Rings And Modules With Krull Dimension

by

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To Dad

"ABANDON ALL HOPE YE WHO ENTER HERE"

Abstract

This thesis concerns the interplay between the structures of modules and those of their overlying rings; investigating finite annihilation, prime submodules, boundedness and the relationships between various forms of Krull dimension.

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Statement

Apart from well-known results and unless stated otherwise, Section 2.1 and in particular Theorem 2.1.12, is original work by the author in collaboration with Professor P. F. Smith at Glasgow University. A version of Section 2.1 has appeared in the Bulletin of the London Mathematical Society [30]. Similarly, apart from well-known results and unless stated otherwise, Section 2.2 is original work by the author in collaboration with Professor P. F. Smith. A version of Section 2.2 has appeared in the Mediterranean Journal of Mathematics [31]. A version of Chapter 3 has appeared in the Journal of Algebra [32] and again apart from well-known results and unless stated otherwise, the contents of Chapter 3 is original work by the author in collaboration with Professor P. F. Smith. Apart from well-known results and unless stated otherwise Sections 4.2, 4.3, 4.4, 4.6 and 4.7 of Chapter 4 consist of original work by the author. Chapter 5 combines new work by the author with previously known results, in particular giving detailed explanation and development of results of R. N. Roberts [27] and D. Kirby [15]. Front- and end-piece quotes are taken from [6].

Contents

Chapter 1

Introduction

1.1 Overview

The overall theme of this thesis is the interplay between the structures of rings and their modules. We investigate various aspects of how information about one can be used to gain information about the other. Central to this work will be the concept of finite annihilation, which provides a straightforward way to move between rings and modules and is used both explicitly and implicitly throughout the thesis. Krull dimension will also play an important role, in particular in its most common, modern, module-theoretic form.

Chapter 1, the Introduction, provides this overview of the work presented in the following chapters and then introduces some of the most important concepts that we use throughout the thesis, giving definitions and some basic properties and results. In particular we consider Krull dimension, boundedness, finite annihilation and the H-condition.

The thesis proper begins with Chapter 2, "Finitely Annihilated Modules and Artinian Rings", in which we consider a number of characterisations of right Artinian rings using finitely annihilated modules. Firstly, in Section 2.1, we prove that a ring is right Artinian if and only if every countably generated right module is finitely annihilated (Theorem 2.1.12). This extends a known result that a ring is right Artinian if and only if every right module is finitely annihilated and is also related to the well-known result of Cauchon that over a right Noetherian ring every finitely generated module is finitely annihilated if and only if the ring is right fully bounded.

In Section 2.2, "On Families of Finitely Annihilated Modules", we then go on to look at further extensions of the above result of Cauchon, considering the circumstances under which various restricted families of modules are finitely annihilated. We prove results similar to the main theorem of Section 2.1, but involving, for the most part, Artinian properties of quotients of the ring.

We first consider simple and semisimple modules and prove in Theorem 2.2.2 that if J is the Jacobson radical of the ring R then R/J is Artinian if and only if every (countably generated) semisimple right R-module is finitely annihilated.

We next consider hereditary torsion theories and, in particular, give a related result involving singular modules (Theorem 2.2.6) showing that if R is a ring with right socle $\mathrm{Soc}(R_R)$ then $R/\mathrm{Soc}(R_R)$ is right Artinian if and only if every (countably generated) singular right R-module is finitely annihilated.

Section 2.2.3 considers when injective modules are finitely annihilated and in Theorem 2.2.12 we show that a commutative Noetherian ring is Artinian if and only if every (indecomposable) injective module is finitely annihilated. In Theorem 2.2.21 we show that a Noetherian ring is Artinian if and only if every injective module (on either side) is finitely annihilated. Note that the example of Section 2.2.5 shows that these results concerning injective modules do not necessarily hold for non-Noetherian rings.

In Section 2.2.4 we look at the case when uniform and finite (Goldie) dimensional modules are finitely annihilated and prove in Theorem 2.2.24 that a ring with right Krull dimension is right Artinian if and only if every uniform right module is finitely annihilated. An example is given in the following section to show that this result is not necessarily true for rings without right Krull dimension.

Finally, in Section 2.2.5 we detail the aforementioned example of a ring over which every injective module and every finite dimensional module is finitely annihilated but which is not Artinian. This ring is commutative but does not have Krull dimension and shows that the extra conditions in Theorems 2.2.12, 2.2.21 and 2.2.24 are in fact necessary.

In Chapter 3, "Krull Dimension of Bimodules" we investigate a theorem of Lambek and Michler [20, Theorem 3.6], which says that a ring is right Artinian if and only if it is right Noetherian and every irreducible prime right ideal is maximal and consider, in particular, how analogous results might be developed for modules.

We give an example to show that Lambek and Michler's result fails for one-sided modules (Example 3.1.4), but go on in Theorem 3.2.12 to prove the following bimodule analogue of the result. We show that if R and S are rings and M is a left S-, right R-bimodule such that M has Krull dimension and M/N has the same Krull dimension as a left S- and as a right R-module for every sub-bimodule N of M and if, moreover, M is a finitely generated left S-module then the Krull dimension of the right R-module M is the Krull dimension of a k-critical right R-module M/K for some prime submodule K of the right R-module M.

In Section 3.3 we consider Artinian bimodules and prove another bimodule analogue of Lambek and Michler's result, showing in Theorem 3.3.1 that if M is a Noetherian bimodule over rings R and S then the right R-module M is Artinian if and only if every irreducible prime R-submodule is maximal.

Many properties of Noetherian rings can be extended to the wider class of rings with Krull dimension and in Chapter 4, "Right Fully Bounded Rings with Right Krull Dimension", we consider an extension of the concept of right fully bounded right Noetherian rings (right FBN rings), looking at right fully bounded rings with right Krull dimension (we call such rings right FBK rings). We investigate how results on right FBN rings can be extended to right FBK rings and in particular consider whether such right fully bounded rings with right Krull dimension satisfy the H-condition, giving a necessary and sufficient condition for them to do so and an example to show that this is not the case for all such rings.

Our main result of this chapter is Theorem 4.3.10, where we prove that a ring with right Krull dimension satisfies the H-condition if and only if every homomorphic image of the ring is right bounded. Our original question was whether right FBK rings satisfy the H-condition and following this result this question becomes: is every factor ring of a right FBK ring right bounded? Note that the answer is yes for right FBN rings. The question is answered in the negative in Section 4.4, where we detail an example of a ring with right Krull dimension which is right fully bounded but is not itself right bounded.

Section 4.5 deals with the Gabriel correspondence between the prime ideals of a ring and its indecomposable injective modules. It is well known that the Noetherian rings which satisfy the Gabriel correspondence are precisely the right FBN rings and in Theorem 4.5.10 we give a proof of a result of Gordon and Robson that, in fact, right FBK rings satisfy the Gabriel correspondence.

A module is said to satisfy the bimodule condition if it has Krull dimension as both a left module and a right module and these dimensions are equal. In Section 4.6 we investigate the bimodule condition and find that for rings with Krull dimension, as with Noetherian rings, it is closely related to the H-condition and boundedness. Proposition 4.6.2 gives a result of McConnell and Robson concerning the equality of the Krull dimension and deviation of a bimodule which is finitely generated and has Krull dimension on one side over a ring with the H-condition. A corollary of this result (Corollary 4.6.3) gives that a finitely generated bimodule with Krull dimension over rings with the H-condition satisfies the bimodule condition. A ring is called Krull symmetric if it has right and left Krull dimensions and these dimensions are equal. Well-known results show that Noetherian rings which are Artinian on either side and rings which are FBN are Krull symmetric. Our result on the bimodule condition, when applied to rings, shows that a ring with Krull dimension and the H-condition on each side is Krull symmetric (Proposition 4.6.4). In Chapter 5, using the concept of classical Krull dimension and in particular a result of Gordon and Robson, we push this work further to conclude in Theorem 5.3.4 that an FBK ring is Krull symmetric.

Finally in this chapter, in Section 4.7 we briefly consider the Jacobson conjecture and FBK rings. It is well known that FBN rings satisfy the Jacobson conjecture (that is, the intersection of the powers of the Jacobson radical is zero), but that one-sided FBN rings do not. However, in Example 4.7.1 we detail an example which shows that, even in the two-sided case, FBK rings do not necessarily satisfy the Jacobson conjecture.

In Chapter 5, "Krull Dimension and Classical Krull Dimension and their Duals", we consider various types of "Krull dimension" and investigate the relationships between them. We look at (module-theoretic) Krull dimension, dual Krull dimension, classical Krull dimension and dual classical Krull dimension and briefly consider what we term f-Krull dimension.

In Section 5.2 we give a brief history of the development of Krull dimension, from a measure of the lengths of chains of prime ideals in commutative Noetherian rings, to a module-theoretic dimension measuring how close to being Artinian a module is. We then focus on ordinal valued classical Krull dimension and prove some useful basic properties, including, in particular, that a ring has classical Krull dimension if and only if it satisfies the ascending chain condition on prime ideals.

We go on, in Section 5.3, to investigate the relationship between classical Krull dimension and module-theoretic Krull dimension. We give an example to show that a ring with classical Krull dimension need not have Krull dimension. We show, however, that a ring with right Krull dimension necessarily has classical Krull dimension (and the classical Krull dimension is bounded above by the Krull dimension). We also consider some important properties of prime ideals in rings with right Krull dimension. Finally we prove a result of Gordon and Robson (Proposition 5.3.3) that the Krull dimension and classical Krull dimension of a right FBK ring are equal.

In Section 5.4 we look at the relationship between Krull dimension and dual classical Krull dimension. Using the result of the previous section that Krull dimension and classical Krull dimension are equal for right FBK rings (Proposition 5.3.3) we show in Theorem 5.4.3 that a module with Krull dimension over a right fully bounded ring with right Krull dimension has dual classical Krull dimension and that the dual classical Krull dimension of the module is bounded above by the Krull dimension of the ring.

In Section 5.5 we look in more detail at dual classical Krull dimension, considering in particular quasi-local rings and proving some basic properties and results. In Section 5.6 we define polynomial functions and prove some of their basic properties, before looking in more detail in Section 5.7 at polynomial functions in relation to chain conditions and graded modules, presenting some relatively technical results, which we then use to prove a dual of the Artin-Rees Lemma and an analogue of Nakayama's Lemma for Artinian modules.

Using the preparatory technical work of the previous sections, Section 5.8 then details a result of R. N. Roberts [27] and D. Kirby [15], which says that if R is a quasi-local commutative ring and M is an Artinian R-module then M has dual Krull dimension and dual classical Krull dimension and these dimensions are equal. In Section 5.9 we go on to detail an extension of this result to Artinian modules over arbitrary commutative rings.

Finally, in Section 5.10 we consider the relationship between Krull dimension and dual Krull dimension, conjecturing that if M is a module with Krull dimension over a commutative ring R with Krull dimension then the dual Krull dimension of M is bounded above by the Krull dimension of the ring. We prove that the conjecture is true in case M has Krull dimension zero.

1.2 Radicals

Throughout this thesis all rings will be associative and have an identity element and all modules will be unital.

Given a module or ring there are many different types of radical which can be considered and there is often a lack of consistency between different books and authors as to nomenclature and notation. In this section we will define the various different radicals and related constructs that will be used in this thesis and consider some of their basic properties and the relations between them.

The *socle* of a module M is the sum of all the simple submodules of M and will be denoted by $\mathrm{Soc}(M)$. In case M has no simple submodules we define $\mathrm{Soc}(M) = 0$. A module M will be called *semisimple* if $\mathrm{Soc}(M) = M$. Note that a module M is semisimple if and only if every submodule of M is a direct summand of M (see [12, Proposition 3.2]). A ring R will be called *semisimple* if the right R-module R_R is semisimple, or equivalently if every right R-module is semisimple. Some further equivalent conditions are given in [12, Theorem 3.4]. Note that semisimple modules and rings are often referred to as being *completely reducible* by many authors.

An ideal P of a ring R is called *prime* if $P \neq R$ and for all ideals A and B of R such that $AB \subseteq P$, either $A \subseteq P$ or $B \subseteq P$. Some equivalent conditions are given in [12, Proposition 2.1]. A ring R is called *prime* if 0 is a prime ideal of R. An ideal of a ring R is called *semiprime* if it is an intersection of prime ideals of R. A ring R is called *semiprime* if 0 is a semiprime ideal of R. The *prime radical* of a ring R is the intersection of all the prime ideals of R and will be denoted by $\mathrm{P}(R)$. Thus a ring R is semiprime if and only if $\mathrm{P}(R) = 0$. Note that it is clear that a semiprime ring has no nonzero nilpotent ideals. By [12, Corollary 2.8], the converse is also true.

For a right module M over a ring R, the *(right) annihilator* of a non-empty subset $S \subseteq M$ is defined to be the set $\mathrm{ann}_R(S) = \{r \in R | Sr = 0\}$. This is clearly a right ideal of R. If S is a finite subset, $S = \{s_1, \ldots, s_n\}$ say for some integer $n \geq 1$, then we write $\mathrm{ann}_R(S) = \mathrm{ann}_R(s_1, \ldots, s_n)$. An ideal I of a ring R is called *right primitive* if $I = \mathrm{ann}_R(M)$ for some simple right R-module M. A ring R is called *right primitive* if 0 is a right primitive ideal of R. Note that every maximal ideal is right primitive and that every right primitive ideal is prime. The *Jacobson radical* of a ring R, which will be denoted by $\mathrm{J}(R)$, is the intersection of all the right primitive ideals of R. Equivalently $\mathrm{J}(R)$ is the intersection of all the maximal right ideals of R, or can also be defined equivalently as the intersection of all the left primitive ideals of R or as the intersection of all the maximal left ideals of R (see [12, Proposition 2.16]). The Jacobson radical can also be characterised as the set of all elements $r \in R$ such that $1 - rs$ is right invertible for all $s \in R$ (see [19, Section 3.2 Proposition 3]). A ring R is called *semiprimitive* if $\mathrm{J}(R) = 0$. Since right primitive ideals are prime it follows that for a ring R, $\mathrm{P}(R) \subseteq \mathrm{J}(R)$. Therefore a semiprimitive ring is semiprime.

Proposition 1.2.1. *Let R be a right Artinian ring. Then the Jacobson radical of R is nilpotent.*

Proof. Let J denote the Jacobson radical of R (that is $J = \mathrm{J}(R)$). Since R is right Artinian there exists an integer $n \geq 1$ such that $J^n = J^{n+1}$. Let $B = J^n$. Then $B = B^2$. Suppose that $B \neq 0$ and let A be minimal in the set of right ideals of R contained in B such that $AB \neq 0$ (such a right ideal exists since R is right Artinian and $B^2 \neq 0$). Then $aB \neq 0$ for some element $a \in A$. Now $aB \subseteq AB \subseteq B$ and $(aB)B = aB^2 = aB \neq 0$, so $aB = A$ by the minimal choice of A. Therefore $ab = a$ for some element $b \in B$. Now $b \in B \subseteq J$ so there exists an element $c \in R$ such that $(1-b)c = 1$. Hence $a = a(1-b)c = 0$, a contradiction. Thus $B = 0$, so J is nilpotent. □

Corollary 1.2.2. *Let R be a right Artinian ring. Then $P(R) = J(R)$.*

Proof. This follows by Proposition 1.2.1 since every nilpotent ideal of R is contained in the prime radical $\mathrm{P}(R)$ of R. □

It follows that if R is a right Artinian ring then we may refer simply to the radical of R, without having to specify which of the prime and Jacobson radicals we mean. It also means of course that "semiprime" and "semiprimitive" are equivalent for a right Artinian ring R. In this case, however, more can be said, as such rings are also semisimple Artinian (often referred to as completely reducible) and, further, the Artinian condition is symmetric. Such rings are well studied and understood, in particular using the famous Artin-Wedderburn Theorem.

1.3 Singularity and Torsion

Let R be a ring and let M be a right R-module. A submodule N of M will be called an *essential submodule* of M if N has nonzero intersection with every nonzero submodule of M. A right ideal E of R will be said to be an *essential right ideal* of R if E is essential as a right R-submodule of R, that is if it has nonzero intersection with every nonzero right ideal of R. A module U is said to be *uniform* if U is nonzero and every nonzero submodule of U is an essential submodule.

Definition. Let R be a ring and let M be a right R-module. Then the set

$$\mathrm{Z}(M) = \{m \in M | mE = 0 \text{ for some essential right ideal } E \text{ of } R\}$$

is a submodule of M and is called the *singular submodule* of M. M is called *singular* if $\mathrm{Z}(M) = M$ and *nonsingular* if $\mathrm{Z}(M) = 0$.

We note that by the above definition of the singular submodule, for an element $m \in M$, $m \in \mathrm{Z}(M)$ if and only if $\mathrm{ann}_R(m)$ is an essential right ideal of R.

