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An Online Saddle Point optimization algorithm with Regularization

Yue Xu[1*], Ying Jiang[1], Xin Xie[2], Dequan Li[1]

[1]School of Mathematics and Big Data, Anhui University of Science and Technology, Huainan, Anhui 232001, China

[2]School of Mathematics and Statistics, Huangshan University, Huangshan, Anhui 245041, China

[*]xyxuyue7788@163.com

Abstract. This paper presents an online saddle-point optimization algorithm (OSP) to solve the optimal decision-making problem in economic games. In this setting, the two mutual competing players choose a pair of optimal decisions (Nash equilibrium) at each iteration. Firstly, the Follow the Leader (FTL) algorithm is proposed to update the decisions, and the regularization term is added to stabilize the Nash equilibrium for both players. Secondly, the saddle-point regret (SP-Regret) is used to measure the gap between the cumulative payoffs and the saddle point value of the aggregate payoff functions. To this end, this paper aims to minimize it. Finally, the simulation results show that, under the proposed OSP algorithm, the SP-Regret can still be sublinear with regularization and the decision variables of both players can be constrained to fluctuate within a certain range by adding regularization, which can effectively make the Nash equilibrium stable.

1. Introduction

Oligopoly is a market structure dominated by only a few manufacturers, who tightly depend on each other when making decisions. Under the assumption that the competitors try to act optimally, all manufacturers will make the decision of maximizing their profits. Consequently, these decisions of the manufacturers reach the equilibrium of oligopoly market (called Nash equilibrium)[1-2].

Saddle-point (SP) method[3-4] presented in the literature [5] is essentially a primitive-dual method. In other words, it can effectively tackle the optimization problem with the equality or inequality constraints by alternately updating the decision variables and Lagrangian multiplier variables. Different from the literature [5] with the online gradient descent, this paper updates the above variables by means of the Follow the Leader (FTL). The FTL is mentioned here, as the name implies:

$$x_{t+1} \leftarrow arg \min_{x\in X} \sum_{\tau=1}^{t} L_\tau(x_\tau) \tag{1}$$

Motivated by [5], the literature [6] further proves that the regret of the algorithm is not greater than $O(\sqrt{T})$. The SP-FTL algorithm developed by [7] is used to update the leader's decision variables for seeking the saddle point value of the sum of the payoff functions of the Lagrangian dual transformation, that is, the Nash equilibrium point. Besides, it is verified in [7] that saddle-point regret (SP-Regret) and individual regret (Ind-Regret) cannot simultaneously be sublinear regret.

Motivated by the interesting results of [7] and the equivalence between economic game theory and SP optimization, this paper adds regularization term to the SP-FTL algorithm aiming at achieving stable decision-making, and presents an OSP-RFTL algorithm to solve the optimal decision-making problem in a two-player economic game. In order to more clearly verify the role of regularized FTL, this paper operates an experiment with the comparison of regularization and no regularization.

The paper is organized as follows. Section 2 provides description of symbols. Section 3 models the equivalence between economic game theory and SP optimization. Section 4 introduces and analyzes OSP-RFTL algorithm. Section 5 proves the effectiveness of regularization and the bound of SP-regret. Section 6 contains numerical simulations of the proposed algorithms. Finally, Section 7 draws the conclusion.

2. Notation

The notation $\arg \min_{x^{p_k} \in X} \max_{y^{q_k} \in Y} \sum_{\tau=1}^{t} L_{\tau,(\mathrm{p,q})}(x_\tau^{p_k}, y_\tau^{q_k})$ means that, in a two-player economic game, player q chooses the value (decision) of $y_\tau^{q_*}$ to maximize its sum of payoff functions from the sequence $\mathrm{Y} = \{y_t^{q_1}, y_t^{q_2}, \cdots, y_t^{q_k}\}$, while player p chooses the value of $x_{t+1}^{p_*}$ from the sequence $\mathrm{X} = \{x_t^{p_1}, x_t^{p_2}, \cdots, x_t^{p_k}\}$ to minimize its sum of payoff functions. Each component of this matrix represents the payoff function at time τ, and the pair of decisions $(x_\tau^{p_k}, y_\tau^{q_k})$ are negotiated by player p and player q. The player p has k kinds of decisions and correspondingly player q also has. To simplify the notation, in the following of this paper, $L_{\tau,(\mathrm{p,q})}(x_\tau^{p_k}, y_\tau^{q_k})$ can also be denoted as L.

