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Dynamical Analysis on the Model of Tuberculosis Spread with Vaccination and Saturated Incident Rate

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Abstract. This research concern with dynamical analysis of a SVLIT (Susceptible Vaccination Latent Infective Treatment) model. It represents the spread of tuberculosis with vaccination and saturated incident rate. The incident rate is considered because of the barriers effect due to changes in susceptible individuals behavior. This model has two equilibrium points, namely disease-free equilibrium point which always exists and an endemic equilibrium point that exists under some certain conditions. The local stability of the equilibrium points is investigated by using Routh-Hurwitz criteria. The method of next generation matrix is applied to determine the basic reproduction number R_0 . It can be shown numerically that disease-free equilibrium point is local asymptotically stable when $R_0 < 1$, while the endemic equilibrium point exist and local asymptotically stable when $R_0 > 1$. Numerical simulations are given to illustrate the theoretical results.

Keywords: SVLIT epidemic model, saturated incident rate, equilibrium point, local stability.

1. Introduction

Tuberculosis (TB) is an airborne bacterial infection caused by *Mycobacterium tuberculosis*, and kills over 4.000 people each day [1,2]. This disease usually attacks the lungs, but can attack almost any part of the body [2]. TB is spread from individuals who have active TB through airborne infection [3]. However, it is not easy to become infected TB or usually spread among family in close spaces over a long period of time [3].

The most effective way to prevent TB transmission is by applying vaccination. Vaccines for TB are known as BCG (Bacille Calmette-Guérin). The BCG vaccine contains weak form of bacteria that cause TB. Because this bacterium is weakened, it does not cause TB in healthy people, on the contrary it is useful to form protection (immunity) against TB [4].

Mathematical model that can describe the behavior of disease spread is called epidemic model. A lot of mathematical model of tuberculosis spread have been investigated, such as Liu and Zhang [5], Gao and Huang [6], Setiawan [7], Fathoni [8], and Khan [9]. Liu and Zhang [5] proposed and analyzed a mathematical model of tuberculosis spread with vaccination and treatment which partitioned a population into five classes, namely susceptible, vaccination, latent, infective, and treatment. In 2017, the research was continued by Gao and Huang [6] by adding the reduction in risk of infection due to vaccination. Setiawan [7] analyzed the dynamical and optimal control of TB epidemic model which divides the population into four subpopulations namely susceptible, exposed,



infective, and treatment. Optimal control is applied to minimize the number of active TB and medical expenses.

Incident rate is a function that states the number of individuals that are infected due to direct contact with infective individuals. Liu and Zhang [5], Gao and Huang [6], and Setiawan [7] use bilinear incident rate in their model. There is possible that the incident rate is not bilinear because it considered the inhibition effect from the behavioral change of the healthy population. The incident rate which is not bilinear is called the saturated incident rate that first time has been introduced by Capasso and Serio [10]. This incident rate can be seen in Fathoni [8], Khan et al. [9], etc. Fathoni [8], investigated a dynamical analysis the model of TB spread with saturated incident rate that divided a population into four subpopulation who are SLIT (Susceptible, Latent, Infective and Treatment). This paper modified the SLIT model [8] into SVLIT model [5] with saturated incident rate. The model formulation is presented in section 2. The equilibrium point of the model, their local stability analysis and the basic reproduction number are given in section 3. In section 4 we presented the numerical simulations. Finally in section 5 we give conclusion and some discussions.

