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Lengthwise Cracks in Functionally Graded Beams Exhibiting Non-Linear Mechanical Behaviour of the Material

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Lengthwise Cracks in Functionally Graded Beams Exhibiting Non-Linear Mechanical Behaviour of the Material

V Rizov¹

¹ Professor, Department of Technical Mechanics, University of Architecture, Civil Engineering and Geodesy, 1046 - Sofia, Bulgaria

Abstract. The present paper deals with lengthwise fracture analysis of functionally graded four-point bending beams. The Ramberg-Osgood equation is used for treating the material non-linearity. A lengthwise crack located arbitrary along the beam width is studied. A solution to the strain energy release rate is derived. It is assumed that the modulus of elasticity varies exponentially along the beam width. The J -integral is applied for verification. The approach developed is a useful tool for assessing the influence of material non-linearity on the lengthwise fracture in functionally graded beams.

1. Introduction

Functionally graded materials play an important role in aerospace, nuclear reactors, power plants, electronics, optics and biomedicine. That is why functionally graded materials continue to attract the attention of researchers and scientists throughout the world [1-6]. Since the functionally graded materials permit spatial tailoring of the microstructure and composition, one can get maximum benefits from their inhomogeneity. A substantial condition for widening the applicability of the functionally graded materials is the study of their fracture behaviour.

The present paper is focused on the lengthwise fracture in functionally graded beams since functionally graded materials and structures can be built up layer by layer [7] which is a premise for appearance of lengthwise cracks between layers. It should be noted that lengthwise fracture in various functionally graded beams which exhibit material non-linearity has been investigated in previous works of the author mainly by using power law stress-strain relations [8-10]. A lengthwise fracture analysis of functionally graded beams has been developed assuming that the coefficient of the power-law stress-strain relation is distributed linearly along the beam height [8]. A solution to the strain energy release rate has been derived by analysing the complementary strain energy cumulated in the beam [8]. The solution has been applied for studying lengthwise fracture behaviour of functionally graded double cantilever beam configurations with considering the non-linear mechanical behaviour of the material [8]. A power law stress-strain relation has been used also in a fracture analysis of a functionally graded cantilever beam which contains a lengthwise crack [9]. A parabolic function has been adopted for describing the distribution of the coefficient of the power law stress-strain relation [9]. Fracture analysis of a functionally graded beam configuration loaded in eccentric tension has been developed with taking into account the non-linear mechanical behaviour of the material [10]. The J -integral approach has been applied. It has been assumed that the material is functionally graded along the thickness of the beam [10].

In contrast to previous papers, the present paper studies the lengthwise fracture in the functionally graded four-point bending beam by using the Ramberg-Osgood stress-strain relation for describing the



material non-linearity. The four-point bending beam contains a lengthwise vertical crack. The material is functionally graded along the width of the beam. A solution to the strain energy release rate is derived. It is assumed the modulus of elasticity is distributed exponentially in the width direction.

2. Analysis of the Strain Energy Release Rate

The present paper analyses the lengthwise crack in the four-point bending beam shown in figure 1. The material is functionally graded along the beam width. Besides, it is assumed that the beam exhibits non-linear mechanical behaviour of the material which is modelled by the Ramberg-Osgood stress-strain relation. A notch of depth, b_2 , is introduced in the right-hand lateral surface of the beam in order to induce conditions for lengthwise fracture. A lengthwise crack of length, $2a$, is located symmetrically with respect to the mid-span (it should also be mentioned that the crack is located in beam portion, DL , which is loaded in pure bending). The crack is located arbitrary along the beam width (the widths of the left-hand and right-hand crack arms are b_1 and b_2 , respectively). The right-hand crack arm is divided in two symmetric segments by the notch. The beam is loaded by two vertical forces, F , applied at the ends of the beam. It is obvious that the two segments of the right-hand crack arm are free of stresses. The beam cross-section is a rectangle of width, b , and height, h . The beam length is $2(l_1 + l_2)$. Only half of the beam, $l_1 + l_2 \leq x_3 \leq 2(l_1 + l_2)$, is considered in the analysis because of the symmetry.

