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The b-coloring of infinite graphs

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Abstract. We consider the b-coloring number of infinite graphs. We prove that the parameters of the infinite square, triangular and hexagonal lattices are 5, 7 and 4 on the plane. We also obtain that the parameters of infinite square lattices and its induction graph of n-dimensional are $2n+1$ and n^2+n+1.

1. Introduction

The proper k-vertex coloring number $\chi(G)$ of a graph G is the minimum k such that V has a partition $V_1,V_2,...,V_k$ into independent sets. On the basis of define of the proper k-vertex coloring number, the achromatic coloring has been introduced by Garey et al. [1]. The achromatic number $\psi(G)$ of a graph G is the maximum k such that V has a partition $V_1,V_2,...,V_k$ into independent sets, the union of no pair of which is dependent. In 1999, Robert et al. [2] showed that $\psi(G)$ can be viewed as the maximum over all minimal elements of a partial order defined on the set of all coloring of G.

Similarly, Robert et al. [2] put forward concept of a b-coloring. And they proved that determining the b-chromatic number is NP-hard for genera graph, and the b-chromatic number of trees is $m(T)$ or $m(T)-1$ (This metric was upper bounded by the largest integer $m(G)$ for which G has at least $m(G)$ vertices with degree at least $m(G)-1$).

The theory of b-chromatic index attracted many researchers. In 2015 Victor et al. [3] proved that computing the b-chromatic index of a graph G is NP-hard, even complexity of the problem restricted to trees, more specifically, they solved the problem for caterpillars graphs. In 2015, Campos et al. [4] proved that every graph with girth at least 7 has b-chromatic number at least $m(G)-1$. In 2002, Mouider and Maheo [5] proved the determination of two lower bounds for the b-chromatic number of the Cartesian product of two graphs. Marko and Iztok determined lower bound for the b-chromatic number of the Lexicographic product (see [6]).In 2108, Chuan and Mike [7] showed that $K_m \square K_n$ has a upper bound of b-chromatic number, and give different approaches that come close to this bound. We also consider Cartesian powers of general graphs, and show that the Cartesian product of d graphs each with b-chromatic number n is at least $d(n-1)+1$.

Let G be a simple graph; the vertex-set of G, denoted by $V(G)$; the edge-set of G, denoted by $E(G)$; the maximum degree of G, denoted by $\Delta(G)$; denoted by $[k]=\{0,1,\cdots,k-1\}$, $(x)_k = x \bmod k$.

Definition 1[2] For a graph G, suppose that vertices of G are ordered $v_1,v_2,...,v_n$ so that $d(v_1)\geq d(v_2)\geq ...d(v_n)$. Then the $m-$degree, $m(G)$, of G is define by

$m(G) = \max\{1 \le i \le n : d(v_i) \ge i-1\}$.

Definition 2 Let G be a simple graph. A b-coloring is a proper coloring of the vertices of G that has a central set. The b-chromatic number, denoted $\varphi(G)$, is the largest number of colors in any b-chromatic of G.

Lemma 1[2] For any graph G, then $\varphi(G) \le m(G)$

We have straightforward bounds for $\varphi(G)$. For any graph G, then $\chi(G) \le \varphi(G) \le \Delta(G)+1$.

The Cartesian (or box) product of any two graphs G and H, denoted by $G \square H$. The vertex set of $G \square H$ is the Cartesian product $V(G)\times V(H)$ of G and H. There is an edge between two vertices of $G \square H$ if and only if they are adjacent in exactly one coordinate and agree in the other (see [8]).

In this paper, the neighbor sum distinguishing edge coloring of the infinite square, hexagonal and triangular lattices on the plane, and the infinite square lattices and this induction graph of n-dimensional are studied.

The following definition is about the infinite lattices on the plane.

Definition 3[9] Let P_∞ be a path of infinite order. The infinite square lattices on the plane, L, is define by

$$L = P_\infty \square P_\infty .$$

Then for any two vertices (x, y) and (x', y') are adjacent in L if and only if $x = x'$ and $|y - y'| = 1$, or $y = y'$ and $|x - x'| = 1$. For any two vertices (x, y) and (x', y') are adjacent in H if and only if $y = y'$ and $|x - x'| = 1$, or $x = x'$ and $(x+y)_2 = 1$, $|y - y'| = 1$. For any two vertices (x, y) and (x', y') are adjacent in T_r if and only if $x = x'$ and $|y - y'| = 1$, or $y = y'$ and $|x - x'| = 1$, or $x - x' = 1$ and $y - y' = 1$, or $x' - x = 1$ and $y' - y = 1$.