A module is said to have *finite uniform dimension* (or *finite Goldie dimension*) if it does not contain a direct sum of an infinite number of nonzero submodules. We will often refer to such modules as being simply *finite dimensional.* If a module has finite uniform dimension then it can be shown that it contains an essential submodule which is a finite direct sum of uniform submodules and further that the number of summands in such an essential direct sum of uniform submodules is an invariant of the module (see [25, Lemma 2.2.8 and Theorem 2.2.9]). This nonnegative integer is called the *uniform dimension* (or *Goldie dimension*) of the module and for a module M is denoted by $\mathrm{u}(M)$, where we write $\mathrm{u}(M) = \infty$ if M fails to have finite uniform dimension. A ring is said to have finite right uniform dimension if it is finite dimensional as a right module over itself. For a ring R, by a *right annihilator* we mean a right ideal of R of the form $\mathrm{ann}_R(T)$ for some non-empty subset T of R. A ring R is called a *right Goldie ring* if it has finite right uniform dimension and satisfies the ascending chain condition on right annihilators. Prime and semiprime right Goldie rings are of crucial significance in Ring Theory, in particular due to the following classic theorem of Goldie. For the definition of the right quotient ring of a ring R see [25, 2.1.14].

Theorem 1.3.1 (Goldie's Theorem). *Let R be a ring. Then the following statements are equivalent.*

(i) R is semiprime right Goldie.

(ii) R is semiprime with finite right uniform dimension and $Z(R) = 0$.

(iii) R has a semisimple Artinian right quotient ring.

Proof. See [25, Theorem 2.3.6]. □

Let R be a ring and let x be an element of R. Then x is called *right regular* if $xr = 0$ for $r \in R$ implies that $r = 0$. Similarly x is called *left regular* if $sx = 0$ for $s \in R$ implies that $s = 0$. The element $x \in R$ is called *regular* if it is both right and left regular. Note that a (right or left) regular element is necessarily nonzero. In general a right or left regular element is not necessarily regular, however if R is a commutative ring or a semiprime right Goldie ring then right regular elements of R are regular (see [5, Corollary 1.13]).

Definition. Let R be a ring and let M be a right R-module. Then the set

$$\mathrm{T}(M) = \{m \in M | mc = 0 \text{ for some regular element } c \in R\},$$

is called the *torsion subset* of M. M is called *torsion* if $\mathrm{T}(M) = M$ and *torsion-free* if $\mathrm{T}(M) = 0$.

Note that for an arbitrary ring and module the torsion subset is not necessarily a submodule of the module. However, if R is a commutative ring or a semiprime right Goldie ring and M is a (right) R-module, then $\mathrm{T}(M)$ is a submodule of M and is called the *torsion submodule* of M. When considering torsion we will be mostly concerned with semiprime right Goldie rings, in which case the following important result shows that the singular submodule and the torsion submodule coincide.

Proposition 1.3.2 (Goldie). *Let R be a semiprime right Goldie ring and let I be a right ideal of R. Then I is an essential right ideal of R if and only if I contains a regular element of R.*

Proof. See [12, Proposition 5.9]. □

Corollary 1.3.3. *Let R be a semiprime right Goldie ring and let M be a right R-module. Then $Z(M) = T(M)$.*

Proof. By Proposition 1.3.2. □

1.4 Krull Dimension

By "Krull dimension" we will mean the module-theoretic Krull dimension (defined below), introduced by Rentschler and Gabriel [26] and Krause [17] and studied extensively by Gordon and Robson [13] amongst others. We will discuss the history of Krull dimension in its various forms in Section 5.2, where we also consider the relationships between the different types of "Krull dimension".

Definition. Let R be a ring and let M be a right R-module. The *Krull dimension* of M, denoted by $\mathrm{k}(M)$, if it exists, is defined as follows. $\mathrm{k}(M) = -1$ if and only if $M = 0$. If $\alpha \geq 0$ is an ordinal such that all modules with Krull dimension strictly less than α are known, then $\mathrm{k}(M) \leq \alpha$ if for every chain $M = M_0 \supseteq M_1 \supseteq M_2 \supseteq \cdots$ of submodules of M there is a positive integer n such that $\mathrm{k}(M_i/M_{i+1}) < \alpha$ for all $i \geq n$.

Note that $\mathrm{k}(M) = 0$ if and only if M is nonzero Artinian. In this sense, the Krull dimension of a module can be thought of as a measure of how far the module is from being Artinian. It is interesting, however, that many properties of modules with Krull dimension are similar (or identical) to those of Noetherian modules.

A ring R will be said to have *right Krull dimension* if the right R-module R_R has Krull dimension and the right Krull dimension of R shall be the Krull dimension of R_R, denoted by $\mathrm{k}(R)$.

We now go on to detail some results on rings and modules with Krull dimension which we will use later in the thesis. All of these are well known and we often omit the proofs, instead referring to the literature for further information.

Lemma 1.4.1. *Let R be a ring, let M be a right R-module and let N be a submodule of M. Then $k(M) = \sup\{k(M/N), k(N)\}$ if either side exists.*

Proof. See [25, Lemma 6.2.4]. □

Corollary 1.4.2. *Let R be a ring with right Krull dimension and let M be a finitely generated right R-module. Then M has Krull dimension and $k(M) \leq k(R)$.*

Proof. This follows by repeated applications of Lemma 1.4.1. □

Lemma 1.4.3. *Let R be a ring and let M be a Noetherian right R-module. Then M has Krull dimension.*

Proof. Suppose that the result is false. Using the Noetherian property we may assume that all proper factor modules of M have Krull dimension. Let

$$\alpha = \sup\{\mathrm{k}(M/N) | N \text{ is a nonzero submodule of } M\}.$$

Let $M = M_0 \supseteq M_1 \supseteq M_2 \supseteq \cdots$ be any descending chain of nonzero submodules of M. Then the factors in this chain have Krull dimension and satisfy $\mathrm{k}(M_i/M_{i+1}) \leq \alpha$ for each $i \geq 0$. It follows that M has Krull dimension with $\mathrm{k}(M) \leq \alpha + 1$, a contradiction. □

The following is one of many Noetherian-like properties of modules with Krull dimension.

Lemma 1.4.4. *A module with Krull dimension has finite uniform dimension.*

Proof. Suppose that the result is false. Amongst the counterexamples choose one, M say, with minimal Krull dimension, say $\mathrm{k}(M) = \alpha$. Since M does not have finite uniform dimension there exist nonzero submodules N_i of M such that $M \supseteq \oplus_{i=1}^{\infty} N_i$. Set $M_n = \oplus_{j=1}^{\infty} N_{2^n j}$ for each integer $n \geq 0$. Then $M_0 \supseteq M_1 \supseteq M_2 \supseteq \cdots$ is a descending chain of submodules of M such that each factor M_n/M_{n+1} contains an infinite direct sum and so has infinite uniform dimension. Since $\mathrm{k}(M_n/M_{n+1}) \leq \mathrm{k}(M)$ it follows, by the minimality of α, that $\mathrm{k}(M_n/M_{n+1}) = \alpha$ for all $n \geq 0$. Thus $\mathrm{k}(M) > \alpha$, a contradiction. □

Lemma 1.4.5. *Let R be a ring and let M be a right R-module with Krull dimension such that M is a sum of submodules each of which has Krull dimension at most α for some ordinal α. Then $k(M) \leq \alpha$.*

Proof. See [25, Lemma 6.2.17]. □

Compare the following result with Corollary 1.4.2.

Lemma 1.4.6. *Let R be a ring with right Krull dimension and let M be a right R-module. If M has Krull dimension then $k(M) \leq k(R)$.*

Proof. This follows from Lemma 1.4.5, since M is the sum of its cyclic submodules, each of which is isomorphic to a factor module of R_R. □

An important result concerning rings with Krull dimension is that a semiprime ring with right Krull dimension is semiprime right Goldie. In order to prove this we first need the concept of k-critical modules.

Definition. Let R be a ring. For an ordinal $\alpha \geq 0$, a right R-module M is called *α-k-critical* if M has Krull dimension α and $\mathrm{k}(M/N) < \alpha$ for all nonzero submodules N of M. A module is called *k-critical* if it is α-k-critical for some ordinal α.

Note that if a module M is α-k-critical for some ordinal $\alpha \geq 0$, then Lemma 1.4.1 gives that $\mathrm{k}(N) = \alpha$ for every nonzero submodule N of M. In fact, for any ordinal $\alpha \geq 0$ every nonzero submodule of an α-k-critical module is also α-k-critical.

Lemma 1.4.7. *Any nonzero module with Krull dimension contains a k-critical submodule.*

Proof. Let R be a ring and let M be a nonzero right R-module with Krull dimension. Among the nonzero submodules of M choose one with minimal Krull dimension, $\mathrm{k}(N) = \alpha$ say for some ordinal $\alpha \geq 0$. If N is not α-k-critical then it contains a nonzero submodule

N_1 with $\mathrm{k}(N/N_1) = \alpha$. By the minimality of α, $\mathrm{k}(N_1) = \alpha$. Applying this same argument to N_1 and so on we obtain a chain $N = N_0 \supseteq N_1 \supseteq N_2 \supseteq \cdots$ of submodules of M with $\mathrm{k}(N_i/N_{i+1}) = \alpha$ for all $i \geq 0$. Since $\mathrm{k}(N) = \alpha$ this chain must terminate, which only happens when we reach an α-k-critical submodule of M. □

Note that the k-critical submodule need not necessarily have the same Krull dimension as the original module. For Noetherian modules we have the following related result.

Lemma 1.4.8. *Let R be a ring and let M be a Noetherian right R-module. If β is an ordinal such that $0 \leq \beta \leq k(M)$ then there exists a submodule $M' \subseteq M$ such that M/M' is a β-k-critical right R-module.*

Proof. Since M is Noetherian and $\mathrm{k}(M) \geq \beta$, we can choose a submodule $M' \subseteq M$ maximal with respect to $\mathrm{k}(M/M') \geq \beta$. Then each proper factor of M/M' must have Krull dimension strictly less than β. Thus, in any descending chain of submodules of M/M' each factor, except possibly one, has Krull dimension strictly less than β and therefore $\mathrm{k}(M/M') \leq \beta$. It follows that $\mathrm{k}(M/M') = \beta$ and that M/M' is β-k-critical. □

We are now able to prove the important result that a semiprime ring with right Krull dimension is semiprime right Goldie.

Proposition 1.4.9. *Let R be a semiprime ring with right Krull dimension. Then R is a right Goldie ring.*

Proof. By Lemma 1.4.4, R has finite right uniform dimension, so it suffices, by Theorem 1.3.1, to show that $\mathrm{Z}(R) = 0$. If not then, by Lemma 1.4.7, $\mathrm{Z}(R)$ contains a k-critical right ideal C. Since R is semiprime, $C^2 \neq 0$. Let $c \in C$ be such that $cC \neq 0$. Define a map $\theta : C \to C$ by $\theta(x) = cx$ for all $x \in C$. Then $\theta \neq 0$, so that $\mathrm{k}(C/\ker\theta) = \mathrm{k}(\mathrm{im}\theta) = \mathrm{k}(C)$. Thus $\ker\theta = 0$, that is $\mathrm{ann}_R(c) \cap C = 0$. This contradicts $\mathrm{ann}_R(c)$ being an essential right ideal of R. □

Recall that an object (an element of a ring or a ring itself say) is called *nilpotent* if some power of it is zero and that a set is called *nil* if each of its elements is nilpotent.

Proposition 1.4.10. *Let R be a ring with right Krull dimension. Then every nil subring of R is nilpotent.*

Proof. See [25, Theorem 6.3.7]. □

Another important Noetherian-type property of rings with Krull dimension is the following.

Proposition 1.4.11. *Let R be a ring with right Krull dimension and let I be a proper ideal of R. Then there are only finitely many prime ideals of R minimal over I and some finite product of these is contained in I.*

Proof. Since there is a one-to-one correspondence between prime ideals of R minimal over I and minimal prime ideals of the ring R/I, we may assume that $I = 0$ and consider the case of minimal primes. If $N = \mathrm{P}(R)$ is the prime radical of R then R/N is a semiprime ring with Krull dimension so is a semiprime right Goldie ring, by Proposition 1.4.9. Hence there are only finitely many minimal primes in R/N and their intersection is zero (see [12, Proposition 6.1]). It follows that there are only finitely many minimal primes in R and their intersection is N (since N is contained in every prime ideal of R). Now N is a nil ideal of R, so is nilpotent, by Proposition 1.4.10 and hence some finite product of the minimal primes is zero. $\square$

One immediate consequence of this proposition is that a ring with right Krull dimension has only a finite number of distinct minimal prime ideals.

Proposition 1.4.12. *Let R be a ring with right Krull dimension. Then $k(R) = k(R/P)$ for some prime ideal P of R.*

Proof. By Proposition 1.4.11, there is an integer $n \geq 1$ and prime ideals $P_1, \ldots, P_n$ of R such that $P_1 \cdots P_n = 0$. Consider the descending chain $R \supseteq P_1 \supseteq P_1P_2 \supseteq \cdots \supseteq P_1P_2 \cdots P_n = 0$. For each $2 \leq i \leq n$ the factor $(P_1 \cdots P_{i-1})/(P_1 \cdots P_{i-1}P_i)$ is an (R/P_i)-module and so, by Lemma 1.4.6,

$$\mathrm{k}(((P_1 \cdots P_{i-1})/(P_1 \cdots P_{i-1}P_i))_R) = \mathrm{k}(((P_1 \cdots P_{i-1})/(P_1 \cdots P_{i-1}P_i))_{R/P_i}) \leq \mathrm{k}(R/P_i).$$

It follows, by Lemma 1.4.1, that $\mathrm{k}(R) = \sup\{\mathrm{k}(R/P_1), \ldots, \mathrm{k}(R/P_n)\}$ and hence, $\mathrm{k}(R) = \mathrm{k}(R/P_i)$ for some $1 \leq i \leq n$. $\square$

Lemma 1.4.13. *Let R be a semiprime ring with right Krull dimension. Then*

$$k(R) = \sup\{k(R/E) + 1 | E \text{ is an essential right ideal of } R\}.$$

Proof. Let $\alpha = \sup\{\mathrm{k}(R/E) + 1 | E$ is an essential right ideal of $R\}$. Suppose that $\mathrm{k}(R) > \alpha$. Then there is an infinite strictly descending chain of right ideals of R, $R = I_0 \supsetneqq$

$I_1 \supsetneqq I_2 \supsetneqq \cdots$, such that $\mathrm{k}(I_i/I_{i+1}) \geq \alpha$ for all $i \geq 0$. By Lemma 1.4.4, the ring R has finite right uniform dimension, so there is an integer $n \geq 0$ such that I_n and I_{n+1} have the same uniform dimension. Choose a right ideal A of R maximal with respect to the property that $A \cap I_n = 0$. Then $A \oplus I_{n+1}$ is an essential right ideal of R. Moreover, $I_n/I_{n+1} \cong (A \oplus I_n)/(A \oplus I_{n+1}) \subseteq R/(A \oplus I_{n+1})$, so $\mathrm{k}(I_n/I_{n+1}) + 1 \leq \alpha$, a contradiction. Therefore $\mathrm{k}(R) \leq \alpha$.

Now let E be an essential right ideal of R. By Proposition 1.4.9, R is a semiprime right Goldie ring, so E contains a regular element, x say. Then for any integer $n \geq 0$, $x^n R/x^{n+1}R \cong R/xR$, so that $\mathrm{k}(x^n R/x^{n+1}R) = \mathrm{k}(R/xR) \geq \mathrm{k}(R/E)$. The infinite strictly descending chain of right ideals of R, $R \supsetneqq xR \supsetneqq x^2R \supsetneqq \cdots$, then shows that $\mathrm{k}(R) \geq \mathrm{k}(R/E) + 1$. It follows that $\mathrm{k}(R) \geq \alpha$. □

Note that an adaptation of the first part of this proof holds for any module with Krull dimension over an arbitrary ring (see [13, Corollary 1.5]). That is, if R is a ring and M is a right R-module with Krull dimension, then

$$\mathrm{k}(M) \leq \sup\{(\mathrm{k}(M/E)) + 1 | E \text{ is an essential submodule of } M\}.$$

Note also that, by Lemma 1.4.1 and Lemma 1.4.13, for a semiprime ring R with right Krull dimension, $\mathrm{k}(R) = \mathrm{k}(E)$ for any essential right ideal E of R.

