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Mechanical Modeling the Processes of Additive Manufacturing Bearing Pillars of Viscoelastic Aging Materials by Thickening over the Lateral Surface under Axial Loading

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Abstract. A nonclassical mechanical model for the process of additive manufacturing a bearing circular conical pillar by means of its layer-by-layer thickening over the lateral surface using a viscoelastic aging material under simultaneous action of an axial variable central force is developed. The exact in the sense of Saint-Venant principle analytical solution of the stated corresponding quasi-static boundary value problem is constructed. The obtained solution allows to retrace the evolution of the stress-strain state of the manufactured pillar after the start and on completing the piecewise continuous additive process under consideration.

1. Introduction

Technological processes of additive manufacturing elements of structures are accompanied by an increase in the size and shape of the solids obtained in these processes due to the addition of new material layers to their surfaces. A distinctive feature of such kind of processes is the inherent fact that the considered solid does not exist in its final form before the beginning of its deformation process, as it is always considered in the classical mechanics of deformable solids: the solid is formed already in the course of this process. Therefore, in the mechanical analysis of additive processes, it is necessary to simultaneously take into account both external loads acting in the simulated technological process on the solid under consideration, including loads acting on the additional material being attached to it, and the geometric, kinematic and force characteristics of the process of gradual attachment of new material elements to the solid. Due to the absence in principle for the entire additive-manufactured solid any unstressed configuration which could act as a reference configuration for traditional strain measures, the mentioned account cannot be correctly carried out within the framework of classical solid mechanics, even if the traditional equations and conditions are formulated for a time variable spatial domain. Thus, the problems on modeling mechanical behaviour of additive-manufactured solids should have a number of specific features and constitute a special class of problems of solid mechanics. The state-of-the-art of the corresponding nonclassical field of mechanics and particular results achieved in the framework of Russian science school in mechanics of growth created by Professor Manzhirov can be examined, for example, on the basis of works [1–7]. The present study is devoted to a nonclassical quasi-static problem for an additively manufactured solid formulated with the satisfaction of static boundary conditions on a parts of the solid boundary surface in the integral sense. The main obstacle to construct the solution of such problems is the time expansion of the surface part with the integral conditions due to influx of the additional material to another surface part adjacent to the former one. This obstacle is overcome in the study.

2. Material relations for the viscoelastic aging material used

Many artificial and natural materials used in additive manufacturing processes exhibit pronounced rheological properties, in particular, the properties of deformation heredity (viscoelasticity) and aging (weakening over time the deformation properties of the material regardless of the mechanical loads acting on it). Additive processes in which solids are formed from such materials are quite difficult to simulate, since the time-evolving deformation reaction of such a solid to the impacts applied to it continuously interacts with its mechanical response to the process of adding new material elements, and during pauses in the growth process, as well as after the final termination of this process, the formed solid continues to change its stress-strain state. However, the study of this kind of processes is relevant from the point of view of a variety of engineering and physical applications.

We will consider the viscoelastic aging material described by the following linear isotropic relationship between the stress tensor $\mathbf{T}$ and the small strain tensor $\mathbf{E}$ at arbitrary point $\mathbf{r}$ of the considered solid at the time instant t ($\mathbf{1}$ is the unit tensor of the second rank) [8]:

$$\mathbf{T}(\mathbf{r},t) = G(t)\left[\mathbf{T}^{\circ}(\mathbf{r},t) + \int_{\tau_0}^{t} \mathbf{T}^{\circ}(\mathbf{r},\tau)R(t,\tau)\,\mathrm{d}\tau\right] \tag{1}$$

where we use the notation

$$\mathbf{T}^{\circ}(\mathbf{r},t) = 2\mathbf{E}(\mathbf{r},t) + 2\nu(1-2\nu)^{-1}\mathbf{1}\mathbf{1}\bullet\bullet\,\mathbf{E}(\mathbf{r},t)\,. \tag{2}$$

