Analysis of dynamic responses in a composite poroelastic body using the time-stepping boundary element scheme built on the Lobatto method nodes

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Analysis of dynamic responses in a composite poroelastic body using the time-stepping boundary element scheme built on the Lobatto method nodes

S M Aizikovich[1], A N Petrov[1] and S A Tsvetkov[2]

[1]Research and Education Center "Materials", Don State Technical University,1 Gagarin Sq., 344010, Rostov-on-Don, Russia

[2]National Research Lobachevsky State University of Nizhny Novgorod, 23 Prospekt Gagarina, 603950, Nizhny Novgorod, Russia

andrey.petrov@mech.unn.ru

Abstract. In this paper, the dynamic analysis of piecewise-homogeneous poroelastic bodies on the example of a composite three-dimensional column is presented. Biot's model of the material, consisting of a solid phase and two fluid phases (liquid and gas) that filling the pores, is used to describe the poroelastic medium. The boundary element method is used in combination with the time-step method for the numerical inversion of the Laplace transform. The points in the complex plane for calculating the solution are chosen in a special way using the Lobatto method. The two- and the three-stage schemes are considered.

1. Introduction

The research continues a series of the works devoted to the application of the boundary elements method combined with the time-step schemes built on the Runge-Kutta methods nodes in problems of the poroelastic dynamics. The results of applying this approach in problems of the dynamics of homogeneous and composite fully saturated poroelastic bodies for the cases of two-stage schemes Radau and Lobatto, were published previously [1, 2]. As well as for problems of the dynamics of homogeneous partially saturated poroelastic bodies for the cases of two and three-stage schemes Radau and Lobatto [3, 4]. In this paper, the boundary element method is generalized for composite partially saturated poroelastic bodies. Two- and three-stage Lobatto methods are used for the iterative process organization.

In general, the approach is based on the convolution quadrature method (CQM) proposed by Lubich in 1988 for discretization of the convolution integral [5, 6]. The method has enlisted the considerable interest as a technique for applying BEM in dynamic problems (CQ-BEM), in which the classic time-stepping schemes show instability and numerical damping [7-10]. Later in various studies it was shown that the time-stepping BEM scheme, based on the Runge-Kutta methods, provides better accuracy than the scheme based on the linear multi-step method [11-15]. Banjai and Sauter also proposed a formulation for uncoupled problems in the Laplace transform domain, within which the technique actually works as an inverse transform method. The approach posses the original method benefits related to the numerical stability and has the time step value as the only parameter, which is determined by the physical characteristics of the problem.

2. Models and methods

A set of fully coupled governing differential equations of a porous medium saturated by two compressible fluids (water and air) subjected to dynamic loads is considered. In this formulation the solid skeleton displacements u_i, water pressure p^w and air pressure p^a are presumed to be independent variables [16]. The final differential equations in the Laplace domain yield

$$\begin{bmatrix} B_1\delta_{ij}+B_2\partial_i\partial_j & B_3\partial_i & B_4\partial_i \\ B_5\partial_j & B_6 & B_7 \\ B_8\partial_j & B_9 & B_{10} \end{bmatrix} \begin{bmatrix} \hat{u}_i \\ \hat{p}^w \\ \hat{p}^a \end{bmatrix} = \begin{bmatrix} -\hat{F}_i \\ -\hat{I}^w \\ -\hat{I}^a \end{bmatrix}, \mathbf{x}\in\Omega,\ \Omega\subset\mathbb{R}^3, \quad (1)$$

where $\hat{F}_i$, $\hat{I}^w$, $\hat{I}^a$ are bulk body forces and symbol "^" denotes Laplace transform with complex variable s. The expressions for the differential operator coefficients are presented in Appendix A. The parameters of the model of a poroelastic material and their values used in further calculations are listed in table A1. Equation (1), supplemented by boundary conditions, fully describes the boundary value problem in the representations of the 3D isotropic dynamic theory of poroelasticity. When a problem involves more than two poroelastic bodies in full contact, equation (1) is written for each body in partitioned form and the resulting equations are coupled together through equilibrium and compatibility at their interface [17].

