliminaries***

In this section, we present some notation, assumptions and results needed in our proofs later.

IOP Conf. Series: Materials Science and Engineering **571** (2019) 012022 doi:10.1088/1757-899X/571/1/012022

***Definition (2.1)*, [10]:**

The fractional integral of order $\alpha > 0$ with the lower limit 0 for a function h is defined as:

$$I^{\alpha} h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{h(s)}{(t-s)^{1-\alpha}} ds, \ t > 0, \ \alpha > 0$$

where $\Gamma(u) = \int_0^{\infty} e^{-s} s^{u-1} ds$, $u > 0$ (gamma function).

***Definition (2.2)*, [10], [12]:**

The Riemann - Liouvill derivative of order $\alpha > 0$ with lower limit 0 for a function h can be written as:

$${}^{R}D^{\alpha} h(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{h(s)}{(t-s)^{\alpha+1-n}}, t > 0, \ n-1 < \alpha \leq n$$

***Definition (2.3)*, [10]:**

The Mainardi's function is defined by:

$$M_{\alpha}(u) = \sum_{k=0}^{\infty} \frac{(-1)^k u^k}{k! \, \Gamma(-\alpha(\alpha+1)+1)} \quad \text{, where } 0 < \alpha < 1 , u \in \not{C}$$

***Remark (2.1)*, [7]:**

It's clear that, The Mainardi's function satisfies the following:

a. $\int_0^{\infty} M_{\alpha}(u) du = 1$

b. $\int_0^{\infty} u^n M_{\alpha}(u) dr = \frac{\Gamma(n+1)}{\Gamma(\alpha n+1)}$, $n \in N^+$.

c. For any $\lambda \in \not{C}$ and $0 < \beta < 1$ then :

$$e^{-\lambda^{\beta}} = \int_0^{\infty} \frac{\beta}{r^{\beta+1}} M_{\beta}(r^{-\beta}) e^{-\lambda r} dr$$

***Remark (2.2)*, [9]:**

If A is generator of a strongly continuous semigroup $S(t), t \geq 0$ then,

a) for $y \in X$; $\lim_{g\to 0} \frac{1}{g} \int_t^{t+g} S(s) y ds = S(t)y$

b) for $y \in X$; $\int_0^t S(s) y ds \in D(A)$ and $A \int_0^t S(s) y ds = S(t)y - y$

c) for $y \in D(A)$ then $S(t)y \in D(A)$ and $\frac{d}{dt} S(t)y = A\, S(t)y = S(t)Ay$

for $\lambda > 0$ *and* $y \in X$, then $R(\lambda; A) = \int_0^{\infty} e^{-\lambda s} S(s) dt$

***Definition (2.4)* [5]:**

A family one parameter $\{S(t), t \geq 0\}$ of bounded linear operators on a Banach space X is called a semigroup of bounded linear operators on X if:-

i. $S(0) = I$

ii. $S(t + s) = S(t) S(s)$, for every $s, t \geq 0$.

***Definition (2.5)* ,[9]:**

A semigroup $S(t), t \geq 0$ of bounded linear operators on a Banach space X is called uniformly continuous if $\lim_{t\to 0} ||S(t) - I|| = 0$

***Definition (2.6),* [9]:**

A semigroup S(t), $t \geq 0$ of bounded linear operators on a Banach space X is a strongly continuous semigroup of bounded linear operators if, for each $x \in X$ then $\lim_{t\to 0} S(t)x = x$.

***Definition (2.7),* [9]:**

Let S(t), $t \geq 0$, be a semigroup of bounded linear operators on a Banach space X.

The linear operator L defined by:

$Lx = lim_{t\to 0}\frac{T(t)x-x}{t} = \frac{dT(t)x}{dt}\Big|_{t=0}$, for $x \in D(A)$

Where, $D(L) = \left\{x \in X\ lim_{t\to 0}\frac{T(t)x-x}{t}\ \text{exists}\right\}$.

is the generator of semigroup T(t), $t \geq 0$, where D(L) is the domain of L .

***Definition (2.8),* [10]:**

The Laplace transform, of the Riemann-Liouville fractional derivation of order $\alpha > 0$ gives as:

$\mathcal{L}\{\,{}^{R}D_t^{\alpha}\, h(t)\,\} = \lambda^{\alpha}\ \mathcal{L}(h(t))\ (\lambda) - \sum_{k=0}^{n-1}\lambda^k[\,{}^{R}D_t^{\alpha-k-1}\ h(t)]|_{t=0})\ (\lambda)$, *n-1*$< \alpha \leq$*n.*

Lemma (2.1), [11]:

If $F(t)$ is a linear operator and $I^{1-\alpha}F(t)x \in C^1([0,T]), T > 0$ then, for $0<\alpha<1,$ we get

$${}^{R}D_t^{\alpha}\int_0^t F(t-s)xds = \int_0^t {}^{R}D_t^{\alpha}F(t-s)xds + \lim_{t\to 0^+} I^{1-\alpha}F(t)x\,, x \in X, t \in [0,T]$$

Lemma (2.2), [11]:

If h is a continuous function and $I^{1-\alpha}h(t) \in C^1([0,T]), T > 0$ and $F(t)$ is continuous, then for $0<\alpha<1,$ we have

$${}^{R}D_t^{\alpha}\int_0^t F(t-s)h(s)ds = \int_0^t F(t-s)\,{}^{R}D_t^{\alpha}h(s)ds\,, \qquad t \in [0,T]$$

Definition (2.9)

A function z is called a mild solution of the system (1.4) if it is continuous and ${}^{R}D_t^{\nu}z$ ($\nu = \max\{\alpha,\gamma\}$) exists and also it is continuous on $[0,T]$ and the following equation holds

$$z(t,x) = S_\alpha(t)\,t^{\alpha-1}\tilde{z}_0 + \lambda S_\alpha(t)\,t^{\alpha-1}\tilde{y}_0$$
$$+\int_0^t S_\alpha(t-s)\,(t-s)^{\alpha-1}\left[P^{-1}f\left(s, z(s,x), {}^{R}D_t^{\gamma}z(s,x)\right)\right]ds\,,\ \ t \in [0,T].$$

We assume the following conditions:

(A_1) $P = (I - \beta A^{\alpha})$ and $\tilde{A}^{\alpha} = \left(\beta_1 + \beta_2 M\left[\|A^{\alpha/2}z\|^2\right]\right)A^{\alpha} - A^{2\alpha}$, are closed linear operators, $D(P) \subset D(\tilde{A}^{\alpha})$, P is bijective and $P^{-1}: X \to D(P)$ is continuous.

(A_2) $P^{-1}\tilde{A}^{\alpha}$ infinitesimal generate a strongly continuous semigroup $S_\alpha(t)$, $t \in \mathbb{R}$, the adjoint operator $\left(P^{-1}\tilde{A}^{\alpha}\right)^*$ is densely defined $\left(\overline{D\left(P^{-1}\tilde{A}^{\alpha}\right)^*} = C_\nu^{RL}([0,T],X)^*\right)$.

(A_3) $S_\alpha(t)$, $t > 0$ is a compact operator.

IOP Conf. Series: Materials Science and Engineering **571** (2019) 012022 doi:10.1088/1757-899X/571/1/012022

(A_4)

(i) $f(t,.,.): C_\nu^{RL}([0,T],X) \times C_\nu^{RL}([0,T],X) \to C_\nu^{RL}([0,T],X)$ is continuous for a.e. $t \in J$,

(ii) for every $(z, {}^R D_t^\gamma z) \in C_\nu^{RL}([0,T],X) \times C_\nu^{RL}([0,T],X)$, the function $f(.,z, {}^R D_t^\gamma z): J \to C_\nu^{RL}([0,T],X)$ is strongly measurable .

