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Analysis of the mean transition times in the Markov birth-death chains for calculation of the reliability indices of the technical systems

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Abstract. This scientific paper deals with the generalized Markov birth-death chain and its application in the reliability analysis of the technical systems. The special types of reductions of the generalized Markov birth-death chain and the obtained by the author formulas for calculation of the mean time of forward transitions from one state to another and mean time of backward transitions from one state to another are also presented. Finally, an example of using the generalized Markov birth-death chain and obtained formulas of the mean time of forward and backward transitions for calculation of the reliability indices of the specialized technical system is also given.

1. Introduction

It is almost impossible to imagine the modern world without the technical systems that have become an integral part of everyday life and professional activity of a person, as well as key elements of activity of various enterprises and industries.

Manufactures and end users are mostly interested in functionality and basic specifications of the technical systems, such as performance, power, capacity, etc. However, the reliability indices are also very important, as they directly affect to the efficiency and safety of operation of the technical systems [1, 2]. Ignoring of the reliability indices at the best case leads to the additional material costs and loss of time, in more severe case to the loss of important information and in the worst case to fatalities and large-scale disasters. Accordingly, development of the models and methods for calculation of the reliability indices of the technical systems (availability factor, mean time to failure, mean time to repair) is quite urgent scientific task.

Nowadays, there are many books [3-8] dedicated to the basis of the reliability theory, general and specialized models and methods for calculation of the reliability indices of the technical systems, as well as methods for improvement of the fault-tolerance of the technical systems.

One of the best known models of the repairable technical systems is the models based on the Markov chains [3-8] that allows evaluation of such reliability indices of the technical systems as availability factor, mean time to failure and mean time to repair. If the technical system consists of



several identical elements, such as fault-tolerant data storages based on the redundant disk arrays [9, 10], then the Markov birth-death chains can be applied for the reliability analysis. Moreover, if the elements have an exponential distribution of the time to failure and time to repair, then it is often possible to obtain analytical formulas for calculation of the reliability indices. However, definite difficulties might arise in this case, since the technical system may have additional restrictions on the simultaneity of failures (one, several or all elements are under load), restrictions on the simultaneity of repair (one, several or unlimited number of the repair brigades), as well as the dependence of the load and failure rate of elements on the specific state of the system. Besides, the system may have specific system failure thresholds (the minimum number of the failed elements at which the system becomes inoperable) for the integer and fractional multiplicity of redundancy. In each case, the Markov birth-death chain with the corresponding transition rates can be used. Accordingly, in each case it is necessary to form and solve a specialized Kolmogorov-Chapman differential equations system for calculation of the reliability indices of the technical system. It is obvious that it would be much more convenient to have an analytical solution for the generalized Markov death-birth chain with arbitrarily given transition rates and appropriate analytical formulas for calculation of the reliability indices. This would simplify the analysis of the reliability indices of the systems described by the Markov birth-death chain.

A number of books on reliability theory [3-6] consider the generalized Markov birth-death chain for an arbitrary number of elements n of the system and arbitrarily given transition rates $\lambda_0 \dots \lambda_{n-1}$ and $\mu_1 \dots \mu_n$, but they are limited to the derivation of the formula for calculation of only the stationary availability factor. A number of other books [7, 8] limit themselves to the discussion of the reliability models using the Markov birth-death chain for the systems with very small number of elements (duplex systems $n = 2$ and triple-modular redundancy systems $n = 3$).

In recent years, the research work in the field of reliability models of the modern data storage systems [9, 10], consisting of a set of identical elements, was provided by the author. Moreover, within the author's practical experience of implementation and maintenance of the different systems and analysis of their reliability indices, there was a need to assess not only the stationary availability factor of the systems. Also it was necessary to estimate the mean time of transition from the arbitrary state u to another state v , where $u < v$, and the mean time of transition from the arbitrary state v to the other state u in the reliability models based on the Markov birth death chain. Accordingly, within this scientific paper the author presents the obtained formulas for calculation of the mean time of forward and backward transitions in the generalized Markov birth-death chain in order to use the obtained formulas for calculation of the reliability indices of the specialized technical systems.

