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Mathematical model of the impact response of a linear viscoelastic auxetic plate

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Abstract. The aim of the present paper is to construct a mathematical model describing the impact of a rigid sphere upon a linear Kirchhoff-Love plate, made of viscoelastic auxetic with fractional viscosity. Auxetic's Poisson's ratio is a time-dependent value changing from negative to positive magnitudes with time. In the case of a linear plate, the solution out of the contact domain is found through the Green function, and within the contact zone via the modified Hertz contact theory. Integral equations for the contact force and local indentation have been obtained.

1. Introduction

Nowadays material engineers are looking for methods rising physical and mechanical characteristics of traditional materials via creating the structures possessing essential abnormal deformational features. As demonstrated by experimental results, materials with negative Poisson's ratio (auxetics), belong to such abnormal materials. When stretched, they become thicker perpendicular to the applied force [1]. This occurs due to their particular internal structure. Auxetics present enhanced mechanical properties, which make them superior to classical materials for many practical applications.

There are a lot of papers devoted to auxetic materials, however, the majority of them deal with the internal structure of auxetics, experimental determination of Poisson's ratios, as well as with the description of features of different auxetics [1-5]. Papers discussing the mathematical models describing the behavior of viscoelastic auxetics are rare, and there are practically no studies devoted to the solution of boundary-value dynamic problems with such materials [6,7]. With the accelerated progress in studies of auxetics made in last few years, there are new potential opportunities for researchers. Auxetic materials are still in developing stage and require intensive research [8,9].

In the present paper, a mathematical model describing the impact of a rigid sphere upon a linear viscoelastic plate is suggested, in doing so the Poisson's ratio is considered to be a time-dependent operator described by the fractional derivative model.

2. Problem formulation and governing equations

Let us consider a rigid spherical impactor of mass m and radius R moving along the z -axis with the velocity V_0 towards a simply-supported linear Kirchhoff-Love plate with the dimensions a and b . The origin of the Cartesian coordinate system x, y is at the plate's center. The plate's viscoelastic



properties are described by the Kelvin-Voigt model with fractional derivatives [10]. The impact occurs at the moment $t = 0$ at the point N with Cartesian coordinates x_0, y_0 .

In this case, the equation of motion of the spherical impactor is

$$m\ddot{z} = -P(t), \quad (1)$$

subjected to the initial conditions

$$\dot{z}(0) = V_0, \quad z(0) = 0, \quad (2)$$

where $P(t)$ is the contact force, $z(t)$ is the displacement of the spherical impactor, and over dots denote time-derivatives.

The equation of motion of a simply-supported viscoelastic plate has the form

$$\frac{\tilde{E}h^2}{12(1-\tilde{\nu}^2)} \left(\frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} \right) = -\rho \ddot{w} + h^{-1} P(t) \delta(x-x_0) \delta(y-y_0), \quad (3)$$

where \tilde{E} and $\tilde{\nu}$ are, respectively, time-dependent Young's operator and Poisson's ratio operator, h , $w(x, y, t)$ and ρ are the plate's thickness, deflection and density, respectively, $\delta(x-x_0)$ and $\delta(y-y_0)$ are the Dirac delta-functions.

Equation (3) is subjected to the following boundary and initial conditions:

$$w = 0 \quad \text{at} \quad x = 0, a \quad \text{and} \quad y = 0, b. \quad (4)$$

$$\dot{w}|_{t=0} = 0. \quad (5)$$

The values $z(t)$ and $w(x_0, y_0, t)$ are coupled as follows

$$z(t) = \alpha(t) + w(x_0, y_0, t), \quad (6)$$

where $\alpha(t)$ is the value characterizing the local bearing of the plate's material within the contact domain, which is connected with the contact force by the generalized Hertzian law

$$P(t) = \tilde{k} [\alpha(t)]^{3/2}, \quad (7)$$

where $\tilde{k} = \frac{4\sqrt{R}}{3} \tilde{d}$, and $\tilde{d} = \frac{\tilde{E}}{1-\tilde{\nu}^2}$.

3. Method of solution

The solution to equation (3) satisfying the initial conditions (4) has the form

$$w(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \xi_{mn}(t) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}, \quad (8)$$

where $\xi_{mn}(t)$ are the generalized displacements, which are yet unknown functions.

Substituting (8) in equation (3) and considering the orthogonality of $\sin \frac{m\pi x}{a}$ within the segment

$[0, a]$ and $\sin \frac{n\pi y}{b}$ within the segment $[0, b]$, as a result we obtain

$$\ddot{\xi}_{mn}(t) + \frac{h^2 \pi^4}{12\rho} \frac{\tilde{E}}{(1-\tilde{\nu}^2)} \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right)^2 \xi_{mn}(t) = P(t) F_{mn}, \quad (9)$$

where $F_{mn} = \frac{4}{\rho h a b} \sin \frac{m\pi x_0}{a} \sin \frac{n\pi y_0}{b}$.