The fracture is analyzed in terms of the strain energy release rate, G , by using the following formula [9]:

$$G = \frac{dU^*}{hda}, \quad (1)$$

where U^* is the complementary strain energy cumulated in half of the beam, da is an elementary increase of the crack length.

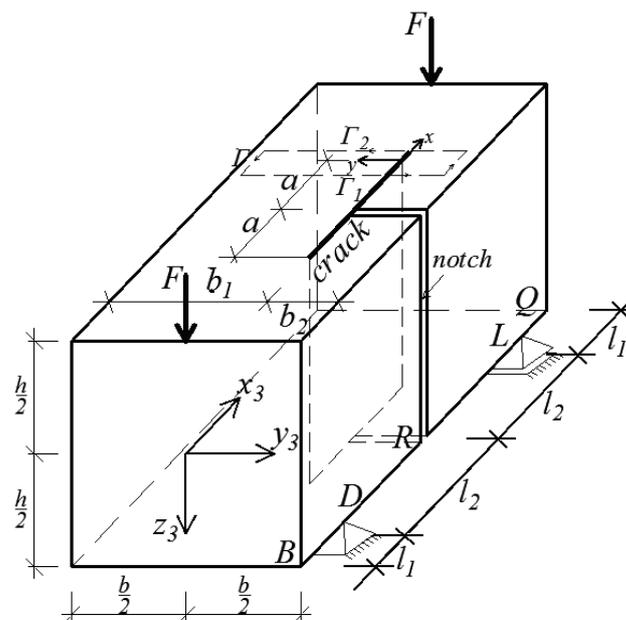


Figure 1. Functionally graded beam configuration.

Since the complementary strain energy in portions, BD and LQ , of the beam (figure 1) does not depend on a , it is enough to calculate the complementary strain energy in the portion, RL , of the beam. Therefore, U^* is written as

$$U^* = a \int_{\frac{h}{2}}^{\frac{h}{2}} \left(\int_{\frac{b_1}{2}}^{\frac{b_1}{2}} u_l^* dy_1 \right) dz_1 + (l_2 - a) \int_{-h}^h \left(\int_{\frac{b}{2}}^{\frac{b}{2}} u_u^* dy_2 \right) dz_2, \quad (2)$$

where u_l^* and u_u^* are, respectively, the complementary strain energy densities in the left-hand crack arm and the un-cracked beam portion, $l_1 + l_2 + a \leq x_3 \leq l_1 + 2l_2$.

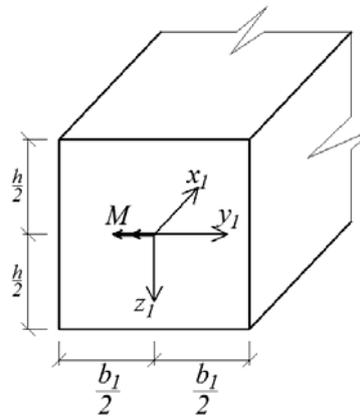


Figure 2. Cross-section of the left-hand crack arm.

In (2), y_1 and z_1 are the centroidal axes of the left-hand crack arm cross-section (figure 2), y_2 and z_2 are the centroidal axes of the cross-section of the un-cracked beam portion. The Ramberg-Osgood stress-strain relation that is applied to model the non-linear mechanical behaviour of the material is written as

$$\varepsilon = \frac{\sigma}{E} + \left(\frac{\sigma}{H} \right)^n, \quad (3)$$

where ε is the lengthwise strain, σ is the normal stress, E is the modulus of elasticity, H and n are material properties. The modulus of elasticity varies continuously along the beam width according to the following exponential law:

$$E = E_0 e^{\frac{q_0}{b} \left(y_3 + \frac{b}{2} \right)}, \quad (4)$$

where E_0 is the value of the modulus of elasticity at the left-hand lateral surface of the beam, q_0 is a material property which governs the material gradient, the y_3 -axis is shown in figure 1. By using (4), the distribution of E in the cross-section of the left-hand crack arm is written as

$$E = E_0 e^{\frac{q_0}{b} \left(y_1 + \frac{b_1}{2} \right)}. \quad (5)$$

The complementary strain energy density in the left hand crack arm is expressed as [11]

$$u_l^* = \frac{\sigma^2}{2E} + \frac{n\sigma^{\frac{1+n}{n}}}{(1+n)H^{\frac{1}{n}}}. \quad (6)$$

