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Problems of mathematical modelling of elastic boundary value in the stress-strain state of car body elements

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Abstract. This article considers mathematical modeling problems of elastic boundary value. The stress-strain state mathematical modeling is used to assess the adequacy of theoretical solutions. The conditions of stress-strain state elements of the study mathematical modeling of the car body are given. The obtained analytical dependences make it possible to simulate the stress-strain state of the car body elements in real operating conditions and to compare the experimental and calculated values of stresses, strains and displacements arising under load.

1. Introduction

Models of static and dynamic testing aimed to determine the stress-strain state automotive products made up of elements are important. This issue becomes especially relevant in the development of new technology when materials are subject to increasing loads. The resulting stress concentration usually determines the product strength in the areas of its formation. The method of mathematical modeling is very effective for establishing such zones in a full-scale sample to determine the nominal stresses values, strains and displacements.

The dimensions and similarity analysis of the theoretical methods are a basis for the mathematical modeling of the stress-strain state. Experimental studies of stress fields, strains and displacements arising under load in the full-scale model, geometrically similar to the prototype, are used in the solid mechanics modeling.

The working condition of the stress-strain state friction system significantly depends on the distribution of contact stresses, which in experimental mechanics are modeled by polarization-interference methods [1 – 6]. The methods of photomechanics are related to the methods of physical modeling. Therefore the accuracy results, obtained during their application, depend significantly on how fully the similarity issues are developed in the elastic fields of stresses, strains and displacements. The experimental solutions of contact problems [3 – 5] agree with the obtained theoretical results [7 – 9] and numerical [3] methods, which indicates a sufficient degree of similarity between photoelastic experiments and the validity of the used modeling conditions.

2. Materials and methods

In this article we generalize an approach by deriving identical similarity conditions from the equations of the boundary-value main problem of elasticity theory [10-12] during deriving the conditions of mathematical modeling from particular contact problems [13-17].

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The volume of the static boundary value problem in formulation of elasticity theory is determined by the systems of equilibrium equations:

$$\sigma_{ij}(x_i)+\rho F_i(x_i)=0\,, \qquad (1)$$

where σ_{ij} – the stress tensor ($i, j=1,2,3$), F_i – forces. The Cauchy relations:

$$2\varepsilon(x_i)=u_{ij}+u_{ji}, \qquad (2)$$

where ε_{ij} – the strain tensor; x_i is the displacement vector; u_{ij} – conditions of compatibility, u_{ji} – Saint-Venant:

$$\eta^{ij}\equiv\Psi_{ikl}\Psi_{imn}\varepsilon_{lm};kn=0, \qquad (3)$$

where Ψ_{ikl} – the alternating tensor; Ψ_{imn} – the relation of the law of connection between stresses and strains; ε_{im} – adopted in the form of Hooke's law:

$$\sigma_{ij}(x_i)=\lambda\theta\delta_{ij}+2\nu\varepsilon_{ij}, \qquad (4)$$

where θ – volume deformation; $\theta=3\varepsilon=\varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33}=\varepsilon_{ij}$; λ, ν – lame constants.

Boundary conditions must be known for the formulation of the boundary value problem in linear elasticity theory. Let the surface bounding the test body consist of two parts and one part of the surface S_σ have external forces (loads):

$$\sigma_{ijni}\big|_{S_\sigma}=\sigma_{io}. \qquad (5)$$

On the other side of the surface S_u, the displacements are set:

$$u_i\big|_{S_u}=u_{io}. \qquad (6)$$

The problem system equations in a closed form contain the equations of equilibrium (1), at small strains, the ratio of the Cauchy problem (2), the conditions of deformations compatibility (3), the equations of Hooke's law (4) and boundary conditions (5) and (6). More general boundary conditions are "contact" boundary conditions [18, 19, 20]:

$$\alpha_{im}^{(x)}\sigma_{mjnj}+\frac{\beta_{im}^{(x)}u_m E_x}{l}=N_i^{(x)}, \qquad (7)$$

where E_x – modules of elasticity of contacting elastic bodies; $\alpha_{im}^{(x)}$ and $\beta_{im}^{(x)}$ – constants; l – the characteristic linear size of the body; $N_i^{(x)}$ – the so-called “contacting” forces with the dimension of stress; x – the index of sites; S_x – components in the sum of the body surface.

