Approximate analytical solution to the stationary two-dimensional heat conduction problem on infinite bar with the source of heat

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Approximate analytical solution to the stationary two-dimensional heat conduction problem on infinite bar with the source of heat

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Abstract. An approximate analytical solution to the boundary- value heat conduction problem for an infinite bar with a heat source was obtained with the use of the integral method of heat balance, by introducing a complementary required function and complementary boundary conditions. The boundary - value problem for a partial differential equation is reduced to an ordinary differential equation with respect to this function due to the complementary required function that characterizes the change in temperature along the axis of symmetry in the cross-section of the bar. The complementary boundary conditions determined by the initial differential equation and the given boundary conditions are found so that their satisfaction is equivalent to the solution of the initial equation of the boundary value problem at the boundary points. The fulfilment of the equation at the boundary points as well as the heat balance integral results in the fulfilment of the initial equation inside the domain. The approximate analytical solution obtained can be used to identify the amount of internal heat generated by various production processes (vibration and deformation loads, electromagnetic fields effects, etc.) in thermal and nuclear power plants, in the rocket and space industry and other industrial facilities.

1. Introduction

Among all approximate analytical solutions for boundary value problems of heat conduction, the integral heat balance method related to the group of orthogonal weighted residual methods, has been widely used [1–4]. It helps to obtain approximate analytical solutions for boundary value problems, the accurate solution to which can not be found (nonlinearity of the problem, variable physical properties of the medium, etc.) [5–9].

However, the main disadvantage of these methods is low accuracy as these methods require solving the averaged initial equation (heat balance integral). The application of the complementary desired function and complementary boundary conditions provides satisfying the initial differential equation with a specified degree of accuracy which depends on the number of approximations of the initial solution. Note that the methods based on the solving the equation at the boundary points have been also considered in papers [10–13].

2. Theoretical justification and application of the method

Consider the use of the integral heat balance method with complementary boundary conditions and a complementary required function for solving a stationary 2D heat conduction problem in an infinite bar of square cross section with a constant heat source (figure 1) [14–17].

Assume that the temperature on the lateral faces of the bar is known and is equal to T_w. It is necessary to determine the temperature distribution in the cross section of the bar. Due to the symmetry of the boundary value problem, only a quarter of the cross section can be considered fixing the boundary conditions of the first kind along EF and AF curves, and the conditions of the absence of heat transfer along OE and AO curves. The mathematical setting of the problem in this case is as follows (figure 1)

$$\frac{\partial^2 T(x,y)}{\partial x^2}+\frac{\partial^2 T(x,y)}{\partial y^2}=-\frac{\nu}{\lambda} \qquad (0<x<\delta;\, 0<y<\delta); \tag{1}$$

$$T(\delta,y)=T_w;\ \frac{\partial T(0,y)}{\partial x}=0;\ \frac{\partial T(x,0)}{\partial y}=0;\ T(x,\delta)=T_w \tag{2}$$

where T - temperature, K; x, y - coordinates, m; ν — power of the internal heat source, W / m^3; λ — thermal conductivity coefficient, W / (mK); T_w - wall temperature, K; δ - half width of the bar face, m. Finding a simple type of analytical solutions for 2D boundary-value problems based on the Poisson equation (1) is of considerable practical interest because of its wide use when analyzing many physical processes (heat conduction, fluid flow, theory of elasticity, thermoelasticity, torsion of prismatic objects, etc.).

The difficulty in constructing a solution is caused by the two-dimensionality of the problem and the non-homogeneity of the initial differential equation. Exact analytical solutions to such problems with complex infinite functional series were obtained only in individual particular cases and, moreover, with substantial assumptions (only under boundary conditions of the first kind and with a heat source to be constant in space). Note that regardless of the simplified mathematical problem setting (1), (2) the method described below provides obtaining solutions with variable heat sources, without regard to the type of boundary conditions and the cross section shape (rectangle, square) [18–21].

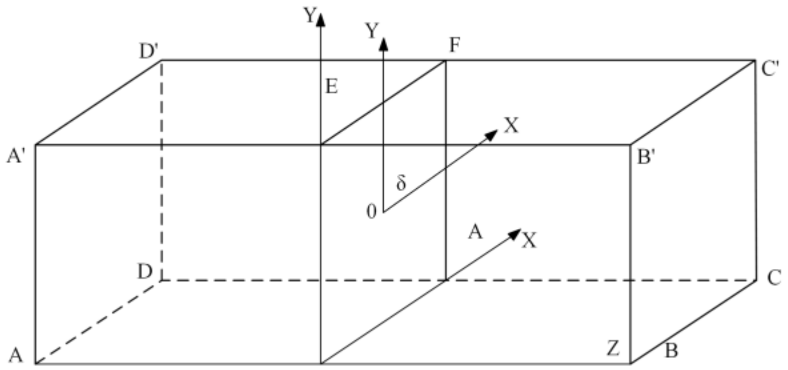

Figure 1. The computational domain for the cross-section of the infinite rectangular bar.