Lemma 1.4.14. *Let R be a ring with right Krull dimension. Then R has the ascending chain condition on prime ideals.*

Proof. Suppose that $P_1 \subsetneqq P_2$ are distinct prime ideals of R. Then P_2/P_1 is a nonzero ideal of the ring R/P_1, which is a prime ring with right Krull dimension. Hence P_2/P_1 is an essential right ideal of R/P_1 and it follows, by Lemma 1.4.13, that $\mathrm{k}(R/P_2) < \mathrm{k}(R/P_1)$. Therefore a strictly ascending chain of prime ideals of R, $P_1 \subsetneqq P_2 \subsetneqq \cdots$, gives rise to a decreasing sequence of ordinals $\mathrm{k}(R/P_1) > \mathrm{k}(R/P_2) > \cdots$ and the result follows. □

Lemma 1.4.15. *Let R be a semiprime right Goldie ring and let M be an R-module which is not singular. Then M has a uniform submodule isomorphic to a right ideal of R.*

Proof. Let $x \in M$ be an element which is not in the singular submodule of M. Then $\mathrm{ann}_R(x)$ is a non-essential right ideal of R, so there is a nonzero right ideal I of R such that $I \cap \mathrm{ann}_R(x) = 0$. Since R_R has finite uniform dimension, I contains a uniform right ideal of R, J say. Then $J \cap \mathrm{ann}_R(x) = 0$, so that $J \cong xJ$. Hence xJ is the required uniform submodule of M. □

Proposition 1.4.16. *Let R be a prime ring with right Krull dimension and let M be an R-module with Krull dimension. If M is not singular then $k(M) = k(R)$. Suppose further that M is finitely generated or has finite Krull dimension. Then M is singular if and only if $k(M) < k(R)$.*

Proof. Suppose that M is not singular. By Lemma 1.4.15, M has a submodule, U say, isomorphic to a uniform right ideal of R, I say. All uniform right ideals in a prime right Goldie ring are subisomorphic (that is each contains an isomorphic copy of the other), so have the same Krull dimension (see [25, Lemma 3.3.4]). By Proposition 1.4.9, R_R has finite uniform dimension so there is an essential finite direct sum of uniform right ideals in R, $I_1 \oplus \cdots \oplus I_n$ say for some integer $n \geq 1$. But $\mathrm{k}(R) = \mathrm{k}(I_1 \oplus \cdots \oplus I_n) = \mathrm{k}(I) = \mathrm{k}(U) \leq \mathrm{k}(M)$. By Lemma 1.4.6, it follows that $\mathrm{k}(M) = \mathrm{k}(R)$.

Now suppose that M is singular and further that M is finitely generated or has finite Krull dimension. Let $x \in M$. Then $\mathrm{ann}_R(x)$ is an essential right ideal of R. Now $xR \cong R/\mathrm{ann}_R(x)$ so, by Lemma 1.4.13, $\mathrm{k}(xR) = \mathrm{k}(R/\mathrm{ann}_R(x)) < \mathrm{k}(R)$. It follows, by Lemma 1.4.5, that $\mathrm{k}(M) < \mathrm{k}(R)$. $\square$

1.5 Boundedness

Definition. A ring R is called *right bounded* if every essential right ideal contains a two-sided ideal which is essential as a right ideal.

For example, commutative rings are right bounded. Semisimple rings are also right bounded, since such rings have no proper essential right ideals (every right ideal of a semisimple ring R is a direct summand of R_R). Note that a prime ring is right bounded if and only if every essential right ideal contains a nonzero two-sided ideal, since a nonzero two-sided ideal of a prime ring is essential as a right ideal.

Definition. A ring R is called *right fully bounded* if the ring R/P is right bounded for all prime ideals P of R.

It is worth noting that a right fully bounded ring is not necessarily right bounded. In Section 4.4 of Chapter 4 we detail an example of a ring which is right fully bounded but not right bounded. Conversely, a right bounded ring is not necessarily right fully bounded. For example let F be a field, let S be a simple F-algebra which is not Artinian (for example $F = \mathbb{C}$ and $S = A_1(\mathbb{C})$, the first Weyl algebra (see [25, Section 1.3])) and let

R be the "matrix ring"

$$R = \begin{pmatrix} S & S \\ 0 & F \end{pmatrix} = \{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} | a, b \in S \text{ and } c \in F \}.$$

Then R is a right bounded ring which is not right fully bounded.

Definition. A ring R is called a *right FBN* ring if R is right fully bounded and right Noetherian.

1.6 Finite Annihilation and the H-Condition

Definition. Let R be a ring and let M be a right R-module. M is said to be *finitely annihilated* if $\text{ann}_R(M) = \text{ann}_R(F)$ for some finite subset F of M.

Note that modules which are finitely annihilated are said to satisfy the "H-condition" by some authors. This definition is due we believe to P. Gabriel [8] and was studied by G. Cauchon, among others (see [3], [12]). This condition is often considered in terms of rings over which every finitely generated module is finitely annihilated, such rings also being said to satisfy the "H-condition". It is in this latter manner that we will use the term.

Definition. A ring R is said to satisfy the *H-condition* if every finitely generated (right) R-module is finitely annihilated.

Unless otherwise stated we will always be considering the H-condition on the right, whereby every finitely generated right module is finitely annihilated.

Over a commutative ring, every finitely generated module is finitely annihilated (by the annihilator of the generators of the module) and hence commutative rings satisfy the H-condition.

If R is a ring and M is a right R-module then it is easy to prove that M is finitely annihilated if and only if $R/\text{ann}_R(M)$ embeds in a finite direct sum of copies of M. This result allows us to move between the module structure and the structure of the overlying ring and is perhaps the main reason why finite annihilation is such a useful and important property to study. Note that if R is a simple non-Artinian ring and U is a simple right R-module then U is not finitely annihilated because $\text{ann}_R(U) = 0$ and R_R does not embed in a finite direct sum of copies of U. Thus simple non-Artinian rings do not satisfy the H-condition.

Chapter 2

Finitely Annihilated Modules and Artinian Rings

2.1 Finitely Annihilated Modules and Artinian Rings

It is clear that if R is a commutative ring then every finitely generated R-module is finitely annihilated. It is a well-known result of Cauchon that over a right Noetherian ring every finitely generated module is finitely annihilated if and only if it the ring is right fully bounded [5, Proposition 7.6 and Theorem 7.8]. There is also a lesser known result, proved by C. Faith [7, Theorem 17A], A. Ghorbani [9], C. R. Hajarnavis (unpublished), T. H. Lenagan (unpublished) and essentially also by J. A. Beachy [2], that a ring R is right Artinian if and only if every right R-module is finitely annihilated. In this section we prove the following extension of this result.

Theorem. *The following statements are equivalent for a ring R.*

(i) R is right Artinian.

(ii) Every right R-module is finitely annihilated.

(iii) Every countably generated right R-module is finitely annihilated.

(iv) R satisfies the descending chain condition on (two-sided) ideals and every cyclic right R-module is finitely annihilated.

We shall prove the theorem by way of a number of lemmas. The first lemma is well known.

Lemma 2.1.1. *Let R be a right Artinian ring. Then every right R-module is finitely annihilated.*

Proof. Let M be any right R-module. Let F be a finite subset of M such that $\mathrm{ann}_R(F)$ is minimal in the collection of annihilators of finite subsets of M. Let $m \in M$. Then $\mathrm{ann}_R(F \cup \{m\}) \subseteq \mathrm{ann}_R(F)$ gives $\mathrm{ann}_R(F \cup \{m\}) = \mathrm{ann}_R(F)$, by the minimal choice of $\mathrm{ann}_R(F)$. Hence $m.\mathrm{ann}_R(F) = 0$. It follows that $M.\mathrm{ann}_R(F) = 0$, so that $\mathrm{ann}_R(M) = \mathrm{ann}_R(F)$. □

Let R be any ring. We shall say that a right R-module M is *weakly finitely annihilated* provided $\mathrm{ann}_R(M) = \mathrm{ann}_R(N)$ for some finitely generated submodule N of M. It is not difficult to see that if a module is finitely annihilated then it is also weakly finitely annihilated.

Lemma 2.1.2. *The following statements are equivalent for a ring R.*

(i) R satisfies the descending chain condition on ideals.

(ii) Every right R-module is weakly finitely annihilated.

(iii) Every countably generated right R-module is weakly finitely annihilated.

(iv) Every left R-module is weakly finitely annihilated.

(v) Every countably generated left R-module is weakly finitely annihilated.

Proof. (i)⇒(ii) Adapt the proof of Lemma 2.1.1.

(ii)⇒(iii) Clear.

(iii)⇒(i) Let $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$ be any descending chain of ideals of R. Let M denote the right R-module $(R/I_1) \oplus (R/I_2) \oplus (R/I_3) \oplus \cdots$. Clearly M is countably generated. By hypothesis, there exists a finitely generated submodule N of M such that $\mathrm{ann}_R(M) = \mathrm{ann}_R(N)$. There exists a positive integer k such that $N \subseteq (R/I_1) \oplus \cdots \oplus (R/I_k)$. Then

$$I_k = \mathrm{ann}_R((R/I_1) \oplus \cdots \oplus (R/I_k)) \subseteq \mathrm{ann}_R(N) = \mathrm{ann}_R(M) = \bigcap_{n \geq 1} I_n,$$

so that $I_k = I_{k+1} = I_{k+2} = \cdots$.

(i)⇔(iv)⇔(v) By symmetry. □

Let R be any ring. For any right ideal A and element r of R, $(A : r)$ will denote the right ideal $\{s \in R | rs \in A\}$ of R. It is well known that if A is an essential right ideal of R

then so too is $(A : r)$ for any $r \in R$ (see, for example, [5, Lemma 1.1]), but this result is used several times throughout the thesis so its proof is included below.

Lemma 2.1.3. *Let R be a ring, let r be an element of R and let A be an essential right ideal of R. Then $(A : r)$ is an essential right ideal of R.*

Proof. Let I be a nonzero right ideal of R. If $rI = 0$ then $I \subseteq (A : r)$ so $I \cap (A : r) \neq 0$. Suppose now that $rI \neq 0$. Then $rI \cap A \neq 0$, so $0 \neq rx \in A$ for some $x \in I$. Clearly x is nonzero and $x \in (A : r)$, so that $I \cap (A : r) \neq 0$. Thus $(A : r)$ is an essential right ideal of R. $\square$

Lemma 2.1.4. *Let R be a ring such that every cyclic right R-module is finitely annihilated. Then R is right bounded.*

Proof. Let E be any essential right ideal of R. Let $I = \mathrm{ann}_R(R/E)$ and note that I is an ideal of R such that $I \subseteq E$. By hypothesis, there exists a positive integer k such that $I = \cap_{i=1}^{k} \mathrm{ann}_R(a_i + E)$ for some elements $a_i \in R$ $(1 \leq i \leq k)$. Clearly $\mathrm{ann}_R(a_i + E) = (E : a_i)$ so that, by Lemma 2.1.3, $\mathrm{ann}_R(a_i + E)$ is an essential right ideal of R for each $1 \leq i \leq k$. It follows that I is an essential right ideal of R. $\square$

Lemma 2.1.5. *Let R be a right bounded ring which satisfies the descending chain condition on ideals. Then R has essential right socle.*

Proof. Let I be an ideal of R which is minimal in the collection of ideals of R which are essential as right ideals. Let E be a right ideal of R contained in I such that E is an essential submodule of the right R-module I. Then E is an essential right ideal of R. By hypothesis, there exists an ideal J of R such that $J \subseteq E$ and J is an essential right ideal of R. Now $J \subseteq E \subseteq I$ gives $J = I$. Hence the right R-module I has no proper essential submodules. This implies that the right R-module I is semisimple and hence that I is contained in the right socle of R (see, for example, [1, Proposition 9.7]). $\square$

Lemma 2.1.6. *Let R be a ring such that every cyclic uniform right R-module is finitely annihilated and such that R satisfies the descending chain condition on ideals. Then the right R-module R has finite uniform dimension.*

Proof. Let $\mathcal{S}$ denote the collection of ideals A of R such that the right R-module R/A has finite uniform dimension. Note that $R \in \mathcal{S}$. By hypothesis, $\mathcal{S}$ has a minimal member, I say. Let $A \in \mathcal{S}$. Note that the right R-module $R/(A \cap I)$ embeds in the right R-module

$(R/A) \oplus (R/I)$ which has finite uniform dimension and hence so too does $R/(A \cap I)$. Thus $A \cap I \in \mathcal{S}$ and $A \cap I \subseteq I$. By the choice of I we have $A \cap I = I$ and hence $I \subseteq A$. Thus $I = \cap\{A | A \in \mathcal{S}\}$.

Suppose that $I \neq 0$. Let $0 \neq a \in I$. By Zorn's Lemma there exists a right ideal E of R maximal with respect to the property $a \notin E$. Then every nonzero submodule of the right R-module R/E contains the nonzero element $a + E$. Thus R/E is a cyclic uniform right R-module. By hypothesis, $\text{ann}_R(R/E) = \text{ann}_R(F)$ for some finite subset F of R/E. If $B = \text{ann}_R(R/E)$ then B is an ideal of R such that $B \subseteq E$ and the right R-module R/B embeds in the right R-module $(R/E)^n$, where $n = |F|$ (the cardinality of F). Thus $B \in \mathcal{S}$ and we obtain the contradiction $a \in I \subseteq B \subseteq E$. It follows that $I = 0$ and thus the right R-module R has finite uniform dimension. □

In order to prove our main theorem of this section we will require a result of J. A. Beachy concerning quasi-Artinian modules [2]. A module is said to be *quasi-Artinian* if it contains an essential Artinian submodule. This notion coincides with that of finitely embedded modules, which were introduced by P. Vámos [34]. A module is said to be *finitely embedded* if its injective hull is (isomorphic to) a finite direct sum of injective hulls of simple modules. Both of these are equivalent to the socle of the module being essential and finitely generated (equivalently, finite dimensional). A ring R is said to be *right quasi-Artinian* if the module R_R is quasi-Artinian. Vámos proved [34, Proposition 2*] that a module is Artinian if and only if each of its factor modules is finitely embedded. Beachy improved this result for rings and proved that a ring is right Artinian if and only if each of its factor rings is right quasi-Artinian [2, Proposition 5]. In order to prove this proposition we first require a number of preliminary lemmas taken from Beachy's and Vámos' papers.

A right module M over a ring R is said to be *co-faithful* if, for some integer n, M^n has a submodule isomorphic to R_R. A family $\{M_i\}_{i \in I}$ of submodules of a module M will be called an *inverse system* if for any finite number $i_1, \ldots, i_k$ of elements of I there is an element $i_0 \in I$ such that $M_{i_0} \subseteq M_{i_1} \cap \ldots \cap M_{i_k}$.

Lemma 2.1.7. *A module M is quasi-Artinian if and only if every inverse system of nonzero submodules of M is bounded below by a nonzero submodule of M.*

Proof. See [34, Proposition 1*]. □

Corollary 2.1.8. *Let R be a ring such that every factor ring of R is right quasi-Artinian. Then R satisfies the descending chain condition on two-sided ideals.*

Proof. Let $R = A_0 \supseteq A_1 \supseteq A_2 \supseteq \cdots$ be any descending chain of two-sided ideals of R. Let $A = \cap_{n\geq 0} A_n$. Then A is a two-sided ideal of R, so R/A is right quasi-Artinian. If the chain of ideals of R does not terminate then $\{R/A, A_1/A, A_2/A, \cdots\}$ is an inverse system of nonzero ideals of R/A, so, by Lemma 2.1.7, must be bounded below by a nonzero (right) ideal of R/A. But $\cap_{n\geq 0}(A_n/A) = (\cap_{n\geq 0} A_n)/A = A/A = 0$. This contradiction means that the original chain of ideals of R must terminate. □

The following couple of lemmas are taken from Beachy's paper [2, Proposition 2 and Proposition 3]. Note that [2, Proposition 3] in fact proves an improved version of Lemma 2.1.9.

Lemma 2.1.9. *Let R be a right quasi-Artinian ring. Then every faithful right R-module is co-faithful.*

Proof. Let M be a faithful right R-module. For any $m \in M$, let $f_m : R \to M$ be the R-homomorphism defined by $f_m(r) = mr$ for all $r \in R$. Since M is faithful, the intersection of the kernels of the homomorphisms f_m is zero. Consider the system S consisting of all finite intersections of the kernels of the homomorphisms f_m. This is an inverse system of submodules of the right R-module R. However, the intersection of all its elements is zero so it is not bounded below by a nonzero submodule of R_R. Since R is right quasi-Artinian it follows, by Lemma 2.1.7, that S does not consist of nonzero submodules of R_R. Hence some finite intersection of the kernels of the homomorphisms f_m is zero. The corresponding homomorphisms give an embedding of R_R into M^n, for corresponding n. Thus M is co-faithful. □

Lemma 2.1.10. *Let R be a right quasi-Artinian ring. Then R is semiprime if and only if R is semiprimitive and right Artinian.*

Proof. Sufficiency is clear, since a semiprimitive ring is semiprime. Conversely, suppose that R is semiprime. Now $\mathrm{Soc}(R_R).\mathrm{ann}_R(\mathrm{Soc}(R_R)) = 0$ implies that $\mathrm{Soc}(R_R) \cap \mathrm{ann}_R(\mathrm{Soc}(R_R)) = 0$, since R is semiprime. Thus $\mathrm{ann}_R(\mathrm{Soc}(R_R)) = 0$, since $\mathrm{Soc}(R_R)$ is essential, so $\mathrm{Soc}(R_R)$ is faithful and hence co-faithful, by Lemma 2.1.9. This shows that R_R can be embedded in $(\mathrm{Soc}(R_R))^n$ for some integer n, so R_R is a finite direct sum of minimal right ideals. Therefore R semisimple and hence semiprimitive and right Artinian. □

If R is a ring, M is a left R-module and A is a non-empty subset of M, then we will denote the left annihilator of A in R (defined analogously to the right annihilator of a non-empty subset of a right R-module) by $\mathrm{l.ann}_R(A)$.

