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To cite this article: Mohammed Kadhim and Mohsin Almamoori 2019 *IOP Conf. Ser.: Mater. Sci. Eng.* **571** 012018

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Solvability and Controller Action on Local Saddle-Node Bifurcation of Reduced Order Two-Dimensional Differential-Algebraic System

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Abstract. This paper focuses on the solvability and local Saddle-Node bifurcation ensurance of two-dimensional differential-algebraic equations of index-1 with its feedback controller. The feedback controller action has been designed to satisfy the necessary requirements for Saddle-Node bifurcation ensurance. The mathematical assumptions and proofs have proposed with systematic illustrations to demonstrate the applicability of the proposed approach.

1. Introduction

Differential-algebraic systems (DAS) are those dynamics governed by a mixture of algebraic and differential equations. During the past decade, DAS have attracted much attention due the comprehensive applications in chemical (physical) engineering, control theory, electrical. Some of perturbations may happen in DAS by changes of manufacturing process of the variations of constructive elements, components or due to replacement act. Therefore, the changing in the behavior of system from-to stability and instability of its solutions may call Bifurcation. The Bifurcation occurs in linear infinite dimensional systems and non-linear system of equations which have changing parameter [2][4][5][6][9][14][15][16][17][11][19].

The aim of this paper is to study the solvability and local Saddle-Node bifurcation ensurance for forced/unforced two dimensional differential-algebraic systems using reduced order approach. When the system is forced, a feedback controller action is designed to satisfy the necessary mathematical requirements for Saddle-Node bifurcation. Step-by-step illustrations have been developed to study these phenomena.

2. Problem Formulation-I:

Consider the semi-explicit DAS:

$$\dot{x}_1 = F_1(x_1, x_2; \mu) \quad (1a)$$

$$0 = F_2(x_1, x_2; \mu) \quad (1b)$$

where $(x_1; \mu) \in D \subset R^{1+1}$ which is an open subset, $x_2 \in R, \mu \in R$. The function $F_1(x_1, x_2; \mu)$ is needed to be in the class $C^1(D \times R; R)$, while $F_2(x_1, x_2; \mu) \in C^2(D \times R; R)$.

Remarks (1)[17]:

- (1) It may seen like all x_1 is the state, which is holding the information about the past, this is not true generally, where the derivative $\left(\frac{\partial F_2}{\partial x_2}(x_1, x_2; \mu)\right)$, at least locally nonzero [17][18].
- (2) The class of initial conditions should be determined later on, based on the nature of $F_2(x_1, x_2; \mu)$.

2.1 The Solvability of Problem Formulation-I:



Consider problem formulation-I, and assume that there exist an open subset $\tilde{\Omega}_{x_1} \subset D$ such that for all $(\tilde{x}_1; \mu) \in \tilde{\Omega}_{x_1; \mu} \triangleq \{(x_1; \mu) \in R^{1+1}, x_1 \in \tilde{\Omega}_{x_1}, \mu \in R\} \subset D \subset R^{1+1}$, such that, it may be solve $F_2(\tilde{x}_1, \tilde{x}_2; \tilde{\mu}) = 0$ for \tilde{x}_2 . The solution manifold set is define as:

$$\tilde{\Omega} = \{(x_1, x_2; \mu) : (x_1; \mu) \in \tilde{\Omega}_{x_1; \mu}, x_2 \in R^1 \mid F_2(x_1, x_2; \mu) = 0\},$$

which is not necessary be open. For this purpose, one can assume that, $\frac{\partial F_2}{\partial x_2}(x_1, x_2; \mu) \neq 0$ for

$(x_1, x_2; \mu) \in \tilde{\Omega} \subset R^{1+1+1}$ by the help of implicit function theorem (IFT) one have that for every $(\tilde{x}_1; \tilde{\mu}) \in \tilde{\Omega}_{x_1, \mu}$ (open subset of R^{1+1}), there exist a neighborhood $\tilde{O}_{x_1, \mu}$ of $(\tilde{x}_1; \tilde{\mu})$ and the corresponding \tilde{O}_{x_2} of \tilde{x}_2 (since R^1 is open) such that for each point $(\tilde{x}_1; \tilde{\mu}) \in \tilde{O}_{x_1, \mu}$, a unique solution $x_2 \in \tilde{O}_{x_2}$ exists and $x_2 = \varphi_{\tilde{x}_1; \tilde{\mu}}(x_1; \mu)$ locally [17][18].

Therefore, to find the class of initial conditions (consistency class) as $t = 0$ (for simplicity) given an initial gest $x_1(0)$ for a given μ_0 , compute $x_2(0) = \varphi(x_1(0); \mu_0)$ and therefore set the class of consistency as:

$$\mathcal{W}_k = \{(x_1(0), x_2(0); \mu_0) \mid (x_1(0); \mu_0) \in \tilde{\Omega}_{x_1; \mu}, x_2(0) = \varphi(x_1(0); \mu_0), \text{ for } \mu_0 \in R\} \subset \tilde{\Omega} \subset R^{1+1+1}$$

Hence the following reduced order system (equivalent) DAS are found as follows:

$$\dot{x}_1 = F_1(x_1, \varphi(x_1; \mu); \mu) \quad (2a)$$

$$\text{with } x_2 = \varphi(x_1; \mu) \quad (2b)$$

$$\mathcal{W}_k = \{(x_1(0), x_2(0); \mu_0) \mid (x_1(0); \mu_0) \in \tilde{\Omega}_{x_1; \mu}, x_2(0) = \varphi(x_1(0); \mu_0), \text{ for } \mu_0 \in R\} \subset \tilde{\Omega} \subset R^{1+1+1} \quad (2c)$$

where F_1 and F_2 are assumed to satisfy the assumption of Problem Formulation-I.

Thus, the solvability of Problem Formulation-I is equivalent to solvability of the ROS (2), locally, under the condition $\frac{\partial F_2}{\partial x_2}(x_1, x_2; \mu) \neq 0$ for $(x_1, x_2; \mu) \in \tilde{\Omega}$, and the condition (2b).

Lemma (1) (Locally Lipschitz Property on D):

Consider the Problem Formulation-I. Then if $F_1 \in C^1(D \times R; R)$ and $\varphi \in C^2(D; R)$, then F_1 of (2a) is Locally Lipschitz on D .

Proof: Since D is an open subset of R^{1+1} , given $(x_{10}; \mu^0) \in D$, then there is an $\epsilon > 0$ such that

$$N_\epsilon(x_{10}; \mu^0) = \{(x_1, \mu) \in R^{1+1} \mid |x_1 - x_{10}| + |\mu - \mu^0| < \epsilon\}.$$

$$\text{Let } M_{\frac{\epsilon}{2}}(x_{10}; \mu^0) = \{(x_1; \mu) \in R^{1+1} \mid |(x_1, \mu) - (x_{10}; \mu^0)| \leq \frac{\epsilon}{2}\}$$

Define, $F_1 \in C^1(D \times R; R)$ on closed and bounded subset $M_{\frac{\epsilon}{2}}(x_{10}; \mu^0)$ of R^{1+1} , one can have:

$$k = \max_{(x_1, \mu) \in M_{\frac{\epsilon}{2}}(x_{10}; \mu^0)} \left| \frac{\partial F_1}{\partial x_1}(x_1, \varphi(x_1; \mu); \mu) \right| \leq \max_{(x_1; \mu) \in M_{\frac{\epsilon}{2}}(x_{10}; \mu^0)} \left[\left| \frac{\partial F_1}{\partial x_1}(x_1; \mu) \right| + \left| \frac{\partial F_1}{\partial \varphi}(x_1; \mu) \right| \left| \frac{\partial \varphi}{\partial x_1}(x_1; \mu) \right| \right].$$

Hence, the maximum of continuous functions $\frac{\partial F_1}{\partial x_1}(x_1, \varphi; \mu)$ and $\frac{\partial F_1}{\partial \varphi}(x_1, \varphi; \mu) \frac{\partial \varphi}{\partial x_1}(x_1; \mu)$ on the compact set $M_{\frac{\epsilon}{2}}(x_{10}; \mu^0)$ of R^{1+1} . Then for $x_1, y \in M_{\frac{\epsilon}{2}}(x_{10}; \mu^0)$ set $v = y - x_1$. It follows that

$x_1 + sv \in M_{\frac{\epsilon}{2}}(x_{10}; \mu^0)$ for $0 \leq s \leq 1$ since $M_{\frac{\epsilon}{2}}(x_{10}; \mu^0)$ is a convex set. Define $F[0,1] \rightarrow R^{1+1+1}$ by

$$F(s) \triangleq F_1(x_1 + sv, \varphi(x_1 + sv; \mu); \mu)$$

By chain rule will get:

$$\begin{aligned} F'(s) &\triangleq \frac{\partial F_1}{\partial x_1} \frac{d(x_1 + sv)}{ds} + \frac{\partial F_1}{\partial \varphi} \frac{\partial \varphi}{\partial x_1} \frac{d(x_1 + sv)}{ds} \\ &= \frac{\partial F_1}{\partial x_1}(x_1 + sv, \varphi(x_1 + sv; \mu); \mu)v + \frac{\partial F_1}{\partial \varphi}(x_1 + sv, \varphi(x_1 + sv; \mu); \mu) \frac{\partial \varphi}{\partial x_1}(x_1 + sv; \mu)v. \end{aligned} \quad (3)$$

