Force driven nonlinear vibrations of a thin plate with 1:1 internal resonance in a fractional viscoelastic medium

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Force driven nonlinear vibrations of a thin plate with 1:1 internal resonance in a fractional viscoelastic medium

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Abstract. In the present paper, force driven vibrations of a physically nonlinear plate are studied by the method of multiple time scales when the plate is subjected to the conditions of the one-to-one internal resonance. The damping features of the surrounding medium are described by the fractional derivative Kelvin-Voight model. The influence of viscosity on the energy exchange mechanism between interacting nonlinear modes has been analyzed. For the one-to-one internal resonance, the nonlinear set of resolving equations in terms of amplitudes and phase differences has been obtained, and a comparative analysis of numerical calculations of free and forced vibrations are presented.

1. Introduction

It is well known that the analysis of forced nonlinear vibrations of plates is the important area of applied mechanics, since plates, which could be made of different materials, are used as structural elements in many fields of industry and technology [1]. Moreover, nonlinear vibrations could be accompanied by such a phenomenon as the internal resonance, resulting in multimode response with a strong interaction of the modes involved [2]. The internal resonance could be found within a certain combination of natural frequencies of one and the same type of vibrations, when vibratory motions are described by one equation [3], or between natural frequencies belonging to different types of vibrations, when two or more equations are used for the description of dynamic behaviour of a structure [4].

Nonlinear free damped vibrations of a rectangular plate described by three nonlinear differential equations have been studied in [5], wherein the procedure resulting in decoupling linear parts of equations has been proposed with the further utilization of the generalized method of multiple scales [6] for solving nonlinear governing equations of motion, in so doing the amplitude functions are expanded into power series in terms of the small parameter and depend on different time scales. All possible types of the internal resonance within the considered model of nonlinear plate's behaviour have been revealed in [5].

In the present paper, the approaches proposed in [5] for free vibrations of plates and in [7] for forced vibrations of a nonlinear oscillator with weak fractional damping have been extended to the force driven vibrations of a thin plate under the one-to-one internal resonance with the force frequency approximately equal to a certain natural frequency of vertical vibrations.

2. Problem formulation and the method of solution

Let us consider the dynamic behavior of a free supported nonlinear thin rectangular plate (figure 1), forced vibrations of which in a viscoelastic fractional derivative medium are described by the following three differential equations in the dimensionless form (free damped equations presented in [5] are supplemented herein by the vertical harmonic force applied at the point with the coordinates x_0, y_0):

$$u_{xx}+\frac{1-\nu}{2}\beta_1^2u_{yy}+\frac{1+\nu}{2}\beta_1v_{xy}+w_x\left(w_{xx}+\frac{1-\nu}{2}\beta_1^2w_{yy}\right)+\frac{1+\nu}{2}\beta_1^2w_yw_{xy}=\ddot{u}+æ_1D^\gamma u, \quad (1)$$

$$\beta_1^2v_{yy}+\frac{1-\nu}{2}v_{xx}+\frac{1+\nu}{2}\beta_1u_{xy}+\beta_1w_y\left(\beta_1^2w_{yy}+\frac{1-\nu}{2}w_{xx}\right)+\frac{1+\nu}{2}\beta_1w_xw_{xy}=\ddot{v}+æ_2D^\gamma v, \quad (2)$$

$$\begin{aligned}\frac{\beta_2^2}{12}\left(w_{xxxx}\right. &+ 2\beta_1^2w_{xxyy}+\left.\beta_1^4w_{yyyy}\right)-w_{xx}\left(u_x+\nu\beta_1v_y\right)-w_x\left(u_{xx}+\nu\beta_1v_{xy}\right)\\ &- \frac{1-\nu}{2}\beta_1\left[w_{xy}\left(\beta_1u_y+v_x\right)+w_y\left(\beta_1u_{xy}+v_{xx}\right)\right]\\ &- \beta_1^2\left[w_{yy}\left(\nu u_x+\beta_1v_y\right)+w_y\left(\nu u_{xy}+\beta_1v_{yy}\right)\right]\\ &- \frac{1-\nu}{2}\beta_1\left[w_{xy}\left(\beta_1u_y+v_x\right)+w_x\left(\beta_1u_{yy}+v_{xy}\right)\right]\\ &- F\cos\left(\Omega_Ft\right)\delta(x-x_0)\delta(y-y_0)=-\ddot{w}-æ_3D^\gamma w, \end{aligned} \quad (3)$$

