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Development of the Interfaces for the Nonlinear Reaction-Diffusion equation with Convection

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ABSTRACT

We study the initial development and asymptotics of the interfaces and local solutions near the interfaces for the nonlinear reaction diffusion convection equation with compactly supported initial function. Depending on the relative strength of three competing terms such as diffusion, advection or absorption, the interface may shrink, expand or remain stationary. In this paper we focus only on two cases when the diffusion dominates and the interface expands and the other case when absorption term dominates and the interface shrinks. The significant methods that we used are rescaling and blow-up techniques.

Keywords: nonlinear degenerate parabolic PDEs, interfaces, nonlinear scaling methods.

1. Introduction

Consider the Cauchy problem(CP) for nonlinear degenerate parabolic PDEs.

$$\frac{\partial u}{\partial t}-\frac{\partial}{\partial x}\left[\frac{\partial u^m}{\partial x}-cu^\gamma\right]=bu^\beta \qquad (1)$$

$$u(x;0)=u_0(x) \quad , \quad x\in\mathbb{R} \qquad (2)$$

With $m>1$; c is negative and b,β,γ are nonnegative; $0<T<+\infty$.The initial function u_0 is nonnegative and continuous. Equation(1.1)is usually called a reaction diffusion convection equation. It is a simple and widely used model for various physical, chemical and biological problems involving diffusion with a source or absorption, and accompanied with additional convective flow as for instance in modeling filtration in porous media, transport of thermal energy in a plasma, flow of a chemically reacting fluid from a flat surface, evolution of populations etc. There has been a considerable amount of published work on this subject during the last five decades. For a general list of references we can refer to books [6, 10], and various survey articles such as [5, 8, 7, 11, 12, 9] etc.

The goal of this paper is to apply the results of the general theory which developed in [1] to analyze the behavior of interfaces which are separating the regions where $u=0$ from the region where $u>0$. Due to invariant equation (1.1) with respect to translation, we shall investigate the case when $\eta(0)=0$; where $\eta(t)=\sup\{x: u(x;t)>0\}$ is called an interface function. We are interested in the short time behavior of the interface function $\eta(t)$ and local solution $u(x;t)$ near the interface. We consider the following initial conditions in two cases, first the local case

$$u_0(x)\asymp C(-x)_+^\alpha \quad ,\text{as } x\to 0^- \qquad (3)$$

for some $C > 0,\ \alpha > 0$. The movement of the interface and its asymptotic is directed depending on the competition between three factors, such as diffusion, convection and reaction and depends on the parameters $c,\ b,\ \gamma,\ \beta,\ \alpha$ and C. Since the results of this paper are local in nature, without loss of generality we assume that u_0 either is bounded or satisfied

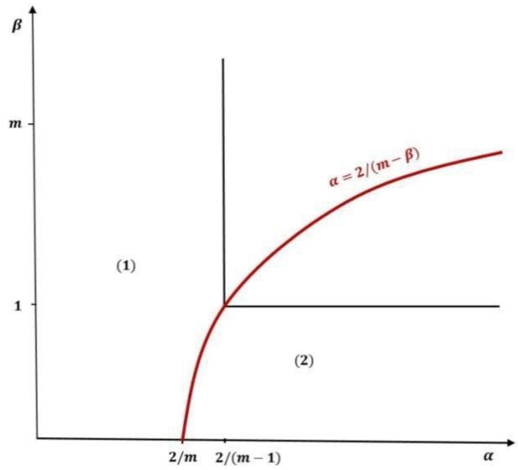

Figure 1. Classification of the interface movement in the plane (α, β) for CP (1.1)-(1.3).

some restrictions on its growth rate when $x \downarrow -\infty$ that is suitable for the results of the general theory. The special global case

$$u_0(x) \equiv C(-x)_+^{\alpha} \quad , \quad x \in \mathbb{R} \qquad (4)$$

If $c = 0$, then the initial development for the interface and structure of local solution near the interface is completely understood in the cases of the reaction diffusion equation

$$u_t = (u^m)_{xx} - bu^{\beta}, \quad x \in \mathbb{R}, 0 < t < T. \qquad (5)$$

General theory of this equation and full classification of the evolution of interface and the local solution near the interface in CP (1.5), (1.2), (1.3) was presented in [2,3].

The major obstacle to solve the interfaces development problem for nonlinear degenerate parabolic equations is the non-uniform asymptotic in singular perturbations theory, that the dominant balance as t goes to 0^+ between the terms in (1.1), (1.5) on curves that close to the boundary of the support. The rigorous proof method developed in [3, 4] is based on a barriers techniques using special comparison theorem in non-smooth domains with the characteristic boundary curves. In this paper we apply the general theory developed in [1] to solve the interface problem in the cases when either diffusion or reaction are dominating forces. The methods used are rescaling method and blow up technique for the identification of the asymptotics of solutions along the class of interface type curves, creation of the barriers and application of the comparison theorem in non-cylindrical or irreqular domains with boundary characteristic curves.

2. Main Results

In the following theorem we identify the kind of parameters when diffusion dominates over absorption and convection, and accordingly the interface expands (see Figure 1, region (1)).