Conditions (5) and (6) are a special boundary condition case when it takes the value of *1* and *2*, at that:

$$\alpha_{im}^{(1)}=\delta_{im};\ \ \beta_{im}^{(1)}=0;N_i^{(1)}=P_{io};\ \ \alpha_{im}^{(2)}=0;\beta_{im}^{(2)}=\delta_{im};\ \ \frac{N_i^{(2)}l}{E_x}=u_{io}.$$

After performing the invariant transformations with respect to the linear-unambiguous correspondence $x_N=k_x x_M$ of the equations closed system to the elasticity theory problem of the mixed basic boundary value (1-6), we obtain the following system of similarity indicators:

$$\frac{k_p k_g kl}{k_e k_\varepsilon}=1;\ \ k_u k_\varepsilon kl=1;\ \ \frac{k_{uo}k_e kl}{k_p}=1;\ \ \frac{k_p}{kl^2 k_e k_\varepsilon}=1, \qquad (8)$$

where $kl=l_N/l_M$ – the scale of geometric similarity; k_p – the similarity relations scale of the body in nature and the model densities, k_g – the scale similarity of the gravity acceleration, etc. The first equation of equilibrium does not consider the body forces for a two-dimensional problem in a coordinate form, as in (1):

$$\frac{\partial \sigma_x}{\partial x} = -\frac{\partial \tau_{xy}}{\partial y}. \tag{9}$$

Let us introduce in the dependence (9) the extent of similarity:

$$\frac{k\sigma_x}{k_x}\frac{\partial \sigma_x}{\partial x} = -\frac{k\tau_{xy}}{k_y}\frac{\partial \tau_{xy}}{\partial y}. \tag{10}$$

So that the dependence (10) was identical to nature and the model, you need to take:

$$\frac{k\sigma_x}{k_x} = \frac{k\tau_{xy}}{k_y} = 1. \tag{11}$$

Taking into account equality $k_x = k_y = kl$ from (11), we have:

$$k\sigma_x = k\tau_{xy}. \tag{12}$$

That is, the normal and tangential components of the stress tensor ($i, j = 1,2,3$) are modeled on the same similarity scale. The equality (12) follows also from dimensional analysis, the involved variables.
The expression of the components of the stress state in an isotropic body through the strain components has the form:

$$\tau_{xy} = \frac{E\varepsilon_{xy}}{2(1+\mu)}. \tag{13}$$

The scale of such transformation is introduced in (13):

$$k_{\tau_{xy}}\tau_{xy} = \frac{k_e k_{\varepsilon xy}}{k(1+\mu)}\frac{E}{2(1+\mu)}\varepsilon_{xy}. \tag{14}$$

Similarly to the previous one, after omitting indexes:

$$\frac{k}{\tau} = \frac{k_e k_\varepsilon}{k(1+\mu)} = 1. \tag{15}$$

Taking into account (12), the dependence between the degree of similarity, (15) can be represented in the form:

$$k_\sigma = \frac{k_e k_\varepsilon}{k(1+\mu)} = 1 \text{ or } k_\sigma = \frac{k_\varepsilon k_u}{k_l k(1+\mu)} = 1. \tag{16}$$

From the boundary contact condition (7) the similarity indicator is:

$$k_\sigma = k \text{ and } \frac{k_e}{kl} = 1, \tag{17}$$

coinciding with the second indicator (16).
According to Hooke's law in form (4) for a flat problem, we have:

$$\varepsilon_x = \frac{\sigma_x - \mu\sigma_y}{E} \tag{18}$$

or given that the similarity scales are introduced in (18):

$$k_{\varepsilon_x}\varepsilon_x = \frac{1}{k_e}\frac{1}{E}\left(k_{\sigma_x}\sigma_x - \mu k_{\sigma_y}\sigma_y\right). \tag{19}$$

