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Counting independent sets in two class of edge corona product of graphs

Counting independent sets in two class of edge corona product of graphs

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Abstract. We consider the number of independent sets $\sigma(G)$. In general, the problem of determining the value of $\sigma(G)$ is *NP*-complete. For a graph G, the *M-S* index is defined as the total number of its independent sets. In this paper, we mainly discusses the *M-S* index of two classes of edge corona product graphs, and the specific expressions are given.

1. Introduction

In this work we only consider undirected simple connected graphs. Let G be a graph with n vertices. Two vertices of G are said to be independent if they are not adjacent in G. A $k-$independent set of G is a set of k mutually independent vertices. Denote by $\sigma(G,k)$ the number of the $k-$independent set of G. By definition, the empty vertex set is an independent set. Then $\sigma(G,0)=1$ for any graph G. The *M-S* index of a graph G, denoted by $\sigma(G)$, was introduced by Merrifield and Simmons[1] in 1989, which was defined as $\sigma(G)=\sum_{k=0}^{n}\sigma(G,k)$. So $\sigma(G)$ is equal to the total number of independent vertex sets of G. This indices has been important topological parameters in combinatorial chemistry. It is closely related to the boiling point of the substance. For detailed information on the chemical applications, please refer to Gutman and Polansky 1986. For related applications, see the literature[1-2].

Let us first introduce some notation and terminology. Let $G=(V,E)$ be a graph with the vertex set $V=V(G)$ and edge set $E=E(G)$. If a subset V' of $V(G)$, we denote by $G-V'$ the subgraph of G obtained by deleting the vertices of V' and the edge incident with them. Similarly, if a subset E' of $E(G)$, we denote by $G-E'$ the subgraph of G obtained by deleting the edges of E'. If $V'=\{v\}$ and $E'=\{xy\}$ consist of a single element, we use the abbreviations $G-\{v\}$ and $G-\{xy\}$, respectively. We will frequently make use of the following formulas that can be used to compute the *M-S* index recursively. We write $N(v)=\{u \mid uv \in E(G)\}$ and $N[v]=N(v)\cup\{v\}$ for the open and the closed neighborhood of a vertex v in a graph G. We denote by P_n and C_n the path and the cycle on n vertices, respectively. The Fibonacci numbers, 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55,... are defined recursively by $f_0=0, f_1=1$, and for $n\geq 2$, $f_n=f_{n-2}+f_{n-1}$. f_n is extended to negative values of n via Bennet's formula $f_n=\frac{1}{\sqrt{5}}(\phi^n-(-\phi)^{-n})$, where $\phi=\frac{1+\sqrt{5}}{2}$. Similarly, The Lucas numbers are $l_n=f_{n-1}+f_{n+1}$ and $l_n=\phi^n+(-\phi)^{-n}$ for $n\geq 1$. Prodinger and Tichy have confirmed in the literature[3] that in all n order trees T_n, the star S_n has the largest *M-S* index ($\sigma(S_n)=2^{n-1}+1$), and the path P_n has the smallest *M-S* index ($\sigma(P_n)=f_{n+2}$). For instance, Prodinger and Tichy[3] proved, by induction, that $\sigma(P_n)$ and $\sigma(C_n)$, respectively, is the sequence of Fibonacci and Lucas numbers. We refer to the books [2-6] for graph theory terminology and notation not defined in this paper.

Let G and H be simple graph. The corona product[7], denoted by $G\circ H$, is the graph obtained by each vertex of the graph G are connected to all the vertices of a copy of the graph H, respectively. The edge corona product[8], denoted by $G\Diamond H$, is the graph obtained by two end-vertices of each edge of the graph G are connected to all the vertices of a copy of the graph H, respectively. It follows from definition of the edge corona product that for two graphs G and H with $|V(G)|=n_1$, $|E(G)|=m_1$, $|V(H)|=n_2$ and $|E(H)|=m_2$, the graph $G\Diamond H$ has $n_1+m_1n_2$ vertices and $m_1(1+m_2+2n_2)$ edges. We denote by P_n denotes a path on n vertices and C_n a cycle on n vertices. As an example of edge corona of two graphs, it see Figure 1 and Figure 2.