subjected to the initial

$$u|_{t=0} = v|_{t=0}=w|_{t=0}=0, \quad (4)$$

$$\dot{u}\Big|_{t=0} = \varepsilon U^0(x,y), \quad \dot{v}\Big|_{t=0}=\varepsilon V^0(x,y), \quad \dot{w}\Big|_{t=0}=\varepsilon W^0(x,y), \quad (5)$$

as well as the boundary conditions

$$w|_{x=0}=w\Big|_{x=1}=0, \quad v|_{x=0}=v\Big|_{x=1}=0, \quad u_x\Big|_{x=0}=u_x\Big|_{x=1}=0, w_{xx}\Big|_{x=0}=w_{xx}\Big|_{x=1}=0, \quad (6)$$

$$w|_{y=0}=w\Big|_{y=1}=0, \quad u|_{y=0}=u\Big|_{y=1}=0, \quad v_y\Big|_{y=0}=v_y\Big|_{y=1}=0, w_{yy}\Big|_{y=0}=w_{yy}\Big|_{y=1}=0, \quad (7)$$

where $u=u(x,y,t)$, $v=v(x,y,t)$, and $w=w(x,y,t)$ are the displacements of points located in the plate's middle surface in the $x-$, $y-$, and $z-$directions, respectively, ν is the Poisson's ratio, $\beta_1=a/b$ and $\beta_2=h/a$ are the parameters defining the dimensions of the plate, a and b are the plate's dimensions along the $x-$ and $y-$axes, respectively, h is the thickness, t is the time, an overdot denotes the time-derivative, lower indices label the derivatives with respect to the corresponding coordinates, $U^0(x,y)$, $V^0(x,y)$ and $W^0(x,y)$ are functions describing the distribution of the initial velocities of the points lying in the middle surface of the plate, ε is a small dimensionless parameter of the same order of magnitude as the amplitudes, F is the amplitude of the harmonic force with the frequency Ω_F, δ is the Dirac delta function, $æ_i=\varepsilon\mu_i\tau_i^\gamma$ $(i=1,2,3)$ are damping coefficients, μ_i are finite values, τ_i are the relaxation times, and D_{0+}^γ is the Riemann-Liouville fractional derivative of the γ–order [8].

The set of equations (1)-(3) admits the solution of the Navier type for the simply-supported plate (figure 1), which could be represented in terms of the time-dependent generalized

