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Stress-strains state calculation of a rod at uniaxial tension with non-local effects

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Abstract. A numerical calculation of a uniaxial extension of a rod is carried out in terms of the nonlocal integral theory of elasticity using the finite element method for various parameters of the material, taking into account its microstructure, and contributions of nonlocal effects. The dependencies of the strain energy are presented with increasing number of finite elements, which confirm the convergence and correctness of the mathematical model and method.

1. Introduction

The creation of new structurally sensitive materials based on nanotechnology is an important direction in the development of modern materials science. Consolidated structurally sensitive materials are usually obtained by compacting nanopowders, deposition on the substrate, crystallization of amorphous alloys and other methods. Such materials have unique physicomechanical properties that allow them to be used effectively in structures exposed to high-intensity external influences [1, 2].

An important step in the creation and usage of new structurally sensitive materials is the construction of mathematical models that allow describing their behavior in a wide range of changes in external influences.

Currently, there are at least three main approaches to the theoretical modeling of materials with a complex internal structure: 1) quantum mechanical approaches, molecular dynamics methods and Monte Carlo models [3, 4]; 2) generalized continuum mechanics approaches that include theories of effective properties [5 — 8], mixture models [9, 10], models based on averaging procedures [11], Cosserat-type micropolar media [4, 5] and statistical models [12]; 3) hybrid molecular-mechanical approach combining molecular dynamics methods and macroscale models [3, 13].

It is known that the usage of methods of continuum mechanics for materials with a microstructure is limited to scale and boundary effects. Direct application of continuum mechanics methods for materials modified by micro- and nanostructured inclusions is incorrect [5, 14 — 16]. For these reasons, there is an interesting theory where, on the one hand, is taken into account the presence of the microstructure, and on the other hand the equations have the form of the usual equations of continuum mechanics, integro-differential in the general case, for solutions which are applied the methods of continuum mechanics. Approaches of classical continuum mechanics media with micro- and nanostructure is called the method of continuous approximation [14, 15]. The field of science in which the behavior of materials with micro- and nanostructures is studied using the method of continuous approximation is sometimes called generalized continuum mechanics. The key points in this method are the establishment of connections between the characteristics of the micro (nano-) level and the macro level, as well as taking into account the effects of spatial and temporal non-locality of the media.

2. Mathematical model

Let us consider a mathematical model of a uniaxial tension of a rod (fig. 1) with a length L, a cross-sectional area S and a concentrated force on the P right-hand side.

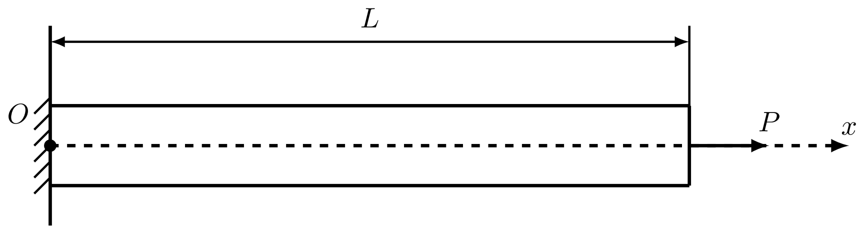

Figure 1. Uniaxial tension of a rod.

The formulation of the boundary value problem of the mechanics of a deformed body in terms of small deformations has the form [11]:

$$\frac{\partial\sigma(x)}{\partial x} + b(x) = 0,\ u(0) = 0,\ \sigma(L) = \frac{P}{S} \tag{1}$$

where $b(x)$ is the force distributed over the volume, $\sigma(x)$ and $u(x)$ are the stress and displacement along the axis Ox. However, according to the non-local theory of elasticity [12], the relation between stresses and strains can be represented as

$$\sigma(x) = p_1 E\varepsilon(x) + p_2 \int\limits_0^L \varphi(|x' - x|)E\varepsilon(x')dx', \tag{2}$$

where $\varepsilon(x)$ — the strains along the axis Ox, E — the Young modulus, p_1 and p_2 are the proportions of the influence of local and non-local effects such as $p_1 + p_2 = 1$; $\varphi(|x' - x|)$ is an influence function that determines spatial nonlocality, while

$$\int\limits_{-\infty}^{+\infty} \varphi(|x' - x|)dx' = 1.$$

The influence function was used by A.C. Eringen [4, 5, 14, 15] to solve problems in the theory of elasticity and based on the idea that long-range forces that responsible for the non-local deformation of the material at a given point of the space x are adequately described using the decreasing distance function $\varphi(|x' - x|)$, while increasing $|x' - x|$.