(iii) there exists a nonnegative continuous function $K_f(t)$ and a continuous nondecreasing positive function Ω_f such that

$\|f(t,z, {}^R D_t^\gamma z)\| \le K_f(t)\Omega_f(\|z\| + \| {}^R D_t^\gamma z\|)$,

for $(t,z, {}^R D_t^\gamma z) \in J \times C_\nu^{RL}([0,T],X) \times C_\nu^{RL}([0,T],X)$.

(A_5) let

(i) $\mathrm{K}_1 = \{(\tilde{\mathrm{N}} + \widetilde{M}_\alpha e^{wt^\alpha} \| t^{\alpha-1}\|)\|\tilde{z}_0\| + \lambda(\tilde{\mathrm{N}} + \widetilde{M}_\alpha e^{wt^\alpha} \| t^{\alpha-1}\|)\|\tilde{y}_0\|.$

(ii) $\mathrm{K}_3 = max\left\{1, \frac{T^{\nu-\gamma}}{\Gamma(\nu-\gamma+1)}\right\}$ and $\mathrm{K}_2 = \mathrm{K}_1\mathrm{K}_3$.

(iii) $L(s) = \|P^{-1}\|(\widetilde{M}_\alpha e^{w(t-s)^\alpha}\|(t-s)^{\alpha-1}\| + \tilde{\mathrm{N}})K_f(s).$

3. Existence of mild solutions

In this section we prove the existence of a mild solution of system (1.4) in the space $C_\nu^{RL}([0,T],X) = \{z \in C([0,T]): {}^R D_t^\nu z \in C([0,T])\}$ with the norm $\|z\|^* = \|z\|_C + \| {}^R D_t^\nu z\|_C$ where $\|.\|_C$ is sup norm in $C([0,T],X)$ and $\nu = max\{\alpha , \gamma\}$.

***Lemma (3.1)*:**

If $P^{-1}\tilde{A}^\alpha$ is an infinitesimal generator, of strongly continuous semigroup $S_\alpha(t)$, $t \ge 0$ in Z and $\lambda^\alpha \in \rho(P^{-1}\tilde{A}^\alpha)$. Then we get

i) for any function $z(t,x) \in$ L([0,T] ;X), we have

$$(\lambda^\alpha I - P^{-1}\tilde{A}^\alpha)^{-1}\mathcal{L}(z(t))(\lambda) = \int_0^\infty e^{-\lambda t}\left[\int_0^t \tilde{S}_\alpha(t-s)\,(t-s)^{\alpha-1}z(s)ds\right]dt \tag{3.1}$$

ii) For any $\tilde{z}_0 \in D(P^{-1}\tilde{A}^\alpha)$, we have

$$(\lambda^\alpha I - P^{-1}\tilde{A}^\alpha)^{-1}\tilde{z}_0 = \int_0^\infty e^{-\lambda t}\tilde{S}_\alpha(t)(t)^{\alpha-1}\tilde{z}_0 dt \tag{3.2}$$

iii) For any $\tilde{z}_0 \in D(P^{-1}\tilde{A}^\alpha)$, $\lambda^\alpha \in \rho(P^{-1}\tilde{A}^\alpha)$ and λ complex number, we have

$$\lambda(\lambda^\alpha I - P^{-1}\tilde{A}^\alpha)^{-1}\tilde{z}_0 = \lambda\int_0^\infty e^{-\lambda t}\tilde{S}_\alpha(t)(t)^{\alpha-1}\tilde{z}_0 dt \tag{3.3}$$

where, $1 < \alpha \le 2$ and the operator

$$\tilde{S}_\alpha(t)x = \int_0^\infty \alpha r M_\alpha(r)\, S_\alpha(t^\alpha r)x\, dr \tag{3.4}$$

where M_α is defined in Definition (2.3).

Proof:

i) From remark (2.2), for every $z \in D(P^{-1}\tilde{A}^\alpha)$, we get

$$\left(\lambda^\alpha I - P^{-1}\tilde{A}^\alpha\right)^{-1}\mathcal{L}(z(t))(\lambda) = \int_0^\infty e^{-\lambda^\alpha u}\, S_\alpha(u)(\mathcal{L}z(t))(\lambda)du$$

Applying Laplace transform, we get

$$\left(\lambda^{\alpha}I-P^{-1}\tilde{A}^{\alpha}\right)^{-1}\left(\mathcal{L}z(t)\right)(\lambda)=\int_0^{\infty}\int_0^{\infty}e^{-\lambda^{\alpha}u}\,S_{\alpha}(u)\,e^{-st}z(s)dsdu$$

Suppose that $u=x^{\alpha}$

$$\left(\lambda^{\alpha}I-P^{-1}\tilde{A}^{\alpha}\right)^{-1}\left(\mathcal{L}z(t)\right)(\lambda)=\int_0^{\infty}\int_0^{\infty}e^{-(\lambda x)^{\alpha}}S_{\alpha}(x^{\alpha})\alpha x^{\alpha-1}\;e^{-st}z(s)dsdx$$

from remark (2.1), we get

$e^{-(\lambda x)^{\alpha}}=\int_0^{\infty}\frac{\alpha}{r^{\alpha+1}}\,M_{\alpha}(r^{-\alpha})e^{-\lambda rx}dr.$ then we obtain

$$\left(\lambda^{\alpha}I-P^{-1}\tilde{A}^{\alpha}\right)^{-1}\left(\mathcal{L}z(t)\right)(\lambda)$$
$$=\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}e^{-\lambda rx}\frac{\alpha}{r^{\alpha+1}}M_{\alpha}(r^{-\alpha})S_{\alpha}(x^{\alpha})\alpha x^{\alpha-1}e^{-\lambda s}z(s)drdsdx$$

now suppose $rx=t$ we get

$$\left(\lambda^{\alpha}\mathrm{I}-P^{-1}\tilde{A}^{\alpha}\right)^{-1}\left(\mathcal{L}z(t)\right)(\lambda)$$
$$=\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}e^{-\lambda t}\;\frac{\alpha^2}{r^{\alpha+1}}M_{\alpha}(r^{-\alpha})\;S_{\alpha}(t^{\alpha}r^{-\alpha})\;t^{\alpha-1}r^{-\alpha}e^{-\lambda s}z(s)dr\;ds\;dt$$

Let $y=r^{-\alpha}$ which implies $r=y^{\frac{-1}{\alpha}}$, we get

$$\left(\lambda^{\alpha}I-P^{-1}\tilde{A}^{\alpha}\right)^{-1}\left(\mathcal{L}z(t)\right)(\lambda)$$
$$=\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}e^{-\lambda(t+s)}\;\frac{\alpha^2}{y^{\frac{-1}{\alpha}-1}}M_{\alpha}(y)\;S_{\alpha}(t^{\alpha}y)\;\;t^{\alpha-1}yz(s)\frac{-1}{\alpha}y^{\frac{-1}{\alpha}-1}dy\;ds\;dt$$
$$=\int_0^{\infty}e^{-\lambda t}[\;\int_0^{t}\int_0^{\infty}\alpha yM_{\alpha}(y)(t-s)^{\alpha-1}\;\;S_{\alpha}\big((t-s)y\big)z(s)dy\;ds\;]dt$$
$$=\int_0^{\infty}e^{-\lambda t}\left[\int_0^{t}\tilde{S}_{\alpha}(t-s)\;(t-s)^{\alpha-1}z(s)ds\right]dt \tag{3.5}$$

ii) From remark (2.2), for every $\tilde{z}_0\in D(P^{-1}\tilde{A}^{\alpha})$, we get