The boundary-element technique is based on the use of a regularized boundary integral equation (BIE) direct approach:

$$\int_\Gamma \left(\mathbf{T}(\mathbf{x},\mathbf{y},s)\mathbf{u}(\mathbf{x},s)-\mathbf{T^0}(\mathbf{x},\mathbf{y})\mathbf{u}(\mathbf{y},s)\right)d\Gamma = \int_\Gamma \mathbf{U}(\mathbf{x},\mathbf{y},s)\mathbf{t}(\mathbf{x},s)d\Gamma, \mathbf{x},\mathbf{y}\in\Gamma, \Gamma=\partial\Omega, \quad (2)$$

where $\mathbf{U}(\mathbf{x},\mathbf{y},s)$ and $\mathbf{T}(\mathbf{x},\mathbf{y},s)$ are matrices of fundamental and singular solutions, respectively, $\mathbf{T^0}(\mathbf{x},\mathbf{y})$ contains isolated singularities, $\mathbf{x}$ is integration point, $\mathbf{y}$ is observation point, $\mathbf{u}$ is generalized displacement vector, $\mathbf{t}$ is generalized force vector.

To solve equation (2), the boundary surface Γ is divided into the generalized eight-node quadrangular elements. Generalized boundary functions of the first kind are approximated bilinearly, and generalized boundary functions of the second kind are assumed to be constant over the element.

The discrete representation of BIE (2) is constructed at the interpolation nodes of unknown boundary functions (collocation points) and has the following matrix form

$$[\Delta\mathbf{G}]\{\mathbf{T}\}=[\Delta\mathbf{F}]\{\mathbf{U}\}. \quad (3)$$

Matrices $\Delta\mathbf{G}$ and $\Delta\mathbf{F}$ contain integrals of the components of matrices $\mathbf{U}(\mathbf{x},\mathbf{y},s)$ and $\mathbf{T}(\mathbf{x},\mathbf{y},s)$, multiplied by the shape functions. The choice of the numerical integration scheme for computing the integral depends on its type. When a collocation point lies on integration element, the procedure for revealing the feature is performed. To improve the accuracy of integration on an element that does not contain a collocation point, in addition to the Gauss integration formulas, a hierarchical integration algorithm is applied, wherein the element is subdivided until the specified accuracy is achieved.

The solution in the time domain is obtained using the time-step method of numerical inversion of the Laplace transform. This method is close in its formulation to the CQM, but, in contrast to it, is based on the operational calculus of integrating original $f(s)$ of representation $\hat{f}(s)$. In general, the integral

$$y(t)=\int_0^t f(\tau)d\tau \quad (4)$$

is approximated as follows [11]:

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$$y(0)=0, y(n\Delta t)=\mathbf{b}^T\mathbf{A}^{-1}\sum_{k=1}^{n}\boldsymbol{\omega}_k^{\Delta t}, n=1,...N \tag{5}$$

where N is number of equal time steps. In this expression the quadrature weights $\boldsymbol{\omega}_k^{\Delta t}$ are determined using Laplace representation $\hat{f}(s)$ and the Runge-Kutta method. The quadrature weights can be expressed by the Cauchy integral form and approximated by using a trapezoidal rule with the number of steps L as follows:

$$\boldsymbol{\omega}_n^{\Delta t}=\frac{R^{-n}}{L}\sum_{l=0}^{L-1}\hat{f}\left(\frac{\psi(z)}{\Delta t}\right)\frac{\psi(z)}{\Delta t}e^{-nl\frac{2\pi}{L}\mathrm{i}}\ ,\ z=Re^{l\frac{2\pi}{L}\mathrm{i}}\ , \tag{6}$$

$$\psi(z)=\mathbf{A}^{-1}-z\mathbf{A}^{-1}\mathbf{1}\mathbf{b}^T\mathbf{A}^{-1}\ , \tag{7}$$

where $\psi(z)$ is characteristic function of the Runge-Kutta method and $\mathbf{1}=(1,1,...,1)^T$. The parameter R can be calculated by

$$R^L=\sqrt{\varepsilon}\ , \tag{8}$$

where ε is the error of the numerical calculation of equation (6).

The approximation used in deriving formulas (6), (7) is based on the m-stage Runge-Kutta method written down employing Butcher's table:

$$\frac{\mathbf{c}\,|\,\mathbf{A}}{\mathbf{b}^T}\ ,\ \mathbf{A}\in\mathbb{R}^{m\times m}\ ,\ \mathbf{b},\mathbf{c}\in\mathbb{R}^m\ . \tag{9}$$

A correct formulation of a time-step scheme requires that the method be A-stable and L-stable. In the assumption of $\mathbf{b}^T\mathbf{A}^{-1}=(0,...,0,1)$, the method is automatically L-stable. It is also important to note that the quadrature weights $\boldsymbol{\omega}_k^{\Delta t}$ and the characteristic function $\psi(z)$ are m-order matrices.