As shown in [18], problem (1) can be reformulated into the Fredholm integral equation of the second kind with respect to strains, which does not have a strict analytical solution in the general case.

3. Nonlocal FEM-formulation

From the varational approach, the solution of problem (1) is the equivalent to minimizing a functional, which, in its physical sense, is the total potential energy of a loaded rod [11]:

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$$\Pi = \frac{1}{2}\int\limits_V \varepsilon\sigma d\mathrm{v} - \int\limits_V ub d\mathrm{v} - \int\limits_S uP ds, \tag{3}$$

where V is the volume of the rod and S is the cross-sectional area of the right side. Using the discretization procedure, this problem can be solved in terms of isoparametric finite elements in displacements [13], then the original functional (3) can be rewritten in view of the relations in the form of FEM

$$\Pi[\mathbf{u}] = \frac{1}{2}p_1\sum_{n=1}^{N_e}\mathbf{u}_n^T\mathbf{K}_n^{loc}\mathbf{u}_n + \frac{1}{2}p_2\sum_{n=1}^{N_e}\sum_{m=1}^{N_e}\mathbf{u}_n^T\mathbf{K}_{nm}^{nl}\mathbf{u}_m - \sum_{n=1}^{N_e}\mathbf{u}_n^T(\mathbf{f}_n^{(b)} + \mathbf{f}_n^{(P)}), \tag{4}$$

where N_e is the number of elements, $\mathbf{u}_n$ and $\mathbf{u}_m$ are the displacement field in the element with number n and m, $\mathbf{K}_n^{loc}$ is the stiffness matrix of the element with the number n in terms of the classical theory of elasticity, $\mathbf{K}_{nm}^{nl}$ is matrix of non-local effects [27] that represents the influence of the m-th element on the n-th one (fig. 2), $\mathbf{f}_n^{(b)}$ and $\mathbf{f}_n^{(P)}$ is vector-columns of forces distributed over the volume and distributed load over the surface of the element with the number n. The relations included in (2) can be represented as

$$\mathbf{K}_n^{loc} = \int\limits_{V_n}\mathbf{B}_n^T E\mathbf{B}_n S dx, \qquad \mathbf{K}_{nm}^{nl} = \int\limits_{V_n}\int\limits_{V_m}\varphi(|x'-x|)\mathbf{B}_n^T E\mathbf{B}_m S dx' dx, \tag{5}$$

$$\mathbf{f}_n^{(b)} = \int\limits_{V_n}\mathbf{N}_n^T b S dx, \qquad \mathbf{f}^{(P)} = \int\limits_S \mathbf{N}_n^T P dx, \tag{6}$$

where $\mathbf{N}_n$ is the matrix of shape functions on the element with number n, $\mathbf{B}_n$ and $\mathbf{B}_m$ are the matrices of connection strains and stresses of the elements with numbers n and m, containing derivatives of the shape functions, V_n is the length of the element with number n.

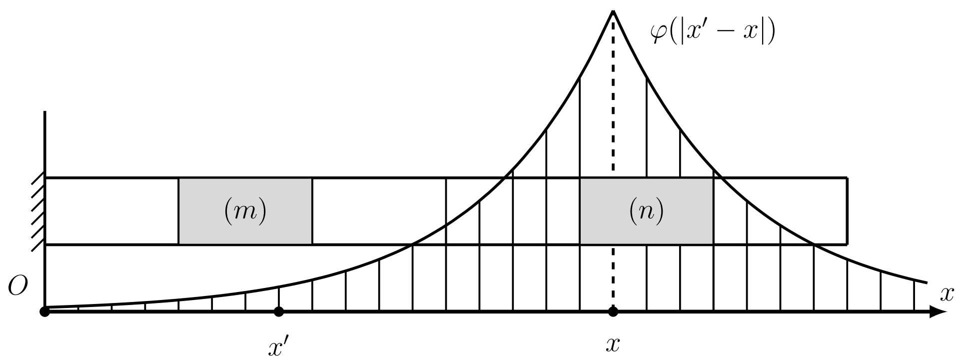

Figure 2. Influence of field of the strains at the element with a number m at the field of the strains at the element with number n.