2. Model Formulation

This section describe a modified the model of Liu and Zhang [5] by adding the saturated incident rate to the susceptible and infective individuals. The assumption in the model is the vaccine efficacy 100%, so there is no risk of vaccination failure ($\rho_2 = 0$). The total population at time (t) denoted by $N(t)$, which is divided into five subpopulations who are susceptible ($S(t)$), vaccinated ($V(t)$), latent ($L(t)$), infective ($I(t)$) and treatment ($T(t)$). Λ is represent the recruitment into susceptible class. $\frac{\beta SI}{1+\omega I}$ is the contact rate of susceptible with infective individuals where β is the disease contact rate and ω represent the saturation constant. The natural death rate is μ and the vaccination rate is p . A fraction of susceptible individuals who acquire TB infection move to the latent class denoted by l . The parameter $\rho_1 < 1$ accounts for the reduction in infectiousness among individuals with TB active who are treated. δ is represent the rate transfer of latent class to infective. ρ is the rate of individuals who are successfully treated. Compartment I has additional disease death rate with α , and γ is represent the rate transfer of compartment I to treatment class. More precisely, we studied the following system:

$$\begin{aligned}\frac{dS}{dt} &= \Lambda - \beta S \left(\frac{I}{1 + \omega I} + \rho_1 T \right) - (\mu + p)S \\ \frac{dV}{dt} &= pS - \mu V \\ \frac{dL}{dt} &= l\beta S \left(\frac{I}{1 + \omega I} + \rho_1 T \right) - (\mu + \delta)L + \rho T \\ \frac{dI}{dt} &= \frac{\beta SI}{1 + \omega I} (1 - l) + \rho_1 \beta ST (1 - l) + \delta L - (\mu + \alpha + \gamma)I \\ \frac{dT}{dt} &= \gamma I - \mu T - pT\end{aligned}\tag{2.1}$$

with initial condition:

$$S(0) \geq 0, V(0) \geq 0, L(0) \geq 0, I(0) \geq 0, \text{ and } T(0) \geq 0,$$

the time “ t ” has been considered in “days”, where the coefficients $\Lambda, \beta, \mu, p, \rho_1, l, \delta, \rho, \alpha$ and γ are assumed to be positive. Compartment V does not appear in other equations and only depends on compartment S , so the model can be reduced to four compartments and notice that:

$$b_1 = \mu + p,$$

$$b_2 = \mu + \delta,$$

$$b_3 = \mu + \alpha + \gamma,$$

$$b_4 = \mu + p.$$

Then we can obtained the reduced system as follows:

$$\begin{aligned}\frac{dS}{dt} &= \Lambda - \beta S \left(\frac{I}{1 + \omega I} + \rho_1 T \right) - b_1 S \\ \frac{dL}{dt} &= l\beta S \left(\frac{I}{1 + \omega I} + \rho_1 T \right) - b_2 L + \rho T \\ \frac{dI}{dt} &= \frac{\beta SI}{1 + \omega I} (1 - l) + \rho_1 \beta ST (1 - l) + \delta L - b_3 I \\ \frac{dT}{dt} &= \gamma I - b_4 T\end{aligned}\tag{2.2}$$

3. Model Analysis

3.1 Basic Reproduction Number

Basic reproduction number (R_0), is the threshold for the spread of disease. The basic reproduction number obtained by using next generation matrix [11] that can be formed based on compartment L and I . So we can define matrix F and V from those compartment as follows:

$$F = \begin{bmatrix} 0 & \frac{l\beta\Lambda}{b_1} \\ \delta & \frac{\beta\Lambda}{b_1} \end{bmatrix}$$

and

$$V = \begin{bmatrix} b_2 & 0 \\ 0 & \frac{l\beta\Lambda}{b_1} + b_3 \end{bmatrix}.$$

The next generation matrix (H) stated with

$$H = FV^{-1},$$

thus, basic reproduction number is the spectral radius of next generation matrix (H), we have :

$$R_0 = \frac{1}{2} \frac{b_2\beta\Lambda + \sqrt{4b_2\beta^2\Lambda^2\delta l^2 + 4b_1b_2b_3\beta\delta l\Lambda + b_2^2\beta^2\Lambda^2}}{(l\beta\Lambda + b_1b_3)b_2}.$$