The kernel $R(t,\tau)$ is called relaxation kernel and is the resolvent of the creep kernel $K(t,\tau)$ which can be determined by the expression

$$K(t,\tau) = G(\tau)\,\partial\Delta(t,\tau)/\partial\tau\,. \tag{3}$$

The quantity $G(\tau)$ is the elastic shear modulus increasing with the material age $\tau > 0$ (we calculate the material age from the material origination instant) due to aging of the material; ν is the constant Poisson ratio for elastic and creep strain. The so called specific strain function for pure shear $\Delta(t,\tau)$ describes the actual total strain, elastic and creeping, to the time instant $t \ge \tau$ of a material specimen under the constant unit pure shear loading which is acting starting from the time instant $\tau > 0$:

$$\Delta(t,\tau) = G(\tau)^{-1} + \omega(t,\tau) \tag{4}$$

where $\omega(t,\tau)$ is the creep measure for pure shear, $\omega(\tau,\tau) \equiv 0$, see figure 1. It is to be noted that the quantity $\omega(+\infty,\tau)$ has meaning of creep resource of the material at its age $\tau > 0$. This material characteristic decreases with the age augmentation due to the aging process going on regardless of any loadings acting on the material. The creep kernel $K(t,\tau)$ can be also expressed through other characteristics of the material, describing its behavior in different elementary stress states. Given expression (3) uses a variant with the characteristics for the elementary state of pure shear.

As we can see, state equation (1) contains as a specific case the state equation of isotropic elastic material. The latter equation follows from (1) by taking $\omega(t,\tau) \equiv 0$, $G(\tau) \equiv \mathrm{const}$.

The time instant τ_0 in (1) is the instant of occurrence of stresses at the solid points. If state equation (1) is used to describe the mechanical behaviour of a growing solid which is formed additively by attaching new material layers to the current solid surface then the time τ_0 of stresses occurrence must depend on the variable radius-vector $\mathbf{r}$ of the solid points: $\tau_0 = \tau_0(\mathbf{r})$. As in the considered growing process the additional material is being loaded simultaneously with its attaching to the solid, the corresponding function $\tau_0(\mathbf{r})$ is to be determined in the following way. In the original part of the considered conical pillar this function should be identically equal to the time moment t_0 of loading of this part, and in the formed due to growth additional part of the solid under consideration

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this function should coincide with the distribution $\tau_*(\mathbf{r})$ of moments of attaching of particles $\mathbf{r}$ of the additional material to the solid.

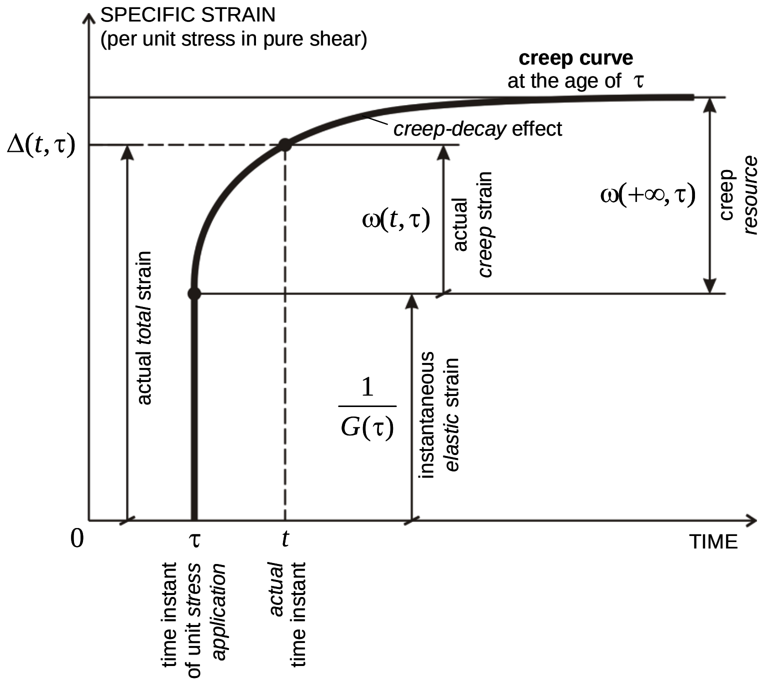

Figure 1. A conceptual creep curve of the used viscoelastic aging material.