Since $\frac{\partial F_1}{\partial \varphi} = \frac{\partial F_1}{\partial x_2}$, where $x_2 = \varphi(x_1; \mu)$ which is bounded in the class $C^2(D; R)$, $D \subseteq R^{1+1}$. Therefore

$$F_1(y, \varphi(y; \mu); \mu) - F_1(x_1, \varphi(x_1; \mu); \mu) \triangleq F(1) - F(0) = \int_0^1 \frac{d}{ds} F(s) ds.$$

(F is continuously differentiable), and $y = x_1 + sv$. Therefore from (3), one gets

$$\begin{aligned} & F_1(y, \varphi(y; \mu); \mu) - F_1(x_1, \varphi(x_1; \mu); \mu) \\ &= \int_0^1 \left(\frac{\partial F_1}{\partial x_1}(x_1 + sv, \varphi(x_1 + sv; \mu); \mu) v + \frac{\partial F_1}{\partial \varphi}(x_1 + sv, \varphi(x_1 + sv; \mu); \mu) \frac{\partial \varphi}{\partial x_1}(x_1 + sv; \mu) v \right) ds. \end{aligned}$$

$$\text{Thus, } |F_1(y, \varphi(y; \mu); \mu) - F_1(x_1, \varphi(x_1; \mu); \mu)| \leq \int_0^1 \left| \frac{\partial F_1}{\partial x_1} v \right| ds + \int_0^1 \left| \frac{\partial F_1}{\partial \varphi} \right| \left| \frac{\partial \varphi}{\partial x_1} v \right| ds.$$

By the assumption of the Problem Formulation-I one can get that $\left| \frac{\partial F_1}{\partial x_1} \right| < B_{x_1}$, $\left| \frac{\partial F_1}{\partial \varphi} \right| < B_\varphi$ and $\left| \frac{\partial \varphi}{\partial x_1} \right| < k_\varphi$ on $M_\varepsilon(x_{10}; \mu^0)$ for a $B_{x_1} \geq 0$, $B_\varphi \geq 0$, $k_\varphi \geq 0$, and therefore

$$\begin{aligned} |F_1(y, \varphi(y; \mu); \mu) - F_1(x_1, \varphi(x_1; \mu); \mu)| &\leq B_{x_1} |v| + B_\varphi k_\varphi |v| \leq (B_{x_1} + B_\varphi k_\varphi) |v| \\ &\leq (B_{x_1} + B_\varphi k_\varphi) |y - x_1| \end{aligned}$$

which makes F_1 satisfying the Lipschitz condition, with Lipschitz constant $k = (B_{x_1} + B_\varphi k_\varphi)$

Now, consider the Problem Formulation-I which is equivalent to the following one

$$\frac{dz}{dt} = G(z; \mu) \quad (4a)$$

$$z(0) = z_0 \quad (4b)$$

where $z_0 \in R^{1+1}$, $z_0 = (x_{10}, x_{20}; \mu^0) \in \mathcal{W}_k \subset \tilde{\Omega} \subset R^{1+1+1}$ and

$$G(z; \mu) = \begin{pmatrix} F_1(x_1, \varphi(x_1; \mu); \mu) \\ \frac{\partial \varphi}{\partial x_1} F_1(x_1, \varphi(x_1; \mu); \mu) \end{pmatrix} \in \begin{pmatrix} C^1(D \times R; R) \\ C^2(D; R) \times C^1(D \times R; R) \end{pmatrix}$$

on $x_2 \in \tilde{\Omega}_{x_2}$ and $(\tilde{x}_1; \tilde{\mu}) \in \tilde{\Omega}_{x_1; \mu}$, where z is the initial condition variable. Since $F_1(x_1, \varphi(x_1; \mu); \mu)$ satisfied the Lipschitz condition with constant $k = (B_{x_1} + k_\varphi B_\varphi)$ by Lemma (1) and $\frac{\partial \varphi}{\partial x_1}$ is bounded on the compact (convex) set. While, $\varphi \in C^2(D; R)$ on $x_2 \in \tilde{\Omega}_{x_2}$ by IFT result. Set

$$x = (x_1, \varphi(x_1; \mu); \mu)^T = (x_1, x_2; \mu)^T \text{ for } \mu \in R, x_2 = \varphi(x_1; \mu) \text{ is the solution of } F_2(x_1, x_2; \mu) = 0.$$

$$\text{Let } \dot{x} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} \dot{x}_1 \\ \frac{\partial \varphi}{\partial x_1} \dot{x}_1 \end{pmatrix} = \begin{pmatrix} F_1(x_1, \varphi(x_1; \mu); \mu) \\ \frac{\partial \varphi}{\partial x_1} F_1(x_1, \varphi(x_1; \mu); \mu) \end{pmatrix} = G(x_1, x_2; \mu)$$

$$\text{Then, } \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} \triangleq \begin{pmatrix} F_1(x_1, \varphi(x_1; \mu); \mu) \\ \frac{\partial \varphi}{\partial x_1} F_1(x_1, \varphi(x_1; \mu); \mu) \end{pmatrix} \quad (5)$$

$$\text{and } \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} x_1(0) \\ \varphi(x_1(0); \mu^0) \end{pmatrix}, \mu^0 \in R, \text{ where } (x_1(0), x_2(0)) \in \mathcal{W}_k \text{ for } \mu^0 \in R \text{ see (2c). The system (5)}$$

$$\begin{aligned} \text{is then equivalent to } & \frac{dz}{dt} = G(z; \mu), & G \text{ is defined by (4a)} \\ & z(0) = z_0 \in R^{1+1} & \text{with } z = (x_1, x_2)^T \text{ for } \mu \in R \end{aligned}$$

Therefore, by Theorem (1), P. (80) (Dependence on Initial Condition) in [11], there exist a unique solution $z(t, z_0; \mu_0) \in C^1(H)$, where $H = [-a, a] \times N_\delta(z_0) \times N_\delta(\mu_0)$, $N_\delta(x_0) \subset R^{1+1}$ and

$$N_\delta(\mu_0) \subset R, \text{ where } a \text{ defined by } 0 < a < \min \left[\frac{b}{M_0 + M_1}, \frac{1}{B_{x_1} + k_\varphi B_\varphi} \right], \text{ where}$$

$$\max_{[-a, a]} |v_k(t) - x_{10}| \leq b, \{v_k(t)\} \text{ is a Cauchy sequence of continuous function in } C[-a, a]$$

$\Rightarrow v_k(t) \rightarrow v(t)$, v_k convergence to continuous function $v(t)$ uniformly for all $t \in [-a, a]$, which clarifies the differential equations as a sequence of parameter equations:

$$\begin{aligned} \max_{|x_1-x_{10}| \leq \frac{\epsilon}{2}} |F_1| = M_0, \quad \max_{|x_1-x_{10}| \leq \frac{\epsilon}{2}} |DF_1| = M_1, \quad \max_{|x_1-x_{10}| \leq \frac{\epsilon}{2}} \left| \frac{\partial F_1}{\partial x_1} \right| < B_{x_1}, \quad \max_{|x_1-x_{10}| \leq \frac{\epsilon}{2}} \left| \frac{\partial \varphi}{\partial x_1} \right| < k_\varphi \quad \text{and} \\ \max_{|x_1-x_{10}| \leq \frac{\epsilon}{2}} \left| \frac{\partial F_1}{\partial \varphi} \right| < B_\varphi. \end{aligned}$$

Hence, the problem formulation-I is solvable and has a unique solution defined locally.

3. Bifurcation of Problem Formulation-I:

The study of bifurcation theory of the Problem Formulation-I must start with the following important discussion: that will begin with consider the ROS system (2)

$$\dot{x}_1 = F_1(x_1, \varphi(x_1; \mu); \mu)$$

with

$$x_2 = \varphi(x_1; \mu)$$

$\mathcal{W}_k = \{(x_1(0), x_2(0); \mu_0) | (x_1(0); \mu_0) \in \tilde{\Omega}_{x_1; \mu}, x_2(0) = \varphi(x_1(0); \mu_0), \text{ for } \mu_0 \in R\} \subset \tilde{\Omega} \subset R^{1+1+1}$
such that the function F_1 is a C^r -function on some open set in $R^1 \times R^1 \times R^1 \rightarrow R^3$ and it is sufficient at $r = 5$ from the illustrated examples in [14].

Definition (1):

The equilibrium point $(x_1, x_2; \mu) = (0, 0; 0)$, $x_2 = \varphi(0; 0) = 0$ of a one-parameter family of vector field (2) is said to undergo a bifurcation at $\mu = 0$ if the flow for μ near zero and x_1 near zero is not qualitatively the same as the flow near $x_1 = 0$ at $\mu = 0$.

