Verification of Solutions of Equations Governing Fluid Flows

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Verification of Solutions of Equations Governing Fluid Flows

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Abstract. We have carried out the verifications of the solutions for the equations governing fluid flows. The equations solved are the similar equation of the boundary layer flow and the two-dimensional Navier-Stokes equations in the formulation of the stream function and the vorticity. The methods of the verification are the interval analysis and the Krawczyk operator based on the fixed point theorem. A modified Gaussian elimination method is introduced to solve the system of linear interval equations. The Krawczyk operator presents the results with a high accuracy. The modified elimination method gives the results with an accuracy of one or two order(s) of magnitude higher than those obtained by the original interval analysis.

1. Introduction

Numerical simulations have given useful results in many areas. Some of the results, however, predict feasible phenomena different from those found in the real world. In order to find and reduce these contradictions, the standards of the verification and validation [1] give guidelines to investigate the quality of the predictions. Numerical calculation suffers from numerical errors of the truncation error, loss of trailing digits, cancellation of significant digits and round-off error. When we numerically solve the differential equations in the real space, we also have a problem of the discretization error. These errors mainly result from the conversion of the real values to the floating-point representations defined by IEEE 754 [2]. In this paper, we pick up the round-off error in the numerical calculations. Two major methods, the interval analysis and the method based on the fixed-point theorem, help to estimate the round-off error [3]. These methods enclose a real number in an interval with bounds of floating-point values. The interval method ensures the intervals before and after the arithmetic operations and guarantees the accuracy. The method based on the fixed-point theorem accepts the constraints defining the phenomena and then it makes the intervals shorter and identify the narrowest interval that includes the exact value. We apply these methods to the two problems in fluid dynamics. One is the two-point boundary value problem of the self-similar boundary layer flow. The other is the two-dimensional lid-driven square cavity flow formulated by the stream function and the vorticity. The discretization method is the finite difference method. The solution method of the similar equation uses the Gaussian elimination method to resolve the system of linear equations presented by the Newton iteration method. In the elimination process, we modify the interval operation of the pivoting process and improve the results of the interval analysis. An iterative procedure of the Jacobi method is adopted to solve the Poisson equation with a diagonally dominant coefficient matrix in the problem of the cavity flow. The results obtained by these methods show that the method based on the fixed-point theorem gives a reasonable results with high accuracy.

2. Formulation

2.1. Interval analysis
In this subsection, we review the interval analysis. In this analysis, the interval X that includes the real value x has a representation of $[a,b]=\{x\,|\,a\le x\le b\}$ [4, 5]. Here, a and b denote the lower bound $\underline{x}$ and the upper bound $\overline{x}$, respectively. For simplicity, we express $[\underline{x},\overline{x}]$ as $[x]$.

In arithmetic between intervals X and Y,

$$X+Y=[\underline{x}+\underline{y},\overline{x}+\overline{y}], \tag{1}$$

$$X-Y=[\underline{x}-\overline{y},\overline{x}+\underline{y}], \tag{2}$$

$$X\times Y=[\min(\underline{x}\underline{y},\underline{x}\overline{y},\overline{x}\underline{y},\overline{x}\overline{y}),\max(\underline{x}\underline{y},\underline{x}\overline{y},\overline{x}\underline{y},\overline{x}\overline{y})], \tag{3}$$

$$X/Y=[\min(\underline{x}/\underline{y},\underline{x}/\overline{y},\overline{x}/\underline{y},\overline{x}/\overline{y}),\max(\underline{x}/\underline{y},\underline{x}/\overline{y},\overline{x}/\underline{y},\overline{x}/\overline{y})]. \tag{4}$$

When the division has an interval denominator including 0.0, it is assumed that the division returns an error. The midpoint of the interval $[x]$ is given by

$$mid([x])=mid([\underline{x},\overline{x}])=\frac{\underline{x}+\overline{x}}{2}. \tag{5}$$

In the manipulation of the intervals expressed in the floating-point representation, we need to convert the real values into floating-point values. In order to determine the lower and upper bounds of the result of the interval arithmetic, we first obtain the candidates of these bounds obtained from the arithmetic between two intervals. Then, we apply the round downward and round upward operations to these candidates and specify the superior of the lower bound and the inferior of the upper bound in the floating-point representation.