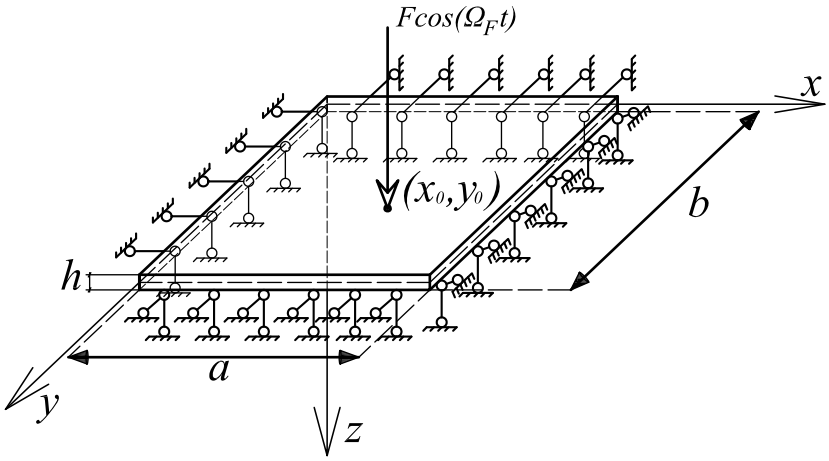

Figure 1. Scheme of a freely supported rectangular plate

displacements $x_{imn}(t)$ $(i = 1, 2, 3)$ and eigen-functions of the mn-th natural modes of linear vibrations with the natural frequencies ω_{imn} of the plate. The procedure resulting in decoupling linear parts of nonlinear equations of motion has been proposed in [5] with the further utilization of the generalized method of multiple scales [6] for solving nonlinear governing equations of motion, in so doing the amplitude functions are expanded into power series in terms of the small parameter and depend on different time scales with $T_0 = t$ as a fast time characterizing oscillatory motions with eigenfrequencies, and $T_1 = \varepsilon t$, $T_2 = \varepsilon^2 t$ as slow scales characterizing the modulation of amplitudes and phases of nonlinear vibrations.

It has been revealed [5] that nonlinear vibrations of the plate could be accompanied by different types of the internal resonance when two or more modes could be coupled. Moreover, its type depends on the order of smallness of the viscosity involved into consideration. Thus, at the ε^2-order, damped vibrations could be accompanied by the one-to-one, one-to-one-to-one, and combinational resonances of the additive and difference types .

In [5] it has been shown that the one-to-one internal resonance could be of two types: (1) when the natural frequencies of two modes of in-plane vibrations are close to each other, i.e., when $\omega_1 = \omega_2 = \omega$, and (2) when the natural frequency of the out-of-plane mode is close to the natural frequency of one of the in-plane modes, i.e., $\omega_3 = \omega_1 = \omega$ or $\omega_3 = \omega_2 = \omega$. From hereafter the indices mn are omitted for the ease of presentation.

2.1. One-to-one internal resonance, when $\omega_1 = \omega_2 = \omega$, but $\omega_3 \neq \omega$

Now let us consider the case of the one-to-one internal resonance $\omega_1 = \omega_2 = \omega$ when coupled with the external resonance $\omega_3 \approx \Omega_F \neq \omega$, in so doing the external force, following [7], is assumed to be $F \sin \pi x_0 \sin \pi y_0 = \varepsilon^3 f$.

Using the system of solvability equations to eliminate secular terms in the case of free vibrations [5], and adding the external resonance, we obtain the following solvability equations for the case of force driven vibrations:

$$2i\omega_1 D_2 A_1 + \mu_1 (i\omega_1 \tau_1)^\gamma A_1 + 2\zeta_1 \left[(k_5 + k_7) A_1 A_3 \bar{A}_3 + (k_6 + k_8) A_2 A_3 \bar{A}_3 \right] = 0, \quad (8)$$

$$2i\omega_2 D_2 A_2 + \mu_2 (i\omega_2 \tau_2)^\gamma A_2 + 2\zeta_2 \left[(k_6 + k_8) A_2 A_3 \bar{A}_3 + (k_5 + k_7) A_1 A_3 \bar{A}_3 \right] = 0, \quad (9)$$

$$2i\omega_3 D_2 A_3 + \mu_3 (i\omega_3 \tau_3)^\gamma A_3 + \zeta_{13} \left[(k_5 + k_7) A_1 \bar{A}_1 A_3 + (k_1 + 2k_2) A_3^2 \bar{A}_3 \right] + \zeta_{23} \left[(k_6 + k_8) A_2 \bar{A}_2 A_3 + (k_3 + 2k_4) A_3^2 \bar{A}_3 \right] + (\zeta_{13} k_6 + \zeta_{23} k_7) \bar{A}_1 A_2 A_3$$

$$+ \quad (\zeta_{13}k_8 + \zeta_{23}k_5)\, A_1\bar{A}_2A_3 - 2f = 0, \tag{10}$$

where $A_{j\,mn}(T_2)$ and $\bar{A}_{j\,mn}(T_2)(j = 1, 2, 3)$ are complex conjugate functions to be found, ζ_2, ζ_{13}, and ζ_{23} are coefficients depending on the plate dimensions and numbers of excited modes [5], while coefficients k_i $(i = 1, 2, ..., 8)$ have the form