Proposition 2.1.11. *Let R be a ring. Then R is right Artinian if and only if every factor ring of R is right quasi-Artinian.*

Proof. If R is right Artinian then every factor ring of R is right Artinian and hence right quasi-Artinian.

Conversely, suppose that R/A is right quasi-Artinian for every two-sided ideal A of R. By Corollary 2.1.8, R satisfies the descending chain condition on two-sided ideals. In particular, if J is the Jacobson radical of R, then the descending chain $R = J^0 \supseteq J \supseteq J^2 \supseteq J^3 \supseteq \cdots$ must become stationary after a finite number of steps, n say. Let $B = J^n = J^{n+1} = \cdots$. Suppose that $B \neq 0$. Let $A = B \cap \mathrm{l.ann}_R(B)$, then A is a two-sided ideal of R and $A \subsetneq B$, since $B^2 = B \neq 0$ so $B \nsubseteq \mathrm{l.ann}_R(B)$. By hypothesis, R/A is right quasi-Artinian, so $B/A \neq 0$ must contain a minimal nonzero right ideal. Let C be the inverse image in R of this minimal right ideal. Then $A \subsetneq C \subseteq B$, so there exists an element $0 \neq c \in C$ such that $cB \neq 0$. But $cB \subseteq C$ and $(cB)B = cB^2 = cB \neq 0$ so $cB \nsubseteq A$. Since C/A is minimal, we must have $cB \equiv C \mod A$. Therefore there exists an element $b \in B$ such that $c - cb \in A$. Now, $b \in B = J^n \subseteq J$ implies that $1 - b$ has a right inverse, say $(1-b)b' = 1$ for some $b' \in R$. Then $c = c(1-b)b' = (c-cb)b' \in A$, which contradicts the fact that $cB \neq 0$. It follows that $B = 0$, so J is nilpotent.

R/J is semiprimitive and, by assumption, is right quasi-Artinian. By Lemma 2.1.10, it is right Artinian, so is semisimple and hence every right (R/J)-module is a direct sum of simple modules. In particular, this is true for J^i/J^{i+1} for $i = 0, 1, 2, \ldots$, so as an R-module each of these is a direct sum of simple R-modules. Regarding J^i/J^{i+1} as a right ideal of the right quasi-Artinian ring R/J^{i+1}, it is a sum of minimal right ideals, so is contained in the socle of R/J^{i+1}, which, by assumption, has a composition series.

This shows that, for $i = 0, 1, 2, \ldots$, J^i/J^{i+1} has a composition series as a right (R/J^{i+1})-module and hence as a right R-module. Since J is nilpotent, this implies that R_R has a composition series and thus R is right Artinian. □

We are now able to prove our main theorem of this section.

Theorem 2.1.12. *The following statements are equivalent for a ring R.*

(i) R is right Artinian.

(ii) Every right R-module is finitely annihilated.

(iii) Every countably generated right R-module is finitely annihilated.

(iv) R satisfies the descending chain condition on (two-sided) ideals and every cyclic right R-module is finitely annihilated.

Proof. (i)$\Rightarrow$(ii) By Lemma 2.1.1.

(ii)$\Rightarrow$(iii) Clear.

(iii)$\Rightarrow$(iv) By Lemma 2.1.2

(iv)$\Rightarrow$(i) By Lemmas 2.1.4 and 2.1.5, R has essential right socle and by Lemma 2.1.6 the right socle of R is finitely generated. Let I be any ideal of R. Clearly R/I inherits (iv) from R so that R/I has finitely generated essential right socle. Thus every ring homomorphic image of R has finitely generated essential right socle and hence R is right Artinian by 2.1.11. □

Note that one consequence of the theorem is that a right Artinian ring R is left Artinian if and only if every cyclic left R-module is finitely annihilated.

2.2 On Families of Finitely Annihilated Modules

Having considered the case when every module over a ring is finitely annihilated, we consider in this section the circumstances under which various restricted families of modules are finitely annihilated. In particular we consider simple and semisimple modules, hereditary torsion theories, uniform and finite dimensional modules and injective modules, proving results similar to the main theorem of the previous section, but involving, for the most part, Artinian properties of quotients of the ring.

2.2.1 Simple and Semisimple Modules

First we consider the case when every simple or every semisimple module is finitely annihilated. Recall that an ideal is called right primitive if it is the annihilator of a simple module and that the Jacobson radical of a ring can be characterised as the intersection of the right primitive ideals of the ring. Before our main result of this section we have a preparatory lemma.

Lemma 2.2.1. *The following statements are equivalent for a ring R.*

(i) R/P is Artinian for every right primitive ideal P of R.

(ii) Every simple right R-module is finitely annihilated.

Proof. (i)⇒(ii) Let M be a simple right R-module. Then $P = \mathrm{ann}_R(M)$ is a right primitive ideal of R and M can be considered as a right (R/P)-module. Since R/P is Artinian, $M_{R/P}$ is finitely annihilated, by Theorem 2.1.12, and it follows that M is finitely annihilated as a right R-module.

(ii)⇒(i) Let P be a right primitive ideal of R. Then $P = \mathrm{ann}_R(U)$ for some simple right R-module U. By hypothesis, U is finitely annihilated, so $P = \mathrm{ann}_R(U) = \mathrm{ann}_R(u_1, \ldots, u_n)$ for some finite subset $\{u_1, \ldots, u_n\}$ of U. It follows that $R/P \hookrightarrow U^n$ as right R-modules, via the map $r + P \mapsto (u_1 r, \ldots, u_n r)$ $(r \in R)$. Hence R/P is right Artinian and hence also left Artinian, being a prime ring. Thus R/P is Artinian. □

Theorem 2.2.2. *The following statements are equivalent for a ring R with Jacobson radical J.*

(i) R/J is Artinian.

(ii) Every semisimple right R-module is finitely annihilated.

(iii) Every countably generated semisimple right R-module is finitely annihilated.

Proof. (i)⇒(ii) Let M be a semisimple right R-module. Then $MJ = 0$, so M can be considered as a right (R/J)-module. Since R/J is right Artinian, $M_{R/J}$ is finitely annihilated, by Theorem 2.1.12, and it follows that M is finitely annihilated as a right R-module.

(ii)⇒(iii) Clear.

(iii)⇒(i) By hypothesis every simple right R-module is finitely annihilated, so, by Lemma 2.2.1, R/P is Artinian for every right primitive ideal P of R. It follows that every right primitive ideal of R is maximal, since right primitive ideals are prime.

Suppose that R contains an infinite number of right primitive ideals and let $P_1, P_2, \ldots$ be distinct right primitive ideals of R. Then $M = (R/P_1) \oplus (R/P_2) \oplus \cdots$ is a countably generated semisimple right R-module, so is finitely annihilated, and thus $\mathrm{ann}_R(M) = \mathrm{ann}_R(m_1, \ldots, m_n)$ for some finite subset $\{m_1, \ldots, m_n\}$ of M. There exists an integer $k \geq 1$ such that $\{m_1, \ldots, m_n\} \subseteq R/P_1 \oplus \cdots \oplus R/P_k$ and hence

$$\begin{aligned} P_1 \cap \ldots \cap P_k &= \mathrm{ann}_R((R/P_1) \oplus \cdots \oplus (R/P_k)) \\ &\subseteq \mathrm{ann}_R(m_1, \ldots, m_n) = \mathrm{ann}_R(M) = \bigcap_{i \geq 1} P_i \subseteq P_{k+1}, \end{aligned}$$

which implies that $P_i = P_{k+1}$ for some $1 \leq i \leq k$, a contradiction. It follows that J is a finite intersection of maximal ideals, say $J = Q_1 \cap \ldots \cap Q_m$. Then the map $R \to (R/Q_1) \oplus \cdots \oplus (R/Q_m)$ given by $r \mapsto (r + Q_1, \ldots, r + Q_m)$ $(r \in R)$, gives rise to an embedding $R/J \hookrightarrow (R/Q_1) \oplus \cdots \oplus (R/Q_m)$ of right R-modules. It follows that R/J is right Artinian and hence also left Artinian, being a semiprime ring. Thus R/J is Artinian. $\square$

Note that condition (i) is symmetric and so left-handed versions of conditions (ii) and (iii) are also equivalent.

2.2.2 Hereditary Torsion Theories

Let R be a ring and let τ be a hereditary torsion theory on Mod-R. For the definition and basic properties of hereditary torsion theories see [33, Chapter VI]. Let M be a right R-module. A submodule N of M is called a *τ-dense submodule* of M if M/N is a τ-torsion right R-module. We define $\mathrm{Rej}_\tau(M) = \cap\{N | N \text{ is a } \tau\text{-dense submodule of } M\}$. Before our main theorem of this section we require a couple of lemmas.

Lemma 2.2.3. *Let R be a ring, let τ be a hereditary torsion theory on Mod-R and let M be any right R-module. Then $\mathrm{Rej}_\tau(M) = \cap\{\ker f | f : M \to T$ an R-homomorphism for some τ-torsion right R-module $T\}$.*

Proof. Suppose that $x \in \mathrm{Rej}_\tau(M)$ and let $f : M \to T$ be an R-homomorphism for some τ-torsion right R-module T. Then $M/\ker f \cong \mathrm{im} f \subseteq T$, so $\ker f$ is a τ-dense submodule of M. Thus $x \in \ker f$. Conversely, suppose that $x \in \cap\{\ker f | f : M \to T$ an R-homomorphism for some τ-torsion right R-module $T\}$ and let N be a τ-dense submodule of M. Consider the natural homomorphism $g : M \to M/N$. Then $x \in \ker g = N$. Thus $x \in \mathrm{Rej}_\tau(M)$. $\square$

Lemma 2.2.4. *Let R be a ring and let τ be a hereditary torsion theory on Mod-R. Then $\mathrm{Rej}_\tau(R_R)$ is a two-sided ideal of R.*

Proof. It is clear that $\mathrm{Rej}_\tau(R_R)$ is a right ideal of R. Suppose that $a \in \mathrm{Rej}_\tau(R_R)$ and let $r \in R$. Define a map $f : R \to R$ by $f(x) = rx$ for all $x \in R$. Then f is an R-homomorphism. Let T be a τ-torsion right R-module and let $g : R \to T$ be an R-homomorphism. Then $gf : R \to T$ is an R-homomorphism so, by Lemma 2.2.3, $a \in \ker(gf)$. Thus $f(a) \in \ker g$, that is, $ra \in \ker g$. Hence, again by Lemma 2.2.3, $ra \in \mathrm{Rej}_\tau(R_R)$. It follows that $\mathrm{Rej}_\tau(R_R)$ is a two-sided ideal of R. $\square$

Theorem 2.2.5. *Let R be a ring and let τ be a hereditary torsion theory on Mod-R. Then the following statements are equivalent.*

(i) $R/Rej_\tau(R_R)$ is right Artinian.

(ii) R satisfies the descending chain condition on τ-dense right ideals.

(iii) Every τ-torsion right R-module is finitely annihilated.

(iv) Every countably generated τ-torsion right R-module is finitely annihilated.

(v) R satisfies the descending chain condition on τ-dense two-sided ideals and every cyclic τ-torsion right R-module is finitely annihilated.

Proof. (i)⇒(ii) Since, by definition, $\mathrm{Rej}_\tau(R_R) = \cap\{B|B$ is a τ-dense right ideal of $R\}$, this is clear.

(ii)⇒(iii) Let M be a τ-torsion right R-module. Let F be a finite subset of M and let $A = \mathrm{ann}_R(F)$, say $A = \mathrm{ann}_R(f_1, \ldots, f_k)$ for some $k \geq 1$ and $f_1, \ldots, f_k \in M$. Then R/A embeds in M^k as a right R-module via the map $r + A \mapsto (f_1 r, \ldots, f_k r)$ for all $r \in R$. It follows that R/A is a τ-torsion right R-module, that is A is a τ-dense right ideal of R. By hypothesis we can choose $A = \mathrm{ann}_R(F)$ minimal amongst the annihilators of finite subsets of M. It is then easy to show that $\mathrm{ann}_R(M) = \mathrm{ann}_R(F)$ (see Lemma 2.1.1).

(iii)⇒(iv) Clear.

(iv)⇒(v) Let $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$ be any descending chain of τ-dense ideals of R. Let $X = R/I_1 \oplus R/I_2 \oplus R/I_3 \oplus \cdots$. Then X is a countably generated τ-torsion right R-module so, by hypothesis, X is finitely annihilated. By the proof of (iii)⇒(i) in Theorem 2.2.2, there exists a positive integer k such that $I_1 \cap \ldots \cap I_k \subseteq \cap_{j \geq 1} I_j$ and hence $I_k = I_{k+1} = I_{k+2} = \cdots$. This proves (v).

(v)⇒(i) Let A be a minimal τ-dense ideal of R. If B is any τ-dense ideal of R then $A \cap B$ is a τ-dense ideal of R and $A \supseteq A \cap B$, so that $A = A \cap B \subseteq B$. Let E be any τ-dense right ideal of R and let $C = \mathrm{ann}_R(R/E)$. Note that the right R-module R/E is cyclic and τ-torsion. By hypothesis, $C = \mathrm{ann}_R(u_1, \ldots, u_n)$ for some integer $n \geq 1$ and finite subset $\{u_1, \ldots, u_n\}$ of R/E and hence R/C embeds in the right R-module $(R/E)^n$. It follows that C is a τ-dense ideal of R and we deduce that $A \subseteq C \subseteq E$. Thus $A = \mathrm{Rej}_\tau(R_R)$. In particular, $\mathrm{Rej}_\tau(R_R)$ is a τ-dense ideal of R.

Now, (v) gives that R/A satisfies the descending chain condition on two-sided ideals. Moreover, if V is a cyclic right (R/A)-module then V is a cyclic right R-module such that

$VA = 0$. Let $V = vR$, for some $v \in V$, and let $E = \mathrm{ann}_R(v)$. Then $V = vR \cong R/E$, as right R-modules. Now $A \subseteq E$, so R/E is isomorphic to a quotient of R/A and hence is τ-torsion. Thus V is a cyclic τ-torsion right R-module and, by hypothesis, V is finitely annihilated. Thus every cyclic right (R/A)-module is finitely annihilated. By Theorem 2.1.12, the ring R/A is right Artinian. □

Note that Theorem 2.2.5 implies Theorem 2.1.12, by taking τ to be the hereditary torsion theory in which every right R-module is torsion, in which case $\mathrm{Rej}_\tau(R_R) = 0$ and every right ideal of R is τ-dense. However, we believe it is worthwhile to consider the proof of Theorem 2.1.12 separately.

By adapting the proof of Theorem 2.2.5, the following result concerning singular modules and the right socle of a ring can be proved.

Theorem 2.2.6. *The following statements are equivalent for a ring R with right socle $Soc(R_R)$.*

(i) $R/Soc(R_R)$ is right Artinian.

(ii) R satisfies the descending chain condition on essential right ideals.

(iii) Every singular right R-module is finitely annihilated.

(iv) Every countably generated singular right R-module is finitely annihilated.

(v) R satisfies the descending chain condition on two-sided ideals which are essential as right ideals and every cyclic singular right R-module is finitely annihilated.

It might be expected that Theorem 2.2.6 could be deduced from Theorem 2.2.5 by means of the Goldie torsion theory, since this well-known torsion theory relates singular modules and essential right ideals. However, the following example shows that it is not clear how this could be done.

For the Goldie torsion theory, the torsion-free modules are precisely the nonsingular modules and the torsion modules are the modules M such that $\mathrm{Z}_2(M) = M$, where $Z_2(M)$ is defined to be the submodule of M such that $Z_2(M)/\mathrm{Z}(M) = \mathrm{Z}(M/\mathrm{Z}(M))$. For details of the Goldie torsion theory see [11] or [33].

Example 2.2.7. Let F be a field and let V be an infinite dimensional vector space over F. Let R denote the "matrix ring" consisting of all matrices of the form $\begin{pmatrix} a & v \\ 0 & a \end{pmatrix}$, where

$a \in F$ and $v \in V$, with the usual addition and multiplication of matrices. Then R is a commutative ring and $\mathrm{Soc}(R) = \begin{pmatrix} 0 & V \\ 0 & 0 \end{pmatrix}$, so $R/\mathrm{Soc}(R) \cong F$ is Artinian.

Consider the Goldie torsion theory on R, which we will denote by τ_G. For all $0 \neq v \in V$, $\mathrm{ann}_R(\begin{pmatrix} 0 & v \\ 0 & 0 \end{pmatrix}) = \begin{pmatrix} 0 & V \\ 0 & 0 \end{pmatrix}$, which is an essential ideal of R. It follows that R is Goldie torsion, that is 0 is a τ_G-dense ideal, and hence $\mathrm{Rej}_{\tau_G}(R) = 0$. Thus $R/\mathrm{Rej}_{\tau_G}(R) \cong R$, which is not Artinian because V does not have finite uniform dimension. We conclude that for this ring R, every singular module is finitely annihilated but not every Goldie torsion module is finitely annihilated.