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A necessary condition for the extremum of the functional (4) is the fulfillment of the relation $\frac{\partial \Pi}{\partial \mathbf{U}} = 0$ [13], where $\mathbf{U}$ is the vector of nodal displacements on the entire rod. Using this, one can obtain a system of linear algebraic equations for displacements with help of (5) and (6)

$$\mathbf{KU} = \mathbf{F},$$

where $\mathbf{K} = \sum_{n=1}^{N_e} \left(p_1 \mathbf{K}_n^{loc} + p_2 \sum_{m=1}^{N_e} \mathbf{K}_{nm}^{nl} \right)$ and $\mathbf{F} = \sum_{n=1}^{N_e} \left(\mathbf{f}_n^{(b)} + \mathbf{f}_n^{(P)} \right)$. The global stiffness matrix $\mathbf{K}$ would be completely filled, whereas in terms of the classical theory of elasticity it would have a three-diagonal structure, and the global vector of the right side $\mathbf{F}$ would have one single nonzero component corresponding to the node number coinciding with the right side of the rod.

Like the matrix $\mathbf{K}_n^{loc}$, the matrix of nonlocal effects $\mathbf{K}_{nm}^{loc}$ is calculated with isoparametric relations taking into account

$$\mathbf{K}_{nm}^{nl} = \int_{-1}^{1} \int_{-1}^{1} \varphi(|x'(\eta) - x(\xi)|) \mathbf{B}_n^T E \mathbf{B}_m S \det \mathbf{J} d\xi d\eta,$$

where ξ and η is local coordinates, $\mathbf{J}$ is the Jacobi matrix of the derivatives connection in the local coordinate system with the derivatives in global coordinate system, while $\mathbf{B}_n = \mathbf{B}_n\left(x(\xi)\right)$ and $\mathbf{B}_m = \mathbf{B}_m\left(x'(\eta)\right)$.

Similarly with the classical theory of elasticity, as shown in [13], the strain energy (7) in terms of the non-local theory of elasticity is finite in its physical meaning

$$W = \frac{1}{2} \int_V \varepsilon \sigma d\mathrm{v}. \tag{7}$$

At the same time, turning to a finite element formulation, the strain energy should tend to its limit as the number of elements increases and the number of their characteristic size decreases, that is $\lim_{h \to 0} W_h = W$, where W_h is strain energy, computing at the current amount of elements, which can be represented as

$$W_h = \frac{1}{2} \mathbf{u}^T \mathbf{K} \mathbf{u},$$

h is the maximum characteristic size of a finite element, which in the one-dimensional case is the length of the element.

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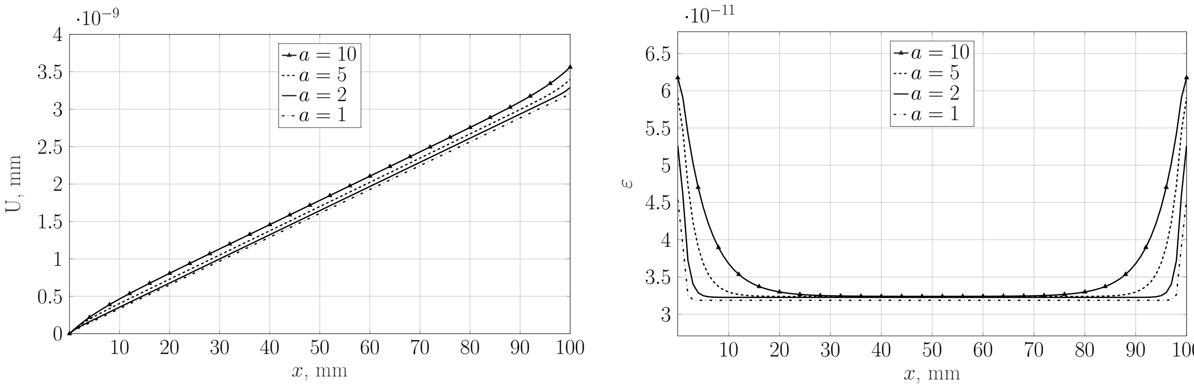
4. Numerical results