3.2 Equilibrium point and their local stability analysis

System (2.2) has two equilibrium points namely disease free equilibrium point $E_0 = (S_0, L_0, I_0, T_0) = \left(\frac{\Lambda}{b_1}, 0, 0, 0\right)$ and endemic equilibrium point $E^* = (S^*, L^*, I^*, T^*)$, where $S^* = \frac{\Lambda}{\left(\frac{\beta I^*}{1 + \omega I^*} + \frac{\beta \rho_1 \gamma I^*}{b_4} + b_1\right)}$, $L^* =$

$$\frac{l\beta \frac{\Lambda}{\left(\frac{\beta I^*}{1 + \omega I^*} + \frac{\beta \rho_1 \gamma I^*}{b_4} + b_1\right)} \left(\frac{I^*}{1 + \omega I^*} + \frac{\rho_1 \gamma I^*}{b_4}\right) + \frac{\rho \gamma I^*}{b_4}}{b_2}, \quad T^* = \frac{\gamma I^*}{b_4} \text{ and } I^* \text{ is positive root of the quadratic equation } KI^{*2} + LI^* + M, \text{ where :}$$

$$K = \beta \rho_1 \omega \gamma (\delta \rho \gamma - b_2 b_3 b_4)$$

$$L = \Lambda \beta \rho_1 \omega \gamma b_4 (b_2 (1 - l) + \delta) + \delta \rho \gamma (\beta b_4 + \beta \rho_1 \gamma + \omega b_1 b_4) - b_2 b_3 b_4 (\beta b_4 + \beta \rho_1 \gamma + \omega b_1 b_4)$$

$$M = \Lambda\beta b_2 b_4 (b_4(1-l) + \rho_1\gamma(1-l)) + \Lambda\beta\delta l b_4 (b_4 + \rho_1\gamma) + b_1 b_4 (\delta\rho\gamma - b_2 b_3 b_4).$$

Furthermore, from the above equation can be obtained some possibilities that I^* is a positive real number if and only if $D \geq 0$, or $L^2 - 4KM \geq 0$.

The local stability of equilibrium point will be analyzed by linearizing system [12] resulting Jacobian matrix as follows

$$J = \begin{bmatrix} -\frac{\beta I}{1+\omega I} - \beta\rho_1 T - b_1 & 0 & -\frac{\beta S}{(1+\omega I)^2} & -\beta\rho_1 S \\ \frac{l\beta I}{1+\omega I} + l\beta\rho_1 T & -b_2 & \frac{l\beta S}{(1+\omega I)^2} & l\beta\rho_1 S + \rho \\ \frac{\beta I}{1+\omega I}(1-l) + \beta\rho_1 T(1-l) & \delta & \frac{\beta S(1-l)}{(1+\omega I)^2} - b_3 & \beta\rho_1 S(1-l) \\ 0 & 0 & \gamma & -b_4 \end{bmatrix}.$$

When all of eigenvalues the Jacobian matrix are negative then the equilibrium point is stable otherwise, it is unstable

3.2.1 Behaviour of disease free equilibrium point

Substituted the disease free equilibrium point E_0 into the Jacobian matrix, we have

$$J(E_0) = \begin{bmatrix} -b_1 & 0 & -\frac{\beta\Lambda}{b_1} & -\frac{\beta\Lambda\rho_1}{b_1} \\ 0 & -b_2 & \frac{l\beta\Lambda}{b_1} & \frac{l\beta\Lambda\rho_1}{b_1} + \rho \\ 0 & \delta & \frac{\beta\Lambda}{b_1}(1-l) - b_3 & \frac{\beta\Lambda\rho_1}{b_1}(1-l) \\ 0 & 0 & \gamma & -b_4 \end{bmatrix}.$$

