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BEM modeling of a 3D homogeneous anisotropic elastic half space under dynamic load

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Abstract. In this paper, we consider a problem of a homogeneous anisotropic and linearly elastic half space subjected to dynamic loading. Zero body forces and vanishing initial conditions are assumed. The problem is solved using the Boundary Element Method in the Laplace transformed domain. Integral expressions of the three-dimensional dynamic fundamental solutions for displacements and tractions are utilized. We employ the displacement boundary integral equations which are regularized using the static part of the dynamic anisotropic traction fundamental solution. For the spatial discretization of the boundary integral equations mixed boundary elements are adopted. The geometry of the considered domain is approximated with quadrilateral quadratic eight-noded elements. On the boundary elements displacements and tractions are interpolated using linear and constant shape functions, respectively. Time-domain solutions are obtained using suitable scheme for numerical inverse Laplace transform. The boundary-element solutions for the illustrative problem of an anisotropic elastic half space subjected to a Heaviside-type load are provided.

1. Introduction

Modeling the elastic wave propagation in large unbounded domains is often required in a number of engineering problems such as dynamic soil-structure interaction, in the fields of geotechnical engineering, foundation engineering, seismology and so on. Particularly, Rayleigh surface wave propagation [1] is of significant importance. Most naturally occurring soils are rarely isotropic and in practical cases can be modelled by an anisotropic and linearly elastic medium. Taking into account anisotropy of the elastic properties brings considerable difficulties even for the relatively simple configurations of the problems involving unbounded domains and renders analytical examination of the dynamic wave propagation impossible. Thus numerical methods play the crucial role in studying the wave fields in infinite and semi-infinite anisotropic elastic regions.

Though the nature of the boundary element methods (BEM) makes them particularly advantageous for the problems involving infinite domains, formulations for anisotropic materials is rather scarce (see e.g. [2-8]). This is due to the unavailability of the explicit closed form of dynamic fundamental solutions (Green's functions) for anisotropic elastic media. Wang and Achenbach [9, 10] obtained integral expressions for singular (static) and regular (dynamic) parts of anisotropic elastodynamic Green's function. Evaluation of the regular term of fundamental solution requires numerical integration of the two-dimensional integral.



In this paper, to investigate a problem of the wave propagation in an anisotropic elastic semi-infinite region we employ a conventional direct boundary element formulation based on the regularized displacement boundary integral equations (BIEs) in Laplace domain. Spatial discretization of the BIEs is based on the idea of mixed boundary elements. The geometry of the considered domain is approximated with quadrilateral quadratic eight-noded elements. On the boundary elements displacements and tractions are interpolated using linear and constant shape functions, respectively. After obtaining results in the Laplace domain the time-domain solutions are obtained using modified Durbin's method for numerical inverse Laplace transform. The presented BEM formulation is applied to a 3D dynamic problem of anisotropic elastic half space subjected to a Heaviside-type traction.

2. Problem statement and BEM formulation

We consider a homogeneous, anisotropic and linearly elastic solid occupying volume $\Omega \subset R^3$ with boundary $\Gamma = \partial\Omega$. In the absence of the body forces and with vanishing initial conditions the Laplace-transformed equations of motion can be expressed as

$$C_{ijkl}\bar{u}_{k,lj}(\mathbf{x},s) = \rho s^2 \bar{u}_i(\mathbf{x},s), \quad \mathbf{x} \in \Omega, \quad (1)$$

where \bar{u}_i is the displacement vector, C_{ijkl} is the elastic stiffness tensor, ρ is the mass density and s is the Laplace transform parameter.

The boundary conditions are given as follows

$$\bar{u}_i(\mathbf{x},s) = \bar{u}_i^*(\mathbf{x},s), \quad \mathbf{x} \in \Gamma_u, \quad (2)$$

$$\bar{t}_i(\mathbf{x},s) = \bar{t}_i^*(\mathbf{x},s), \quad \mathbf{x} \in \Gamma_t, \quad (3)$$

where \bar{t}_i denote the traction vector; \bar{u}_i^* and \bar{t}_i^* are the given displacements and tractions, correspondingly.