2.2.3 Injective Modules

Let R be a ring. A right R-module E is called *injective* if for all right R-modules A and B such that A embeds in B, via an R-monomorphism $\psi : A \to B$ say, any R-homomorphism $f : A \to E$ can be extended to an R-homomorphism $g : B \to E$ such that $g\psi = f$. An equivalent condition is that for all right ideals I of R, any R-homomorphism $f' : I \to E$ can be extended to an R-homomorphism $g' : R \to E$ which agrees with f' on I (Baer's Lemma).

For any module M, there is a unique (up to isomorphism) essential injective extension of M, which we will call the *injective hull* of M and will denote by $\mathrm{E}(M)$. A nonzero module is called *indecomposable* if it has no direct summands other than 0 and itself. In this section we consider the case when, for certain rings, every (indecomposable) injective module is finitely annihilated. For more details on injective modules see [28].

An element r of a ring R is said to be a *left zero-divisor* if it is not right regular, that is if $rs = 0$ for some nonzero element s of R. Let R be a ring and let E be a right R-module. An element e of E is said to be *divisible* if for every element r of R which is not a left zero-divisor, there exists an element e' of E such that $e = e'r$. If every element of E is divisible, then E is said to be a *divisible module*. Alternatively, E is divisible if $E = Er$ for every element r of R which is not a left zero-divisor. We begin with a couple of well-known results.

Lemma 2.2.8. *Every injective module is divisible.*

Proof. Let R be a ring and let E be an injective right R-module. Let $e \in E$ and let r be an element of R which is not a left zero-divisor. Let $f : rR \to E$ be the map defined by $f(rs) = es$ for all $s \in R$. Since r is not a left zero-divisor, f is a well-defined R-homomorphism. Hence f can be extended to an R-homomorphism $g : R \to E$. Thus $e = f(r) = g(r) = g(1.r) = g(1)r$. It follows that E is divisible. □

Lemma 2.2.9. *Let R be a commutative Noetherian ring, let P be a maximal ideal of R and let $E = E(R/P)$. Then*

(i) For each $e \in E$ there exists a positive integer n such that $eP^n = 0$.

(ii) $ann_R(E) = \cap_{k=1}^{\infty} P^k$.

Proof. See [28, Proposition 4.23]. □

Lemma 2.2.10. *A ring R is right Artinian if and only if R is right Noetherian and R/P is right Artinian for all prime ideals P of R.*

Proof. It is well known that a right Artinian ring is right Noetherian (see, for example, [28, Theorem 3.25 Corollary]), so the necessity is clear. Conversely, suppose that R is right Noetherian and that R/P is right Artinian for every prime ideal P of R. By Proposition 1.4.11, there are prime ideals $P_1, \ldots, P_k$ of R for some integer $k \geq 1$ such that $P_1 \cdots P_k = 0$. Consider the chain $R \supseteq P_1 \supseteq P_1P_2 \supseteq \cdots \supseteq P_1 \cdots P_k = 0$. For each $2 \leq i \leq k$, the factor $P_1 \cdots P_{i-1}/P_1 \cdots P_{i-1}P_i$ is a finitely generated module over the prime Artinian ring R/P_i and, as such, is itself Artinian. Thus each factor in this chain is Artinian as a right R-module and hence the ring R is right Artinian. □

Note that a consequence of this result is that in a right Artinian ring every prime ideal is maximal. In general the converse does not necessarily hold, however for commutative rings we have the following corollary.

Corollary 2.2.11. *A commutative ring R is Artinian if and only if R is Noetherian and every prime ideal of R is maximal.*

Proof. This follows by Lemma 2.2.10, since a (right) Artinian ring is prime if and only if it is simple (see [12, Corollary 3.18]). □

Theorem 2.2.12. *A commutative Noetherian ring R is Artinian if and only if every indecomposable injective R-module is finitely annihilated.*

Proof. ($\Rightarrow$) By Theorem 2.1.12.

($\Leftarrow$) Let P be a maximal ideal of R and let $E = \mathrm{E}(R/P)$. Then E is an indecomposable injective R-module. Let $A = \mathrm{ann}_R(E)$. By Lemma 2.2.9, $A = \bigcap_{k=1}^{\infty} P^k$. By hypothesis, E is finitely annihilated, so $A = \mathrm{ann}_R(e_1, \ldots, e_n)$ for some $n \geq 1$ and $e_i \in E$ $(1 \leq i \leq n)$. Again by Lemma 2.2.9, there exists $m \geq 1$ such that $e_i P^m = 0$ for $1 \leq i \leq n$, so $P^m \subseteq A$. Thus $P^m = P^{m+1} = \cdots$. Now let Q be a prime ideal of R such that $Q \subseteq P$. By Krull's Intersection Theorem $P^m = \bigcap_{k=1}^{\infty} P^k \subseteq Q$ (see, for example, [35, p. 216 Corollary 1]), so $P \subseteq Q$ and thus $P = Q$. It follows that every prime ideal of R is maximal and hence R is Artinian by Corollary 2.2.11. $\square$

The example in the following section shows that this result does not hold for an arbitrary commutative ring. However we are able to prove some partial results for right Noetherian rings. We first require a number of lemmas and begin by outlining some basic properties of injective modules.

Lemma 2.2.13. *Let R be a ring, let A be a right R-module and let a be a nonzero element of A. Then there is a simple right R-module S and an R-homomorphism $\phi : A \to \mathrm{E}(S)$ such that $\phi(a) \neq 0$.*

Proof. Consider the right ideal $\mathrm{ann}_R(a)$ of R. Since $a \neq 0$, this is a proper right ideal of R, so is contained in a maximal right ideal M of R. Define a mapping $\phi' : aR \to \mathrm{E}(R/M)$ by $\phi'(ar) = r + M$ for all $r \in R$. Note that ϕ' is well-defined, since $\mathrm{ann}_R(a) \subseteq M$. Further, ϕ' is an R-homomorphism and $\phi'(a) = 1 + M \neq 0$. Put $S = R/M$, so S is a simple right R-module. Since $\mathrm{E}(S)$ is injective, there is a homomorphism $\phi : A \to \mathrm{E}(S)$ extending ϕ' such that $\phi(a) = \phi'(a) \neq 0$. $\square$

Lemma 2.2.14. *Let R be a ring and let $\{E_\lambda\}_{\lambda \in \Lambda}$ be a family of right R-modules for some index set Λ. Then the direct product $\prod_{\lambda \in \Lambda} E_\lambda$ is injective if and only if each E_λ is injective.*

Proof. Put $E = \prod_{\lambda \in \Lambda} E_\lambda$ and let $\phi_\lambda : E_\lambda \to E$ and $\pi_\lambda : E \to E_\lambda$ be the injections and projections, respectively, associated with this direct product.

Suppose first that E is injective and let $\lambda \in \Lambda$. Let A and B be right R-modules such that there is an R-monomorphism $\psi : A \to B$ and let $\mu : A \to E_\lambda$ be an R-homomorphism. Then $\phi_\lambda \mu$ gives an R-homomorphism from A to E. Since E is injective, this can be extended to an R-homomorphism $h : B \to E$ such that $h\psi = \phi_\lambda \mu$. Now define a map

$h' : B \to E_\lambda$ by $h' = \pi_\lambda h$. Then h' is an R-homomorphism and $h'\psi = \pi_\lambda h\psi = \pi_\lambda\phi_\lambda\mu = \mu$. It follows that E_λ is injective.

Conversely, suppose that each E_λ is injective. Let A and B be right R-modules such that there is an R-monomorphism $\psi : A \to B$ and let $f : A \to E$ be an R-homomorphism. For each λ, since E_λ is injective, there is an R-homomorphism $g_\lambda : B \to E_\lambda$ such that $g_\lambda\psi = \pi_\lambda f$. Define a map $g : B \to E$ by $g = \{g_\lambda\}_{\lambda\in\Lambda}$. Then g is an R-homomorphism and $g\psi = \{g_\lambda\psi\}_{\lambda\in\Lambda} = \{\pi_\lambda f\}_{\lambda\in\Lambda} = f$. It follows that E is injective. □

Corollary 2.2.15. *Let R be a ring and let $\{E_\lambda\}_{\lambda\in\Lambda}$ be a family of right R-modules for some index set Λ.*

(i) If $\oplus_{\lambda\in\Lambda}E_\lambda$ is injective, then each E_λ is injective.

(ii) If the index set Λ is finite and each E_λ is injective, then $\oplus_{\lambda\in\Lambda}E_\lambda$ is injective.

Proof. This follows by Lemma 2.2.14, since the direct product and the direct sum of a finite family of modules coincide. □

Note that over an arbitrary ring, although a direct product of injective modules is injective, it is not necessarily true that a direct sum of injective modules is injective, unless, as in Lemma 2.2.15, the sum is finite. For Noetherian rings, however, we have the following result.

Lemma 2.2.16. *Let R be a right Noetherian ring. Then every direct sum of injective right R-modules is injective.*

Proof. Let $\{E_i\}_{i\in I}$ be a family of injective right R-modules and let $E = \oplus_{i\in I}E_i$. Let B be a right ideal of R and let $f : B \to E$ be an R-homomorphism. Since R is right Noetherian, B is finitely generated and it follows that there exists a finite subset J of I such that $f(B) \subseteq E' = \oplus_{j\in J}E_j$. As a finite direct sum of injective modules, E' is itself injective by Corollary 2.2.15, so there exists an R-homomorphism $\phi' : R \to E'$ which agrees with f on B. Then f is extended by the R-homomorphism $\phi : R \to E$ given by ϕ' followed by the inclusion mapping. Hence E is injective. □

Corollary 2.2.15 also has the following consequence.

Lemma 2.2.17. *Let R be a ring. A right R-module E is injective if and only if E is a direct summand of every extension of itself.*

Proof. Suppose that E is injective and let E' be an extension of E. The identity map $\iota : E \to E$ can be extended to an R-homomorphism $\theta : E' \to E$. Let $e' \in E'$. Then $\theta(e') \in E$, so that $\theta(\theta(e')) = \iota(\theta(e')) = \theta(e')$. Thus $\theta(e') - e' \in \ker\theta$, so $e' \in E + \ker\theta$ and hence $E' = E + \ker\theta$. However, $E \cap \ker\theta = 0$, so $E' = E \oplus \ker\theta$.

Conversely suppose that E is a direct summand of every extension of itself. In particular, E is a direct summand of its injective hull $\mathrm{E}(E)$, so, by Corollary 2.2.15, is itself injective. □

We are now able to prove the following partial result concerning the finite annihilation of injective modules in right Noetherian rings.

Lemma 2.2.18. *Let R be a right Noetherian ring such that every injective right R-module is finitely annihilated. Then R has essential right socle.*

Proof. Let $E = \oplus\mathrm{E}(R/M)$ where the direct sum is taken over all the maximal right ideals M of R. Consider the ideal $\mathrm{ann}_R(E)$ of R. Suppose that $\mathrm{ann}_R(E) \neq 0$ and let $0 \neq a \in \mathrm{ann}_R(E)$. By Lemma 2.2.13, there is a maximal right ideal M of R and an R-homomorphism $\phi : R \to \mathrm{E}(R/M)$ such that $\phi(a) \neq 0$. Now $\phi(a) = \phi(1.a) = \phi(1)a = 0$. This contradiction shows that E must be a faithful right R-module.

By Lemma 2.2.16, E is injective, so is finitely annihilated and thus the right R-module R_R embeds in E^n for some integer $n \geq 1$. It follows that R embeds in some finite direct sum of injective hulls of simple right R-modules and hence R has essential right socle. □

In order to prove our final result of this section we require the following well-known result of Lenagan concerning bimodules.

Proposition 2.2.19 (Lenagan). *Let R and S be rings and let ${}_SM_R$ be a bimodule such that M_R is Noetherian and ${}_SM$ is both Noetherian and Artinian. Then M_R is Artinian.*

Proof. Suppose on the contrary that M_R is not Artinian. Since M_R is Noetherian we can choose a maximal sub-bimodule N of ${}_SM_R$ such that $(M/N)_R$ is not Artinian. Therefore, without loss of generality, we may assume that $(M/K)_R$ is Artinian for all nonzero sub-bimodules K of ${}_SM_R$. Since ${}_SM$ is Artinian we can choose a minimal nonzero sub-bimodule N' of ${}_SM_R$. Then ${}_SN'_R$ is a simple bimodule and, since $(M/N')_R$ is Artinian, N'_R is not Artinian. Therefore, without loss of generality, we may assume that ${}_SM_R$ is a simple bimodule. By factoring out the ideals $\mathrm{l.ann}_S(M)$ of S and $\mathrm{ann}_R(M)$ of R, we may also assume, without loss of generality, that both ${}_SM$ and M_R are faithful.

Let A and B be ideals of R such that $AB = 0$. Then $MAB = 0$. But MA is a sub-bimodule of ${}_SM_R$ so $MA = 0$ or $MA = M$ and in the latter case $MB = 0$. Thus $A = 0$ or $B = 0$. Therefore R is a prime ring. Similarly S is a prime ring. Since ${}_SM$ is Noetherian, $M = Sm_1 + \cdots + Sm_n$ for some positive integer n and elements $m_i \in M$ $(1 \leq i \leq n)$. Then $\cap_{i=1}^{n} \mathrm{ann}_R(m_i) = \mathrm{ann}_R(M) = 0$ so the mapping defined by $r \mapsto (m_1 r, \ldots, m_n r)$ for all $r \in R$ gives an embedding of right R-modules, $R \hookrightarrow (M_R)^n$. Since M_R is Noetherian it follows that R is a right Noetherian ring.

Let $\mathcal{C}(0)$ denote the set of regular elements of R (note that, since R is a prime right Noetherian ring, right regular elements of R are regular (see [5, Corollary 1.13])). Let $N = \{x \in M | xc = 0 \text{ for some } c \in \mathcal{C}(0)\} = \mathrm{T}(M)$. Then, since R is a prime right Noetherian ring and so satisfies the right Ore condition, N is a right R-submodule of M. But N is clearly a left S-submodule of M, so in fact N is a sub-bimodule of ${}_SM_R$ and hence either $N = 0$ or $N = M$. If $N = M$ then $m_i c_i = 0$ for some $c_i \in \mathcal{C}(0)$ for each $1 \leq i \leq n$. Then, by the Ore Condition, there exists $c \in \mathcal{C}(0)$ such that $m_i c = 0$ for all $1 \leq i \leq n$ and hence $Mc = 0$, which contradicts M_R being faithful. Thus $N = 0$. It follows that for any $c \in \mathcal{C}(0)$, the map $M \to Mc$ given by $m \mapsto mc$ for all $m \in M$ is an S-monomorphism. Therefore, as S-modules, the composition length of M must equal the composition length of its submodule Mc. Since ${}_SM$ is Artinian, it follows that $M = Mc$ for any $c \in \mathcal{C}(0)$.

Now R is a prime right Noetherian ring, so is prime right Goldie and thus has a right quotient ring Q, where Q is a simple Artinian ring. Let $m \in M$, $r \in R$ and $c \in \mathcal{C}(0)$. Then $mr \in M = Mc$, so $mr = m'c$ for some $m' \in M$. Define a right action of Q on M by $m.(rc^{-1}) = m'$, where m, r, c and m' are as above. If $mr = m'c = m''c$ for some $m', m'' \in M$ then $(m' - m'')c = 0$, so $m' - m'' \in N = 0$ and thus $m' - m'' = 0$, that is $m' = m''$. It follows that the right action of Q on M is well-defined. It can be checked that this gives a right module action of Q on M. We shall denote this module structure in the usual way by M_Q. Since M_R is Noetherian, M is finitely generated as a right R-module and hence as a right Q-module. Since Q is Artinian, it follows that M_Q is Artinian.

Since Q is a simple Artinian ring, there is a unique (up to isomorphism) simple right Q-module U. Since both M_Q and Q_Q are Artinian and semisimple, we have $Q \cong U^n$ and $M \cong U^m$ for some positive integers n and m. This gives an embedding $Q \hookrightarrow M^n$ as right Q-modules and hence as right R-modules. Thus, Q_R is isomorphic to a submodule of $(M_R)^n$. Since M_R is Noetherian it follows that Q_R is Noetherian.

Now, let $c \in \mathcal{C}(0)$. Then the ascending chain of R-submodules of Q,

$$c^{-1}R \subseteq c^{-2}R \subseteq \cdots$$

must terminate, so there exists an integer $n \geq 1$ such that $c^{-n}R = c^{-n-1}R$. Thus $c^{-n-1} = c^{-n}b$ for some $b \in R$ and hence $1 = cb$, so $cR = R$. It follows that $cR = R$ for all $c \in \mathcal{C}(0)$.