In the present study, we use PROFIL/BIAS [6] to perform the interval analysis, which is a numerical library including the operations among the interval scalars, interval vectors and interval matrices.

2.2. Self-similar equation of the boundary layer on a flat plate
The following equation derived by the Falkner-Skan transformation governs the self-similar boundary layer flow on a flat plate [7],

$$f'''+\frac{m+1}{2}ff''+m(1-(f')^2)=0, \tag{6}$$

with the two-point boundary conditions

$$\eta=0.0:\quad f=0.0,\quad f'=0.0, \tag{7}$$

$$\eta=\eta_e:\quad f'=1.0. \tag{8}$$

Here, η is the normalized normal coordinate from the flat plate and f is the normalized stream function. The prime denotes the derivative with respect to η. The streamwise velocity component is $U_e f'$, where U_e is the velocity component of the external stream at the edge of the boundary layer. With x that is the distance from the leading edge of the flat plate, the pressure gradient parameter m is $(x/U_e)(dU_e/dx)$, and it is assumed to be constant in the present study. The surface of the flat plate and the edge of the boundary layer are at $\eta=0$ and $\eta=\eta_e$, respectively.

We use the Keller's box method [8] to solve the differential equations. The third-order equation (6) has the following system of the first-order equations with new dependent variables of u and v,

$$f'=u, \tag{9}$$

$$u'=v, \tag{10}$$

$$v'=-\frac{m+1}{2}fv-m(1-u^2). \tag{11}$$

The boundary conditions are

$$\eta=0.0:\quad f=0.0,\quad u=0.0, \tag{12}$$

$$\eta = \eta_e : \quad u = 1.0 . \tag{13}$$

When we take $NJ+1$ grid points from $\eta_0 = 0$ to $\eta_{NJ} = \eta_e$ on the η axis and derive the central difference equations of equations (9), (10) and (11), we obtain the $3 \times NJ$ system of nonlinear equations. Substituting $f_j^{n+1} = f_j^n + \Delta f_j$, $u_j^{n+1} = u_j^n + \Delta u_j$ and $v_j^{n+1} = v_j^n + \Delta v_j$ into these nonlinear equations and omitting the second-order terms of Δ, we obtain a system of linear equations with the boundary conditions of $\Delta f_0 = 0$, $\Delta u_0 = 0$ and $\Delta u_{NJ} = 0$, which uses the *n*-th approximate solutions of f_j^n, u_j^n and v_j^n and determines Δf_j, Δu_j and Δv_j with j from 0 to *NJ*. Updating the *n*-th approximate solutions with Δf_j, Δu_j and Δv_j gives the Newton method determining (*n*+1)-th approximate solutions. In the present study, the Gaussian elimination method is used to solve the system of linear equations.

2.3. Lid-driven square cavity flow

The lid-driven square cavity flow studied in the pioneering work by Ghia et al. [9] presents one of the benchmark problems. Here, we consider the cavity flow given by the following Navier-Stokes equations in the stream function ψ and vorticity ω formulation in the Cartesian coordinate $(x, y)^T$ with $0.0 \le x \le 1.0$ and $0.0 \le y \le 1.0$

$$\frac{\partial \omega}{\partial t} + \frac{\partial \psi}{\partial y}\frac{\partial \omega}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial \omega}{\partial y} = \frac{1}{\mathrm{Re}}\left(\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2} \right), \tag{14}$$

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = -\omega . \tag{15}$$

The boundary conditions are the unity velocity component in x coordinate direction on the driving lid at $y = 1.0$ and the no slip condition on the stationary walls at $x = 0.0$, $x = 1.0$ and $y = 0.0$.