3. Problem Statement

In a two-player economic game, two players mutually choose a pair of decisions $(x_t, y_t) \in X \times Y$, which can make objective function be minimum for one player but maximize for another player. At each iteration, this pair of decisions is as close as possible to the Nash equilibrium. Coincidentally, the method of SP-FTL can update two interdependent variables.

$$(x_{t+1}, y_{t+1}) \leftarrow arg \min_{x \in X} \max_{y \in Y} \sum_{\tau=1}^{t} L_\tau(x_\tau, y_\tau) \tag{2}$$

It has been shown that the Nash equilibrium decision-making problem is equivalent to the online saddle point optimization problem[8]. In this paper, we apply the optimization method of OSP to seek the optimal pair of decisions in economic game. In order to make the whole market stable, each pair of decisions made by the two players are expected to be able to fluctuate within a given range.

Many literatures have extended the saddle point optimization problem to online learning environment[9]. In order to measure the gap between the cumulative payoffs and the saddle point value of the aggregate payoff functions, the saddle-point regret is introduced and defined as:

$$\mathrm{SP-Regret(T)} = \left| \sum_{t=1}^{T} L_t(x_t, y_t) - \min_{x \in X} \max_{y \in Y} \sum_{t=1}^{T} L_t(x, y) \right| \tag{3}$$

The goal of OSP problem is to select the decisions made by both players jointly, so that the profits of both players are close to Nash equilibrium[10]. In addition, when only one player's benefit is optimized at a time, the standard online convex optimization environment is applicable to OSP problem. Specifically, in order to measure each player's own regret while fixing the other player's decisions, the individual-regret of both players are defined as:

$$\mathrm{Ind-Regret}_x\mathrm{(T)} = \sum_{t=1}^{T} L_t(x_t, y_t) - \min_{x \in X} \sum_{t=1}^{T} L_t(x, y_t) \tag{4}$$

$$\mathrm{Ind-Regret}_y\mathrm{(T)} = \max_{y \in Y} \sum_{t=1}^{T} L_t(x_t, y) - \sum_{t=1}^{T} L_t(x_t, y_t) \tag{5}$$

4. Algorithm Introduction

4.1. Preliminaries

The function f is called H-strongly convex about X → R, if for any $x_1, x_2 \in X$, the following holds:

$$\mathrm{f}(x_1) \geq \mathrm{f}(x_2) + \nabla \mathrm{f}(x_2)^T (x_1 - x_2) + \frac{H}{2} \|x_1 - x_2\|^2 \tag{6}$$

where $\nabla \mathrm{f}(x)$ represents the sub-gradient of f at x. Strong convexity means the problem of $\min_{x \in X} \mathrm{f}(x)$ has a unique solution. Similarly, if L is H-strongly convex-concave[7], then for any fixed $y_t^{q_0} \in Y$, the function $L_{t,(\mathrm{p,q})}(x_\tau^{p_k}, y_\tau^{q_0})$ is H-strongly convex in x, and for any fixed $x_t^{p_0} \in X$, the function $L_{t,(\mathrm{p,q})}(x_\tau^{p_0}, y_\tau^{q_k})$ is H-strongly concave in y. In this case, there exists a unique saddle point for L.