Introduce the following dimensionless variables and parameters:

$$\Theta=\frac{T-T_w}{T_w};\ \xi=\frac{x}{\delta};\ \eta=\frac{y}{\delta};\ B=\frac{\nu\delta^2}{\lambda T_w},$$

where Θ - dimensionless temperature; ξ, η — dimensionless coordinates; B –dimensionless parameter.

Taking into account notations introduced, the heat exchange diagram in the problem (1), (2) will be (figure 2)

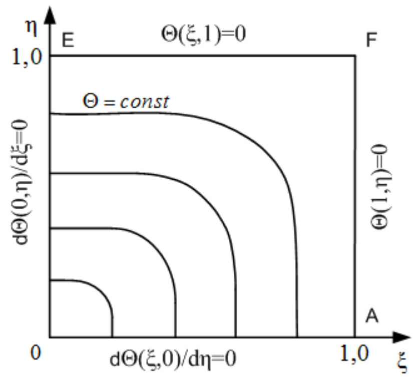

Figure 2. Heat exchange diagram for the cross section of an infinite rectangular bar.

The approximate analytical solution of the stationary 2D heat conduction problem for an infinite bar with a heat source is described by the following relations

$$\frac{\partial^2\Theta(\xi,\eta)}{\partial\xi^2}+\frac{\partial^2\Theta(\xi,\eta)}{\partial\eta^2}+B=0 \quad (0<\xi<1;\ 0<\eta<1)\,; \tag{3}$$

$$\Theta(1,\eta)=0\,; \tag{4}$$

$$\frac{\partial\Theta(0,\eta)}{\partial\xi}=0\,; \tag{5}$$

$$\frac{\partial\Theta(\xi,0)}{\partial\eta}=0\,; \tag{6}$$

$$\Theta(\xi,1)=0\,; \tag{7}$$

where $\Theta=\frac{T-T_{cr}}{T_{cr}}$; $\xi=\frac{x}{\delta}$; $\eta=\frac{y}{\delta}$; $B=\frac{\nu\delta^2}{\lambda T_{cr}}$ - dimensionless complex; Θ, ξ, η - dimensionless temperature and spatial variables, respectively.

Introduce the additional required function that characterizes the temperature distribution along the OA curve (Fig. 2) by

$$q(\xi)=\Theta(\xi,0)\,. \tag{8}$$

An approximate solution to the problem (3) - (7) may be found in the form

$$\Theta(\xi,\eta)=\sum_{k=1}^{n}b_k(q)\varphi_k(\eta)\,, \tag{9}$$

where $b_k(q)$ – required coefficients; $\varphi_k(\eta)=\cos\left(\frac{r\pi\eta}{2}\right)$ – coordinate functions $(r=2k-1,\ k=\overline{1,n})$.

Relation (7) due to the adopted system of coordinate functions satisfies the boundary conditions (6), (7). To obtain a solution to the problem (3) - (7) in the first approximation we substitute (9) (taking into account only one term of the series with k = 1) to the relation (8)

$$b_1(q)\cos\left(\frac{\pi\eta}{2}\right)\bigg|_{\eta=0}=q(\xi)\,. \tag{10}$$

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Relation (10) with respect to the required coefficient $b_1(q)$ offers an algebraic linear equation, solving which one can find $b_1(q)=q(\xi)$. By taking into account the value found of the required coefficient $b_1(q)$, the relation (9) is written as

$$\Theta(\xi,\eta)=q(\xi)\cos\left(\frac{\pi\eta}{2}\right). \tag{11}$$

We will demand that relation (11) to be fulfilled the averaged equation (3) with respect to the coordinate η (heat balance integral).

$$\int_0^1\left[\frac{\partial^2\Theta(\xi,\eta)}{\partial\xi^2}+\frac{\partial^2\Theta(\xi,\eta)}{\partial\eta^2}+B\right]d\eta=0\,. \tag{12}$$

