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The Martingale as an Orthogonal Projection

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Abstract
This paper studies martingale as an orthogonal projection. It is proved that martingale achieves infimum relation if and only if it satisfies orthogonality relation. It is also proved that orthogonal martingale in Hilbert space achieves minimum relation. It is observed that martingale differences are orthogonal in Hilbert space and martingale can be represented as the sum of orthogonal difference.

Keywords: Orthogonal projection, square-integrable, orthogonality of martingale, martingale differences.

1. Introduction

This paper is concerned with the study of martingale and it's orthogonality. Therefore, the definition of norm in $L^P(\Omega, F, P)$, the inner product and L^2-martingale are needed. If L^2 -martingale is closed subspace of a Hilbert space $L^P(\Omega, F, P)$ then satisfies the property of infimum if and only if it satisfies the orthogonality relation.
It is concluded that the projection of X_{n+1} belongs to $L^2(\Omega, F, P)$ has a unique representation on L^2 -martingale. There are many previous studies which studied martingale. These studies have in some way or another a relation to the current study, like Mansuy's that studied the origins of the word "martingale" [1] and Merkouris's which tackled the transform martingale estimating functions.[2]

1.1 Definitions

Definition 1.2 [3]: Let $X_1, X_2, ...$ be a sequence of integrable a random variable on a probability space (Ω, F, P) and $F_n \subset F_m$ an increasing sequence of sub $\sigma-$ fields of F , X_n is assumed F_n - measurable that is $X_n : (\Omega, F) \to (R, B(R))$, $\{X_n, F_n\}$ is said to be Martingale if and only if $E[X_{n+1} | F_n] = X_n$ a.e, for all n=1,2,… .

Definition 1.3 [4]: A martingale $\{X_n, F_n\}$ is said to be L^2 -- martingale or (square-integrable) if $X_n \in L^2$, for all. $n \in N$.

Definition 1.4 [5]: For $X, Y \in L^2(\Omega, F, P)$ the inner product or (scalar) is defined as:

$$\langle X, Y \rangle = E(XY).$$

Definition 1.5 [5]: If $E(|Y|^P) < \infty$ it's said $Y \in L^P(\Omega, F, P)$ for P∈ $[1, \infty)$ and then the norm defined as:

$$\|Y\|_P = \left\{E(|Y|^P)\right\}^{\frac{1}{P}}.$$

Orthogonality 1.6 (Pythagoras theorem) [5]: Pythagoras theorem takes the form $\|W+Z\|_2^2=\|W\|_2^2+\|Z\|_2^2$ if. $\langle W,Z\rangle=0$ .It is said that , W and Z are orthogonal or perpendicular if $\langle W,Z\rangle=0$ thus, it is written as follows: $W\perp Z$.

Now, we need the following theorem to prove the main results.
Theorem 1.7 (completeness of L^P) [6]: If $X_1,X_2,...$ form a Cauchy sequence in L^P where $1\leq P<\infty$, that is,

$$\lim_{t\to\infty}\sup_{u,v\geq l}\|X_u-X_v\|_P=0.$$

Then there exist $X\in L^P$ such that $X_u\to X$ in,th L^P at is

$$\lim_{l\to\infty}\|X_u-X\|_P=0$$

2 Main Results

Theorem 2.1: If X_n is L^2 - martingale and orthogonal and $Y\in L^2(\Omega,F,P)$ is orthogonal with X_j for all j=1,2,…,n then $\left\|Y-\sum_{j=1}^n b_jX_j\right\|_2$ is minimized when $E(X_j^2)=\frac{1}{n(b_j^2)}, b_j\neq 0$.

Proof: $0\leq\left\|Y-\sum_{j=1}^n b_jX_j\right\|_2^2=\left\langle Y-\sum_{j=1}^n b_jX_j, Y-\sum_{j=1}^n b_jX_j\right\rangle$

$$=E[(Y-\sum_{j=1}^n b_jX_j)^2]$$

$$=E[(Y^2-2Y\sum_{j=1}^n b_jX_j+(\sum_{j=1}^n b_jX_j)^2]$$

$$=E(Y^2)-2\sum_{j=1}^n b_jE(YX_j)+\sum_{j=1}^n E(b_j^2X_j^2)$$

$$=E(Y^2)+1. \blacksquare$$

Our next proposition shows that for each finite $P\geq 1$ the space $L^P(\Omega,F,P)$ is a Banach space for the norm $\|.\|_P$. Further, $L^2(\Omega,F,P)$ is Hilbert space for the inner product $\langle X,Y\rangle=E(XY)$.
Proposition 2.2 [5]: For each $P\geq 1$ and σ −field F , the space $L^P(\Omega,F,P)$ is a Banach space for the norm $\|.\|_P$. Further, $L^2(\Omega,F,P)$ is Hilbert space.

If L^2 - martingale is a closed subspace of, $L^2(\Omega,F,P)$ the X_n is called the projection of X_{n+1} on L^2 - martingale. The following result helps to justify terminology.