Theorem 2.1. Let $m > 1$; $c < 0$; b, β, γ are nonnegative, and

$$\alpha < \min[1/(m - \min\{\gamma, (m+1)/2\};\ 2/(m - \min\{\beta, 1\})]\ . \qquad (6)$$

Then we initially have an expanding interface and

$$\eta(t) \sim t^{\frac{1}{2+\alpha(1-m)}}\xi_* \quad \text{as} \quad t \to 0^+, \qquad (7)$$

where

$$\xi_* = C^{\frac{m-1}{2+\alpha(1-m)}}\xi_*' \ , \qquad (8)$$

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and $\xi'_* > 0$ depends on m and α (only see the lemma 3.2). For $\rho < \xi_*$ there exists $s(\rho) > 0$ depending on C, m, and α such that

$$u(x,t) \sim t^{\frac{\alpha}{2+\alpha(1-m)}} s(\rho) \quad \text{as} \quad t \to 0^+ , \qquad (9)$$

along the curve $x = \xi_\rho(t) = t^{\frac{1}{2+\alpha(1-m)}}\rho$, and s is a shape function of the self-similar solutions for (1.1), (1.4) with b=c=0 (see lemma 1)

$$u_*(x,t) = t^{\frac{\alpha}{2+\alpha(1-m)}} s(\xi) \quad , \quad \xi = t^{\frac{-1}{2+\alpha(1-m)}} x \ . \qquad (10)$$

In fact, uniqueness of the solution s of the following nonlinear Ordinary DE problem is satisfied,

$$\frac{\alpha}{2+\alpha(1-m)} s - \frac{1}{2+\alpha(1-m)} \xi \frac{ds}{d\xi} - \frac{d^2 s^m}{d\xi^2} = 0 \,, \xi < \xi_* \ (11a)$$

$$s(-\infty) \sim C(-\xi)^\alpha; \qquad s(\xi) \equiv 0, \ \ \xi \geq \xi_*. \qquad (11b)$$

It depends on C and concluded the following relation,

$$s(\rho) = s_0\left(C^{-\frac{m-1}{2+\alpha(1-m)}}\rho\right) C^{\frac{2}{2+\alpha(1-m)}}, \qquad (12a)$$

$$s_0(\rho) = \omega(\rho, 1), \qquad \xi'_* = \sup\{\rho : s_0(\rho) > 0\}, \qquad (12b)$$

where ω is a solution of (1.1), (1.4) with $b = 0, c = 0, C = 1$. According to [3] we also have that

$$\xi'_* = W_0^{\frac{m-1}{2}} \left[\frac{m(2+\alpha(1-m))}{m-1}\right]^{\frac{1}{2}} \xi''_* \ , \qquad (13)$$

where $W_0 = \omega(0,1)$ and ξ''_* is a number in $[\xi_1, \xi_2]$, such that

$$\xi_1 = \left(\sqrt{\alpha(m-1)}\right)^{-1}, \quad \xi_2 = 1 \ \text{ if } 1/(m-1) \leq \alpha < 2/(m-1),$$

$$\xi_1 = 1, \qquad \xi_2 = \left(\sqrt{\alpha(m-1)}\right)^{-1} \quad \text{if } \ 0 < \alpha \leq 1/(m-1). \ (14)$$

The following upper and lower estimations for f are proved in [3],

$$C_4 t^{\frac{\alpha}{2-\alpha(m-1)}} (\xi_3 - \xi)_+^{\frac{1}{m-1}} \leq u \leq C_5 t^{\frac{\alpha}{2-\alpha(m-1)}} (\xi_3 - \xi)_+^{\frac{1}{m-1}}, \qquad (15)$$

where $C_4 = W_0 C^{\frac{2}{2+\alpha(1-m)}} \xi_3^{\frac{1}{m-1}}$, $\quad C_5 = W_0 C^{\frac{2}{2+\alpha(1-m)}} \xi_3^{-\frac{1}{m-1}}$.

In the following theorem, we discuss the parameter range where absorption term dominates over both diffusion and advection terms (see Figure 1, region (2)).

Theorem 2.2. Let $0 < \beta < 1$, and one of the following cases be hold:
(a) $\beta < \gamma \leq (m+\beta)/2, \quad \alpha > 1/(\gamma - \beta)$,
(b) $\gamma \geq (m+\gamma)/2, \ \ \alpha > 2/(m-\beta)$.

If u_0 is satisfied the initial condition (1.3), then the interface is shrinking and

$$\eta(t) \sim -t^{\frac{1}{\alpha(1-\beta)}}\lambda_* \quad \text{as} \ \ t \to 0^+, \qquad (16)$$

where $\lambda_* = C^{-\frac{1}{\alpha}}\left(b(1-\beta)\right)^{\frac{1}{\alpha(1-\beta)}}$, for arbitrary $\lambda > \lambda_*$ we have

$$u(x,t) \sim \left[C^{1-\beta}(-x)_+^{\alpha(1-\beta)} - (1-\beta)bt\right]_+^{1/(1-\beta)} \quad \text{as} \ \ t \to 0^+, \ \ (17)$$

along the curve $x = \eta_\lambda(t) = -\lambda t^{\frac{1}{\alpha(1-\beta)}}$.