Substituting (11) into (12), after determining the integrals with respect to the required function $q(\xi)$, we obtain the following ordinary differential equation

$$\frac{d^2q(\xi)}{d\xi^2}-\frac{\pi^2q(\xi)}{4}+\frac{\pi B}{2}=0\,. \tag{13}$$

The general integral of the equation (13) is as follows

$$q(\xi)=C_1\exp\left(\frac{\pi\xi}{2}\right)+C_2\exp\left(\frac{-\pi\xi}{2}\right)+\frac{2B}{\pi}, \tag{14}$$

where C_1, C_2 - constants of integration. Substituting (14) into (11) we have

$$\Theta(\xi,\eta)=\left(C_1e^{\frac{\pi\xi}{2}}+C_2e^{\frac{-\pi\xi}{2}}+\frac{2B}{\pi}\right)\cos\left(\frac{\pi\eta}{2}\right). \tag{15}$$

The constants of integration C_1 and C_2 are found from boundary conditions (4), (5). Substituting (15) into (4), (5) with respect to C_1 and C_2 we will have a system of two algebraic linear equations, and solving them we obtain

$$C_1=C_2=\frac{-2B\exp\left(\frac{\pi}{2}\right)}{\pi+\pi\exp\pi}$$

Taking into account values found C_1 and C_2 the relation (15) is as follows

$$\Theta(\xi,\eta)=\frac{2B}{\pi}\left[\frac{\exp\frac{\pi}{2}}{1+\exp\pi}\exp\frac{\pi\xi}{2}+\exp\frac{-\pi\xi}{2}+1\right]\cos\frac{\pi\eta}{2}\,. \tag{16}$$

The solution of the problem (3) - (7) in the first approximation is given by relation (16). It exactly satisfies the boundary conditions (4) - (7), the heat balance integral (12) and approximately satisfies

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the equation (3) (in the first approximation). The standard deviation of the solution (16) compared to the exact one [22] according to Gauss form is 0.7%.

To increase the accuracy of the solution, it is necessary to enlarge the number of terms of series (9). To determine the required coefficients it is necessary to apply complementary boundary conditions (CBC). These conditions are found in such a form that their fulfillment by the desired solution is equivalent to the fulfillment of equation (3) at the borders of OE and OA of the bar section (Fig. 2). To obtain the first condition, let us differentiate the boundary condition (6) twice with respect to the variable ξ

$$\frac{\partial}{\partial\eta}\left(\frac{\partial^2\Theta(\xi,0)}{\partial\xi^2}\right)=0\,. \tag{17}$$

The relation (17) taking into account equation (3) is reduced to the following CBC along OA curve

$$\frac{\partial^3\Theta(\xi,0)}{\partial\eta^3}=0\,. \tag{18}$$

To obtain the second CBC (for the border set OA), let us differentiate (18) twice with respect to ξ

$$\frac{\partial^3}{\partial\eta^3}\left(\frac{\partial^2\Theta(\xi,0)}{\partial\xi^2}\right)=0\,. \tag{19}$$

Relation (19) taking into account equation (3) is reduced to CBC in the form

$$\frac{\partial^5\Theta(\xi,0)}{\partial\eta^5}=0 \tag{20}$$

Similarly, we can get any arbitrary number of CBCs on the axis of symmetry (along the OA curve). The general formula for them will be

$$\frac{\partial^{2i-1}\Theta(\xi,0)}{\partial\eta^{2i-1}}=0\ \ (i=2,3,4\ldots) \tag{21}$$

Note that due to the adopted system of coordinate functions $\varphi_k(\eta)$ $k=\overline{1,n}$, all CBCs determined by formula (19) by relation (9) are fulfilled.

To determine the CBC along the axis of symmetry η (along OE curve), we differentiate the boundary condition (5) twice with respect to the variable η

$$\frac{\partial}{\partial\xi}\left(\frac{\partial^2\Theta(0,\eta)}{\partial\eta^2}\right)=0\,. \tag{22}$$

Relation (22) taking into account equation (3) is reduced to the following CBC

$$\frac{\partial^3\Theta(0,\eta)}{\partial\xi^3}=0\,. \tag{23}$$

If we differentiate (23) twice with respect to variable η, we can find

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$$\frac{\partial^3}{\partial \xi^3}\left(\frac{\partial^2 \Theta(0,\eta)}{\partial \eta^2}\right) = 0 . \tag{24}$$

