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The Existence and Upper Bound of Periodic Solutions for Two-Coupled-Oscillator Model in Optics Chiral Molecular Medium

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Abstract. In this paper, we focus on the two-coupled-oscillator model in optics chiral molecular medium. We perform scale transformations for variables and study the existence of periodic solutions in detail for the two-coupled-oscillator system. We obtain the Melnikov function by establishing the curvilinear coordinate transformation and constructing a Poincaré map. Then the existence of periodic solutions of this oscillator system is analyzed when unperturbed system is Hamiltonian system. We apply them to discuss the upper bound of periodic solutions of this oscillator system and give the configuration of the phase diagram by numerical simulation. It has great theoretical significance to study the non-planar motion of the two-coupled-oscillator system for analyzing dynamic characteristics in optics chiral molecular medium.

1. Introduction

Chiral material is a class of asymmetric substance that exists widely in nature. It has been widely studied because of its importance in many fields, such as biology, physics, chemistry and pharmacology, chiral phenomena. Chiral materials show some optical unique second-order nonlinearity. Characterization of chiral molecules and their media using nonlinear dynamics have significance for studying the molecular chiral structure and exploring new type of functional materials theoretically and practically. A class of chiral molecule have the following characteristics: there are two spatially separated but coupled clusters. The positive and negative electric centers of the two clusters do not coincide and vibrate under the action of the external field. This kind of molecule can be simplified as the two-coupled-oscillator structure. Condon [1] has successfully used the two-coupled-oscillator model to explain the linear optical rotation of molecules. Yin et al. [2] realized and discussed exact plasmonic analog in a system of corner-stacked gold nanorods. Manevitch et al. [3] considered the most simple nonlinear problem of energy transfer in the system of two weakly coupled nonlinear oscillators with cubic restoring forces. Zheng et al. [4] derived the hyperpolarizabilities of chiral molecules applied to the two-coupled-oscillator model and gave the expressions of hyperpolarizabilities with microscopic parameters.

The description of the two-coupled-oscillator trajectories had been more investigated, but the vibrational behavior of the two-coupled-oscillator model is less studied. When there is no dissipative factor, the two-coupled-oscillator system is a conservative nonlinear dynamic system that has inherent dynamic characteristics. Therefore, it has great significance to study the periodic motion of the two-coupled-oscillator model. When $\varepsilon = 0$, the two-coupled-oscillator system degenerates to Hamiltonian system. Some scholars studied them and and got some results [5-14]. Kovačič [5-7] used the Melnikov method and the geometric singular perturbation theory to analyze the homoclinic orbits of singularities in resonance of Hamiltonian systems and near integrable dissipative systems. Yagasaki [8-9] developed the global perturbation method of Melnikov and studied the existence of two homoclinic

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orbits for multiple degrees of freedom Hamiltonian perturbed systems. Li et al. [10] studied the periodic orbits and homoclinic orbits of dynamical systems by using the generalized Melnikov method. Llibre et al. [11] developed the high-order averaging method to calculate the number of periodic solutions of higher order equations and extended the averaging theory to the number of periodic solutions of arbitrary dimensional continuous differential equations. Han et al. [12] obtained a class of cubic Hamiltonian systems that have nine limit cycles under cubic polynomial perturbation. Li et al. [13] used the first-order Melnikov function method to study the upper bound of limit cycles of a class of third-order isochronous centers system. Li et al. [14] studied the existence and bifurcation of subharmonic solutions of a four dimensional slow-fast system with time-dependent perturbations for the unperturbed system in two cases: one is a Hamiltonian system and the other has a singular periodic orbit, respectively.

In this paper, the existence and the upper bound of periodic solutions has been studied based on the improved the two-coupled-oscillator model. We give the nonlinear motion equation by the method of multiple scales and perform scale transformations for variables in detail for the two-coupled-oscillator system. Furthermore, the numbers and relative positions of the periodic solutions can be clearly found from the numerical results.