3.1 Saddle-Node Bifurcation (SNB)

To study the **SNB**, consider problem formulation-I

$$\dot{x}_1 = F_1(x_1, \varphi(x_1; \mu); \mu)$$

$$x_2 = \varphi(x_1; \mu)$$

with $\mathcal{W}_k = \{(x_1(0), x_2(0); \mu_0) | (x_1(0); \mu_0) \in D, \mu \in R, x_2(0) = \varphi(x_1(0); \mu_0)\}$

where $(x_1; \mu) \in D, \mu \in R, x_2 \in R$ and $D \subset R^{1+1}$ is an open subsets,

$\varphi \in C^2(D; R)$ and $F_1 \in C^1(D \times R; R)$ are continuously differentiable functions, with $(x_1; \mu) \in \tilde{\Omega}_{x_1; \mu}$ and $x_2 \in \tilde{\Omega}_{x_2}$. So, the following important assumption is:

Assumptions A:

The orbit structure near $(x_1^0, \varphi(x_1^0; \mu^0); \mu^0)$ is determined by the associated center manifold equation with

$$\text{a) } F_1(0, \varphi(0; 0); 0) = F_1(0, 0; 0) = (0, \varphi(0; 0) = 0 \quad (6a)$$

is the equilibrium point condition at origin.

$$\text{b) } \frac{\partial F_1}{\partial x_1} + \frac{\partial F_1}{\partial \varphi} \frac{\partial \varphi}{\partial x_1} = 0 \text{ at } (0, 0, 0) \quad (6b)$$

the zero eigenvalue condition. The equilibrium point

$(x_1, \varphi(x_1; \mu); \mu) = (0, 0; 0)$ is not hyperbolic.

Knowing that an equilibrium point has zero eigenvalue for $\mu = 0$ is not sufficient to determine the orbit structure for μ near zero.

So, a unique curve of equilibrium points parameterized by x_1 , passed through $(0, 0; 0)$ is denoted by $\mu(x_1)$. The curve $\mu(x_1)$ is satisfying two properties [19]:

$$\text{a) } \text{It was tangent to the line } \mu = 0 \text{ at } x_1 = 0 \text{ i.e., } \frac{d\mu}{dx_1}(0) = 0$$

$$\text{b) } \text{It was laid entirely to one side of } \mu = 0 \text{ locally; this will be satisfied if } \frac{d^2\mu}{dx_1^2}(0) \neq 0.$$

From the above discussions one can have that: If a unique curve of equilibrium points passed through the origin and lays entirely on one side of $\mu = 0$ in $\mu - x_1$ plane, then the origin is a SNB point.

From the geometry of the curve in $\mu - x_1$ plane in a neighborhood of bifurcation point may help us to derive conditions on the ROS evaluated at the bifurcation point.

If we have that $\frac{\partial F_1}{\partial \varphi} \frac{\partial \varphi}{\partial \mu} + \frac{\partial F_1}{\partial \mu} \neq 0$ at $(0,0,0)$. By IFT, there exists a unique function $\mu = \mu(x_1)$ with $\mu(0) = 0$, defined on the manifold for x_1 sufficiently small such that $F_1(x_1, \varphi(x_1; \mu(x_1)); \mu(x_1)) = 0$, which is locally in same neighborhood.

The purpose is to derive conditions under a general one-parameter family of the system (2) will undergo a **SNB** in term of derivatives of $F_1(x_1, \varphi(x_1; \mu(x_1)); \mu(x_1)) = 0$ evaluated at $(0,0,0)$.

Theorem (1): Consider problem formulation-I with

- 1) $\frac{\partial F_1}{\partial x_1} + \frac{\partial F_1}{\partial \varphi} \frac{\partial \varphi}{\partial x_1} = 0$, at $(x_1, \varphi(x_1; \mu(x_1)); \mu(x_1)) = (0, \varphi(0; \mu(0)); \mu(0)) = (0, \varphi(0,0), 0) = (0,0,0)$
- 2) $\frac{\partial F_1}{\partial \varphi} \frac{\partial \varphi}{\partial \mu} + \frac{\partial F_1}{\partial \mu} \neq 0$ at $(0,0,0)$.
- 3) $\frac{\partial^2 F_1}{\partial x_1^2} + 2 \frac{\partial F_1}{\partial x_1 \partial \varphi} + \frac{\partial^2 F_1}{\partial \varphi^2} \frac{\partial^2 \varphi}{\partial x_1^2} + \frac{\partial F_1}{\partial \varphi} \frac{\partial^2 \varphi}{\partial x_1^2} \neq 0$ at $(0,0,0)$

Then saddle-node bifurcation occurs for the problem (2.4) \Leftrightarrow problem formulation-I

Proof: By Assumption (A). Consider $F_1 = (x_1, \varphi(x_1; \mu(x_1)); \mu(x_1))$ along the curve of equilibrium points $(x_1, \varphi(x_1; \mu(x_1)); \mu(x_1)) = (0,0,0)$. By IFT and the condition $\frac{\partial F_1}{\partial \mu}(0,0,0) \neq 0$, this impels that, there is $\mu = \mu(x_1)$ with $\mu(0) = 0$ and

$$F_1 = (x_1, \varphi(x_1; \mu(x_1)); \mu(x_1)) = 0 \quad (7)$$

Differentiating (7) with respect to x_1 :

$$\frac{\partial F_1}{\partial x_1} (x_1, \varphi(x_1; \mu(x_1)); \mu(x_1)) = 0 \Rightarrow \frac{\partial F_1}{\partial x_1} + \frac{\partial F_1}{\partial \varphi} \frac{\partial \varphi}{\partial x_1} + \frac{\partial F_1}{\partial \mu} \frac{d\mu}{dx_1} + \frac{\partial F_1}{\partial \varphi} \frac{\partial \varphi}{\partial \mu} \frac{d\mu}{dx_1} = 0 \quad (8)$$

at $(x_1, \varphi(x_1; \mu(x_1)); \mu(x_1)) = (0,0,0)$, let $F_1(0,0,0) \triangleq F_1^0$ for simplicity, so we have got the following

$$F_{1x_1}^0 + F_{1\varphi}^0 \varphi_{x_1}^0 + F_{1\mu}^0 \mu_{x_1}^0 + F_{1\varphi}^0 \varphi_{\mu}^0 \mu_{x_1}^0 = 0$$

Therefore, $\frac{d\mu^0}{dx_1} = \frac{-[F_{1x_1}^0 + F_{1\varphi}^0 \varphi_{x_1}^0]}{F_{1\varphi}^0 \varphi_{\mu}^0 + F_{1\mu}^0}$

Then, from conditions (1) and (2) with **Assumption (A)** gives:

$$\frac{d\mu(0)}{dx_1} = 0 \quad (9)$$

which implies that the curve of equilibrium points is tangent to the line $x_1 = 0$ at $\mu = 0$.

Next, the second derivative of (7) with respect to x_1 is:

$$\frac{d^2 F_1}{dx_1^2} + \frac{\partial^2 F_1}{\partial \varphi \partial x_1} \frac{\partial \varphi}{\partial x_1} + \frac{\partial^2 F_1}{\partial \varphi^2} \left(\frac{\partial \varphi}{\partial x_1}\right)^2 + \frac{\partial F_1}{\partial \varphi} \frac{\partial^2 \varphi}{\partial x_1^2} + \frac{\partial^2 F_1}{\partial x_1 \partial \varphi} \frac{\partial \varphi}{\partial x_1} + \left(\frac{\partial F_1}{\partial \varphi} \frac{\partial \varphi}{\partial \mu} + \frac{\partial F_1}{\partial \mu}\right) \frac{d^2 \mu}{dx_1^2} = 0.$$

By continuity properties have $\frac{\partial^2 F_1}{\partial \varphi \partial x_1} + \frac{\partial^2 F_1}{\partial x_1 \partial \varphi} = 2 \frac{\partial^2 F_1}{\partial x_1 \partial \varphi}$. By evaluating at

$(x_1, \varphi(x_1; \mu(x_1)); \mu(x_1)) = (0,0,0)$ with (9) one can get:

$$F_{1x_1x_1}^0 + 2 F_{1x_1\varphi}^0 \varphi_{x_1}^0 + F_{1\varphi\varphi}^0 (\varphi_{x_1}^0)^2 + F_{1\varphi}^0 \varphi_{x_1x_1}^0 + (F_{1\varphi}^0 \varphi_{\mu}^0 + F_{1\mu}^0) \mu_{x_1x_1}^0 = 0.$$

This implies that: $\frac{d^2 \mu(0)}{dx_1^2} = \frac{-[F_{1x_1x_1}^0 + 2F_{1x_1\varphi}^0 \varphi_{x_1}^0 + F_{1\varphi\varphi}^0 (\varphi_{x_1}^0)^2 + F_{1\varphi}^0 \varphi_{x_1x_1}^0]}{F_{1\varphi}^0 \varphi_{\mu}^0 + F_{1\mu}^0}$

Therefore by conditions (2) and (3) the above equation will gives

$$\frac{d^2 \mu(0)}{dx_1^2} \neq 0 \quad (10)$$

Thus, Assumption A decides that the curve lie locally on one side of $\mu = 0$ as shown in Figure (1). Moreover, the sign of $\frac{d^2\mu(0)}{dx_1^2} \neq 0$ determine on which side of $\mu = 0$ the curve lies. Therefore, from (9) and (10), by Assumption (A), and from [19] one can conclude the SNB occur as shown briefly in the following Figure (1). Therefore, the proof is completed.

