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Weak Convergence of Iterative Approximation for Fixed Points of A Monotone α-nonexpansive Mapping

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Abstract: In this paper, a modified approximation method is introduced for finding the fixed point of α-nonexpansive mapping in a uniformly convex Banach space with a partial order. Moreover, the weak convergence theorems of the fixed point of α-nonexpansive mapping are established under Opial and weak-Opial conditions, respectively.
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1. Introduction

Let E be a Banach space whose norm and partial order are expressed as $\|\cdot\|$ and $\leq$, respectively. Let C be a nonempty, closed, and convex subset of the Banach $(E,\leq)$ space. Let $T:C\to E$ be a nonlinear mapping, and $F(T)$ denote the fixed point set of T, i.e., $F(T)=\{x\in C, Tx=x\}$.

For any $x,y\in C$,

① If $Tx\leq Ty$ whenever $x\leq y$, then T is called monotone;

② If T is monotone and $\|Tx-Ty\|\leq\|x-y\|$ whenever $x\leq y$, then T is called monotone nonexpansive;

③ If $\|Tx-p\|\leq\|x-p\|$ for all $p\in F(T)$ and $x\leq p$ or $p\leq x$, then T is called a monotone quasi-nonexpansive mapping;

④ If T is monotone and there exists a constant $\alpha<1$, which makes

$$\|Tx-Ty\|^2\leq\alpha\|Tx-y\|^2+\alpha\|Ty-x\|^2+(1-2\alpha)\|x-y\|^2 \tag{1}$$

whenever $x\leq y$, then T is called a monotone α-nonexpansive mapping. Obviously, monotone α-nonexpansive mappings contain monotone non-expansive mappings (0-monotone nonexpansive mappings) , and monotone α- nonexpansive mappings with nonempty fixed point sets must be monotone quasi-nonexpansive mappings.

In 1965, Browder proved that each nonexpansive mapping has a fixed point in the uniformly convex Banach space [1]. After that, the fixed point theory and method of nonlinear mapping have been continuously developed and improved. The basic methods for approximating the fixed points of a nonlinear mapping are Mann iteration and Ishikawa iteration [2-5]. In 2004, Ran-Reurings [6] introduced partial order in the metric space to study the principle of compression mapping under Lipschitz conditions, and fixed point method is successfully applied to solve the positive solution problem of matrix equations. A new research field of the fixed-point problem of nonexpansive mapping is thus expanded. However, monotone mapping based on partial order definition may be not continuous, which will make problems, such as the existence of the fixed point of monotone mapping, the positive and negative solutions to the fixed point problem, and the convergence analysis of the approximation method more difficult.

In 2015, Bin Dehaish-Khamsi [7] investigated the Mann iterations for approximating the fixed points of monotone nonexpansive mappings:

$$x_{n+1} = (1-a_n)x_n + a_n T x_n, \qquad (2)$$

where $\alpha_n \in [0,1]$. And the existence and weak convergence theorems of fixed points of monotone nonexpansive mappings are proved in Banach space. In 2016, Song et al. [8] further extended Mann iterations to monotone α -nonexpansive mappings. However, in the convergence analysis, the particularity of the proved techniques is limited, and the approximation of the fixed point of monotone α -nonexpansive mapping is not extended to the multi-step iterative method such as Ishikawa.

Besides, Agrawal et al . [9] introduced an iterative method similar to Ishikawa's, which improves the convergence rate of approximating fixed points of nearly asymptotically nonexpansive mappings. Based on this, Abbas-Nazir [10] further introduces a faster iteration method for optimization and feasible problems:

$$\begin{cases} z_n = (1-\gamma_n)x_n + \gamma_n T x_n, \\ y_n = (1-\beta_n)Tx_n + \beta_n T z_n, \\ x_{n+1} = (1-\alpha_n)Ty_n + \alpha_n T z_n, \end{cases} \qquad (3)$$

where $\alpha_n, \beta_n, \gamma_n \in [0,1]$.

This paper introduces the partial order into the uniformly convex Banach space, and applies equation (3) into the investigation of the numerical method for approximating the fixed point of the monotone α -nonexpansive mapping. The weak convergence theorems of the monotone α -nonexpansive mapping fixed point are established under the Opial condition and the weak-Opial condition, respectively. The obtained results will improve corresponding conclusions in the literature [7-10].