The following definitions are about the infinite square lattices and this induction graph of n-dimensional.

Definition 4 Let P_∞ be a path of infinite order. The infinite square lattices of n-dimensional, L_n, is define by

$$L_n = P_\infty \square P_\infty \square \ldots \square P_\infty \square P_\infty .$$

Then for any two vertices $u = (x_1, x_2, \ldots, x_n)$ and $v = (x_1', x'_2, \ldots, x_n')$ are adjacent in L_n if and only if $v = (x_1, x_2, \ldots, x_i \pm 1, \ldots, x_n)$, where $i \in \{1, 2, \ldots, n\}$. For any two vertices $u = (x_1, x_2, \ldots, x_n)$ and $v = (x_1', x'_2, \ldots, x_n')$ are adjacent in $\tilde{L}_n$ if and only if

$v = (x_1, x_2, \ldots, x_i \pm 1, \ldots, x_n)$ or

$v = (x_1, x_2, \ldots, x_j - 1, \ldots, x_m - 1, \ldots x_n)$ or

$v = (x_1, x_2, \ldots, x_j + 1, \ldots, x_m + 1, \ldots x_n)$,

where $i \in \{1, 2, \ldots, n\}$, $j \in \{1, 2, \ldots, n-1\}$, $m \in \{2, 3, \ldots, n\}$ and $j < m$.

2. Main results and proofs

The following theorem is about the b-coloring of L.

Theorem 1 $\varphi(L) = 5$.

Proof The degree of any vertex of L is 4, then $\Delta(L) = 4$. It is known by Lemma2, that $\varphi(L) \le \Delta(L) + 1 = 5$. Now we prove that $\varphi(L) \ge 5$.

Each vertex u of L is defined by its coordinates, i.e., $u = (x, y)$. Let us define the following coloring: each vertex $u = (x, y)$ is assigned color

$$\sigma(x, y) = (x + 2y)_5 .$$

Clearly, this coloring uses no more than 5 colors.

Let us first show that this is a proper coloring. For this, assume that two neighbors u and v are

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assigned the same color. Form the definition of L, then $u=(x,y)$ and $v=(x\pm1,y)$ or $v=(x,y\pm1)$, by definition of $\sigma(u)$ and $\sigma(v)$ we have

$$(x+2y)_5=(x\pm1+2y)_5 \text{ or } (x+2y)_5=(x+2(y\pm1))_5 .$$

Thus we end up with $0=(\pm j)_5$. However, $j\in\{1,2\}$ hence this is impossible.

Now let us prove that this is a b-chromatic index. For every vertex $u=(x,y)$ of L have four incident vertices, by definition of $\sigma(x,y)$, then $\sigma(v_i)=(x+2y+i)_5$, where $i=1,2,3,4$. If $i\neq j$, then $\sigma(v_i)\neq\sigma(v_j)$, where $j=1,2,3,4$. The colors appearing on incident vertex are different about coloring of σ. Then σ is the b-coloring of L, then $\varphi(L)\geq5$.

Corollary 1 $\varphi(T_r)=7$ and $\varphi(H)=4$.

The proof is similar to proof of theorem1. There is only one different for the construction of b-coloring when each vertex $u=(x,y)$ of T_r (or H) is assigned color

$$\sigma(x,y)=(x+2y)_7 \text{ (or } \sigma(x,y)=(x+2y)_4\text{)}.$$

The following theorem is about the b-chromatic index of L_n.

Theorem 2 $\varphi(L_n)=2n+1$.

Proof The degree of any vertex of L_n is $2n$, then $\Delta(L_n)=2n$. It is known by Lemma2, that $\varphi(L_n)\leq\Delta(L_n)+1=2n+1$. Now we prove that $\varphi(L_n)\geq2n+1$.

Each vertex u of L_n is defined by its coordinates, i.e., $u=(x_1,x_2,...,x_n)$. Let us define the following coloring: each vertex $u=(x_1,x_2,...,x_n)$ is assigned color

$$\sigma(u)=(\sum_{i=1}^{n} ix_i)_{2n+1} .$$

Clearly, this coloring uses no more than $2n+1$ colors.

Let us first show that this is a proper coloring. For this, assume that two neighbors u and v are assigned the same color. Assume also that the coordinates of u and v differ on the j th dimension. Since $u=(x_1,x_2,...,x_n)$ and $v=(x_1,x_2,...,x_j\pm1,...x_n)$, by definition of $\sigma(u)$ and $\sigma(v)$ we have

$$(jx_j+\sum_{i=1,i\neq j}^{n} ix_i)_{2n+1}=(j(x_j\pm1)+\sum_{i=1,i\neq j}^{n} ix_i)_{2n+1} .$$

Thus we end up with $0=(\pm j)_{2n+1}$. However, $j\in[1,n]$ hence this is impossible.

