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To cite this article: Alyaa A. Alwan *et al* 2019 *IOP Conf. Ser.: Mater. Sci. Eng.* **571** 012019

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IOP Conf. Series: Materials Science and Engineering **571** (2019) 012019 doi:10.1088/1757-899X/571/1/012019

The Solutions of Mixed Hemiequilibrium Problem with Application in Sobolev Space

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Abstract. The present work deals with the existence and uniqueness of solutions for new class of hemiequilibrium problems

$$\boldsymbol{\vartheta}(\mathbf{x},\mathbf{y}) + v(\mathbf{y}) - v(\mathbf{x}) + \mathbf{P^0}\big(\boldsymbol{\Omega}\mathbf{x}\,;\boldsymbol{\Omega}\,\boldsymbol{\delta}(\mathbf{x},\mathbf{y})\big) \geq \mathbf{0}.$$

The proof of the first result is based on arguments of Tarafdar's theorem involving ψ – monotone bifunction. Moreover, an application to the existence of solution for a differential inclusion is given.

Keywords: hemiequilibrium problem; Tarafdar's fixed point Theorem; Monotone bi-function; differential inclusion.
Mathematics Subject Classification: 35J85; 47H05; 47J20; 49J53; 26D15.

1. Introduction and preliminaries

Equilibrium and hemiequilibrium problems have witnessed an explosive growth in theoretical advances and algorithmic developments across almost all disciplines of pure and applied sciences. For more details on a variety of mechanical, economics and optimal control problems related to this incident the reader can be referred to ([10], [16], [20], [24] and [27]). In the papers above, Researchers studied the existence and uniqueness of solutions depended on Clarke's generalized gradient and directional derivative for locally Lipchitz functions, by using different tools such as fixed point theorems, KKM theorems, critical point theory, surjectivity theorems for pseudomonotone and coercive operators (see [1], [2], [8], [11], [13] and [26]). The monotonicity and generalized monotonicity play an important role in the study of equilibrium and hemiequilibrium problems. Recently, a substantial number of papers on existence results for solving equilibrium problems, mixed equilibrium problems and hemiequilibrium problems based on different generalization of monotonicity such as quasimonotonicity, semimonotonicity, relaxed monotonicity, α-monotonicity and η-monotonicity are published (see [23] and [28- 30]). Variational inequalities are such important were introduced in 1964, it has been a powerful tool to investigate a wide class of unrelated problems arising in the industrial, as a unified and general framework. Variational inequalities have been generalized and extended in several directions using novel techniques (see [3], [14] and [21]).

In this article, unless stated otherwise, assume that E is Banach space and E^* is a topological dual space of E, whereas $\langle .\,;.\rangle$ and $\|\,.\,\|$ denote the duality pairing between E and E^* and norm in E^*, respectively. For the convenience of the reader, we mention some basic definitions and results that will help us to prove our main results.

We recall that a function $P : E \rightarrow R$ is called locally Lipschitz if for every $u \in E$ there exists a neighborhood U of u and a constant $L_u > 0$ such that

$$|P(w)\quad P(v)| \leq L_u \parallel w - v \parallel E;\ for\,each\,v, w \in U.$$

Definition 1.1. Assume that $P : E \rightarrow R$ is a locally Lipschitz functional. The generalized derivative of P at $u \in E$ in the direction of $v \in X$, denoted by $P^0(u;\,v)$, is defined as

$$P^0(u;\,v) = \lim_{\substack{w\to u\\ \lambda\downarrow 0}} Sup\,\frac{P(w+\lambda v) - P(w)}{\lambda}.$$

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Lemma 1.2. Let $P: E \to R$ be locally Lipschitz function of rank L_u near the point $u \in E$. Then

(i) The function $v \mapsto P^0(u;\, v)$ is finite, positively homogeneous, subadditive and satisfies that $|P^0(u;\, v)| \leq L_u \|v\|_E$;
(ii) $P^0(u;\, v)$ is upper semicontinuous.
(iii) $P^0(u;\, -v) = (-P)^0(u;\, v)$.
A detailed proof is found in [6].