$$k_1 = \frac{\zeta_1}{4\omega_3^2 - \omega_1^2}, \quad k_2 = -\frac{\zeta_1}{\omega_1^2}, \quad k_3 = \frac{\zeta_2}{4\omega_3^2 - \omega_2^2}, \quad k_4 = -\frac{\zeta_2}{\omega_2^2}, \quad k_5 = \frac{\zeta_{13}}{\omega_1(\omega_1 + 2\omega_3)},$$

$$k_6 = \frac{\zeta_{23}}{\omega_2(\omega_2 + 2\omega_3)}, \quad k_7 = \frac{\zeta_{13}}{\omega_1(\omega_1 - 2\omega_3)}, \quad k_8 = \frac{\zeta_{23}}{\omega_2(\omega_2 - 2\omega_3)}.$$

Let us multiply equations (8)–(10), respectively, by $\bar{A}_1$, $\bar{A}_2$, and $\bar{A}_3$ and find their complex conjugates. Adding every pair of the mutually adjoint equations with each other and subtracting one from another, and considering that the functions $A_i(T_2)$ could be represented in the polar form $a_i(T_2)e^{i\,\varphi_i(T_2)}$ $(i = 1, 2, 3)$, where a_i and φ_i are amplitudes and phases, as a result we arrive at a set of six coupled nonlinear differential equations

$$\left(a_1^2\right)^{\cdot} + s_1a_1^2 + 2\omega_1^{-1}\zeta_1\,(k_6 + k_8)\, a_1a_2a_3^2 \sin\delta = 0, \tag{11}$$

$$\left(a_2^2\right)^{\cdot} + s_2a_2^2 - 2\omega_2^{-1}\zeta_2\,(k_5 + k_7)\, a_1a_2a_3^2 \sin\delta = 0, \tag{12}$$

$$\left(a_3^2\right)^{\cdot} + s_3a_3^2 = -2f\omega_3^{-1}a_3 \sin\varphi_3, \tag{13}$$

$$\dot{\varphi}_1 - \frac{1}{2}\,\sigma_1 - \omega_1^{-1}\zeta_1\,(k_5 + k_7)\, a_3^2 \; - \omega_1^{-1}\zeta_1\,(k_6 + k_8)\, a_1^{-1}a_2a_3^2 \cos\delta = 0, \tag{14}$$

$$\dot{\varphi}_2 - \frac{1}{2}\,\sigma_2 - \omega_2^{-1}\zeta_2\,(k_6 + k_8)\, a_3^2 - \omega_2^{-1}\zeta_2\,(k_5 + k_7)\, a_2^{-1}a_1a_3^2 \cos\delta \; = 0, \tag{15}$$

$$\begin{aligned}\dot{\varphi}_3 &- \frac{1}{2}\,\sigma_3 - \frac{1}{2}\,\omega_3^{-1}\{\zeta_{13}\,(k_1 + 2k_2) + \zeta_{23}\,(k_3 + 2k_4)\}a_3^2 + f(\omega_3\, a_3)^{-1}\cos\varphi_3 \\ &- \frac{1}{2}\,\omega_3^{-1}\zeta_{23}\,(k_6 + k_8)\, a_2^2 - \frac{1}{2}\,\omega_3^{-1}\zeta_{13}\,(k_5 + k_7)\, a_1^2 = 0,\end{aligned} \tag{16}$$

where $\delta = \varphi_2 - \varphi_1$ is the phase difference, an overdot denotes the differentiation with respect to T_2, $s_i = \mu_i\tau_i^{\gamma}\omega_i^{\gamma-1}\sin\psi$, $\sigma_i = \mu_i\tau_i^{\gamma}\omega_i^{\gamma-1}\cos\psi$, and $\psi = \pi\gamma/2$ $(i = 1, 2, 3)$.