Remark that constitutive equations similar to (1) are widely used to describe the mechanical behavior of various natural and artificial stones (in particular, concrete), polymers, soils, ice, wood. Typical experimental creep curves for such materials can be borrowed, e.g., from [9].

3. The mechanical problem under consideration

Let us consider a conical significantly high axisymmetric pillar made from the material subordinated to constitutive equation (1). At the moment $t = t_0 > 0$ an external loading is applied to the upper (bearing) end of the pillar. We assume that at every moment of time $t \geq t_0$ this loading is statically equivalent to a central axial force $P(t)$ which can arbitrarily vary with the time t. A compressing force $P(t)$ will be considered as positive.

At the instant $t = t_1 \geq t_0$, some time after the loading application, the process of gradual, layer-by-layer, axisymmetric thickening the considered pillar is starting. Thickening is implemented via piecewise-continuous adding the additional material to the pillar lateral surface. The added material is identic to the material of the originally existing pillar and is initially free of any stresses. Thickening occurs in such a way that at each time moment t the pillar maintains a conical shape. We denote the with time increasing radius of the bearing end of the pillar by $a(t)$, and we denote $a_0 = a(t_0) = a(t_1)$. The current angle $\alpha(t)$ of the pillar tapering can arbitrarily change due to pillar thickening. The model of piecewise-continuous thickening supposes that the process consists of $N \geq 1$ consecutive stages $t \in (t_{2k-1}, t_{2k})$, $k = \overline{1, N}$, of the pillar continuous accreting. These arbitrary long stages are separated by pauses $t \in [t_{2k}, t_{2k+1}]$, $k = \overline{1, N-1}$, of arbitrary duration when the influx of additional material to the pillar lateral surface does not take place. At the stages of continuous accreting an infinitely thin layer of additional material attaches to the lateral pillar surface each infinitely small period of time. In the whole process of piecewise-continuous thickening, $t \in (t_1, t_{2N})$, and after this process completion, $t \in [t_{2N}, +\infty)$, a time-varying central axial force $P(t)$ continues to act on the bearing end of the pillar. The modelled process is schematically illustrated in figure 2.

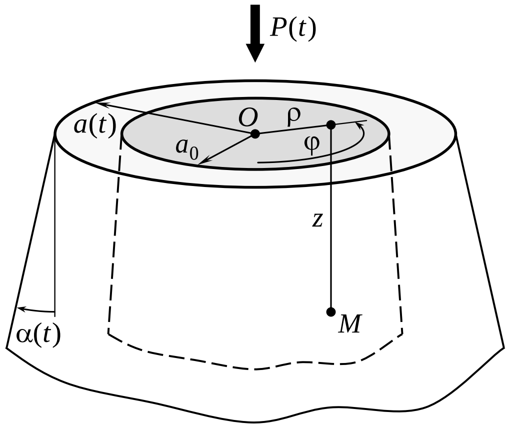

Figure 2. A scheme of the modelled process. The coordinate system used.

We investigate the evolution of the stress-strain state of the considered pillar under specified loading before the start, during and upon the completion of the described piecewise-continuous process of its thickening in the case of small strain. The deforming process is assumed quasi-static.

We place the origin O of coordinates into the geometric center of the bearing end of the pillar. We direct the coordinate axis Oz perpendicular to this end inside the pillar. Denote the polar radius and the polar angle of the polar cylindrical coordinate system with the reference axis Oz by ρ and φ respectively (see figure 2). Let the vector system $\{\mathbf{e}_z, \mathbf{e}_\rho(\varphi), \mathbf{e}_\varphi(\varphi)\}$ be the local orthonormal basis of the introduced cylindrical coordinate system (z, ρ, φ). It is clear that the radius-vector $\mathbf{r} = \overrightarrow{OM}$ of an arbitrary point M of the considered solid can be represented in the form $\mathbf{r} = \mathbf{e}_z\, z + \mathbf{e}_\rho(\varphi)\,\rho$.