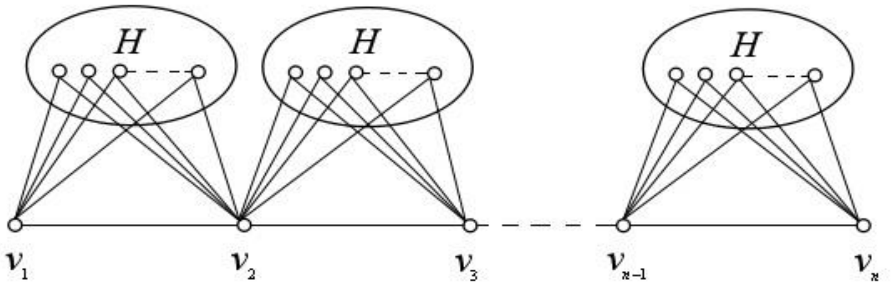

Figure 1: The graph is $P_n\Diamond H$.

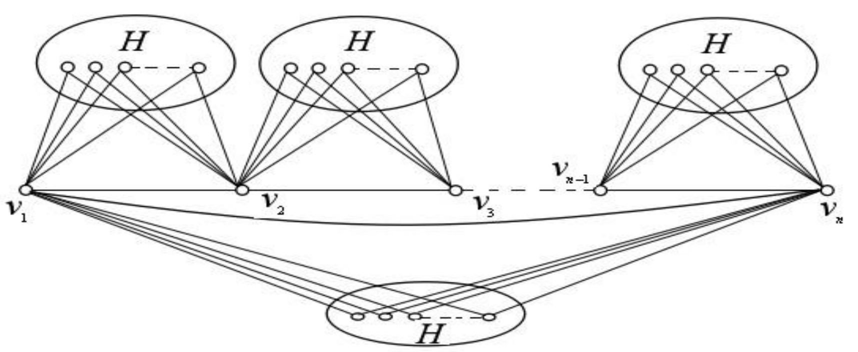

Figure 2: The graph is $C_n\Diamond H$.

According to the definitions of the *M-S* index, to obtain our main results, we first give some lemmas as necessary preliminaries.

Lemma 1[3] Let G be a graph, if $v\in V(G)$, then

$$\sigma(G)=\sigma(G-v)+\sigma(G-N[v]).$$

Lemma 2[3] Let G be a graph, if $uv\in E(G)$, then

$$\sigma(G)=\sigma(G-uv)-\sigma(G-\{N[u]\bigcup N[v]\}).$$

Lemma 3[3] If $G_1,G_2,\cdots,G_t$ are the connected components of G, then

$$\sigma(G)=\prod_{i=1}^{t}\sigma(G_i).$$

Lemma 4 Let H be a simple connected graphs with order m, for the graph A_n shown in Figure 3, we have

$$\sigma(A_n)=\left[\frac{(-\lambda-1)\cdot\sqrt{\lambda^2+4}-\lambda^2-\lambda-2}{\lambda\cdot(-\lambda-\sqrt{\lambda^2+4})-4}\right]\times\left[\frac{\lambda+\sqrt{\lambda^2+4}}{2}\right]^n+\left[\frac{(\lambda+1)\cdot\sqrt{\lambda^2+4}-\lambda^2-\lambda-2}{\lambda\cdot(-\lambda+\sqrt{\lambda^2+4})-4}\right]\times\left[\frac{\lambda-\sqrt{\lambda^2+4}}{2}\right]^n.$$

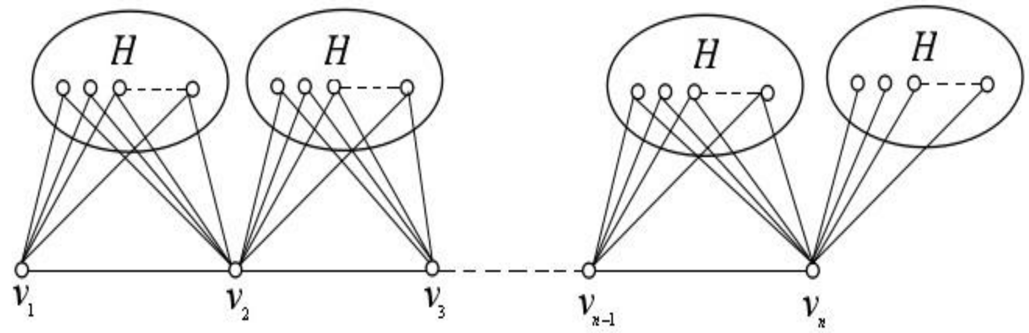

Figure 3: The graph is A_n.

