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Bayesian Reliability Analysis of Exponential Distribution Model Based on Zero-failure Data and Scale Squared Error Loss Function

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Abstract. Based on the zero failure data, this paper studies the Bayes estimation of the failure rate and reliability of the exponential distribution model under the scale squared error loss function. When the prior distribution of the parameters is the Gamma distribution, the Bayes estimators and E-Bayes estimators of the failure rate and reliability of the exponential model can be obtained. Finally, a practical example is utilized to illustrate the effectiveness of the estimation.

1. Introduction

With the rapid development of high technology, advanced manufacturing technology and high end machinery production, the reliability of products becomes higher and higher, which makes it difficult to observe the failure data. Zero-failure or no-failure data often occurs at the end of the truncation life test. The reliability research based on zero failure data is a meaningful work, which has attracted the attention of many scholars. Based on zero failure data, Han[1] proposed a weighted hierarchical Bayesian estimation method to estimate the reliability parameter of products in the case of zero-failure sampling. Under the condition of a known or unknown probability distribution of product lifetime, Xia[2] proposed a grey bootstrap method based on the information poor theory for the reliability analysis of zero-failure data. The grey bootstrap method can help generate many simulated zero-failure data, and then the generated data is used to estimate the lifetime probability distribution by means of an empirical failure probability function. Zhang[3] proposed a reliability analysis method for time truncated zero-failure data coming from Weibull distribution, which can obtain a high confidence assessment for the reliability of products when the lower limit of the shape and reliability parameters is provided. Jiang et al. [4] developed an interval estimation method to estimate the failure probability for the Weibull model by using the concavity or convexity and property of Weibull distribution function. They also proposed a method by p-value hypothesis testing to determine an approximate value of the shape parameter. For zero-failure data and small number of failure data, Jia et al. [5] developed a match distribution curve method to compute the estimator and confidence interval of parameters for Weibull distribution.

Xu and Chen[6]studied the interval estimation of failure rate and the reliability for exponential distribution with zero-failure data with the help of two-sided Modified Bayesian (M-Bayesian) credible limit. They discussed the properties of two-sided M-Bayesian credible limits which include the impact of the upper bound value of hyper parameter, and the influence of different prior distribution of hyper parameter on two-sided M-Bayesian credible limits. They also discussed the relationship among three kinds of two-sided M-Bayesian credible limits and two-sided classical confidence limits. Yin et al.[7] studied the E-Bayesian estimation for the exponential distribution to deal with the zero-failure life testing data. For the problem of product reliability evaluation under the condition of zero-failure data, Cai et al.[8] combined the weight-least-square with the expected-Bayesian reliability evaluation methods to deal with the reliability analysis of Weibull distribution. Han[9] developed the E-Bayesian estimation and hierarchical Bayesian estimation methods for the estimation of the reliability of Binomial distribution products. For more references about zero-failure data, one can refer to[10-15].

The exponential distribution is the most important continuous distribution. The statistical analysis and application of the model have penetrated into various fields of engineering science, such as reliability, meteorology, hydrology, quality control, and so on[16-18]. For the failure rate estimation of the zero failure data model, a reduction function determination method for the prior distribution of the failure rate is proposed in the document[19]. The existing Bayes estimation problem of the exponential distribution failure rate of zero- failure data is mainly discussed under the square error loss function. Until recently, some scholars have studied the Bayes estimation of the failure rate under other loss functions such as LINEX loss and p, q-symmetric entropy loss functions[20-22]. As an important loss function, the scale squared error loss function has been applied to the Bayes statistical inference in recent years. Under the scale squared error loss function, Song et al.[23] discussed the Bayes estimation of the Poisson distribution parameters, and discussed the admissibility of the estimation, and gave the mathematical expression of the hierarchical Bayes estimation. Yan et al. [24] studied the Bayes estimation of the reliability and failure rate parameters of the Burr- distribution model based on the gradual increase of the type II truncated life test. He and You[25]studied the empirical Bayes estimation of the parameter of the scale exponent distribution family based on positive associate samples, and the asymptotic optimality of the estimation was also discussed.

This article will study the Bayes estimation problem of the failure rate and reliability for the exponential distribution model under the scale squared error loss function with zero-failure data.