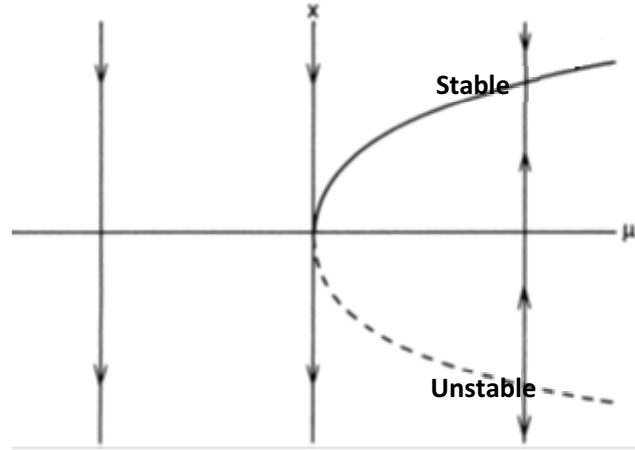


Figure (1): The SNB diagram

Illustration (1):

Consider the semi-explicit DAEs in normal form:

$$\dot{x}_1 = \mu - \frac{1}{2}x_1^2 - \frac{x_2}{\cos(x_1)} \quad (11a)$$

$$0 = x_1^2 - 2x_2 \sec(x_1) \quad (11b)$$

where $x_1 \in R, x_2 \in R, \mu \in R$ is the parameter.

Step (1): The domain of (11) is $x_1 \in (-\frac{\pi}{2}, \frac{\pi}{2})$ to ensure the local solvability of IFT.

Step (2): The point (0,0,0) is one of equilibrium points of (11).

Step (3): Applying the IFT i.e., $\frac{\partial F_2}{\partial x_2} = -2 \sec(x_1) = -2 \neq 0, x_1 \in (-\frac{\pi}{2}, \frac{\pi}{2})$, so on the neighborhood of

$(0) \in (-\frac{\pi}{2}, \frac{\pi}{2}), \frac{dF_2}{dx_2} \neq 0$. Then, there exist $\varphi(x_1; \mu)$ such that $x_2 = \varphi(x_1; \mu)$ which is continuously differentiable

$$x_2 = \frac{x_1^2}{2 \sec(x_1)} \equiv \varphi(x_1, \mu) \quad (12)$$

$$x_1 \in N_{x_1}(0) = \{x_1 \in R \mid |x_1 - 0| < \frac{\pi}{2}\} \text{ i.e., } x_1 \in (0,0,0),$$

$$\text{Step (4): The ROS is: } F_1 = \mu - \frac{1}{2}x_1^2 - \varphi(x_1; \mu) \left(\frac{1}{\cos(x_1)} \right), \varphi = \frac{x_1^2}{2 \sec(x_1)} \quad (13)$$

Step (4.1): Drive the conditions: $F_1(0, \varphi(0,0), 0) = 0, \frac{\partial F_1}{\partial x_1} + \frac{\partial F_1}{\partial \varphi} \frac{\partial \varphi}{\partial x_1} = 0$

$$\frac{\partial F_1}{\partial x_1} \frac{\partial \varphi}{\partial \mu} + \frac{\partial F_1}{\partial \mu} = \frac{1}{\cos(x_1)}(0) + 1 = 1 \neq 0, F_{1x_1x_1} = -1, 2 F_{1x_1\varphi} = 0, \varphi_{x_1} = 0,$$

$$\Rightarrow -[F_{1x_1x_1} + 2 F_{1x_1\varphi} \varphi_{x_1} + F_{1\varphi\varphi} (\varphi_{x_1})^2 + F_{1\varphi} \varphi_{x_1x_1}] \neq 0$$

$$\text{Step (5): } \mu_{x_1}^0 = \frac{-[F_{1x_1}^0 + F_{1\varphi}^0 \varphi_{x_1}^0]}{F_{1\varphi}^0 \varphi_{\mu}^0 + F_{1\mu}^0} = \frac{0}{1} = 0 \quad (14)$$

Thus the curve of the equilibrium point (0,0,0) is tangent to the line $\mu = 0$ at $x_1 = 0$.

$$\text{Step (6): } \mu_{x_1 x_1}^0 = \frac{-[F_{1x_1 x_1}^0 + 2F_{1x_1 \varphi}^0 \varphi_{x_1}^0 + F_{1\varphi \varphi}^0 (\varphi_{x_1}^0)^2 + F_{1\varphi}^0 \varphi_{x_1 x_1}^0]}{F_{1\varphi}^0 \varphi_{\mu}^0 + F_{1\mu}^0} = \frac{1}{1} \neq 0 \quad (15)$$

Thus, the curve lies locally on one side of $\mu = 0$. Therefore, the qualitative bifurcation of the system (13) has been changed around $\mu = 0$ ($\mu < 0$, $\mu = 0$ and $\mu > 0$) as shown in the following Figure (2).

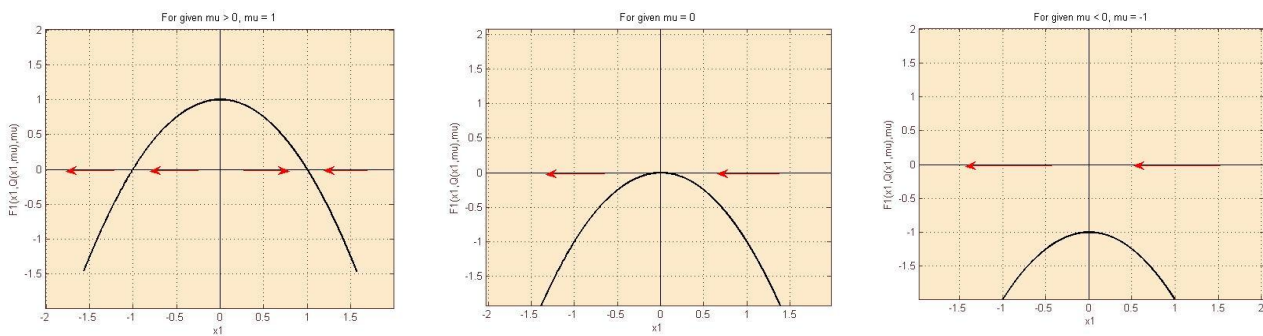


Figure (2): The function $F_1(x_1, x_2; \mu) = \mu - \frac{1}{2}x_1^2 - \varphi(x_1; \mu) \left(\frac{1}{\cos(x_1)} \right)$, versus x_1 , where $x_2 = \varphi(x_1; \mu) = \frac{x_1^2}{2 \sec(x_1)}$ for the values ($\mu = -1, \mu = 0$ and $\mu = 1$)

We observe that for $\mu = 0$ there is a unique equilibrium point which is not stable.

For $\mu > 0$ there are two equilibrium points, one is stable and the other unstable. For $\mu < 0$, $F_1(x_1, \varphi(x_1; \mu); \mu) = 0$ does not admit any solution, so there are no equilibrium points. This behavior can be visualized with the help of the bifurcation diagram, shown in Figure (1).

The directional derivatives of the system (13) have been used to show the stability and instability under the above conditions as shown in the Figure (3).