3. Preliminary Results

Lemma 3.1. [3] If $b = 0, c = 0$ and $0 < \alpha < 2/(m-1)$, then the solution u of the CP (1.1), (1.4) has a self-similar form (2.10), where the self-similarity function s satisfies (2.12). If u_0 satisfies (1.3), then the solution to CP (1.1), (1.2) satisfies (2.7)-(2.9).

We identify the parameter plane in the next lemma when diffusion dominates over both advection and absorption forces, and the approximated solution to the nonlinear diffusion with reaction and convection equation coincides with the asymptotic properties of the classical diffusion equation.

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Lemma 3.2.
Let u be any solution to the problem (1.1), (1.2) and u_0 satisfies (1.3). Let one of the following cases be valid:

(a) $0 < \gamma < (m+1)/2, 0 < \beta < 1, \alpha < \min\{2/(m-\beta), 1/(m-\gamma)\}$;
(b) $(m+1)/2 \le \gamma, 0 < \beta < 1, \alpha < 2/(m-\beta)$;
(c) $(m+\beta)/2 < \gamma < (m+1)/2, \beta \ge 1, \quad \alpha < 1/(m-\gamma)$;
(d) $(m+1)/2 \le \gamma, \ \beta \ge 1, \quad \alpha < 2/(m-1)$;

then u satisfies (2.9).
Proof: Let u_0 satisfy (1.3), then for appropriately small value $\epsilon > 0$, there is an $x_\epsilon < 0$, $x_\epsilon \le x \le +\infty$ such that

$$(C-\epsilon/2)(-x)_+^\alpha \le u_0(x) \le (C+\epsilon/2)(-x)_+^\alpha, \quad (18)$$

The cases (a), (c) and cases (b), (d) with $\gamma \le m$ are valid. Then from results from (Theorem 2.1.1, 2.2.1 ; [1]) then the existence and uniqueness of the (1.1), (1.2) with $u_0 = (C \pm \epsilon)(-x)_+^\alpha, T = +\infty$ hold. Let $u_\epsilon(x,t)$ (respectively, $u_{-\epsilon}(x,t)$) be a solution to the problem (1.1), (1.2) with initial condition $(C+\epsilon)(-x)_+^\alpha$(respectively, $(C-\epsilon)(-x)_+^\alpha$). Because of the continuity of the solution, there exists a number $\sigma = \sigma(\epsilon) > 0$ such that

$$u_{-\epsilon}(x_\epsilon,t) \le u(x_\epsilon,t) \le u_{+\epsilon}(x_\epsilon,t), \quad 0 \le t \le \sigma \quad (19)$$

From (1.18),(1.19) and a comparison principle from (Theorem 2.3.1, [1]) , we have

$$u_{-\epsilon} \le u \le u_{+\epsilon}, \quad x_\epsilon \le x \le +\infty, \ 0 \le t \le \sigma. \quad (20)$$

Suppose that

$$u_k^{\pm\epsilon}(x,t) = k u_{\pm\epsilon}\left(k^{-1/\alpha}x, k^{(\alpha(m-1)-2)/\alpha}t\right), k > 0, \quad (21)$$

then $u_k^{\pm\epsilon}(x,t)$ satisfies the problem

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[\frac{\partial u^m}{\partial x} - ck^{\frac{\alpha(m-\gamma)-1}{\alpha}}u^\gamma\right] - bk^{\frac{\alpha(m-\beta)-2}{\alpha}}u^\beta, \quad (22a)$$

$$u(x,0) = (C \pm \epsilon)(-x)_+^\alpha, \quad x \in R. \quad (22b)$$

There exists a unique solution to problem (3.22), which satisfies a comparison theorem. Since $\alpha(m-\gamma)-1 < 0$ and $\alpha(m-\beta)-2 < 0$, from [1] it follows that

$$\lim_{k\to+\infty} u_k^{\pm\epsilon}(x,t) = v_{\pm\epsilon}(x,t), \quad x \in R, \ t > 0, \quad (23)$$

According to lemma 3.1, $v_{\pm\epsilon}$ is a solution to the problem (1.1), (1.2) with $b = c = 0$, $u_0 = (C \pm \epsilon)(-x)_+^\alpha, T = +\infty$. Thus, $v_{\pm\epsilon}$ satisfies (3.20). If we take $x = \xi_\rho(t)$, where ρ is any fixed number satisfying $\rho < \xi_*$, then from (3.23), we have

$$\lim_{k\to+\infty} k u_{\pm\epsilon}\left(k^{-1/\alpha}x, k^{(\alpha(m-1)-2)/\alpha}t\right) = v_{\pm\epsilon}(x,t).$$

If we take $\tau = k^{(\alpha(m-1)-2)/\alpha}t$, then (3.23) implies

$$u_{\pm\epsilon}\left(\xi_\rho(\tau),\tau\right) \sim \tau^{\alpha/(2-\alpha(m-1))}s(\rho; C \pm \epsilon), \text{as } \tau \to 0^+. \quad (24)$$

Therefore (2.9) follows from (3.20), (3.24). Let consider the cases (b) and (d) with $\gamma > m$. Assume that $u_{\pm\epsilon}$ is a solution of DP

