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Analysis of Periodic Solutions for Composite Laminated Circular Cylindrical Shell with 1:2 Internal Resonance

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Abstract. In this paper, we are concerned with periodic motions of composite laminated circular cylindrical shell with 1:2 internal resonance. The existence condition and number estimation of periodic solutions of the governing system are studied by discussing the Poincaré map and the displacement function. To illustrate the effectiveness of our results, the method of numerical simulations is performed to detect the phase portraits of the periodic solutions under a set of parameter condition.

1. Introduction

Due to the excellent thermal properties [1-2], composite materials are successfully applied in many fields of industry. Circular cylindrical shell, one of the important structures, is extensively applied to aeronautic and aerospace engineering. Owing to the complexity of the working conditions, various vibrations may occur to circular cylindrical shell [3], which can cause damages to the devices. Hence, the investigation for the complicated vibrations of the shell is very important, which is of significance in both theory and application.

The nonlinear oscillations of the composite shell structure have been widely investigated. The motions of the shell can often be expressed by nonlinear partial differential equations, and often be transformed into high-dimensional dynamic systems by Galerkin method [4-5]. Because of the phenomenon of modal interactions of these high-dimensional systems, the theoretical study on the nonlinear dynamic behaviours is very complex. Hence, many results on nonlinear dynamic behaviours, such as periodic vibrations, chaotic motions, etc., are often obtained by numerical and experimental methods [6-7]. Periodic vibrations are essential forms of vibrations, which attracts much interest of scholars. To provide a comprehensive understanding on the periodic motions with a wide range of parameters, proposing a method for the theoretical analysis is urgent for our study.

The theoretical study on the periodic solutions of nonlinear dynamic systems is an important but difficult topic. Some classic and important methods were widely used to discuss the bifurcation of periodic solutions of nonlinear systems [8-9]. The existence of periodic solutions and their distribution configurations of some nonlinear dynamic systems, such as near-Hamilton systems, slow-fast systems, etc., have been widely studied [10-11]. However, the results could not be widely applied to some other types of engineering models.

We outline the structure of this paper. In section 2, the system of circular cylindrical shell is presented. In section 3, the periodic solutions are studied by discussing the Poincaré map and the displacement function. In section 4, the phase portraits of periodic solutions are obtained by numerical simulations. In section 5, the conclusions of our paper are shown.



2. System Description

A simplified model shown in figure 1 is used to model the laminated circular cylindrical shell, where l_0 , K_i and ζ_i are respectively pre-stretched length, stiffness coefficients and damping coefficients, where $i = 1, 2$.

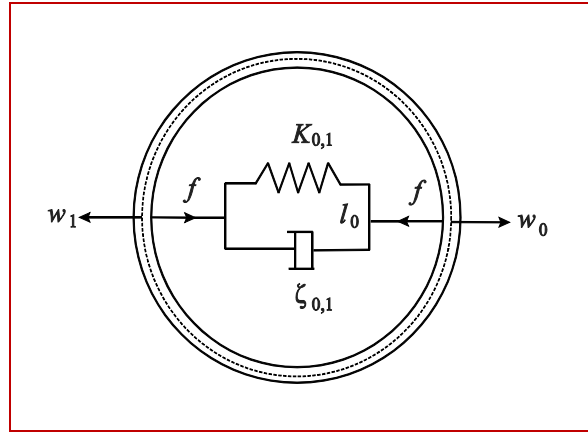


Figure 1. The simplified model of composite shell

Under temperature excitation, the governing system can be obtained as follows [4],

$$\begin{aligned} \ddot{v}_1 + (\omega_1^2 + \beta_{11}T \cos(\Omega t))v_1 + (\beta_{12} + \beta_{13}T \cos(\Omega t))v_2 + f_1(v_1, v_2, \dot{v}_1, \dot{v}_2, \ddot{v}_1, \ddot{v}_2) &= 0 \\ \ddot{v}_2 + (\omega_2^2 + \beta_{23}T \cos(\Omega t))v_2 + (\beta_{21} + \beta_{22}T \cos(\Omega t))v_1 + f_2(v_1, v_2, \dot{v}_1, \dot{v}_2, \ddot{v}_1, \ddot{v}_2) &= 0 \end{aligned} \quad (1)$$

