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Fuzzy Topological Spectrum of a KU-Algebra

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Abstract. In the present paper, discuss the concept of fuzzy topological spectrum of a bounded commutative KU-algebra and study some of the characteristics of this topology. Also, we show that the fuzzy topological spectrum of this structure is compact and T_1-space.

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1. Introduction

Iseki [2] introduced BCK-algebras. The topology on M э $\mathrm{M}=\{\text{all fuzzy prime ideal of a commutative BCK-algebra}\}$ was introduced by [1]. A new algebraic structure which is a KU-algebra was introduced by [5]. In [4] a topology on the set of all prime ideals of a commutative KU-algebra was studied. Now, the purpose of this paper is to define the concept of fuzzy spectrum and discusses some properties of this topological space over a bounded commutative KU-algebra.

2. Preliminaries:

Definition 2.1 [3,5]. Let $\mathrm{M} \neq \phi$ with a binary operation $\bullet$ and a constant 0, then $(\mathrm{M},\bullet,0)$ is a KU - algebra, if

(ku_1) $(s\bullet w)\bullet[(w\bullet z))\bullet(s\bullet z)]=0$,

(ku_2) $s\bullet 0=0$,

(ku_3) $0\bullet s=s$,

(ku_4) $s\bullet w=0$ and $w\bullet s=0$ implies $s=w$,

(ku_5) $s\bullet s=0$,

For all $s,w,z\in \mathrm{M}$.

On a KU-algebra $(\mathrm{M},\bullet,0)$, a binary relation $\leq$ on M define by:

$u\leq v \Leftrightarrow v\bullet u=0$. Therefore $(\mathrm{M},\leq)$ is a partially ordered set also 0 is its smallest element. Hence $(\mathrm{M},\bullet,0)$ It meets the conditions as it comes: $\forall s,w,z\in \mathrm{M}$

$(ku_{1'})$: $(w*z)*(s*z)\leq(s*w)$

$(ku_{2'})$: $0\leq s$

$(ku_{3'})$: $s\leq w, w\leq s$ implies $s=w$,

$(ku_{4'})$: $w\bullet s\leq s$.

Theorem 2.2 [3]: Let $(\mathrm{M},\bullet,0)$ be a KU-algebra, the following axioms are satisfied:

$\forall s,w,z\in \mathrm{M}$,

(1): $s\leq w$ imply $w\bullet z\leq s\bullet z$,

(2): $s\bullet(w\bullet z)=w\bullet(s\bullet z)$, $\forall s,w,z\in \mathrm{M}$,

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(3): $((w \bullet s) \bullet s) \leq w$.

Definition 2.3 [4]: A KU-algebra M is KU-commutative if it satisfies: $(w \bullet s) \bullet s = (s \bullet w) \bullet w$, $\forall s, w$ in M. For all s, w
$(w * s) * s$ is denoted by $s \dot{\wedge} w$.

Definition 2.4 [4]: A KU-algebra M is bounded if $\exists e \in \mathrm{M}$ satisfying $u \leq e \ \forall u \in \mathrm{M}$. In this case, $\forall u \in \mathrm{M} \ \ u * e$ is denoted by N_u.

Definition 2.5 [3]: A subset J of a KU-algebra $(\mathrm{M}, \bullet, 0)$ is a KU- ideal of M, if
(I_1) $0 \in J$
(I_2) $\forall s, w, z \in \mathrm{M}$, if $s * (w * z) \in J$ and $w \in J$, imply $s * z \in J$.

Theorem 2.6 [4]: Any bounded commutative KU- algebra M with respect to $(\mathrm{M}, \leq)$ is a KU- lattice.

Definition 2.7 [4]: Let M be a KU-lattice and $\Pr$ a proper ideal of M. Then $\Pr$ is a prime if $u \dot{\wedge} v \in \Pr$ implies $u \in \Pr$ or $v \in \Pr$ $\forall u, v$ in M.