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[\frac{\partial u^m}{\partial x} - cu^\gamma\right] - bu^\beta, |x| < |x_\epsilon|, \ 0 < t < \sigma \quad (25a)$$

$$u(x,0) = (C \pm \epsilon)(-x)_+^\alpha, |x| \le |x_\epsilon|, \quad (25b)$$

$$u(x_\epsilon,t) = (C \pm \epsilon)(-x_\epsilon)_+^\alpha, u(-x_\epsilon,t) = 0, \quad 0 \le t \le \sigma \quad (25c)$$

The function $u_k^{\pm\epsilon}$ is defined as in (2.9) which satisfies the DP

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[\frac{\partial u^m}{\partial x} - ck^{\frac{\alpha(m-\gamma)-1}{\alpha}}u^\gamma\right] - bk^{\frac{\alpha(m-\beta)-2}{\alpha}}u^\beta, \text{ in } D_\epsilon^k \quad (26a)$$

$$u(x,0) = (C \pm \epsilon)(-x)_+^\alpha, \quad |x| \le k^{1/\alpha}|x_\epsilon|, \quad (26b)$$

$$u\left(k^{1/\alpha}x_\epsilon,t\right) = k(C \pm \epsilon)(-x_\epsilon)_+^\alpha, u\left(-k^{1/\alpha}x_\epsilon,t\right) = 0, \quad (26c)$$

where $D_\epsilon^k = \{(x,t): |x| \le k^{1/\alpha}|x_\epsilon|, \ 0 \le t \le k^{\frac{2+\alpha(1-m)}{\alpha}}\sigma\}$.
We get from [1] that there is a number $\sigma > 0$ which does not depend on k such that problems (3.25a)-(3.25c) and (3.26a)-(3.26c) have unique solutions. From the concept of finite speed of propagation a $\sigma = \sigma(\epsilon) > 0$ may be chosen such that

$$u(-x_\epsilon,t) = 0, \quad 0 \le t \le \sigma \quad (26)$$

By applying comparison principle (Theorem 2.3.1,[1]), from (3.18), (3.19) and (3.26),(3.20) follows. The next step we focus on the proof of convergence of the sequences $\{u_k^{\pm\epsilon}\}$.Consider a function $f(x,t) = (C+1)(1-\nu t)^{\frac{1}{1-m}}(1+x^2)^{\frac{\alpha}{2}}$, where $0 < t \le t_0$ and $t_0 = \frac{1}{2\nu}$, such that

$$\nu = H_* + 1,\ \ H_* = H_*(\alpha; m) = \max_{x\in R} H(x), \text{ and}$$

$$H(x) = \alpha m(m-1)(C+1)^{m-1}(1+x^2)^{\frac{\alpha(m-1)-4}{2}}\left((\alpha m-1)x^2+1\right),$$

then we have $$L_k f = (C+1)(1+x^2)^{\frac{\alpha}{2}}(1-\nu t)^{\frac{m}{1-m}}(m-1)^{-1}E$$

where $E = \nu - H(x) + E_1 + E_2$

$$E_1 = k^{\frac{\alpha(m-\gamma)-1}{\alpha}} c\alpha\gamma(C+1)^{\gamma-1}(1+x^2)^{\frac{\gamma-m}{1-m}}(1-\nu t)^{\frac{\alpha(\gamma-1)}{2}-1},$$

$$E_2 = k^{\frac{\alpha(m-\beta)-2}{\alpha}} b(C+1)^{\beta-1}(1+x^2)^{\frac{\beta-m}{1-m}}(1-\nu t)^{\frac{\alpha(\beta-1)}{2}},$$

$$E \ge 1 + E_1 + E_2 \quad \text{in} \quad D_{0\epsilon}^k = D_\epsilon^k \cup \{0 < t \le t_0\}.$$

Since we have $E_1 = O\left(k^{(\alpha(m-\gamma)-1)/\alpha}\right)$ uniformly for $(x,t) \in D_{0\epsilon}^k$ as $k \to \infty$, and $E_2 = O\left(k^{\theta}\right)$ uniformly for $(x,t) \in D_{0\epsilon}^k$ as $k \to \infty$, where $\theta = (\alpha(m-1)-2)/\alpha$ if $\beta \ge 1$ and $\theta = (\alpha(m-\beta)-2)/\alpha$ if $0 < \beta < 1$.

Let $0 < \epsilon \ll 1$

$$f(x,0) \ge u_k^{\pm\epsilon}(x,0) \quad \text{for} \quad |x| \le k^{1/\alpha}|x_\epsilon|\,, \tag{27}$$

$$f\left(\pm k^{1/\alpha}x_\epsilon, t\right) \ge u_k^{\pm\epsilon}\left(\pm k^{1/\alpha}x_\epsilon, t\right) \quad \text{for} \quad 0 < t < t_0\,. \tag{28}$$

Hence, $\exists k_0 = k_0(\alpha;\gamma)$ such that for $\forall k \ge k_0$ the comparison principle implies

$$0 \le u_k^{\pm\epsilon}(x,t) \le f(x,t) \ \text{ in } \ \overline{D}_{0\epsilon}^k. \tag{29}$$