As a result, (x^*, y^*) is called the saddle point of L if there holds:

$$\mathrm{L}(x^*, y) \le \mathrm{L}(x^*, y^*) \le \mathrm{L}(x, y^*) \tag{7}$$

If the following holds, then f is called G-Lipschitz:

$$|\mathrm{f(a)} - \mathrm{f(b)}| \le \mathrm{G}|\mathrm{a} - \mathrm{b}| \tag{8}$$

If the SP-Regret is called sublinear, there holds:

$$|\sum_{t=1}^{T} L_t(x_t, y_t) - \min_{x\in X} \max_{y\in Y} \sum_{t=1}^{T} L_t(x, y)| < \mathrm{O}(T) \tag{9}$$

4.2. Online Saddle-Point FTL algorithm with regularization

When the equivalence between Nash equilibrium point and saddle point has been modelled, the OSP-FTL algorithm update decisions according to the following rule:

$$(x_{t+1}^{p_*}, y_{t+1}^{q_*}) \leftarrow \arg \min_{x^{p_k}\in X} \max_{y^{q_k}\in Y} \sum_{\tau=1}^{t} L_{\tau,(\mathrm{p,q})}(x_\tau^{p_k}, y_\tau^{q_k}) \tag{10}$$

In order to make the market stable, each decision of the two players are expected to be able to fluctuate within a given range. However, the following example shows that, the saddle point obtained by the algorithm (10) of [7] is not always stable. For $x \in [-1,1]$, let $f_1(x) = \frac{1}{2}x$, and for $\tau = 2, \cdots, T$, let f_τ alternate between $-x$ and x.

$$\sum_{\tau}^{t} f_\tau(x) = \begin{cases} \frac{1}{2}x & , t \text{ } is \text{ } odd \\ -\frac{1}{2}x & , otherwise \end{cases} \tag{11}$$

it can be seen that the decision change between $x_t = -1$ and $x_t = 1$, thus the obtained equilibrium is unstable. So how to modify the FTL algorithm so that it does not change the decision frequently, but it can still achieve lower regret? This motivates us to introduce a regularization term[11] for the OSP-FTL algorithm.

Because the Nash equilibrium point is different at each iteration, the regularization is needed to perform on the respective decision. This paper presents an OSP-RFTL algorithm, which is iteratively updated according to the following rule:

$$(\bar{x}_{t+1}^{p_*}, \bar{y}_{t+1}^{q_*}) \leftarrow (x_{t+1}^{p_*}, y_{t+1}^{q_*}) + R_{\tau,(\mathrm{p,q})}(x_\tau^{p_k}, y_\tau^{q_k}) \tag{12}$$

where $R_{\tau,(\mathrm{p,q})}(x_\tau^{p_k}, y_\tau^{q_k})$ is a regularization function which aims to prevent the function $L_{\tau,(\mathrm{p,q})}(x_\tau^{p_k}, y_\tau^{q_k})$ from overfitting.

Algorithm 1 OSP-RFTL

1: Initialize (define memory).

2: Compute $L_t(x, y) = \begin{bmatrix} L_t(x_t^{p_1}, y_t^{q_1}) & \cdots & L_t(x_t^{p_1}, y_t^{q_k}) \\ \vdots & \ddots & \vdots \\ L_t(x_t^{p_k}, y_t^{q_1}) & \cdots & L_t(x_t^{p_k}, y_t^{q_k}) \end{bmatrix}$.

3: Solve $\min_{x^{p_k}\in X} \max_{y^{q_k}\in Y} \sum_{\tau=1}^{t} L_{\tau,(\mathrm{p,q})}(x_\tau^{p_k}, y_\tau^{q_k})$ find the corresponding saddle point$(x_{t+1}^{p_*}, y_{t+1}^{q_*})$.

4: Regularize $\begin{cases} \bar{x}_{t+1}^{p_*} \leftarrow x_{t+1}^{p_*} + \alpha_1 \|x_t^{p_k}\|_2^2 \\ \bar{y}_{t+1}^{q_*} \leftarrow y_{t+1}^{q_*} + \alpha_2 \|y_t^{q_k}\|_2^2 \end{cases}$.