One of eigenvalue is obtained $\lambda_1 = -b_1 < 0$, and the others are the roots of the characteristic equation $a_0\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0$,

where

$$a_0 = 1$$

$$a_1 = -\frac{\beta\Lambda}{b_1}(1-l) + b_2 + b_3 + b_4$$

$$a_2 = \beta\Lambda\left(-\frac{b_2}{b_1}(1-l) - \frac{b_4}{b_1}(1-l) - \frac{\rho_1\gamma}{b_1}(1-l) - \frac{\delta l}{b_1}\right) + b_2 b_3 + b_3 b_4 + b_2 b_4$$

$$a_3 = -\frac{b_2 b_4 \beta\Lambda}{b_1}(1-l) - \frac{b_2 \rho_1 \gamma \beta\Lambda}{b_1}(1-l) - \frac{\delta l \beta\Lambda}{b_1}(\rho_1\gamma + b_4) + b_2 b_3 b_4 - \delta\rho\gamma.$$

to verified that $\lambda_2, \lambda_3, \lambda_4$ have negative real parts it used Routh-Hurwitz criteria [13]. The above equation will have a negative real part roots if and only if $a_1 > 0$, $a_3 > 0$ and $a_1 a_2 - a_3 > 0$.

3.2.2 Behaviour of endemic equilibrium point

The Jacobian matrix of (2.2) at E^* is given by

$$J(E^*) = \begin{bmatrix} -\frac{\beta I^*}{1+\omega I^*} - \beta\rho_1 T^* - b_1 & 0 & -\frac{\beta S^*}{(1+\omega I^*)^2} & -\beta\rho_1 S^* \\ \frac{l\beta I^*}{1+\omega I^*} + l\beta\rho_1 T^* & -b_2 & \frac{l\beta S^*}{(1+\omega I^*)^2} & l\beta\rho_1 S^* + \rho \\ \frac{\beta I^*}{1+\omega I^*}(1-l) + \beta\rho_1 T^*(1-l) & \delta & \frac{\beta S^*(1-l)}{(1+\omega I^*)^2} - b_3 & \beta\rho_1 S^*(1-l) \\ 0 & 0 & \gamma & -b_4 \end{bmatrix},$$

where $S^* = \frac{\Lambda}{\left(\frac{\beta I^*}{1+\omega I^*} + \frac{\beta \rho_1 \gamma I^*}{b_4} + b_1\right)}$, $L^* = \frac{l\beta \frac{\Lambda}{\left(\frac{\beta I^*}{1+\omega I^*} + \frac{\beta \rho_1 \gamma I^*}{b_4} + b_1\right)} \left(\frac{I^*}{1+\omega I^*} + \frac{\rho_1 \gamma I^*}{b_4}\right) + \rho \gamma I^*}{b_2}$, $T^* = \frac{\gamma I^*}{b_4}$,

we obtained

$$J(E^*) = \begin{bmatrix} -\frac{\beta I^*}{1+\omega I^*} - \beta \rho_1 \frac{\gamma I^*}{b_4} - b_1 & 0 & -\frac{\beta \Lambda}{(1+\omega I^*)^2 \left(\frac{\beta I^*}{1+\omega I^*} + \frac{\beta \rho_1 \gamma I^*}{b_4} + b_1\right)} & \frac{-\beta \rho_1 \Lambda}{\left(\frac{\beta I^*}{1+\omega I^*} + \frac{\beta \rho_1 \gamma I^*}{b_4} + b_1\right)} \\ \frac{l\beta I^*}{1+\omega I^*} + l\beta \rho_1 \frac{\gamma I^*}{b_4} & -b_2 & \frac{l\beta \Lambda}{(1+\omega I^*)^2 \left(\frac{\beta I^*}{1+\omega I^*} + \frac{\beta \rho_1 \gamma I^*}{b_4} + b_1\right)} & \frac{l\beta \rho_1 \Lambda}{\left(\frac{\beta I^*}{1+\omega I^*} + \frac{\beta \rho_1 \gamma I^*}{b_4} + b_1\right)} + \rho \\ \frac{\beta I^*}{1+\omega I^*} (1-l) + \beta \rho_1 \frac{\gamma I^*}{b_4} (1-l) & \delta & \frac{\beta (1-l) \Lambda}{(1+\omega I^*)^2 \left(\frac{\beta I^*}{1+\omega I^*} + \frac{\beta \rho_1 \gamma I^*}{b_4} + b_1\right)} - b_3 & \frac{\beta \rho_1 \Lambda (1-l)}{\left(\frac{\beta I^*}{1+\omega I^*} + \frac{\beta \rho_1 \gamma I^*}{b_4} + b_1\right)} \\ 0 & 0 & \gamma & -b_4 \end{bmatrix}$$