Definition 1.3. The generalized gradient of a locally Lipschitz function $P: E \to R$ at a point $u \in E$ (subset of a dual space E^*) is defined by

$$\partial P(u) = \{\zeta \in E^* : P^0(u;\, v) \geq (\zeta, v)_E \ for\ each\ v \in E\}.$$

Herewith, we point out that for each $u \in E$, we have $\partial J(u) \neq \emptyset$. In order to see that it suffices to apply the Hahn-Banach theorem (see e.g. [5]). Let $\vartheta, \tau : K \times K \to R$ be bi-functions where K is a nonempty subset of Banach space E such that $\vartheta(x;\, x) = \tau(x;\, x) = 0$. Then In 1998 Riahi [22] introduced a new type of monotone bi-function. They called it ϑ -monotone bi -function, as follows:

Definition 1.4. ϑ is called ψ - monotone iff

$$\vartheta(x;\, y) + \vartheta(y;\, x) \leq \psi(x;\, y) + \psi(y;\, x) for\ each\ x;\, y \in K.$$

Note that, the monotonicity implies ψ- monotonicity. However, the converse is not true in general, as shown in the example below.

Example 1.5. Let $X = \ell^p$ be a reflexive Banach space, $1 < p < \infty$, where

$$X = \left\{ x = \{x_n\} \subseteq R:\ \| x \| = \left(\sum_{n \geq 1} |x_n|^p \right)^{\frac{1}{p}} \right\}.$$

Define a set $\{K = x \in \ell^p : \| x \| \leq 1\}$ which is non-empty closed and convex subset of ℓ^p.
$\vartheta : K \times K \to R$ such that $\vartheta\,(x;\, y) = \| y - x \|^2$. Then,
$\vartheta\,(x;\, y) + \vartheta\,(y;\, x) = 2 \| y - x \|^2 > 0$, for $x \neq y$, in which ϑ is not monotone bi-function. But one can choose $\psi : K \times K \to R$ such that $\psi(x;\, y) = 2 \| y - x \|^2$. Then, ϑ is ψ- monotone bi-function.

The next notion of Tarafdar's theorem plays an important role in the proof of our main results.

Theorem 1.6. [25] Let K be a non empty and convex of a Hausdorff Topological vector space E and that $\pi : K \multimap K$ be a set valued map. Then, $\exists x^0 \in K$ such that $x^0 \in \pi(x^0)$ If the following are satisfied

(i) For any $x \in K, \pi(x)$ is a non empty convex subset of K,
(ii) For any $y \in K, \pi^{-1}(y) = \{x \in K, y \in \pi(x)\}$ contains an open set O_y which may be empty.
(iii) $\bigcup_{y \in K} O_y = K$.
(iv) There exists a nonempty set U_0 contained in a compact convex subset U_1 of K in which $G = \bigcap_{y \in K} O_y^c$ is either empty or compact.

Definition 1.7. [7] Let ϑ be a real-valued function, defined on a convex subset K of E, is said to be hemi continuous, if

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$$\lim_{t\to 0+} \vartheta\ (tx + (1-t)y) = g(y) for\ each\ x, y \in K.$$

Definition 1.8. Let x_n be a sequence of a Banach space E such that $x_n \to x_0$. Then, the mapping $\wedge$: $E \to R$ is said to be

(i) lower semi continuous (for short, (u.s.c)) at $x_0 \in E$, if
$$\wedge (x_0) \leq \lim_n\ inf\ \wedge (x_n)$$

(ii) upper semi continuous (for short, (u.s.c)) at $x_0 \in E$, if
$$\wedge (x_0) \leq \lim_n\ sup\ \wedge (x_n)$$

Definition 1.9. [22] Let E be a Banach space and that $\vartheta : K \to E^*$ is a proper function. Then, $x^* \in E^*$ is a ψ -subdifferential of $\vartheta\ in\ x \in dom\vartheta = \{ x : \vartheta(x) < \infty\}$, if

$$\vartheta^{\psi}\vartheta(x) = \{x^* \in E^*: \vartheta(y) - \vartheta(x) \geq \langle x^*, y - x\rangle - \psi(x, y) for\ each\ y \in K\}$$

Remark 1.10. Throughout this paper, let us assume that for each $\lambda \in (0, 1)$
$$\lim_{\lambda\to 0} \frac{\psi(x, x_\lambda)}{\lambda} + \psi(x, y) = 0.$$

In the following, we consider a hemiequilibrium problem on a nonempty subset K of a Banach space E.