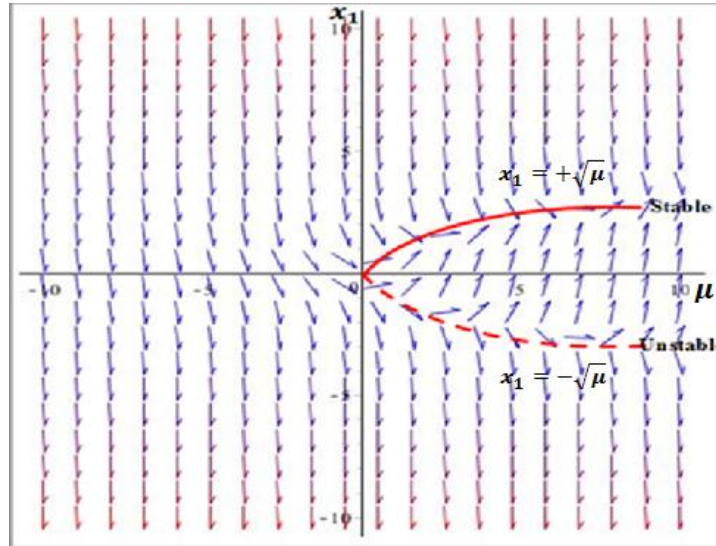
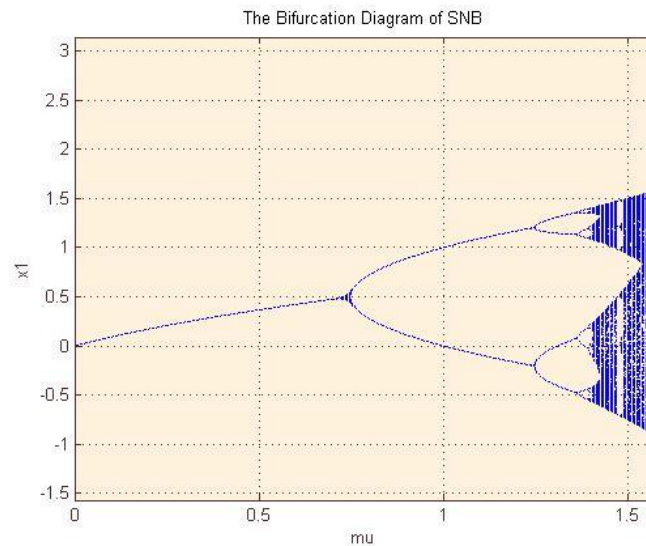


Figure (3): The solution $x_1(\mu)$ of $\dot{x}_1 = F_1(x_1, x_2; \mu) - \frac{1}{2}x_1^2 - \varphi(x_1; \mu) \left(\frac{1}{\cos(x_1)} \right)$, where $x_2 = \varphi(x_1; \mu) = \frac{x_1^2}{2 \sec(x_1)}$ in the neighborhood of $(-\frac{\pi}{2}, \frac{\pi}{2})$ versus μ .

When the bifurcation diagram for the solution x_1 versus the bifurcation parameter μ have been shown in the following Figure (4).



Discretization of the results

Figure (4): The bifurcation diagram of the solution x_1 of $\dot{x}_1 = F_1(x_1, x_2; \mu)$ where $x_2 = \varphi(x_1; \mu)$ in the neighborhood of $(0, \frac{\pi}{2})$ versus μ with $N = 500$.

4. Problem Formulation-II:

Consider the semi-explicit DAS with control:

$$\dot{x}_1 = F_1(x_1, x_2; \mu, u) \quad (16a)$$

$$0 = F_2(x_1, x_2; \mu, u) \quad (16b)$$

where, $(x_1; \mu) \in D \subset R^{1+1}$, D is an open subset, $x_2 \in R^1$,

$u \in R$ is a feedback controller, $\mu \in R$. The function $F_1(x_1, x_2; \mu, u)$ is needed to be in the class $C^1(D \times R \times R; R)$, while $F_2(x_1, x_2; \mu, u) \in C^2(D \times R \times R; R)$ with consider the Remarks (1) in suction 2.

4.1. The Solvability of Problem Formulation-II:

Consider problem formulation-II, and assume that there exist an open subset $\tilde{\Omega}_{(x_1, \mu)} \subset D$ such that for all

$$((\tilde{x}_1; \tilde{\mu}), \tilde{u}) \in \tilde{\Omega}_{(x_1, \mu), u} \triangleq \{(\tilde{x}_1; \tilde{\mu}), \tilde{u}) \in R^{1+1+1}, (x_1; \mu) \in \tilde{\Omega}_{(x_1, \mu)}, u \in R, \mu \in R\} \subset D \subset R^{1+1+1}$$

such that, it is possible to solve $F_2(\tilde{x}_1, \tilde{x}_2; \tilde{\mu}, \tilde{u}) = 0$ for \tilde{x}_2 . The solution manifold set is then defined as:

$$\tilde{\Omega} = \{(x_1, x_2; \mu, u) : ((x_1; \mu), u) \in \tilde{\Omega}_{(x_1, \mu), u}, \mu \in R, x_2 \in R | F_2(x_1, x_2; \mu, u) = 0\}$$

which is not necessary be open. The Jacobian matrix of $F_2(x_1, x_2; \mu, u)$ i.e. $\frac{\partial F_2}{\partial x_2}(x_1, x_2; \mu, u)$ must be not zero for $(x_1, x_2; \mu, u) \in \tilde{\Omega} \subset R^{1+1+1+1}$ by IFT one can have for every $((\tilde{x}_1; \tilde{\mu}), \tilde{u}) \in \tilde{\Omega}_{(x_1, \mu), u}$ (open subset of R^{1+r+1}) there exist a neighborhood $\tilde{O}_{((\tilde{x}_1; \tilde{\mu}), \tilde{u})}$ of $((\tilde{x}_1; \tilde{\mu}), \tilde{u})$ and the corresponding $\tilde{O}_{\tilde{x}_2}$ of \tilde{x}_2 (since R^1 is open) such that for each point $((x_1; \mu), u) \in \tilde{O}_{(x_1; \mu), u}$, a unique solution $x_2 \in \tilde{O}_{\tilde{x}_2}$ exists and $x_2 = \varphi_{((\tilde{x}_1; \tilde{\mu}), \tilde{u})}(x_1; \mu, u)$ locally. To find the class of initial conditions (consistency class) as $t = 0$ (for simplicity) given an initial gest $x_1(0)$ for a given μ_0 , and the numerical u_0 and therefore solve $F_2(x_1(0), x_2(0); \mu_0, u_0) = 0$, to get $x_2(0) = \varphi(x_1(0); \mu_0, u_0)$ and therefore set the class of consistency is defined:

$$\mathcal{W}_k = \{(x_1(0), x_2(0); \mu_0, u_0) | (x_1(0); \mu_0, u_0) \in \tilde{\Omega}_{x_1, \mu, u}, x_2(0) = \varphi(x_1(0); \mu_0, u_0), u_0 \in R^1, \mu_0 \in R\} \\ \subset \tilde{\Omega} \subset R^{1+1+1+1}$$

Hence the following reduced order system (equivalent) DAS are found as follows:

$$\dot{x}_1 = F_1(x_1, \varphi(x_1; \mu, u); \mu, u) \quad (17a)$$

$$\text{with } x_2 = \varphi(x_1; \mu, u) \quad (17b)$$

$$\mathcal{W}_k = \{(x_1(0), x_2(0); \mu_0, u_0) | (x_1(0); \mu_0, u_0) \in D \times R, u_0 \in R, \mu_0 \in R, x_2(0) = \varphi(x_1(0); \mu_0, u_0)\} \quad (17c)$$

where F_1 and F_2 are assumed to satisfy the assumption of Problem Formulation-II. Thus, the solvability of Problem Formulation-II is equivalent to solvability of the ROS (17) locally under $\frac{\partial F_2}{\partial x_2}(x_1, x_2; \mu, u) \neq 0$ for $(x_1, x_2; \mu, u) \in \tilde{\Omega}$, and as discussed in suction (2), one can have

Lemma (2) (Locally Lipschitz Property on $D \times R$)

Consider the Problem Formulation-II. Then if $F_1 \in C^1(D \times R \times R; R)$ and $\varphi \in C^2(D \times R; R)$.

Then F_1 is Locally Lipschitz on $D \times R$, with Lipschitz constant $\hat{k} = (B_{x_1} + B_{\varphi} k_{\varphi} + B_u k_u)$.

Proof: The proof is of direct manipulation of lemma (1) in suction 2, with taking into accounts the effect of the control action as a feedback controller.

Now, consider the Problem Formulation-II and equivalent to

$$\frac{dz}{dt} = G(z; \mu) \quad (18a)$$

$$z(0) = z_0 \quad (18b)$$

where $z_0 \in R^{1+1+1}$, $z_0 = (x_{10}, x_{20}; \mu^0, u^0) \in \mathcal{W}_k \subset \tilde{\Omega} \subset R^{1+1+1+1}$ and

$$G(z; \mu, u) = \begin{pmatrix} F_1(x_1, \varphi(x_1; \mu, u); \mu, u) \\ \frac{\partial \varphi}{\partial x_1} F_1(x_1, \varphi(x_1; \mu, u); \mu, u) \end{pmatrix} \in \begin{pmatrix} C^1(D \times R; R) \\ C^2(D \times R; R) \times C^1(D \times R \times R; R) \end{pmatrix}$$

on $x_2 \in \tilde{O}_{x_2}$ and $((\tilde{x}_1; \tilde{\mu}), \tilde{u}) \in \tilde{O}_{(x_1; \mu), u}$, where z is the initial condition variable. Since $F_1(x_1, \varphi(x_1; \mu, u); \mu, u)$ satisfied Lipschitz with constant $(B_{x_1} + B_\varphi k_\varphi + B_u k_u)$ by Lemma (2) and $\frac{\partial \varphi}{\partial x_1}$ is bounded on the compact (convex) set. Since, $\varphi \in C^2(D \times R; R)$ on $x_2 \in \tilde{O}_{x_2}$. Set $x = (x_1, \varphi(x_1; \mu, u); \mu, u)^T = (x_1, x_2; \mu, u)^T$ for $\mu \in R$, $x_2 = \varphi(x_1; \mu, u)$ is the solution of $F_2(x_1, x_2; \mu, u) = 0$. Let

$$\dot{x} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} \dot{x}_1 \\ \frac{\partial \varphi}{\partial x_1} \dot{x}_1 \end{pmatrix} = \begin{pmatrix} F_1(x_1, \varphi(x_1; \mu, u); \mu, u) \\ \frac{\partial \varphi}{\partial x_1} F_1(x_1, \varphi(x_1; \mu, u); \mu, u) \end{pmatrix} = G(x_1, x_2; \mu, u)$$

$$\text{Then, } \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} \triangleq \begin{pmatrix} F_1(x_1, \varphi(x_1; \mu, u); \mu, u) \\ \frac{\partial \varphi}{\partial x_1} F_1(x_1, \varphi(x_1; \mu, u); \mu, u) \end{pmatrix} \quad (19)$$

where $F_1 \in C^1(D \times R \times R; R)$, $F_2 \in C^2(D \times R \times R; R)$ and $\varphi \in C^2(D \times R; R)$. Then

$$\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} x_1(0) \\ \varphi(x_1(0); \mu^0, u^0) \end{pmatrix},$$