Theorem 2.8 [4]: In a KU-lattice M, a proper ideal $\Pr$ of M is a prime if $A \bigcap B \subseteq \Pr$ implies $A \subseteq \Pr$ or $B \subseteq \Pr$ $\forall A, B$ ideals in M.

Lemma 2.9 [4]: Every maximal KU-ideal is prime.

Lemma 2.10: Every KU-ideal is contained in a maximal KU-ideal.

Proof. Straightforward

Definition 2.11 [6]: A fuzzy subset σ of M is a function $o : \mathrm{M} \to [0,1]$. Let σ and η be fuzzy sets in a set fuzzy sets, we define:
(1) $\sigma = \eta \Leftrightarrow (\forall s \in \mathrm{M})(\sigma(s) = \eta(s))$,
(2) $\sigma \subseteq \eta \Leftrightarrow (\forall s \in \mathrm{M})(\sigma(s) \leq \eta(s))$.
The union is defined by $(\sigma \bigcup \eta)(s) = \max\{\sigma(s), \eta(s)\}, \forall s \in \mathrm{M}$.
The intersection is defined by $(\sigma \bigcap \eta)(s) = \min\{\sigma(s), \eta(s)\}, \forall u \in \mathrm{M}$.
More generally, for a family of fuzzy sets $\{\sigma_j : j \in \Lambda\}$ in a set M, the union and the intersection are defined by
$(\bigcup_{j \in \Lambda} \sigma_j)(s) = \sup_{j \in \Lambda} \sigma_j(s)$, $(\bigcap_{j \in \Lambda} \sigma_j)(s) = \inf_{j \in \Lambda} \sigma_j(s)$, respectively $\forall s \in \mathrm{M}$.
M is a KU- lattice unless otherwise indicated.

<u>**Definition 2.12:**</u> For any fuzzy subsets μ and η in M. $\mu\eta$ is defined as

$$\mu\eta(u) = \sup_{u = v \dot{\wedge} w} \{\min(\mu(v), \eta(w))\}, \ \forall u, v, w \in \mathrm{M}.$$

Definition 2.13: Let $u \in \mathrm{M}$, then u_t the fuzzy point of M is a fuzzy subset of M, which is defined by $u_t(v) = \begin{cases} t & u = v \\ 0 & u \neq v. \end{cases}$, where $t \in (0,1]$.

Definition 2.14: Let μ be a fuzzy subset of M, if $\mu(u) = 0 \ \ \forall u \in \mathrm{M}$, then μ is called empty fuzzy set denoted by Φ.

Definition 2.15: If μ is a fuzzy subset of M and u_t is a fuzzy point of M. For $t \in [0,1]$, the set $\mu_t = \{u \in \mathrm{M} : \mu(u) \geq t\}$ is a level subset of μ and if $u \in \mu_t$ then $u_t \subseteq \mu$.

Definition 2.16 [3]: A fuzzy set σ of a KU-algebra M is fuzzy KU-ideal if
(F_1) $\sigma(0) \geq \sigma(s), \forall s \in \mathrm{M}$.

(F_2) $\sigma(s \bullet z) \geq \min\{\sigma(s \bullet (wz)), \sigma(w)\}, \forall s, w, z \in \mathrm{M}$.

Lemma 2.17 [3]: Let μ be fuzzy KU-ideal of M. $\forall u, v \in \mathrm{M}$, if $u \leq v$, then $\mu(u) \geq \mu(v)$.

Definition 2.18: The fuzzy KU-ideal generated by μ (μ is a fuzzy subset of M) which is denoted by $\langle\mu\rangle$ is the intersection of all fuzzy KU-ideals η of M . (i.e)

$\langle\mu\rangle = \bigcap\{\eta : \mu \subseteq \eta, \eta$ is fuzzy KU-ideal of $\mathrm{M}\}$.