2. Preliminaries

Let E be a Banach space, for any $x, y \in E$, if $\|x\| = \|y\| = 1$ and $x \neq y$, $\left\|\frac{1}{2}(x+y)\right\| < 1$ holds, then E is said to be strictly convex; for any $x, y \in E$ and $\varepsilon \in (0,2]$, if $\|x\| = \|y\| = 1$ and $\|x-y\| \geq \varepsilon$, $\frac{1}{2}\|x+y\| < 1-\delta$ holds for some $\delta > 0$, then E is said to be uniformly convex.

Let $\{x_n\}$ be any sequence in the order Banach space $(E, \leq)$, and "$\rightharpoonup$" and "$\rightarrow$" respectively represent the weak convergence and strong convergence of the sequence $\{x_n\}$. Suppose $\{x_n\}$ weakly converges to x, if $\limsup_{n\to\infty}\|x_n - x\| < \limsup_{n\to\infty}\|x_n - y\|$, $\forall y \in E, y \neq x$ is set up, then E is said to satisfy the Opial condition; if $\liminf_{n\to\infty}\|x_n - x\| < \liminf_{n\to\infty}\|x_n - y\|$, $\forall y \in E, y \neq x$ is set up, then E is said to satisfy the weak-Opial condition. At the same time, for any $a, b \in E$, define interval $[a,b] = \{x \in E : a \leq x \leq b\} = [a,+) \cap (-,b]$, where $[a,+) = \{x \in E : a \leq x\}$ and $(-,b] = \{x \in E : x \leq b\}$. For any $\lambda \in [0,1]$, if interval $[a,b]$ contains $a \leq \lambda a + (1-\lambda)b \leq b$, then the order interval $[a,b]$ is called convex. This paper assumes that the interval $[a,b]$ is closed convex, and $F_{\leq}(T)=\{p\in F(T):x_1\leq p\}$ and $F_{\geq}(T)=\{p\in F(T):p\leq x_1\}$.

Lemma 1[8] Let C be a nonempty closed convex subset of the order Banach space $(E,\leq)$. Let $T:C\to C$ be a monotone α-nonexpansive mapping, then for any $x,y\in C$ and $x\leq y$, we have

$$\|Tx-Ty\|^2\leq\|x-y\|^2+\frac{2\alpha}{1-\alpha}\|Tx-x\|^2+\frac{2|a|}{1-a}\|Tx-x\|(\|x-y\|+\|Tx-Ty\|)$$

The necessary and sufficient condition for Lemma 2[11] for any $x,y\in B_r(0)=\{x\in E:\|x\|\leq r,r>0\}$, Banach space E uniformly convex is that there is a strictly convex continuous function $f:[0,+\infty)\to[0,+\infty)$ that satisfies $f(0)=0$ and

$$\|\lambda x+(1-\lambda)y\|^2\leq\lambda\|x\|^2+(1-\lambda)\|y\|^2-\lambda(1-\lambda)f(\|x-y\|),\ \forall\lambda\in[0,1]$$

Lemma 3[12] Let E be a uniformly convex Banach space, and sequence $\{\lambda_n\}$ satisfies condition $0<\liminf_{n\to\infty}\lambda_n\leq\limsup_{n\to\infty}\lambda_n<1$. If sequence $\{x_n\},\{y_n\}$ in E satisfy $\limsup_{n\to\infty}\|x_n\|\leq r,\limsup_{n\to\infty}\|y_n\|\leq r$ and $\lim_{n\to\infty}\|\lambda_n x_n+(1-\lambda_n)y_n\|=r$, then $\lim_{n\to\infty}\|x_n-y_n\|=0$.

Lemma 4 Let C be a nonempty closed convex subset of uniformly convex Banach spaces $(E,\leq)$, and $T:C\to C$ is a monotone α-nonexpansive mapping. If $x_1\leq Tx_1$, the sequence $\{x_n\}$ defined by equation (3) satisfies:

(i) $x_n\leq z_n\leq Tx_n\leq y_n\leq Tz_n\leq x_{n+1}\leq Ty_n$;

(ii) If $x_n\leq x$, the sequence $\{x_n\}$ weakly converges to point $x\in C$, that is $x_n\rightharpoonup x$.