Let F be an arbitrary compact subset of $P = \{(x,t): x \in \mathbb{R},\ \ 0 < t \le t_0\}$.
We take k_0 so large that $F \subset D_{0\epsilon}^k$ for $k \ge k_0$. From (3.29) it follows that $\{u_k^{\pm\epsilon}\}$, $k \ge k_0$, are uniformly bounded sequences in F.Therefore we proved the sequences are uniformly Hölder continuous in F and there exist a function $v_{\pm\epsilon}$ such that for some subsequence k'

$$\lim_{k'\to+\infty} u_{k'}^{\pm\epsilon}(x,t) = v_{\pm\epsilon}(x,t), \qquad (x,t) \in P, \tag{30}$$

we can easily check that $v_{\pm\epsilon}$ is a solution to the problem (1.1), (1.2) with $c = 0, b = 0,\ T = t_0,\ u_0 = (C \pm \epsilon)(-x)_+^{\alpha}$. From (3.23), (3.24) and (3.20), the estimation (2.9) follows.

Lemma 3.3.

Let u be any solution to the problem (1.1), (1.3). If $0 < \beta < 1$ and one of the following cases be valid:

(a) $\alpha > 1/(\gamma-\beta)$, $\beta < \gamma \le (m+\beta)/2$;
(b) $\alpha > 2/(m-\beta)$, $\gamma \ge (m+\beta)/2$;

then for arbitrary $\lambda > \lambda_*$ (see the value of λ_* from (2.16)) the asymptotic formula (2.17) is valid along $x = \eta_\lambda(t) = -\lambda t^{1/(\alpha(1-\beta))}$.

Proof: As before, (3.18) and (3.19) follow from (1.3). Suppose the functions $u_{\pm\epsilon}$ solve the problem

$$\frac{\partial v}{\partial t} = \frac{\partial}{\partial x}\left[\frac{\partial v^m}{\partial x} - cv^{\gamma}\right] - bv^{\beta}, \qquad |x| < |x_\epsilon|, \qquad 0 < t < \sigma$$

$$v(x,0) = (C \pm \epsilon)(-x)_+^{\alpha}, \qquad |x| \le |x_\epsilon|,$$

$$v(x_\epsilon, t) = (C \pm \epsilon)(-x_\epsilon)_+^{\alpha}, \qquad v(-x_\epsilon, t) = u(-x_\epsilon, t), \qquad 0 \le t \le \sigma$$

Applying a comparison theorem, from (3.18) and (4.19), (4.20) follows for $|x| \le |x_\epsilon|, 0 \le t \le \sigma$. If we rescale

$$u_k^{\pm\epsilon}(x,t) = ku_{\pm\epsilon}\left(k^{-1/\alpha}x, k^{\beta-1}t\right), \qquad k > 0,$$

which satisfies the DP,

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v^m}{\partial x^2}k^{\frac{2-\alpha(m-\beta)}{\alpha}} - c\frac{\partial v^{\gamma}}{\partial x}k^{\frac{1-\alpha(\gamma-\beta)}{\alpha}} - bv^{\beta}, \ \text{ in } \qquad D_\epsilon^k$$

$$v(x,0) = (C \pm \epsilon)(-x)_+^{\alpha}, \qquad |x| \le k^{1/\alpha}|x_\epsilon|,$$

$$v\left(k^{1/\alpha}x_\epsilon, t\right) = k(C \pm \epsilon)(-x_\epsilon)_+^{\alpha}, \quad v\left(-k^{1/\alpha}x_\epsilon, t\right) = ku\left(-x_\epsilon, k^{\beta-1}t\right),$$

in $0 \le t \le t^{1-\beta}\sigma$ where $D_\epsilon^k = \left\{(x,t): |x| \le k^{1/\alpha}|x_\epsilon|,\ 0 \le t \le t^{1-\beta}\sigma\ \right\}$.

Now, we prove the convergence of the sequences $\{u_k^{\pm\epsilon}\}$ as $k \to +\infty$. Let consider a new function $f(x,t) = (C+1)e^t(1+x^2)^{\frac{\alpha}{2}}$,

$$\bar{L}_k f = \frac{\partial f}{\partial t} - \frac{\partial^2 f^m}{\partial x^2} k^{\frac{2-\alpha(m-\beta)}{\alpha}} + c\frac{\partial f^\gamma}{\partial x} k^{\frac{1-\alpha(\gamma-\beta)}{\alpha}} + bf^\beta \geq$$
$$f[1 - e^{(m+1)t}(C+1)^{m-1}\alpha m(1+x^2)^{\frac{\alpha(m-1)}{2}-2} k^{\frac{2-\alpha(m-\beta)}{\alpha}} \times$$
$$(1+(\alpha m-1)x^2) + ce^{(\gamma-1)t}(C+1)^{\gamma-1}\alpha\gamma x(1+x^2)^{\frac{\alpha(\gamma-1)}{2}-1} \times$$
$$k^{\frac{1-\alpha(\gamma-\beta)}{\alpha}} + be^{(\beta-1)t}(C+1)^{\beta-1}(1+x^2)^{\frac{\alpha(\beta-1)}{2}} \;] \quad \text{in } D_\epsilon^k. \tag{31}$$