$(\lambda^{\alpha}I-P^{-1}\tilde{A}^{\alpha})^{-1}\tilde{z}_0=\int_0^{\infty}e^{-\lambda^{\alpha}u}\;S_{\alpha}(u)\tilde{z}_0du$, Suppose that $u=x^{\alpha}$. we get

$$(\lambda^{\alpha}I-P^{-1}\tilde{A}^{\alpha})^{-1}\tilde{z}_0=\int_0^{\infty}e^{-(\lambda x)^{\alpha}}S_{\alpha}(x^{\alpha})\alpha x^{\alpha-1}\,\tilde{z}_0dx$$

from remark (2.1), we get

$$(\lambda^{\alpha}I-P^{-1}\tilde{A}^{\alpha})^{-1}\tilde{z}_0=\int_0^{\infty}\int_0^{\infty}\frac{\alpha}{r^{\alpha+1}}e^{-\lambda rx}M_{\alpha}(r^{-\alpha})S_{\alpha}(x^{\alpha})\alpha x^{\alpha-1}\tilde{z}_0drdx$$

suppose $rx=t$. then we get

$$(\lambda^{\alpha}I-P^{-1}\tilde{A}^{\alpha})^{-1}\tilde{z}_0=$$
$$\int_0^{\infty}\int_0^{\infty}e^{-\lambda t}\;\frac{\alpha^2}{r^{\alpha+1}}M_{\alpha}(r^{-\alpha})\;S_{\alpha}(t^{\alpha}r^{-\alpha})\;t^{\alpha-1}r^{-\alpha}\;\tilde{z}_0drdt$$

assume that $y=r^{-\alpha}$ then, $r=y^{\frac{-1}{\alpha}}$. Thus, we get

$$(\lambda^{\alpha}I-P^{-1}\tilde{A}^{\alpha})^{-1}\tilde{z}_0=\int_0^{\infty}\int_0^{\infty}e^{-\lambda t}\;\alpha yM_{\alpha}(y)\;S_{\alpha}(t^{\alpha}y)t^{\alpha-1}\tilde{z}_0dydt$$
$$=\int_0^{\infty}e^{-\lambda t}\tilde{S}_{\alpha}(t)\;t^{\alpha-1}\tilde{z}_0dt \tag{3.6}$$

iii) From remark (2.2), for every $\tilde{z}_0 \in D(P^{-1}A^{\alpha})$, we can follow the same steps in the prove of (ii) to get that

$$(\lambda^{\alpha} I - P^{-1}\tilde{A}^{\alpha})^{-1}\tilde{z}_0 = \int_0^{\infty} e^{-\lambda t}\tilde{S}_{\alpha}(t)\, t^{\alpha-1}\tilde{z}_0 dt$$

multiplying both sides by a complex number λ to get

$$\lambda(\lambda^{\alpha} I - P^{-1}A^{\alpha})^{-1}\tilde{y}_0 = \lambda \int_0^{\infty} e^{-\lambda t} S_{\alpha}(t)\, t^{\alpha-1}\tilde{y}_0 dt\ . \tag{3.7}$$

Now, equation (1.4) is an abstract Cauchy form can be written in the form

$$\begin{cases} {}^{R}D_t^{\alpha} z(t,x) = P^{-1}\tilde{A}^{\alpha} z(t,x) + P^{-1} f\left(t, z(t,x),\ {}^{R}D_t^{\gamma} z(t,x)\right), \\ \left[{}^{R}D_t^{\alpha-1} z(t,x)\right]\Big|_{t=0} = \tilde{z}_0, \quad \left[{}^{R}D_t^{\alpha-2} z(t,x)\right]\Big|_{t=0} = \tilde{y}_0\ . \end{cases} \tag{3.8}$$

By definition (2.8)

$$\lambda^{\alpha}\mathcal{L}\big(z(t,x)\big)(\lambda) - \sum_{k=0}^{n-1} \lambda^{k}\left[{}^{R}D_t^{\alpha-k-1} z(t,x)\right]\Big|_{t=0}(\lambda)$$
$$= P^{-1}\tilde{A}^{\alpha}\mathcal{L}(z(t,x))(\lambda) + \mathcal{L}\left[P^{-1} f\left(t, z(t,x),\ {}^{R}D_t^{\gamma} z(t,x)\right)\right](\lambda)$$

Where $n-1 < \alpha \leq n$, and $n = 2$ then

$$\lambda^{\alpha}\mathcal{L}\big(z(t,x)\big)(\lambda) - \left[{}^{R}D_t^{\alpha-1} z(t,x)\right]\Big|_{t=0}(\lambda) - \lambda\left[{}^{R}D_t^{\alpha-2} z(t,x)\right]\Big|_{t=0}(\lambda)$$
$$= P^{-1}\tilde{A}^{\alpha}\mathcal{L}(z(t,x))(\lambda) + \mathcal{L}\left[P^{-1} f\left(t, z(t,x),\ {}^{R}D_t^{\gamma} z(t,x)\right)\right](\lambda)$$

$$\big(\lambda^{\alpha} I - P^{-1}\tilde{A}^{\alpha}\big)\mathcal{L}\big(z(t,x)\big)(\lambda) = \tilde{z}_0$$
$$+\lambda\tilde{y}_0 + \mathcal{L}\left[P^{-1} f\left(t, z(t,x),\ {}^{R}D_t^{\gamma} z(t,x)\right)\right](\lambda) \tag{3.9}$$

Then from (A_2) for $\lambda^{\alpha} \in \rho\big(P^{-1}\tilde{A}^{\alpha}\big)$ we get $\big(\lambda^{\alpha} I - P^{-1}\tilde{A}^{\alpha}\big)^{-1}$

Multiplying both sides of (3.9) by $\big(\lambda^{\alpha} I - P^{-1}\tilde{A}^{\alpha}\big)^{-1}$,

$$\mathcal{L}\big(z(t,x)\big)(\lambda) = \big(\lambda^{\alpha} I - P^{-1}\tilde{A}^{\alpha}\big)^{-1}\tilde{z}_0 + \lambda\big(\lambda^{\alpha} I - P^{-1}\tilde{A}^{\alpha}\big)^{-1}\tilde{y}_0$$
$$+\big(\lambda^{\alpha} I - P^{-1}\tilde{A}^{\alpha}\big)^{-1}\left(\mathcal{L}\left[P^{-1} f\left(t, z(t,x),\ {}^{R}D_t^{\gamma} z(t,x)\right)\right]\right)(\lambda) \tag{3.10}$$