Proof. By Lemma 1-3, we have

$$\sigma(A_n)=\sigma(A_n-v_n)+\sigma(A_n-N_{A_n}[v_n])=\lambda\cdot\sigma(A_{n-1})+\sigma(A_{n-2}),$$

where $\lambda=\sigma(H)$, the characteristic equation is $x^2-\lambda x-1=0$, then the root of the equation is

$$x_1=\frac{\lambda+\sqrt{\lambda^2+4}}{2},\ x_2=\frac{\lambda-\sqrt{\lambda^2+4}}{2}.$$

Therefore, the general solution of the recursive relationship is $\sigma(A_n)=c_1x_1^n+c_2x_2^n$. It is easy to see $\sigma(A_1)=\lambda+1$, $\sigma(A_2)=\lambda^2+\lambda+1$, we have

$$\begin{cases}c_1x_1+c_2x_2=\lambda+1\\ c_1x_1^2+c_2x_2^2=\lambda^2+\lambda+1\end{cases},$$

Hence, we have

$$c_1=\frac{(-\lambda-1)\cdot\sqrt{\lambda^2+4}-\lambda^2-\lambda-2}{\lambda\cdot(-\lambda-\sqrt{\lambda^2+4})-4},c_2=\frac{(\lambda+1)\cdot\sqrt{\lambda^2+4}-\lambda^2-\lambda-2}{\lambda\cdot(-\lambda+\sqrt{\lambda^2+4})-4},$$

Then

$$\sigma(A_n)=\left[\frac{(-\lambda-1)\cdot\sqrt{\lambda^2+4}-\lambda^2-\lambda-2}{\lambda\cdot(-\lambda-\sqrt{\lambda^2+4})-4}\right]\times\left[\frac{\lambda+\sqrt{\lambda^2+4}}{2}\right]^n+\left[\frac{(\lambda+1)\cdot\sqrt{\lambda^2+4}-\lambda^2-\lambda-2}{\lambda\cdot(-\lambda+\sqrt{\lambda^2+4})-4}\right]\times\left[\frac{\lambda-\sqrt{\lambda^2+4}}{2}\right]^n.$$

Lemma 5 Let H be a simple connected graphs with order m, for the graph B_n shown in Figure 4, we have

$$\sigma(B_n)=\left[\frac{(-\lambda^2-1)\cdot\sqrt{\lambda^2+4}-\lambda^3-3\lambda}{\lambda\cdot(-\lambda-\sqrt{\lambda^2+4})-4}\right]\times\left[\frac{\lambda+\sqrt{\lambda^2+4}}{2}\right]^n+\left[\frac{(\lambda^2+1)\cdot\sqrt{\lambda^2+4}-\lambda^3-3\lambda}{\lambda\cdot(-\lambda+\sqrt{\lambda^2+4})-4}\right]\times\left[\frac{\lambda-\sqrt{\lambda^2+4}}{2}\right]^n.$$

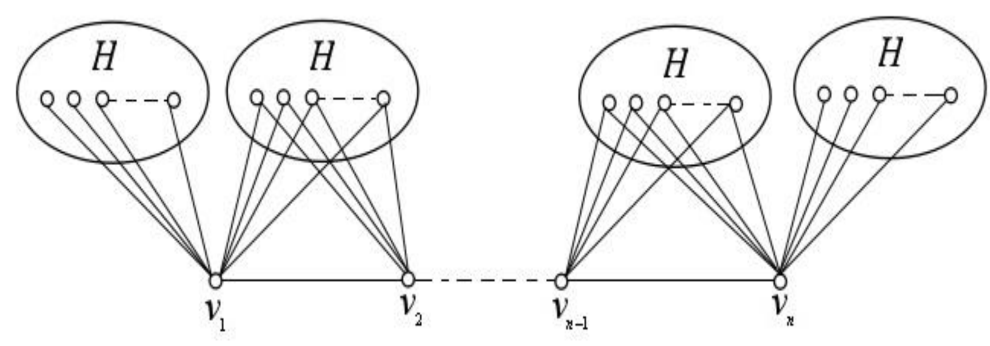

Figure 4: The graph is B_n.