By substituting of (5) in (6), one arrives at

$$u_l^* = \frac{\sigma^2}{2E_0 e^{\frac{q_0}{b} \left(y_1 + \frac{b_1}{2} \right)}} + \frac{n\sigma^{\frac{1+n}{n}}}{(1+n)H^{\frac{1}{n}}}. \quad (7)$$

The span to height ratio of the beam under consideration is large. Thus, the distribution of longitudinal strains in the cross-section of the left-hand crack arm is analyzed by applying the Bernoulli's hypothesis for plane sections. Concerning the applicability of the Bernoulli's hypothesis, it should also be noted that since the beam portion, DL , is loaded in pure bending, the only non-zero strain is ε . Thus, according to the small strains compatibility equations, ε is distributed linearly in the beam cross-section:

$$\varepsilon = k_l z_1, \quad (8)$$

where the curvature of the left-hand crack arm, k_l , is determined from the following equation for equilibrium of the cross-section of the left-hand crack arm:

$$M = \int_{-\frac{h}{2}}^{\frac{h}{2}} \left(\int_{-\frac{b_1}{2}}^{\frac{b_1}{2}} \sigma z_1 dy_1 \right) dz_1, \quad (9)$$

where $M = Fl_1$ is the bending moment in beam portion, DL .

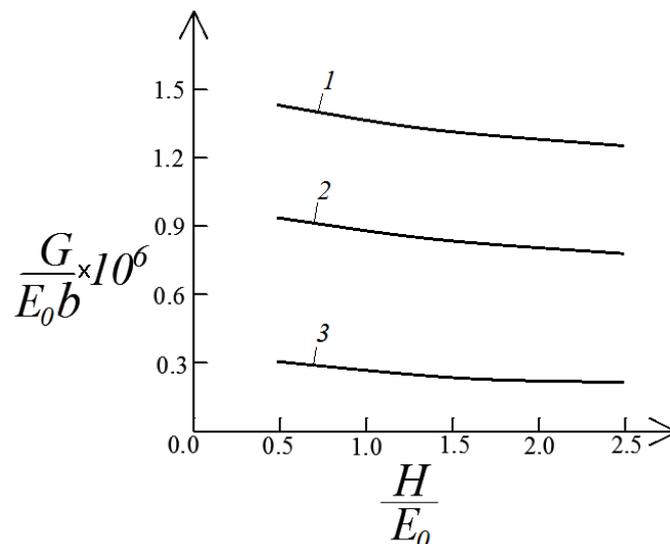


Figure 3. The strain energy release rate presented as a function of H/E_0 ratio (curve 1 – at $b_1/b = 0.25$, curve 2 – at $b_1/b = 0.50$ and curve 3 – at $b_1/b = 0.75$).

Equation (9) can be used also to obtain the curvature, k_u , of the un-cracked beam portion, RL . For this purpose, b_1 , σ , y_1 and z_1 have to be replaced, respectively, with b , σ_u , y_2 and z_2 (here σ_u is the normal stress in the un-cracked beam portion).

By substituting of (2) in (1), one obtains:

$$G = 2 \left[\frac{1}{h} \int_{\frac{h}{2}}^{\frac{b_1}{2}} \int_{\frac{-b_1}{2}}^{\frac{b_1}{2}} u_1^* dy_1 dz_1 - \frac{1}{h} \int_{-h}^{\frac{h}{2}} \int_{\frac{b}{2}}^{\frac{b}{2}} u_u^* dy_2 dz_2 \right]. \quad (10)$$

It should be noted that the term in the square brackets in (10) is doubled in view of the symmetry (figure 1). The integration in (10) is performed by using the MatLab computer program.