From lemma (3.1) we can write the equation (3.10) by the following form:

$$\mathcal{L}\big(z(t,x)\big)(\lambda) = \int_0^{\infty} e^{-\lambda t}\tilde{S}_{\alpha}(t)\, t^{\alpha-1}\tilde{z}_0 dt + \lambda\int_0^{\infty} e^{-\lambda t}\tilde{S}_{\alpha}(t)\, t^{\alpha-1}\tilde{y}_0 dt$$
$$+\int_0^{\infty} e^{-\lambda t}\left[\int_0^t \tilde{S}_{\alpha}(t-s)\,(t-s)^{\alpha-1}\left[P^{-1} f\left(s, z(s,x),\ {}^{R}D_t^{\gamma} z(s,x)\right)\right] ds\right] dt \tag{3.11}$$

taking the Laplace inverse transform, we have the mild solution

$$z(t,x) = \tilde{S}_{\alpha}(t)\, t^{\alpha-1}\tilde{z}_0 + \lambda\tilde{S}_{\alpha}(t)\, t^{\alpha-1}\tilde{y}_0$$
$$+\int_0^t \tilde{S}_{\alpha}(t-s)\,(t-s)^{\alpha-1}\left[P^{-1} f\left(s, z(s,x),\ {}^{R}D_t^{\gamma} z(s,x)\right)\right] ds \tag{3.12}$$

Lemma (3.2):

Let $P^{-1}\tilde{A}^{\alpha}$ is an infinitesimal generator of strongly continuous semigroup $S_{\alpha}(t)$, $t \geq 0$ in $Z = C_{v}^{RL}([0,T], X)$. the family of operators $\{\tilde{S}_{\alpha}(t), t \geq 0\}$ and $1 < \alpha \leq 2$ satisfy the following.

i) For any $t \geq 0$, $\tilde{S}_\alpha(t)$ is bounded linear operator (for any $x \in Z$ there exists $\widetilde{M}_\alpha > 1$, $w \geq 0$ then $\|\tilde{S}_\alpha(t)\| \leq \widetilde{M}_\alpha e^{wt^\alpha}$).

ii) The family $\{\tilde{S}_\alpha(t), t \geq 0\}$ is strongly continuous, which means that for every $y \in Z$ and $0 \leq t_1 < t_2 \leq T$, we have
$\|\tilde{S}_\alpha(t_2)x - \tilde{S}_\alpha(t_1)y\|_Z \to 0$ if $t_2 \to t_1$.

iii) If $S_\alpha(t)$ is a compact then the operator $\tilde{S}_\alpha(t)$ is a compact in Z, $t > 0$.

proof:

i. For any fixed $t \geq 0$, since $S_\alpha(t)$ is a linear operator , then $\tilde{S}_\alpha(t)$ is also linear operator. For any $y \in Z$, from (3.4), we have

$$\tilde{S}_\alpha(t)y = \int_0^\infty \alpha r M_\alpha(r)\, S_\alpha(t^\alpha r) y\, dr$$

$$\| \tilde{S}_\alpha(t)y\|_Z = \left\|\int_0^\infty \alpha r M_\alpha(r)\, S_\alpha(t^\alpha r) y\, dr\right\|_Z$$

From properties of strongly continuous semigroup $S_\alpha(t)$, there exists $\widetilde{M} > 1$ and $w \geq 0$ such that

$$\begin{aligned}\|\tilde{S}_\alpha(t)y\|_Z &\leq \alpha\widetilde{M} \int_0^\infty r M_\alpha(r)\, e^{wt^\alpha r}\| y \|_X dr \\ &\leq \alpha\widetilde{M} \int_0^\infty r M_\alpha(r) \sum_{n=0}^\infty \frac{(wt^\alpha r)^n}{n!} \|y\|_X\, dr \\ &\leq \alpha\widetilde{M} \sum_{n=0}^\infty \frac{(wt^\alpha)^n}{n!} \int_0^\infty r^{n+1}\, M_\alpha(r)\|y\|_X dr \\ &\leq \alpha\widetilde{M} \sum_0^\infty \frac{(wt^\alpha)^n}{n!} \frac{\Gamma(n+2)}{\Gamma(\alpha(n+1)+1} \|y\|_X \\ &\leq \alpha\widetilde{M} \sum_0^\infty \frac{(wt^\alpha)^n}{n!} \frac{(n+1)\Gamma(n+1)}{\alpha(n+1)\Gamma(\alpha(n+1))} \|y\|_X \\ &\leq \widetilde{M}\, E_{\alpha,\alpha}(wt^\alpha)\|y\|_X\end{aligned}$$

, where $E_{\alpha,\alpha}$ is a Mittag - Leffler function.

From properties of Mittag - Leffler function, we get

$\|\tilde{S}_\alpha(t)y\|_Z \leq \widetilde{M}\widehat{M}_\alpha e^{wt^\alpha}\|y\|_Z$, where $1 < \widehat{M}_\alpha = \sup_{n\in N} \frac{n!}{\Gamma(\alpha n+\alpha)}$

Assume that $\widetilde{M}_\alpha = \widetilde{M}\widehat{M}_\alpha > 1$, we have

$$\|\tilde{S}_\alpha(t)y\|_Z \leq \widetilde{M}_\alpha e^{wt^\alpha}\|y\|_Z .$$

ii. For any $y \in Z$ and $0 \leq t_1 < t_2 \leq T$, we have

$$\begin{aligned}\|\tilde{S}_\alpha(t_2)x - \tilde{S}_\alpha(t_1)y\|_Z &= \left\|\int_0^\infty \alpha r M_\alpha(r)[S_\alpha((t_2)^\alpha r) - S_\alpha((t_1)^\alpha r)]y dr \right\|_Z \\ &\leq \int_0^\infty \alpha r M_\alpha(r)\|S_\alpha((t_2)^\alpha r) - S_\alpha((t_1)^\alpha r)\|_Z\|y\|_Z dr\end{aligned}$$

According to the strongly continuity of $S_\alpha(t)$, $t \geq 0$, then $\|S_\alpha(t_2)z - S_\alpha(t_1)y\|_X$ tends to zero as $t_2 \to t_1$, which mean that $\{\tilde{S}_\alpha(t), t \geq 0\}$ is a strongly continuous.

iii. To prove that $\tilde{S}_\alpha(t)$ is a compact operator in Z for every $t > 0$. For each positive constant L, the set $Z_L = \{ x \in Z : \|x\| \leq L \}$ is clearly a bounded subset in Z . We only need prove that for any

IOP Conf. Series: Materials Science and Engineering **571** (2019) 012022 doi:10.1088/1757-899X/571/1/012022

positive constant L and t > 0 , the set $W(t) = \{\tilde{S}_\alpha(t)x, x \in Z_L\} = \{\int_0^\infty \alpha r M_\alpha(r) S_\alpha(t^\alpha r)x\,dr, x \in Z_L\}$ is a relatively compact in Z. Let t > 0 be a fixed. For all $\delta > 0$, define the subset in Z by $W_\delta(t) = \{\int_\delta^\infty \alpha r M_\alpha(r) S_\alpha(t^\alpha r)x\,dr, x \in Z_L\}$, Then for any $x \in Z_L$, we have

$$\int_\delta^\infty \alpha r M_\alpha(r) S_\alpha(t^\alpha r)x\,dr = S_\alpha(t^\alpha \delta) \int_\delta^\infty \alpha r M_\alpha(r)\, S_\alpha(t^\alpha r - t^\alpha \delta)x\,dr$$

Since $S_\alpha(t^\alpha\delta)$, $t^\alpha\delta > 0$ is a compact, we obtain that for every $t > 0$ the set $W_\delta(t)$ is a relatively compact in Z. Moreover, for every $x \in Z_L$, we have

$$\left\|\int_0^\infty \alpha r M_\alpha(r) S_\alpha(t^\alpha r)x\,dr - \int_\delta^\infty \alpha r M_\alpha(r) S_\alpha(t^\alpha r)x\,dr\right\|$$
$$= \left\|\int_0^\delta \alpha r M_\alpha(r) S_\alpha(t^\alpha r)x\,dr\right\| \le \alpha M \int_0^\delta e^{t^\alpha r} r M_\alpha(r)\,dr$$

Therefore, there are relatively compact sets arbitrary close to the set W(t), t > 0. Hence the set W(t), t > 0 is also relatively compact in Z.