Proof. By Lemma 1-3, we have

$$\sigma(B_n)=\sigma(B_n-v_n)+\sigma(B_n-N_{B_n}[v_n])=\lambda\cdot\sigma(B_{n-1})+\sigma(B_{n-2}),$$

where $\lambda=\sigma(H)$, the characteristic equation is $x^2-\lambda x-1=0$, then the root of the equation is

$$x_1=\frac{\lambda+\sqrt{\lambda^2+4}}{2},\ x_2=\frac{\lambda-\sqrt{\lambda^2+4}}{2}.$$

Therefore, the general solution of the recursive relationship is $\sigma(B_n)=c_1x_1^n+c_2x_2^n$. It is easy to see $\sigma(B_1)=\lambda^2+1, \sigma(B_2)=\lambda^3+2\lambda$, we have

$$\begin{cases} c_1x_1+c_2x_2=\lambda^2+1 \\ c_1x_1^2+c_2x_2^2=\lambda^3+2\lambda \end{cases},$$

Hence, we have

$$c_1=\frac{(-\lambda^2-1)\cdot\sqrt{\lambda^2+4}-\lambda^3-3\lambda}{\lambda\cdot(-\lambda-\sqrt{\lambda^2+4})-4}, c_2=\frac{(\lambda^2+1)\cdot\sqrt{\lambda^2+4}-\lambda^3-3\lambda}{\lambda\cdot(-\lambda+\sqrt{\lambda^2+4})-4},$$

Then

$$\sigma(B_n)=\left[\frac{(-\lambda^2-1)\cdot\sqrt{\lambda^2+4}-\lambda^3-3\lambda}{\lambda\cdot(-\lambda-\sqrt{\lambda^2+4})-4}\right]\times\left[\frac{\lambda+\sqrt{\lambda^2+4}}{2}\right]^n+\left[\frac{(\lambda^2+1)\cdot\sqrt{\lambda^2+4}-\lambda^3-3\lambda}{\lambda\cdot(-\lambda+\sqrt{\lambda^2+4})-4}\right]\times\left[\frac{\lambda-\sqrt{\lambda^2+4}}{2}\right]^n.$$

2. Results and discussion

Theorem 1 Let H be a simple connected graphs of order m, we have

$$\sigma(P_n\lozenge H)=\left[\frac{(-\lambda-1)\cdot\sqrt{\lambda^2+4}-\lambda^2-\lambda-2}{\lambda\cdot(-\lambda-\sqrt{\lambda^2+4})-4}\right]\times\left[\frac{\lambda+\sqrt{\lambda^2+4}}{2}\right]^{n-1}+\left[\frac{(\lambda+1)\cdot\sqrt{\lambda^2+4}-\lambda^2-\lambda-2}{\lambda\cdot(-\lambda+\sqrt{\lambda^2+4})-4}\right]\times\left[\frac{\lambda-\sqrt{\lambda^2+4}}{2}\right]^{n-1}$$
$$+\left[\frac{(-\lambda-1)\cdot\sqrt{\lambda^2+4}-\lambda^2-\lambda-2}{\lambda\cdot(-\lambda-\sqrt{\lambda^2+4})-4}\right]\times\left[\frac{\lambda+\sqrt{\lambda^2+4}}{2}\right]^{n-2}+\left[\frac{(\lambda+1)\cdot\sqrt{\lambda^2+4}-\lambda^2-\lambda-2}{\lambda\cdot(-\lambda+\sqrt{\lambda^2+4})-4}\right]\times\left[\frac{\lambda-\sqrt{\lambda^2+4}}{2}\right]^{n-2}$$

where $\lambda=\sigma(H)$.