2. The generalized Markov birth-death chain and stationary probability of its states

Let the generalized Markov birth-death chain is given (figure 1) with $n + 1$ states, forward transition rates $\lambda_0 \dots \lambda_{n-1}$ and backward transitions rates $\mu_1 \dots \mu_n$ (figure 1):

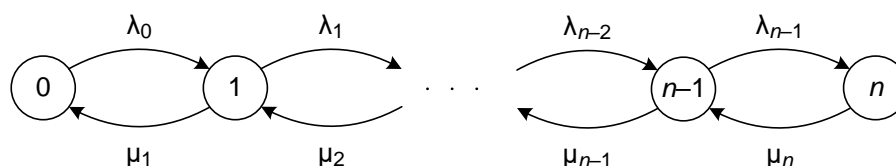


Figure 1. Generalized Markov birth-death chain.

Accordingly, the mathematical model (Kolmogorov-Chapman equations system) is as follow:

$$\left\{ \begin{array}{l} P_0(0) = 1; \quad P_1(0) = 0 \quad \dots \quad P_n(0) = 0; \\ P_0(t) + P_1(t) + \dots + P_n(t) = 1; \\ dP_0/dt = -\lambda_0 P_0(t) + \mu_1 P_1(t); \\ dP_1/dt = \lambda_0 P_0(t) - (\mu_1 + \lambda_1) P_1(t) + \mu_2 P_2(t); \\ \vdots \\ dP_{n-1}/dt = \lambda_{n-2} P_{n-2}(t) - (\mu_{n-1} + \lambda_{n-1}) P_{n-1}(t) + \mu_n P_n(t); \\ dP_n/dt = \lambda_{n-1} P_{n-1}(t) - \mu_n P_n(t). \end{array} \right. \quad (1)$$

It is possible to obtain the stationary probabilities P_k of the Markov chain states at $t \rightarrow \infty$ by solving the system of linear algebraic equations, taking into account that in the infinite perspective the Markov process becomes steady, and the derivatives tend to zero $dP_i(t)/dt \rightarrow 0$.

Accordingly, the well-known formulas for calculation of the stationary probabilities of states of the Markov birth-death chain are as follows [1-4]:

$$P_k(\infty) = \left(\prod_{i=0}^{k-1} \lambda_i \prod_{j=k+1}^n \mu_j \right) / \sum_{q=0}^n \left(\prod_{i=0}^{q-1} \lambda_i \prod_{j=q+1}^n \mu_j \right); \quad 0 \leq k \leq n. \quad (2)$$

3. Analysis of the mean time of forward transitions in the Markov birth-death chain

To obtain a formula for calculation of the mean time of forward transition from an arbitrary state u to the state v , $0 \leq u < v \leq n$, in the generalized Markov chain we need to make the state u to be initial, and the state v absorbing. In this case we obtain the following reduced Markov chain (figure 2):

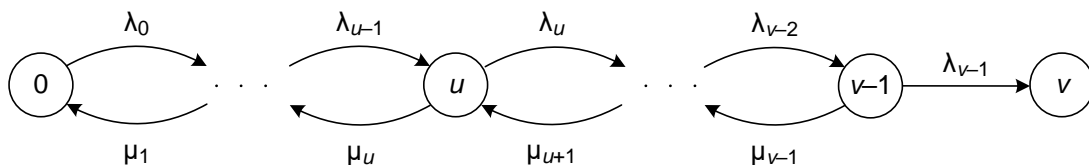


Figure 2. Reduced Markov chain for calculation of the mean times of forward transitions.

Accordingly, the mathematical model (Kolmogorov-Chapman equations system) is as follow:

$$\left\{ \begin{array}{l} P_0(0) = 0; \quad \dots \quad P_u(0) = 1; \quad \dots \quad P_v(0) = 0; \\ P_0(t) + \dots + P_u(t) + \dots + P_{v-1}(t) + P_v(t) = 1; \\ dP_0(t)/dt = -\lambda_0 P_0(t) + \mu_1 P_1(t); \\ dP_1(t)/dt = \lambda_0 P_0(t) - (\mu_1 + \lambda_1) P_1(t) + \mu_2 P_2(t); \\ \vdots \\ dP_u(t)/dt = \lambda_{u-1} P_{u-1}(t) - (\mu_u + \lambda_u) P_u(t) + \mu_{u+1} P_{u+1}(t); \\ \vdots \\ dP_{v-1}(t)/dt = \lambda_{v-2} P_{v-2}(t) - (\mu_{v-1} + \lambda_{v-1}) P_{v-1}(t); \\ dP_v(t)/dt = \lambda_{v-1} P_{v-1}(t). \end{array} \right. \quad (3)$$

