

PAPER • OPEN ACCESS

Energy equations for the temperature three-dimensional boundary layer for the flow within boundary conditions of turbo machinery

To cite this article: A A Zuev *et al* 2019 *IOP Conf. Ser.: Mater. Sci. Eng.* **537** 022008

View the [article online](#) for updates and enhancements.



IOP | ebooks™

Bringing you innovative digital publishing with leading voices to create your essential collection of books in STEM research.

Start exploring the **collection** - download the first chapter of every title for free.

Energy equations for the temperature three-dimensional boundary layer for the flow within boundary conditions of turbo machinery

A A Zuev¹, A A Kishkin², D A Zhuikov¹, M I Tolstopyatov¹ and M V Savelyeva²

¹ Department of Aircraft Engines, Reshetnev Siberian State University of Science and Technologies, 31 Krasnoyarsky Rabochy Av., 660037, Krasnoyarsk, Russian Federation

² Department of Refrigeration, Cryogenic Equipment and Air Conditioning, Reshetnev Siberian State University of Science and Technologies, 31 Krasnoyarsky Rabochy Av., 660037, Krasnoyarsk, Russian Federation

E-mail: dla2011@inbox.ru

Abstract. The research generates an integral relation of the energy equations for the temperature three-dimensional boundary layer allowing to integrate surfaces of any shape to determine thickness of energy loss. An equation to determine thickness of energy loss is necessary to specify heat transfer law and local heat transfer coefficients within boundary conditions of turbo machinery cavities.

1. Introduction

In theory of turbo machinery of both compressor and expansion ones, while developing mathematical models, several design-boundary elements are focused: inlet and outlet devices in the stator housing; impeller channel; auxiliary hydraulic path formed by the gap between a rotor and a stator.

While determining the functional relation, individual elements are located in a general turbo machinery model. For the compressed working bodies, the mechanical issue about changing the kinetic energy does not separate from the thermal energy; considering irreversibility and non-adiabatic flow in turbo machinery elements requires to determine functions for the local friction stress and heat transfer coefficient. Semi-empirical integral methods of boundary layer theory (dynamic and temperature) increasingly consider flat two-dimensional models for the linear problems [1-7]. Rotor spinning is used as the main technical motion in turbo machinery, trajectories and flow streamlines have got a shape of a spiral or a circle. If the flow lines are curved, then, in addition to the longitudinal differential pressure, there is also transverse pressure drop in the flow, balancing the action of centrifugal forces. In the boundary layer, where the pressure of the external flow is transmitted unchanged, this equilibrium is disturbed, since the centrifugal force, due to a decrease in speed, reduces. Equilibrium is restored by the action of friction forces of the secondary flow in the boundary layer directed opposite to the transverse pressure gradient that is from the concave side of the streamline of the external flow. The secondary flow velocities varying in thickness and directed to the centre of curvature of the streamlines cause transverse shear stresses in the body and on the surface of the body. Therefore, the total tangential stress on the surface of the body in the general case does not coincide with the direction of the streamlines of



the external flow, as it is the case in flat or axially symmetrical boundary layers. The velocity field of the secondary flow can have a complex structure and change its direction through the layer thickness.

2. Deriving an energy equation

To solve the problem of local heat, transfer with transverse pressure gradient of flow at the external boundary of the boundary layer requires, as a rule, to use integral relations of the dynamic [8-9] and temperature spatial boundary layer (SBL). In the classical formulation, the integral relation of the energy equation of the temperature SBL is a differential equation in two unknowns: the thickness of the energy loss and the local heat transfer coefficient.

For the case of the flow of an incompressible fluid, it is sufficient to solve together the equations of motion [8–9] and energy in the boundary conditions SBL, for a compressible fluid, the state equation is necessary to add to the system. Recording and integrating the energy equation of the temperature SBL is a separate, but significant task.

The general view of the energy equation in the operator form [10] is:

$$\rho C_p \frac{dT}{d\tau} = \text{div}q + \mu\Phi + p\text{div}\bar{c} + \varepsilon, \quad (1)$$

where regarding to $\rho = \text{const}$, absolute speed divergence is:

$$\bar{c} = \bar{u} + \bar{v} + \bar{w},$$

equal to zero, accordingly, the work of pressure forces is not considered in the energy equation (1):

$$p\text{div}\bar{c} = 0. \quad (2)$$

The divergence of the specific heat flow in natural curvilinear coordinates is:

$$\begin{aligned} \text{div}q &= \text{div}(\text{grad}\lambda T) = \nabla^2(\lambda T) = \\ &= \frac{1}{H_\phi H_y H_\psi} \left[\frac{\partial}{\partial \phi} \left(\lambda \frac{\partial T}{\partial \phi} \frac{H_y H_\psi}{H_\phi} \right) + \frac{\partial}{\partial y} \left(\lambda \frac{\partial T}{\partial y} \frac{H_\psi H_\phi}{H_y} \right) + \frac{\partial}{\partial \psi} \left(\lambda \frac{\partial T}{\partial \psi} \frac{H_\phi H_y}{H_\psi} \right) \right] \end{aligned} \quad (3)$$