The characteristic equation of $J(E^*)$ is

$$A_0 \lambda^4 + A_1 \lambda^3 + A_2 \lambda^2 + A_3 \lambda + A_4 = 0,$$

where

$$A_0 = 1$$

$$A_1 = -\frac{1}{(c_1+b_1)} \left[(-c_1-b_1) \left(b_3+b_4 - \frac{\beta \lambda (1-l)}{(1+\omega I^*)^2 (c_1+b_1)} \right) \right]$$

$$A_2 = -\frac{1}{b_2(c_1+b_1)} \left(b_2 \left[(-c_1-b_1) \left(\frac{-\beta \lambda b_4 (1-l)}{(1+\omega I^*)^2 (c_1+b_1)} + b_3 b_4 - \frac{\rho_1 \beta \lambda \gamma (1-l)}{c+b_1} \right) - (1-l) c_1 \left(\frac{-\beta \lambda}{(1+\omega I^*)^2 (c_1+b_1)} \right) \right] - \delta \left[(-c_1-b_1) \left(\frac{l\beta \lambda}{(1+\omega I^*)^2 (c_1+b_1)} \right) \right] \right)$$

$$A_3 = -\frac{1}{b_2(c_1+b_1)} \left(b_2 \left[-(1-l) c_1 \left(\frac{-\beta \lambda b_4}{(1+\omega I^*)^2 (c_1+b_1)} + \frac{\rho_1 \beta \lambda \gamma}{c+b_1} \right) \right] - \delta \left[(-c_1-b_1) \left(\frac{l\beta \lambda b_4}{(1+\omega I^*)^2 (c_1+b_1)} + \frac{l\rho_1 \beta \lambda \gamma}{c+b_1} + \rho \gamma \right) - \frac{l c_1 \beta \lambda}{(1+\omega I^*)^2 (c_1+b_1)} \right] \right)$$

$$A_4 = \frac{\delta l c_1}{b_2(c_1+b_1)} \left[\left(\frac{\beta \lambda b_4}{(1+\omega I^*)^2 (c_1+b_1)} + \frac{\rho_1 \beta \lambda \gamma}{c+b_1} \right) \right]$$

Hence, according to the Routh-Hurwitz criteria, the roots of the characteristic equation of $J(E^*)$ have negative real part if and only if $A_1 > 0$, $A_4 > 0$, $A_1 A_2 - A_3 > 0$ and $A_1 A_2 A_3 - A_3^2 - A_1^2 A_4 > 0$.

4. Numerical Simulations

In this section, the dynamics models (2.2) is studied numerically using the parameters values are set as follows : $\Lambda = 143$, $\beta = 0.005$, $\mu = \frac{1}{70}$, $p = 1$, $\alpha = 0.17$, $\delta = 0.005$, $\gamma = 0.11$, $l = 0.9$, $\rho = 0.2$, $\rho_1 = 0.25$, $\omega = 0.25$. Based on the previous analysis, the disease free equilibrium point $E_0 = (140.99, 0, 0, 0)$ is local asymptotically stable because $R_0 = 0.95 < 1$ and an endemic equilibrium point E^* does not exist.