2.2. The case of free damped vibrations

In the absence of the external force equations (13) and (16) are reduced to

$$\left(a_3^2\right)^{\cdot} + s_3a_3^2 = 0, \tag{17}$$

$$\begin{aligned}\dot{\varphi}_3 &= \frac{1}{2}\,\sigma_3 + \frac{1}{2}\,\omega_3^{-1}\left[\zeta_{13}\,(k_1 + 2k_2) + \zeta_{23}\,(k_3 + 2k_4)\right]a_3^2 \\ &+ \frac{1}{2}\,\omega_3^{-1}\zeta_{23}\,(k_6 + k_8)\, a_2^2 - \frac{1}{2}\,\omega_3^{-1}\zeta_{13}\,(k_5 + k_7)\, a_1^2.\end{aligned} \tag{18}$$

From equation (17) it is seen that it is independent of all other five equations, and its solution has the form

$$a_3^2 = c_3e^{-s_3T_2}, \tag{19}$$

where c_3 is a constant of integration to be determined from the initial conditions, while reference to equation (18) shows that the phase φ_3 depends only on squared amplitudes of all three interacting modes of vibrations.

Introducing new functions $\xi_1(T_2)$ and $\xi_2(T_2)$ such that

$$a_1^2 = \frac{\zeta_1\,(k_6+k_8)}{\omega}\,\xi_1\exp(-s_1T_2), \quad a_2^2 = \frac{\zeta_2\,(k_5+k_7)}{\omega}\,\xi_2\exp(-s_2T_2), \tag{20}$$

and adding equations (11) and (12) with due account for (20) yield

$$\dot{\xi}_2 + \dot{\xi}_1 e^{(s_2-s_1)T_2} = 0, \tag{21}$$

while subtracting (14) from (15) we obtain

$$\begin{aligned}\dot{\delta} = \dot{\varphi}_2 - \dot{\varphi}_1 = \Sigma + \omega^{-1}\left[\zeta_2\,(k_6+k_8) - \zeta_1\,(k_5+k_7)\right]a_3^2 \\ +\omega^{-1}\left[\zeta_2\,(k_5+k_7)\,a_1^2 - \zeta_1\,(k_6+k_8)\,a_2^2\right]\frac{a_3^2}{a_1a_2}\,\cos\delta,\end{aligned} \tag{22}$$

where $\Sigma = \frac{1}{2}(\sigma_2 - \sigma_1)$.

Considering (20), equation (22) could be reduced to

$$\dot{\delta} - \Sigma = Ke^{-s_3T_2} + \frac{1}{2}\left(\frac{\dot{\xi}_1}{\xi_1} + \frac{\dot{\xi}_2}{\xi_2}\right)\cot\delta, \tag{23}$$

where $K = \omega^{-1}\left[\zeta_2\,(k_6+k_8) - \zeta_1\,(k_5+k_7)\right]c_3$.

2.3. The case $K = 0$

For this case at $\Sigma = 0$, the first integral of the set of equations (11)–(16) has been found in [5] in a form of the stream-function $G(\xi,\delta)$

$$G_1(\xi,\delta) = \xi^{1/2}(1-\xi)^{1/2}\cos\delta = G_1^0(\xi_0,\delta_0). \tag{24}$$

where ξ_0 and δ_0 are the initial magnitudes of the relative amplitude and phase difference, respectively.

The stream-lines of the phase fluid in the phase plane $\xi - \delta$ are presented in figure (2). Magnitudes of G are indicated by digits near the curves which correspond to the stream-lines; the flow direction of the phase fluid elements are shown by arrows on the stream-lines.

Reference to figure (2) shows that the phase fluid flows within the circulation zones, which tend to be located around the perimeter of the rectangles bounded by the lines $\xi = 0$, $\xi = 1$, and $\delta = \pm(\pi/2) \pm 2\pi n$ $(n = 0, 1, 2, ...)$. As this takes place, the flow in each such rectangle becomes isolated. On all four rectangle sides $G = 0$ and inside it the value G preserves its sign. The function G attains its extreme magnitudes ± 0.5 at the points with the coordinates $\xi = 0.5$, $\delta = \pm\pi n$ $(n = 0, 1, 2, ...)$.