Considering that when analyzing the scale of quantities, the author [10] leaves only the terms with the coordinate of the orthogonal surface - the terms with $\frac{\partial}{\partial y}$, then the equation (3) regarding $\lambda = \text{const}$ will be written as:

$$\nabla^2(\lambda T) = \frac{\lambda}{H_\phi H_y H_\psi} \frac{\partial}{\partial y} \left(\frac{\partial T}{\partial y} \frac{H_\psi H_\phi}{H_y} \right). \quad (4)$$

The total derivative with respect to temperature in natural curvilinear coordinates obtains the form:

$$\frac{dT}{d\tau} = \frac{\partial T}{\partial \tau} + \frac{1}{H_\phi} \frac{\partial T}{\partial \phi} \frac{d\phi}{dt} + \frac{1}{H_y} \frac{\partial T}{\partial y} \frac{dy}{dt} + \frac{1}{H_\psi} \frac{\partial T}{\partial \psi} \frac{d\psi}{dt},$$

finally, we have got an equation for the total derivative:

$$\frac{dT}{d\tau} = \frac{\partial T}{\partial \tau} + \frac{U}{H_\phi} \frac{\partial T}{\partial \phi} + \frac{v}{H_y} \frac{\partial T}{\partial y} + \frac{w}{H_\psi} \frac{\partial T}{\partial \psi}. \quad (5)$$

The dissipative function in natural curvilinear coordinates is:

$$\Phi = 2 \left[\left(\frac{1}{H_\phi} \frac{\partial u}{\partial \phi} \right)^2 + \left(\frac{1}{H_y} \frac{\partial v}{\partial y} \right)^2 + \left(\frac{1}{H_\psi} \frac{\partial w}{\partial \psi} \right)^2 \right] + \left(\frac{1}{H_y} \frac{\partial u}{\partial y} + \frac{1}{H_\phi} \frac{\partial v}{\partial \phi} \right)^2 + \left(\frac{1}{H_\phi} \frac{\partial w}{\partial \phi} + \frac{1}{H_\psi} \frac{\partial u}{\partial \psi} \right)^2 + \left(\frac{1}{H_\psi} \frac{\partial v}{\partial \psi} + \frac{1}{H_y} \frac{\partial w}{\partial y} \right)^2 \quad (6)$$

Considering the results of value scales, in the dissipative term, the author [10] leaves only terms with $\frac{\partial u}{\partial y}$ and $\frac{\partial w}{\partial y}$, then the equation (6) simplifies:

$$\Phi = \left(\frac{1}{H_y} \frac{\partial u}{\partial y} \right)^2 + \left(\frac{1}{H_y} \frac{\partial w}{\partial y} \right)^2. \quad (7)$$

Taking into account (2; 4; 5; 7) and the absence of internal heat sources, which means $\varepsilon = 0$, the equation (1) will be:

$$\rho C_p \left(\frac{\partial T}{\partial \tau} + \frac{u}{H_\phi} \frac{\partial T}{\partial \phi} + \frac{v}{H_y} \frac{\partial T}{\partial y} + \frac{w}{H_\psi} \frac{\partial T}{\partial \psi} \right) = \frac{\lambda}{H_\phi H_y H_\psi} \frac{\partial}{\partial y} \left(\frac{\partial T}{\partial y} \frac{H_\psi H_\phi}{H_y} \right) + \mu \left[\left(\frac{1}{H_y} \frac{\partial u}{\partial y} \right)^2 + \left(\frac{1}{H_y} \frac{\partial w}{\partial y} \right)^2 \right]. \quad (8)$$

We need to take into account that $H_y = 1$, and Lamé coefficient is $H_\psi = \text{const}$, $H_\phi = \text{const}$, while integrating along the y axis; the streamline flow is $\partial T / \partial \tau = 0$, then the final energy equation for the spatial boundary layer in the natural curvilinear coordinate system will take the form:

$$\rho C_p \left(\frac{u}{H_\phi} \frac{\partial T}{\partial \phi} + v \frac{\partial T}{\partial y} + \frac{w}{H_\psi} \frac{\partial T}{\partial \psi} \right) = \lambda \frac{\partial^2 T}{\partial y^2} + \mu \left[\left(\frac{\partial u}{\partial y} \right)^2 + \left(\frac{\partial w}{\partial y} \right)^2 \right] \quad (9)$$