In the present study, Lobatto (Lobatto IIIC) scheme was chosen as a particular example of the Runge-Kutta schemes meeting the formulated conditions.

3. Results and discussion

The problem of the impact of force $F(t)=1N/m^2$ on compound prismatic poroelastic column with a rigidly fixed end is considered. The problem setting is shown in figure 1. Boundary-element meshes for each of the parts Ω_1 and Ω_2 contain 512 elements. Plots of displacement and pore pressure at the midpoint of the body are shown in figures°2–3. Comparison of the results, obtained using nodes of 2- and 3-stage Lobatto methods, was carried out.

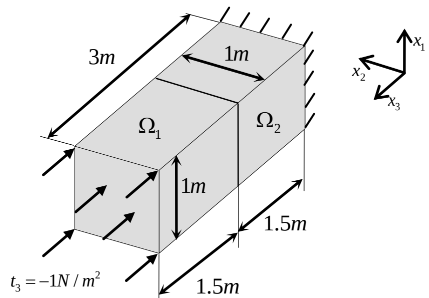

Figure 1. Geometry and boundary conditions of a partially saturated poroelastic column.

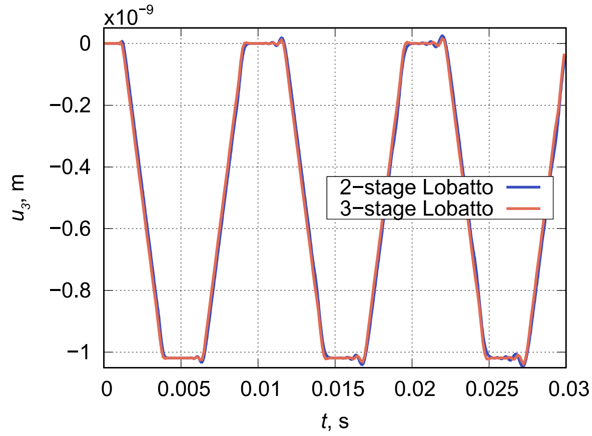

Figure 2. Displacement u_3 versus time at the midpoint of column.

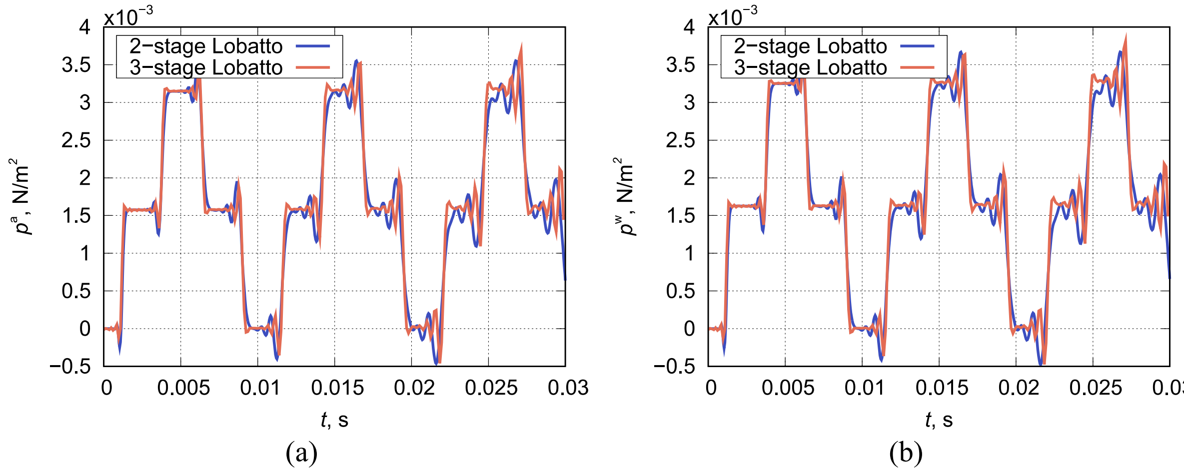

Figure 3. Pore pressures p^a (a) and p^w (b) versus time at the midpoint of the column.