Find an element $x \in K$ such that
$$\vartheta(x, y) + v(y) - v(x) + P^0\big(\Omega x, \Omega s(x, y)\big) \geq 0\ \ for\ each\ y \in K, \tag{1.1}$$
where $\Omega : E \to E$ is a linear compact operator, $\delta : E \times E \to E$ single-valued function, $P : E \to R$ locally Lipschitz functional and $v : X \to R\,\{+\infty\}$, where $domv = \{x \in E: v(x) < +\infty\}$ is the effective domain of v. In order to solve problem (1.1), we assume that the following is fulfilled:

H1: The mapping $\delta(.;.) : E \times E \to E$ satises the following assumptions

(i) $\delta(u; u) = 0$ for all $u \in E$,

(ii) $\delta(u; .)$ is linear operator for all $u \in E$,

(iii) for any $v \in E$, $\delta(u^m; v) \rightharpoonup \delta(u; v)$, whenever $u^m \rightharpoonup u$.

H2: ϑ is ψ −monotone bi function on $K\ of\ E^*$.

H3: v is a convex on $K, K \cap domv \neq v$.

Remark 1.11. From the convexity of $P^0(u, v)$ and linearity of $\theta(u, .)$ for all $u \in E$. One can get that $v \mapsto P^0(u, \theta(u, v))$ is convex function.

Remark 1.12. It is clear that Ωu_n converges strongly to some $\Omega_u \in K$ because Ω is a linear compact operator. Hence, $\Omega\delta(u_n, v)$ converges strongly to $\Omega(u, v)$ in which $v \in K$. By applying this fact, together with Lemma 1.2 (*ii*), one can get that

$$\limsup_n\ P^0\,(\Omega u_n; \Omega\delta(u_n, v)) \leq P^0\big(\Omega u\,; \Omega\delta(u\,, v)\big).$$

Upcoming, an example of linear compact operator which satisfies hypothesis H_1.

Example 1.13. Let $g : X \to X$ be a functional such that $\pi(x) := r\Omega(x) + s$ and $r > 0$; $s \in X$ and assume that $\Omega : X \to X$ be a linear compact operator. Define the function $\delta: X \times X \to X$ as follows:
$$\delta(u, v) \coloneqq \pi(u) - \pi(v),\ \ for\ each\ u, v \in X.$$
In this case, $\theta(v;\ u)$ satises the assumations (i), (ii) and (iii) from H_1.

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Next, some special cases of a problem 1.1 are recalled.

(i) $\vartheta(x,y) = \langle Ax, y-x\rangle$ and $P, v \equiv 0$ then problem 1.1 is reduces to the standard variational inequality (see [9]).

(ii) $\vartheta(x,y) = \langle Ax, y-x\rangle$, $\delta(x,y) = y - x$, Ω is surjective and $v \equiv 0$ then problem 1.1 is reduces to the hemivariational inequality (see [30]).

(iii) $If\ P, v \equiv 0$ then problem 1.1 is reduces to the classical equilibrium problem (for short, (EP)), for which an element $x \in K$ is found such that
$\vartheta(x;\ y) \geq 0$ for each $y \in K$ (see [4]).

(iv) $P \equiv 0$ then problem 1.1 is reduces to the Mixed equilibrium problem (for short, (MEP)), for which an element $x \in K$ is found such that

$$\vartheta(x;\ y) + v(y) - v(x) \geq 0\ for\ each\ y \in K\ (see\ [17]).$$

(v) $If\ v \equiv 0$ then problem 1.1 is reduces to the hemivariational-like inequality (for short, (HLI)), for which an element $x \in K$ is found such that
$\vartheta(x,y)+P^0(\Omega x; \Omega\delta(x,y)) \geq 0$ for each $y \in K$ (see [19]).

The chief aim of this work is to give a new contribution in this field. In particular, we study new type of hemiequilibrium problem, comprising a kind of generalized monotonicity, so called -monotone operator in reflexive and non-reflexive Banach spaces. Hereby, we would like to mention that we do not deal with a classical technique to proof our results. Thus, several difficulties occur in finding an application to the main results, because the classical methods fail to be applied directly. In order to achieve the aim, the study is divided into the following sections. In Section 2, under suitable conditions, we provide sufficient conditions for existence and uniqueness of solutions for the problem on convex and closed sets (either bounded or unbounded). In Section 3, we illustrate the applicability of our approach by a differential inclusion in the special case of our main results. We point out the fact that the results of this work can be viewed as generalization of many known results in the literature (see [12], [15] and [18]).