5: Solve $\sum_{\tau=1}^{t} L_{\tau,(\mathrm{p,q})}(\bar{x}_\tau^{p_*}, \bar{y}_\tau^{q_*})$.

6: Get $\mathrm{SP-Regret(T)} = |\sum_{t=1}^{T} L_t(\bar{x}_t^{p_*}, \bar{y}_t^{q_*}) - \min_{x\in X} \max_{y\in Y} \sum_{t=1}^{T} L_t(x_t^{p_k}, y_t^{q_k})|$

5. Convergence Analysis

Supposing $\{L_{t,(p,q)}(x, y)\}_{t=1}^{T}$ are a series of H-strongly convex-concave functions and satisfy the G-Lipschitz condition, we will verify that the SP-Regret can still be sublinear by adding the regularization. Firstly, the following Lemma 1 will analyse a quantity which is similar to SP-Regret, but with decisions $(x_t^{p_*}, y_t^{q_*})$ replaced by $(x_{t+1}^{p_*}, y_{t+1}^{q_*})$.

Lemma 1: Let $\{(x_t^{p_*}, y_t^{q_*})\}_{t=1}^{T}$ be the iterates of OSP-RFTL, the following holds:

$$-\mathrm{G}\sum_{t=1}^{T}\left\|x_t^{p_*} - x_{t+1}^{p_*}\right\|$$
$$\leq \sum_{t=1}^{T} L_{t,(p,q)}\left(x_{t+1}^{p_*}, y_{t+1}^{q_*}\right) - \min_{x^{p_k}\in X}\max_{y^{q_k}\in Y}\sum_{t=1}^{T} L_{t,(\mathrm{p,q})}\left(x_t^{p_k}, y_t^{q_k}\right) + R_{t,(\mathrm{p,q})}\left(x_t^{p_k}, y_t^{q_k}\right)$$
$$\leq -\mathrm{G}\sum_{t=1}^{T}\left\|y_t^{q_*} - y_{t+1}^{q_*}\right\| . \tag{13}$$

Proof: We next use the inductive method to prove the above inequality. When $t = 1$, the following can be established:

$$L_{1,(p,q)}\left(x_2^{p_*}, y_2^{q_*}\right) - \mathrm{G}\left\|y_1^{q_*} - y_2^{q_*}\right\|$$
$$\leq L_{1,(p,q)}\left(x_2^{p_*}, y_2^{q_*}\right)$$
$$= \min_{x_1^{p_k}\in X}\max_{y_1^{q_k}\in Y}\sum_{t=1}^{T} L_{1,(\mathrm{p,q})}\left(x_1^{p_k}, y_1^{q_k}\right) + R_{1,(\mathrm{p,q})}\left(x_1^{p_k}, y_1^{q_k}\right) \tag{14}$$

where the equation on the right is due to the following iterative rule:

$$(x_2^{p_*}, y_2^{q_*}) \leftarrow \arg\min_{x_1^{p_k}\in X}\max_{y_1^{q_k}\in Y} L_{1,(\mathrm{p,q})}\left(x_1^{p_k}, y_1^{q_k}\right) + R_{1,(\mathrm{p,q})}\left(x_1^{p_k}, y_1^{q_k}\right) . \tag{15}$$

Assume that when $t = T - 1$, the following is established:

$$\min_{x^{p_k}\in X}\max_{y^{q_k}\in Y}\sum_{t=1}^{T-1} L_{t,(\mathrm{p,q})}\left(x_t^{p_k}, y_t^{q_k}\right) + R_{t,(\mathrm{p,q})}\left(x_t^{p_k}, y_t^{q_k}\right)$$
$$\geq \sum_{t=1}^{T-1} L_{t,(\mathrm{p,q})}\left(x_{t+1}^{p_*}, y_{t+1}^{q_*}\right) - \mathrm{G}\sum_{t=1}^{T-1}\left\|y_t^{q_*} - y_{t+1}^{q_*}\right\| . \tag{16}$$