Theorem (3.1):

Consider the assumptions (A_1)-(A_5) hold

and $$\int_0^t L(s)\,ds < \frac{1}{K_3}\int_{K_2}^\infty \frac{ds}{\Omega_f(s)} \tag{3.13}$$

where K_1, K_2, K_3 and $L(s)$ defined in assumption (A_5)

Then the problem (3.8) has a mild solution $z \in C_\nu^{RL}([0,T],X)$.

proof:

From our assumptions and $z \in C_\nu^{RL}([0,T],X)$ by (3.12) and lemma (2.2) we obtain the maps

$$\Phi(z,y)(t) = \tilde{S}_\alpha(t)\, t^{\alpha-1}\tilde{z}_0 + \lambda\tilde{S}_\alpha(t)\, t^{\alpha-1}\tilde{y}_0$$
$$+\int_0^t \tilde{S}_\alpha(t-s)\,(t-s)^{\alpha-1}\left[P^{-1}f\left(s, z(s,x), I_t^{\nu-\gamma}y(s,x)\right)\right]ds \tag{3.14}$$

For $t \in [0,T]$, and

$$\Psi(z,y)(t) = {}^R D^\nu \tilde{S}_\alpha(t)\, t^{\alpha-1}\tilde{z}_0 + \lambda\, {}^R D^\nu \tilde{S}_\alpha(t)\, t^{\alpha-1}\tilde{y}_0$$
$$+ {}^R D^\nu \int_0^t \tilde{S}_\alpha(t-s)\,(t-s)^{\alpha-1}\left[P^{-1}f\left(s, z(s,x), I_t^{\nu-\gamma}y(s,x)\right)\right]ds \tag{3.15}$$

are well-defined, and map $[C([o,T])]^2 \to C([o,T])$. we use the Schaefer's theorem which means that, firstly we show that the set (z_α, y_α) of

$$(z_\alpha, y_\alpha) = \tau\big(\Phi(z_\alpha,y_\alpha), \Psi(z_\alpha,y_\alpha)\big), \qquad 0 < \tau < 1 \tag{3.16}$$

is bounded. Secondly, we show that $F: C_\nu^{RL}([0,T],X) \to C_\nu^{RL}([0,T],X)$ is completely continuous.

Now , from (3.14)

$$\|z_\alpha(t,x)\| \le \widetilde{M}_\alpha e^{wt^\alpha}\|\, t^{\alpha-1}\|\|\tilde{z}_0\| + \lambda\widetilde{M}_\alpha e^{wt^\alpha}\|\, t^{\alpha-1}\|\|\tilde{y}_0\|$$
$$+\|P^{-1}\|\int_0^t \widetilde{M}_\alpha e^{w(t-s)^\alpha}\|(t-s)^{\alpha-1}\|\left\|f\left(s, z(s,x), I_t^{\nu-\gamma}y(s,x)\right)\right\|\,ds$$
$$\le \widetilde{M}_\alpha e^{wt^\alpha}\|\, t^{\alpha-1}\|\|\tilde{z}_0\| + \lambda\widetilde{M}_\alpha e^{wt^\alpha}\|\, t^{\alpha-1}\|\|\tilde{y}_0\|$$

IOP Conf. Series: Materials Science and Engineering **571** (2019) 012022 doi:10.1088/1757-899X/571/1/012022

$$+\|P^{-1}\| \int_0^t \widetilde{M}_\alpha e^{w(t-s)^\alpha} \|(t-s)^{\alpha-1}\| K_f(s)\Omega_{\mathrm{f}}$$
$$\left(\|z_\alpha(s)\| + \frac{s^{\nu-\gamma}}{\Gamma(\nu-\gamma+1)} sup_{0\le\tau\le s}\|y_\alpha(\tau)\|\right) ds$$

from (3.15)

$$\|y_\alpha(t,x)\| \le \left\| {}^{R}D^{\nu}\tilde{S}_\alpha(t)\, t^{\alpha-1}\tilde{z}_0\right\| + \left\|\lambda\, {}^{R}D^{\nu}\tilde{S}_\alpha(t)\, t^{\alpha-1}\tilde{y}_0\right\|$$
$$+ \left\| {}^{R}D^{\nu} \int_0^t \tilde{S}_\alpha(t-s)\,(t-s)^{\alpha-1}\left[P^{-1} f\left(s, z(s,x), I_t^{\nu-\gamma} y(s,x)\right)\right] ds\right\|$$

Let $\widetilde{\mathrm{N}} = \max\left\| {}^{R}D^{\nu}\tilde{S}_\alpha(t)\, t^{\alpha-1}\right\|$, and from lemma (2.2) we get

$$\|y_\alpha(t,x)\| \le \widetilde{\mathrm{N}}\|\tilde{z}_0\| + \lambda\widetilde{\mathrm{N}}\|\tilde{y}_0\|$$
$$+ \int_0^t \left\| {}^{R}D^{\nu}\tilde{\tilde{S}}_\alpha(t-s)\right\| \|P^{-1}\| \left\| f\left(s, z(s,x), I_t^{\nu-\gamma} y(s,x)\right)\right\| ds$$
$$\le \widetilde{\mathrm{N}}\|\tilde{z}_0\| + \lambda\widetilde{\mathrm{N}}\|\tilde{y}_0\|$$
$$+\|P^{-1}\|\widetilde{\mathrm{N}} \int_0^t K_f(s)\Omega_{\mathrm{f}}\left(\|z_\alpha(s)\| + \frac{s^{\nu-\gamma}}{\Gamma(\nu-\gamma+1)} \sup_{0\le\tau\le s}\|y_\alpha(\tau)\|\right) ds$$

Clearly $\|z_\alpha(t,x)\| + \|y_\alpha(t,x)\|$

$$\le \left(\widetilde{\mathrm{N}} + \widetilde{M}_\alpha e^{wt^\alpha}\| t^{\alpha-1}\|\right)\|\tilde{z}_0\| + \lambda\left(\widetilde{\mathrm{N}} + \widetilde{M}_\alpha e^{wt^\alpha}\| t^{\alpha-1}\|\right)\|\tilde{y}_0\|$$
$$+\|P^{-1}\| \int_0^t \left(\widetilde{M}_\alpha e^{w(t-s)^\alpha}\|(t-s)^{\alpha-1}\| + \widetilde{\mathrm{N}}\right) K_f(s)\Omega_{\mathrm{f}}\left(\|z_\alpha(s)\| + \frac{s^{\nu-\gamma}}{\Gamma(\nu-\gamma+1)} \sup_{0\le\tau\le s}\|y_\alpha(\tau)\|\right) ds$$