2. Main results

In this section we establish some existence results for hemiequilibrium problem 1.1. It is worth mentioning that through the results of this section, we prove the existence of a solution of the problem 1.1 without the assumption of boundedness of the set K. In the next lemma, we assume that K is a nonempty subset of a real reflexive Banach space E.

Lemma 2.1. Let $\vartheta, \psi : K \times K \to R$ be two bi functions and that ϑ, ψ are hemi continuous in the first argument and convex in the second argument. If the conditions (H_1) - (H_3) are fulfilled, then a hemiequilibrium problem 1.1 is equivalent to the following problem:

Find an element $x \in K$ such that

$$\vartheta(y,x) - \psi(x,y) - v(y) + v(x) \leq P^0(\Omega x ; \Omega\, \delta(x,y)) + \psi(y,x) \text{ for each } y \in K \quad (2.1)$$

Proof. Suppose that the 1.1 has a solution. Then, exists element $x \in K$ such that

$$\vartheta(x,y) + v(y) - v(x) + P\text{^}0\ (\Omega x ; \Omega\ \delta(x,y)) \geq 0 \text{ for each } y \in K.$$

By ψ_ monotonicity of ϑ, we have

$$\vartheta(y,x) + \vartheta(x,y) \leq \psi(x,y) + \psi(y,x) \text{ for each } x,y \in K, \text{ then}$$

$$\vartheta(y,x) - \psi(x,y) \leq -\vartheta(x,y) + \psi(y,x)$$
$$\leq v(y) - v(x) + P^0\big(\Omega x ; \Omega\delta(x,y)\big) + \psi(y,x).$$

Therefore, $x \in K$ is a solution of problem 2.1. Conversely, assume that $x \in K$ is a solution of problem 2.1 and fix $y \in K$.

Letting $x_\lambda = x - \lambda(x-y)$, $\lambda \in (0,1)$. Then $x_\lambda \in K$, since K is a convex, and

$$\vartheta(x_\lambda, x) - v(x_\lambda) + v(x) - \psi(x, x_\lambda) - \psi(x_\lambda, x) \leq P^0(\Omega x ; \Omega\delta(x_\lambda, x))$$

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$$= \lambda P^0\big(\Omega x\,;\Omega\delta(y,x)\big). \quad (2.2)$$

Since ϑ is convex in the second argument, one can obtain

$$0 = \vartheta(x_\lambda, x_\lambda) \le \vartheta(x_\lambda, x) - \lambda[\vartheta(x_\lambda, x) - \vartheta(x_\lambda, y)].$$

Then,

$$\lambda\,[\vartheta(x_\lambda, x) - \vartheta(x_\lambda, y)] \le \vartheta(x_\lambda, x) \quad (2.3)$$

The convexity of v and ψ in the second argument implies that

$$\lambda[\,v(x) - v(y)] \le v(x) - v(x_\lambda). \quad (2.4)$$

and

$$\lambda[\psi(x_\lambda, x) - \psi(x_\lambda, y)] \le \psi(x_\lambda, x) \quad (2.5)$$

From (2.2)-(2.5), one can get

$$\begin{aligned}&\lambda[\vartheta(x_\lambda, x) - \vartheta(x_\lambda, y) + v(x) - v(y) + \psi(x_\lambda, x) - \psi(x_\lambda, y)]\\ &\le \vartheta(x_\lambda, x) - v(x_\lambda) + v(x) + \psi(x_\lambda, x)\\ &\le \lambda P^0\big(\Omega x\,;\Omega\delta(y,x)\big) + 2\psi(x_\lambda, x) + \psi(x, x_\lambda).\end{aligned}$$

The hemi continuousity of ϑ and ψ in first argument implies that

$$\begin{aligned}&-\lambda\big[\vartheta(x,y) - v(x) + v(y) + P^0\big(\Omega x\,;\,\Omega\delta(y,x)\big) + \psi(x,y)\big]\\ &\le \psi(x, x_\lambda) + 2\lambda\psi(x,x).\end{aligned}$$