The regularized displacement boundary integral equations can be expressed as

$$\alpha \bar{u}_j(\mathbf{x},s) + \int_{\Gamma} [\bar{u}_k(\mathbf{y},s) \bar{h}_{jk}(\mathbf{r},s) - \bar{u}_k(\mathbf{x},s) h_{jk}^s(\mathbf{r}) - \bar{t}_k(\mathbf{y},s) \bar{g}_{jk}(\mathbf{r},s)] d\Gamma(\mathbf{y}) = 0, \quad \mathbf{x} \in \Gamma, \quad (4)$$

where $\mathbf{r} = \mathbf{y} - \mathbf{x}$, $\bar{g}_{jk}(\mathbf{r},s)$ and $\bar{h}_{jk}(\mathbf{r},s)$ are the 3D displacement and traction fundamental solutions in Laplace domain, respectively; $h_{jk}^s(\mathbf{r})$ is the static term of $\bar{h}_{jk}(\mathbf{r},s)$.

Spatial discretization of the BIEs (4) is based on the idea of mixed boundary elements [11]. The quadrilateral quadratic elements are employed to approximate the geometry of the boundary Γ . The linear and constant variation of the displacements and tractions, respectively, is assumed over each element. After the nodal collocation procedure and some rearrangements we obtain a complex-valued set of linear algebraic equations for the fixed value of Laplace transform parameter s

$$[\bar{\mathbf{A}}(s)] \{\bar{\mathbf{p}}(s)\} = \{\bar{\mathbf{f}}(s)\}, \quad (5)$$

where $\{\bar{\mathbf{p}}(s)\}$ is the vector of unknown field variables, vector $\{\bar{\mathbf{f}}(s)\}$ contains prescribed boundary values and $[\bar{\mathbf{A}}(s)]$ is the system matrix.

3. Fundamental solutions

Laplace transformed dynamic fundamental solutions for anisotropic media can be represented as a sum of singular (static, superscript "S") and regular (dynamic, superscript "R") terms as follows

$$\bar{g}_{jk}(\mathbf{r},s) = g_{jk}^S(\mathbf{r}) + \bar{g}_{jk}^R(\mathbf{r},s), \quad (6)$$

$$\bar{h}_{jk}(\mathbf{r}, s) = C_{ijkl} \bar{g}_{kp,l}(\mathbf{r}, s) n_l(\mathbf{y}) = h_{jk}^S(\mathbf{r}) + \bar{h}_{jk}^R(\mathbf{r}, s), \quad (7)$$

where $n_i(\mathbf{y})$ is the unit normal vector to the boundary at the point \mathbf{y} .

Singular and regular terms of the fundamental solution are expressed in the form of a one-dimensional integral over a unit circle and in the form of a two-dimensional integral over the surface of a unit hemi-sphere as follows [9, 10]

$$g_{ij}^S(\mathbf{r}) = \frac{1}{8\pi^2 |\mathbf{r}|} \int_{|\mathbf{d}|=1} \Gamma_{ij}^{-1}(\mathbf{d}) dL(\mathbf{d}), \quad (8)$$

$$\bar{g}_{ij}^R(\mathbf{r}, s) = -\frac{1}{8\pi^2} \int_{\substack{|\mathbf{n}|=1 \\ \mathbf{n} \cdot \mathbf{r} > 0}} \sum_{m=1}^3 \frac{k_m E_{im} E_{jm}}{\rho c_m^2} e^{-k_m |\mathbf{n} \cdot \mathbf{r}|} dS(\mathbf{n}), \quad (9)$$

with

$$c_m = \sqrt{\lambda_m / \rho}, \quad k_m = s / c_m, \quad \Gamma_{ij}(\mathbf{d}) = C_{kijl} d_k d_l, \quad \Gamma_{ij}(\mathbf{n}) = C_{kijl} n_k n_l, \quad (10)$$

$$dL(\mathbf{d}(\varphi)) \in D^S = \{0 \leq \varphi \leq 2\pi\}, \quad dS(\mathbf{n}(b, \varphi)) \in D^D = \{0 \leq b \leq 1; 0 \leq \varphi \leq 2\pi\}, \quad (11)$$

$$\mathbf{n}(b, \varphi) = \sqrt{1-b^2} \mathbf{d} + b \mathbf{e}, \quad \mathbf{e} = \mathbf{r}/|\mathbf{r}|, \quad \mathbf{e} = [e_1, e_2, e_3], \quad (12)$$

$$\mathbf{d}(\varphi) = [e_2 \cos \varphi + e_1 e_3 \sin \varphi, -e_1 \cos \varphi + e_2 e_3 \sin \varphi, -(1-e_3^2) \sin \varphi] / \sqrt{1-e_3^2}. \quad (13)$$

where λ_m are eigenvalues of $\Gamma_{jk}(\mathbf{n})$ and E_{jm} are the corresponding eigenvectors.