One can describe changing the geometry of the considered circular conical pillar due to its thickening over the lateral surface by defining the shape function $\rho = L(z,t)$ for the current lateral surface of the cone: $L(z,t) = a(t) + z\,\mathrm{tg}\,\alpha(t)$. The trace of the lateral surface passing in space on the time interval $t \in (t_1, t_{2N})$ forms obviously the additional part of the solid under consideration. At time moments $t \in (t_{2k-1}, t_{2k})$, $k = \overline{1, N}$, the lateral pillar surface represents the actual accretion surface of this solid. We can also consider the actual accretion surface during the continuous accretion stages as the level surface of the level t for the scalar function $\tau_*(\mathbf{r}) = \tau_*(z, \rho)$ (see Section 2):

$$\rho = L(z,t) \quad \Leftrightarrow \quad \tau_*(z, \rho) = t. \tag{5}$$

The external (relative to the accreted solid) unit normal vector to this surface can be calculated as

$$\mathbf{n}(\mathbf{r}) = \nabla \tau_*(\mathbf{r})\,|\,\nabla \tau_*(\mathbf{r})\,|^{-1} = \mathbf{e}_\rho(\varphi)\cos\alpha(\tau_*(z,\rho)) - \mathbf{e}_z \sin\alpha(\tau_*(z,\rho)). \tag{6}$$

4. The accretion problem statement and solution

Due to the objective lack of any unstressed configuration for an accreted solid the traditional kinematic description of the process of its deformation is not suitable for this solid. However, we can observe that all the particles of the added material after its attaching to the accretion surface of the solid continue to move in space as the material constituents of a solid continuum. That means that in the region of space occupied by the instantly existing accreted solid a smooth displacement velocities field $\mathbf{v}(\mathbf{r},t)$ of the particles deformation movement is uniquely determined. Therefore, the mechanical problem for the accreted solid can be stated in terms of velocities.

Let us introduce the strain velocity tensor and the so called operator stress velocity tensor

$$\mathbf{D} = (\nabla\mathbf{v} + \nabla\mathbf{v}^{\mathrm{T}})/2\,, \tag{7}$$

$$\mathbf{S} = \partial\mathbf{T}^{\circ}/\partial t\,, \tag{8}$$

respectively. Using these tensors we can rewrite material relation (2) in the velocity form

$$\mathbf{S}(\mathbf{r},t) = 2\mathbf{D}(\mathbf{r},t) + 2\nu(1-2\nu)^{-1}\mathbf{11}\bullet\bullet\,\mathbf{D}(\mathbf{r},t)\,. \tag{9}$$

One can prove [10] that the following analogue of the standard equilibrium equation

$$\nabla\bullet\mathbf{T}(\mathbf{r},t) = \mathbf{0} \tag{10}$$

for the operator stress velocity tensor is fair in the considered case of the accreted solid deformation:

$$\nabla\bullet\mathbf{S}(\mathbf{r},t) = \mathbf{0}\,. \tag{11}$$

We consider the unstressed additional material being attached to the surface of the accreted pillar in the process of its accretion. So we are to put the following condition for the initial stresses at all points of the additional part of this solid:

$$\mathbf{T}(\mathbf{r},\tau_*(\mathbf{r})) = \mathbf{0}\,. \tag{12}$$

We can mathematically show that specific nonclassical initial condition (12) taken into account coupled with equilibrium equation (10) generates in our problem the following boundary condition for the operator stress vector velocity at points of the current accretion surface, $t\in(t_{2k-1},t_{2k})$, $k=\overline{1,N}$:

$$\mathbf{n}(\mathbf{r})\bullet\mathbf{S}(\mathbf{r},t) = \mathbf{0},\quad \rho = F(z,t)\,. \tag{13}$$

It is evident that this boundary condition remains in force also in the pauses between the continuous accretion stages and after the accretion completion, because the lateral surface of the pillar is stress-free on these time segments (condition (13) has, however, a completely different mechanical nature on these time segments).