Let t_0 be fixed and let $D_{0\epsilon}^k = D_\epsilon^k \cap \{(x,t): 0 < t \leq t_0\}$. From (3.31) it follows that

$$\bar{L}_k f \geq f[1 - H(x) + E] \quad \text{in} \quad D_{0\epsilon}^k, \quad \text{where}$$
$$E = ce^{(\gamma-1)t}(C+1)^{\gamma-1}\alpha\gamma x(1+x^2)^{\frac{\alpha(\gamma-1)}{2}-1} k^{\frac{1+\alpha(\beta-\gamma)}{\alpha}}$$
$$+be^{(\beta-1)t}(C+1)^{\beta-1}(1+x^2)^{\frac{\alpha(\beta-1)}{2}},$$

which satisfies $E \equiv O(k^\vartheta)$ uniformly for $(x,t) \in D_{0\epsilon}^k$ as $k \to \infty$, where $\vartheta = (1+\alpha(\beta-\gamma))/\alpha$ if $\alpha < 1/(\gamma-1)$ and $\vartheta = \beta - 1$ if $\alpha \geq 1/(\gamma-1)$. And,

$$H(x) = e^{(m+1)t}(C+1)^{m-1}\alpha m(1+x^2)^{\frac{\alpha(m-1)}{2}-2} k^{\frac{2+\alpha(\beta-m)}{\alpha}} (1+(\alpha m-1)x^2),$$

which satisfies $H(x) \equiv O(k^\theta)$ uniformly for $(x,t) \in D_{0\epsilon}^k$ as $k \to \infty$, where $\theta = \beta - 1$ if $\alpha \geq 2/(m-1)$ and $\theta = (2+\alpha(\beta-m))/\alpha$ if $\alpha < 2/(m-1)$.

Let $0 < \epsilon < 1$, we have

$$f(x,0) = u_k^{\pm\epsilon}(x,0) \quad \text{for} \quad |x| \leq k^{1/\alpha}|x_\epsilon| \quad ,$$
$$u_k^{\pm\epsilon}(\pm k^{1/\alpha}x_\epsilon, t) = o(k) \quad \text{for} \quad 0 \leq t \leq t_0 \quad \text{as } k \to \infty.$$

We also have $f(\pm k^{1/\alpha}x_\epsilon, t) \geq u_k^{\pm\epsilon}(\pm k^{1/\alpha}x_\epsilon, t)$ for $0 \leq t \leq t_0$, if we choose k is large enough. Thus, as in the proof of lemma 3.1, a comparison Theorem (see [1]) implies (3.29) in $D_{0\epsilon}^k$, where the functions $u_k^{\pm\epsilon}$ and G apply in the context of the our proof. As before, from the interior regularity results [7] it follows that the sequence of non-negative and local bounded solutions $\{u_k^{\pm\epsilon}\}$ is local uniform Hölder continuous on the compact subsets of P. Hence, the functions $v_{\pm\epsilon}$ for some subsequence k' ,(3.30) is valid. It may be easily be established that the limit functions $v_{\pm\epsilon}$ are satisfied the solutions of the problem

$$u_t + bu^\beta = 0, x \in R, 0 < t \leq t_0; \qquad u(x,0) = (C \pm \epsilon)(-x)_+^\alpha \quad , x \in R,$$

such that $v_{\pm\epsilon}(x,t) = \left[(C \pm \epsilon)^{1-\beta}(-x)_+^{\alpha(1-\beta)} - bt(1-\beta)\right]_+^{\frac{1}{1-\beta}}$.

Let $\lambda > \lambda_*$ and $\epsilon > 0$ be chosen such that $b(1-\beta) < (C-\epsilon)^{1-\beta}\lambda^{\alpha(1-\beta)}$. If we take $x = \eta_\lambda(t)$ and $\tau = tk^{\beta-1}$, then from (3.30) we have

$$u_{\pm\epsilon}(x,t) \sim \tau^{\frac{1}{1-\beta}}\left[(C \pm \epsilon)^{1-\beta}\lambda^{\alpha(1-\beta)} - b(1-\beta)\right]^{\frac{1}{1-\beta}}, \quad \text{as } \tau \to +\infty. \tag{32}$$

Since $\epsilon > 0$is an arbitrary number, From (3.20) and (3.32), then the desired formula (2.17) follows.

4. Proof of the Main Results

Proof of Theorem 2.1. From lemma 3.2, then the formula (2.9) follows. Since ρ is any arbitrary value, from lemma 3.2, it follows

$$u(\xi_\rho(t), t) \sim t^{\alpha/(2+\alpha(1-m))}s(\rho) \quad \text{as } t \to 0^+, \forall \rho < \xi_*, \tag{33}$$

where $\xi_\rho(t) = \rho t^{1/(2+\alpha(1-m))}$. For $\epsilon > 0$, take $\rho + \epsilon = \xi_*$, then we get

$$\lim_{t \to 0^+} \inf t^{1/(\alpha(m-1)-2)}\, \eta(t) \geq \xi_*. \tag{34}$$

Take sufficiently small value $\epsilon > 0$. Let u_ϵ be a solution of the Cauchy problem (1.1), (1.4) with $b = c = 0$ and C replaced by $C + \epsilon$. The first inequality of (3.19) and the second inequality of (3.18) follow from (1.4). Suppose that $b > 0$. Since we have $\mathcal{L}u_\epsilon = c\frac{\partial u_\epsilon^\gamma}{\partial x} + bu_\epsilon^\beta$, then u_ϵ is a suppersolution if $c\frac{\partial u_\epsilon^\gamma}{\partial x} + bu_\epsilon^\beta \geq 0$ and since $bu_\epsilon^\beta \geq 0$. Build a family of solutions $u_{0,n}(x)$ such that