2. The Two-coupled-oscillator System and Averaged Equation

In this section, we investigate the multiple periodic solutions of the two-coupled-oscillator model. The oscillator vibrates under the force of an external field. The two-coupled-oscillator model is shown in figure 1, where m is the mass of two-coupled-oscillator model, x_0, y_0 is the natural length of the two-coupled-oscillator model in x, y direction, l_0 is the distance between the two-coupled-oscillator model in equilibrium position and $l_0 = \sqrt{x_0^2 + y_0^2 + d^2}$, d is the space distance of the two-coupled-oscillator model.

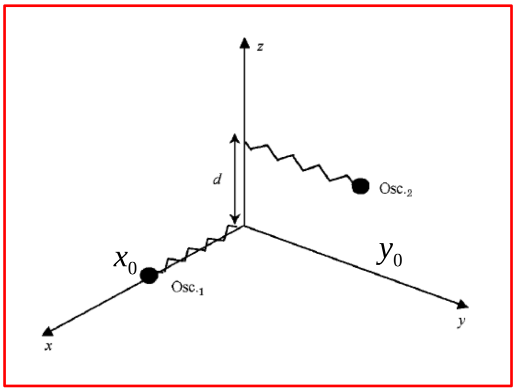

Figure 1. The two-coupled-oscillator model of the chiral molecules

The vibration equations of the two-coupled-oscillator system is as follows [4]

$$\ddot{v} + 2\gamma\dot{v} + \omega_1^2 v + a_0\cos(\Omega_2 t)v + a_1 w + a_2 v^2 + a_3 w^2 + 2a_4 v w + a_5 v^3 - a_6 w^3 + a_7 w^2 v - a_8 v^2 w = f_1\cos(\Omega_1 t) \tag{2.1a}$$

$$\ddot{w} + 2\gamma\dot{w} + \omega_2^2 w + b_0\cos(\Omega_2 t)w + a_1 v + b_2 w^2 + a_4 v^2 + 2a_3 v w + b_5 w^3 - a_6 v^3 + a_7 v^2 w - a_8 v^2 w = f_2\cos(\Omega_1 t) \tag{2.1b}$$

where v and w are the direction of amplitude, γ is the damping constant, ω_1^2 and ω_2^2 are the frequency of oscillation respectively, a_0 and b_0 are the elastic coefficients of two-coupled-oscillator model respectively, f_1 and f_2 are the incident electric field, k is the coupling elastic coefficient. The other parameters are defined as follows

$$\omega_1^2=(a_0+kc_0^2)/m,\omega_2^2=(b_0+kd_0^2)/m,\ \ a_1=kc_0d_0/m,a_2=3kc_0(1-c_0^2)/2ml_0$$
$$b_2=3kd_0(1-d_0^2)/2ml_0,a_3=kc_0(1-3d_0^2)/2ml_0,a_4=kd_0(1-3c_0^2)/2ml_0$$
$$a_5=k(1-6c_0^2)/2ml_0^2,b_5=k(1-3d_0^2)/2ml_0^2,a_6=3kc_0d_0/2ml_0^2$$
$$a_7=(1-3c_0^2-3d_0^2)k/2ml_0^2,a_8=9kc_0d_0/2ml_0^2=3a_6$$

In this case of 1:1 internal resonance and primary parametric resonance ,

$$\omega_1^2=\Omega_1^2+\varepsilon\sigma_1,\omega_2^2=\Omega_2^2+\varepsilon\sigma_2,\omega_1\approx\omega_2$$

where ε is a small parameter, σ_1 and σ_2 are two detuning parameters, and we assume that $\Omega=\Omega_1=\Omega_2=1$. Then we perform scale transformations for variables are as follows.

$$\gamma_1\to\varepsilon\gamma_1,\ \gamma_2\to\varepsilon\gamma_2,\ k_1\to\varepsilon k_1,\ k_2\to\varepsilon k_2,\ a_1\to\varepsilon a_1,\ a_2\to\varepsilon a_2\ f_1\to\varepsilon f_1,\ f_2\to\varepsilon f_2$$