2. Zero-failure data model of exponential distribution

Assume that the life of a product distributed with an exponential distribution with the following probability density function

$$f(t)=\lambda e^{-\lambda t}\ ,t>0 \tag{1}$$

In this formula, $\lambda>0$ is the unknown parameter, which is also called the failure rate. $R=\exp(-\lambda t)$ is the reliability parameter. The time truncation test is carried out on the product with an exponential distribution of life (1), t_i (i=1,2,…,m) denotes the censored times under the type-I censored life test; t_i (i=1,2,…,m) satisfies the condition $t_1<t_2<\ldots,t_m$, and n_i is the corresponding sample size of time t_i .The derivation of the likelihood function of exponential distribution parameters for zero failure data is given below:

Assuming that in the i-th time truncation test, a total of X_i samples fail, known by the literature [26], X_i is a random variable distributed with the Poisson distribution with the following mathematical expression

$$P_i(X_i=r_i)=\frac{(n_it_i\lambda)^{r_i}}{(r_i)!}\exp(-n_it_i\lambda) \tag{2}$$

Here $r_i=0,1,2,\cdots,n_i,i=1,2,\cdots,m$

Then the likelihood function of parameter λ is

$$L(x\mid\lambda)=\prod_{i=1}^{m}p_i(X_i=r_i\mid\lambda)=\left[\prod_{i=1}^{m}\frac{(n_it_i\lambda)^{r_i}}{(r_i)!}\right]\exp(-N\lambda)\ , \tag{3}$$

Here $N = \sum_{i=1}^{m} n_i t_i$

In particular, when $r_i = 0$ ($i = 1, 2, \cdots, m$), i.e. zero-failure data case, we have ,

$$L(0 \mid \lambda) = \exp(-N\lambda) \tag{4}$$

That is the likelihood function of parameter λ based on zero failure sample data.

3. Bayes and E-bayes estimation of failure-rate and reliability

In this part, the Bayes and E-Bayes estimation problems of the failure rate and reliability parameters of an exponential distribution model based on the scale squared error loss function will be studied.
The mathematical expression of scale squared error loss function is

$$L(\hat{\theta}, \theta) = \frac{(\theta - \hat{\theta})^2}{\theta^k} \tag{5}$$

Here k is a positive integer. Under the loss function (5), the unique Bayes estimator of parameter θ can be obtained as follows:

$$\hat{\theta}_B = \frac{E(\theta^{1-k} \mid X)}{E(\theta^{-k} \mid X)} \tag{6}$$

Theorem 1. The time truncation test is carried out on the product with an exponential distribution (1), and t_i (i=1,2,…,m) denotes the censored times under the type-I censored life test, and t_i (i=1,2,…,m) satisfies the condition $t_1 < t_2 < \ldots, t_m$ and n_i is the corresponding sample size of time t_i. The measurement data is $(t_i, n_i), i = 1, 2, \cdots, m$. Note $N = \sum_{i=1}^{m} n_i$ and suppose that the prior distribution of failure rate λ is the Gamma distribution with the following density function:

$$\pi(\lambda; a, b) = \frac{b^a}{\Gamma(a)} \lambda^{a-1} \exp(-b\lambda), \quad \lambda > 0 \tag{7}$$

Then under the scale squared error loss function (5), we can get
(i) The Bayes estimator of failure rate λ is

$$\hat{\lambda}_B = \frac{a - k}{b + N} \tag{8}$$

(ii) The Bayes estimator of reliability is

$$\hat{R}_B = \exp(-\hat{\lambda}_B t) \tag{9}$$

Proof. According to formula (4), (7) and Bayes Theorem, the posterior density function of failure rate can be derived as follows:

$$\begin{aligned} h(\lambda \mid 0) &\propto L(0 \mid \lambda) \cdot \pi(\lambda) \\ &\propto e^{-N\lambda} \cdot \frac{b^a}{\Gamma(a)} \lambda^{a-1} e^{-b\lambda} \\ &\propto \lambda^{a-1} e^{-(b+N)\lambda} \end{aligned} \tag{10}$$

According to Eq. (10), it is known that the posterior distribution of λ is Gamma distribution $\Gamma(\alpha, \beta + N)$, and the probability is as follows:

$$h(\lambda \mid 0) = \frac{(b+N)^a}{\Gamma(a)} \lambda^{a-1} e^{-(b+N)\lambda} \tag{11}$$