Calculations using the 2-stage Lobatto method were performed with the time step value $\Delta t = 0.00003s$, and using the 3-stage Lobatto method with $\Delta t = 0.00012s$. Despite the significant difference in the time step values, the differences between the dynamic responses of the displacements obtained for each case are almost not noticeable on the plots. Fluctuating change of pore pressures allows comparing of the results in conditions of additional difficulties of approximation. The solution shown in figure 3, which obtained using three-stage method, is sufficiently smooth and retains the stable behavior on the considered time interval. In turn, the solution obtained using two-stage method, although smoother, but shows significant oscillations at the wave front points.

4. Conclusion

A boundary-element approach for solving the initial-boundary-value problems of the three-dimensional poroelastic bodies dynamics, which based on the application of the boundary elements method combined with the time-step scheme built on the Lobatto method nodes, is presented. Numerical studies of a model problem of the Heaviside function impact on a three-dimensional prismatic composite poroelastic body are carried out. A good agreement between the numerical boundary-element solution and the analytical solution is demonstrated.

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Appendix A

Table 1. Parameters of Massilon sandstone

Parameter type	Symbol	Value	Unit
Porosity	φ	0.23	-
Density of the solid skeleton	ρ_s	2650	kg/m^3
Density of the water	ρ_w	997	kg/m^3
Density of the air	ρ_a	1.10	kg/m^3
Drained bulk modulus of the mixture	K	1.02×10^9	N/m^2
Shear modulus of the mixture	G	1.44×10^9	N/m^2
Bulk modulus of the solid grains	K_s	3.5×10^{10}	N/m^2
Bulk modulus of the water	K_w	2.25×10^9	N/m^2
Bulk modulus of the air	K_a	1.10×10^5	N/m^2
Intrinsic permeability	k	2.5×10^{-12}	m^2
Viscosity of the water	η_w	1.0×10^{-3}	Ns/m^2
Viscosity of the air	η_a	1.8×10^{-5}	Ns/m^2
Gas entry pressure	p^d	5.0×10^4	Ns/m^2
Saturation degree	S_w	0.9	-
Residual water saturation	S_{rw}	0	-
Residual air saturation	S_{ra}	1	-
Pore size distribution index	ϑ	1.5	-

$$\kappa_a = K_{ra}k/\eta_a,\ \kappa_w = K_{rw}k/\eta_w,\ K_{rw} = S_e^{(2+3\vartheta)/\vartheta},\ K_{ra} = (1-S_e)^2\left(1-S_e^{(2+\vartheta)/\vartheta}\right) \tag{1}$$

$$S_e = \frac{S_w - S_{rw}}{S_{ra} - S_{rw}},\ S_u = -\frac{\vartheta(S_{ra} - S_{rw})}{p^d}\left(\frac{S_w - S_{rw}}{S_{ra} - S_{rw}}\right)^{\frac{\vartheta+1}{\vartheta}},\ \beta = \frac{\kappa_w\varphi\rho_w s}{\varphi S_w + \kappa_w\rho_w s},\ \gamma = \frac{\kappa_a\varphi\rho_a s}{\varphi S_a + \kappa_a\rho_a s}, \tag{2}$$

$$B_1 = G\nabla^2 - (\rho - \beta S_w\rho_w - \gamma S_a\rho_a)s^2,\ B_2 = K + \frac{G}{3},\ B_3 = -(\alpha-\beta)S_w,\ B_4 = -(\alpha-\gamma)S_a, \tag{3}$$

$$B_5 = -(\alpha-\beta)S_w s,\ B_6 = -\left(\zeta S_{ww}S_w + \frac{\varphi}{K_w}S_w - S_u\right)s + \frac{\beta S_w}{\rho_w s}\nabla^2,\ B_7 = -\left(\zeta S_{aa}S_w + S_u\right)s, \tag{4}$$

$$B_8 = -\left(\alpha-\gamma\right)S_a s,\ B_9 = -\left(\zeta S_{ww}S_a + S_u\right)s,\ B_{10} = -\left(\zeta S_{aa}S_a + \frac{\varphi}{K_a}S_a - S_u\right)s + \frac{\gamma S_a}{\rho_a s}\nabla^2, \tag{5}$$

$$\alpha = 1 - K / K_s\,,\ \zeta = (\alpha - \varphi) / K_s\,,\ S_{ww} = S_w - \vartheta(S_w - S_{rw})\,,\ S_{aa} = S_a + \vartheta(S_w - S_{rw})\,. \tag{6}$$

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