Relation (24) taking into account equation (3) is reduced to the following CBC

$$\frac{\partial^5 \Theta(0,\eta)}{\partial \xi^5} = 0 . \tag{25}$$

The general formula of CBC along OE curve will be

$$\frac{\partial^{2i-1} \Theta(\xi,0)}{\partial \xi^{2i-1}} = 0 \ (i = 2,3,4...) \tag{26}$$

To determine the CBC along AF curve, we differentiate the boundary condition (3) twice with respect to the variable η

$$\frac{\partial^2 \Theta(1,\eta)}{\partial \eta^2} = 0 . \tag{27}$$

Equation (3) taking into account (27) is reduced to the following CBC

$$\frac{\partial^2 \Theta(1,\eta)}{\partial \xi^2} + B = 0 . \tag{28}$$

If we differentiate (28) twice with respect to the variable η, we find

$$\frac{\partial^2}{\partial \xi^2}\left(\frac{\partial^2 \Theta(1,\eta)}{\partial \eta^2}\right) = 0 . \tag{29}$$

Relation (29) taking into account equation (3) is reduced to the CBC in the form

$$\frac{\partial^4 \Theta(1,\eta)}{\partial \xi^4} = 0 . \tag{30}$$

The general formula for all subsequent CBCs along the AF curve will be

$$\frac{\partial^{2i} \Theta(1,\eta)}{\partial \xi^{2i}} = 0 \ (i = 2,3,4...) \tag{31}$$

Let us differentiate relation (28) twice with respect to the variable η

$$\frac{\partial^2 q(\xi)}{\partial \xi^2} = \frac{\partial^2 \Theta(\xi,0)}{\partial \xi^2} \tag{32}$$

Relation (32) taking into account equation (3) is reduced to the following CBC

$$\frac{\partial^2 q(\xi)}{\partial \xi^2} + \frac{\partial^2 \Theta(\xi,0)}{\partial \eta^2} + B = 0 . \tag{33}$$

Let us differentiate (33) twice with respect to the variable ξ

$$\frac{\partial^4 q(\xi)}{\partial \xi^4}+\frac{\partial^2}{\partial \eta^2}\left(\frac{\partial^2 \Theta(\xi,0)}{\partial \xi^2}\right)=0\,. \tag{34}$$

Relation (34) taking into account equation (3) is reduced to the CBC in the form

$$\frac{\partial^4 q(\xi)}{\partial \xi^4}+\frac{\partial^4 \Theta(\xi,0)}{\partial \eta^4}=0\,. \tag{35}$$

The general formula for all subsequent CBCs, starting with CBC (35), will be

$$\frac{\partial^{2i} q(\xi)}{\partial \xi^{2i}}\pm\frac{\partial^{2i} \Theta(\xi,0)}{\partial \eta^{2i}}=0\ \ (i=2,3,4,...) \tag{36}$$

where «+» sign is for even i, and «–» sign is for odd i.

To obtain a solution to the problem (3) - (7) in the second approximation, substitute (9) (taking into account two terms of the series) to relations (8) and (33). With respect to $b_1(q)$ and $b_2(q)$ we will have a system of two algebraic linear equations, solving which we can find

$$b_1(q)=-\frac{4q'-9\pi^2 q+4}{8\pi^2}\,;\ b_2(q)=\frac{4q'-\pi^2 q+4}{8\pi^2}\,; \tag{37}$$

Relation (9) taking into account (37) is as follows

$$\Theta(\xi,\mathrm{Fo})=\frac{(-q'+2{,}25\pi^2 q-1)\cos\left(\dfrac{\pi\eta}{2}\right)+(q'-0{,}25\pi^2 q+1)\cos\left(\dfrac{3\pi\eta}{2}\right)}{\pi^2} \tag{38}$$

Substituting (38) into the heat balance integral (12), with respect to the required function q(Fo) we will have the following ordinary differential equation

$$\frac{4}{3\pi^3}q^{IV}-\frac{10}{3\pi}q''+\frac{3\pi}{4}q=0\,, \tag{39}$$

where $q^{IV}=\dfrac{d^4 q(\xi)}{d\xi^4}$; $q''=\dfrac{d^2 q(\xi)}{d\xi^2}$

The general integral of equation (39) is as follows

$$q(\xi)=\frac{4B(\pi+1)}{3\pi^2}+C_1 e^{\frac{\pi\xi}{2}}+C_2 e^{\frac{3\pi\xi}{2}}+C_3 e^{\frac{-\pi\xi}{2}}+C_4 e^{\frac{-3\pi\xi}{2}}\,, \tag{40}$$