The transition of fluid elements from the points $\xi = 0$, $\delta = \pi/2 \pm 2\pi n$ to the points $\xi = 0$, $\delta = -\pi/2 \pm 2\pi n$ $(n = 0, 1, 2...)$ and from the points $\xi = 1$, $\delta = -\pi/2 \pm 2\pi n$ to the points $\xi = 1$, $\delta = \pi/2 \pm 2\pi n$ $(n = 0, 1, 2...)$ proceeds instantly, because according to the distribution of the phase velocity $\mathbf{v}$ along the section $\delta = 0$ its magnitude tends to infinity as $\xi \to 0$ and $\xi \to 1$, respectively, [5].

The transition of fluid elements from the points $\xi = 1$, $\delta = -\pi/2 \pm 2\pi n$ to the points $\xi = 1$, $\delta = \pi/2 \pm 2\pi n$ $(n = 0, 1, 2...)$ proceeds instantly as well, because according to the distribution of the phase velocity $\mathbf{v}$ along the section $\delta = 0$ its magnitude [5] tends to infinity as $\xi \to 1$.

Along the lines $\delta = \pm(\pi/2) \pm 2\pi n$ $(n = 0, 1, 2, ...)$ pure amplitude modulated aperiodic motions are realized, since with an increase in time t the value ξ increases from ξ_0 to 1 (along

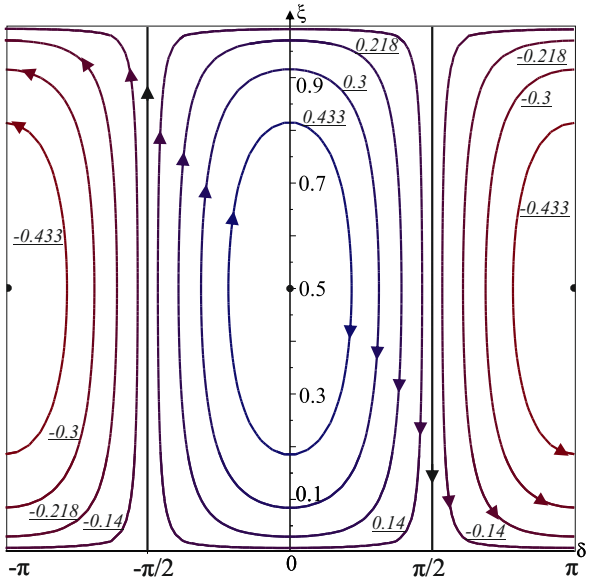

Figure 2. Phase portrait for the case of 1:1 internal resonance $\omega_1 = \omega_2$ at $K = 0$

the line $\delta = -\pi/2$) or decreases from ξ_0 to 0 (along the line $\delta = \pi/2$), and the solution could be written in the form of a soliton-like [5].

Stream-lines give a pictorial estimate of the connection of G with all types of the energy-exchange mechanism. Thus, in the case of undamped vibrations, i.e., when the damping coefficient is equal to zero and $s = 0$, the points with the coordinates $\xi_0 = 1/2$, $\delta_0 = \pm\pi n$ $(n = 0, 1, 2, ...)$ correspond to the stationary regime, since $\dot{\xi} = 0$ and $\dot{\delta} = 0$. The stationary points $\xi_0 = 1/2$, $\delta_0 = \pm\pi n$ are centers, as with a small deviation from a center, a phase element begins to move around the stationary point along a closed trajectory. Closed stream-lines correspond to the periodic change of both amplitudes and phases.

3. Numerical investigations

Equations (11)–(16) have been solved numerically using the Runge-Kutta fourth-order algorithm. The results calculated for the cases of free ($f = 0$) and force driven ($f \neq 0$) vibrations are presented in figure 3 for different magnitudes of the fractional parameter γ: 0 (the case of undamped vibrations), 0.25 and 0.5 (fractionally damped vibrations).

From figure 3, wherein the T_2-dependence of the amplitudes a_1 (dashed lines) and a_2 (solid lines) is shown for the initial amplitudes $a_{i0} = 0.5$, it is seen that the presence of small viscosity results in damping of the energy exchange between two coupled modes of in-plane vibrations due to the one-to-one internal resonance $\omega_{1m_1n_1} \approx \omega_{2m_2n_2}$ and in the increase of the energy cycle with time. In so doing the damping character of the energy interchange is amplified with the increase in the fractional parameter. The points of tangency of the envelopes in figure 3a allows one to calculate the value characterizing the energy decay as $\ln(a_{i+1}/a_i) / (T_{i+1} - T_i)$.