and the notation,

$$\begin{aligned} f_i(v_1, v_2, \dot{v}_1, \dot{v}_2, \ddot{v}_1, \ddot{v}_2) = & \beta_{i4}\dot{v}_1 + \beta_{i5}\dot{v}_2 + \beta_{i6}\ddot{v}_1 + \beta_{i7}\ddot{v}_2 + \beta_{i8}v_1\ddot{v}_2 + \beta_{i9}\ddot{v}_1v_2 + \beta_{i10}\dot{v}_1^3 + \beta_{i11}\dot{v}_2^3 \\ & + \beta_{i12}\dot{v}_1^2\dot{v}_2 + \beta_{i13}\dot{v}_1\dot{v}_2^2 + \beta_{i14}\dot{v}_1^2 + \beta_{i15}\dot{v}_2^2 + \beta_{i16}\dot{v}_1\dot{v}_2 + \beta_{i15}\dot{v}_2^2 + \beta_{i16}\dot{v}_1\dot{v}_2 \\ & + \beta_{i17}v_1^3 + \beta_{i18}v_2^3 + \beta_{i19}v_1^2v_2 + \beta_{i20}v_1v_2^2 + \beta_{i21}v_1^2 + \beta_{i22}v_2^2 + \beta_{i23}v_1v_2 + \beta_{i24} \end{aligned}$$

where $i = 1, 2$. Considering 1:2 internal resonance of system (1), we have

$$\omega_1^2 = \frac{1}{4}\Omega^2 + \varepsilon\sigma_1, \quad \omega_2^2 = \Omega^2 + \varepsilon\sigma_2,$$

where $0 < |\varepsilon| \ll 1$. Supposing $\Omega = 2$, based on the method of asymptotic perturbation, we can obtain the following averaged equations,

$$\begin{aligned} \dot{x}_1 &= -\frac{1}{2}\beta_{14}x_1 - \frac{1}{4}(\beta_{11}T - 4\sigma_1)x_2 + F_1(x_1, x_2, x_3, x_4) \\ \dot{x}_2 &= -\frac{1}{4}(\beta_{11}T + 4\sigma_1)x_1 - \frac{1}{2}\beta_{14}x_2 + F_2(x_1, x_2, x_3, x_4) \\ \dot{x}_3 &= \frac{1}{2}\beta_{25}\sigma_2x_4 + F_3(x_1, x_2, x_3, x_4) \\ \dot{x}_4 &= -\frac{1}{2}\beta_{25}\sigma_2x_3 + F_4(x_1, x_2, x_3, x_4) \end{aligned} \quad (2)$$

where $F_k(x_1, x_2, x_3, x_4)$, $k = 1, 2, 3, 4$ are polynomials.

3. Periodic Solutions

To study the periodic solutions of system (2), we will introduce the following notations for convenience if $\sigma_1 \neq 0$,

$$G_1(x_1, x_2, x_3, x_4) = -\frac{1}{2}\beta_{14}\sigma_1^{-1}x_1 - \frac{1}{4}\beta_{11}T\sigma_1^{-1}x_2 + \sigma_1^{-1}F_1(x_1, x_2, x_3, x_4)$$

$$G_2(x_1, x_2, x_3, x_4) = -\frac{1}{4}\beta_{11}T\sigma_1^{-1}x_1 - \frac{1}{2}\beta_{14}\sigma_1^{-1}x_2 + \sigma_1^{-1}F_2(x_1, x_2, x_3, x_4)$$

$$G_k(x_1, x_2, x_3, x_4) = \sigma_1^{-1}F_k(x_1, x_2, x_3, x_4), \quad k = 3, 4$$

The following results can be obtained.