Then, for $t = T$,

$$\min_{x^{p_k}\in X}\max_{y^{q_k}\in Y}\sum_{t=1}^{T-1} L_{t,(\mathrm{p,q})}\left(x_t^{p_k}, y_t^{q_k}\right) + R_{t,(\mathrm{p,q})}\left(x_t^{p_k}, y_t^{q_k}\right)$$
$$= \sum_{t=1}^{T-1} L_{t,(\mathrm{p,q})}\left(x_{T+1}^{p_*}, y_{T+1}^{q_*}\right) + L_{T,(\mathrm{p,q})}\left(x_{T+1}^{p_*}, y_{T+1}^{q_*}\right) .$$

By saddle point property inequality (7),

$$\geq \sum_{t=1}^{T-1} L_{t,(\mathrm{p,q})}\left(x_{T+1}^{p_*}, y_T^{q_*}\right) + L_{T,(\mathrm{p,q})}\left(x_{T+1}^{p_*}, y_T^{q_*}\right)$$
$$\geq \sum_{t=1}^{T-1} L_{t,(\mathrm{p,q})}\left(x_T^{p_*}, y_T^{q_*}\right) + L_{T,(\mathrm{p,q})}\left(x_{T+1}^{p_*}, y_T^{q_*}\right) .$$

According to the assumptions (16),

$$\geq \sum_{t=1}^{T-1} L_{t,(\mathrm{p,q})}\left(x_{t+1}^{p_*}, y_{t+1}^{q_*}\right) - \mathrm{G}\sum_{t=1}^{T-1}\left\|y_t^{q_*} - y_{t+1}^{q_*}\right\| + L_{T,(\mathrm{p,q})}\left(x_{T+1}^{p_*}, y_T^{q_*}\right)$$
$$= \sum_{t=1}^{T} L_{t,(\mathrm{p,q})}\left(x_{t+1}^{p_*}, y_{t+1}^{q_*}\right) - \mathrm{G}\sum_{t=1}^{T-1}\left\|y_t^{q_*} - y_{t+1}^{q_*}\right\| + L_{T,(\mathrm{p,q})}\left(x_{T+1}^{p_*}, y_T^{q_*}\right) - L_{T,(\mathrm{p,q})}\left(x_{T+1}^{p_*}, y_{T+1}^{q_*}\right)$$
$$\geq \sum_{t=1}^{T} L_{t,(\mathrm{p,q})}\left(x_{t+1}^{p_*}, y_{t+1}^{q_*}\right) - \mathrm{G}\sum_{t=1}^{T-1}\left\|y_t^{q_*} - y_{t+1}^{q_*}\right\| - \mathrm{G}\left\|y_t^{q_*} - y_{t+1}^{q_*}\right\|$$
$$= \sum_{t=1}^{T} L_{t,(\mathrm{p,q})}\left(x_{t+1}^{p_*}, y_{t+1}^{q_*}\right) - \mathrm{G}\sum_{t=1}^{T}\left\|y_t^{q_*} - y_{t+1}^{q_*}\right\| . \tag{17}$$

Similarly, by inductive method, the left side of the inequality (12) can also be obtained.

$$\min_{x^{p_k}\in X}\max_{y^{q_k}\in Y}\sum_{t=1}^{T} L_{t,(\mathrm{p,q})}\left(x_t^{p_k}, y_t^{q_k}\right) + R_{t,(\mathrm{p,q})}\left(x_t^{p_k}, y_t^{q_k}\right)$$
$$\leq \sum_{t=1}^{T} L_{t,(\mathrm{p,q})}\left(x_{t+1}^{p_*}, y_{t+1}^{q_*}\right) - \mathrm{G}\sum_{t=1}^{T}\left\|x_t^{p_*} - x_{t+1}^{p_*}\right\| \tag{18}$$

Then, based on Lemma 1, the main result of convergence analysis will be presented in the following theorem.

