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# Bayesian Statistical Analysis of Lifetime Performance Index of Exponential Product Under Precautionary Loss Function

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**Abstract.** The purpose of this paper is to study the Bayesian estimation of life performance index when product's life obeys exponential distribution based on complete samples. The Bayesian estimator of the life performance index is obtained when the prior distribution of the unknown parameter is the Gama prior distribution and the loss function used in this paper is the precautionary loss function. Furthermore, a new Bayesian testing procedure is developed using the obtained Bayesian estimator. Finally, a practical example is used to illustrate the effectiveness and feasibility of the results.

## 1. Introduction

In practice, effective management and evaluation of product quality and performance is an important task of modern enterprises, and process capability analysis is an important means to measure enterprise performance. Process capability index is a simple and effective index to detect product quality and monitor product production process. So far, many process capability indices have been proposed one after another. The most famous of them are the following process capability indices,  $C_P$ ,  $C_{pk}$ ,  $C_{pm}$  and  $C_{pmk}$ . These four process capability indices are process capability indices for evaluating product quality characteristics with expected shape. They are not applicable to the characteristics of the produce, whose life is the larger-the-better. For this reason, Montgomery [1] puts forward the expected large process capability index  $C_L$ , which can better reflect product quality characteristics with longer product life. Therefore, this index is also called life performance index. In recent years, much attention has been paid to the statistical inference of life performance index. For example, in Liao and Zhang [2], the Bayesian estimation and hypothesis test of life performance index are discussed based on the entropy loss function, and the feasibility and validity of the test method are illustrated by practical examples. When the life of products is exponentially distributed, in Wu and Lin [3], based on the progressive type I interval censoring samples, a new hypothesis testing algorithm for life performance index is proposed by using the maximum likelihood estimation method to estimate the life performance index of products and under the condition of known specifications lower limit. Using data transformation, El-Sagheer [4] studied the maximum likelihood estimation of life performance index of extreme value distribution under progressively type II censored samples. Based on conjugate prior distribution assumption, they further studied the Bayesian estimation of life performance index under square error loss function. Jiang et al.[5] developed a Bayesian posterior probability ratio test procedure for lifetime performance index under LINEX loss function when



product's life distributed with exponential distribution. Fan [6] established a Bayesian test procedure under symmetric entropy loss function for the lifetime performance index of exponential distribution.

In recent years, many scholars have studied Bayesian statistical inference of reliability model parameters based on precautionary loss function. Under the precautionary loss function, Golparvar et al. [7] discussed the Bayesian, robust Bayesian and minimax predictions for a subfamily of scale parameters under a precautionary loss function. Karimnezhad [8] considered the prediction problem of for a scale distribution family under a class of precautionary loss function using Bayesian and robust Bayesian approaches. Ren and Wang [9] discussed Bayesian estimation of the traffic intensity in an M/M/1 queue under a precautionary loss function. More information about precautionary loss function can be found in references [10-12].

In this paper, the Bayesian estimation and Bayesian testing procedure of product life performance index with exponential distribution are studied under a precautionary loss function assuming the conjugate gamma prior distribution of unknown parameter.

## 2. Preliminary Knowledge

Let the life of an electronic product, noted by  $X$  obeys the exponential distribution with unknown parameter  $\lambda$  ( $\lambda > 0$ ), that is, the corresponding probability density function (pdf) is

$$f(x; \lambda) = \lambda e^{-\lambda x}, \quad x > 0 \quad (1)$$

Since most electronic products are expected to be large-scale, that is, to meet the characteristics of longer life and better quality, Montgomery [1] proposes life performance indicators for such products:

$$C_L = \frac{\mu - L}{\sigma} \quad (2)$$

to measure the performance of such products, where  $L$  is the lower bound of specifications,  $\mu$  is the mean of product life and  $\sigma$  is the standard deviation of product life.