where C_1, C_2, C_3, C_4 - constants of integration determined from the basic (4), (5) and complementary (23), (28) boundary conditions, which for the function $q(\xi)=\Theta(\xi,0)$ are written as $q(1)=0$; $\dfrac{dq(0)}{d\xi}=0$; $\dfrac{d^3 q(0)}{d\xi^3}=0$; $\dfrac{d^2 q(1)}{d\xi^2}+B=0$. Substituting (40) into all these boundary conditions with respect to C_1, C_2, C_3, C_4 we will have a system of four algebraic equations, solving which we can find

$$C_1 = C_3 = -\frac{(3\pi + 2)e^{\frac{\pi}{2}}}{2\pi^2(e^{\pi} + 0{,}5)}; \; C_2 = C_4 = \frac{(\pi + 2)e^{\frac{3\pi}{2}}}{6\pi^2(e^{3\pi} + 1)}. \tag{41}$$

Relation (33) taking into account (40), (41) offers the solution to the problem (3) - (7) in the second approximation. It exactly satisfies the boundary conditions (4) - (7), the heat balance integral (12) and approximately (in the second approximation) the equation (3). The results of calculations by the formula (38) are shown in figure 3

Compared to the exact solution [8], it follows that the standard deviation according to the Gauss formula is 1.3%.

To increase the accuracy of the solution, it is necessary to enlarge the number of terms of the series (9). To find the required coefficients $b_k(q)$, we use the condition (8) and complementary boundary conditions obtained by the general formula (36). To find the constants of integration C_k, ($k = 2n$) the initial boundary conditions (2), (3), CBC (21) and CBCs obtained by the general formulas (26) and (31) are used.

It should be noted that the adopted solution method has rather rapid convergence. So, already in the second approximation, the maximum ratio error of the solution obtained (Chebyshev norm) compared to the exact solution [8] decreases from 33% to 10%.

Note also that the use of an exact analytical solution (which is an infinite series) is difficult in the cases where the solution of the problem (3) - (7) is an intermediate stage of the researches for other problems, such as thermoelasticity, inverse boundary value problems, control problems, etc [23-24]. In these cases, a solution that contains a limited number of terms of the series and has sufficient accuracy for engineering applications will be more effective.

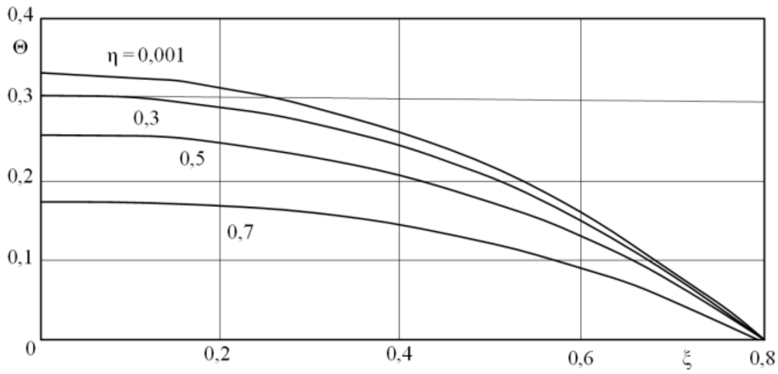

Figure 3. Temperature distribution in the bar section in the second approximation.

3. Conclusions.

1. Based on the use of a complementary required function and complementary boundary conditions in the integral heat balance method, an approximate analytical solution to the stationary two-dimensional heat conduction problem for an infinite bar with a uniformly distributed heat source is obtained. The use of the complementary required function provided reducing the solution of the partial differential equation to the integration of an ordinary differential equation.

2. To find the required coefficients and the constants of integration we used complementary boundary conditions determined in such a way that their fulfilling by the required solution is equivalent to the fulfillment of the initial differential equation at domain borders. It is shown that the fulfillment of the equation at the boundary points also leads to its fulfillment within the considered domain, with the accuracy depending on the number of approximations.

3. The use of the heat balance integral provides applying this method to solving boundary value problems for the Poisson equation with space-wise variables by physical properties of the medium and internal heat source.

4. Using the analytical solution (15) found, by solving the inverse heat conduction problem, one can fulfill quantitative identification of the internal heat sources generated in solids due to the processes of different nature (chemical reactions, heating by electromagnetic fields, vibration loads, deformation, friction, etc.) when included an experimental temperature value at any point in the structure.

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