Lemma 1 If $\sigma_1 \neq 0$, considering system (2) and writing $t = \sigma_1^{-1}\tau$, $\beta_{ij} = \varepsilon\beta_{ij}$, we obtain

$$\frac{d\mathbf{x}}{d\tau} = A\mathbf{x} + \varepsilon\mathbf{G}(\mathbf{x}) \quad (3)$$

where $i = 1, 2$, $j = 1, 2, 3, 4, 6, 7, \dots, 24$, $\mathbf{x} = (x_1, x_2, x_3, x_4)^T$, $A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$, $A_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $A_2 = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}$,

$$a = \frac{1}{2}\beta_{25}\sigma_1^{-1}\sigma_2, \quad \mathbf{G} = (G_1, G_2, G_3, G_4)^T.$$

Lemma 2 Consider system (3) and suppose $\beta_{25}\sigma_1^{-1}\sigma_2 = 2$. Based the transformations,

$$x_1 = r \cos \theta, \quad x_2 = r \sin \theta, \quad x_3 = \rho \cos(\theta + s), \quad x_4 = \rho \cos(\theta + s) \quad (4)$$

we obtain

$$\frac{d\mathbf{y}}{d\theta} = \varepsilon\mathbf{H}(\theta, \mathbf{y}) + O(\varepsilon^2) \quad (5)$$

where

$$\mathbf{y} = (r, \rho, s)^T, \quad \mathbf{H} = (H_1, H_2, H_3)^T$$

$$H_1(\theta, \mathbf{y}) = G_1(\theta, \mathbf{y})\cos \theta + G_2(\theta, \mathbf{y})\sin \theta$$

$$H_2(\theta, \mathbf{y}) = G_3(\theta, \mathbf{y})\cos(\theta + s) + G_4(\theta, \mathbf{y})\sin(\theta + s)$$

$$H_3(\theta, \mathbf{y}) = \frac{1}{\rho}(G_4(\theta, \mathbf{y})\cos(\theta + s) - G_3(\theta, \mathbf{y})\sin(\theta + s)) - \frac{1}{r}(G_2(\theta, \mathbf{y})\cos \theta - G_1(\theta, \mathbf{y})\sin \theta)$$

Proof When $\beta_{25}\sigma_1^{-1}\sigma_2 = 2$, based on transformations (4), we obtain

$$\frac{d\theta}{d\tau} = 1 - \frac{\varepsilon(G_1(\theta, \mathbf{y})\cos \theta - G_2(\theta, \mathbf{y})\sin \theta)}{r}$$

$$\frac{d\mathbf{y}}{d\tau} = 1 + \varepsilon\mathbf{H}(\theta, \mathbf{y})$$

Considering θ as the independent variable, lemma 2 holds.

Suppose $\mathbf{y}(\theta, \mathbf{z}, \varepsilon)$ is the solution of (5), where $\mathbf{y}(0, \mathbf{z}, \varepsilon) = \mathbf{z}$, $\mathbf{z} = (r_0, \rho_0, s_0)$. The Poincaré map and displacement function are shown as $P(\mathbf{z}, \varepsilon)$ and $\Phi(\mathbf{z}, \varepsilon)$ respectively, where

$$P(\mathbf{z}, \varepsilon) : \mathbf{y}(0, \mathbf{z}, \varepsilon) \rightarrow \mathbf{y}(2\pi, \mathbf{z}, \varepsilon)$$

$$\Phi(\mathbf{z}, \varepsilon) = \frac{1}{\varepsilon} (\mathbf{y}(2\pi, \mathbf{z}, \varepsilon) - \mathbf{y}(0, \mathbf{z}, \varepsilon))$$

$$\Phi = (\Phi_1, \Phi_2, \Phi_3)^T$$

With the aid of Maple and Matlab software, we obtain

$$\Phi(\mathbf{z}, \varepsilon) = \mathbf{h}(\mathbf{z}) + \mathbf{O}(\varepsilon) \quad (6)$$

where $\mathbf{h} = (h_1, h_2, h_3)^T$,

$$h_1(\mathbf{z}) = -\frac{1}{4} r_0 \pi (2\alpha_{16} \rho_0^2 \cos^2(s_0) + 8\mu_1 - 3\alpha_{16} \rho_0^2 + 8\alpha_{13} r_0^2)$$