Then,

$$\vartheta(x,y) - v(x) + v(y) + P^0\big(\Omega x\,;\Omega\delta(y,x)\big) \ge \frac{-\psi(x,x_\lambda)}{\lambda} - \psi(x,y). \quad (2.6)$$

From Remark (1.10) and (2.6), we have

$$\vartheta(x,y) - v(x) + v(y) + P^0(\Omega x\,;\Omega\delta(y,x)) \ge 0 \text{ for each } y \in K.$$

Therefore, the problem 1.1 admits at least one solution. □

Theorem 2.2. Let K be a non-empty closed bounded convex subset of a reflexive Banach space E. Assume that $\vartheta, \psi : K \times K \to R$ be two bi-functions in which the conditions $(H_1) - (H_2)$ and that the following hypotheses are hold. Then 1.1 admits at least one solution.

(i) $\vartheta(.,y)$ and $\psi(.,y)$ are hemi continuous;
(ii) $\vartheta(x,.)$ and $\psi(x,.)$ are convex;
(iii) $\vartheta(x,.)$ and v are l.s.c;
(iv) $\lim_n \sup\{\psi(x_n, y) + \psi(y, x_n)\} \le \psi(x,y) + \psi(y,x)$.

Proof. Arguing by contradiction suppose that 1.1 has no solution. Then for each x ∈ K there exist y ∈ K such that

$$\vartheta(x,y) + v(y) - v(x) + P^0\big(\Omega x\,;\Omega\delta(x,y)\big) < 0.$$

This implies, by Lemma 2.1, that, for each x ∈ K , there exist y ∈ K such that

$$\vartheta(x,y) - \psi(x,y) - v(y) + v(x) > P^0\big(\Omega x\,;\,\Omega\delta(x,y)\big) + \psi(y,x). \quad (2.7)$$

Let $\eta: K \multimap K$ be a set valued mapping as defined follows:

$$\eta(x) = \big\{\, y \in K :\ \vartheta(x,y) + v(y) - v(x) + P^0\big(\Omega x\,;\,\Omega\delta(x,y)\big) < 0 \,\big\}.$$

Our claim that η satisfies the hypotheses of Tarafdar's fixed point theorem. It is clear that η is a non-empty set for x ∈ K. Let x ∈ K be arbitrary fixed and $w = (1-\lambda)y_1 + \lambda y_2$, with $y_1, y_2 \in \eta(x)$, $t \in [0,1]$. Taking into the account the effect of the convexity v and $\vartheta(x,.)$, then the remark (1:11) illustrates that

$$\begin{aligned}0 &> (1-\lambda)\big[\vartheta(x,y_1) + v(y_1) - v(x) + P^0\big(\Omega x\,;\,\Omega\delta(x,y_1)\big)\big]\\ &\quad + \lambda\big[\vartheta(x,y_2) + v(y_2) - v(x) + P^0\big(\Omega x\,;\,\Omega\delta(x,y_2)\big)\big]\\ &\ge \vartheta(x,w) + v(w) - v(x) + P^0\big(\Omega x\,;\,\Omega\delta(x,w)\big).\end{aligned}$$

It means that $w \in \eta(x)$, so , $\eta(x)$ is convex for any x ∈ K . For y ∈ K , then

$$\eta^{-1}(y) = \{\, x \in K : y \in \eta(x)\}$$

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$$= \{ x \in K : \vartheta(x,y) + \nu(y) - \nu(x) + P^0(\Omega x ; \Omega\delta(x,y)) < 0 \}$$
$$\supseteq \{ x \in K : \vartheta(x,y) - \psi(x,y) - \nu(y) + \nu(x) > P^0(\Omega x ; \Omega\delta(x,y)) + \psi(y,x)\}$$
$$= O_y.$$

From lemma (2:1) one can prove that the above inclusion. It follows that, we will show that O_y^c is a weakly closed. Adding condition iv and remark (1:11) into account and since $\vartheta(y,.)$ and v are l.s.c. one can obtain

$$\vartheta(y,x) + \nu(x_n) \leq \lim_n \inf[\vartheta(y,x_n) + \nu(x_n)]$$
$$\leq \lim_n \sup \{ \psi(x_n,y) + \nu(y) + P^0(\Omega x_n ; \Omega\delta(x_n,y)) + \psi(y,x_n)\}$$
$$\leq \psi(x,y) + \nu(y) + P^0(\Omega x ; \Omega\delta(x,y)) + \psi(y,x).$$