Reference to figures 3b-d shows the influence of the external force on the amplitudes of vibrations. It is evident that even in the case under consideration when the frequency of the external vertical harmonic force is close to one of the frequencies of plate's vertical vibrations $\omega_F \approx \omega_3$ which is not involved in the internal resonance $\omega_1 \approx \omega_2$, the external force increases the amplitudes of plate's in-plane vibrations which are coupled due to the internal resonance between the frequencies of horizontal modes. With the increase of the magnitude of force amplitude, the

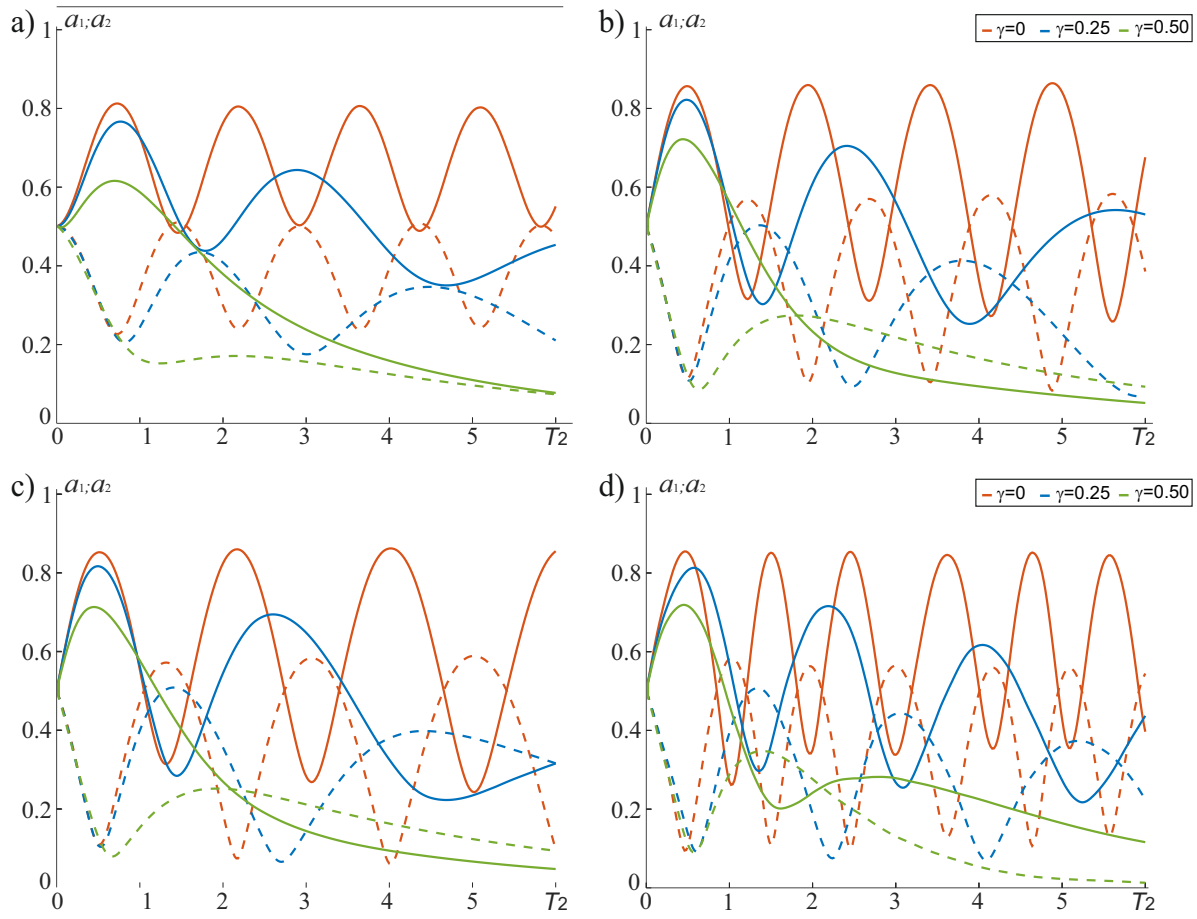

Figure 3. The T_2-dependence of the amplitudes a_1 (dashed lines) and a_2 (solid lines) at $\nu = 0.27$, $\beta_1 = 4.75$, $\beta_2 = 0.067$, $m_1 = m_2 = m_3 = 1$, $n_1 = n_3 = 3$, $n_2 = 5$, $\omega_{1m_1n_1} = 44.88$, $\omega_{2m_2n_2} = 45.11$, and $\omega_{3m_3n_3} \approx \Omega_F = 22.45$: (a) free vibrations, and forced vibrations (b) $f = 3$, (c) $f = 10^3$, (d) $f = 5 \cdot 10^3$

energy exchange process has been amplified, and the periods of energy interchange cycles have been decreased.

The case of stationary vibrations corresponding to the stationary center-like point with the coordinates $\xi_0 = 0.5$, $\delta_0 = 0$ in the phase portrait (figure 2) is shown in figure 4 for the initial amplitudes $a_{10} = 0.05$ and $a_{20} = 0.03$. In