The solution to the strain energy release rate (10) is verified by applying the J -integral approach [12]. The integration is carried-out along the integration contour, Γ , shown by a dashed line in figure 1. Since the lateral surfaces of the beam are free of stresses, the J -integral solution is obtained by addition of J_{Γ_1} and J_{Γ_2} (here J_{Γ_1} and J_{Γ_2} are the J -integral values, respectively, in segments Γ_1 and Γ_2 of the integration contour). It should be noted that Γ_1 and Γ_2 coincide with the cross-sections of the left-hand crack arm and the un-cracked beam portion, respectively.

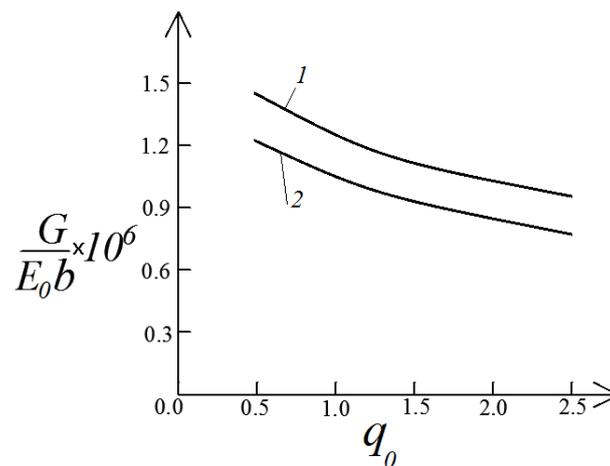


Figure 4. The strain energy release rate presented as a function of q_0 (curve 1 – at non-linear behaviour of the material, curve 2 – at linear-elastic behaviour).

The integration of the J -integral is performed by using the MatLab computer program. The J -integral values obtained match exactly the strain energy release rates. This fact is a verification of the fracture analysis developed in the present paper.

The influence of the material gradient, the lengthwise crack location along the beam width and the non-linear mechanical behaviour of the material on the fracture in the functionally graded four-point bending beam (figure 1) is investigated. For this purpose, calculations of the strain energy release rate are performed by applying (10). The results obtained are presented in non-dimensional form by using the formula $G_N = G/(E_0 b)$. It is assumed that $b = 0.025$ m, $h = 0.008$ m, $l_1 = 0.100$ m and $F = 50$ N. The crack location along the beam width is characterised by b_1/b ratio. The strain energy release rate in non-dimensional form is presented as a function of H/E_0 ratio in figure 3 at three b_1/b ratios. It can be observed in figure 3 that the strain energy release rate decreases with increasing

b_1/b ratio. This behaviour is due to the increase of the left-hand crack arm stiffness. The curves in figure 3 indicate also that the strain energy release rate decreases with increasing of H/E_0 ratio. The effect of q_0 on the strain energy release rate is elucidated in figure 4. One can observe in figure 4 that the strain energy release rate decreases with increasing of q_0 . The influence of material non-linearity on the fracture behaviour is elucidated too. For this purpose, the strain energy release rate obtained assuming linear-elastic behaviour of the functionally graded material is plotted in figure 4 for comparison with the non-linear solution. At $H \rightarrow \infty$ the Ramberg-Osgood stress-strain relation (3) transforms into the Hooke's law. Thus, the linear-elastic solution to the strain energy release rate is derived by substituting of $H \rightarrow \infty$ in (7) and (10). It can be observed in figure 4 that the material non-linearity leads to increase of the strain energy release rate.

3. Conclusions

An approach for analysis of lengthwise fracture in functionally graded four-point bending beam is developed. The Ramberg-Osgood equation is used for modelling the non-linear mechanical behaviour of the material. A lengthwise crack located arbitrary along the beam width is considered. It is assumed that the material is functionally graded in the width directions (the modulus of elasticity varies continuously along the beam width according to exponential law). The fracture is studied in terms of the strain energy release rate. The solution derived is verified by the applying the J -integral approach. The influence of the material gradient, the lengthwise crack location and the material non-linearity on the fracture behaviour is investigated. The approach developed in the present paper can be used to evaluate the effect of material non-linearity in design of functionally graded beam structures with considering the fracture behaviour.

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