Theorem 2.3: If X_n is an L^2- martingale, let L^2- martingale be a closed subspace of a Hilbert space $L^2(\Omega,F,P)$ and $X_{n+1}\in L^2(\Omega,F,P)$ then $\|X_{n+1}-X_n\|_2=\inf\{\|X_{n+1}-Y\|_2 : Y\in L^2-martingale\}$. ...(1)
If and only if it satisfies the orthogonality relations

$$E[(X_{n+1}-X_n)Z]=0 \quad \text{…(2)}$$

for all $Z\in L^2$ - martingale. Then it's said that X_n is orthogonal projection of X_{n+1} .

Proof: If $X_n \in L^2$ - martingale satisfies (1) then considering $Y = X_n + \alpha Z$ for any $Z \in L^2$ - martingale and $\alpha \in R$,

$$0 \le \|X_{n+1} - Y\|_2^2 - \|X_{n+1} - X_n\|_2^2$$
$$= \|X_{n+1} - X_n - \alpha Z\|_2^2 - \|X_{n+1} - X_n\|_2^2$$
$$= \|X_{n+1} - X_n\|_2^2 - 2\alpha\langle X_{n+1} - X_n, Z\rangle + \|\alpha Z\|_2^2 - \|X_{n+1} - X_n\|_2^2$$
$$= \alpha^2 E(Z^2) - 2\alpha E[(X_{n+1} - X_n)Z]$$

By elementary calculus, this inequality holds for all $\alpha \in R$ if and only if $E[(X_{n+1} - X_n)Z] = 0$

Conversely: suppose $X_n \in L^2$ - martingale satisfies (2) and fix $Y \in L^2$-martingale then, considering $Z = Y - X_n$ we have

$$\|X_{n+1} - Y\|_2^2 = \|X_{n+1} - X_n - Z\|_2^2$$
$$= \|X_{n+1} - X_n\|_2^2 - 2\langle X_{n+1} - X_n, Z\rangle + \|Z\|_2^2$$
$$= \|X_{n+1} - X_n\|_2^2 - 2E[(X_{n+1} - X_n)Z] + \|Y - X_n\|_2^2$$
$$= \|X_{n+1} - X_n\|_2^2 + \|Y - X_n\|_2^2 \ge \|X_{n+1} - X_n\|_2^2$$

Therefore $\|X_{n+1} - Y\|_2 \ge \|X_{n+1} - X_n\|_2$, thus. ▌

$$\|X_{n+1} - X_n\|_2 = \inf\{\|X_{n+1} - Y\|_2\}$$

We may give still another way of characterizing the projection of X_{n+1} on L^2 - martingale.

Theorem 2.4: Let L^2 - martingale be a closed subspace of the Hilbert space $L^2(\Omega, F, P)$ .if $X_{n+1} \in L^2(\Omega, F, P)$ then X_{n+1} has a unique representation $X_{n+1} = X_n + Y$ where $X_n \in L^2$ - martingale and $Y \perp L^2$ - martingale .Furthermore , X_n is the projection of X_{n+1} on L^2 - martingale.

Proof: Let X_n^* be the projection of X_{n+1} on L^2 - martingale, $X_n = X_n^*$, $Y = X_{n+1} - X_n^*$.

By theorem (2.3), $Y \perp L^2$ - martingale ,proving the existence of the desired representation

To proof uniqueness , let $X_{n+1} = X_n + Y = X_n' + Y'$ where $X_n, X_n' \in L^2$ - martingale, $Y, Y' \perp L^2$ - martingale. Then $X_n - X_n' \in L^2$ - martingale since L^2 - martingale is a closed subspace and $X_n - X_n' \perp L^2$ - martingale .

Since. $X_n - X_n' = Y' - Y$ Thus $X_n - X_n'$ is orthogonal to itself that is, $E(X_n - X_n')^2 = 0$ then $\|X_n - X_n'\|_2^2 = 0$, implies that, $X_n - X_n' = 0$ hence $X_n = X_n'$ and $Y' - Y = 0$ proving uniqueness. ▌

2.5 Definition: If A any subset of $L^2(\Omega, F, P)$, the set

$$A^{\perp} = \{X \in L^2(\Omega, F, P) \mid E(XY) = 0, \forall Y \in A\}$$

is called the orthogonal complement of $L^2(\Omega, F, P)$, and

(2.4) theorem is expressed by saying that if denoted by A to the set of L^2 -martingale then $L^2(\Omega, F, P)$ is the orthogonal direct sum of A and $A^{\perp}$, written

$$L^2(\Omega, F, P) = A \oplus A^{\perp}.$$

Theorem 2.6: If $E(X_n^2) < \infty$ for all n and $\{X_1, X_2 \ldots\}$ is a martingale, implies that the martingale differences $X_1, X_2 - X_1, \ldots, X_n - X_{n-1}, \ldots$ are orthogonal.