According to formula (2), it is easy to calculate life performance indicators of exponential distribution as follows

$$C_L = \frac{\mu - L}{\sigma} = \frac{1/\lambda - L}{1/\lambda} = 1 - \lambda L \quad (3)$$

And failure rate function:

$$r(x) = \frac{f(x; \lambda)}{1 - F(x; \theta)} = \frac{\lambda e^{-\lambda x}}{e^{-\lambda x}} = \lambda \quad (4)$$

Because of the average life of the product  $\mu = \frac{1}{\lambda} = \frac{1}{r(x)}$ , and according to formula (3) and (4), the

index  $C_L$  can better reflect the life of the product. Let  $X$  be the life of the product and  $L$  be the lower limit of the specification of the life, that is, if  $X \geq L$ , then the product is considered to be a qualified product. The qualified rate of products is defined as follows:

$$P_r = P(X \geq L) \quad (5)$$

For products whose life is exponentially distributed, the qualified rate can be deduced as follows:

$$P_r = P(X \geq L) = \int_L^{\infty} \lambda e^{-\lambda x} dx = e^{-\lambda L} = e^{C_L - 1}, \quad -\infty < C_L < 1 \quad (6)$$

According to formula (6), the qualified rate corresponds to the life performance index one by one, so as long as the value is known, the qualified rate value of the product can be calculated. But literature [1] pointed out that large samples are usually needed to estimate accurately, which cannot be achieved in practice because of the limited cost, manpower and time. Bayesian method can deal well with the estimation of model parameters in the case of small samples, and can also accurately estimate the value of the product eligibility rate in the case of small samples. Therefore, this paper will study

Bayesian estimation of life performance index of exponentially distributed products under the precautionary loss function.

### 3. Estimation of Lifetime Performance Index

Let  $X_1, X_2, \dots, X_n$  be a sample from the exponential distribution (1),  $x = (x_1, x_2, \dots, x_n)$  and  $t = \sum_{i=1}^n x_i$  are the observations of  $X = (X_1, X_2, \dots, X_n)$  and  $T = \sum_{i=1}^n X_i$ . Then the likelihood function of  $\theta$  is given by

$$L(x; \lambda) = \prod_{i=1}^n f(x; \lambda) = \prod_{i=1}^n \lambda e^{-\lambda x_i} = \lambda^n e^{-\lambda t} \quad (7)$$

The maximum likelihood estimator (MLE) of  $\theta$  is the solution of the log-likelihood equation  $\frac{dLn[l(\theta; x)]}{d\theta} = 0$ , then

$$\hat{\lambda}_{MLE} = \frac{n}{T} \quad (8)$$

It is easy to prove that  $T$  is a random variable distributed with Gamma distribution  $\Gamma(n, \theta)$ , with the following pdf:

$$f_T(t; \theta) = \frac{\theta^n}{\Gamma(n)} t^{n-1} e^{-\theta t}, \quad t > 0, \theta > 0 \quad (9)$$

Now, we will discuss the Bayesian estimation of the lifetime performance index of exponential distribution (1) under precautionary loss function (Norstorm [10])

$$L(\hat{\lambda}, \lambda) = \frac{(\hat{\lambda} - \lambda)^2}{\hat{\lambda}}, \quad (10)$$

Where,  $\hat{\lambda}$  is an estimator of  $\lambda$ . Under the precautionary loss function (10), the Bayesian estimator of  $\lambda$  is

$$\hat{\lambda}_B = [E(\lambda^2 | X)]^{1/2} \quad (11)$$

Assume that the parameter's prior distribution is the Gamma prior distribution  $\Gamma(\alpha, \beta)$ , with pdf

$$\pi(\lambda; \alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\beta\lambda}, \quad \alpha, \beta > 0, \lambda > 0 \quad (12)$$

Then combining (7) with (12), the posterior pdf of  $\lambda$  can be obtained using Bayes' Theorem as follows:

$$h(\lambda | x) \propto L(\lambda; x) \cdot \pi(\lambda) \propto \lambda^n e^{-\lambda t} \cdot \frac{\beta^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\beta\lambda} \propto \lambda^{n+\alpha-1} e^{-(\beta+t)\lambda} \quad (13)$$

That is  $\lambda | X \sim \Gamma(n + \alpha, \beta + T)$ . Then under the precautionary loss function (10), the Bayesian estimator of  $\lambda$  can be derived as follows

$$\hat{\lambda}_B = [E(\lambda^2 | X)]^{1/2} = \left[ \frac{(n + \alpha)(n + \alpha + 1)}{(\beta + T)^2} \right]^{1/2} = \frac{1}{\beta + T} \sqrt{(n + \alpha)(n + \alpha + 1)} \quad (14)$$

Further the Bayesian estimator of  $C_L$  is

$$\hat{C}_B = 1 - \hat{\lambda}_B L = 1 - \frac{L}{\beta + T} \sqrt{(n + \alpha)(n + \alpha + 1)} \quad (15)$$

#### 4. Bayesian Test of Lifetime performance Index

This section will propose a new testing procedure to assess whether the  $C_L$  adheres to the required level. Assume that the required index value of  $C_L$  is larger than  $c$ , where  $c$  is the target value. First, we construct the following hypothesis:

$$H_0 : C_L \leq c \leftrightarrow H_1 : C_L > c. \quad (16)$$

Let  $Y = 2\lambda(\beta + T) | X$ , then for given significance level  $\gamma$ , we can easily shown that  $\lambda | X \sim \Gamma(2n + \alpha, \beta + T)$ . Let  $\chi_{1-\gamma}^2(2(n + \alpha))$  be the  $1 - \gamma$  percentile of  $\chi^2(2(n + \alpha))$ .