The numerical experiment was carried out without influence of volume forces and with the following values of the material parameters: $L = 100$ mm, $S = 10$ mm^2, $E = 2.1 \cdot 10^6$ MPa and $P = 10$ N. The influence function selected for calculations was looked like

$$\varphi(|x' - x|) = \frac{1}{2a} \exp\left(\frac{|x' - x|}{a}\right),$$

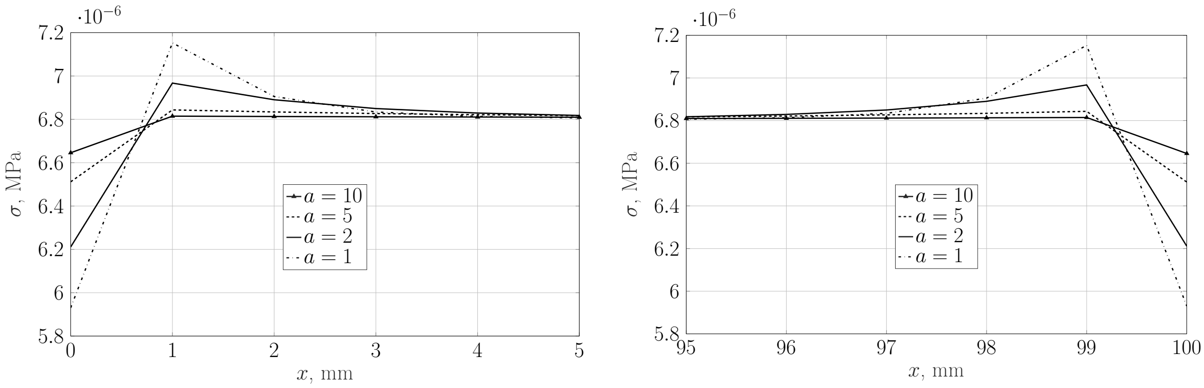
where a is the characteristic of the material, which describes its microstructure. To calculate the matrices $\mathbf{K}_n^{loc}$ and $\mathbf{K}_n^{nl}$ numerical integration with Gauss quadratures over two points was used. Each calculation used linear finite elements of the same length in the amount of 100 elements.



a) b)

Figure 3. Various nodal distributions depending on parameter a with non-local contribution parameter $p_1 = 0.25$: a) displacements; b) strains.

Figure 3 shows the dependencies of the distributions of displacements, strains, and stresses for various parameters a and the contribution parameter $p_1 = 0.25$. The dependencies of strains and stresses are of the greatest interest, since there is a sharp increase and change in gradients near the boundary of a rod.



a) b)

Figure 4. Various nodal stress distributions depending on parameter a with non-local contribution parameter $p_1 = 0.25$: a) at the left side of a rod; b) at the right side of a rod.