Now let us prove that this is a b-chromatic index. For every vertex $u=(x,y)$ of L_n have $2n$ incident vertices, by definition of $\sigma(x,y)$, then $\sigma(v_i)=(x+2y+i)_{2n+1}$, where $i=1,2,\cdots 2n$. If $i\neq j$, then $\sigma(v_i)\neq\sigma(v_j)$, where $j=1,2,\cdots 2n$. The colors appearing on incident vertex are different about coloring of σ. Then σ is the b-coloring of L_n, then $\varphi(L_n)\geq2n+1$.

The following theorem is about the b-coloring of $\tilde{L}_n$.

Theorem 3 $\varphi(\tilde{L}_n)=n^2+n+1$.

Proof The degree of any vertex of $\tilde{L}_n$ is n^2+n, then $\Delta(\tilde{L}_n)=n^2+n$. It is known by le , that $\varphi(\tilde{L}_n)\leq\Delta(\tilde{L}_n)+1=n^2+n+1$. Now we prove that $\varphi(\tilde{L}_n)\geq n^2+n+1$.

Each vertex u of $\tilde{L}_n$ is defined by its coordinates, i.e., $u=(x_1,x_2,...,x_n)$. Let us define the following coloring: each vertex $u=(x_1,x_2,...,x_n)$ is assigned color

$$\sigma(u)=(\sum_{i=1}^{n} ix_i)_{n^2+n+1} .$$

Clearly, this coloring uses no more than n^2+n+1 colors.

Let us first show that this is a proper coloring. For this, assume that two neighbors u and v are assigned the same color. Assume also that the coordinates of u and v differ on the j th dimension. Since $u=(x_1,x_2,...,x_n)$ and $v=(x_1,x_2,...,x_j\pm1,...x_n)$. Or assume also that the coordinates of u and v differ on the j th and m th dimension. Since $u=(x_1,x_2,...,x_n)$ and $v=(x_1,x_2,...,x_j-1,...,x_m-1,...x_n)$ or $v=(x_1,x_2,...,x_j+1,...,x_m+1,...x_n)$. By definition of $\sigma(u)$ and $\sigma(v)$ we have

$$(jx_j+\sum_{i=1,i\neq j}^{n}ix_i)_{n^2+n+1}=(j(x_j\pm1)+\sum_{i=1,i\neq j}^{n}ix_i)_{n^2+n+1}\text{, or}$$

$$(jx_j+mx_m+\sum_{i=1,i\neq j}^{n}ix_i)_{n^2+n+1}=(j(x_j-1)+m(x_m-1)+\sum_{i=1,i\neq j,i\neq m}^{n}ix_i)_{n^2+n+1}\text{, or}$$

$$(jx_j+mx_m+\sum_{i=1,i\neq j}^{n}ix_i)_{n^2+n+1}=(j(x_j+1)+m(x_m+1)+\sum_{i=1,i\neq j,i\neq m}^{n}ix_i)_{n^2+n+1}.$$

Thus we end up with $0=(\pm j)_{n^2+n+1}$ or $0=(\pm(j+m))_{n^2+n+1}$. However, $j\in[1,n-1],m\in[2,n]$ and $j>m$ hence this is impossible.

Now let us prove that this is a b-chromatic index. For every vertex $u=(x,y)$ of $\tilde{L}_n$ have n^2+n incident vertices, by definition of $\sigma(x,y)$, then $\sigma(v_i)=(x+2y+i)_{n^2+n+1}$, where $i=1,2,\cdots,n^2+n+1$. If $i\neq j$, then $\sigma(v_i)\neq\sigma(v_j)$, where $j=1,2,\cdots,n^2+n+1$. The colors appearing on incident vertex are different about coloring of σ. Then σ is the b- coloring of $\tilde{L}_n$, then $\varphi(\tilde{L}_n)\geq n^2+n+1$.

3. Conclusion

For the b-coloring of the infinite graphs on the plane square, triangular and hexagonal lattices, we obtained the b-coloring number of the infinite square, triangular and hexagonal lattices as Theorem 1 and Corollary 1. Similarly, the b-coloring of the infinite graphs on the n- dimensional, we obtained the parameters of infinite square lattices and its induction graph as Theorem 2 and Theorem 3.This paper only considers the b-coloring of the common lattices graphs, and can also other lattices graphs.
infinite square, triangular and hexagonal lattices

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