$\mu^0 \in R$ and $u^0 \in R$, where $(x_1(0), x_2(0)) \in \mathcal{W}_k$ for $\mu^0 \in R$ and $u^0 \in R$. The system (19) is equivalent to

$$\frac{dz}{dt} = G(z; \mu), \quad G \text{ is defined by (18)}$$

$$z(0) = z_0 \in R^{1+1+1} \quad \text{and } z = (x_1, x_2)^T \text{ for } \mu \in R \text{ and } u \in R,$$

and as discussed earlier the solution is guaranteed uniquely as $z(t, z_0; \mu_0, u_0) \in C^1(H)$, where $H = [-a, a] \times N_\delta(z_0) \times N_\delta(\mu_0) \times N_\delta(u_0)$, where $N_\delta(z_0) \subset R^{1+1+1}$, $N_\delta(\mu_0) \subset R$ and $N_\delta(u_0) \subset R$, where a defined by $0 < a < \min \left[\frac{b}{M_0 + M_1}, \frac{1}{B_{x_1} + k_\varphi B_\varphi + k_u B_u} \right]$, where $\max_{[-a, a]} |v_k(t) - x_{10}| \leq b$, $\{v_k(t)\}$ is a Cauchy sequence of continuous function in $C[-a, a] \Rightarrow v_k(t) \rightarrow v(t)$, v_k convergence to continuous function $v(t)$ uniformly for all $t \in [-a, a]$, which clarifies the differential equations with effecting of the feedback controller u as a sequence of parameter equations:

$$\max_{|x_1 - x_{10}| \leq \frac{\epsilon}{2}} |F_1| = M_0, \quad \max_{|x_1 - x_{10}| \leq \frac{\epsilon}{2}} |DF_1| = M_1, \quad \max_{|x_1 - x_{10}| \leq \frac{\epsilon}{2}} \left| \frac{\partial F_1}{\partial x_1} \right| < B_{x_1}, \quad \max_{|x_1 - x_{10}| \leq \frac{\epsilon}{2}} \left| \frac{\partial F_1}{\partial \varphi} \right| < B_\varphi,$$

$$\max_{|x_1 - x_{10}| \leq \frac{\epsilon}{2}} \left| \frac{\partial F_1}{\partial u} \right| < B_u, \quad \max_{|x_1 - x_{10}| \leq \frac{\epsilon}{2}} \left| \frac{\partial F_1}{\partial \varphi} \right| < k_\varphi \text{ and } \max_{|x_1 - x_{10}| \leq \frac{\epsilon}{2}} \left| \frac{du}{dx_1}(x_1) \right| < k_u.$$

Hence, the problem formulation-II is solvable and has a unique solution defined locally.

5. Bifurcation of Problem Formulation-II

The study of bifurcation of Problem Formulation-II, will consider the same strategy of bifurcation theory in Section 3. Moreover, consider the ROS (16):

$$\dot{x}_1 = F_1(x_1, \varphi(x_1; \mu, u); \mu, u)$$

with

$$x_2 = \varphi(x_1; \mu, u)$$

$$\mathcal{W}_k = \{(x_1(0), x_2(0); \mu_0, u_0) \mid (x_1(0); \mu_0, u_0) \in D \times R, u_0 \in R, \mu_0 \in R \text{ (given)}, x_2(0) = \varphi(x_1(0); \mu_0, u_0)\}$$

where F_1 is a C^r -function on some open set in $R^1 \times R^1 \times R^1 \rightarrow R^3$ and it is sufficient at $r = 5$ from the illustration (1).

Remarks (2) [19]

Consider the Assumption A, with

- 1) The bifurcation is extremely general. The orbit structure near nonhyperbolic equilibrium point

$$\begin{cases} F_1(0,0,0,0) = 0 \\ \frac{\partial F_1}{\partial x_1}(0,0,0,0) = 0 \end{cases} \quad (20)$$

The orbit structure in all cases (the three types of bifurcation, in [19][11]) near $\mu = 0$ was different.

- 2) Knowing that an equilibrium point has zero eigenvalue for $\mu = 0$ is not sufficient to determine the orbit structure.

Definition (2):

The equilibrium point $(x_1, x_2; \mu, u) = (0,0,0,0)$, $x_2 = \varphi(0,0,0) = 0$ of a one-parameter family of the ROS is said to undergo a bifurcation at $\mu = 0$ if the flow for μ near zero and x_1 near zero is not qualitatively the same as the flow near $x_1 = 0$ at $\mu = 0$.

5.1. Saddle-Node Bifurcation (SNB)

To study the SNB of problem formulation-II

$$\dot{x}_1 = F_1(x_1, \varphi(x_1; \mu, u); \mu, u) \quad (21a)$$

$$x_2 = \varphi(x_1; \mu, u) \quad (21b)$$

$$\mathcal{W}_k = \{(x_1(0), x_2(0); \mu_0, u_0) | (x_1(0); \mu_0, u_0) \in D \times R, x_2(0) = \varphi(x_1(0); \mu_0, u_0), \text{ for } \mu_0 \in R, u_0 \in R\} \quad (21c)$$

where $F_1 \in C^1(D \times R \times R; R)$, $\varphi \in C^2(D \times R; R)$ are continuously differentiable with $(x_1; \mu) \in D$, $\mu \in R$, $x_2 \in R$, $u \in R$ and $D \subset R^{1+1}$ is an open subset. The derivation of conditions with control will undergo SNB as in the following theorem.

Theorem (2)

Consider problem formulation-II with

$$1) \left[\frac{\partial F_1}{\partial x_1} + \frac{\partial F_1}{\partial \varphi} \frac{\partial \varphi}{\partial x_1} + \frac{\partial F_1}{\partial \varphi} \frac{\partial \varphi}{\partial u} \frac{du}{dx_1} + \frac{\partial F_1}{\partial u} \frac{du}{dx_1} \right] = 0 \quad \text{at} \quad (0, \varphi(x_1(0); \mu(0), u(0)); \mu(0), u(0)) = (0, \varphi(0,0,0), 0,0) = (0,0,0,0)$$

$$2) \frac{\partial F_1}{\partial \varphi} \frac{\partial \varphi}{\partial \mu} + \frac{\partial F_1}{\partial \mu} \neq 0 \text{ at } (0,0,0,0).$$

$$3) \left\{ F_{1x_1x_1}^0 + 3 F_{1x_1\varphi}^0 (2\varphi_{x_1}^0 u_{x_1}^0 + \varphi_{x_1}^0) + 2F_{1x_1u}^0 u_{x_1}^0 + F_{1\varphi u}^0 [\varphi_{x_1}^0 u_{x_1}^0 + \varphi_u^0 (u_{x_1}^0)^2] + F_{1\varphi}^0 [\varphi_u^0 u_{x_1x_1}^0 + u_{x_1}^0 (\varphi_{u x_1}^0 + \varphi_{uu}^0 u_{x_1}^0)] + (F_{1\varphi}^0 + F_{1\varphi\varphi}^0 \varphi_{x_1}^0 + F_{1\varphi\varphi}^0 \varphi_u^0 u_{x_1}^0 + F_{1u\varphi}^0 u_{x_1}^0)(\varphi_{x_1}^0 + \varphi_u^0 u_{x_1}^0) + F_{1u}^0 u_{x_1x_1}^0 \right\} \neq 0 \text{ at } (0,0,0,0).$$

- 4) The curve $u = u(x_1)$ is used smoothly such that (1) – (3) are satisfied (in such a way).

Then, the saddle-node bifurcation occurs.