Obviously, we get $\mu \subseteq \langle\mu\rangle$, and if μ is fuzzy KU-ideal of M, then $\mu = \langle\mu\rangle$.

Lemma 2.19: If λ, η are two fuzzy KU-ideal of M, then $\lambda\eta = \lambda \bigcap \eta$.

Proof. Let $u \in \mathrm{M}$, $u = b \dot{\wedge} c$ and λ, η be fuzzy KU-ideals. Since $b \dot{\wedge} c \leq b$ So,
$\lambda(b) \leq \lambda(b \dot{\wedge} c) = \lambda(u)$ and $\eta(c) \leq \eta(b \dot{\wedge} c) = \eta(u)$ by Lemma 2.17. Hence,
$\min\{\lambda(b), \eta(c)\} \leq (\lambda \bigcap \eta)(u)$. Therefore, $\lambda\eta \leq (\lambda \bigcap \eta)(u)$ or equivalently $\lambda\eta \subseteq \lambda \bigcap \eta$.
Conversely, $\lambda\eta(u) = \sup_{u=v\dot{\wedge}w}\{\min\{\lambda(v), \eta(w)\}\} \geq \min\{\lambda(u), \eta(u)\} = (\lambda \bigcap \eta)(u)$.
So $\lambda\eta \supseteq \lambda \bigcap \eta$. Thus $\lambda\eta = \lambda \bigcap \eta$.

Corollary 2.20: If λ, η are two fuzzy KU-ideal of M, then $\lambda\eta$ is a fuzzy KU-ideal of M.

Lemma 2.21: If λ, η are two fuzzy KU-ideal of M and $\lambda\eta$ is a fuzzy KU-ideal of M. Then $\lambda\eta \subseteq \eta, \lambda\eta \subseteq \nu$.

Proof. If η is a fuzzy KU-ideal of M. If $s = w \dot{\wedge} z$, then $\eta(z) \leq \eta(s)$ by Lemma2.17
Hence,

$$\lambda\eta(s) = \sup_{s=w\dot{\wedge}z}\{\min\{\lambda(w), \eta(z)\}\} \leq \sup_{s=w\dot{\wedge}z}\{\eta(z)\} \leq \eta(s).$$

Therefore $\lambda\eta \subseteq \eta$.

Definition2.22: A non-constant fuzzy KU-ideal σ of a bounded commutative KU-algebra M is called prime if $\sigma(s \dot{\wedge} w) \leq \max\{\sigma(s), \sigma(w)\}$ $\forall s, w \in \mathrm{M}$.

Definition 2.23: A non-constant fuzzy KU-ideal μ of a bounded commutative KU-algebra M is called prime if, for all fuzzy KU-ideals λ and σ such that $\lambda\sigma \subseteq \mu$, then either $\lambda \subseteq \mu$ or $\sigma \subseteq \mu$.

Lemma 2.24: Let σ be a fuzzy prime KU-ideal of a commutative KU-algebra M. Then $\sigma_* = \{s \in \mathrm{M} : \sigma(s) = \sigma(0)\}$ is a prime KU-ideal of M.

Proof. Let $s, w \in \mathrm{M}$ be such that $\sigma(0) = \sigma(s \dot{\wedge} w) \leq \max\{\sigma(s), \sigma(w)\} = \sigma(s)$ or $\sigma(w)$. It follows from (F_1) that $\sigma(s) = \sigma(0)$ or $\sigma(w) = \sigma(0)$. Hence $s \in \sigma_*$ or $w \in \sigma_*$. Therefore σ_* is a prime KU-ideal of M.

Corollary 2.25. Let μ be a fuzzy KU-ideal of a commutative KU-algebra M, then $pr = \{u \in \mathrm{M} : \mu(u) = 1\}$ is either $pr = \Phi$ or a prime ideal of M.

3. Fuzzy spectrum:

In this section we define a topology on $spec(\mu)$ and we give some properties about the induced space, we prove this space is T_1-space and we can define a nontrivial base for this topology.