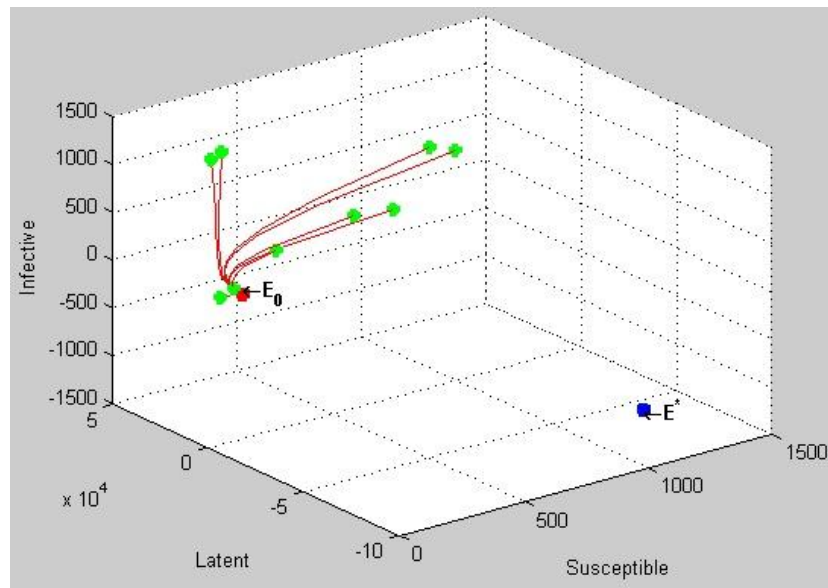


Figure 1. The disease-free equilibrium point E_0 is local asymptotically stable when $R_0 < 1$

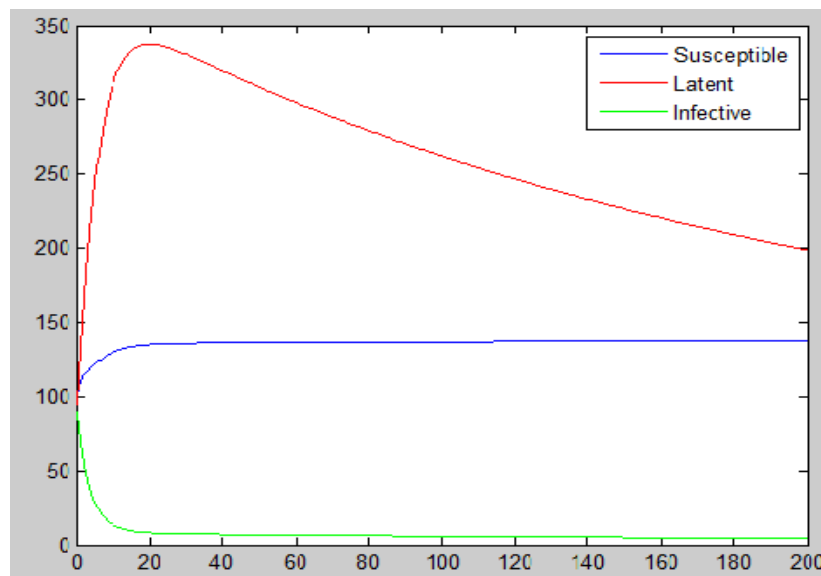


Figure 2. Numerical simulations of system (2.2) when $R_0 < 1$

The red dot denotes the disease free equilibrium point, and the other is an endemic equilibrium point. It can be seen in Figure 1. with some initial conditions, the solution of system (2.2) converge to the disease free equilibrium point. Otherwise, the endemic equilibrium point does not exist.

Then, we consider the same parameters value except $\beta = 0.05$ and $\delta = 0.05$, it can be seen that system (2.2) has two equilibrium points, that are $E_0 = (140.99, 0, 0, 0)$ and $E^* = (3.9, 2774.2, 518.6, 266.2)$.