We integrate equation (9) along the y coordinate within the boundaries of the boundary layer thickness. In this occasion, we take into account the equation for speed v (of the normal boundary surface), derived from the continuity equation:

$$v = -\frac{1}{H_\phi H_\psi} \left(\int_0^y \frac{\partial(H_\psi u)}{\partial \phi} dy + \int_0^y \frac{\partial(H_\phi w)}{\partial \psi} dy \right).$$

We consistently integrate the terms of equation (9), starting from the left. We need to account that we inherently consider the function $(T - T_0)$, where T is temperature in the boundary layer; T_0 is a wall temperature.

$$\int_0^\delta \frac{u}{H_\phi} \frac{\partial(T - T_0)}{\partial \phi} dy = \frac{1}{H_\phi} \int_0^\delta \frac{\partial(u(T - T_0))}{\partial \phi} dy - \frac{1}{H_\phi} \int_0^\delta \left((T - T_0) \frac{\partial u}{\partial \phi} \right) dy = A1. \quad (10)$$

When integrating the second term, the integration method is used in parts:

$$\int_0^\delta v \frac{\partial(T - T_0)}{\partial \phi} dy = -\frac{1}{H_\phi H_\psi} \left[\int_0^y \frac{\partial(H_\psi u)}{\partial \phi} dy + \int_0^y \frac{\partial(H_\phi w)}{\partial \psi} dy \right] (T - T_0) \Big|_0^\delta - \int_0^\delta (T - T_0) \left(\frac{\partial(H_\psi u)}{\partial \phi} + \frac{\partial(H_\phi w)}{\partial \psi} \right) dy.$$

After the transformations, we obtain the equation for the second term.

$$\begin{aligned}
\int_0^\delta \nu \frac{\partial T}{\partial \varphi} dy &= -\frac{(T_\delta - T_0)}{H_\varphi} \frac{\partial}{\partial \varphi} \left(\int_0^\delta u dy \right) - \frac{(T_\delta - T_0)}{H_\varphi H_\psi} \frac{\partial H_\psi}{\partial \varphi} \int_0^\delta u dy - \frac{(T_\delta - T_0)}{H_\psi} \frac{\partial}{\partial \psi} \left(\int_0^\delta w dy \right) - \\
&- \frac{(T_\delta - T_0)}{H_\varphi H_\psi} \frac{\partial H_\varphi}{\partial \psi} \int_0^\delta w dy + \frac{1}{H_\varphi H_\psi} \frac{\partial H_\psi}{\partial \varphi} \int_0^\delta (T - T_0) u dy + \frac{1}{H_\varphi} \int_0^\delta (T - T_0) \frac{\partial u}{\partial \varphi} dy + \\
&+ \frac{1}{H_\varphi H_\psi} \frac{\partial H_\varphi}{\partial \psi} \int_0^\delta (T - T_0) w dy + \frac{1}{H_\psi} \int_0^\delta (T - T_0) \frac{\partial w}{\partial \psi} dy = A2,
\end{aligned} \tag{11}$$

where T_δ is temperature at the external boundary of the boundary layer.

The integral of the third member is determined by the equation:

$$\int_0^\delta \frac{w}{H_\psi} \frac{\partial (T - T_0)}{\partial \psi} dy = \frac{1}{H_\psi} \int_0^\delta \frac{\partial (w(T - T_0))}{\partial \psi} dy - \frac{1}{H_\psi} \int_0^\delta (T - T_0) \frac{\partial w}{\partial \psi} dy = A3. \tag{12}$$

Providing the equation to the specific heat flow is:

$$q = \lambda \frac{\partial (T - T_0)}{\partial y},$$

due to Newton – Richman law $q = dQ/dS = \alpha(T_\delta - T_0)$, the equation for the integral of the fourth term will be:

$$\lambda \int_0^\delta \frac{\partial^2 (T - T_0)}{\partial y^2} dy = \int_0^\delta \frac{\partial q}{\partial y} dy = q|_0^\delta = q_\delta - q_0 = -q_0 = -\alpha(T_\delta - T_0) = A4. \tag{13}$$