Definition 3.1: If M is a bounded commutative KU-algebra and μ is a fuzzy set of M. The set L of all prime fuzzy KU-ideals of σ is spectrum of σ and denoted by $spec(\sigma)$.
(i.e.) $L = spec(\sigma) = \{\rho : \rho$ is fuzzy prime KU-ideal of $\sigma\}$. And for each proper fuzzy KU-ideal λ of σ, let $D(\lambda) = \{\rho \in spec(\sigma) : \lambda \subseteq \rho\}$.

Notation 3.2: $L(\lambda) = L \setminus D(\lambda) = \{\rho \in spec(\sigma) : \lambda \not\subset \rho\}$, $\ni L \setminus D(\lambda)$ is the complement of $O(\lambda)$ in K.

Clearly $D(\langle\lambda\rangle) = D(\lambda)$, for all fuzzy subset of σ.

Theorem 3.3: If M is a bounded KU-commutative algebra and σ is a fuzzy set of M, $\tau = \{L(\lambda) : \lambda$ is a fuzzy KU-ideal of $\sigma\}$. Then the pair (L,τ) is a topological space, it is called fuzzy spectrum of σ.

Proof. $L(\sigma) = Spec(\sigma) = L$ and $L(\Phi) = \phi$, where Φ is the fuzzy empty set. Thus $\mathrm{K}, \phi \in \tau$.
Now, prove that $\boldsymbol{\tau}$ is closed under finite intersection.
If λ and $\boldsymbol{\eta}$ are two fuzzy ideals of $\boldsymbol{\mu}$. We claim that $D(\lambda) \bigcup D(\eta) = D(\lambda\eta)$. Let $\mu \in D(\lambda\eta)$. Then $\lambda\eta \subseteq \mu$. Since $\mu \in L$, we have $\lambda \subseteq \mu$ or $\eta \subseteq \mu$. Hence $\mu \in D(\lambda) \bigcup D(\eta)$.
Conversely, if $\mu \in D(\lambda) \bigcup D(\eta)$, then $\mu \in D(\lambda)$ or $\mu \in D(\eta)$.
$\lambda\eta = \lambda \bigcap \eta \subseteq \lambda$ and $\lambda\eta = \lambda \bigcap \eta \subseteq \eta$. Thus, $\lambda\eta \subseteq \mu$ and so $\mu \in D(\lambda\eta)$. It follows that $D(\lambda) \bigcup D(\eta) \subseteq D(\lambda\eta)$. We get $D(\lambda) \bigcup D(\eta) = D(\lambda\eta)$, are equivalently,
$L(\lambda) \bigcap L(\eta) = (L \setminus D(\lambda)) \bigcap (L \setminus D(\eta)) = (L \setminus (D(\lambda) \bigcup D(\eta)) = (L \setminus D(\lambda\eta)) = L(\lambda\eta)$. By Corollary (2.18), $\lambda\eta$ is fuzzy ideal and so $L(\lambda) \bigcap L(\eta) = L(\lambda\eta) \in \tau$.
Finally, if $\{\lambda_j : j \in \Lambda\}$ is a collection of fuzzy ideals of M. Now, prove that $\bigcap_{j\in\Lambda} D(\lambda_j) = D(\bigcup_{j\in\Lambda} \lambda_j)$.
Let $\rho \in \bigcap_{j\in\Lambda} D(\lambda_j)$, then for any $j \in \Lambda$, $\rho \in D(\lambda_j)$ and so $\lambda_j \subseteq \rho$. Hence, $\bigcup_{j\in\Lambda} \lambda_j \subseteq \rho$ and thus $\rho \in D(\bigcup_{j\in\Lambda} \lambda_j)$. Conversely, let $\rho \in D(\bigcup_{j\in\Lambda} \lambda_j)$, then $\bigcup_{j\in\Lambda} \lambda_j \subseteq \rho$. Thus, for any $j \in \Lambda$, $\lambda_j \subseteq \bigcup_{j\in\Lambda} \lambda_i \subseteq \rho$. Hence $\rho \in D(\lambda_j)$ for all $j \in \Lambda$ and so $\rho \in \bigcap_{j\in\Lambda} D(\lambda_j)$.
This shows that $\bigcap_{j\in\Lambda} D(\lambda_j) = D(\bigcup_{j\in\Lambda} \lambda_j)$. By Notation (3.2), we get $D(\bigcup_{j\in\Lambda} \lambda_j) = D(\langle\bigcup_{j\in\Lambda} \lambda_j\rangle)$ and so $\bigcap_{j\in\Lambda} D(\lambda_j) = D(\langle\bigcup_{j\in\Lambda} \lambda_j\rangle)$.
Furthermore, we get
$\bigcup_{j\in\Lambda} L(\lambda_j) = \bigcup_{j\in\Lambda} (L \setminus D(\lambda_j)) = L \setminus \bigcap_{j\in\Lambda} D(\lambda_j) = L \setminus D(\langle\bigcup_{j\in\Lambda} \lambda_j\rangle) = L(\langle\bigcup_{j\in\Lambda} \lambda_j\rangle) \in \tau$.
Hence (L,τ) is a topological space.