If we put $\vartheta_\alpha(t) = max\left\{1, \frac{T^{\nu-\gamma}}{\Gamma(\nu-\gamma+1)}\right\} \sup_{0\le\tau\le t}(\|z_\alpha(\tau)\| + \|y_\alpha(\tau)\|)$, then

$$\vartheta_\alpha(t) \le max\left\{1, \frac{T^{\nu-\gamma}}{\Gamma(\nu-\gamma+1)}\right\}\left\{\left(\widetilde{\mathrm{N}} + \widetilde{M}_\alpha e^{wt^\alpha}\| t^{\alpha-1}\|\right)\|\tilde{z}_0\| + \lambda\left(\widetilde{\mathrm{N}} + \widetilde{M}_\alpha e^{wt^\alpha}\| t^{\alpha-1}\|\right)\|\tilde{y}_0\|\right.$$
$$\left.+\|P^{-1}\| \int_0^t \left(M_\alpha e^{w(t-s)^\alpha}\|(t-s)^{\alpha-1}\| + \widetilde{\mathrm{N}}\right) K_f(s)\Omega_{\mathrm{f}}\left(\vartheta_\alpha(s)\right) ds\right\}$$

$$\vartheta_\alpha(t) \le max\left\{1, \frac{T^{\nu-\gamma}}{\Gamma(\nu-\gamma+1)}\right\}\left\{\mathrm{K}_1 + \int_0^t L(s)\left(\Omega_{\mathrm{f}}\left(\vartheta_\alpha(s)\right)\right) ds\right\}$$
$$\le \mathrm{K}_3\left\{\mathrm{K}_1 + \int_0^t L(s)\left(\Omega_{\mathrm{f}}\left(\vartheta_\alpha(s)\right)\right) ds\right\}$$
$$\le \mathrm{K}_2 + \mathrm{K}_3\int_0^t L(s)\left(\Omega_{\mathrm{f}}\left(\vartheta_\alpha(s)\right)\right) ds$$

where

$$\mathrm{K}_1 = \left\{\left(\widetilde{\mathrm{N}} + \widetilde{M}_\alpha e^{wt^\alpha}\| t^{\alpha-1}\|\right)\|\tilde{z}_0\| + \lambda\left(\widetilde{\mathrm{N}} + \widetilde{M}_\alpha e^{wt^\alpha}\| t^{\alpha-1}\|\right)\|\tilde{y}_0\|\right.$$

$\mathrm{K}_3 = max\left\{1, \frac{T^{\nu-\gamma}}{\Gamma(\nu-\gamma+1)}\right\}$ and $\mathrm{K}_2 = \mathrm{K}_1\mathrm{K}_3$

$$L(s) = \|P^{-1}\|\left(\widetilde{M}_\alpha e^{w(t-s)^\alpha}\|(t-s)^{\alpha-1}\| + \widetilde{\mathrm{N}}\right)K_f(s)$$

Let $\tilde{\vartheta}_\alpha(t) = \mathrm{K}_2 + \mathrm{K}_3\int_0^t L(s)\left(\Omega_{\mathrm{f}}\left(\vartheta_\alpha(s)\right)\right) ds$ then

$\vartheta_\alpha(t) \le \tilde{\vartheta}_\alpha(t)$ and

$\tilde{\vartheta}'_\alpha(t) \le \mathrm{K}_3 L(t)\left(\Omega_{\mathrm{f}}\left(\vartheta_\alpha(t)\right)\right)$, $t \in J$

IOP Conf. Series: Materials Science and Engineering **571** (2019) 012022 doi:10.1088/1757-899X/571/1/012022

We infer that

$$\int_{\mathrm{K}_2}^{\tilde{\vartheta}_\alpha(t)} \frac{ds}{\Omega_{\mathrm{f}}(s)} \le \mathrm{K}_3 \int_0^t L(s)\, ds$$

From (3.13), we get

$$\int_{\mathrm{K}_2}^{\tilde{\vartheta}_\alpha(t)} \frac{ds}{\Omega_{\mathrm{f}}(s)} \le \mathrm{K}_3 \int_0^t L(s)\, ds < \int_{\mathrm{K}_2}^{\infty} \frac{ds}{\Omega_{\mathrm{f}}(s)}$$

This mean that $\vartheta_\alpha(t)$ is bounded and thereafter the set of solutions of (3.16) is bounded in $[C([0,T],X)]^2$.

Now, we need to prove that the operator $F\colon C_\nu^{RL}([0,T],X) \to C_\nu^{RL}([0,T],X)$ is a completely continuous where

$$(Fz)(\mathrm{t}) = \tilde{S}_\alpha(t)\, t^{\alpha-1}\tilde{z}_0 + \lambda\tilde{S}_\alpha(t)\, t^{\alpha-1}\tilde{y}_0$$
$$+ \int_0^t \tilde{S}_\alpha(t-s)\,(t-s)^{\alpha-1}\left[P^{-1}f\left(s, z(s,x),\, {}^{R}D_t^{\gamma} z(s,x)\right)\right] ds.$$

Let $H_l = \{z \in C_\nu^{RL}([0,T],X) :\ \|z\|^* \le l, l \ge 1\}$.

We first do that F maps H_l into an equicontinuous family. Let $z \in H_l$ and $t_1, t_2 \in [0,T]$ then if $0 < t_1 < t_1 \le T$.

$$\|(Fz)(t_1) - (Fz)(t_2)\| \le \left\|\tilde{S}_\alpha(t_1)t_1^{\alpha-1} - \tilde{S}_\alpha(t_2)t_2^{\alpha-1}\right\|\|\tilde{z}_0\|$$
$$+\lambda\left\|\tilde{S}_\alpha(t_1)t_1^{\alpha-1} - \tilde{S}_\alpha(t_2)t_2^{\alpha-1}\right\|\|\tilde{y}_0\|$$
$$+ \int_0^{t_1} \left\|\left[\tilde{S}_\alpha(t_1-s)(t_1-s)^{\alpha-1} - \tilde{S}_\alpha(t_2-s)(t_2-s)^{\alpha-1}\right]P^{-1}f\left(s, z(s),\, {}^{R}D_t^{\gamma} z(s)\right)\right\| ds$$
$$+ \int_{t_1}^{t_2} \left\|\left[\tilde{S}_\alpha(t_2-s)(t_2-s)^{\alpha-1}\right]P^{-1}f\left(s, z(s),\, {}^{R}D_t^{\gamma} z(s)\right)\right\| ds$$
$$\le \left\|\tilde{S}_\alpha(t_1) - \tilde{S}_\alpha(t_2)\right\|\|\tilde{z}_0\| + \lambda\left\|\tilde{S}_\alpha(t_1) - \tilde{S}_\alpha(t_2)\right\|\|\tilde{y}_0\|$$
$$+ \int_0^{t_1} \left\|\left[\tilde{S}_\alpha(t_1-s) - \tilde{S}_\alpha(t_2-s)\right]P^{-1}f\left(s, z(s), I_s^{\nu-\gamma} y(s)\right)\right\| ds$$
$$+ \int_{t_1}^{t_2} \left\|\left[\tilde{S}_\alpha(t_2-s)\right]P^{-1}f\left(s, z(s), I_s^{\nu-\gamma} y(s)\right)\right\| ds$$
$$\le \left\|\tilde{S}_\alpha(t_1) - \tilde{S}_\alpha(t_2)\right\|\|\tilde{z}_0\| + \lambda\left\|\tilde{S}_\alpha(t_1) - \tilde{S}_\alpha(t_2)\right\|\|\tilde{y}_0\|$$
$$+ \int_0^{t_1}\left\|\tilde{S}_\alpha(t_1-s) - \tilde{S}_\alpha(t_2-s)\right\|\|P^{-1}\|K_f(s)\Omega_{\mathrm{f}}\left(\|z(s)\| + \left\|I_s^{\nu-\gamma} y(s)\right\|\right) ds$$
$$+ \int_{t_1}^{t_2}\left\|\left[\tilde{S}_\alpha(t_2-s)\right]\right\|\, \|P^{-1}\|K_f(s)\Omega_{\mathrm{f}}\left(\|z(s)\| + \left\|I_s^{\nu-\gamma} y(s)\right\|\right) ds$$
$$\le \left\|\tilde{S}_\alpha(t_1) - \tilde{S}_\alpha(t_2)\right\|\|\tilde{z}_0\| + \lambda\left\|\tilde{S}_\alpha(t_1) - \tilde{S}_\alpha(t_2)\right\|\|\tilde{y}_0\|$$
$$+ \int_0^{t_1}\left\|\tilde{S}_\alpha(t_1-s) - \tilde{S}_\alpha(t_2-s)\right\|\|P^{-1}\|K_f(s)\Omega_{\mathrm{f}}\left(r + \frac{T^{\nu-\gamma}}{\Gamma(\nu-\gamma+1)}r\right) ds$$
$$+ \int_{t_1}^{t_2}\left\|\left[\tilde{S}_\alpha(t_2-s)\right]\right\|\, \|P^{-1}\|K_f(s)\Omega_{\mathrm{f}}\left(r + \frac{T^{\nu-\gamma}}{\Gamma(\nu-\gamma+1)}r\right) ds \tag{3.17}$$