$$h_2(\mathbf{z}) = -\frac{1}{2} \rho_0 \pi (-2\alpha_{23} r_0^2 \cos^2(s_0) + \alpha_{23} r_0^2 + \alpha_{21} \rho_0^2)$$

$$h_3(\mathbf{z}) = \frac{1}{4} \pi (\alpha_{16} \rho_0^2 \sin(2s_0) + 2\alpha_{23} r_0^2 \sin(2s_0) + 8\alpha_{14} r_0^2 - 8\alpha_{22} \rho_0^2)$$

$$\alpha_{16} = 4\beta_{113}, \quad \mu_1 = \frac{\beta_{14}}{2}, \quad \alpha_{13} = \frac{3\beta_{110}}{2}, \quad \alpha_{23} = \beta_{212}, \quad \alpha_{21} = -\frac{7\beta_{29}\beta_{215}\beta_{222}}{9} - 6\beta_{211}$$

$$\alpha_{14} = \frac{\beta_{114}\beta_{121}}{3} - \frac{7\beta_{16}\beta_{121}}{6} + \frac{3\beta_{117}}{2} + \frac{\beta_{121}^2}{3} + \frac{5\beta_{16}^2}{6} - \frac{2\beta_{114}^2}{3} + \frac{\beta_{114}\beta_{16}}{6}$$

$$\alpha_{22} = \frac{5\beta_{29}^2}{6} + \frac{\beta_{29}\beta_{215}}{3} + \frac{\beta_{215}\beta_{222}}{6} + \frac{\beta_{222}^2}{24} + \frac{3\beta_{218}}{2} \quad (7)$$

Theorem 1 Suppose that $\mathbf{h}(\mathbf{z})$ is not zero. If the solution $\mathbf{z}^* = (r_0^*, \rho_0^*, s_0^*)$ of $\mathbf{h}(\mathbf{z}^*) = 0$ satisfies $r_0^* \geq 0$, $\rho_0^* \geq 0$, $r_0^* \rho_0^* \neq 0$, $s_0^* \in (0, 2\pi]$ and the Jacobian $\frac{\partial \mathbf{h}}{\partial \mathbf{z}}|_{\mathbf{z}^*} \neq 0$, system (3) exists a periodic solution in the neighbourhood of $(r_0^* \cos \tau, r_0^* \sin \tau, \rho_0^* \cos(\tau + s_0^*), \rho_0^* \sin(\tau + s_0^*))$.

Proof If $\mathbf{h}(\mathbf{z}^*) = 0$ and $\frac{\partial \mathbf{h}}{\partial \mathbf{z}}|_{\mathbf{z}^*} \neq 0$, the expressions $\Phi(\mathbf{z}^*, 0) = 0$ and $\frac{\partial \Phi}{\partial \mathbf{z}}|_{(\mathbf{z}^*, 0)} \neq 0$ hold. By implicit function theorem, there exists $(\mathbf{z}^*(\varepsilon), \varepsilon)$ in the a neighbourhood of $(\mathbf{z}^*, 0)$ such that $\Phi(\mathbf{z}^*(\varepsilon), \varepsilon) = 0$ for $0 < |\varepsilon| < 1$. Hence, system (5) has a periodic solution in the neighbourhood of $\mathbf{z}^*$.

Considering the coordinate transformations shown in lemma 2, only when the solution $\mathbf{z}^*$ satisfies $r_0^* \geq 0$, $\rho_0^* \geq 0$ but $r_0^* \rho_0^* \neq 0$, system (3) exists a periodic solution in the neighbourhood of $(r_0^* \cos \tau, r_0^* \sin \tau, \rho_0^* \cos(\tau + s_0^*), \rho_0^* \sin(\tau + s_0^*))$.