4. Laplace transform inversion

The inverse Laplace transform is defined as the follows

$$f(t) = \frac{1}{2\pi i} \int_{\alpha-i\infty}^{\alpha+i\infty} \bar{f}(s) e^{st} ds, \quad (14)$$

where $\alpha > 0$ is a real number greater than the real parts of all singularities of $\bar{f}(s)$.

It is impossible to directly use equation (14) in situations when values of $\bar{f}(s)$ can be computed only at the discrete set of values of the Laplace transform parameter s . Following the work of Durbin [12], with the known Laplace transform $\bar{f}(s)$ we have for the $f(t)$

$$s = \alpha + i\omega, \quad f(0) = \frac{1}{\pi} \int_0^\infty \text{Re}[\bar{f}(\alpha + i\omega)] d\omega, \quad (15)$$

$$f(t) = \frac{e^{\alpha t}}{\pi} \int_0^\infty \left\{ \text{Re}[\bar{f}(\alpha + i\omega)] \cos \omega t - \text{Im}[\bar{f}(\alpha + i\omega)] \sin \omega t \right\} d\omega, \quad t > 0, \quad (16)$$

For more reliable and stable long time inversion we employ modification of the Durbin's method proposed by Zhao [13].

With R being some large real number and defining $0 = \omega_1 < \omega_2 < \dots < \omega_n < \omega_{n+1} = R$ we obtain

$$f(0) = \frac{1}{\pi} \sum_{k=1}^n \int_{\omega_k}^{\omega_{k+1}} \text{Re}[\bar{f}(\alpha + i\omega)] d\omega, \quad (17)$$

$$f(t) = \frac{e^{\alpha t}}{\pi} \sum_{k=1}^n \int_{\omega_k}^{\omega_{k+1}} \left\{ \operatorname{Re}[\bar{f}(\alpha + i\omega)] \cos(\omega t) - \operatorname{Im}[\bar{f}(\alpha + i\omega)] \sin(\omega t) \right\} d\omega, \quad t > 0. \quad (18)$$

On each interval $[\omega_k, \omega_{k+1}]$, $k = \overline{1, n}$, the $\operatorname{Re}[\bar{f}(s)]$ and $\operatorname{Im}[\bar{f}(s)]$ are approximated as follows

$$\operatorname{Re}[\bar{f}(\alpha + i\omega)] \approx F_k + \frac{1}{2}(Z_{k+1} + Z_k)(\omega - \tau_k) + \frac{1}{2\Delta_k}(Z_{k+1} - Z_k)(\omega - \tau_k)^2, \quad (19)$$

$$\operatorname{Im}[\bar{f}(\alpha + i\omega)] \approx F_k + \frac{1}{2}(Y_{k+1} + Y_k)(\omega - \tau_k) + \frac{1}{2\Delta_k}(Y_{k+1} - Y_k)(\omega - \tau_k)^2, \quad (20)$$

$$\Delta_k = \omega_{k+1} - \omega_k, \quad \tau_k = (\omega_{k+1} + \omega_k)/2, \quad F_k = \operatorname{Re}[\bar{f}(\alpha + i\tau_k)], \quad G_k = \operatorname{Im}[\bar{f}(\alpha + i\tau_k)]. \quad (21)$$

The following system of linear algebraic equations is solved to determine Z_k $k = \overline{1, n+1}$

$$\begin{cases} 3\Delta_1 Z_1 + \Delta_1 Z_2 = 8(F_1 - F_0), \\ \dots \\ \Delta_{k-1} Z_{k-1} + 3(\Delta_k + \Delta_{k-1})Z_k + \Delta_k Z_{k+1} = 8(F_k - F_{k-1}), \\ \dots \\ 3\Delta_n Z_{n+1} + \Delta_n Z_n = 8(F_{n+1} - F_n). \end{cases} \quad (22)$$

Likewise for the Y_k we have the following system

$$\begin{cases} 3\Delta_1 Y_1 + \Delta_1 Y_2 = 8(G_1 - G_0), \\ \dots \\ \Delta_{k-1} Y_{k-1} + 3(\Delta_k + \Delta_{k-1})Y_k + \Delta_k Y_{k+1} = 8(G_k - G_{k-1}), \\ \dots \\ 3\Delta_n Y_{n+1} + \Delta_n Y_n = 8(G_{n+1} - G_n), \end{cases} \quad (23)$$

with $F_0 = \operatorname{Re}[\bar{f}(\alpha + i\omega_1)]$, $F_{n+1} = \operatorname{Re}[\bar{f}(\alpha + i\omega_{n+1})]$, $G_0 = \operatorname{Im}[\bar{f}(\alpha + i\omega_1)]$, $G_{n+1} = \operatorname{Im}[\bar{f}(\alpha + i\omega_{n+1})]$.