and similarly

$$\left\|\,{}^{R}D_t^{\nu}(Fz)(t_1) - {}^{R}D_t^{\nu}(Fz)(t_2)\right\| \le \left\|\,{}^{R}D_t^{\nu}\tilde{S}_\alpha(t_1)t_1^{\alpha-1} - {}^{R}D_t^{\nu}\tilde{S}_\alpha(t_2)t_2^{\alpha-1}\right\|\|\tilde{z}_0\|$$
$$+\lambda\left\|\,{}^{R}D_t^{\nu}\tilde{S}_\alpha(t_1)t_1^{\alpha-1} - {}^{R}D_t^{\nu}\tilde{S}_\alpha(t_2)t_2^{\alpha-1}\right\|\|\tilde{y}_0\|$$

$+\int_0^{t_1}\left\|\left[{}^RD_t^\nu\tilde{S}_\alpha(t_1-s)(t_1-s)^{\alpha-1}-{}^RD_t^\nu\tilde{S}_\alpha(t_2-s)(t_2-s)^{\alpha-1}\right]P^{-1}f\left(s,z(s),{}^RD_t^\gamma z(s)\right)\right\|ds$

$+\int_{t_1}^{t_2}\left\|\left[{}^RD_t^\nu\tilde{S}_\alpha(t_2-s)(t_2-s)^{\alpha-1}\right]P^{-1}f\left(s,z(s),{}^RD_t^\gamma z(s)\right)\right\|ds$

$\leq\left\|{}^RD_t^\nu\tilde{S}_\alpha(t_1)t_1^{\alpha-1}-{}^RD_t^\nu\tilde{S}_\alpha(t_2)t_2^{\alpha-1}\right\|\|\tilde{z}_0\|+\lambda\left\|{}^RD_t^\nu\tilde{S}_\alpha(t_1)t_1^{\alpha-1}-{}^RD_t^\nu\tilde{S}_\alpha(t_2)t_2^{\alpha-1}\right\|\|\tilde{y}_0\|$

$+\int_0^{t_1}\left\|\left[{}^RD_t^\nu\tilde{S}_\alpha(t_1-s)(t_1-s)^{\alpha-1}-{}^RD_t^\nu\tilde{S}_\alpha(t_2-s)(t_2-s)^{\alpha-1}\right]P^{-1}f\left(s,z(s),I_s^{\nu-\gamma}y(s)\right)\right\|ds$

$+\int_{t_1}^{t_2}\left\|\left[{}^RD_t^\nu\tilde{S}_\alpha(t_2-s)(t_2-s)^{\alpha-1}\right]P^{-1}f\left(s,z(s),I_s^{\nu-\gamma}y(s)\right)\right\|ds$

$\leq\left\|{}^RD_t^\nu\tilde{S}_\alpha(t_1)t_1^{\alpha-1}-{}^RD_t^\nu\tilde{S}_\alpha(t_2)t_2^{\alpha-1}\right\|\|\tilde{z}_0\|$

$+\lambda\left\|{}^RD_t^\nu\tilde{S}_\alpha(t_1)t_1^{\alpha-1}-{}^RD_t^\nu\tilde{S}_\alpha(t_2)t_2^{\alpha-1}\right\|\|\tilde{y}_0\|$

$+\int_0^{t_1}\left\|{}^RD_t^\nu\tilde{S}_\alpha(t_1-s)(t_1-s)^{\alpha-1}-{}^RD_t^\nu\tilde{S}_\alpha(t_2-s)(t_2-s)^{\alpha-1}\right\|$

$\|P^{-1}\|K_f(s)\Omega_{\mathrm{f}}\left(\|z(s)\|+\left\|I_s^{\nu-\gamma}y(s)\right\|\right)ds$

$+\int_{t_1}^{t_2}\left\|{}^RD_t^\nu\tilde{S}_\alpha(t_2-s)(t_2-s)^{\alpha-1}\right\|\|P^{-1}\|K_f(s)\Omega_{\mathrm{f}}\left(\|z(s)\|+\left\|I_s^{\nu-\gamma}y(s)\right\|\right)ds$

$\leq\left\|{}^RD_t^\nu\tilde{S}_\alpha(t_1)t_1^{\alpha-1}-{}^RD_t^\nu\tilde{S}_\alpha(t_2)t_2^{\alpha-1}\right\|\|\tilde{z}_0\|$

$+\lambda\left\|{}^RD_t^\nu\tilde{S}_\alpha(t_1)t_1^{\alpha-1}-{}^RD_t^\nu\tilde{S}_\alpha(t_2)t_2^{\alpha-1}\right\|\|\tilde{y}_0\|$

$+\int_0^{t_1}\left\|{}^RD_t^\nu\tilde{S}_\alpha(t_1-s)(t_1-s)^{\alpha-1}-{}^RD_t^\nu\tilde{S}_\alpha(t_2-s)(t_2-s)^{\alpha-1}\right\|$

$\|P^{-1}\|K_f(s)\Omega_{\mathrm{f}}\left(r+\frac{T^{\nu-\gamma}}{\Gamma(\nu-\gamma+1)}r\right)ds$

$$+\int_{t_1}^{t_2}\left\|{}^RD_t^\nu\tilde{S}_\alpha(t_2-s)(t_2-s)^{\alpha-1}\right\|\|P^{-1}\|K_f(s)\Omega_{\mathrm{f}}\left(r+\frac{T^{\nu-\gamma}}{\Gamma(\nu-\gamma+1)}r\right)ds \tag{3.18}$$

The right-hand side of equations (3.17) and (3.18) are independent of $z\in H_l$ and tend to zero as $t_1\to t_2$.

Since $\tilde{S}_\alpha(t)$, ${}^RD_t^\nu\tilde{S}_\alpha$ are uniformly continuous for $t\in[0,T]$ and the compactness of $S_\alpha(t)$, ${}^RD_t^\nu\tilde{S}_\alpha$ for $t>0$ imply the continuity in the uniform operator topology.