The partial differential equations are discretized on the collocated square grid, and no singular problem at the corners between the moving and stationary walls [10] is considered. The Poisson equation (15) gives a system of linear equations whose coefficient matrix has a diagonal component of - 4.0 and other four non-diagonal components of 1.0. That is, at the grid points j and k in the x and y coodinates, respectively, the difference equation centered for $\psi_{j,k}$ can be expressed as

$$\psi_{j-1,k} + \psi_{j,k-1} - 4\psi_{j,k} + \psi_{j,k+1} + \psi_{j+1,k} = -h^2 \omega_{j,k} . \tag{16}$$

Here, h is the uniform grid spacing in x and y directions. While the direct method such as the Gaussian elimination method is applicable to solve the system of linear equations (16), the iterative method such as the Jacobi method is effective to resolve a large system in two or three-dimensional space. The diagonal dominancy of the coefficient matrix assures the convergence of the iterative process. Therefore, we assume that the *n*-th iteration solutions are similar to the (*n*+1)-th iteration solutions and

$$4\psi_{j,k}^n + 4\psi_{j,k}^{n+1} \cong 8\psi_{j,k}^{n+1} . \tag{17}$$

Then, we rewrite equation (16) as follows

$$\psi_{j-1,k}^{n+1} + \psi_{j,k-1}^{n+1} - 8\psi_{j,k}^{n+1} + \psi_{j,k+1}^{n+1} + \psi_{j+1,k}^{n+1} = -h^2 \omega_{j,k} - 4\psi_{j,k}^n . \tag{18}$$

This rewriting makes the coefficient matrix diagonally dominant and a favorable convergence is expected. Let denote the coefficient matrix, the constant vector and the unknown vector of equation (18) by A, b and x, respectively, and have an expression

$$Ax = b . \tag{19}$$

We decompose the coefficient matrix into the diagonal matrix D and the matrix A_{LR} with zeros on its diagonal

$$A = D + A_{LR}. \tag{20}$$

The n-th iteration solutions x^n give the (n+1)-th iteration solutions x^{n+1} by

$$x^{n+1} = D^{-1}(-A_{LR}\, x^n + b). \tag{21}$$

3. Solution Method

3.1. Krawczyk operator based on the fixed-point theorem

One of the methods to assure the accuracy of the numerical solution is the one that uses the interval analysis described in subsection 2.1. The other method regards the problem described by equations as a constraint satisfaction problem and uses the operator that makes the intervals of the solutions narrower. We express the governing equations with their boundary conditions as $f(x) = 0$, where each component of the vector function f represents one equation or one boundary condition. When approximate interval solution vector is assumed to be given by $[x]$, the iterative operator based on the Krawczyk method is given as follows [3, 5],

$$x_0 = mid([x]), \tag{22}$$

$$M = J_f^{-1}(x_0), \tag{23}$$

$$[J_k] = I - M\, J_f([x]), \tag{24}$$

$$[r] = x_0 - M\, f(x_0) + [J_k]([x] - x_0), \tag{25}$$

$$[x] = [x] \cap [r], \tag{26}$$

where J_f and J_f^{-1} are the Jacobian matrix of f and its approximated inverse matrix, respectively, and I is the identity matrix. The operation $\cap$ gives an intersection of two intervals. The interval $[x]$ obtained by equation (26) presents new iterative solutions.

3.2. Modification of forward process in Gaussian elimination method

The forward process of the Gaussian elimination method consists of the reduction of unknown variables. We now consider two equations

$$[a_{jp}][x_p] + \cdots + [a_{jq}][x_q] + \cdots = [b_j], \tag{27}$$

$$[a_{kp}][x_p] + \cdots + [a_{kq}][x_q] + \cdots = [b_k]. \tag{28}$$

In order to eliminate the unknown variable $[x_p]$, a new equation will be derived from, for example, equation (27)

$$\frac{[a_{kp}]}{[a_{jp}]}[a_{jp}][x_p] + \cdots + \frac{[a_{kp}]}{[a_{jp}]}[a_{jq}][x_q] + \cdots = \frac{[a_{kp}]}{[a_{jp}]}[b_j]. \tag{29}$$