Proof (i) Since the sequence interval $[c_1,c_2]$ is convex, so for any $\lambda\in[0,1]$, $c_1\leq\lambda c_1+(1-\lambda)c_2\leq c_2$ is established, where $c_1,c_2\in C$ and $c_1\leq c_2$. On the other hand, since $x_1\leq Tx_1$, by using the monotonicity of T, it is easy to obtain $x_n\leq Tx_n$, and then

$$\begin{aligned}x_n&\leq(1-\gamma_n)x_n+\gamma_nTx_n=z_n\\&\leq(1-\gamma_n)Tx_n+\gamma_nTx_n=Tx_n,\end{aligned}$$

Using the monotonicity of T, it is obtained that $x_n\leq z_n\leq Tx_n\leq Tz_n$. Meanwhile, from formula (3) we have

$$\begin{aligned}x_n&\leq(1-\beta_n)Tx_n+\beta_nTx_n=Tx_n\\&\leq(1-\beta_n)Tx_n+\beta_nTz_n=y_n\\&\leq(1-\beta_n)Tz_n+\beta_nTz_n=Tz_n,\end{aligned}$$

Hence $x_n\leq z_n\leq Tx_n\leq y_n\leq Tz_n\leq Ty_n$. Similarly, combining formula (3) further we have

$$\begin{aligned}x_n&\leq(1-\alpha_n)Tz_n+a_nTz_n=Tz_n\\&\leq(1-a_n)Ty_n+\alpha_nTz_n=x_{n+1}\\&\leq(1-\alpha_n)Ty_n+a_nTy_n=Ty_n\end{aligned}$$

Finally $x_n\leq z_n\leq Tx_n\leq y_n\leq Tz_n\leq x_{n+1}\leq Ty_n$ is obtained, which contains $x_{n+1}\leq Tx_{n+1}$,

(ii) Lecture 3.1 by (i) and Bin Dehaish-Khamsi [7], similarly available (ii).

3. Main Results

Theorem 1 sets C as a non-empty closed convex subset of uniformly convex Banach spaces $(E,\leq)$ and E satisfies the Opial condition. Let $T:C\to C$ be a monotone α-nonexpansive mapping and $F_{\leq}(T)\neq\varnothing$. For a given $x_1\leq Tx_1$, if $\alpha_n\in[a,b]\subset(0,1),\beta_n,\gamma_n\in(0,1)$ and $0<\liminf_{n\to\infty}\gamma_n\leq\limsup_{n\to\infty}\gamma_n<1$, then by equation (3) The defined sequence $\{x_n\}$ weakly converges to $q\in F_{\leq}(T)$.

First, prove that $\{x_n\}$ is bounded. Take $p\in F_{\leq}(T)=\{p\in F(T):x_1\leq p\}$, then the monotony of T yields

$$x_1\leq Tx_1\leq Tp=p. \tag{4}$$

From equations (3) and (4) we can get

$$z_1=(1-\gamma_1)x_1+\gamma_1Tx_1\leq p,\ Tz_1\leq Tp=p,$$
$$y_1=(1-\beta_1)\mathrm{T}x_1+\beta_1Tz_1\leq p,\ Ty_1\leq Tp=p,$$
$$x_2=(1-\alpha_1)Ty_1+\alpha_1Tz_1\leq p,\ Tx_2\leq Tp=p,$$

Assume $x_n\leq p,Tx_n\leq Tp=p,$ and $y_n\leq p,Ty_n\leq Tp=p,z_n\leq p,Tz_n\leq Tp=p.$ By Lemma 4 we can get

$$x_n\leq z_n\leq Tx_n\leq y_n\leq Tz_n\leq x_{n+1}\leq Ty_n\leq p, \tag{5}$$

This implies $x_{n+1}\leq p$. Therefore, the sequence $\{x_n\}$ is bounded. Similarly, the sequences $\{y_n\}$ and $\{z_n\}$ are bounded.