The mean time of forward transition $T_{u \rightarrow v}$ from the state u to the state v , $0 \leq u < v \leq n$, is equal to the mean time of staying of the system in the states $0 \dots v-1$ before it reaches the final state v taking into account that state u is initial:

$$T_{u \rightarrow v} = \int_0^\infty t \frac{dP_v(t)}{dt} dt = \int_0^\infty (1 - P_v(t)) dt = \sum_{k=0}^{v-1} \left(\int_0^\infty P_k(t) dt \right); \quad P_u(0) = 1. \quad (4)$$

By using the advanced mathematical analysis, the author has obtained the following formula for calculation of the mean time of forward transition from the state u to the state v , $0 \leq u < v \leq n$:

$$T_{u \rightarrow v} = \sum_{k=0}^{v-1} \left(\sum_{q=\max(0, u-k)}^{v-1-k} \left(\frac{1}{\lambda_q} \prod_{j=1}^k \left(\frac{\mu_{q+j}}{\lambda_{q+j}} \right) \right) \right). \quad (5)$$

4. Analysis of the mean time of backward transitions in the Markov birth-death chain

To obtain a formula for calculation of the mean time of backward transitions from an arbitrary state v to the state u , $0 \leq u < v \leq n$, in the generalized Markov chain we need to make the state v to be initial, and the state u absorbing. In this case we obtain the following reduced Markov chain (figure 3):

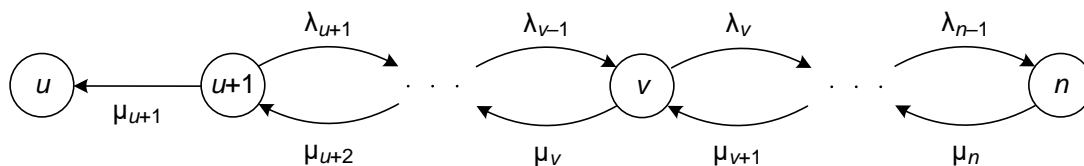


Figure 3. Reduced Markov chain for calculation of the mean times of backward transitions.

Accordingly, the mathematical model (Kolmogorov-Chapman equations system) is as follow:

$$\left\{ \begin{array}{l} P_u(0) = 0; \quad \dots \quad P_v(0) = 1; \quad \dots \quad P_n(0) = 0; \\ P_u(t) + P_{u+1}(t) + \dots + P_v(t) + \dots + P_n(t) = 1; \\ dP_u(t)/dt = \mu_{u+1}P_{u+1}(t); \\ dP_{u+1}(t)/dt = -(\mu_{u+1} + \lambda_{u+1})P_{u+1}(t) + \mu_{u+2}P_{u+2}(t); \\ \vdots \\ dP_v(t)/dt = \lambda_{v-1}P_{v-1}(t) - (\mu_v + \lambda_v)P_v(t) + \mu_{v+1}P_{v+1}(t); \\ \vdots \\ dP_{n-1}(t)/dt = \lambda_{n-2}P_{n-2}(t) - (\mu_{n-1} + \lambda_{n-1})P_{n-1}(t) + \mu_nP_n(t); \\ dP_n(t)/dt = \lambda_{n-1}P_{n-1}(t) - \mu_nP_n(t). \end{array} \right. \quad (6)$$

The mean time of backward transition $T_{v \rightarrow u}$ from the state v to the state u , $0 \leq u < v \leq n$, is equal to the mean time of the system staying in the states $u+1 \dots n$ before it reaches the final state u taking into account that state v is initial:

$$T_{v \rightarrow u} = \int_0^\infty t \frac{dP_u(t)}{dt} dt = \int_0^\infty (1 - P_u(t)) dt = \sum_{k=u+1}^n \left(\int_0^\infty P_k(t) dt \right); \quad P_v(0) = 1. \quad (7)$$