Since R is a prime right Goldie ring, a right ideal E of R is an essential right ideal of R if and only if $cR \subseteq E$ for some $c \in \mathcal{C}(0)$ and this holds if and only if $E = R$. Therefore the module R_R is semisimple (see [12, Corollary 3.24]) and hence R is a semisimple Artinian ring. In particular R is right Artinian. Now M_R is Noetherian, so finitely generated, and hence M_R is Artinian. This is a contradiction, so M_R must in fact be Artinian. $\square$

We also require the following result of Ginn and Moss.

Lemma 2.2.20. *Let R be a left and right Noetherian ring and suppose that the right socle of R is essential as a right ideal or as a left ideal. Then R is left and right Artinian.*

Proof. See [5, Theorem 4.6]. $\square$

Theorem 2.2.21. *The following statements are equivalent for a right and left Noetherian ring R.*

(i) R is right Artinian.

(ii) R is left Artinian.

(iii) Every injective right R-module is finitely annihilated.

(iv) Every injective left R-module is finitely annihilated.

Proof. By Proposition 2.2.19, R is right Artinian if and only if it is left Artinian, so (i)$\Leftrightarrow$(ii). Since (i)$\Rightarrow$(iii) is clear by Theorem 2.1.12, it suffices to show that (iii)$\Rightarrow$(i). Suppose that every injective right R-module is finitely annihilated. By Lemma 2.2.18, R has essential right socle. Lemma 2.2.20 then shows that R is right and left Artinian. $\square$

Note that we do not know an example of a right Noetherian ring R such that every injective right R-module is finitely annihilated but R is not right Artinian.

2.2.4 Uniform and Finite Dimensional Modules

We now consider the case when, over certain rings, the uniform modules and modules with finite uniform dimension are finitely annihilated. We show that over a ring with right Krull dimension these conditions are equivalent to each other and to the ring being right Artinian. We later give an example which shows that this is not the case in general. We begin with a couple of lemmas. The first is taken from [5, Theorem 1.24].

Lemma 2.2.22. *Let R be a prime right Goldie ring and suppose that R has nonzero right socle. Then R is a simple Artinian ring.*

Proof. Let $S = \mathrm{Soc}(R_R)$. Because R is prime, every nonzero two-sided ideal of R is essential as a one-sided ideal and it follows that for every nonzero ideal I of R, $S \subseteq I$ (since, by [5, Lemma 1.2], the socle of a module M can also be characterised as the intersection of all the essential submodules of M). Now, being an essential right ideal, S contains a regular element of R, c say, and we have $S = cS$. Clearly $cS \subseteq cR \subseteq S$ and therefore $cS = cR$. Hence $S = R$, so R is Artinian. Thus R is prime Artinian and hence R is simple. □

Lemma 2.2.23. *Let R be a semiprime ring. Then the left and right socles of R coincide.*

Proof. Consider a simple right ideal I of R. Since R is semiprime, $I^2 \neq 0$, so there exists an element $x \in I$ such that $xI \neq 0$. Since I is simple, it follows that $I \cap \mathrm{ann}_R(x) = 0$ and that $xI = I$. Thus $x = xe$ for some element $e \in I$. Then $x = xe = xe^2$, so $x(e - e^2) = 0$ and $e - e^2 \in I \cap \mathrm{ann}_R(x)$. Hence $e - e^2 = 0$, so $e = e^2$, that is e is an idempotent. Since $0 \neq eR \subseteq I$, it follows that $I = eR$. Note that a similar proof works for simple left ideals. Therefore, every simple right or left ideal of R is generated by an idempotent.

Given an idempotent f of a semiprime ring S it can be shown that fS is a simple right ideal of S if and only if fSf is a division ring (that is, a non-commutative field) if and only if Sf is a simple left ideal of S (see [19, Section 3.4 Proposition 2]).

Now, let I be a simple right ideal of R. Then $I = eR$ for some idempotent $e \in I$. As above, Re is a simple left ideal of R, so $Re \subseteq \mathrm{Soc}({}_RR)$ and, in particular $e \in \mathrm{Soc}({}_RR)$. Since $\mathrm{Soc}({}_RR)$ is a two-sided ideal of R, $I = eR \subseteq \mathrm{Soc}({}_RR)$. It follows that $\mathrm{Soc}(R_R) \subseteq \mathrm{Soc}({}_RR)$. A symmetric argument shows that $\mathrm{Soc}({}_RR) \subseteq \mathrm{Soc}(R_R)$. Therefore $\mathrm{Soc}({}_RR) = \mathrm{Soc}(R_R)$. □

We now prove our main result of this section.

Theorem 2.2.24. *The following statements are equivalent for a ring R with right Krull dimension.*

(i) R is right Artinian.

(ii) Every finite dimensional right R-module is finitely annihilated.

(iii) Every uniform right R-module is finitely annihilated.

Proof. (i)$\Rightarrow$(ii) By Theorem 2.1.12.

(ii)$\Rightarrow$(iii) Clear.

(iii)$\Rightarrow$(i) Suppose first that R is prime. Then, by Lemma 1.4.9, R is a prime right Goldie ring. Let U be a simple right R-module and let $A = \mathrm{ann}_R(\mathrm{E}(U))$. If $A \neq 0$ then the ideal A is essential as a right ideal of R, so contains a regular element. But, by Lemma 2.2.8, $\mathrm{E}(U)$ is divisible, so $\mathrm{E}(U) = \mathrm{E}(U)A = 0$, which is a contradiction. Thus $A = 0$. Now $\mathrm{E}(U)$ is uniform (see [28, Proposition 2.28 Corollary 2]), so is finitely annihilated and hence the right R-module R_R embeds in $\mathrm{E}(U)^n$ for some integer $n \geq 1$. Thus R_R has essential socle and, in particular, $\mathrm{Soc}(R_R)$ is nonzero. By Lemma 2.2.22, R is simple Artinian and, in particular, R is right Artinian.

In general, R/P is right Artinian for all prime ideals P of R. By Lemma 1.4.12, it follows that R is right Artinian. □

A variation on this result shows that if the uniform left R-modules over a ring with right Krull dimension are finitely annihilated then R can still be shown to be right Artinian.

Proposition 2.2.25. *Let R be a ring with right Krull dimension such that every uniform left R-module is finitely annihilated. Then R is right Artinian.*

Proof. Suppose first that R is prime. Then R is a prime right Goldie ring. Let U be a simple left R-module and let $A = \mathrm{l.ann}_R(\mathrm{E}({}_RU))$. If $A \neq 0$ then the ideal A is essential as a right ideal of R, so contains a regular element. But, by Lemma 2.2.8, $\mathrm{E}({}_RU)$ is divisible, so $\mathrm{E}({}_RU) = A\mathrm{E}({}_RU) = 0$, which is a contradiction. Thus $A = 0$. Now $\mathrm{E}({}_RU)$ is a uniform left R-module, so is finitely annihilated and the left R-module ${}_RR$ embeds in $\mathrm{E}({}_RU)^n$ for some integer $n \geq 1$. Thus ${}_RR$ has essential socle, that is $\mathrm{Soc}({}_RR)$ is an essential left ideal of R. By Lemma 2.2.23, R has nonzero right socle and hence, by Lemma 2.2.22, R is right Artinian.

In general, R/P is right Artinian for every prime ideal P of R and, by Lemma 1.4.12, R is right Artinian. □

2.2.5 An Example

We conclude this chapter by giving an example which shows that for a general ring, even if every uniform module or every injective module is finitely annihilated, this is not enough to guarantee that the ring is (right) Artinian. This shows that the extra conditions in Theorems 2.2.12, 2.2.21 and 2.2.24 are necessary.

Before considering our example we require a preparatory lemma. A ring R is called *von Neumann regular* if for all elements $r \in R$ there exists an element $x \in R$ such that $r = rxr$.

Lemma 2.2.26. *Let R be a commutative von Neumann regular ring. Then every simple R-module is injective.*

Proof. Let M be a simple R-module. Let A be an ideal of R and let $\varphi : A \to M$ be a nonzero R-homomorphism. Let $B = \ker\varphi$. Then B is an ideal of R and $B \subsetneq A$. Choose an element $a \in A \setminus B$. Then $a = axa$ for some element $x \in R$, so $ax = (ax)^2$. Thus $e = ax$ is an idempotent of R and $ea = axa = a$, so $aR = eR$. Define a map $\theta : R \to M$ by $\theta(r) = \varphi(er)$ for all $r \in R$. Now $A/B \cong M$, so B is a maximal R-submodule of A. It follows that $A = aR + B = eR + B$. Let r be an element of A, then $r = es + b$ for some $s \in R$ and $b \in B$. Thus, $\varphi(r) = \varphi(es + b) = \varphi(es) + \varphi(b) = \varphi(es)$, since $b \in B = \ker\varphi$. Now $\theta(r) = \varphi(er) = \varphi(e(es+b)) = \varphi(es+eb) = \varphi(es)+\varphi(eb) = \varphi(es)+e\varphi(b) = \varphi(es)$, so $\theta(r) = \varphi(r)$. Therefore θ extends φ. It follows, by Baer's Lemma, that M is injective. $\square$

We now proceed with our example of a ring which is not Artinian, but over which every injective and every uniform module is finitely annihilated. Let K be any field. Let

$$R = \{\{k_n\}_{n=1}^{\infty} | k_n \in K \text{ for all } n \geq 1 \text{ and there exists } N \text{ such that } k_N = k_{N+1} = \cdots\}.$$

Then clearly R is a commutative von Neumann regular ring.

Let

$$S = \{\{k_n\}_{n=1}^{\infty} \in R | k_n \neq 0 \text{ for at most a finite number of } n \geq 1\}.$$

For each $m \geq 1$, let $U_m = \{\{k_n\} \in R | k_n = 0 \text{ for all } n \neq m\}$. Then U_m is a minimal ideal of R for each $m \geq 1$ and $S = \bigoplus_{m\geq 1} U_m$. Thus S is a semisimple R-module and so $S \subseteq \mathrm{Soc}(R)$. Let $0 \neq r = \{k_1, k_2, \ldots\} \in R$ and suppose that $k_i \neq 0$ for some $i \geq 1$. If e_i is the element $\{k_n\}_{n\geq 0}$ of R such that $k_i = 1$ and $k_n = 0$ for all $n \neq i$, then $0 \neq re_i \in U_i \cap rR$. It follows that S is an essential ideal of R. Therefore $S = \mathrm{Soc}(R)$. Define a map $\varphi : R \to K$

by $r = \{k_1, \ldots, k_m, k, k, \ldots\} \mapsto k$ for all $r \in R$. Then φ is a ring epimorphism with kernel S. Thus $R/S \cong K$, so S is a maximal ideal of R.

It is clear that R is not Artinian and in fact it is not even finite dimensional, since it contains the infinite direct sum of nonzero ideals $\mathrm{Soc}(R) = \bigoplus_{m \geq 1} U_m$.

For the remainder of this section, K will be a field and R, S and U_m $(m \geq 1)$ will be as defined above. We first show that every finite dimensional R-module is finitely annihilated.

Lemma 2.2.27. *Every nonzero R-module contains a simple submodule.*

Proof. Let M be any nonzero R-module. If $MS = 0$ then M can be considered as a module over the field R/S and as such is semisimple, so contains a simple submodule. If $MS \neq 0$ then there exists an element $m \in M$ such that $mS \neq 0$. Define a map $\varphi : S \to mS$ by $\varphi(s) = ms$ for all $s \in S$. Then φ is an R-homomorphism, so $mS \cong S/\ker\varphi$ and mS is semisimple. It follows that M contains a simple submodule. □

Lemma 2.2.28. *Every finite dimensional R-module is finitely annihilated.*

Proof. Let U be a uniform R-module. By Lemma 2.2.27, U contains a simple submodule, A say. Since R is von Neumann regular, A is injective, by Lemma 2.2.26, so, by Lemma 2.2.17, is a direct summand of U. Since U is uniform, $A = U$. Thus every uniform R-module is simple.

Now let M be a finite dimensional R-module. There is a finite direct sum of uniform R-modules, $U_1 \oplus \cdots \oplus U_n$, which is essential in M. Each U_i is simple, so injective and thus, by Lemma 2.2.15, $U_1 \oplus \cdots \oplus U_n$ is injective. Hence, by Lemma 2.2.17, $U_1 \oplus \cdots \oplus U_n$ is a direct summand of M. It follows that $M = U_1 \oplus \cdots \oplus U_n$. Therefore, every finite dimensional R-module is a finite direct sum of simple modules and hence is finitely generated. Since R is commutative, it follows that every finite dimensional R-module is finitely annihilated. □

We now show that every injective R-module is finitely annihilated. This requires a number of intermediate lemmas.

Lemma 2.2.29. *An R-module U is a simple R-module if and only if $U \cong R/S$ or $U \cong U_m$ for some $m \geq 1$.*

Proof. Sufficiency is clear, so suppose that U is a simple R-module. Let $0 \neq u \in U$ and let $M = \mathrm{ann}_R(u) = \mathrm{ann}_R(U)$. Then $U \cong R/M$ and M is a maximal ideal of R. If $S \subseteq M$ then $S = M$, since both S and M are maximal, and in this case $U \cong R/S$. Suppose that

$S \nsubseteq M$. Then there exists a positive integer m such that $U_m \nsubseteq M$ and hence $R = U_m \oplus M$ and in this case $U \cong R/M \cong U_m$. $\square$

Lemma 2.2.30. *Let X be the R-module given by $X = (R/S)^\Lambda$ for some non-empty index set Λ. Then X is an injective R-module.*

Proof. Let A be an ideal of R and let $\varphi : A \to X$ be an R-homomorphism. Let $B = A \cap S$. Then $B = B^2 \subseteq SB \subseteq B$ so that $B = SB$ and hence $\varphi(B) = \varphi(SB) \subseteq S\varphi(B) \subseteq SX = 0$. If $A \subseteq S$ then $A = B$ and hence $\varphi = 0$. Clearly, in this case, φ can be lifted to R. If $A \nsubseteq S$ then $R = A + S$ and $1 = a + s$ for some $a \in A$ and $s \in S$. Define a map $\theta : R \to X$ by $\theta(r) = \varphi(a)r$ for all $r \in R$. Then θ is an R-homomorphism and, for $r \in A$, $\theta(r) = \varphi(a)r = \varphi(ar) = \varphi(ar) + \varphi(sr) = \varphi(ar + sr) = \varphi(r)$, so θ lifts φ to R. By Baer's Lemma, X is an injective R-module. $\square$

Lemma 2.2.31. *For integers $i \geq 1$, let X_i be an R-module such that $X_i \cong {U_i}^{\Lambda_i}$ for some non-empty index set Λ_i. Then X_i is an injective R-module.*

Proof. We can write $X_i = \bigoplus_{\lambda \in \Lambda_i} X_{i\lambda}$ where $X_{i\lambda} \cong U_i$ for each $\lambda \in \Lambda_i$. Note that each $X_{i\lambda}$ is simple and hence injective, by Lemma 2.2.26. If Λ_i is finite then X_i is injective, by Lemma 2.2.15. Suppose that Λ_i is infinite. Let $Y_i = \prod_{\lambda \in \Lambda_i} X_{i\lambda}$ be the direct product of the $X_{i\lambda}$; then Y_i is an injective R-module, by Lemma 2.2.14.

Suppose that there is a submodule Z of Y_i such that X_i is an essential submodule of Z. Then $Z \neq 0$ so let $0 \neq z \in Z$. Then $z = \{z_{i\lambda}\}_{\lambda \in \Lambda_i}$ for some $z_{i\lambda} \in X_{i\lambda}$ $(\lambda \in \Lambda_i)$. Since X_i is an essential R-submodule of Z, there exists an element $r \in R$ such that $0 \neq zr \in X_i$. Hence $z_{i\mu}r \neq 0$ for some $\mu \in \Lambda_i$. It follows that for each $\lambda \in \Lambda_i$, either $z_{i\lambda} = 0$ or $z_{i\lambda}r \neq 0$. Thus $z_{i\lambda} \neq 0$ for at most a finite number of $\lambda \in \Lambda_i$ and hence $z \in X_i$. Therefore $X_i = Z$. It follows that X_i is an injective R-module (see [28, Propositions 2.19 and 2.20]). $\square$

Lemma 2.2.32. *Let X be any nonzero injective R-module. Then $X \cong \prod_{i \geq 0} X_i$ where the X_i $(i \geq 0)$ are R-module such that $X_0 = (R/S)^{\Lambda_0}$ and $X_i \cong U_i^{\Lambda_i}$ for $i \geq 1$, for some index sets Λ_i $(i \geq 0)$, where $X_i = 0$ if $\Lambda_i = \emptyset$ $(i \geq 0)$.*

Proof. By Lemma 2.2.27, X has essential socle, so $X = \mathrm{E}(\mathrm{Soc}(X))$. It follows, by the characterisation of simple R-modules given in Lemma 2.2.29, that $X = \mathrm{E}(\bigoplus_{i \geq 0} X_i)$ where the X_i $(i \geq 0)$ are R-module such that $X_0 = (R/S)^{\Lambda_0}$ and $X_i \cong U_i^{\Lambda_i}$ for $i \geq 1$, for some index sets Λ_i $(i \geq 0)$, where $X_i = 0$ if $\Lambda_i = \emptyset$ $(i \geq 0)$.