Theorem 1: Let the component of $\{L_t(x_t^{p_k}, y_t^{q_k})\}_{t=1}^{T}$ be an arbitrary sequence of H-strongly convex-concave, G-Lipschitz function. Then, the OSP-RFTL algorithm guarantees SP-Regret(T)

$$= \left|\sum_{t=1}^{T} L_{t,(p,q)}\left(x_t^{p_*}, y_t^{q_*}\right) - \min_{x^{p_k}\in X}\max_{y^{q_k}\in Y}\sum_{t=1}^{T} L_{t,(\mathrm{p,q})}\left(x_t^{p_k}, y_t^{q_k}\right) + R_{t,(\mathrm{p,q})}\left(x_t^{p_k}, y_t^{q_k}\right)\right|$$
$$\leq \frac{8G^2}{H}(1 + \log T) . \tag{19}$$

Proof: By Lemma 1,

$$\mathrm{SP-Regret(T)} \leq \sum_{t=1}^{T} L_{t,(p,q)}\left(x_t^{p_*}, y_t^{q_*}\right) - \sum_{t=1}^{T} L_{t+1,(p,q)}\left(x_{t+1}^{p_*}, y_{t+1}^{q_*}\right) + \mathrm{G}\sum_{t=1}^{T}\left\|y_t^{q_*} - y_{t+1}^{q_*}\right\| . \tag{20}$$

Since $L_{t,(p,q)}$ is G-Lipschitz, then SP-Regret(T)

$$\leq \sum_{t=1}^{T} G\left\|\left(x_t^{p_*}, y_t^{q_*}\right) - \left(x_{t+1}^{p_*}, y_{t+1}^{q_*}\right)\right\| + \mathrm{G}\sum_{t=1}^{T}\left\|y_t^{q_*} - y_{t+1}^{q_*}\right\|$$
$$\leq G\sum_{t=1}^{T}\left(\left\|x_t^{p_*} - x_{t+1}^{p_*}\right\| + \left\|y_t^{q_*} - y_{t+1}^{q_*}\right\|\right) + \mathrm{G}\sum_{t=1}^{T}\left\|y_t^{q_*} - y_{t+1}^{q_*}\right\| .$$

According to the Lemma 2 of literature [7], SP-Regret(T) further

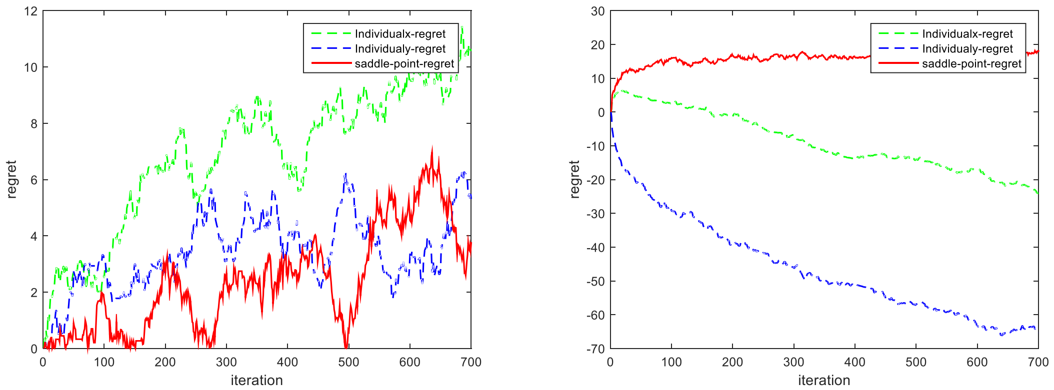
$$\leq G\sum_{t=1}^{T}\frac{4G}{Ht}+G\sum_{t=1}^{T}\frac{4G}{Ht}$$
$$\leq \frac{8G^2}{H}\left(1+\int_1^T\frac{1}{t}dt\right)$$
$$\leq \frac{8G^2}{H}(1+lnT)\ . \tag{21}$$