Secondly, prove that $\lim_{n\to\infty}\|x_n-Tx_n\|=0$. Derived from the quasi-nonexpansion of equations (3) and T we can get

$$\begin{aligned}\|z_n-p\|&=\|(1-\gamma_n)x_n+\gamma_nTx_n-p\|\\&\leq(1-\gamma_n)\|x_n-p\|+\gamma_n\|Tx_n-p\|\end{aligned} \tag{6}$$

$$\leq(1-\gamma_n)\|x_n-p\|+\gamma_n\|Tx_n-p\|$$

Similarly, using equations (3) and (6) we can get

$$\begin{aligned}\|y_n-p\|&\leq(1-\beta_n)\|Tx_n-p\|+\beta_n\|Tz_n-p\|\\&\leq(1-\beta_n)\|x_n-p\|+\beta_n\|z_n-p\|\\&\leq\|x_n-p\|\end{aligned} \tag{7}$$

Combining equations (3), (6) and (7) further we can get

$$\begin{aligned}\|x_{n+1}-p\|&\leq(1-\alpha_n)\|Ty_n-p\|+\alpha_n\|Tz_n-p\|\\&\leq(1-\alpha_n)\|y_n-p\|+a_n\|z_n-p\|\\&\leq\|x_n-p\|\end{aligned} \tag{8}$$

That is, the sequence $\{\|x_n-p\|\}$ is monotonous and bounded, so the limit exists, and denote $\lim_{n\to\infty}\|x_n-p\|=r.$ By the formula (6) we can obtain

$$\limsup_{n\to\infty}\|z_n - p\| \le \limsup_{n\to\infty}\|x_n - p\| = r. \tag{9}$$

From equations (6), (7), and (8) we can get

$$\begin{aligned}\|x_{n+1}-p\| &\le (1-\alpha_n)\|y_n - p\| + \alpha_n\|z_n - p\| \\ &\le (1-\alpha_n)\left[(1-\beta_n)\|x_n - p\| + \beta_n\|z_n - p\|\right] + \alpha_n\|z_n - p\| \\ &\le (1-\alpha_n)\|x_n - p\| + \alpha_n\|z_n - p\|\end{aligned} \tag{10}$$

Finishing formula (10) we can get

$$\|x_{n+1}-p\| - \|x_n - p\| \le \frac{\|x_{n+1}-p\| - \|x_n - p\|}{\alpha_n} \le \|z_n - p\| - \|x_n - p\|,$$

That is $\|x_{n+1}-p\| \le \|z_n - p\|$, because $\alpha_n \in [a,b] \subset (0,1)$, takes the limit

$$r = \liminf_{n\to\infty}\|x_{n+1}-p\| \le \liminf_{n\to\infty}\|z_n - p\| \tag{11}$$

At the same time, by equations (9) and (11)

$$\lim_{n\to\infty}\|z_n - p\| = \lim_{n\to\infty}\|(1-\gamma_n)(x_n - p) + \gamma_n(Tx_n - p)\| = r. \tag{12}$$

On the other hand, using the monotonic non-expansion of $p = Tp$ and T

$$\limsup_{n\to\infty}\|Tx_n - p\| \le \limsup_{n\to\infty}\|x_n - p\| = r. \tag{13}$$

Combine equations (12), (13) and lemma 4, we can get

$$\lim_{n\to\infty}\|x_n - Tx_n\| = 0. \tag{14}$$

Finally, it is proved that $x_n \rightharpoonup q \in F_{\le}(T)$. The sequence $\{x_n\}$ is bounded and the lemma 4 we can know that there is a subsequence $\{x_{n_k}\} \subset \{x_n\}$ weakly converging to $q \in C$, at the same time $x_1 \le x_{n_k} \le q$, from equation (14) and lemma 1 we can get

$$\lim_{k\to\infty}\|Tx_{n_k} - Tq\| \le \lim_{k\to\infty}\|x_{n_k} - q\|. \tag{15}$$

(Proof by contradiction) Assume $q \ne Tq$, from (21) and the Opial condition,

$$\begin{aligned}\lim_{k\to\infty}\|x_{n_k} - q\| &< \limsup_{k\to\infty}\|x_{n_k} - Tq\| \\ &\le \limsup_{k\to\infty}\left[\|x_{n_k} - Tx_{n_k}\| + \|Tx_{n_k} - Tq\|\right] \\ &\le \lim_{k\to\infty}\|x_{n_k} - q\|\end{aligned} \tag{16}$$