The equation of motion in ordinary differential form of two-coupled-oscillator system is obtained as follows

$$\begin{aligned}&\ddot{v}+2\varepsilon\gamma\dot{v}+\omega_1^2v+\varepsilon a_0\cos t\,v+\varepsilon a_1w+\varepsilon a_2v^2+\varepsilon a_3w^2+2\varepsilon a_4vw+\varepsilon a_5v^3-\varepsilon a_6w^3\\&+\varepsilon a_7w^2v-\varepsilon a_8v^2w=\varepsilon f_1\cos t\end{aligned}\tag{2.2a}$$

$$\begin{aligned}&\ddot{w}+2\varepsilon\gamma\dot{w}+\omega_2^2w+\varepsilon b_0\cos t\,w+\varepsilon a_1v+\varepsilon b_2w^2+\varepsilon a_4v^2+2\varepsilon a_3vw+\varepsilon b_5w^3-\varepsilon a_6v^3\\&+\varepsilon a_7v^2w-\varepsilon a_8v^2w=\varepsilon f_2\cos t\end{aligned}\tag{2.2b}$$

By the methods of multiple scales, the averaged equations is obtained as follows

$$\dot{x}_1=(a_{11}-a_{12})x_2+f_1(x_1,x_2,x_3,x_4)\tag{2.3a}$$
$$\dot{x}_2=(-a_{11}-5a_{12})x_1+f_2(x_1,x_2,x_3,x_4)\tag{2.3b}$$
$$\dot{x}_3=(b_{11}-b_{12})x_4+f_3(x_1,x_2,x_3,x_4)\tag{2.3c}$$
$$\dot{x}_4=(-b_{11}-5b_{12})x_3+f_4(x_1,x_2,x_3,x_4)\tag{2.3d}$$

where

$$a_{11}=\frac{1}{8}a_1^2+\frac{1}{2}\gamma^2+\frac{1}{8}\sigma_1^2,a_{12}=\frac{1}{24}a_0^2,b_{11}=\frac{1}{8}a_1^2+\frac{1}{2}\gamma^2+\frac{1}{8}\sigma_2^2,b_{12}=\frac{1}{24}b_0^2.$$

$f_j\ (j=1,2,3,4)$ are polynomials in variables of $x_i\ (i=1,2,3,4)$.

3. Study on Periodic Solutions of the Two-coupled-oscillator System

3.1. Transformations for the System

For convenience, we introduce the following rescaling transformation

$$a_{ij}\to\varepsilon a_{ij},\ b_{uv}\to\varepsilon b_{uv},\ i+j\geq 3,\ u+v\geq 3$$

Then system (2.3) can be rewrite as

$$\dot{x}=Ax+\varepsilon F(x)\tag{3.1}$$

where $x=(x_1,x_2,x_3,x_4)^{\mathrm{T}}\in R^4$, $F(x)=(F_1,F_2,F_3,F_4)^{\mathrm{T}}$ is the vector-valued polynomials in variables of $x_i=(i=1,2,3,4)$. $A=\partial_{1,2}^{4,4}(a_{11})+\partial_{2,1}^{4,4}(-a_{11})+\partial_{3,4}^{4,4}(b_{11})+\partial_{4,3}^{4,4}(-b_{11})$ and $\partial_{i,j}^{m,n}(M)$

denote an $m\times n$ block matrix with the (i,j)-th block M, a smaller matrix, and all other blocks are zero matrices [15]. When $\varepsilon=0$, system (3.1) degenerates to Hamiltonian system on plane x_1x_2 and x_3x_4 with Hamilton function $H(x)=(H_1,H_2)$ and a family of periodic orbits. Where

$$H_1(x_1,x_2)=\frac{1}{2}a_{11}(x_1^2+x_2^2),\, H_2(x_3,x_4)=\frac{1}{2}a_{11}(x_3^2+x_4^2) \tag{3.2}$$

$$\Gamma_{h_1}=\{x_{h_1}|H_1(x_1,x_2)=h_1\},\Gamma_{h_2}=\{x_{h_2}|H_2(x_3,x_4)=h_2\} \tag{3.3}$$