Therefore, O_y^c is a weakly closed for any $y \in K$.
Our claim that $\bigcup_{y\in K} O_y = K$. It suffices to prove

$$\bigcup_{y\in K} O_y \supseteq K.$$

For arbitrary
$x \in K$, using (2.7), there exist $y \in K$ such that $x \in O_y$. Therefore, the desired inclusion holds. Then, $G = \bigcap_{y\in K} O_y^c$ is either empty or weakly closed set as it is the intersection of weakly closed sets O_y^c.

It is clear that K is weakly compact set since it is a non-empty, bounded, closed and convex of reflexive space E. It means that G is weakly compact set. Hence, the hypotheses of Tarafdar′s fixed point theorem are hold. Hence, there exists an element $x^0 \in K$ and $x^0 \in \eta(x^0)$ such that

$$0 = \vartheta(x^0,x^0) + \nu(x^0) - \nu(x^0) + P^0(\Omega x^0 ; \Omega\delta(x^0,x^0)) < 0.$$

This is a contradiction which assured that the problem 1.1 has at least one solution. □

For uniqueness of solutions, we introduce the next result.

Theorem 2.3. Suppose that the hypotheses $(i - iv)$ in Theorem 2.2 are fulfilled as well as the following assumptions:

A_1 : $\exists\, M > 0$ such that $\vartheta(a,b) + \vartheta(b,a) \leq -M \parallel b_2 - b_1 \parallel$ for all $a, b \in K$.

A_2 : $\exists$ apositive constant $S \leq M$ such that $|P^0(a,b)| \leq \frac{S}{2}\|b\|^2$.

Then, 1.1 has a unique solution.

Proof. Assume that $a_1, a_2 \in K$ be two solutions to (1.1). Writting 1.1 for a_2 with $a = a_1$, we have

$$\vartheta(a_1,a_2) + \nu(a_2) - \nu(a_1) + P^0(\Omega A_1 ; \Omega\delta(a_1,a_2)) \geq 0. \quad (2.8)$$

And then for a_1 with $a = a_2$, we have

$$\vartheta(a_2,a_1) + \nu(a_1) - \nu(a_2) + P^0(\Omega a_2 ; \Omega\delta(a_2,a_1)) \geq 0. \quad (2.9)$$

By multiplying each of (2.8) and (2.9) by -1 and summing together, one can get

$$0 \geq -\vartheta(a_1,a_2) - \vartheta(a_2,a_1) - P^0(\Omega a_2 ; \Omega\delta(a_2,a_1)) - P^0(\Omega a_1 ; \Omega\delta(a_1,a_2))$$
$$\geq M \|a_2 - a_1\|^2 - P^0(\Omega a_2 ; \Omega\delta(a_2,a_1)) - P^0(\Omega a_1 ; \Omega\delta(a_1,a_2))$$
$$\geq (M - S)\|a_2 - a_1\|^2.$$

which shows that $\|a_2 - a_1\|2 \leq 0$ since $M - S \geq 0$. Consequently, we have $a_1 = a_2 \in K$. □

3. Application

An important application for theorem 2.2 is the field of partial differential inclusion problems. For doing so, the usual Sobolev space as $W^{1,p}$ $(\Theta)$and the Banach $W^{-1,p}$ (Θ) is dual space of $W^{1,p}(\Theta)$ and the Banach $W^{-1,P'}(\Theta)$ is dual space of $W^{1,p}(\Theta)$ where $\frac{1}{p}+\frac{1}{p'}=1$, $p>1$ is a real constant, and Θ is a bounded domain of R^N, $N \geq 1$ with smooth boundary $\partial\Theta$. Take the partial differential inclusion problem:

$$\begin{cases} 0 \in \xi + l(x) + \partial^{2\psi} G(u), & x \in \Theta \\ u = 0 & \text{on } \partial\Theta \end{cases} \tag{3.1}$$

where $\xi \in \partial P(u), G: K \to \mathbb{R}$ is a continuous convex function, $l: \Omega \to \mathbb{R}$ is continuous function with compact support and K is a bounded convex subset of sobolev space of $W^{1,p}$ (Θ).
For technical reasons. Define $v: W_0^{1,P}(\Theta) \to \mathbb{R}$ as follows:

$$v(\mu) := \int_\Theta -l(x)\mu(x)dx.$$

And assume that K is a non-empty, closed, bounded and convex subset of Sobolev space $W_0^{1,P}(\Theta)$.