For the similarity of the model and the model in (19):

$$\frac{k_e k_\varepsilon}{k_\sigma} = k(1-\mu). \tag{20}$$

Expressions (17) and (20) imply the equality of similarity indicators:

$$\frac{k_u k_e}{kl} = \frac{k_e k_\varepsilon}{k(1+\mu)}. \tag{21}$$

Expressing the ratio (17) scale $k_\varepsilon = k_u / kl$, taking into account the dimensions, we obtain:

$$\frac{k(1-\mu^2)}{k_e} = \frac{k_u k^2 l}{k_p k_l}. \tag{22}$$

Let us introduce the notation $k_1 = (1-\mu_1^2)/E_1$ and $k_2 = (1-\mu_2^2)/E_2$ and rewrite the dependence (22):

$$\frac{k_b^2 kl}{k_p k_r} = \frac{(k_1 + k_2)_u}{(k_1 + k_2)_v}, \tag{23}$$

where k_r – the scale of the characteristic linear dimensions of the bodies in contact (for example, curvature radii), k_b – the geometric scale of similarity of the width of the contact area, kl – the scale of the geometric similarity, k_p – the scale of the force similarity, indices *1* and *2* refer to the bodies in contact:

$$k_j = \frac{(1-\mu_i^2)}{E_j} \quad (j = 1{,}2).$$

In the simulation with the overall scale of geometric similarity for the length of the contacting bodies and the radii of contact areas ($k_{r_1} = k_{r_2} = k_l$), the dependence (23) is simplified:

$$\frac{k_b^2}{k_p} = \frac{(k_1 + k_2)_N}{(k_1 + k_2)_M}. \tag{24}$$

The analysis of dependence (23) allows us to consider three special cases.
1. When $\mu_1 = \mu_2 = \mu = const$, the dependence (23) takes the form:

$$\frac{k_b^2 k_l}{k_p k_r} = \frac{(E_1 + E_2)_N (E_1 E_2)_M}{(E_1 + E_2)_M (E_1 E_2)_N}, \tag{25}$$

where $k_r = k_l$, subject to the equality, follows the condition of similarity for the sites of contact:

$$\left(\frac{E_{1N}}{E_{1M}} + \frac{E_{1N}}{E_{2M}}\right) k_b^2 = k_p \left(1 + \frac{E_{1N}}{E_{2N}}\right). \tag{26}$$

2. When the elastic modulus of interacting bodies in contact is equal to $E_1 = E_2 = E$, the dependence (23) is modified:

$$\frac{k_b^2 k_l k_e}{k_l k_r} = \frac{\left[(1-\mu_1^2) + (1-\mu_2^2)\right]_N}{\left[(1-\mu_1^2) + (1-\mu_2^2)\right]_M} \tag{27}$$

with appropriate simplification $k_r = k_l$.

3. When Poisson's coefficients and Young's modulus $E_1 = E_2 = E$ of contacting bodies are equal, i.e. when they are made of the same material, the formula (23) takes the form $r_1 = r_2 = r$:

$$\frac{k_b^2 k_l}{k_p k_r} = \frac{(1-\mu_N^2)}{(1-\mu_M^2)} \frac{E_M}{E_N}. \tag{28}$$

If the contact conditions of two bodies affect the stresses under study, the model of contact areas with linear dimensions should be modeled according to the geometric similarity scale k_l. This condition is satisfied by the selection of contact zone radii r_{Mi} ($i = 1{,}2$) in the interacting bodies or loads p_M applied to the model. The similarity condition (23) and the resulting special cases satisfy the selection of numerical values of their constituent quantities. In the contact area models can also be applied to the initial radii of contact surfaces, made in accordance with the geometric similarity scale $r_{1N} / r_{1M} = r_{2N} / r_{2M} = k_r = k_l$. In this case the simulation of the contact problems' condition should be performed:

$$\frac{k_b^2 k_l}{k_p k_r} = \frac{(k_1 + k_2)_N}{(k_1 + k_2)_M}. \tag{29}$$

3. Results and discussion

The results of the analysis allow drawing some conclusions.

On the basis of a linear unambiguous correspondence between the similar values of the full-scale sample and the mathematical model, the invariant transformation is obtained. In the elasticity theory of equations systems, the main mixed boundary value problem with contact boundary condition (7) are obtained. Generalized similarity condition (23) for modeling of a wide range of elastic contact problems (8) is obtained. In general, the contacting bodies in the mathematical model and the full-scale sample can be made of different structural materials.

4. Conclusion

Mathematical modeling can be implemented for three special cases:

1) equality of Poisson's ratio of two contacting bodies (metal alloys-glass);
2) the elastic modulus equality of two bodies in the full-scale sample and the mathematical model;
3) the contacting bodies are made of one material.

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