This is a contradictory conclusion, so $q = Tq$, that is, $q \in F_{\le}(T)$ and $\lim_{n\to\infty}\|x_n - q\|$ exist. Suppose $\{x_n\}$ has another subsequence $\{x_{n_j}\} \subset \{x_n\}$ that weakly converges to $z \ne q$. Similarly, $z \in F_{\le}(T)$ and $\lim_{n\to\infty}\|x_n - z\|$ are present, then

$$\begin{aligned}\lim_{n\to\infty}\|x_n - q\| &= \limsup_{k\to\infty}\|x_{n_k} - q\| \\ &< \limsup_{k\to\infty}\|x_{n_k} - z\| = \lim_{n\to\infty}\|x_n - z\| \\ &= \limsup_{j\to\infty}\|x_{n_j} - z\| < \lim_{j\to\infty}\|x_{n_j} - q\| \\ &= \lim_{n\to\infty}\|x_n - q\|.\end{aligned}$$

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This is also the conclusion of contradiction. Therefore, $z = q$, that is $x_n \rightharpoonup q \in F_{\leq}(T)$

Theorem 2 sets C as a non-empty closed convex subset of uniformly convex Banach spaces $(E, \leq)$ and satisfies the Opial condition. Let $T: C \to C$ be a monotone α -nonexpansive mapping and $F_{\geq}(T) \neq \varnothing$. For a given $Tx_1 < x_1$, if $a_n \in [a,b] \subset (0,1), \beta_n, \gamma_n \in (0,1)$ and $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 1$, then the defined sequence $\{x_n\}$ which is gained from formula (3) is weakly converges to $q \in F_{\geq}(T)$.

Certificate because E meets the Opial condition, $Tx_1 \leq x_1$ and $F_{\geq}(T) \neq \varnothing$, it is similar to the theorem 1.

Theorem 3 Let C be a non-empty closed convex weak subset of uniformly convex Banach spaces $(E, \leq)$ and E satisfy the weak-Opial condition. Let $T: C \to C$ be a monotone α -nonexpansive mapping e and $F_{\leq}(T) \neq \varnothing$. For a given $x_1 \leq Tx_1$, if $a_n \in [a,b] \subset (0,1), \beta_n, \gamma_n \in (0,1)$ and $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 1$, then the sequence $\{x_n\}$ defined by the formula (3) weakly converges to $q \in F_{\leq}(T)$.

It is known that $x_1 \leq Tx_1$, from Lemma 4 we can get $x_n \leq z_n \leq Tx_n \leq y_n \leq Tz_n \leq x_{n+1} \leq Ty_n$. ω_1 and ω_2 are respectively weak points of $\{x_n\}$, then there are two sub-sequences $\{x_{n_j}\} \in \{x_n\}$ and $\{x_{n_k}\} \in \{x_n\}$, and satisfy

$$\lim_{j\to\infty} x_{n_j} = \omega_1 \lim_{k\to\infty} x_{n_k} = \omega_2.$$

Since $\{x_n\}$ is a monotonically increasing sequence, and the sequence interval $[x_k, +)$ is closed convex, so $\omega_i \in [x_k, +), i = 1, 2,$ that is $\{x_n\} \subset (-, \omega_i]$,we can further obtain $\omega_i \in (-, \omega_j], i, j = 1, 2$. Therefore, $\omega_1 = \omega_2$ that is, $\{x_n\}$ has at most one weak convergence point. Meanwhile, since C is a weak tightness of $(E, \leq)$ subset, then $\{x_n\}$ is weakly convergent, denote $x_n \rightharpoonup q$. On the other hand, if $Tq \neq q$, by the weak Opial condition of Theorem 1 and E

$$\liminf_{n\to\infty} \|x_n - q\| < \liminf_{n\to\infty} \|x_n - Tq\| = \liminf_{n\to\infty} \|Tx_n - Tq\| \leq \liminf_{n\to\infty} \|x_n - q\|,$$

This is a contradictory conclusion, so $q = Tq$, that is $x_n \rightharpoonup q \in F_{\leq}(T)$.

Theorem 4 Let C be a non-empty closed convex weak subset of the uniformly convex Banach space $(E, \leq)$ and E satisfy the weak Opial condition. Let $T: C \to C$ be a monotone α -nonexpansive mapping and $F_{\geq}(T) \neq \varnothing$, for a given $Tx_1 \leq x_1$, such as $a_n \in [a,b] \subset (0,1), \beta_n, \gamma_n \in (0,1)$ and $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 1$. The sequence $\{x_n\}$ defined by the equation (3) weakly converges to $q \in F_{\geq}(T)$.

Proof since C is a non-empty closed convex weak subset of $(E, \leq)$, $Tx_1 \leq x_1$ and $F_{\geq}(T) \neq \varnothing$, similarly proved by Theorem 3.

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