Finally,

$$f(0) \approx \frac{1}{\pi} \sum_{k=1}^n \left[F_k + \frac{\Delta_k}{24}(Z_{k+1} - Z_k) \right] \Delta_k, \quad (24)$$

$$f(t) \approx \frac{e^{\alpha t}}{\pi} \left\{ \frac{1}{t} g_1(t) + \frac{1}{t^2} g_2(t) - \sum_{k=1}^n \frac{1}{t^3 \Delta_k} [(Z_{k+1} - Z_k)(\sin \omega_{k+1} t - \sin \omega_k t) + (Y_{k+1} - Y_k)(\cos \omega_{k+1} t - \cos \omega_k t)] \right\}, \quad t > 0, \quad (25)$$

where

$$g_1(t) = \left[F_n + \frac{1}{8}(Z_n + 3Z_{n+1})\Delta_n \right] \sin \omega_{n+1} t + \frac{1}{8}(Y_2 + 3Y_1)\Delta_1 - G_1 + \left[G_n + \frac{1}{8}(Y_n + 3Y_{n+1})\Delta_n \right] \cos \omega_{n+1} t, \quad (26)$$

$$g_2(t) = -Z_1 + Z_{n+1} \cos \omega_{n+1} t - Y_{n+1} \sin \omega_{n+1} t. \quad (27)$$

5. Numerical example

An anisotropic homogeneous elastic half space with mass density $\rho = 2216 \text{ kg/m}^3$ is considered. The otherwise traction free half space is subjected to a Heaviside-type tractions $t_3 = t_3^* H(t)$, $t_3^* = -1000 \text{ Pa}$ on a square with an area of 1 m^2 as depicted at figure 1. The elastic stiffness tensor of the half space is given in Voigt notation as follows [14]

$$\mathbf{C} = \begin{bmatrix} 17.77 & 3.78 & 3.76 & 0.24 & -0.28 & 0.03 \\ & 19.45 & 4.13 & -0.41 & 0.07 & 1.13 \\ & & 21.79 & -0.12 & 0.01 & 0.38 \\ & & & 8.30 & 0.66 & 0.06 \\ \text{sym.} & & & & 7.62 & 0.52 \\ & & & & & 7.77 \end{bmatrix} \text{ GPa.} \quad (28)$$

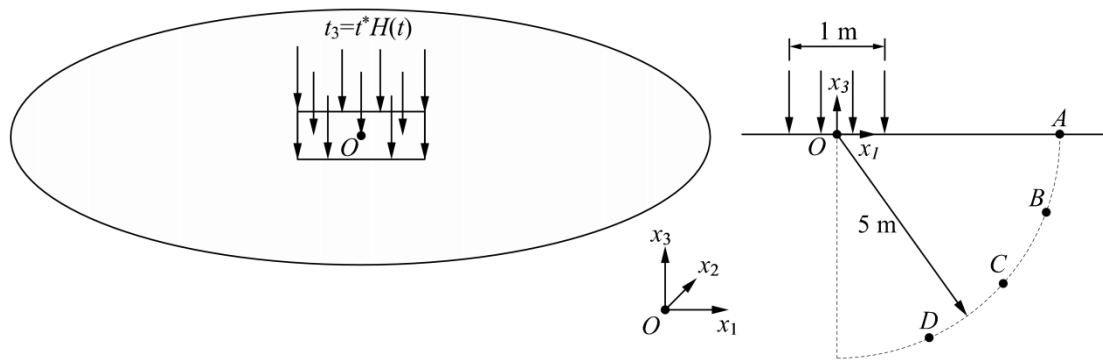


Figure 1. Configuration of the half space problem.

To use Zhao's modification of Durbin's method we need to define dimensionless frequencies ω_k . To this end we employ the following relation

$$\omega_k = e^{(kx)^m} - 1, \quad (29)$$

with $m = 0.5$, $k = 1, 2, \dots, 800$ and $x = (\ln(\omega_{\max} + 1))^{1/m} / k$, where $\omega_{\max} = 400$.

Displacements $u_1(t)$ and $u_3(t)$ at the surface point $A(5, 0, 0) \text{ m}$ and at the points inside half space $B(5 \cos(\pi/8), 0, -5 \sin(\pi/8)) \text{ m}$, $C(5 \cos(\pi/4), 0, -5 \sin(\pi/4)) \text{ m}$, $D(5 \cos(3\pi/8), 0, -5 \sin(3\pi/8)) \text{ m}$, (see schematic representation at figure 1) are shown at the figures 2 and 3, respectively.