By using the advanced mathematical analysis, the author has obtained the following formula for calculation of the mean time of backward transition from the state v to the state u , $0 \leq u < v \leq n$:

$$T_{v \rightarrow u} = \sum_{k=u+1}^n \left(\sum_{q=\max(0, k-v)}^{k-u-1} \left(\frac{1}{\mu_{n-q}} \prod_{j=1}^{n-k} \left(\frac{\lambda_{n-q-j}}{\mu_{n-q-j}} \right) \right) \right). \quad (8)$$

5. Example of using the generalized Markov birth-death chain and formulas of the mean time of forward and backward transitions for calculation of the reliability indices of the technical systems with n identical repairable elements and the failure threshold s

Now, let us overview the technical systems with n identical repairable elements with the same failure and repair rates. The system is inoperable in case of failure of s or more elements, $1 \leq s \leq n$. In other words, s is the «failure threshold» of the system.

In addition, if the state s is reached, the system fails, but does not shutdown. Moreover, the remaining operable elements can continue to work until all the n elements fail, and the failed elements can be repaired. When the state $s-1$ is reached, the system becomes operable again.

Accordingly, the reliability model of the discussed system can be described by the following generalized Markov birth-death chain (figure 4):

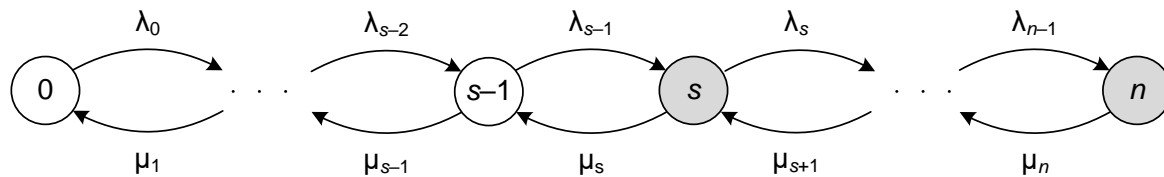


Figure 4. Markov birth-death chain in the reliability model of the system with n identical elements and failure threshold s .

Now let us derive formulas for a number of the most important reliability indices of the system.

The stationary availability factor K of the system is equal to the sum of the stationary probabilities of the states $0...s-1$ in which the system is operable, and these probabilities can be calculated by the formula 2. Accordingly, the formula for calculation of the stationary availability factor K is as follow:

$$K = \sum_{k=0}^{s-1} P_k(\infty) = \sum_{k=0}^{s-1} \left(\prod_{i=0}^{k-1} \lambda_i \prod_{j=k+1}^n \mu_j \right) / \sum_{q=0}^n \left(\prod_{i=0}^{q-1} \lambda_i \prod_{j=q+1}^n \mu_j \right). \quad (9)$$

The mean time to first failure T_{FF} of the system is equal to the mean time of the forward transition from the initial operable state 0 to the border failed state s . Accordingly, the calculation formula for T_{FF} can be derived from the formula 2.3 by substituting $u = 0$ and $v = s$:

$$T_{FF} = T_{0 \rightarrow s} = \sum_{k=0}^{s-1} \left(\sum_{q=0}^k \left(\frac{1}{\lambda_q} \prod_{j=1}^{s-1-k} \left(\frac{\mu_{q+j}}{\lambda_{q+j}} \right) \right) \right). \quad (10)$$

The mean time to failure T_F of the system is equal to the mean time of forward transition from the border operable state $s-1$ to the border failed state s . Accordingly, the calculation formula for T_F can be derived from the formula 5 by substituting $u = s-1$ and $v = s$:

$$T_F = T_{s-1 \rightarrow s} = \sum_{k=0}^{s-1} \left(\frac{1}{\lambda_k} \prod_{j=1}^{s-1-k} \left(\frac{\mu_{k+j}}{\lambda_{k+j}} \right) \right). \quad (11)$$

The mean time to repair T_R of the system is equal to the mean time of backward transition from the border failed state s to the border operable state $s-1$. Accordingly, the calculation formula for T_R can be derived from the formula 8 by substituting $u = s-1$ and $v = s$:

$$T_R = T_{s \rightarrow s-1} = \sum_{k=0}^{n-s} \left(\frac{1}{\mu_{n-k}} \prod_{j=1}^{n-s-k} \left(\frac{\lambda_{n-k-j}}{\mu_{n-k-j}} \right) \right). \quad (12)$$

It should be noted, that the stationary availability factor of the system is associated with the mean time to failure and mean time to repair by the next identity: $K = \frac{T_F}{T_F + T_R}$.