Proof. By Lemma 1-3, we have

$$\sigma(P_n\lozenge H)=\sigma(P_n\lozenge H-v_n)+\sigma(P_n\lozenge H-N_{P_n\lozenge H}[v_n])=\sigma(A_{n-1})+\sigma(A_{n-2})$$

Form Lemma 4, we have

$$\sigma(A_{n-1})=\left[\frac{(-\lambda-1)\cdot\sqrt{\lambda^2+4}-\lambda^2-\lambda-2}{\lambda\cdot(-\lambda-\sqrt{\lambda^2+4})-4}\right]\times\left[\frac{\lambda+\sqrt{\lambda^2+4}}{2}\right]^{n-1}+\left[\frac{(\lambda+1)\cdot\sqrt{\lambda^2+4}-\lambda^2-\lambda-2}{\lambda\cdot(-\lambda+\sqrt{\lambda^2+4})-4}\right]\times\left[\frac{\lambda-\sqrt{\lambda^2+4}}{2}\right]^{n-1}$$

$$\sigma(A_{n-2})=\left[\frac{(-\lambda-1)\cdot\sqrt{\lambda^2+4}-\lambda^2-\lambda-2}{\lambda\cdot(-\lambda-\sqrt{\lambda^2+4})-4}\right]\times\left[\frac{\lambda+\sqrt{\lambda^2+4}}{2}\right]^{n-2}+\left[\frac{(\lambda+1)\cdot\sqrt{\lambda^2+4}-\lambda^2-\lambda-2}{\lambda\cdot(-\lambda+\sqrt{\lambda^2+4})-4}\right]\times\left[\frac{\lambda-\sqrt{\lambda^2+4}}{2}\right]^{n-2}$$

Then

$$\sigma(P_n\lozenge H)=\left[\frac{(-\lambda-1)\cdot\sqrt{\lambda^2+4}-\lambda^2-\lambda-2}{\lambda\cdot(-\lambda-\sqrt{\lambda^2+4})-4}\right]\times\left[\frac{\lambda+\sqrt{\lambda^2+4}}{2}\right]^{n-1}+\left[\frac{(\lambda+1)\cdot\sqrt{\lambda^2+4}-\lambda^2-\lambda-2}{\lambda\cdot(-\lambda+\sqrt{\lambda^2+4})-4}\right]\times\left[\frac{\lambda-\sqrt{\lambda^2+4}}{2}\right]^{n-1}$$
$$+\left[\frac{(-\lambda-1)\cdot\sqrt{\lambda^2+4}-\lambda^2-\lambda-2}{\lambda\cdot(-\lambda-\sqrt{\lambda^2+4})-4}\right]\times\left[\frac{\lambda+\sqrt{\lambda^2+4}}{2}\right]^{n-2}+\left[\frac{(\lambda+1)\cdot\sqrt{\lambda^2+4}-\lambda^2-\lambda-2}{\lambda\cdot(-\lambda+\sqrt{\lambda^2+4})-4}\right]\times\left[\frac{\lambda-\sqrt{\lambda^2+4}}{2}\right]^{n-2}$$

Theorem 2 Let H be a simple connected graphs of order m, we have

$$\sigma(C_n\lozenge H)=\left[\frac{(-\lambda^2-1)\cdot\sqrt{\lambda^2+4}-\lambda^3-3\lambda}{\lambda\cdot(-\lambda-\sqrt{\lambda^2+4})-4}\right]\times\left[\frac{\lambda+\sqrt{\lambda^2+4}}{2}\right]^{n-1}+\left[\frac{(\lambda^2+1)\cdot\sqrt{\lambda^2+4}-\lambda^3-3\lambda}{\lambda\cdot(-\lambda+\sqrt{\lambda^2+4})-4}\right]\times\left[\frac{\lambda-\sqrt{\lambda^2+4}}{2}\right]^{n-1}$$
$$+\left[\frac{(-\lambda^2-1)\cdot\sqrt{\lambda^2+4}-\lambda^3-3\lambda}{\lambda\cdot(-\lambda-\sqrt{\lambda^2+4})-4}\right]\times\left[\frac{\lambda+\sqrt{\lambda^2+4}}{2}\right]^{n-3}+\left[\frac{(\lambda^2+1)\cdot\sqrt{\lambda^2+4}-\lambda^3-3\lambda}{\lambda\cdot(-\lambda+\sqrt{\lambda^2+4})-4}\right]\times\left[\frac{\lambda-\sqrt{\lambda^2+4}}{2}\right]^{n-3}$$

where $\lambda=\sigma(H)$.