For the equation of the fifth term (8) we need to highlight that for a turbulent boundary layer, a viscous layered flow is realized in a thin sublayer δ_1 , where the velocity plot is linear and both $\frac{\partial u}{\partial y}$ and $\frac{\partial w}{\partial y}$ are constant. We take into account that equation for friction stress is:

$$\tau_\phi = \mu \left(\frac{\partial u}{\partial y} \right); \tau_\psi = \mu \left(\frac{\partial w}{\partial y} \right).$$

The integral for the dissipative term transforms into:

$$\int_0^\delta \mu \left(\left(\frac{\partial u}{\partial y} \right)^2 + \left(\frac{\partial w}{\partial y} \right)^2 \right) dy = \int_0^\delta \frac{1}{\mu} (\tau_\phi^2 + \tau_\psi^2) dy. \tag{14}$$

The integral in the boundaries from the wall to δ is divided into two intervals. In the first interval from 0 to δ_1 , the derivative of the velocity is constant and not equal to zero, in the second interval from δ_1 to δ , the derivatives are constant and equal to zero. Similar approximation of the power velocity profiles allows to integrate (14) in a simple way. Due to the friction stresses are constant and equal to friction stress at the wall, and the longitudinal and transverse components are related by the equation $\tau_\phi = \varepsilon \tau_\psi$, we obtain:

$$\int_0^\delta \frac{1}{\mu} (\tau_{\phi_0}^2 + \tau_{\psi_0}^2) dy = \int_0^{\delta_1} \frac{1}{\mu} (\tau_{\phi_0}^2 + \tau_{\psi_0}^2) dy = \frac{\delta_1 \tau_{\phi_0}^2 (1 + \varepsilon^2)}{\mu} = A5. \quad (15)$$

As it will be shown below, the conditional thickness will be included into the equation for friction stress and will not be among the influencing parameters.

We record the sum of the terms (10), (11), (12), (13), (15):

$$\rho C_p (A1 + A2 + A3) = A4 + A5.$$

In this case, we take into account that the four components are mutually destroyed, in the equations

$\int_0^\delta \frac{\partial(u(T-T_0))}{\partial \phi} dy$ and $\int_0^\delta \frac{\partial(w(T-T_0))}{\partial \psi} dy$ the sign of the integral and differential swap:

$$\begin{aligned} \rho C_p \left[\frac{1}{H_\phi} \frac{\partial}{\partial \phi} \left(\int_0^\delta u(T-T_0) dy \right) - \frac{(T_\delta - T_0)}{H_\phi} \frac{\partial}{\partial \phi} \left(\int_0^\delta u dy \right) - \frac{(T_\delta - T_0)}{H_\phi H_\psi} \frac{\partial H_\psi}{\partial \phi} \int_0^\delta u dy - \right. \\ \left. - \frac{(T_\delta - T_0)}{H_\psi} \frac{\partial}{\partial \psi} \left(\int_0^\delta w dy \right) - \frac{(T_\delta - T_0)}{H_\phi H_\psi} \frac{\partial H_\phi}{\partial \psi} \int_0^\delta w dy + \frac{1}{H_\phi H_\psi} \frac{\partial H_\psi}{\partial \phi} \int_0^\delta u(T-T_0) dy + \right. \\ \left. + \frac{1}{H_\phi H_\psi} \frac{\partial H_\phi}{\partial \psi} \int_0^\delta w(T-T_0) dy + \frac{1}{H_\psi} \frac{\partial}{\partial \psi} \left(\int_0^\delta w(T-T_0) dy \right) \right] = -\alpha(T_\delta - T_0) + \frac{\delta_1 \tau_{\phi_0}^2 (1 + \varepsilon)}{\mu}. \end{aligned} \quad (16)$$

According to the authors' recommendations [8-10] we introduce the notions of the energy loss thickness of the temperature boundary layer:

– the thickness of the energy loss of temperature SBL in the longitudinal direction is:

$$\delta_{t\phi}^{**} = \int_0^\delta \frac{u}{U} \left(1 - \frac{T-T_0}{T_\delta - T_0} \right) dy; \quad (17)$$

– the thickness of the energy loss of temperature SBL in the transverse direction is:

$$\delta_{t\psi}^{**} = \int_0^\delta \frac{w}{U} \left(1 - \frac{T-T_0}{T_\delta - T_0} \right) dy. \quad (18)$$

We group the equation terms (16) and divide to $\rho C_p U (T_\delta - T_0)$. Due to (17) and (18) we obtain the equation for the integral relation of the energy loss of the temperature spatial boundary layer:

$$\frac{1}{H_\phi} \frac{\partial(\delta_{t\phi}^{**})}{\partial \phi} + \frac{1}{H_\psi} \frac{\partial(\delta_{t\psi}^{**})}{\partial \psi} + \frac{1}{H_\phi H_\psi} \frac{\partial H_\psi}{\partial \phi} \delta_{t\phi}^{**} + \frac{1}{H_\phi H_\psi} \frac{\partial H_\phi}{\partial \psi} \delta_{t\psi}^{**} = St - \frac{\tau_{\phi_0} (1 + \varepsilon^2)}{\rho C_p (T_\delta - T_0)},$$

where Stanton criterion is $St = \frac{\alpha}{\rho C_p U}$.