In order to find the equation for the Green function $G_{mn}(t)$, we have to substitute the function $P(t)$ in equation (9) with the Dirac delta-function $\delta(x)$, i.e.,

$$\ddot{G}_{mn}(t) + \frac{h^2 \pi^4}{12\rho} \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right)^2 \frac{\tilde{E}}{1-\tilde{\nu}^2} G_{mn}(t) = \delta(t) F_{mn}. \quad (10)$$

We also have to take into consideration that the bulk extension-compression operator \tilde{K} is assumed to be time-independent, i.e., volumetric relaxation is neglected (this assumption is due to the fact that for many viscoelastic materials volumetric relaxation is much smaller than the shear relaxation [11]), i.e.

$$\tilde{K} = K_0,$$

resulting in

$$\frac{\tilde{E}}{1-2\tilde{\nu}} = 3K_0, \quad (11)$$

where $K_0 = \text{const}$ is the bulk modulus.

In addition we have to assume that the shear operator $\tilde{\mu}$ is preassigned using the fractional derivative Kelvin-Voigt model [10]

$$\tilde{\mu} = \mu_0 \left[1 + \left(\tau_\sigma^\mu \right)^\gamma D^\gamma \right], \quad (12)$$

where μ_0 is the relaxed shear modulus, τ_σ^μ is the retardation time during shear deformations, γ ($0 < \gamma \leq 1$) is the fractional parameter,

$$D^\gamma x(t) = \frac{d}{dt} \int_0^t \frac{x(t') dt'}{\Gamma(1-\gamma)(t-t')^\gamma} \quad (13)$$

is the Riemann-Liouville fractional derivative [12], $\Gamma(1-\gamma)$ is the Gamma-function, and $x(t)$ is a certain function.

To find the Poisson's operator, first it is necessary to find the operator \tilde{E} from formula (11). For this purpose using the Volterra correspondence principle, the following formula could be utilized:

$$\tilde{E} = \frac{9K_0 \tilde{\mu}}{3K_0 + \tilde{\mu}}. \quad (14)$$

First using formula (12), we write the operator

$$3K_0 + \tilde{\mu} = (3K_0 + \mu_0)(1 + t_\sigma^\gamma D^\gamma), \quad (15)$$

where $t_\sigma^\gamma = \mu_0 (\tau_\sigma^\mu)^\gamma (3K_0 + \mu_0)^{-1}$.

Then we find the operator reverse to (15), i.e.,

$$(3K_0 + \tilde{\mu})^{-1} = (3K_0 + \mu_0)^{-1} \mathcal{D}_\gamma^*(t_\sigma^\gamma), \quad (16)$$

where $\mathcal{D}_\gamma^*(t_\sigma^\gamma) = (1 + t_\sigma^\gamma D^\gamma)^{-1}$ is the dimensionless Rabotnov's fractional operator [13].

Substituting now (12) and (16) in formula (14) and considering [13]

$$t_\sigma^\gamma D^\gamma \mathcal{D}_\gamma^*(t_\sigma^\gamma) = 1 - \mathcal{D}_\gamma^*(t_\sigma^\gamma), \quad (17)$$

we have

$$\tilde{E} = 9K_0 \left[1 - \frac{E_0}{3\mu_0} \mathcal{D}_\gamma^*(t_\sigma^\gamma) \right], \quad (18)$$

where $E_0 = \frac{9K_0 \mu_0}{3K_0 + \mu_0}$.

Now we can calculate Poisson's operator $\tilde{\nu}$ from formula (11), i.e.,

$$\tilde{\nu} = -1 + \frac{E_0}{2\mu_0} \mathfrak{D}_\gamma^*(t_\sigma^\gamma). \quad (19)$$

When operator $\tilde{\nu}$ will act on the unit Heaviside function $\tilde{\nu}H(t)$, as a result we obtain

$$\nu(t) = -H(t) + \frac{E_0}{2\mu_0} \mathfrak{D}_\gamma^*(t_\sigma^\gamma)H(t). \quad (20)$$

Considering that

$$\lim_{t \rightarrow \infty} \mathfrak{D}_\gamma^*(t_\sigma^\gamma)H(t) = 1,$$

from equation (20) we have

$$\nu(0) = -1, \quad \nu(\infty) = \frac{E_0 - 2\mu_0}{2\mu_0} = \nu_0. \quad (21)$$

From formulas (21) it is evident that the proposed model (12) with $\tilde{K} = K_0 = \text{const}$ describes the behavior of a viscoelastic auxetic material. In so doing, the Poisson's ratio could vary from -1 to its relaxed magnitude ν_0 .