Theorem 3.4: The family $\Psi = \{L(u_t) : u \in \mathrm{M}, t \in (0,1]\}$ of $\boldsymbol{\tau}$ is a base for $\boldsymbol{\tau}$. Where u_t is a fuzzy point of M.

Proof. $\forall u \in \mathrm{M}$, $t \in (0,1]$, $L(u_t) = L(\langle u_t\rangle)$ and so $L(u_t) \in \tau$.
We will prove that Ψ is a base for $\boldsymbol{\tau}$. Let $L(\lambda) \in \tau$, $L(\lambda) = L \setminus D(\lambda)$ and $D(\lambda) = \bigcap_{u_t \subseteq \lambda} D(u_t)$ (since $D(\bigcup_{j\in\Lambda} \lambda_j) = \bigcap_{j\in\Lambda} D(\lambda_j)$). So $L(\lambda) = L \setminus \bigcap_{u_t \subseteq \lambda} D(u_t)$. Hence $L(\lambda) = \bigcup_{u_t \subseteq \lambda} L(u_t)$, therefore Ψ is a base for $\boldsymbol{\tau}$.

Theorem 3.5: The topological space $L = Spec(\sigma)$ is a T_1 space.

Proof. Let $\rho_1, \rho_2 \in L$ and $\rho_1 \neq \rho_2$. Then, either $\rho_1 \not\subset \rho_2$ or $\rho_2 \not\subset \rho_1$.
If $\rho_1 \not\subset \rho_2$, then $\rho_2 \notin D(\rho_1)$ but $\rho_1 \in D(\rho_1)$. Moreover, $\rho_2 \in L(\rho_1)$ but $\rho_1 \notin L(\rho_1)$.
If $\rho_2 \not\subset \rho_1$, similarly we can get $\rho_1 \in L(\rho_2)$ but $\rho_2 \notin L(\rho_2)$. Hence, $L = Spec(\sigma)$ is a T_1 space.

Lemma 3.6: Let $t_1, t_2 \in (0,1]$ and $t_1 \leq t_2$. Then $L(u_{t_1}) \subseteq L(u_{t_2})$ for $u \in \mathrm{M}$.

Proof. Suppose that $\mu \in L(u_{t_1})$, then $u_{t_1} \not\subset \mu \Rightarrow t_1 > \mu(u)$.

But $t_2 \geq t_1$ then $t_2 \geq \mu(u)$ and hence $u_{t_2} \not\subset \mu \Rightarrow \mu \in L(u_{t_2})$. Therefore $L(u_{t_1}) \subseteq L(u_{t_2})$.