Suppose that the family of periodic obits Γ_{h_1} and Γ_{h_2} can be expressed as

$$x_1=\left(\frac{2h_1}{a_{11}}\right)^{1/2}\cos(a_{11}t),\, x_2=\left(\frac{2h_1}{a_{11}}\right)^{1/2}\sin(a_{11}t) \tag{3.4a}$$

$$x_3=\left(\frac{2h_2}{b_{11}}\right)^{1/2}\cos(b_{11}(t+t_1)),\, x_4=\left(\frac{2h_2}{b_{11}}\right)^{1/2}\sin(b_{11}(t+t_1)) \tag{3.4b}$$

Then we get the corresponding period of the orbits are $T_1(h_1)\equiv\frac{2\pi}{a_{11}}$, $T_2(h_2)\equiv\frac{2\pi}{b_{11}}$ respectively.

3.2. Melnikov Function and the Upper Bound of Periodic Solutions

For convenience, we introduce curvilinear coordinates in the neighbourhood of Γ_h. In order to construct the Poincaré map, we establish a curvilinear coordinate frame along Γ_h in R^4. Then we define a global cross section Σ in the phase space, and construct the k th iteration of Poincaré map p^k : $\Sigma\to\Sigma$.

We obtain the Melnikov function by establishing the curvilinear coordinate transformation and the Poincaré map p^k. We need study the number of real solutions of $M=(M_1,M_2,M_3)^{\mathrm{T}}=0$ which have nonzero Jacobian, where

$$M_1=\int_0^{2\pi}(a_{11}x_1\cdot f_1+a_{11}x_2\cdot f_2)dt \tag{3.5a}$$

$$M_2=\int_0^{2\pi}(b_{11}x_3\cdot f_3+b_{11}x_4\cdot f_4)dt \tag{3.5b}$$

$$M_3=\int_0^{2\pi}\left(\frac{b_{11}x_4 f_3-b_{11}x_3 f_4}{2b_{11}h_2}-\frac{a_{11}x_2 f_1-a_{11}x_1 f_2}{2a_{11}h_1}\right)dt \tag{3.5c}$$

3.3. The Upper Bound of Periodic Solutions

We need consider the number of real solution of $M=0$ which have nonzero Jacobian. We discuss the following case. Assuming that $b_{11}=2a_{11}=2$, $a_{26}-a_{25}=0$, $b_{24}=0$. Then we have the period $T_1(h_1)=2T_2(h_2)=2\pi$ and M .

$$M_1=4h_1\pi(-a_{24}\sqrt{h_2}\sin(2t_1)+2a_{31}h_1+a_{33}h_2)=0 \tag{3.6a}$$

$$M_2=4\pi h_2(a_{31}h_2+2a_{33}h_1)=0 \tag{3.6b}$$

$$\begin{aligned}M_3=\frac{\pi}{h_2}(&2a_{24}(h_2^{3/2}(\cos(2t_1)+1)-h_2^{3/2})+4a_{37}(h_1h_2^{3/2}-h_2^2)+3a_{38}h_2^2\\&-2a_{39}h_2^2+a_{41}(3h_2^2+4h_1h_2^{3/2})+4a_{40}h_1h_2^{3/2}+3b_{33}h_2^2)\end{aligned} \tag{3.6c}$$

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When $(a_{33}^2 - a_{31}^2)a_{33} \neq 0$, the equations (3.6) are equivalent to the following equations by transformation.

$$\sqrt{h_1} = \sqrt{\frac{b}{2}} a \sin(2t_1) \tag{3.7a}$$

$$\sqrt{h_2} = a \sin(2t_1) \tag{3.7b}$$

$$(A^2 + B^2 C)\cos^4(2t_1) + (2A^2 + 2B^2 C - D^2)\cos^2(2t_1) + (A^2 + B^2 C - D^2) = 0 \tag{3.7c}$$

where

$$a = \frac{a_{24} a_{33}}{a_{33}^2 - a_{31}^2}, b = -\frac{a_{31}}{a_{33}}, A = \sqrt{9(a_{38} + a_{41} + b_{33}) - 6(2a_{37} - a_{39})}\ a^2, B = \sqrt{6(a_{37} + a_{40} + b_{41})}\ a^2$$

$$C = \sqrt{2b}, D = \sqrt{6a_{24}}\ a^{3/2}$$

We get at most two $\cos^2(2t_1)$ from (3.7). For each solution, we can find at most four $\cos(2t_1)$ from (3.7). The number of solutions for in equation (3.7) is closely related to the number of periodic solutions of system (3.1). There are two h_1^* and h_2^* about the solution $h = (h_1, h_2)$. So the system (3.1) most has four periodic solutions .