Proof: Consider $F_1 = (x_1, \varphi(x_1; \mu(x_1), u(x_1)); \mu(x_1), u(x_1))$ along the curve of equilibrium points

$\mu = \mu(x_1)$. By IFT and the condition $\frac{\partial F_1}{\partial \mu}(0,0,0,0) \neq 0$, this implies that, there is $\mu = \mu(x_1)$ with

$\mu(0) = 0$ and

$$F_1 = (x_1, \varphi(x_1; \mu(x_1), u(x_1)); \mu(x_1), u(x_1)) = 0$$

Differentiating F_1 with respect to x_1 :

$$\begin{aligned} & \frac{\partial F_1}{\partial x_1}(x_1, \varphi(x_1; \mu(x_1), u(x_1)); \mu(x_1), u(x_1)) = 0 \\ \Rightarrow & \frac{\partial F_1}{\partial x_1} + \frac{\partial F_1}{\partial \varphi} \frac{\partial \varphi}{\partial x_1} + \frac{\partial F_1}{\partial \varphi} \frac{\partial \varphi}{\partial \mu} \frac{d\mu}{dx_1} + \frac{\partial F_1}{\partial \varphi} \frac{\partial \varphi}{\partial u} \frac{du}{dx_1} + \frac{\partial F_1}{\partial \mu} \frac{d\mu}{dx_1} + \frac{\partial F_1}{\partial u} \frac{du}{dx_1} = 0 \end{aligned} \quad (22)$$

Evaluating at $(x_1, \varphi(x_1; \mu(x_1), u(x_1)); \mu(x_1), u(x_1)) = (0,0,0,0)$ and let $F_1(0,0,0, u_0) \triangleq F_1^0$ for. Therefore

$$F_{1x_1}^0 + F_{1\varphi}^0 \varphi_{x_1}^0 + F_{1\varphi}^0 \varphi_{\mu}^0 \mu_{x_1}^0 + F_{1\varphi}^0 \varphi_u^0 u_{x_1}^0 + F_{1\mu}^0 \mu_{x_1}^0 + F_{1u}^0 u_{x_1}^0 = 0$$

From (22) with conditions (1) and (2) one can get

$$\frac{d\mu^0}{dx_1} = \frac{-[F_{1x_1}^0 + F_{1\varphi}^0 \varphi_{x_1}^0 + F_{1\varphi}^0 \varphi_{\mu}^0 \mu_{x_1}^0 + F_{1\varphi}^0 \varphi_u^0 u_{x_1}^0 + F_{1\mu}^0 \mu_{x_1}^0 + F_{1u}^0 u_{x_1}^0]}{F_{1\varphi}^0 \varphi_{\mu}^0 + F_{1\mu}^0} = 0 \quad (23)$$

Thus, the curve of equilibrium points is tangent to the line $\mu = 0$ at $x_1 = 0$.

Next, the second derivative of F_1 with respect to x_1 is:

$$\left\{ F_{1x_1x_1}^0 + 3 F_{1x_1\varphi}^0 (2\varphi_{x_1}^0 u_{x_1}^0 + \varphi_{x_1}^0) + 2F_{1x_1u}^0 u_{x_1}^0 + F_{1\varphi u}^0 [\varphi_{x_1}^0 u_{x_1}^0 + \varphi_u^0 (u_{x_1}^0)^2] + F_{1\varphi}^0 [\varphi_{\mu}^0 u_{x_1}^0 + u_{x_1}^0 (\varphi_{\mu}^0 \mu_{x_1}^0 + \varphi_{uu}^0 u_{x_1}^0)] + (F_{1\varphi}^0 + F_{1\varphi\varphi}^0 \varphi_{x_1}^0 + F_{1\varphi\varphi}^0 \varphi_{\mu}^0 \mu_{x_1}^0 + F_{1\varphi\varphi}^0 \varphi_u^0 u_{x_1}^0 + F_{1u\varphi}^0 u_{x_1}^0)(\varphi_{x_1}^0 + \varphi_u^0 u_{x_1}^0) + F_{1u}^0 u_{x_1x_1}^0 \right\} \neq 0 \quad (24)$$

By continuity property get:

$$F_{1\varphi x_1}^0 + F_{1x_1\varphi}^0 + F_{1\varphi x_1}^0 = 3F_{1x_1\varphi}^0, F_{1x_1u}^0 + F_{1ux_1}^0 = 2F_{1x_1u}^0 \text{ and } F_{1u\varphi}^0 + F_{1\varphi u}^0 = 2F_{1\varphi u}^0.$$

Therefore by conditions (2) and (3) one have that

$$\frac{d^2\mu(0)}{dx_1^2} = \frac{-\left\{ F_{1x_1x_1}^0 + 3 F_{1x_1\varphi}^0 (2\varphi_{x_1}^0 u_{x_1}^0 + \varphi_{x_1}^0) + 2F_{1x_1u}^0 u_{x_1}^0 + F_{1\varphi u}^0 [\varphi_{x_1}^0 u_{x_1}^0 + \varphi_u^0 (u_{x_1}^0)^2] + F_{1\varphi}^0 [\varphi_{\mu}^0 u_{x_1}^0 + u_{x_1}^0 (\varphi_{\mu}^0 \mu_{x_1}^0 + \varphi_{uu}^0 u_{x_1}^0)] \right\}}{F_{1\varphi}^0 \varphi_{\mu}^0 + F_{1\mu}^0} \neq 0 \quad (25)$$

Therefore, from (23) and (25) with the effecting of the control action as a feedback controller, and by definition (2), there are two curves of the equilibrium points passing through the bifurcation point in $\mu - x_1$ plane one is stable and the other is not. Therefore, the saddle-node bifurcation occurs. Moreover, one can use $u(x_1)$ in such a way (See the following example) to ensure the saddle-node bifurcation requirements.

Moreover, the sign of (25) determine on which side of $\mu = 0$ the curve lies. As the bifurcation diagram is shown in (subsection 3, Fig. (1)):

Illustration (2):

Consider the semi-explicit DAS with control:

$$\dot{x}_1 = \mu - \frac{1}{4}x_1^2 - \frac{x_2}{\cos(x_1)} - \frac{1}{2}u(x_1) \quad (26a)$$

$$0 = x_1^2 + 4x_2 \sec(x_1) + 2u(x_1) \quad (26b)$$

where $x_1 \in R, x_2 \in R, \mu \in R$ is the parameter and $u \in R$ is feedback controller.

Step (1): The domain of (26) is $x_1 \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

Step (2): The point (0,0,0,0) is one of equilibrium points of (26).

Step (3): Applying the IFT we have that $\frac{\partial F_2}{\partial x_2} = 4 \sec(x_1)|_{(0,0,0,u_0)} = 4 \sec(0) = 4(1) = 4 \neq 0$ for

$x_1 \in (-\frac{\pi}{2}, \frac{\pi}{2})$. That is $\frac{4}{\cos(x_1)}$ must be non-zero, which will happen if we let $x_1 \in (-\frac{\pi}{2}, \frac{\pi}{2})$. So on

the neighborhood of (0) $\in (-\frac{\pi}{2}, \frac{\pi}{2})$, $\frac{\partial F_2}{\partial x_2} \neq 0$. Then, there exist $\varphi(x_1; \mu, u)$ such that

$x_2 = \varphi(x_1; \mu, u)$ which is continuously differentiable. Then

$$x_2 = \frac{-x_1^2 - 2u(x_1)}{4 \sec(x_1)} \equiv \varphi(x_1; \mu, u) \quad (27)$$

Substituting (27) in (26a) will get

$$\begin{aligned} \dot{x}_1 &= \mu - \frac{1}{4}x_1^2 + \left[\frac{-x_1^2 - 2u(x_1)}{4 \sec(x_1)} \right] \frac{1}{\cos(x_1)} - \frac{1}{2}u(x_1) = \mu - \frac{1}{4}x_1^2 + \left[\frac{-x_1^2 - 2u(x_1)}{4 \sec(x_1) \cos(x_1)} \right] - \frac{1}{2}u(x_1) \\ &= \mu - \frac{1}{4}x_1^2 + \left[\frac{-x_1^2 - 2u(x_1)}{4 \cos(x_1)} \right] - \frac{1}{2}u(x_1) = \mu - \frac{1}{4}x_1^2 - \frac{1}{4}x_1^2 - \frac{1}{2}u(x_1) - \frac{1}{2}u(x_1) \end{aligned}$$

$$= \mu - \frac{1}{2}x_1^2 - u(x_1) \quad (28)$$

The equation (28) is the ROS of (26).