Since $\prod_{i\geq 0} X_i$ is injective, it suffices to show that $\bigoplus_{i\geq 0} X_i$ is an essential submodule of $\prod_{i\geq 0} X_i$. Let $0 \neq x \in \prod_{i\geq 0} X_i$. Then $x = \{x_i\}_{i\geq 0}$ for some $x_i \in X_i$ $(i \geq 0)$. If $x_i = 0$ for all $i \geq 1$ then $x \in \bigoplus_{i\geq 0} X_i$. Suppose that $x_j \neq 0$ for some $j \geq 1$. Let e_j be the element $\{k_n\}_{n\geq 0}$ of R such that $k_j = 1$ and $k_n = 0$ for all $n \neq j$. Then $xe_j = \{x_i e_j\}_{i\geq 0}$ where $x_j e_j = x_j$ and $x_i e_j = 0$ for $i \neq 0, i \neq j$. Thus $0 \neq xe_j \in \bigoplus_{i\geq 0} X_i$. It follows that $\bigoplus_{i\geq 0} X_i$ is an essential submodule of $\prod_{i\geq 0} X_i$ and hence $X = \mathrm{E}(\bigoplus_{i\geq 0} X_i) \cong \prod_{i\geq 0} X_i$. □

Lemma 2.2.33. *Every injective R-module is finitely annihilated.*

Proof. Let X be a nonzero injective R-module. Then, by Lemma 2.2.32, $X \cong \prod_{i\geq 0} X_i$ where the X_i $(i \geq 0)$ are R-module such that $X_0 = (R/S)^{\Lambda_0}$ and $X_i \cong U_i^{\Lambda_i}$ for $i \geq 1$, for some index sets Λ_i $(i \geq 0)$, where $X_i = 0$ if $\Lambda_i = \emptyset$ $(i \geq 0)$.

Let $\Lambda' = \{i \geq 0 | \Lambda_i \neq \emptyset\}$. For each $i \in \Lambda'$, $X_i \neq 0$, so let $0 \neq x_i \in X_i$. Then $\mathrm{ann}_R(x_i) = \mathrm{ann}_R(U_i) = \mathrm{ann}_R(X_i)$ for all $i \geq 1, i \in \Lambda'$ and $\mathrm{ann}_R(x_0) = \mathrm{ann}_R(R/S) = \mathrm{ann}_R(X_0)$ if $0 \in \Lambda'$. Let $x_i = 0$ for all $i \geq 0, i \notin \Lambda'$. Let $x = \{x_i\}_{i\geq 0} \in X$. Then $\mathrm{ann}_R(x) = \bigcap_{i\geq 0} \mathrm{ann}_R(x_i) = \bigcap_{i\in\Lambda'} \mathrm{ann}_R(X_i) = \mathrm{ann}_R(X)$. Thus X is finitely annihilated. □

Chapter 3

Krull Dimension of Bimodules

3.1 Introduction

Let R be a ring and let M be a right R-module. The module M is called *prime* if M is nonzero and $\mathrm{ann}_R(N) = \mathrm{ann}_R(M)$ for every nonzero submodule N of M. A submodule K of an arbitrary right R-module M will be called a *prime submodule* of M provided the right R-module M/K is prime. By a *prime right ideal* of R we mean a prime submodule of the right R-module R. For example, an ideal P of R is a prime right ideal of R if and only if P is a prime ideal of R in the usual sense. Examples of prime modules include simple modules and any nonzero module over a simple ring. Recall that a submodule L of a module M is called *irreducible* if M/L is a uniform module, that is $L \neq M$ and whenever L_1 and L_2 are submodules of M such that $L = L_1 \cap L_2$ then either $L = L_1$ or $L = L_2$.

If R and S are rings and M is a left S-, right R-bimodule then we shall say that *M has Krull dimension* in case the left S-module M has Krull dimension and the right R-module M has Krull dimension and in this case $\mathrm{k}({}_SM)$ and $\mathrm{k}(M_R)$ will denote these Krull dimensions.

It is well known that a commutative ring R is Artinian if and only if R is Noetherian and every prime ideal of R is maximal (see Corollary 2.2.11). However, the first Weyl algebra $A_1(\mathbb{C})$ is a simple right and left Noetherian ring which is not Artinian (see [25, Section 1.3]). More generally, it is also well known that a ring R is right Artinian if and only if R is right Noetherian and R/P is a right Artinian ring for every prime ideal P of R (see Lemma 2.2.10). In [20, Theorem 3.6] Lambek and Michler prove that a ring is right Artinian if and only if it is right Noetherian and every irreducible prime right ideal is maximal. In this chapter we investigate this theorem of Lambek and Michler and consider, in particular, how analogous results might be developed for modules.

In our main theorem of this chapter we prove that if R and S are rings and M is a left S-, right R-bimodule such that M has Krull dimension and M/N has the same Krull dimension as a left S- and as a right R-module for every sub-bimodule N of M and if, moreover, M is a finitely generated left S-module, then the Krull dimension of the right R-module M is the Krull dimension of a k-critical right R-module M/K for some prime submodule K of the right R-module M.

Following [29], a submodule N of a module M is said to be *radical* if N is an intersection of prime submodules of M. Note, in particular, that a prime submodule is itself radical. We begin with a preparatory lemma and a simple observation about the Krull dimension of prime modules.

Lemma 3.1.1. *Let R be a ring, let M be a right R-module and let N be a radical submodule of M. Then N is a finite intersection of irreducible prime submodules of M if and only if the right R-module M/N has finite uniform dimension.*

Proof. See [29, Corollary 2.4]. $\square$

Lemma 3.1.2. *Let R be any ring and let M be a prime right R-module with Krull dimension. Then $k(M) = k(M/L)$ for some irreducible prime submodule L of M.*

Proof. By Lemma 1.4.4, M has finite uniform dimension and so, by Lemma 3.1.1, $0 = L_1 \cap \ldots \cap L_n$ for some positive integer n and irreducible prime submodules L_i $(1 \leq i \leq n)$ of M. Clearly M embeds in the right R-module $(M/L_1) \oplus \cdots \oplus (M/L_n)$. By Lemma 1.4.1, $\mathrm{k}(M) = \mathrm{k}(M/L_i)$ for some $1 \leq i \leq n$. $\square$

Corollary 3.1.3. *Let R be a ring with right Krull dimension. Then $k(R) = k(R/A)$ for some irreducible prime right ideal A of R.*

Proof. By Proposition 1.4.12, there exists a prime ideal P of R such that $\mathrm{k}(R_R) = \mathrm{k}((R/P)_R)$. The result follows by Lemma 3.1.2. $\square$

In particular, Corollary 3.1.3 shows that if every irreducible prime right ideal is a maximal right ideal then $\mathrm{k}(R_R) = 0$, that is R is right Artinian (cf. [20, Theorem 3.6]). However, Corollary 3.1.3 and [20, Theorem 3.6] fail spectacularly for right modules, as the following example shows.

Example 3.1.4. For each ordinal $\alpha \geq 1$ there exists a right Noetherian PI algebra R and a cyclic projective uniform right R-module M such that every prime submodule of M is maximal but $\mathrm{k}(M) = \alpha$.

Proof. Let $\alpha \geq 1$ be an ordinal. By [13, Theorem 9.8], for any field F, there exists a commutative Noetherian F-algebra domain S such that $\mathrm{k}(S) = \alpha$. Let R be the "matrix ring"

$$R = \begin{pmatrix} F & S \\ 0 & S \end{pmatrix} = \{\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} | a \in F \text{ and } b, c \in S\}.$$

Then R is a right (but not left) Noetherian ring.

Let M be the right R-module

$$M = \begin{pmatrix} F & S \\ 0 & 0 \end{pmatrix}$$

and note that M is cyclic, projective and uniform. If K is a prime submodule of M then

$$K = \begin{pmatrix} 0 & S \\ 0 & 0 \end{pmatrix}$$

so $M/K \cong F$ is simple. Thus every prime submodule of the right R-module M is maximal. Clearly $\mathrm{k}(M) = \mathrm{k}(S) = \alpha$. □

In the next section we shall prove that for certain bimodules we can recover and in fact improve on Corollary 3.1.3. Note the following simple facts.

Lemma 3.1.5. *Let R be a ring and let M be a prime right R-module. Then $ann_R(M)$ is a prime ideal of R.*

Proof. Let $P = \mathrm{ann}_R(M)$ and suppose that A and B are nonzero ideals of R such that $AB \subseteq P$. If $A \nsubseteq P$, then MA is a nonzero submodule of M, so $\mathrm{ann}_R(MA) = \mathrm{ann}_R(M)$ and hence $B \subseteq \mathrm{ann}_R(MA) = P$. Therefore P is a prime ideal of R. □

Lemma 3.1.6. *Let R be a ring, let M be a right R-module and let $P = ann_R(M)$. Then M is a prime right R-module if and only if N is a faithful right (R/P)-module for every nonzero submodule N of M.*

Proof. Clear by the definition of a prime module. □

Corollary 3.1.7. *Let R be a ring and let M be a right R-module with Krull dimension such that $k(M) = k(M/K)$ for some prime submodule K of M. Then $k(M) = k(M/MP)$ for some prime ideal P of R.*

Proof. Let $P = \mathrm{ann}_R(M/K)$. Then, by Lemma 3.1.5, P is a prime ideal of R such that $MP \subseteq K$. Thus $\mathrm{k}(M) = \mathrm{k}(M/K) \leq \mathrm{k}(M/MP) \leq \mathrm{k}(M)$, by Lemma 1.4.1. $\square$

We mention Corollary 3.1.7 because our strategy in proving that $\mathrm{k}(M) = \mathrm{k}(M/K)$ for a given right R-module M and prime submodule K of M is to first prove that $\mathrm{k}(M) = \mathrm{k}(M/MP)$ for some prime ideal P of R.

3.2 Krull Dimension of Bimodules

Let R and S be rings. A left S-, right R-bimodule M will be called *Noetherian* if M is Noetherian both as a left S-module and as a right R-module. We shall say that a (not necessarily Noetherian) left S-, right R-bimodule ${}_SM_R$ is *strongly Krull symmetric* if M has Krull dimension (that is ${}_SM$ and M_R both have Krull dimension) and $\mathrm{k}({}_S(M/N)) = \mathrm{k}((M/N)_R)$ for all sub-bimodules N of ${}_SM_R$. We shall say that a (not necessarily Noetherian) left S-, right R-bimodule ${}_SM_R$ is *Krull symmetric* or equivalently that ${}_SM_R$ satisfies the *bimodule condition* if M has Krull dimension and $\mathrm{k}({}_SM) = \mathrm{k}(M_R)$. No example is known of a Noetherian bimodule which is not Krull symmetric (see [25, 6.4.11]). Note that, by Lenagan's Theorem, a Noetherian bimodule ${}_SM_R$ is (strongly) Krull symmetric in case M is Artinian either as a left S-module or as a right R-module (see Proposition 2.2.19). Moreover, if S is a left FBN ring and R is a right FBN ring then any Noetherian bimodule ${}_SM_R$ is (strongly) Krull symmetric by [25, 6.4.13]. For more information on Krull symmetric bimodules see [12, Appendix 9 p. 287]. Note that the bimodule condition is discussed further in Section 4.6 of Chapter 4.

Following [10], a right R-module M will be called *cocritical* provided M is nonzero and there exists an hereditary torsion theory τ on Mod-R such that M is τ-torsion-free but M/N is τ-torsion for every nonzero submodule N of M. Cocritical modules are discussed in [10, Section 18]. In [20] a right ideal A of R is called *critical* provided the cyclic right R-module R/A is cocritical. Let M be a module with Krull dimension. Recall that M is called *k-critical* provided M is nonzero and $\mathrm{k}(M/N) < \mathrm{k}(M)$ for every nonzero submodule N of M. Note that every k-critical module M is cocritical with respect to the hereditary torsion theory cogenerated by the injective hull $\mathrm{E}(M)$ of M, as described below.

Given a ring R and a right R-module M, we define sets $\mathcal{T}$ and $\mathcal{F}$ as follows.

$$\mathcal{T} = \{X \in \text{Mod-}R | \text{Hom}_R(X, \text{E}(M)) = 0\}$$

$$\mathcal{F} = \{Y \in \text{Mod-}R | \text{Hom}_R(X, Y) = 0 \text{ for all } X \in \mathcal{T}\}.$$

This defines a hereditary torsion theory with torsion class $\mathcal{T}$ and torsion-free class $\mathcal{F}$. This is the hereditary torsion theory *cogenerated by* $E(M)$.

Lemma 3.2.1. *Let R be a ring and let M be a k-critical right R-module. Then M is cocritical with respect to the hereditary torsion theory cogenerated by* $\text{E}(M)$.

Proof. By definition M is nonzero and, since $M \subseteq \text{E}(M)$, it is clear that M is torsion-free with respect to the hereditary torsion theory cogenerated by $\text{E}(M)$. Now let N be a nonzero submodule of M. Suppose that there is a nonzero R-homomorphism $\varphi : M/N \to \text{E}(M)$. Then $\text{im}\varphi \neq 0$, so $M \cap \text{im}\varphi \neq 0$. Then $\ker\varphi = K/N$ for some submodule K of M with $N \subseteq K$ and there exists some submodule L of M with $K \subsetneqq L \subseteq M$ such that $L/K \cong M \cap \text{im}\varphi$. Since M is k-critical it follows that $\text{k}(L/K) = \text{k}(M)$. But $\text{k}(L/K) \leq \text{k}(M/K) < \text{k}(M)$, a contradiction. Thus, for all nonzero submodules N of M, $\text{Hom}(M/N, \text{E}(M)) = 0$ and hence M/N is torsion with respect to the hereditary torsion theory cogenerated by $\text{E}(M)$. □

To establish our improvement of Corollary 3.1.3, we require a number of lemmas.

Lemma 3.2.2. *Let S and R be rings and let ${}_SM_R$ be a strongly Krull symmetric bimodule. Let A and B be ideals of R. Then*

$$k((M/MAB)_R) = \sup\{k((M/MA)_R), k((M/MB)_R)\}.$$

Proof. Let $b \in B$. Define a mapping $\varphi : M/MA \to (Mb + MAB)/MAB$ by $\varphi(m + MA) = mb + MAB$ for all $m \in M$. Then φ is well-defined and is a left S-epimorphism. It follows, by Lemma 1.4.1, that $\text{k}({}_S((Mb + MAB)/MAB)) \leq \text{k}({}_S(M/MA))$. Now

$$MB/MAB = \sum_{b \in B}((Mb + MAB)/MAB),$$

so that, by Lemma 1.4.5, $\text{k}({}_S(MB/MAB)) \leq \text{k}({}_S(M/MA))$. Again using Lemma 1.4.1, we have

$$\text{k}({}_S(M/MAB)) \leq \sup\{\text{k}({}_S(M/MA)), \text{k}({}_S(M/MB))\}.$$

But ${}_SM_R$ is strongly Krull symmetric, so that we have

$$\mathrm{k}((M/MAB)_R) \leq \sup\{\mathrm{k}((M/MA)_R), \mathrm{k}((M/MB)_R)\}.$$

The result follows since M/MA and M/MB are both isomorphic to factor modules of the right R-module M/MAB. □

A right module M over a ring R is called *fully faithful* if all nonzero submodules of M are faithful right R-modules. Note that, by Lemma 3.1.6, a right R-module M is prime if and only if M is fully faithful as a right module over the ring $R/\mathrm{ann}_R(M)$.

Lemma 3.2.3. *Let R be a prime right Goldie ring and let M be a nonzero nonsingular right R-module. Then M is fully faithful.*

Proof. Let N be a nonzero submodule of M. Then N is also a nonsingular right R-module so, by Lemma 1.4.15, N has a submodule isomorphic to a right ideal A of R. Since R is a prime ring, A is a faithful right R-module and hence N is faithful. Thus M is fully faithful. □

Corollary 3.2.4. *Let R be a prime right Goldie ring and let M be a nonzero nonsingular right R-module. Then M is a prime right R-module.*

Proof. By Lemmas 3.1.6 and 3.2.3. □

Note that we need R to be prime in the preceding corollary, for if K is a field and R is the "matrix ring"

$$R = \begin{pmatrix} K & K \\ 0 & K \end{pmatrix},$$

then R is a right Artinian right nonsingular ring, but the right R-module R_R is not prime.