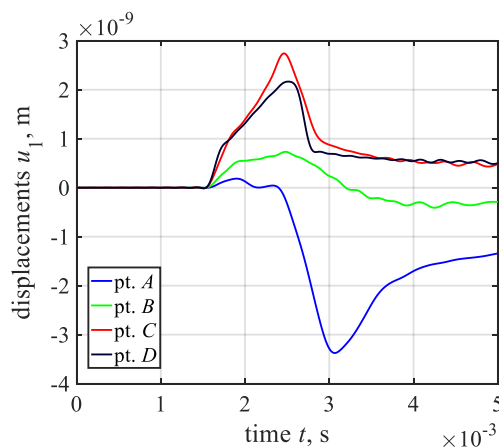


Figure 2. Displacements $u_1(t)$.

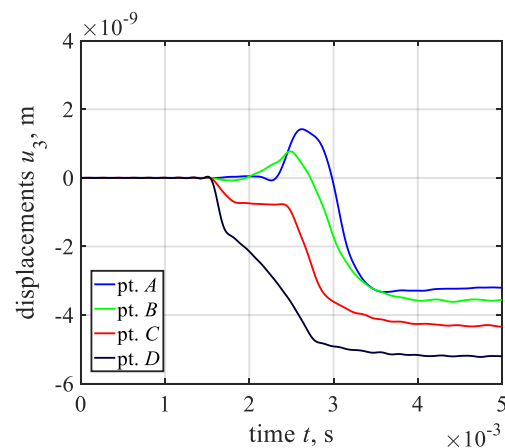


Figure 3. Displacements $u_3(t)$.

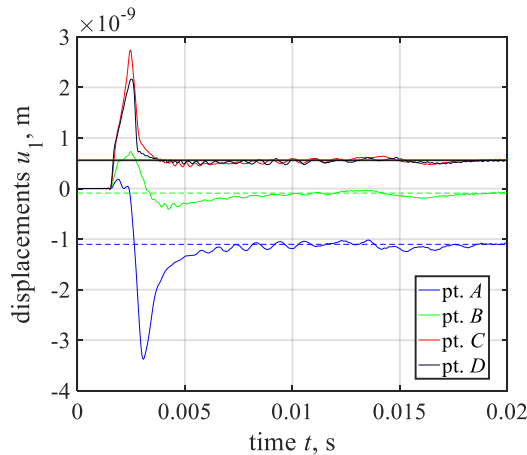


Figure 4. Displacements $u_1(t)$.

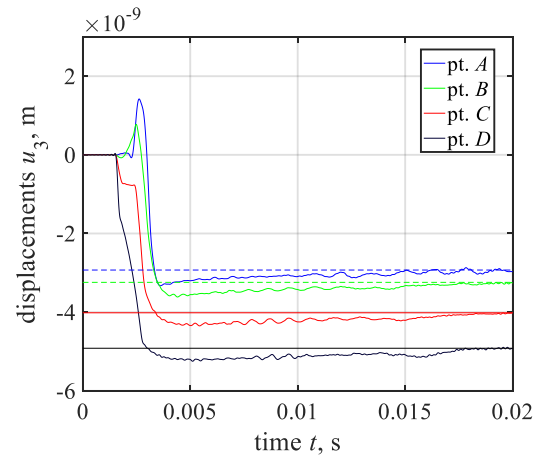


Figure 5. Displacements $u_3(t)$.

6. Conclusions

In this paper BEM formulation in the Laplace transformed domain was considered for problems of a homogeneous anisotropic and linearly elastic half space subjected to dynamic loading. Three-dimensional dynamic anisotropic fundamental solutions are represented as a sum of singular and regular terms which are expressed in the form of a one-dimensional integral over a unit circle and in the form of a two-dimensional integral over the surface of a unit hemi-sphere, respectively. Static term of the dynamic anisotropic traction fundamental solution is used to regularize the boundary integral equations. Spatial discretization rests on the idea of mixed boundary elements.

The described boundary element formulation is then applied to a dynamic problem of an anisotropic elastic half space subjected to a Heaviside-type tractions. Obtained results show that the presented Laplace domain boundary element approach coupled with the modified Durbin's method produce accurate and stable results for the dynamic problems involving the semi-infinite domains and anisotropic materials. On a larger time scale it can be observed that dynamic solutions at all points tend to their respective static counterparts as depicted at figure 4 for the displacements $u_1(t)$ and figure 5 for the displacements $u_3(t)$.

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