Theorem 2 Suppose $\alpha_{13} = 0$, $\beta_{25} \sigma_1^{-1} \sigma_2 = 2$, $\sigma_1 \alpha_{21} \alpha_{23} \neq 0$, where the expressions of α_{13} , α_{21} and α_{23} are shown in equation (7). System (3) exists at most 8 periodic solutions based on displacement function of order ε^0 .

Proof When $\alpha_{13} = 0$ and $\alpha_{21} \neq 0$, the equation $\mathbf{h}(\mathbf{z}) = 0$ has solutions only in the case of $r_0 \rho_0 \neq 0$. If $\alpha_{23} \neq 0$, we obtain

$$\cos^2(s_0) = \frac{\alpha_{23} r_0^2 + 4\alpha_{21} \rho_0^2}{2\alpha_{23} r_0^2}, \quad \sin(2s_0) = \frac{-8(\alpha_{14} r_0^2 - \alpha_{22} \rho_0^2)}{\alpha_{16} \rho_0^2 + 2\alpha_{23} r_0^2} \quad (8)$$

Writing $\lambda = \frac{r_0^2}{\rho_0^2}$, the following equation can be obtained based on $\sin^2(s_0) + \cos^2(s_0) = 1$,

$$b\lambda^4 + c\lambda^3 + d\lambda^2 + e\lambda + f = 0$$

where

$$b = 4\alpha_{23}^2(16\alpha_{14}^2 + 3\alpha_{23}^2), \quad c = 4\alpha_{23}^2(3\alpha_{23}\alpha_{16} - 32\alpha_{14}\alpha_{22} + 16\alpha_{23}\alpha_{21})$$

$$d = \alpha_{23}^2(3\alpha_{16}^2 + 64\alpha_{22}^2 + 64\alpha_{21}\alpha_{16} + 64\alpha_{21}^2), \quad e = 16\alpha_{16}\alpha_{21}\alpha_{23}(\alpha_{16} + 4\alpha_{21}), \quad f = 16\alpha_{21}^2\alpha_{16}^2$$

There are at most four solutions with respect to λ . Based on $h_1(z) = 0$, we obtain

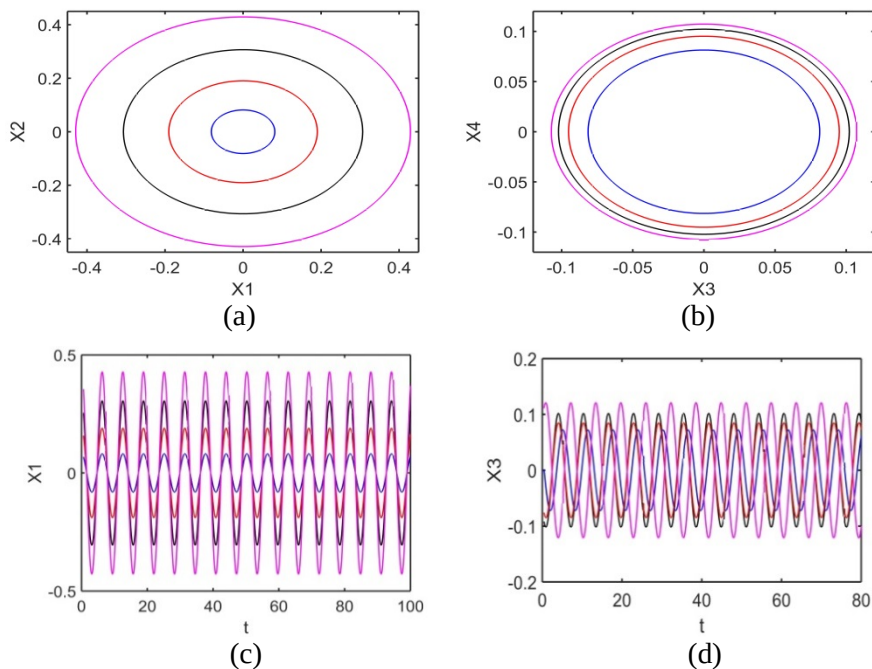
$$\rho_0^2 = \frac{8\mu_1}{3\alpha_{16} - 8\alpha_{13}\lambda - 2\alpha_{16}\cos^2(s_0)} \quad (9)$$

Hence, substituting each solution λ into equations (8) and (9), at most two sets of $\mathbf{z}$ which satisfy $r_0^* > 0$, $\rho_0^* > 0$ can be obtained for $s_0 \in (0, 2\pi]$. Hence, there are at most 8 solutions with respect to $\mathbf{z}$.