We intend to satisfy the static conditions on the expanding due to the material influx bearing end of the considered thickened pillar in integral sense. In concordance with the above formulated pillar loading conditions we have to state so the following integral boundary conditions on the bearing end surface for the stress tensor $\mathbf{T}(\mathbf{r},t) = \mathbf{T}(z,\rho,\varphi,t)$ at any time $t > t_1$ after the thickening process begins:

$$\int_0^{2\pi}\mathrm{d}\varphi\int_0^{a(t)}\mathbf{e}_z\bullet\mathbf{T}(0,\rho,\varphi,t)\,\rho\,\mathrm{d}\rho = -\mathbf{e}_z P(t),\quad \int_0^{2\pi}\mathrm{d}\varphi\int_0^{a(t)}\rho\mathbf{e}_\rho(\varphi)\times[\mathbf{e}_z\bullet\mathbf{T}(0,\rho,\varphi,t)]\,\rho\,\mathrm{d}\rho = \mathbf{0} \tag{14}$$

Notably is that it can be proven the following nontrivial corollary of these integral conditions for the velocities $\mathbf{S}(\mathbf{r},t) = \mathbf{S}(z,\rho,\varphi,t)$ of the operator stresses:

$$\begin{gathered}\int_0^{2\pi}\mathrm{d}\varphi\int_0^{a(t)}\mathbf{e}_z\bullet\mathbf{S}(0,\rho,\varphi,t)\,\rho\,\mathrm{d}\rho = -\mathbf{e}_z\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{P(t)}{G(t)} - \int_{t_0}^{t}\frac{P(\tau)}{G(\tau)}K(t,\tau)\,\mathrm{d}\tau\right],\\ \int_0^{2\pi}\mathrm{d}\varphi\int_0^{a(t)}\rho\mathbf{e}_\rho(\varphi)\times[\mathbf{e}_z\bullet\mathbf{S}(0,\rho,\varphi,t)]\,\rho\,\mathrm{d}\rho = \mathbf{0}.\end{gathered} \tag{15}$$

The proof of this corollary uses specific initial condition (12) and is tedious enough to not be placed within the present constrained paper.

Integral conditions (15) close the mathematical formulation of the stated nonclassical mechanical problem. With these conditions all the above mentioned relations give us the possibility to construct the closed-form analytical solution of the problem in question: we obtain firstly the exact in the sense of Saint-Venant principle analytical solution of the boundary value problem for differential equation (11) supplemented with relations (9) and (7), with boundary conditions (13) and (15); after that we calculate the time evolution of the operator stress tensor $\mathbf{T}^{\circ}$ at each point $\mathbf{r}$ of the ready-made pillar by integrating over time t the obtained values of velocity $\mathbf{S}$ of this tensor, with zero initial values correlated with (12); at last we reconstruct the time evolution of the true stress tensor $\mathbf{T}$ at each point by the found values of the operator stress tensor $\mathbf{T}^{\circ}$. The latter procedure can be conducted with use of formula (1) or, which is more efficient, by solving Volterra equation of the second kind

$$\frac{\mathbf{T}(\mathbf{r},t)}{G(t)}-\int_{\tau_0(\mathbf{r})}^{t}\frac{\mathbf{T}(\mathbf{r},\tau)}{G(\tau)}K(t,\tau)\,\mathrm{d}\tau=\mathbf{T}^{\circ}(\mathbf{r},t) \tag{16}$$

with respect to the desired tensor-function $\mathbf{T}/G$.

The in the present study constructed analytical solution allows to retrace the evolution of the stress-strain state of the considered bearing conical pillar which is manufactured using a viscoelastic aging material under simultaneous action of an axial variable central force, after the start and on completing the piecewise continuous additive process of the pillar thickening over its lateral surface.

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