Then

$$P(2\lambda(\beta + T) \leq \chi_{1-\gamma}^2(2(n + \alpha)) | X) = 1 - \gamma \quad (17)$$

That is

$$P(1 - \lambda L \geq 1 - L \cdot \frac{\chi_{1-\gamma}^2(2(n + \alpha))}{2(\beta + T)} | X) = 1 - \gamma \quad (18)$$

Then the lower confidence limit of  $C_L$  with level  $1 - \gamma$  is:

$$\underline{LB} = 1 - (1 - \hat{C}_B) \cdot \frac{\chi_{1-\gamma}^2(2(n + \alpha))}{2(n + \alpha - 1)} \quad (19)$$

Now we give the new proposed Bayesian testing procedure of  $C_L$  as follows:

- (i) For given sample size  $n$ , determine the lower lifetime specification limit  $L$ .
- (ii) Under precautionary loss function, calculate the Bayesian estimator

$$\hat{C}_B = 1 - \frac{L}{\beta + T} \sqrt{(n + \alpha)(n + \alpha + 1)} \quad (20)$$

Where  $T = \sum_{i=1}^n X_i$ .

(iii) Calculate the  $(1 - \gamma)$  confidence interval  $[\underline{LB}, \infty)$  for  $C_L$ , where  $\underline{LB}$  is defined as equation (19).

(iv) The Bayesian statistical test are given as follows:

If  $c \notin [\underline{LB}, \infty)$ , we reject to  $H_0$  and conclude that the lifetime performance of product meets the required level.

If  $c \in [\underline{LB}, \infty)$ , we accept  $H_0$  and conclude that the lifetime performance does not meet the required level.

#### 5. Numerical Example

A practical example adopt from [13] is used to analyze the performance of the new testing procedure. There are nineteen military personnel carriers failed in service on the mileages and the data set is given as follows:

162, 200, 271, 320, 393, 508, 539, 629, 706, 777, 884, 1008, 1101, 1182, 1463, 1603, 1984, 2355, 2880.

The data set has been proved that it can be modelled by exponential distribution by using least squares test [13]. Now, we will state the proposed testing steps about  $C_L$  as follows:

(i) Calculate  $t = \sum_{i=1}^n x_i = 18963$  and assume that the lower lifetime limit  $L$  is  $L = 171.0144$ . To satisfy the military's concerns, the conforming rate  $P_r$  is required to bigger that 0.80. Then using

formula (6), we know that the value of  $C_L$  is also required to exceed 0.80. Thus, set  $c=0.80$  and construct the testing hypothesis  $H_0 : C_L \leq 0.80 \leftrightarrow H_0 : C_L > 0.80$ .

(ii) Specified a significance level  $\gamma = 0.05$ .

(iii) Assume that the values prior parameters vector is  $(\alpha, \beta) = (2, 3)$ , then  $\hat{C}_B = 0.7184$ , and further we have  $\underline{LB} = 1 - (1 - \hat{C}_B) \cdot \frac{\chi_{1-\gamma}^2(2(n+\alpha))}{2(n+\alpha-1)} = 0.8062$ .

(iv) The  $(1-\gamma)$  one-side confidence interval  $[\underline{LB}, \infty)$  is  $[0.8062, \infty)$ .

(v) Because  $c = 0.80 \notin [\underline{LB}, \infty)$ , then we reject the null hypothesis  $H_0 : C_L \leq 0.80$ . Thus we can conclude that the lifetime performance of products meets the required level.

## 6. Conclusions

For the statistical inference of a very useful process capability index  $C_L$ , which also is named lifetime performance index, this paper studied its estimation by using Bayesian statistical method. Bayesian estimator and Bayesian testing procedure of life performance index are derived under a precautionary loss function. The novel Bayesian testing procedure is easier than classical testing approaches and it easy to operate with the help of computer software, such as Excel and Matlab software. The Bayesian testing procedure can also provide a reference for engineers to assess whether the true lifetime performance of products meets the requirements.

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