Proof: If $m < n$, and $\mathcal{F}_m = \mathcal{F}(X_1, \ldots, X_m)$

$E[(X_m - X_{m-1})(X_n - X_{n-1})]$
$= E[E[(X_m - X_{m-1})(X_n - X_{n-1}) | \mathcal{F}_m]]$
$= E[(X_m - X_{m-1}) E[(X_n - X_{n-1})] | \mathcal{F}_m]$
$= 0$

Since $X_m - X_{m-1}$ is $\mathcal{F}_m$-measurable but,
$E[(X_n - X_{n-1}) | \mathcal{F}_m] = X_m - X_m = 0$ (since X_n is a martingale).[1]

Hence the formula $X_n = X_0 + \sum_{K=1}^{n} (X_k - X_{k-1})$ expresses X_n as the sum of orthogonal terms , and pythagor's theorem yields,

. $E(X_n^2) = E(X_0^2) + \sum_{K=1}^{n} E[(X_k - X_{k-1})^2]$ ▌[3]

Example2.7: Let $\{X_n : n \geq 1\}$ be a sequence of random variables. Assume that $S_n = \sum_{i=1}^{n} X_i$, is a martingale. Then for any $i \neq j$, $E(X_i X_j) = 0$. [7]

Solution: If $i > j$

$S_1 = X_1 \quad , S_2 = X_1 + X_2 \quad , \ldots, S_n = X_1 + X_2 + \ldots + X_n$

$X_i = S_i - S_{i-1}$

$E(X_i X_j) = E[E(X_i X_j | \mathcal{F}_{i-1})]$
$= E[X_j E(X_i | \mathcal{F}_{i-1})]$
$= E[X_j E(S_i - S_{i-1} | \mathcal{F}_{i-1})]$
$= E[X_j [E(S_i | \mathcal{F}_{i-1}) - E(S_{i-1} | \mathcal{F}_{i-1})]]$
$= E[X_j (S_{i-1} - S_{i-1})]$
(by the martingale property)
$= 0$. ▌

$X \in \mathcal{L}^2$ - martingale is closest to Y if and only if $Y - X \perp \mathcal{L}^2$ - martingale.

Theorem 2.8 : Let $\mathcal{L}^2(\Omega, \mathcal{F}, P)$ be a vector space and $\mathcal{L}^2$ - martingale$\subset \mathcal{L}^2(\Omega, \mathcal{F}, P)$ be a subspace and $Y \in \mathcal{L}^2(\Omega, \mathcal{F}, P)$ .If $Y - X \perp \mathcal{L}^2$ - martingale when $X \in \mathcal{L}^2$ - martingale, then $\|Y - X\|_2 \leq \|Y - Z\|_2$ for all $Z \in \mathcal{L}^2$ - martingale and $\|Y - X\|_2 = \|Y - Z\|_2$ if and only if $X = Z$ .Thus X is the member of $\mathcal{L}^2$ - martingale closest to Y.

Proof: $\|Y - Z\|_2^2 = \|(Y - X) + (X - Z)\|_2^2$
$= E[((Y - X) + (X - Z))^2] \quad = E[((Y - X)^2 + 2(Y - X)(X - Z) + (X - Z)^2]$
$= E(Y - X)^2 + 2E[(Y - X)(X - Z)] + E(X - Z)^2$
$= E(Y - X)^2 + E(X - Z)^2$
(Because $Y - X \perp \mathcal{L}^2$ -martingale and $X - Z \in \mathcal{L}^2$ _martingale).
$= \|Y - X\|_2^2 + \|X - Z\|_2^2$
$\geq \|Y - X\|_2^2$ (since $\|X - Z\|_2^2 \geq 0$) …(3)

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So . $\|Y - X\|_2 \le \|Y - Z\|_2$

If $\|Y - X\|_2 = \|Y - Z\|_2$, then we see by using (3) that $\|X - Z\|_2^2 = 0$, hence X=Z .

Conversely: If X=Z , it is clear that $\|Y - X\|_2 = \|Y - Z\|_2$.

Theorem 2.9: Let $L^2(\Omega, F, P)$ be a vector space and L^2 - martingale $\subset L^2(\Omega, F, P)$ and assume that $\{X_1, X_2 \ldots, X_n\}$ is an orthogonal basis for L^2 -martingale .For $Y \in L^2(\Omega, F, P)$, let $X = \sum_{j=1}^{n} \frac{E(YX_j)}{\|X_j\|_2^2} X_j \in L^2$ -martingale .Then $Y - X \perp L^2$ -martingale or (equivalently, X is the vector in L^2 -martingale closest to Y).

Proof: We must prove that $E[(Y - X)X_j] = 0$

$$
\begin{aligned}
E[(Y - X)X_j] &= E(YX_j) - E(XX_j) \\
&= E(YX_j) - E[\sum_{k=1}^{n} \frac{E(YX_k)}{\|X_k\|_2^2} X_k X_j] \\
&= E(YX_j) - \sum_{k=1}^{n} \frac{E(YX_k)}{\|X_k\|_2^2} E[X_k X_j] \\
&= E(YX_j) - \frac{E(YX_j)}{\|X_j\|_2^2} E(X_j^2) \\
&= E(YX_j) - \frac{E(YX_j)}{\|X_j\|_2^2} \|X_j\|_2^2 \\
&= E(YX_j) - E(YX_j) = 0
\end{aligned}
$$

Hence $Y - X \perp X_j$, implies that $Y - X \perp L^2$ -martingale.

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