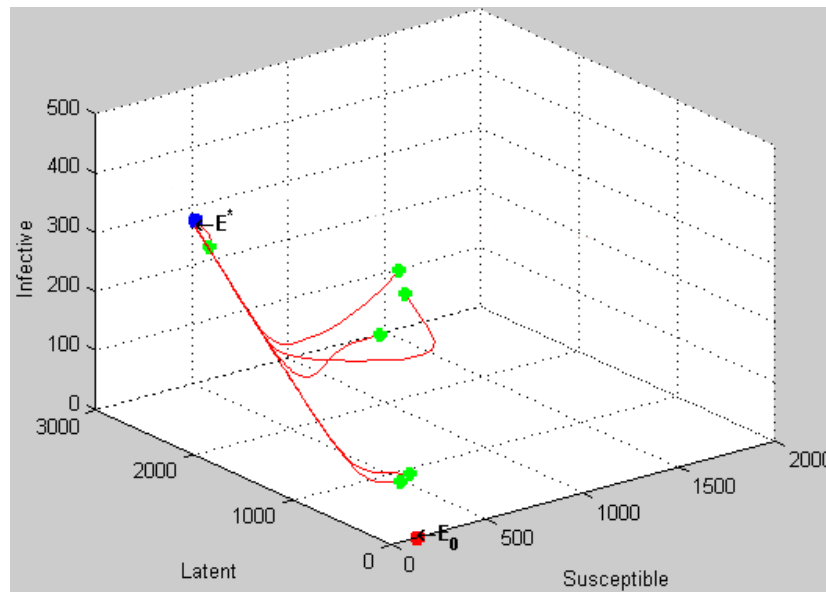


Figure 3. The endemic equilibrium point E^* is local asymptotically stable when $R_0 > 1$

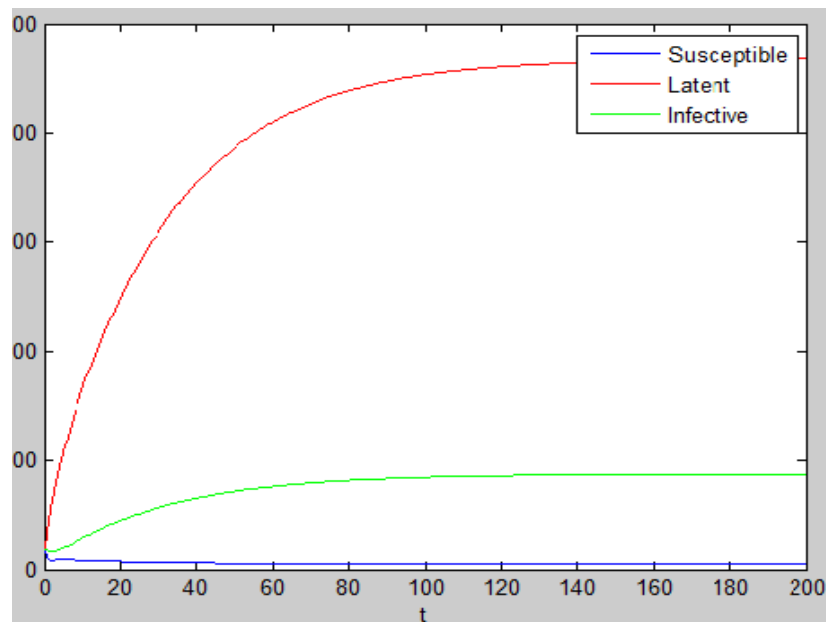


Figure 4. Numerical simulations of system (2.2) when $R_0 > 1$

The numerical simulation confirm that endemic equilibrium point is local asymptotically stable because $R_0 = 1.54 > 1$ while the disease free equilibrium point is not stable. Figure 2. shown that with some initial conditions, the solutions converge to E^* . The Routh Hurwitz criteria of E_0 indicate $a_1 = -6.47 < 0$, $a_3 = -6.47 < 0$, $a_1 a_2 - a_3 = -462.8 < 0$ and by that, numerical simulation verified if E_0 is not stable.

5. Conclusion

In this work, we presented a tuberculosis model with vaccination and saturated incident rate. The dynamical analysis proved that the model has two equilibrium point, named disease free equilibrium point which always exist and an endemic equilibrium point that exist under some certain conditions. The numerical simulations is verified the previous analysis and shown that the disease free equilibrium

point is local asymptotically stable when $R_0 < 1$. Otherwise, the endemic equilibrium point exist and local asymptotically stable when $R_0 > 1$.

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