Then

$$\begin{aligned}E[\lambda^{1-k} \mid 0] &= \int_0^\infty \lambda^{1-k} \frac{(b+N)^a}{\Gamma(a)} \lambda^{a-1} e^{-(b+N)\lambda} d\lambda \\ &= \int_0^\infty \frac{(b+N)^a}{\Gamma(a)} \lambda^{(a+1-k)-1} e^{-(b+N)\lambda} d\lambda \\ &= \frac{(b+N)^a}{\Gamma(a)} \cdot \frac{\Gamma(a+1-k)}{(b+N)^{a+1-k}} \\ &= \frac{\Gamma(a+1-k)}{\Gamma(a)\cdot(b+N)^{1-k}}\end{aligned}$$

$$\begin{aligned}E[\lambda^{-k} \mid 0] &= \int_0^\infty \lambda^{-k} \frac{(b+N)^a}{\Gamma(a)} \lambda^{a-1} e^{-(b+N)\lambda} d\lambda \\ &= \int_0^\infty \frac{(b+N)^a}{\Gamma(a)} \lambda^{(a-k)-1} e^{-(b+N)\lambda} d\lambda \\ &= \frac{(b+N)^a}{\Gamma(a)} \cdot \frac{\Gamma(a-k)}{(b+N)^{a-k}} \\ &= \frac{\Gamma(a-k)}{\Gamma(a)\cdot(b+N)^{-k}}\end{aligned}$$

According to Eq. (6), the Bayes estimator of failure rate λ is

$$\hat{\lambda}_B = \frac{E[\lambda^{1-k} \mid 0]}{E[\lambda^{-k} \mid 0]} = \frac{\dfrac{\Gamma(a+1-k)}{\Gamma(a)\cdot(b+N)^{1-k}}}{\dfrac{\Gamma(a-k)}{\Gamma(a)\cdot(b+N)^{-k}}} = \frac{a-k}{b+N}$$

Then we can get the Bayes estimator of reliability R as follows:

$$\hat{R}_B = \exp(-\hat{\lambda}_B t) .$$

In the case of zero-failure data, Han (1998) proposed the decreasing function method to determine the prior distribution of failure rate. According to Han (1998), a and b should be selected to the guarantee that $\pi(\lambda; a,b)$ is a decreasing function of λ. Because

$$\frac{d\pi(\lambda;a,b)}{d\lambda} = \frac{b^a}{\Gamma(a)} \lambda^{a-1} \exp(-b\lambda)[(a-1)-b\lambda], \tag{12}$$

It is easy to see that, when $\lambda > 0$, $a > 0$ and $b > 0$,then $0 < a \le 1$ and $b > 0$ will result in $\frac{d\pi(\lambda;a,b)}{d\lambda} < 0$. That is to say, $\pi(\lambda; a,b)$ is a decreasing function of λ. Then we choose the prior distributions of hyper parameters a and b as follows:

$$\pi_1(a) = U(0,1),\ \pi_2(b) = U(0,C),\ C \text{ is a constant.} \tag{13}$$

Definition 1 Let $\hat{\theta}_B(a,b)$ as the Bayes estimator of parameter θ , $D = \{(a,b): 0 < a < 1, 0 < b < C\}$, $C > 0$ is a constant, $\pi(a,b)$ is a density function of a and b in D; and then the E-Bayes estimator of parameter θ is defined as follows：

$$\hat{\theta}_{EB} = \iint_D \hat{\theta}_B(a,b)\pi(a,b) da db \tag{14}$$

Theorem 2. The time truncation test is carried out on the product with an exponential distribution(1), and t_i (i=1,2,…,m) s denotes the censored times under the type-I censored life test, t_i (i=1,2,…,m) satisfies the condition $t_1 < t_2 < \dots, t_m$ and n_i is the corresponding sample size of time t_i. The measurement data is $(t_i, n_i), i = 1, 2, \cdots, m$. Note $N = \sum_{i=1}^{m} n_i$ and suppose that the prior distribution of failure rate λ is Gamma distribution (7) and parameters a, b have the prior distribution: $\pi_1(a) = U(0,1),\ \pi_2(b) = U(0, C),$ are constants.