This derivation is required only to make the coefficient of $[x_p]$ consistent between two equations and the application of the interval analysis that may introduces additional inaccuracy is not appropriate. Therefore, instead of equation (29), we use the equation obtained by the pointwise operation with midpoints of $a_{jp} = mid([a_{jp}])$ and $a_{kp} = mid([a_{kp}])$

$$\frac{a_{kp}}{a_{jp}}[a_{jp}][x_p] + \cdots + \frac{a_{kp}}{a_{jp}}[a_{jq}][x_q] + \cdots = \frac{a_{kp}}{a_{jp}}[b_j], \tag{30}$$

and modify the forward elimination process.

All the numerical simulations have been performed by the programs written in gcc and g++ on Linux with double precision calculations.

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4. Result and Discussion

The floating-point calculation, the original interval analysis shown in subsection 2.1, the modified Gaussian elimination method (MGEM) and the Krawczyk operator are used to solve a system of linear equations for the self-similar equation in subsection 2.2. The results are shown in table 1, where the pressure gradient parameter m is 0.0 and - 0.05 and an interval of the form, for example, 0.33 [12, 24] means the interval of [0.3312, 0.3324]. The grid spacing is constant and it is 0.1. The intervals in the table show those of v_0 that corresponds the velocity gradient on the flat plate. The value of IT is the iteration numbers used to obtain a converged solution. Being compared with the result of the floating-point calculation, the original interval analysis has an accuracy of 2 or 3 digits. The result of MGEM gives the errors with one or three orders of magnitude smaller than those of the original interval method. This fact shows an approving effect of the current modification applied to the Gaussian elimination method. The Krawczyk operator offers a much higher accuracy when it is compared with the result of the original interval analysis or MGES. The decimal digits of the floating-point representation is 15.9 at most. Therefore, the result of the Krawczyk operator applied to the present similar equation shows an accuracy comparable with the resolution of the floating-point value.

Table 1. Solutions of the self-similar equation. $\eta_e = 8.0$.

	$m = 0.0$	$m = -0.05$
floating point calculation	0.3320414384213981, IT = 6	0.2135095226597685, IT = 6
interval analysis	0.33[12183451442928, 28570033968403], IT = 3	0.213[5073633921204, 622998266781], IT = 4
interval analysis with MGEM	0.33204[06663353578, 22105826957], IT = 5	0.2135[063433007140, 127020193930], IT = 5
Krawczyk operator	0.33204143842139[79, 84], IT = 4	0.213509522659768[3, 8], IT = 4

For the calculation of the lid-driven square cavity flow, we use the Krawczyk operator and the Jacobi method noted in subsection 2.3. The terms other than those in equation (15) is formulated by the central difference method with the original interval method. Figure 1 shows the contour of the midpoint of the stream function ψ obtained by the Krawczyk operator. The Reynolds number is 300 and the grid points in the x and y coodinates are 100. Figure 2 represents the profiles of the upper and lower bounds of the intervals of the stream function at x = 0.6. The result of the Krawczyk operator is depicted in figure 2 (a). On the end walls at y = 0.0 and 1.0 where the Dirichlet boundary condition of the velocity component is applied, the errors are near zero. At the center of the flow field,