6. Numerical Simulation

The following example is given to further visualize the above theoretical analysis:

$$L_{t,(p,q)}(x,y)=\begin{cases} xy+\frac{1}{2}\|x-2\|^2-\frac{1}{2}\|x+1\|^2 & ,\left(t=1,\cdots,\frac{T}{3}\right)\\ xy+\frac{1}{2}\|x+1\|^2+\frac{1}{2}\|x+2\|^2 & ,\left(t=\frac{T}{3}+1,\cdots,T\right)\end{cases} \tag{22}$$

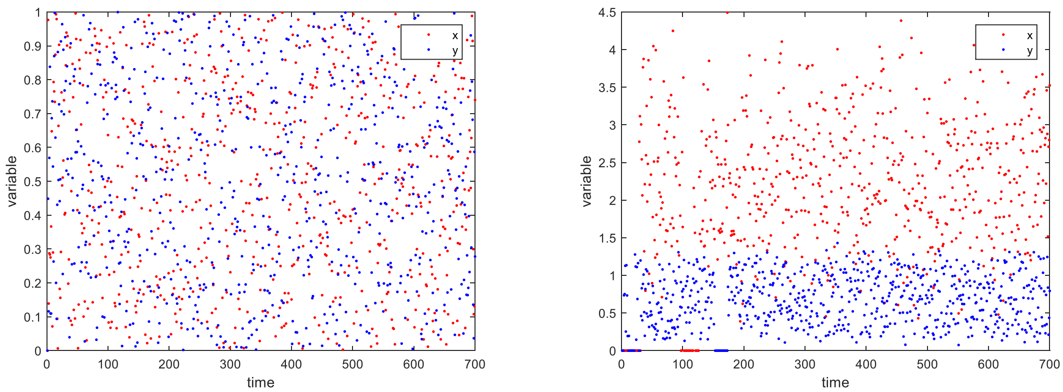
where $L_{t,(p,q)}(x,y)$ represents the payoff function. Let T be the total number of iterations. The initial state values x and y are randomly selected from $(0,1)$. Starting at $t=1$, x and y reach the saddle point value according to the OSP-FTL algorithm in [7].



(a) The regrets without regularization (b) The regrets with regularization

Fig.1 The comparison in regrets

Fig.1(a) clearly shows that, without the regularization, the SP-Regret and the Ind-Regrets intensely fluctuate. As shown in Equation (12), the 2-norm regularization terms $\alpha_1\|x\|_2^2$ and $\alpha_2\|y\|_2^2$ are added to x and y, respectively. The regularization terms are used to control the equilibrium relationship between these two variables: while minimizing the training error, the regularization terms make the Nash equilibrium point stable, which is clearly demonstrated by Fig.1(b).



(a) The decisions of x and y with regularization (b) The decisions of x and y without regularization

Fig.2 The comparison in decision variables

It can be seen from Fig.2(a) that, without regulation, the Nash equilibrium obtained by the OSP-FTL algorithm of [7] fluctuate all the time. In the simulation, we choose the regularization parameters as $\alpha_1=0.9$ and $\alpha_2=0.1$, respectively. Fig.2(b) shows that, by adding regularization, the two decision variables negotiated by the two players are just stable within a certain range. As it can be seen from Fig.2(b), x fluctuates around the value 2, the same as y fluctuates around the value 0.6. By adding regularization term, the Nash equilibrium point can steadily fluctuate within a certain range over time.

7. Conclusion

In this paper, the online saddle point optimization algorithm has been applied to the economic game. First of all, the FTL algorithm has been proposed to solve such decision optimization problems. Besides, regularization has been added to stabilize the decision, and the stability analysis of Nash equilibrium has also been given. In the future, lower regret is expected to ensure better algorithm performance. Combined with the research results on Byzantine-fault tolerance in recent years, our next step is to apply algorithms to economic game so as to effectively monitor and better facilitate the government's anti-monopoly control.

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