Example. A fault-tolerant water supply system consisting of the $n = 5$ pumps is given. The $f = 3$ of pumps are under the load, the rest are in non-loaded spare mode. The number of the repair brigades is $r = 3$. The water supply system provides the required volume of water only if at least 3 pumps are operable. In case of failure of $s = 3$ or more pumps, the system continues to operate, but does not provide the required volume of water. The failure rate of the pumps is $\lambda = 1/720 \text{ h}^{-1}$ (in average one failure in a month). The repair rate is $\mu = 1/12 \text{ h}^{-1}$ (in average one repair in 12 hours).

Let us calculate the reliability indices of the given water supply system.

The reliability model of the water supply system can be described by the following Markov birth-death chain (figure 5), in which each state reflects the number of the failed pumps:

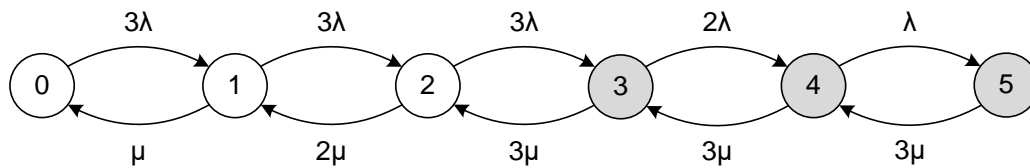


Figure 5. Markov birth-death chain in the reliability model of the water supply system.

In the states 0, 1 and 2, the number of operable pumps is at least 3, and system provides the required volume of water. In the states 3, 4 and 5 the system continues to operate, but the number of operable pumps is less than 3, and system does not provide the required volume of water.

Accordingly, the stationary availability factor K of the system is equal to the sum of the stationary probabilities of the operable states 0, 1 and 2, in which the system provides the required volume of water, and it can be calculated by the formula 9:

$$K = P_0(\infty) + P_1(\infty) + P_2(\infty) = \frac{2\mu^5 + 6\mu^4\lambda + 9\mu^3\lambda^2}{2\mu^5 + 6\mu^4\lambda + 9\mu^3\lambda^2 + 9\mu^2\lambda^3 + 6\mu\lambda^4 + 2\lambda^5} \approx 0.9999799613$$

The mean time to first failure T_{FF} of the system equals to the mean time of forward transition from the initial operable state 0 to the failed state 3, and it can be calculated by the formula 10:

$$T_{FF} = T_{0 \rightarrow 3} = \frac{2\mu^2 + 9\mu\lambda + 27\lambda^2}{27\lambda^3} \approx 207120 \text{ hours.}$$

The mean time to the failure T_F of the system is equal to the mean time of forward transition from the border operable state 2 to the failed state 3, and it can be calculated by the formula 11:

$$T_F = T_{2 \rightarrow 3} = \frac{2\mu^2 + 6\mu\lambda + 9\lambda^2}{27\lambda^3} \approx 201840 \text{ hours.}$$

The mean time to repair T_R of the system is equal to the mean time of backward transition from the failed state 3 to the border operable state 2, and it can be calculated by the formula 12:

$$T_R = T_{3 \rightarrow 2} = \frac{9\mu^2 + 6\mu\lambda + 2\lambda^2}{27\mu^3} \approx 4 \text{ hours.}$$

6. Conclusion

Thus, within the scope of this scientific paper the generalized Markov birth-death chain and its application in the reliability analysis of the technical systems are discussed.

The special types of reductions of the generalized Markov birth-death chain and obtained by the author formulas for calculation of the mean time of forward transitions from one state to another and mean time of backward transitions from one state to another are also presented.

Finally, the example of using the generalized Markov birth-death chain and obtained formulas of the mean time of forward and backward transitions for calculation of the reliability indices of the specialized technical system is also given.

The obtained scientific results were used by the author for development of the specialized simulation software for the reliability analysis of technical systems and laboratory works for studying the reliability theory for the students of technical specialties.

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