$$u_{0,n}(x) = u_0(x) \quad \text{if } x_n > x; \quad u_{0,n}(x) = n^{-1} \quad \text{if } x_n \leq x\,, \quad (35)$$

u_n is a solution of the problem (1.1), (4.35) with $b = c = 0$. By applying the maximum principle, we have $u_n(x,t) \geq u_{n+1}(x,t) \geq \cdots \geq 0$, and the limit $\lim_{t\to\infty} u_n(x,t) = u(x,t)$ exists. In fact, $v_n = (u_n)_x$ is a solution of the following parabolic PDE

$$\begin{cases} (v_n)_t = m(v_n)_{xx}u^{m-1} + m(m-2)(m-1)v_n u^{m-3}(u_x)^2 + 3m(m-1)(v_n)_x u^{m-2}u_x, \\ \quad v_n(x,t) \leq 0, \quad x \in \mathbb{R}, \qquad t = 0\ , \end{cases}$$

in $x \in \mathbb{R}, t > 0$. Apply maximum principle property on the second order parabolic PDE, it follows that $v_n \leq 0$ everywhere. Therefore, $u_x \leq 0$, and for $x \in \mathbb{R}$, $t > 0$ then we have $\mathcal{L}u_\epsilon \geq 0$. From (3.18), (3.19), the second inequality of (3.20) follows. Thus $\forall \epsilon > 0, \exists \sigma > 0$ such that

$$\eta(t) \leq (C+\epsilon)^{(m-1)/(2+\alpha(1-m))} t^{1/(2+\alpha(1-m))} \xi_*' \ , \quad 0 \leq t \leq \sigma,$$

Which implies

$$\lim_{t\to 0^+} \sup t^{1/(\alpha(m-1)-2)} \eta(t) \geq \xi_*. \qquad (36)$$

From (34) and (36), (2.7) follows. Finally (2.13), (2.14) follow from (2.15).

Proof of Theorem 2.2. Let $\epsilon > 0$ be an arbitrary sufficiently small value. From (1.3), then we get (2.18). Consider a function

$$f_\epsilon(x,t) = \left[(C+\epsilon)^{1-\beta}(-x)_+^{\alpha(1-\beta)} - t(1-\beta)(1-\epsilon)\right]_+^{\frac{1}{1-\beta}}. \quad (37)$$

We estimate $\mathcal{L}_f$ in $I_1 = \{(x,t): x_\epsilon < x < \eta_\lambda(t),\ \ 0 < t < \sigma_1\}$,

$$\eta_\lambda(t) = -t^{\frac{1}{\alpha(1-\beta)}}\lambda, \qquad \lambda(\epsilon) = (C+\epsilon)^{-\frac{1}{\alpha}}[b(1-\epsilon)(1-\beta)]^{\frac{1}{\alpha(1-\beta)}},$$

where $\sigma_1 > 0$ is chosen such that $\eta_{\lambda(\epsilon)}(\sigma_1) = x_\epsilon$. We have $\mathcal{L}f_\epsilon = bf_\epsilon^\beta\{\epsilon + E\}$

$$E = -b^{-1}\alpha m(C+\epsilon)^{m-\beta}\,(-x)_+^{\alpha(m-\beta)-2}\left\{\frac{f_\epsilon|x|^{-\alpha}}{C+\epsilon}\right\}^{m+\beta-2} E_1 + E_2, \text{ where}$$

$$E_1 = \left\{(\alpha(1-\beta)-1)\left\{\frac{f_\epsilon|x|^{-\alpha}}{C+\epsilon}\right\}^{1-\beta} + \alpha(m+\beta-1)\right\},$$

$$E_2 = \left\{-\alpha\, b^{-1}c\gamma(C+\epsilon)^{\gamma-\beta}\,(-x)_+^{\alpha(\gamma-\beta)-1}\left\{\frac{f_\epsilon|x|^{-\alpha}}{C+\epsilon}\right\}^{\gamma-1}\right\}.$$

If we have $m+\beta \geq 2$, then we choose $x_\epsilon < 0$, where $|x_\epsilon|$ is sufficiently small such that $|E| < \frac{\epsilon}{2}$ in I_1.

Thus we have $\mathcal{L}f_\epsilon > b\left(\frac{\epsilon}{2}\right)f_\epsilon^\beta$ (respectively , $\mathcal{L}f_\epsilon < -b\left(\frac{\epsilon}{2}\right)f_{-\epsilon}^\beta$) in I_1

$\mathcal{L}f_{\pm\epsilon} = 0$ for $x > \eta_{\ell(\pm\epsilon)}(t)$, $0 < t \leq \sigma_1$,

$f_\epsilon(x,0) \geq u_0(x), x \geq x_\epsilon$ (respectively $f_{-\epsilon}(x,0) \leq u_0(x), x \geq x_\epsilon$) .