Our purpose is to find at least one solution of the problem 1.1 under circumstances $\delta(u,v) := v-u$ and Ω is surjective.

Definition 3.1. Suppose that $u \in K$ is a K-weak subsolution of the problem 3.1 if,

$$(G(v) - G(u) \geq \langle -\xi - l(x), v-u\rangle - 2\psi(u,v) \text{ for each } v \in K.$$

Set $\vartheta(v,u) := G(u) - G(v)$ and $\psi(u,v) = \|v-u\|$. Then, any $v \in K$ one can obtain

$$\vartheta(v,u) - \psi(u,v) + \int_\Omega l(x)v(x)dx - \int_\Omega l(x)u(x)dx + \langle -\xi, v-u\rangle \leq \psi(v,u).$$

Therefore,

$$\vartheta(v,u) - \psi(u,v) - v(v) + v(u) \leq P^0(u; v-u) + \psi(v,u) \quad \text{for each } v \in K.$$

According to the above assumptions, then ϑ is ψ −monotone bi- function. As well as, in Lemma 2.1, we proved that (2.1) and 1.1 are equivalent under some conditions. Therefore, we must prove that ϑ holds all assumptions of Theorem (2.2).
It is clear that the bi-function $\psi(u,v) = \|v-u\|$ satisfies all hypotheses in theorem 2.2. As well as, we notice that ϑ is hemi continuous in first argument, l.s.c. and convex in second argument, because G is convex and continuous function. It remains to prove that v is a convex and l.s.c., we assume that $u_1, u_2 \in W_0^{1,P}(\Theta), t \in (0,1)$,

$$\begin{aligned} v(tu_1 + (1-t)u_2) &= -\int_\Omega l(x)\big(tu_1(x) + (1-t)u_2(x)\big)dx \\ &= t\Big[-\int_\Omega l(x)u_1(x)dx\Big] + (1-t)\Big[-\int_\Omega l(x)u_2(x)dx\Big] \\ &= tv(u_1) + (1-t)v(u_2). \end{aligned}$$

Also, if $u_n \rightharpoonup u \in W_0^{1,P}(\Theta)$

$$
\begin{aligned}
|\nu(u_n) - \nu(u)| &= \left| -\int_{\Omega} l(x)\big(u_n(x) - u(x)\big)dx \right| \\
&\leq (\textstyle\int_{\Omega} \; |l(x)|^{p'})^{\frac{1}{p'}}. \left(\int_{\Omega} |u_n(x) - u(x)|^{p}\right)^{\frac{1}{p}} \\
&\leq M\, \|u_n - u\|_{L^p} \\
&\leq M\, \|u_n - u\| \\
&\rightharpoonup 0.
\end{aligned}
$$

Therefore, all conditions are satisfied.