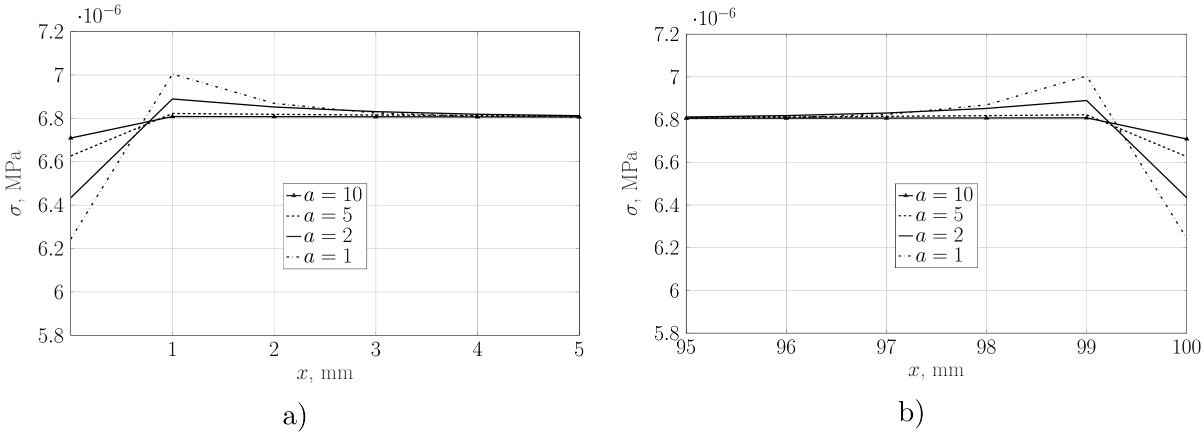


Figure 5. Various nodal stress distributions depending on parameter a with non-local contribution parameter $p_1 = 0.5$: a) at the left side of a rod; b) at the right side of a rod.

The plots of the stress distribution at different values of the parameters a and p_1 are divided into two parts (fig. 4 – 5), since they differ slightly at the points far from the borders, but a sharp drop in stress is noticeable near the boundaries; in this case, the larger value of the parameter a, the smaller drop in stresses near the rod boundaries, whereas for strains (fig. 3b) the opposite is true.

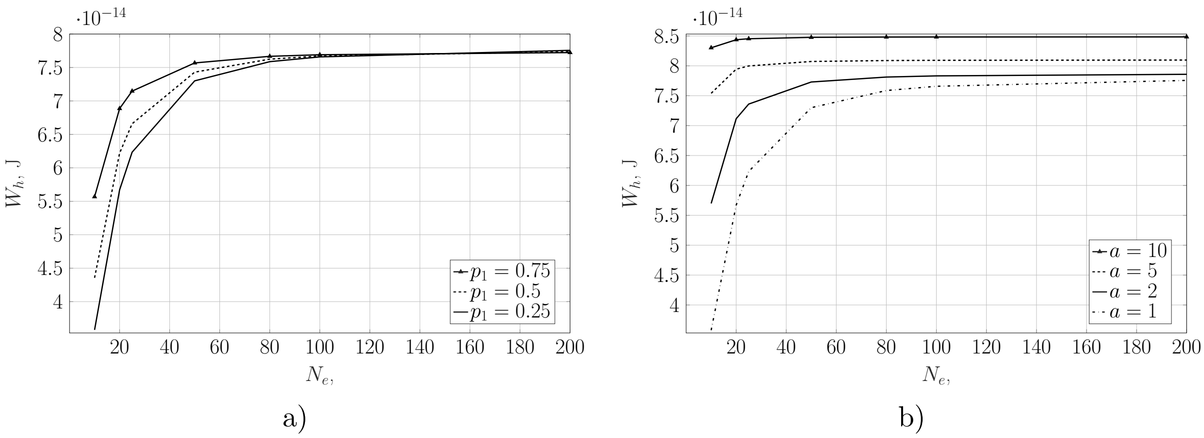


Figure 6. Strain energy by increasing elements amount: a) depending on parameter p_1, when $a = 1$; b) depending on parameter a, when $p_1 = 0.25$.

As can be seen from the plots of the strain energy depending on the number of elements for various values of parameter a and various values of parameter of non-local effects p_1 (fig. 6), there is a convergence of numerical results confirmed.

5. Conclusion

Calculations of the stress-strain state of the rod at uniaxial tension, with non-local effects taking into account, showed: 1) in terms of the non-local integral theory of elasticity, a numerical solution obtained by the finite element method tends to its limit; 2) by increasing the value of parameter a, which determines the microstructure of the rod's material, leads to increasing strains in the zone of growth near the sides of the rod with increasing the maximum values of the strains; 3) by increasing the value of parameter a, the maximum displacement of the rod grows slightly; 4) increasing the value of parameter a, leads to increasing strains drop in the region of the sides of the rod while decreasing the maximum stress values; 5) an increase in the share of the contribution of non-local effects does not have a significant effect on the strain energy of the entire rod.

6. Acknowledgements

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