Then under the scale squared error loss function (5), we can get

(i)The E-Bayes estimator of failure rate λ is

$$\hat{\lambda}_{EB} = \frac{1-2k}{2C}\left[(C+N)\left(\ln(C+N)-1\right) - N(\ln N - 1)\right] \tag{15}$$

(ii) The Bayes estimator of reliability is

$$\hat{R}_{EB} = \exp(-\hat{\lambda}_{EB} t) \tag{16}$$

Proof. From Theorem 1, under the scale squared error loss function and Gamma prior, the Bayes estimator of failure rate λ is $\hat{\lambda}_B = \dfrac{a-k}{b+N}$. Then, according to definition1, the E-Bayes estimation of failure rate λ can be derived as follows:

$$\begin{aligned}
\hat{\lambda}_{EB} &= \iint_D \hat{\lambda}(a,b)\pi(a,b)\,da\,db \\
&= \int_0^1 \int_0^C \frac{a-k}{b+N}\pi(a)\pi(b)\,da\,db \\
&= \frac{1}{C}\int_0^C \left[\int_0^1 \frac{a-k}{b+N}da\right]db \\
&= \frac{1}{C}\int_0^C \frac{1-2k}{2(b+N)}db \\
&= \frac{1-2k}{2C}\left[(C+N)\left(\ln(C+N)-1\right) - N(\ln N - 1)\right]
\end{aligned}$$

Thus, the E-Bayes estimator of reliability R can be obtained as

$$\hat{R}_{EB} = \exp(-\hat{\lambda}_{EB} t).$$

4. Practical example and conclusion

The effectiveness and practicability of the Bayes and E-Bayes estimators obtained in this paper are illustrated by an example in (Han, 1998). Suppose that the life of a certain type of engine obeys the exponential distribution (1) with the failure rate parameter λ, and there is no failure in Type-I lifetime test. Then we only get the zero-failure data given in Table 1, which contains 13 sets of 51 data in total (the unit of test time is second). The aim of this example is to estimate the failure rate and reliability when time T=1000, that is $\hat{R}_0 = e^{-\hat{\lambda}_{EB} \cdot 1000}$.

Table 1. Zero failure data of certain model engine

	(t_i, n_i)						
i	1	2	3	4	5	6	7
	(100. 18,3)	(109. 93, 21)	(115. 01, 2)	(130. 15,1)	(150. 00,3)	(179. 94,8)	(190. 36,1)
i	8	9	10	11	12	13	
	(250. 15,1)	(783. 00, 4)	(849. 94,3)	(870. 03,1)	(909. 77,1)	(1450. 30,2)	

To solve the above example, we first suppose that the scale parameter of scale squared error loss function is $k=0$, and for other values of k,similar results will be easily obtained. The results are calculated by Theorem 1 and theorem 2 , which are shown in Table 2.

Table 2. Calculation results of $\hat{\lambda}$ and $\hat{R}(1000)$

C	300	500	800	1500	3000	Range
$\hat{\lambda}_{EB}$	3.1624e-05	3.1426e-05	3.1137e-05	3.0948e-05	2.9210e-05	2.4140e-06
$\hat{R}_{EB}$	0.9689	0.9691	0.9693	0.9695	0.9712	0.0023
(a,b)	(0.5,0.5)	(1.0,0.5)	(1.5,0.5)	(1.5,1.0)	(1.5,2.0)	
$\hat{\lambda}_{B}$	3.1924e-05	6.3849e-05	9.5773e-05	9.5770e-05	9.5764e-05	6.3849e-05
$\hat{R}_{B}$	0.9686	0.9381	0.9087	0.9087	0.9087	0.0599

As shown in Table 2, C is robust in E-Bayes estimation of failure rate λ and $R(1000)$. Therefore, it is an appropriate choice to use gamma distribution as a prior distribution of failure rate in Bayes estimation.

5. Conslusion

The paper studies the Bayes estimation of the failure rate and reliability of the exponential distribution model under the scale squared error loss function. Draw the conclusion that when the prior distribution of the parameters is the Gamma distribution, the Bayes estimators and E-Bayes estimators of the failure rate and reliability of the exponential model can be obtained. The practical example utilized in the last part to illustrate the effectiveness of the estimation.

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