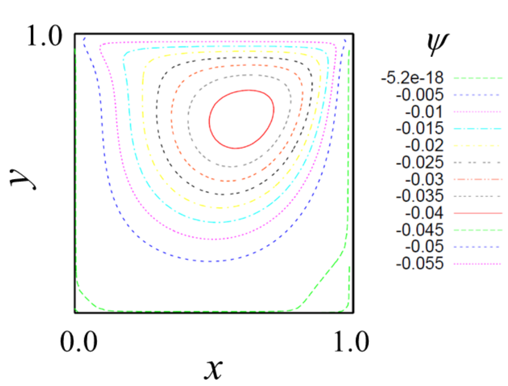
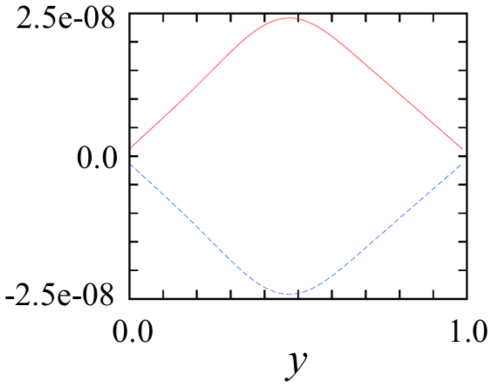
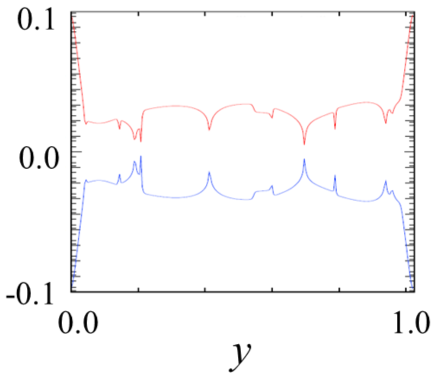


Figure 1. Solution of the lid-driven cavity flow obtained by using Krawczyk operator. The plots are the contours of the midpoints of ψ and the Reynolds number is 300.



(a) Krawczyk method



(b) Jacobi iteration method

Figure 2. Estimated interval of the solution of the lid-driven cavity flow. $y = 0.6$.

the interval is bounded by $\pm 2.5\times 10^{-8}$ and a favorable result is obtained. Figure 2 (b) shows the bounds obtained by the Jacobi method. While the solutions near the center of the flow are bounded within a certain range, the width between the bounds near the walls where the steep velocity gradient appears is large and the accuracy of the solution is low.

5. Conclusion

The interval analysis is applied two fluid dynamics problems. One problem is the ordinary differential equations governing the self-similar boundary layer flow. The other is the lid-driven cavity flow formulated with the stream function and the vorticity. The discretization method is the finite difference method. The Gaussian elimination method with the original interval analysis, the Gaussian elimination method with the modification of the pivoting process (MGEM), the Jacobi iteration method and the Krawczyk operator are used to solve the system of linear equations. For the solution of the self-similar boundary layer, MGEM gives the accuracy with one to three orders of magnitude higher than that of the original interval analysis. The accuracy obtained by the Krawczyk operation is almost comparable with the accuracy of the floating-point representation. For the solution of the lid-driven cavity flow, the Krawczyk operator presents the accuracy digits of half of the decimal digits of the floating-point representation. The result obtained by the interval Jacobi method does not offer any reasonable predictions of the flow near the stationary walls.

References

[1] Oberkampf W L and Roy C J 2010 *Verification and Validation in Scientific Computing*, (Cambridge: Cambridge University Press)
[2] Higham N J 2002 *Accuracy and Stability of Numerical Algorithms* 2nd ed (Philadelphia: SIAM)
[3] Jaulin L, Kieffer M, Didrit O and Walter É 2001 *Applied Interval Analysis* (London: Springer)
[4] Moore R E 1979 *Methods and Applications of Interval Analysis* (Philadelphia: SIAM)
[5] Neumaier A 1990 *Interval Methods for System of Equations* (Cambridge: Cambridge University Press)
[6] Knüppel O 1994 *Computing* 53 pp 277-87
[7] Cebeci T and Bradshaw P 1977 *Momentum Transfer in Boundary Layers* (Washington: Hemisphere)
[8] Cebeci T 2004 *Analysis of Turbulent Flows* 2nd ed (Oxford: Elsevier)
[9] Ghia U Ghia K N and Shin C T 1982 *J. Comput. Phys.* 48 pp 387-411
[10] Auteri F Parolini N and Quartapelle 2002 *J. Comput. Phys.* 183 pp 1-25