Figure 1 demonstrates Prandtl criterion impact on friction and heat transfer according to the research [8].

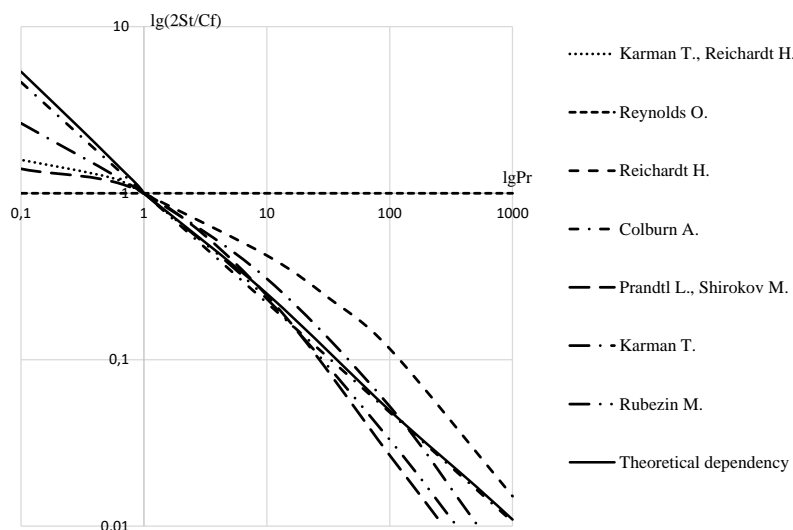


Figure 1. Comparison of various theories of the analogy between friction and heat transfer in turbulent flows when $Re=10^7$.

Within Prandtl number values $Pr < 1$ and $Pr > 1$, the obtained theoretical dependencies agree with the dependencies by other researchers [8] in the case of non-dimensional coefficients of heat transfer in the form of Stanton criteria regarding to the integral relation of energy equation.

3. Conclusion

We have obtained the integral relation of energy equation of the energy equation of the temperature three-dimensional boundary layer; it allows to integrate due to the surface of any form necessary to determine the thickness of the energy loss. The equations to determine energy loss thickness of the temperature three-dimensional boundary layer are required to specify local coefficients of the heat transfer for the specific occasions of flow taking into account the heat transfer.

References

- [1] Stefanyuk Ye V and Kudinov I V 2010 Obtaining an analytical solution of the heat boundary layer equation based on introducing additional boundary conditions *Vestnik of Samara State Aerospace University* **21** 174
- [2] Kudinov I V, Kudinov V A, Kotova E V and Eremin A V 2017 On one method of solving nonstationary boundary-value problems *J. of Eng. Phys. and Therm.* **90** 1317
- [3] Stefanyuk Ye V and Kudinov I V 2010 Approximate analytic solution of heat conduction problems with a mismatch between initial and boundary conditions *Russian Mathematics* **4** 63
- [4] Eremin A V, Kudinov V A and Kudinov I V 2016 Mathematical model of heat transfer in a fluid with account for its relaxation properties *Fluid Dynamics* **51** 33
- [5] Terehov V I and Yarygina N I 2012 Heat transfer in highly turbulent separated flows *XXIII-th seminar (with international participants) to the ejector-type, separated and non-stationary flows* ed G Kuznezov (Tomsk: Tomsk Polytechnic University) pp 328-334
- [6] Valueva E P 2004 Heat transfer and hydrodynamics drag at tube turbulent pulsating flow with great amplitudes *Proc. of the Russ. Acad. of Sci.. Power Engine J.* **5** 140
- [7] Gluschenko G N, Kolina N P, Nersesov G G and Shkirin N N 1990 Calculated analysis of the flow-around and heat transfer for isentropic flow of compression *TsAGI Science Journal* **21(3)** 11
- [8] Romanenko P N 1971 *Heat-mass transfer and friction under the gradient liquid current* (Moscow: Energy)
- [9] Kays W M 1972 *Convective heat and mass transfer* (Moscow: Energy)
- [10] Schlichting G 1969 *Boundary layer theory* (Moscow: Nauka (Science))