Proof. By Lemma 1-3, we have

$$\sigma(C_n\lozenge H)=\sigma(C_n\lozenge H-v_n)+\sigma(C_n\lozenge H-N_{C_n\lozenge H}[v_n])=\sigma(B_{n-1})+\sigma(B_{n-3})$$

Form Lemma 5, we have

$$\sigma(B_{n-1})=\left[\frac{(-\lambda^2-1)\cdot\sqrt{\lambda^2+4}-\lambda^3-3\lambda}{\lambda\cdot(-\lambda-\sqrt{\lambda^2+4})-4}\right]\times\left[\frac{\lambda+\sqrt{\lambda^2+4}}{2}\right]^{n-1}+\left[\frac{(\lambda^2+1)\cdot\sqrt{\lambda^2+4}-\lambda^3-3\lambda}{\lambda\cdot(-\lambda+\sqrt{\lambda^2+4})-4}\right]\times\left[\frac{\lambda-\sqrt{\lambda^2+4}}{2}\right]^{n-1}$$
$$\sigma(B_{n-3})=\left[\frac{(-\lambda^2-1)\cdot\sqrt{\lambda^2+4}-\lambda^3-3\lambda}{\lambda\cdot(-\lambda-\sqrt{\lambda^2+4})-4}\right]\times\left[\frac{\lambda+\sqrt{\lambda^2+4}}{2}\right]^{n-3}+\left[\frac{(\lambda^2+1)\cdot\sqrt{\lambda^2+4}-\lambda^3-3\lambda}{\lambda\cdot(-\lambda+\sqrt{\lambda^2+4})-4}\right]\times\left[\frac{\lambda-\sqrt{\lambda^2+4}}{2}\right]^{n-3}$$

Then

$$\sigma(C_n\lozenge H)=\left[\frac{(-\lambda^2-1)\cdot\sqrt{\lambda^2+4}-\lambda^3-3\lambda}{\lambda\cdot(-\lambda-\sqrt{\lambda^2+4})-4}\right]\times\left[\frac{\lambda+\sqrt{\lambda^2+4}}{2}\right]^{n-1}+\left[\frac{(\lambda^2+1)\cdot\sqrt{\lambda^2+4}-\lambda^3-3\lambda}{\lambda\cdot(-\lambda+\sqrt{\lambda^2+4})-4}\right]\times\left[\frac{\lambda-\sqrt{\lambda^2+4}}{2}\right]^{n-1}$$
$$+\left[\frac{(-\lambda^2-1)\cdot\sqrt{\lambda^2+4}-\lambda^3-3\lambda}{\lambda\cdot(-\lambda-\sqrt{\lambda^2+4})-4}\right]\times\left[\frac{\lambda+\sqrt{\lambda^2+4}}{2}\right]^{n-3}+\left[\frac{(\lambda^2+1)\cdot\sqrt{\lambda^2+4}-\lambda^3-3\lambda}{\lambda\cdot(-\lambda+\sqrt{\lambda^2+4})-4}\right]\times\left[\frac{\lambda-\sqrt{\lambda^2+4}}{2}\right]^{n-3}$$

3. Conclusion

For the *M-S* index of the edge corona product with path P_n and of any connected graph H, we obtain the expression of *M-S* index as Theorem 1. Similarly, the *M-S* index of the edge corona product with cycle C_n and of any connected graph H, we obtain the expression of *M-S* index as Theorem 2. This paper only considers the expression of the *M-S* index, and can also consider the *H* index or the ordering problem.

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