Lemma 3.7: If $F \subseteq (0,1]$ and $L = \bigcup\{L((u_j)_t) : j \in \Lambda, t \in F, u_j \in \mathrm{M}\}$, then $\sup\{t : t \in F\} = 1$.

Proof. Clear.

Theorem 3.8: The topological space $L = Spec(\mu)$ is a compact space.

Proof. Since the set $\Psi = \{L(u_i) : u \in \mathrm{M}, i \in (0,1]\}$ is a base for this topological space, we assume that the set $\{L((u_j)_t) : j \in \Lambda, u_i \in \mathrm{M}, t \in F \subseteq (0,1]\}$ is a cover for L. Let $\beta = \sup\{t : t \in F\}$. By Lemma 3.7, $\beta = 1$ and by Lemma (3.6) the set $\{L((u_j)_1) : j \in \Lambda\}$ is a covering for L. Now we have $L = \bigcup_{j\in\Lambda}(L(u_j)_1) = L(\langle\bigcup_{j\in\Lambda}(u_j)_1\rangle) = L \setminus D(\langle\bigcup_{j\in\Lambda}(u_j)_1\rangle)$. On the other hands we have $D(\langle\bigcup_{j\in\Lambda}(u_j)_1\rangle) = D(\bigcup_{j\in\Lambda}(u_i)_1)$, so $L = L \setminus D(\bigcup_{j\in\Lambda}(u_j)_1)$, hence $D(\bigcup_{j\in\Lambda}(u_j)_1) = \phi$. Let $\Pr$ be any fuzzy prime KU-ideal of M and if $\mu(u) = \begin{cases} 1 & u \in \Pr \\ 0 & u \notin \Pr. \end{cases}$

Clearly, μ is a fuzzy prime KU-ideal of M and $\mu \notin D(\bigcup_{j\in\Lambda}(u_j)_1)$, then $\bigcup_{j\in\Lambda}(u_j)_1 \not\subset \mu$, so $\exists k \in \Lambda$ $\ni (u_k)_1 \not\subset \mu$, therefore $\mu(u_k) < 1$ and hence $u_k \notin \Pr$. Thus, there is no any fuzzy prime KU-ideal consist the set $\{u_j : j \in \Lambda\}$ and then there is no fuzzy KU-ideal consist the set $\{u_j : j \in \Lambda\}$, otherwise $\Lambda \subseteq m$ for some fuzzy maximal ideal m by Lemma2.9 and Lemma2.10, m is fuzzy prime which is contradiction. Hence $\langle\{u_j : j \in \Lambda\}\rangle = \mathrm{M}$, since M is bounded, then $e \in \langle\{u_j : j \in \Lambda\}\rangle$. Now we show that $D(\bigcup_{j=1}^{n}(u_j)_1) = \phi$. Let $\mu \in D(\bigcup_{j=1}^{n}(u_j)_1)$. Then $\bigcup_{j=1}^{n}(u_j)_1 \subseteq \mu$. So $\forall j = 1,2,...,n$, $(u_j)_1 \subseteq \mu$, therefore $\forall j = 1,2,...,n$, $\mu(u_i) \geq 1$ and so $\mu(u_i) = 1$ $\forall j = 1,2,...,n$. Thus $\forall j = 1,2,...,n$, $u_j \in \mu_*$ and hence $\langle\{u_j : j \in \Lambda\}\rangle \subseteq \mu_*$, therefore $e \in \mu_*$, which is a contradiction.

Thus $D(\bigcup_{j=1}^{n}(u_j)_1) = \phi$, and so

$$L = L \setminus D(\bigcup_{j=1}^{n}(u_j)_1) = L \setminus D(\langle\bigcup_{j=1}^{n}(u_i)_1\rangle) = D(\langle\bigcup_{j=1}^{n}(u_j)_1\rangle) = \bigcup_{j=1}^{n}L((u_j)_1).$$

This shows that L is compact.

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