Step (4): Drive the conditions:

$$F_1 = \mu - \frac{1}{4}x_1^2 + \varphi \frac{1}{\cos(x_1)} - \frac{1}{2}u(x_1) = \mu - \frac{1}{2}x_1^2 - u(x_1) \text{ with } \varphi = -\frac{x_1^2+2u(x_1)}{4\sec(x_1)}$$

Then, the derivatives are:

$$F_{1x_1} = -\frac{1}{2}x_1 - \frac{du}{dx_1} \Rightarrow F_{1x_1}^0 = -\frac{du}{dx_1}, F_{1\varphi}^0 = \frac{1}{\cos(0)} = \frac{1}{1} = 1, F_{1u}^0 = -1$$

$$\varphi_{x_1} = -\frac{4\sec(x_1)(2x_1+2\frac{du}{dx_1})+4\sec(x_1)\tan(x_1)(x_1^2+2u(x_1))}{(4\sec(x_1))^2}$$

$$\varphi_{x_1}^0 = -\frac{8\frac{du}{dx_1}-0}{16} = -\frac{1}{2}\frac{du}{dx_1} \text{ and } \varphi_u^0 = -\frac{4\sec(0)(2)}{(4\sec(0))^2} = -\frac{8}{16} = -\frac{1}{2}$$

$$[F_{1x_1}^0 + F_{1\varphi}^0 \varphi_{x_1}^0 + F_{1\varphi}^0 \varphi_u^0 u_{x_1}^0 + F_{1u}^0 u_{x_1}^0] = \left[\frac{du}{dx_1} + \frac{1}{2}\frac{du}{dx_1} + \frac{1}{2}\frac{du}{dx_1} + \frac{du}{dx_1} \right] = 3\frac{du}{dx_1} \quad (29)$$

$$\varphi_\mu^0 = 0, F_{1\mu}^0 = 1 \Rightarrow [F_{1\varphi}^0 \varphi_\mu^0 + F_{1\mu}^0] = 1 \neq 0 \quad (30)$$

$$F_{1x_1x_1}^0 = -\frac{1}{2} - \frac{d^2u}{dx_1^2}, F_{1x_1u}^0 = \frac{d}{du}\left(-\frac{du}{dx_1}\right), F_{1x_1\varphi}^0 = 0, F_{1\varphi u}^0 = 0, F_{1\varphi\varphi}^0 = 0, F_{1u\varphi}^0 = 0, \varphi_{uu}^0 = 0 \text{ and}$$

$$\varphi_{ux_1}^0 = \frac{d}{dx_1}\left(\frac{4\sec(x_1)}{(4\sec(x_1))^2}\right) = \frac{64\sec^2(x_1)\sec(x_1)\tan(x_1)-8(16)\sec(x_1)\sec(x_1)(\sec(x_1)\tan(x_1))}{(4\sec(x_1))^4} = 0$$

$$\begin{aligned} & - \left\{ F_{1x_1x_1}^0 + 3 F_{1x_1\varphi}^0 (2\varphi_{x_1}^0 u_{x_1}^0 + \varphi_{x_1}^0) + 2 F_{1x_1u}^0 u_{x_1}^0 \right. \\ & \quad \left. + F_{1\varphi u}^0 [\varphi_{x_1}^0 u_{x_1}^0 + \varphi_u^0 (u_{x_1}^0)^2] + F_{1\varphi}^0 [\varphi_u^0 u_{x_1x_1}^0 + u_{x_1}^0 (\varphi_{ux_1}^0 + \varphi_{uu}^0 u_{x_1}^0)] \right. \\ & \quad \left. + (F_{1\varphi}^0 + F_{1\varphi\varphi}^0 \varphi_{x_1}^0 + F_{1\varphi\varphi}^0 \varphi_u^0 u_{x_1}^0 + F_{1u\varphi}^0 u_{x_1}^0)(\varphi_{x_1}^0 + \varphi_u^0 u_{x_1}^0) + F_{1u}^0 u_{x_1x_1}^0 \right\} \\ & = \left[\left(\frac{1}{2}\right) + \frac{d^2u}{dx_1^2} + (2)\frac{d}{du}\left(\frac{du}{dx_1}\right) + \left(\frac{1}{2}\right)\frac{d^2u}{dx_1^2} + \frac{du}{dx_1} + \frac{d^2u}{dx_1^2} \right] = \frac{1}{2} + \frac{5}{2}\frac{d^2u}{dx_1^2} + \frac{du}{dx_1} + 2\frac{d}{du}\left(\frac{du}{dx_1}\right) \end{aligned} \quad (31)$$

Step (5): From the conditions (29), (30) and (31) observe that;

if choose $u(x_1)$ as follows:

$$u(x_1) = \frac{1}{2}x_1^2 \quad (32)$$

$$\text{Then from (29) get: } -3\frac{du}{dx_1} = -3x_1 \text{ at } (0,0,0,0) \Rightarrow -3\frac{du}{dx_1} = 0$$

Clearly, (30) is already satisfied, and from (31)

$$\frac{1}{2} + \frac{5}{2}(1) + x_1 + 2\frac{d}{du}(x_1) = \frac{1}{2} + \frac{5}{2}(1) + x_1 + 0 \text{ at } (0,0,0,0) \Rightarrow \frac{1}{2} + \frac{5}{2} = \frac{6}{2} = 3 \neq 0 \quad (33)$$

Therefore, get that:

$$F_1 = \mu - x_1^2 \quad (34)$$

$$\text{Step (6): } \mu_{x_1}^0 = \frac{0}{1} = 0 \quad (35)$$

Thus the curve of the equilibrium point (0,0,0,0) is tangent to the line $\mu = 0$ at $x_1 = 0$, and

$$\text{Step (7): } \mu_{x_1x_1}^0 = \frac{3}{1} = 3 \neq 0 \quad (36)$$

The conditions decide that the curve lie locally on one side of $\mu = 0$ and the sign of (36) determine on which side of $\mu = 0$ the curve lies. The ROS (34) characterizes the SNB where μ represents the bifurcation parameter.

Therefore, the qualitative bifurcation of the system (34) has been changed around $\mu = 0$ ($\mu < 0$, $\mu = 0$ and $\mu > 0$) with effect of the feedback controller $u(x_1) = \frac{1}{2}x_1^2$ as shown in Figure (5) as below.

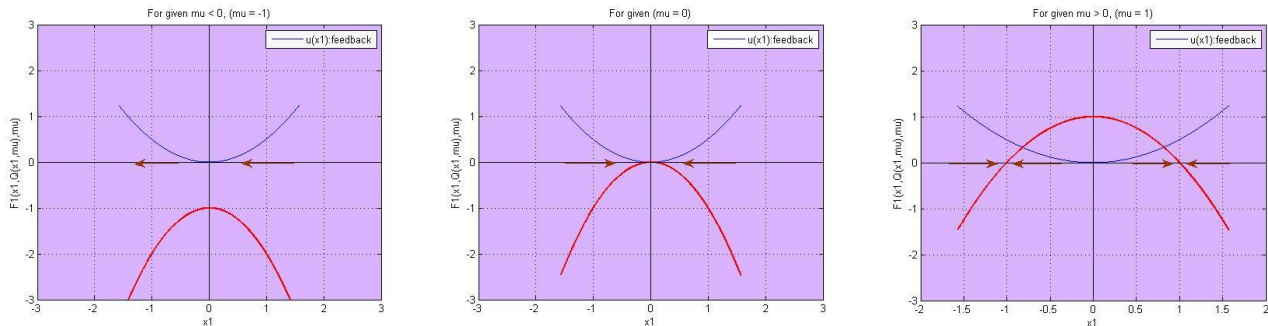


Figure (5): The function $F_1(x_1, x_2; \mu) = \mu - \frac{1}{2}x_1^2 - u(x_1)$ versus x_1 , where $x_2 = \varphi(x_1; \mu) = \frac{-x_1^2 - 2u(x_1)}{4 \sec(x_1)}$ for the values $(\mu = -1, \mu = 0 \text{ and } \mu = 1)$ with $u(x_1) = \frac{1}{2}x_1^2$.

In the figure (5) the arrows along the vertical lines represent the flow generated by (34) along the x_1 -direction. Thus, for $\mu < 0$, (34) has no equilibrium points, and the vector field is decreasing in x_1 .

For $\mu > 0$, (34) has two equilibrium points. A simple linear stability analysis shows that one of the equilibrium points is stable (the solid branch of the parabola), and the other fixed point is unstable (represented by the broken branch of the parabola). So, given a C^r ($r \geq 1$) vector field on R^1 having only two equilibrium points, one must be stable and the other unstable and the parameter value $\mu = 0$ as a bifurcation value.

This particular type of bifurcation (i.e., where on one side of a parameter value there are no fixed points and on the other side there are two fixed points at $x_1 = \pm\sqrt{\mu}$) is referred to as a SNB.

By the above results, the system (34) had the same directional derivatives as well as the bifurcation diagram of the case of SNB without control in the previous chapter.

6. Acknowledgments

Praise be to Allah for what he has bestowed, and he has praise for what he has given, peace and prayer on the brim of his prophet and his divine family, and the perpetual curse of life on their enemies and those who deny their virtues. I would like to express my deep thanks to my supervisor Asst. Prof. Dr. Radhi Ali Zaboon for this invaluable guidance and continuous help throughout the period of research and during the work of this thesis. His insight, patience, backing and encouragement were essential to the successful completion of this thesis. I am sincerely grateful to the deanship of Collage of Science in Mustansiriyah University and headship of mathematics department with all staff members, and to my colleagues, the students of higher studies, for their help and encouragement. Finally, I would like to thank all of those who helped me and gave me their blessing and prayers to finish this study.

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