the case of free vibrations (figure 4a), there is no energy exchange between the modes. However, the presence of the small external vertical harmonic force initiates the process of energy exchange between the modes of in-plane vibrations what is seen in figure 4b.

4. Conclusions

In the present paper, nonlinear force driven vibrations of thin plates in a viscoelastic medium have been studied, when the motion of the plate is described by a set of three coupled nonlinear differential equations subjected to the condition of the one-to-one internal resonance accompanied by the external resonance, resulting in the interaction of three modes corresponding to the mutually orthogonal displacements. Nonlinear sets of resolving equations in terms of amplitudes and phase differences have been obtained. The influence of viscosity and external

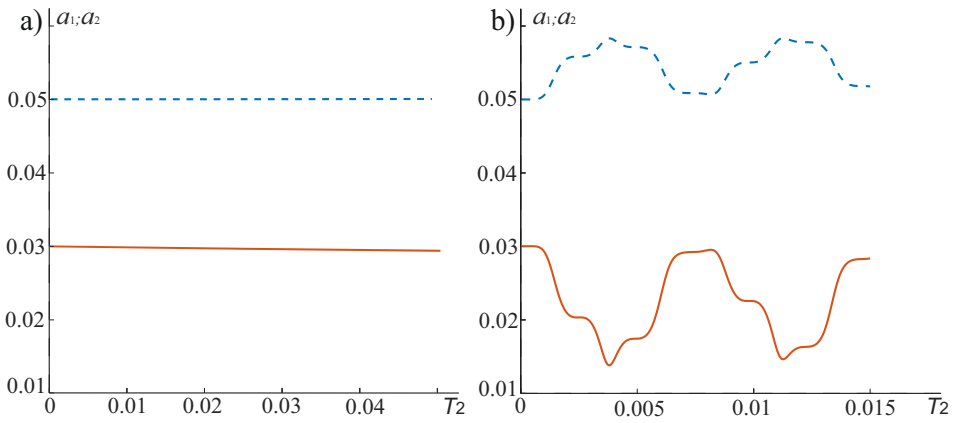

Figure 4. Steady-case vibrations (a) free vibrations, and (b) forced vibrations, at $\nu = 0.27$, $f = 2 \cdot 10^4$, $\beta_1 = 2.75$, and $\beta_2 = 0.0386$, $m_1 = m_2 = m_3 = 1$, $n_1 = n_3 = 3$, $n_2 = 5$, $\gamma = 0$, $\omega_{1m_1n_1} = 26.108$, $\omega_{2m_2n_2} = 26,166$, and $\omega_3 \approx \Omega_F = 7.60$: $a_2\,(T_2)$ - solid line, $a_1\,(T_2)$ - dashed line

vertical harmonic force on the energy exchange mechanism between the coupled modes has been analyzed.

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