Then F maps H_l into an equicontinuous family. Now if we want to show that $\overline{FH_l}$ is a compact. Since we prove that F maps H_l into an equicontinuous family. It is suffices to show that F maps H_l into a precompact set in $C_\nu^{RL}([0,T],X)$ and apply the Arzela- Ascoli theorem.

Let $t\in(0,T]$, $\epsilon\in(0,t)$ for $z\in H_l$, we define

$(F_\epsilon z)(\mathrm{t})=\tilde{S}_\alpha(t)\,t^{\alpha-1}\tilde{z}_0+\lambda\tilde{S}_\alpha(t)\,t^{\alpha-1}\tilde{y}_0$

$+\int_0^{t-\epsilon}\tilde{S}_\alpha(t-s)\,(t-s)^{\alpha-1}\left[P^{-1}f\left(s,z(s,x),{}^RD_t^\gamma z(s,x)\right)\right]ds$

Since $\tilde{S}_\alpha(t)$, ${}^RD_t^\nu\tilde{S}_\alpha$ are compact operator, the set $E_\epsilon(\mathrm{t})=\{(F_\epsilon z)(\mathrm{t}):z\in H_l\}$ is precompact in $C_\nu^{RL}([0,T],X)$ for, $\epsilon\in(0,t)$. and for every $z\in H_l$, we have

$\|(Fz)(t)-(F_\epsilon z)(t)\|$

$\leq\int_{t-\epsilon}^{t}\left\|\left[\tilde{S}_\alpha(t-s)(t-s)^{\alpha-1}\right]P^{-1}f\left(s,z(s),{}^RD_s^\gamma z(s)\right)\right\|ds$

IOP Conf. Series: Materials Science and Engineering **571** (2019) 012022 doi:10.1088/1757-899X/571/1/012022

$$\leq \int_{t-\epsilon}^{t} \left\| \left[\tilde{S}_{\alpha}(t-s) \right] \right\| \|P^{-1}\| K_f(s) \Omega_{\mathrm{f}} \left(r + \frac{T^{\nu-\gamma}}{\Gamma(\nu-\gamma+1)} r \right) ds \tag{3.19}$$

and

$$\left\| {}^{R}D_t^{\nu}(Fz)(t) - {}^{R}D_t^{\nu}(F_{\epsilon}z)(t) \right\|$$

$$\leq \int_{t-\epsilon}^{t} \left\| \left[{}^{R}D_t^{\nu} \tilde{S}_{\alpha}(t-s)(t-s)^{\alpha-1} \right] P^{-1} f\left(s, z(s), {}^{R}D_s^{\gamma} z(s)\right) \right\| ds$$

$$\leq \int_{t-\epsilon}^{t} \left\| {}^{R}D_t^{\nu} \tilde{S}_{\alpha}(t-s)(t-s)^{\alpha-1} \right\| \|P^{-1}\| K_f(s) \Omega_{\mathrm{f}} \left(r + \frac{T^{\nu-\gamma}}{\Gamma(\nu-\gamma+1)} r \right) ds \tag{3.20}$$

Then there are arbitrarily precompact sets close to the set $\{(Fz)(\mathrm{t}): z \in H_l\}$. Hence the set $\{(Fz)(\mathrm{t}): z \in H_l\}$ is precompact in $C_{\nu}^{RL}([0,T],X)$.

Now, to show that $F: C_{\nu}^{RL}([0,T],X) \to C_{\nu}^{RL}([0,T],X)$ is continuous.

Let $\{z_n\}_0^{\infty} \subseteq C_{\nu}^{RL}([0,T],X)$ with $z_n \to z$ in $C_{\nu}^{RL}([0,T],X)$. Then there is integer p such that $\|z_n(t,x)\| \leq p$, $\left\| {}^{R}D_s^{\gamma} z_n(t,x) \right\| \leq p$ for all n and $t \in [0,T]$ so $\|z(t,x)\| \leq p$, $\left\| {}^{R}D_s^{\gamma} z(t,x) \right\| \leq p$, $z(t,x)$, ${}^{R}D_s^{\gamma} z(t,x) \in Z$.

from (A_4) we have $f\left(t, z_n(t), {}^{R}D_s^{\gamma} z_n(t)\right) \longrightarrow f\left(t, z(t), {}^{R}D_s^{\gamma} z(t)\right)$

for each $t \in J$ and since

$$\left\| f\left(t, z_n(t), {}^{R}D_t^{\gamma} z_n(t)\right) - f\left(t, z(t), {}^{R}D_t^{\gamma} z(t)\right) \right\|$$

$$\leq \left\| f\left(t, z_n(t), {}^{R}D_t^{\gamma} z_n(t)\right) \right\| + \left\| f\left(t, z(t), {}^{R}D_t^{\gamma} z(t)\right) \right\|$$

$$\leq \left\| f\left(t, z_n(t), I_t^{\eta-\gamma} y_n(t)\right) \right\| + \left\| f\left(t, z(t), I_t^{\nu-\gamma} y(t)\right) \right\|$$

$$\leq 2K_f(t)\Omega_{\mathrm{f}}\left(\|z\| + \frac{T^{\nu-\gamma}}{\Gamma(\nu-\gamma+1)} \|y\| \right)$$

We have by dominated convergence theorem.

$$\|Fz_n - Fz\|$$

$$= \sup_{\mathrm{t \in J}} \left\| \int_0^t \tilde{S}_{\alpha}(t-s)(t-s)^{\alpha-1} P^{-1} \left[f\left(s, z_n(s), {}^{R}D_s^{\gamma} z_n(s)\right) - f\left(s, z(s), {}^{R}D_s^{\gamma} z(s)\right) \right] ds \right\|$$

$$\leq \left\| \tilde{S}_{\alpha}(t-s)(t-s)^{\alpha-1} \right\| \|P^{-1}\| \int_0^t \left\| f\left(s, z_n(s), {}^{R}D_s^{\gamma} z_n(s)\right) - f\left(s, z(s), {}^{R}D_s^{\gamma} z(s)\right) \right\| ds \to 0$$

and

$$\left\| {}^{R}D_t^{\nu}(Fz_n) - {}^{R}D_t^{\nu}(Fz) \right\|$$

$$= \sup_{\mathrm{t \in J}} \left\| \int_0^t {}^{R}D_t^{\nu} \tilde{S}_{\alpha}(t-s)(t-s)^{\alpha-1} P^{-1} \left[f\left(s, z_n(s), {}^{R}D_s^{\gamma} z_n(s)\right) - f\left(s, z(s), {}^{R}D_s^{\gamma} z(s)\right) \right] ds \right\|$$

$$\leq \left\| {}^{R}D_t^{\nu} \tilde{S}_{\alpha}(t-s)(t-s)^{\alpha-1} \right\| \|P^{-1}\| \int_0^t \left\| f\left(s, z_n(s), {}^{R}D_s^{\gamma} z_n(s)\right) - f\left(s, z(s), {}^{R}D_s^{\gamma} z(s)\right) \right\| ds \to 0$$

Then F is continuous and from (3.17), (3.18) F is equicontinuous and from (3.19), (3.20) F is precompact this means we have F is completely continuous and from first part we proved that the set of solutions $(z_{\alpha}, y_{\alpha}) = \tau\left(\Phi(z_{\alpha}, y_{\alpha}), \Psi(z_{\alpha}, y_{\alpha})\right)$, $0 < \tau < 1$ is bounded. Then F has a fixed point in $C_v^{RL}([0,T],X)$ (Schaefer theorem) therefore every fixed point of F is a mild solution of (3.8) on [0,T].

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