At each of all the solutions, the Jacobian is nonzero. Based on theorem 1, system (3) has at most 8 periodic solutions.

4. Numerical Simulations

The number and distribution configurations of the periodic solutions of nonlinear system (3) are presented based on numerical simulations. Fixing its parameters as $\alpha_{16} = 3.7$, $\alpha_{23} = -1.9$, $\alpha_{14} = 0.5$, $\alpha_{21} = 0.3$, $\alpha_{22} = 0.5$, $\mu_1 = 0.1$, $\alpha_{13} = 0.02$, the periodic solutions can be obtained. Figure 2 (a) and (b) show the periodic solutions on planes (x_1, x_2) and (x_3, x_4) , figure 2 (c) and (d) respectively show the time-history diagrams of x_1 and x_3 with respect to t , and figure 2 (e) and (h) illustrate the relative positions of the periodic solutions in three-dimensional space (x_1, x_2, x_4) and (x_2, x_3, x_4) .



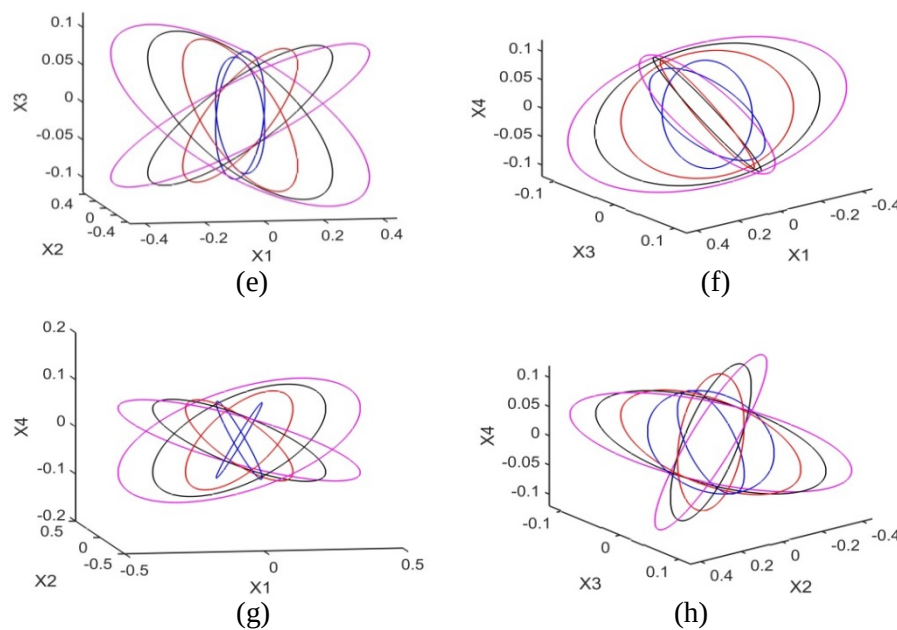


Figure 2. Periodic solutions of system (3)

5. Conclusions

The existence condition and number estimation of periodic orbits of circular cylindrical shell are studied in this paper. With 1:2 internal resonance, the maximum periodic solutions, 8, is obtained when $\alpha_{13} = 0$, $\beta_{25}\sigma_1^{-1}\sigma_2 = 2$ and $\sigma_1\alpha_{21}\alpha_{23} \neq 0$. To illustrate the effectiveness of our results, the phase portraits which provide the relative positions and amplitudes of the periodic solutions are obtained under a group of parameter condition.

Periodic solutions of the averaged equations reveal the complicated periodic vibrations of the shell. Theoretical and numerical results obtained in our paper may provide guidance for the structural designing.

6. Acknowledgements

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