References

[1] M. Alimohammady and A. E. Hashoosh, "Existence theorems for α(x,y)- monotone of non standard hemivariational inequality," *Journal of Advances in Mathematics*, 10, (2015) , 3205–3212.
[2] I. Andrei and N. Costea, Nonlinear hemivariational inequalities and applications to non-smooth mechanics, Adv. Nonlinear Var. Inequal. 13 (2010) 1-17.
[3] J. P. Aubin, and F. H. Clarke, Shadow prices and duality for a class of optimal control problems, SIAM J. Control Optimal. 17 (1979) 567-586.
[4] E. blum and W. Oettli, From optimization and variational inequalities to equilibrium problems, The Mathematics Student, 63 (1994) 123-145.
[5] H. Brezis , Analyse Fonctionnelle Th eorie et Applications, Masson, Paris 1992.
[6] F. E. Browder, The solvability of non-linear functional equations, Duke Math. J. 30, (1963) 557-566.
[7] F. H. Clarke, Optimization and Nonsmooth Analysis, Wiley 1983.
[8] K. Fan, A generalization of Tychonoffs ' fixed point theorem, Math.Ann. 142 (1961) 305-310.
[9] P. Hartman and Stampacchia, On some nonlinear elliptic differential functional equations, Acta Math. 115 (1966) 271-310.
[10] A. E. Hashoosh, EXISTENCE AND UNIQUENESS SOLUTIONS FOR A CLASS OF HEMIVARIATIONAL INEQUALITIES , Journal of Mathematical Inequalities, 11 (2017), 565-576 .
[11] A. E. Hashoosh and M. Alimohammady, B α, β -OPERATOR AND FITZPATRICK FUNCTIONS , Jordan Journal of Mathematics and Statistics, 10 (2017), 33-52.
[12] A. E. Hashoosh and M. Alimohammady, Existence and Uniqueness Results for a Nonstandard Variational-Hemivariational Inequalities with Application, Int. J. Industrial Mathematics, 9 (2017), 241-250.
[13] A. E. Hashoosh and M. Alimohammady, On Well-Posed of Generalized Equilibrium Problems Involving α-Monotone Bifunction, Journal of Hyperstructures, 5 (2016), 151-168.
[14] A. E. Hashoosh, M. Alimohammady and G. A. Almusawi, Existence Results for Nonlinear Quasi-hemivariational Inequality Systems, Journal of Thi-Qar University ,11 (2016) 90-109.
[15] A. E. Hashoosh, M. Alimohammady and M. K. Kalleji, Existence Results for Some Equilibrium Problems involving α-Monotone Bifunction, International Journal of Mathematics and Mathematical Sciences, 2016 (2016) 1-5.
[16] U. Kamraksa and R.Wangkeeree, Generalized equilibrium problems and fixed point problems for nonexpansive semigroups in Hilbert spaces, Journal of Global Optimization, 51 (2011) 689-714.
[17] N. K. Mahato and C. Nahak, Mixed equilibrium problems with relaxed α-monotone mapping in Banach spaces, Rendiconti del Circolo Matematico di Palermo, 2013.
[18] S. Mig' rski, A. Ochal and M. Sofonea, Weak solvability of antiplane frictional contact problems for elastic cylinders, Nonlinear Anal. 1 (2010) 172-183.
[19] A. E. Molnar and O. Vas An existence result for nonlinear hemivariational-like inequality systems 58 (2013) 381-392.

[20] M. A. Noor, HEMIEQUILIBRIUM PROBLEMS, Journal of Applied Mathematics and Stochastic Analysis 2004 (2004) 235-244.
[21] M. A. Noor, Invex equilibrium problems, J. Math. Anal. Appl. 302 (2005) 463-475.
[21] W. Oettli and H. Riahi, On maximal -monotonicity of sums of operators , Communications on Applied Nonlinear Analysis 1988 (1998) 1-15.
[23] J.W. Peng and J. Yao, A viscosity approximation scheme for system of equilibrium problems, nonexpansive mappings and monotone mappings. Nonlinear Anal. 71(2009) 6001-6010.
[24] D. Repovs and C. Varga, A Nash type solutions for hemivariational inequality systems, Nonlinear Analysis 74 (2011), 5585-5590.
[25] E. Tarafdar, A fixed point theorem equivalent to the Fan Knaster Kuratowski Mazurkiewicz Theorem, J. Math. Anal. Appl. 2 (1987) 475-479.
[26] A. Tada and W. Takahashi, Weak and strong convergence theorems for a nonexpansive mapping and equilibrium problem. J. Optim. Theory Appl. 133 (2007) 359-370.
[27] R. U. Verma, On monotone nonlinear variational inequality problems, Commentationes Mathematicae Universitatis Carolinae, 39 (1998) 91-98.
[28] R. U.Verma, On generalized variational inequalities involving relaxed Lipschitz and relaxed monotone operators, Journal of Mathematical Analysis and Applications, 213 (1997) 387-392.
[29] R. U. Verma, A-monotonicity and its role in nonlinear variational inclusions, J. Optim. Theory Appl. 129 (2006) 457-467.
[30] P. D. Panagiotopoulos, M. R Fundo and V Radulescu, Existence theorems of Hartman Stampacchia type for hemivariational inequalities and applications J. Global Optim. 15(1999) 41-54.