3.4. Numerical Simulation

In order to understand more intuitively, we give a set of fixed parameter and numerical simulation to obtain the phase diagram of the periodic solutions. We choose the perturbation parameter $\varepsilon = 0.001$.

$$UP = (\sigma_1, \sigma_2, \Gamma, a_0, b_0, a_1, a_2, b_2, a_3, a_4, a_5, b_5, a_6, a_7, a_8, f_1, f_2)$$
$$= (1,3,1,1,6,\sqrt{3},0.5,0.25,2,2,-1,15,15,2,5,5)$$

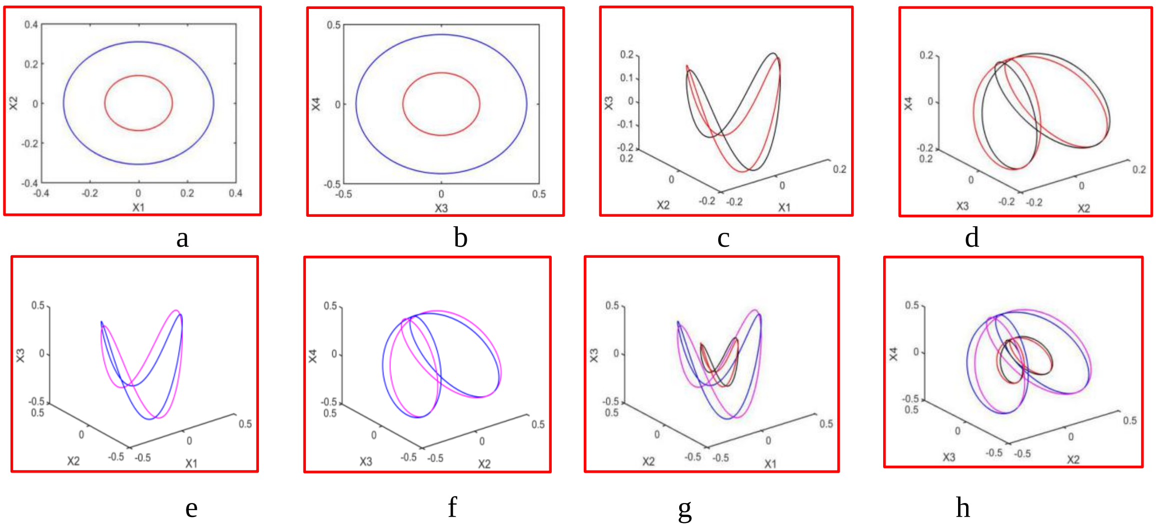

Figure 2. Periodic orbits for UP

Figure 2a-b and g-h, respectively, represent the 4 phase portraits on the plans x_1x_2 and x_3x_4, and the phase portraits in the three-dimensional space $x_1x_2x_3$, $x_2x_3x_4$. Figure 2 c-d and e-f, respectively, represent the phase portraits in the three-dimensional space $x_1x_2x_3$ and $x_2x_3x_4$ when $h = h^{1*}$ and $h = h^{2*}$.

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4. Conclusions

In this paper, we studied the periodic solutions of existence and the upper bound of two-coupled-oscillator model in optics chiral molecular medium. We obtain the Melnikov function by establishing the curvilinear coordinate transformation and constructing a Poincaré map. The maximum number of two-coupled-oscillator system is 4.

The two-coupled-oscillator model is a typical nonlinear dynamic problem of the interaction between oscillators. It has great theoretical significance to study the nonplanar motion of two-coupled-oscillator system for analyzing dynamic characteristics and studying influencing factors and also has great practical significance to solve the kinematic behavior of oscillator model in practice.

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