Because u and $f_{\pm\epsilon}$ are continuous functions, and let choose $\sigma \in (0,\sigma_1]$ such that $f_\epsilon(x,t) \geq u(x_\epsilon,t), 0 \leq t \leq \sigma$. (respectively, $f_{-\epsilon}(x,t) \leq u(x_\epsilon,t), 0 \leq t \leq \sigma$) .

From Comparison Theorem 2.3.1 and lemma 2.3.1 (see [1]), it follows that

$$f_{-\epsilon} \leq u \leq f_\epsilon, \qquad x \geq x_\epsilon, \quad 0 \leq t \leq \sigma, \qquad (38a)$$

$$\eta_{\lambda(-\epsilon)}(t) \leq u \leq \eta_{\lambda(\epsilon)}(t), \qquad 0 \leq t \leq \sigma, \qquad (38b)$$

which imply (2.16) and (2.17).

If $m+\beta < 2$, then the left hand side of (4.38a), (4.38b) can be proved similarly.

To prove the upper estimation , we consider a function

$f(x,t) = C_6(-t^{1/(\alpha(1-\beta))}\zeta_5 - x)_+^\alpha$ in $D_{\lambda,\delta} = \{(x,t): \eta_\lambda(t) < x < +\infty,\ \ 0 < t < \sigma\}$,

where $\lambda \in (\lambda_*, +\infty)$. From (4.17) it implies that for $\lambda > \lambda_*$ and $\epsilon > 0$ there exists a $\sigma = \sigma(\epsilon,\lambda) > 0$ such that

$$u(\eta_\lambda(t),t) \leq \ t^{1/(1-\beta)}\left[C^{1-\beta}\lambda^{\alpha(1-\beta)} - b(1-\epsilon)(1-\beta)\right]_+^{1/(1-\beta)}. \qquad (39)$$

Calculating $\mathcal{L}f$ in $D_{\lambda,\sigma}^+ = \{(x,t): \eta_\lambda(t) < x < -\zeta_5 t^{1/(\alpha(1-\beta))},\ \ 0 < t < \sigma\}$, we have

$$\mathcal{L}f = bf^\beta E, \qquad E = 1 - \big(b(1-\beta)\big)^{-1}\zeta_5 C_6^{\frac{1}{\alpha}}\{f\, t^{1/(\beta-1)}\}^{\frac{\alpha(1-\beta)-1}{\alpha}}$$

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$$-\big(\alpha m(\alpha m-1)\big)b^{-1}C_6^{\frac{2}{\alpha}}f^{\frac{\alpha(m-\beta)-2}{\alpha}}-b^{-1}c\alpha\gamma C_6^{p-\beta}f^{\frac{\alpha(\gamma-\beta)-1}{\alpha}},$$

since $E_x \geq 0$ in $D^+_{\lambda,\sigma}$, we have $E \geq E|_{x=\eta_\ell(t)}$

$$E \geq \epsilon - C_6^{m-\beta}\big(\alpha m(\alpha m-1)\big)b^{-1}\{(\lambda-\zeta_5)t^{1/(\alpha(1-\beta))}\}^{\alpha(m-\beta)-2}$$
$$-b^{-1}c\alpha\gamma C_6^{m-\beta}\{(\lambda-\zeta_5)t^{1/(\alpha(1-\beta))}\}^{\alpha(\gamma-\beta)-1} \text{ in } D^+_{\lambda,\sigma}.$$

Hence we choose $\sigma=\sigma(\epsilon)>0$ so small that $\mathcal{L}f > b\left(\frac{\epsilon}{2}\right)f^\beta$ in $D^+_{\lambda,\sigma}$. Using (39) and Comparison Theorem 2.3.2 (see [1]) in $D'_{\lambda,\sigma}=D_{\lambda,\sigma}\cap\{x<x_0\}, \forall x_0>0$. Then we have $\mathcal{L}f=0$ in $D'_{\lambda,\sigma}\backslash\overline{D}^+_{\lambda,\sigma}$,

$$u(\eta_\lambda(t),t) \leq t^{1/(1-\beta)}\left[C^{1-\beta}\lambda^{\alpha(1-\beta)}-b(1-\beta)(1-\epsilon)\right]_+^{1/(1-\beta)}$$
$$= C_6t^{1/(1-\beta)}(\lambda-\zeta_5)^\alpha = f(\eta_\ell(t),t), \quad 0<t<\sigma,$$

$u(x_0,t)=f(x_0,t)=0, 0<t\leq\sigma;\ u(x,0)=f(x,0)=0, \quad 0\leq x\leq x_0.$

Because $x_0>0$ is an arbitrary number , from above and comparison theorem then we have that for all $\lambda>\lambda_*$ and $\epsilon>0, \exists\sigma=\sigma(\epsilon,\lambda)>0$ such that

$u(x,t)\leq C_6(-t^{1/(\alpha(1-\beta))}\zeta_5-x)_+^\alpha$ in $\overline{D}_{\lambda,\sigma}$. Since (2.17) is valid along $x=\eta_\lambda(t)$, so

$$-t^{1/(\alpha(1-\beta))}\lambda \leq \eta_\lambda(t) \leq -t^{1/(\alpha(1-\beta))}\zeta_5, \quad 0\leq t\leq\sigma, \quad (40)$$

Therefore, (2.16) follows from (4.40).

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