Knowing the form of operators \tilde{E} (18) and $\tilde{\nu}$ (19), we could calculate the operator of rigidity

$$\tilde{d} = \frac{\tilde{E}}{2} \left(\frac{1}{1 + \tilde{\nu}} + \frac{1}{1 - \tilde{\nu}} \right) = \xi + \eta D^\gamma + \varsigma \mathfrak{D}_\gamma^*(T^\gamma), \quad (22)$$

where

$$\xi = \frac{1}{4}(9\lambda_0 + 10\mu_0), \quad \eta = 3(\lambda_0 + \mu_0)t_\sigma^\gamma, \quad \varsigma = -\frac{1}{4} \frac{(3\lambda_0 + 2\mu_0)^2}{\lambda_0 + \mu_0}, \quad T^\gamma = \frac{4(\lambda_0 + \mu_0)}{\lambda_0 + 2\mu_0} t_\sigma^\gamma.$$

The operator \tilde{d}^{-1} , which is reverse to the operator \tilde{d} , could be represented in the following form:

$$\tilde{d}^{-1} = a_1 \mathfrak{D}_\gamma^*(\tau_1^\gamma) + a_2 \mathfrak{D}_\gamma^*(\tau_2^\gamma), \quad (23)$$

where the unknown constants a_1 , a_2 , τ_1 , and τ_2 could be found by utilizing the property of the reciprocal operators $\tilde{d} \tilde{d}^{-1} = \tilde{d}^{-1} \tilde{d} = 1$, resulting in the equation for defining the constants

$$1 = [a_1 \mathfrak{D}_\gamma^*(\tau_1^\gamma) + a_2 \mathfrak{D}_\gamma^*(\tau_2^\gamma)] [\xi + \eta D^\gamma + \varsigma \mathfrak{D}_\gamma^*(T^\gamma)], \quad (24)$$

the solution of which could be found using the algebra of the dimensionless Rabotnov's operators [13].

Knowing the reciprocal rigidity operators (22) and (23), the following integral equations could be obtained: for the determination of the contact force $P(t)$

$$\begin{aligned} -\frac{1}{m} \int_0^t P(t')(t-t')dt' + V_0 t &= \left\{ \frac{3}{4\sqrt{R}} \int_0^t \left[a_1 \mathfrak{D}_\gamma \left(-\frac{t-t'}{\tau_1} \right) + a_2 \mathfrak{D}_\gamma \left(-\frac{t-t'}{\tau_2} \right) \right] P(t') dt' \right\}^{2/3} \\ &+ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sin \frac{m\pi x_0}{a} \sin \frac{n\pi y_0}{b} \int_0^t G_{mn}(t-t') P(t') dt', \end{aligned} \quad (25)$$

and for the local indentation of the plate's material $\alpha(t)$:

$$\begin{aligned}
\alpha(t) = & V_0 t - \frac{4\sqrt{R}}{3m} \xi \int_0^t \left[\alpha^{3/2}(t') + b_1 \frac{d}{dt'} \int_0^{t'} \frac{\alpha^{3/2}(t'') dt''}{\Gamma(1-\gamma)(t'-t'')^\gamma} + b_2 \int_0^{t'} \mathfrak{D}_\gamma \left(-\frac{t'-t''}{T^\gamma} \right) \alpha^{3/2}(t'') dt'' \right] (t-t') dt' \\
& - \frac{4\sqrt{R}}{3m} \xi \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sin \frac{m\pi x_0}{a} \sin \frac{n\pi y_0}{b} \int_0^t G_{mn}(t-t') \\
& \times \left[\alpha^{3/2}(t') + b_1 \frac{d}{dt'} \int_0^{t'} \frac{\alpha^{3/2}(t'') dt''}{\Gamma(1-\gamma)(t'-t'')^\gamma} + b_2 \int_0^{t'} \mathfrak{D}_\gamma \left(-\frac{t'-t''}{T^\gamma} \right) \alpha^{3/2}(t'') dt'' \right] dt'.
\end{aligned} \tag{26}$$

where $b_1 = \eta \xi^{-1}$, $b_2 = \zeta \xi^{-1}$, and

$$\mathfrak{D}_\gamma(-t/\tau_i) = \frac{t^{\gamma-1}}{\tau_i^\gamma} \sum_{n=0}^{\infty} \frac{(-1)^n (t/\tau_i)^{\gamma n}}{\Gamma[\gamma(n+1)]} \tag{27}$$

is the Rabotnov's fractional exponential function [14]. Equations (25) and (26) could be solved numerically.

4. Conclusion

In the present paper, the problem on impact of a rigid spherical impactor upon a linear viscoelastic auxetic Kirchhoff-Love plate, made of viscoelastic auxetic with fractional viscosity, has been formulated for the case, when the shear operator is governed by the fractional derivative Kelvin-Voigt model in conjunction with the assumption that the bulk extension-compression operator \tilde{K} is assumed to be time-independent, i.e., volumetric relaxation is neglected.

The behavior of Kirchhoff-Love plate, made of viscoelastic auxetic is different from the behavior of the same viscoelastic plate. This is due to the fact that the auxetic's Poisson's ratio is a time-dependent value changing from negative to positive magnitudes with time.

The solution out of the contact domain is found through the Green's function, and within the contact domain via the Hertz theory. Using the algebra of dimensionless Rabotnov's fractional operators, integral equations for the contact force and the local indentation have been obtained.

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