Lemma 3.2.5. *Let S be a ring, let R be a prime right Goldie ring and let ${}_SM_R$ be a bimodule such that ${}_SM$ is finitely generated and M_R is faithful. Let $Z = Z(M_R)$. Then Z is a prime submodule of M_R and M/Z is a faithful right R-module.*

Proof. Suppose that $M = Z$. Then there exist a positive integer n and elements $m_i \in M$ $(1 \leq i \leq n)$ such that $M = Sm_1 + \cdots + Sm_n$. For each $1 \leq i \leq n$ there exists an essential right ideal E_i of R such that $m_iE_i = 0$. Then $M(E_1 \cap \ldots \cap E_n) = 0$ and hence $E_1 \cap \ldots \cap E_n = 0$, a contradiction. Thus $M \neq Z$. The right R-module M/Z is nonsingular and hence, by Corollary 3.2.4, Z is a prime submodule of M_R and, moreover, by Lemma 3.1.6, $(M/Z)_R$ is faithful. □

Lemma 3.2.6. *Let S and R be rings and let ${}_SM_R$ be a bimodule such that ${}_SM$ is finitely generated and M_R is faithful and has Krull dimension. Then R has right Krull dimension and $k(R_R) = k(M_R)$.*

Proof. Suppose that $M = Sm_1 + \cdots + Sm_n$ for some positive integer n and elements $m_i \in M$ $(1 \leq i \leq n)$. Then $\cap_{i=1}^{n}\mathrm{ann}_R(m_i) = \mathrm{ann}_R(M) = 0$, so, as right R-modules, R embeds in M^n via the map $r \mapsto (m_1 r, \ldots, m_n r)$ for all $r \in R$. Hence R has right Krull dimension and, by Lemma 1.4.1, $\mathrm{k}(R_R) \leq \mathrm{k}(M_R)$. The result follows by Lemma 1.4.6. □

Lemma 3.2.7. *(i) Every cocritical module is uniform.*

(ii) Every nonsingular uniform module is cocritical.

Proof. (i) Let R be a ring and let M be a right R-module cocritical with respect to a hereditary torsion theory τ. Suppose that X and Y are submodules of M such that $X \cap Y = 0$ but $Y \neq 0$. Then X embeds in the right R-module M/Y via the map given by $x \mapsto x + Y$ for all $x \in X$. Since M is cocritical, M/Y is τ-torsion and hence X is τ-torsion. But M is τ-torsion-free, so it follows that $X = 0$. Hence M is uniform.

(ii) A nonsingular uniform module is cocritical with respect to the Goldie torsion theory, as discussed immediately following Theorem 2.2.6. □

Lemma 3.2.8. *Let R be a ring and let M be a nonzero (prime) nonsingular right R-module with finite uniform dimension. Then there exist a positive integer n and submodules L_i $(1 \leq i \leq n)$ of M such that $0 = L_1 \cap \ldots \cap L_n$ and M/L_i is a (prime) nonsingular uniform right R-module for each $1 \leq i \leq n$.*

Proof. Since M has finite uniform dimension, there exist a positive integer n and uniform submodules U_i $(1 \leq i \leq n)$ of M such that $U_1 \oplus \cdots \oplus U_n$ is an essential submodule of M. For each $1 \leq i \leq n$, Zorn's Lemma gives a submodule L_i of M maximal with the properties $\oplus_{j \neq i} U_j \subseteq L_i$ and $L_i \cap U_i = 0$. It is easy to check that $(L_1 \cap \ldots \cap L_n) \cap (U_1 \oplus \cdots \oplus U_n) = 0$ and hence $L_1 \cap \ldots \cap L_n = 0$. Next, the choice of L_i implies that M/L_i is uniform for all $1 \leq i \leq n$. Moreover, for each $1 \leq i \leq n$, $(U_i \oplus L_i)/L_i \cong U_i$, a submodule of the nonsingular right R-module M, so that $(U_i \oplus L_i)/L_i$ is nonsingular and hence M/L_i is nonsingular.

Now suppose that, in addition, M is a prime right R-module. Let $1 \leq i \leq n$ and let N/L_i be a nonzero submodule of the right R-module M/L_i (where N is a submodule of M with $L_i \subseteq N$). Suppose that $r \in \mathrm{ann}_R(N/L_i)$. Then $Nr \subseteq L_i$, so that $(N \cap$

$U_i)r \subseteq L_i \cap U_i = 0$. Now the maximal choice of L_i means that $N \cap U_i \neq 0$ and hence $r \in \operatorname{ann}_R(N \cap U_i) = \operatorname{ann}_R(M)$, so $r \in \operatorname{ann}_R(M/L_i)$. It follows that M/L_i is a prime right R-module for all $1 \leq i \leq n$. □

Lemma 3.2.9. *Let R be a prime ring with right Krull dimension and let M be a nonsingular uniform right R-module with Krull dimension. Suppose further that M is finitely generated or has finite Krull dimension. Then M is k-critical.*

Proof. By Proposition 1.4.9, R is a right Goldie ring and, by Proposition 1.4.16, $\mathrm{k}(M) = \mathrm{k}(R)$. Suppose that M is a finitely generated module. For any nonzero submodule N of M, M/N is a finitely generated singular right R-module and hence $\mathrm{k}(M/N) < \mathrm{k}(R)$ by Proposition 1.4.16. Thus M is k-critical.

Now suppose that $\mathrm{k}(M)$ is finite. Let N be a nonzero submodule of M. For any $m \in M$, $(mR + N)/N$ is a cyclic singular right R-module. Again by Proposition 1.4.16, $\mathrm{k}((mR + N)/N) < \mathrm{k}(R)$. By Lemma 1.4.5, it follows that $\mathrm{k}(M/N) < \mathrm{k}(R)$. Therefore M is k-critical. □

Lemmas 3.2.8 and 3.2.9 have the following interesting consequence.

Corollary 3.2.10. *Let R be a ring with right Krull dimension. Then $\mathrm{k}(R) = \mathrm{k}(R/A)$ for some right ideal A of R such that R/A is a prime k-critical right R-module.*

Proof. By Proposition 1.4.12, $\mathrm{k}(R) = \mathrm{k}(R/P)$ for some prime ideal P of R. Therefore we may assume that R is a prime ring. Thus, by Proposition 1.4.9, R is a prime right Goldie ring, so is right nonsingular. By Lemma 3.2.8, it follows that $\mathrm{k}(R) = \mathrm{k}(R/A)$ for some right ideal of A of R such that R/A is a prime uniform nonsingular right R-module. But then, by Lemma 3.2.9, R/A is a k-critical right R-module. □

At this point we note that we are now able to recover Lambek and Michler's main result of [20, Theorem 3.6].

Corollary 3.2.11. *Let R be a right Noetherian ring. Then the following statements are equivalent.*

(i) R is right Artinian.

(ii) Every irreducible prime right ideal of R is maximal.

(iii) Every critical prime right ideal of R is maximal.

Proof. (i)⇒(ii) Let A be an irreducible prime right ideal of R. If $P = \text{ann}_R(R/A)$ then P is a prime ideal of R, so that R/P is a simple Artinian ring. It follows that R/A is a simple right R-module, that is A is a maximal right ideal.

(ii)⇒(iii) This is clear since, by Lemma 3.2.7 (i), cocritical modules are uniform, that is, critical right ideals are irreducible.

(iii)⇒(i) This follows by Corollary 3.2.10, since k-critical modules are cocritical with respect to the hereditary torsion theory cogenerated by their injective hull. □

We now prove one of our main theorems of this chapter, where we extend this result of Lambek and Michler to consider the Krull dimension of certain bimodules.

Theorem 3.2.12. *Let S and R be rings and let ${}_SM_R$ be a strongly Krull symmetric bimodule such that the left S-module M is finitely generated. Then $k(M_R) = k(M'_R)$ for some prime cocritical homomorphic image M' of the right R-module M.*

If, in addition, the right R-module M is finitely generated or has finite Krull dimension then $k(M_R) = k(M'_R)$ for some prime k-critical homomorphic image M' of the right R-module M.

Proof. Let $A = \text{ann}_R(M)$. Then A is an ideal of R and, by Lemma 3.2.6, the ring R/A has right Krull dimension. By Proposition 1.4.11, there exist an integer $t \geq 1$ and prime ideals P_i $(1 \leq i \leq t)$ of R such that $P_1 \cdots P_t \subseteq A \subseteq P_1 \cap \ldots \cap P_t$ and hence $MP_1 \cdots P_t = 0$. By Lemma 3.2.2, $\text{k}(M_R) = \text{k}((M/MP_i)_R)$ for some $1 \leq i \leq t$.

Suppose that $\text{k}((M/MP_i)_R) \neq \text{k}((M/K)_R)$ for all prime submodules K of M_R such that $(M/K)_R$ is cocritical. By Lemma 1.4.14, the ring R/A has the ascending chain condition on prime ideals. Thus, there exists a prime ideal P of R maximal such that $\text{k}((M/MP)_R) \neq \text{k}((M/K)_R)$ for all prime submodules K of M_R such that $(M/K)_R$ is cocritical. Let $B = \text{ann}_R(M/MP)$. Then B is an ideal of R and $P \subseteq B$. Suppose that $P \neq B$. By the argument of the first paragraph, $\text{k}((M/MP)_R) = \text{k}((M/MQ)_R)$ for some prime ideal Q of R containing B. The maximal choice of P implies that $\text{k}((M/MP)_R) = \text{k}((M/MQ)_R) = \text{k}((M/K)_R)$ for some prime submodule K of M_R such that $(M/K)_R$ is cocritical, a contradiction. Thus $P = B$ and hence M/MP is a faithful right (R/P)-module. Note that R/P is a prime right Goldie ring. Let Z denote the submodule of M containing MP such that Z/MP is the singular submodule of the right (R/P)-module M/MP. By Lemma 3.2.5, it follows that Z is a prime submodule of M_R and M/Z is a faithful right (R/P)-module. Now Lemma 3.2.6 gives that

$$\mathrm{k}((M/MP)_{R/P}) = \mathrm{k}((R/P)_{R/P}) = \mathrm{k}((M/Z)_{R/P})$$

and hence $\mathrm{k}((M/MP)_R) = \mathrm{k}((M/Z)_R)$. By Lemmas 3.2.8 and 3.2.7(ii), there exist a positive integer n and submodules L_i $(1 \leq i \leq n)$ of M_R such that $Z = L_1 \cap \ldots \cap L_n$ and M/L_i is a prime cocritical right R-module for each $1 \leq i \leq n$. Then there is an embedding of right R-modules $M/Z \hookrightarrow (M/L_1) \oplus \cdots \oplus (M/L_n)$ and so $\mathrm{k}((M/MP)_R) = \mathrm{k}((M/Z)_R) = \mathrm{k}((M/L_i)_R)$ for some $1 \leq i \leq n$, a contradiction. It follows that $\mathrm{k}(M_R) = \mathrm{k}((M/MP_i)_R) = \mathrm{k}((M/K)_R)$ for some prime submodule K of M_R such that $(M/K)_R$ is cocritical.

If, in addition, the right R-module M is finitely generated or has finite Krull dimension then, using Lemma 3.2.9, the result follows similarly, with the supposition for contradiction being that $\mathrm{k}((M/MP_i)_R) \neq \mathrm{k}((M/K)_R)$ for all prime submodules K of M_R such that $(M/K)_R$ is k-critical. □

Theorem 3.2.12 has the following immediate consequence.

Corollary 3.2.13. *Let R be a commutative ring and let M be a finitely generated R-module with Krull dimension. Then $k(M) = k(M')$ for some prime k-critical homomorphic image M' of M.*

Another consequence of Theorem 3.2.12 is the following result.

Corollary 3.2.14. *Let R be a right FBN ring, let S be a left FBN ring and let ${}_SM_R$ be a bimodule such that ${}_SM$ and M_R are both finitely generated. Then $k(M_R) = k(M'_R)$ for some prime k-critical homomorphic image M' of the right R-module M.*

Proof. By [25, Corollary 6.4.13], such a bimodule is strongly Krull symmetric, so the result follows by Theorem 3.2.12. □

3.3 Artinian Bimodules

We now present some results concerning Artinian bimodules. Our first result gives a bimodule analogue of the Lambek–Michler result [20, Theorem 3.6]. Compare with Theorem 3.2.12, where we also extend [20, Theorem 3.6].

Theorem 3.3.1. *Let R and S be rings and let ${}_SM_R$ be a Noetherian bimodule. Then the right R-module M is Artinian if and only if every irreducible prime submodule of M_R is maximal.*

Proof. ($\Rightarrow$) Let $A = \mathrm{ann}_R(M)$. Then A is an ideal of R and, by Lemma 3.2.6, R/A is a right Artinian ring. Let L be an irreducible prime submodule of M_R. If $P = \mathrm{ann}_R(M/L)$ then P is a prime ideal of R and $A \subseteq P$, so that R/P is a simple Artinian ring. It follows that M/L is a simple right R-module.

($\Leftarrow$) Suppose that every irreducible prime submodule of the right R-module M is maximal. By Lemma 3.1.2, M/K is an Artinian right R-module for every prime submodule K of M_R. Suppose that M is not an Artinian right R-module. Because M_R (or ${}_SM$) is Noetherian, we can suppose without loss of generality that M/N is an Artinian right R-module for every nonzero sub-bimodule N of ${}_SM_R$. In addition, we can suppose without loss of generality that M is a faithful right R-module. Let A and B be nonzero ideals of R. Because M_R is faithful, we have $MA \neq 0$ and $MB \neq 0$. By hypothesis, M/MA and M/MB are Artinian right R-modules. Using the method of the proof of Lemma 3.2.2 and Lenagan's Theorem (Proposition 2.2.19), it follows that M/MAB is an Artinian right R-module. This implies that $AB \neq 0$. It follows that R is a prime ring. Moreover, because ${}_SM$ is finitely generated and M_R is faithful Noetherian, R is a right Noetherian ring.

Let $T = \mathrm{Z}(M_R)$. Then T is a sub-bimodule of M. By Lemma 3.2.5, T is a prime submodule of M_R and M/T is a faithful right R-module. Hence M/T is an Artinian right R-module. By Lemma 3.2.6, R is a right Artinian ring. But M is a finitely generated right R-module and hence M_R is Artinian, a contradiction. It follows that M is an Artinian right R-module. $\square$

We now aim to extend Theorem 3.3.1. We first require a lemma.

Lemma 3.3.2. *Let R be any ring. Let $X = X_1 \oplus \cdots \oplus X_n$ be a finite direct sum of right R-modules such that, for all $1 \leq i \leq n$, X_i/K is a simple right R-module for all irreducible prime submodules K of X_i. Then X/L is a simple right R-module for all irreducible prime submodules L of the right R-module X.*

Proof. Let L be any irreducible prime submodule of the right R-module X. Let $1 \leq i \leq n$, then $X_i/(L \cap X_i) \cong (X_i + L)/L$, a submodule of the right R-module X/L, so that $X_i/(L \cap X_i)$ is either zero or a uniform prime right R-module and hence is simple. It follows that $X/(\oplus_{i=1}^n(L \cap X_i))$ is a semisimple right R-module and hence so too is X/L. Clearly X/L is in fact a simple right R-module. □

Theorem 3.3.3. *Let R and S be rings and let ${}_SM_R$ be a Noetherian bimodule. Let N be a submodule of M_R. Then N is an Artinian right R-module if and only if every irreducible prime submodule of the right R-module N is maximal.*

Proof. First consider the sub-bimodule SN of ${}_SM_R$. Now $(SN)_R$ is finitely generated, so $SN = s_1N + \cdots + s_kN$ for some positive integer k and elements s_i $(1 \leq i \leq k)$ of S. Define a map $\varphi : N^k \to SN$ by $\varphi(n_1, \ldots, n_k) = s_1n_1 + \cdots + s_kn_k$ for all $n_i \in N$ $(1 \leq i \leq k)$. Then φ is an R-epimorphism from the right R-module N^k to the right R-module SN.

Now suppose that N is an Artinian right R-module. Then SN is an Artinian right R-module. Let $A = \mathrm{ann}_R(N)$ and note that $A = \mathrm{ann}_R(SN)$. Then A is an ideal of R and, by Lemma 3.2.6, R/A is a right Artinian ring. Let L be an irreducible prime submodule of the right R-module N. If $P = \mathrm{ann}_R(N/L)$ then P is a prime ideal of R and $A \subseteq P$, so that R/P is a simple Artinian ring. It follows that N/L is a simple right R-module.

Conversely, suppose that every irreducible prime submodule of the right R-module N is maximal. By Lemma 3.3.2, every irreducible prime submodule of $(SN)_R$ is maximal. But ${}_S(SN)_R$ is a Noetherian bimodule. By Theorem 3.3.1, $(SN)_R$ is Artinian. Thus N_R is Artinian. □

Corollary 3.3.4. *Let R be a right and left Noetherian ring and let M be a right R-module which embeds in a finitely generated free right R-module. Then M is Artinian if and only if every irreducible prime submodule of M is maximal.*

Proof. Suppose that M embeds in a finitely generated free right R-module F. Suppose that $F \cong R_R^n$ for some positive integer n. Then F is also a finitely generated left R-module. Now $M \hookrightarrow F_R$ as right R-modules, that is M is isomorphic to a right R-submodule M'_R of the left R-, right R-bimodule ${}_RF_R$. Note that both ${}_RF$ and F_R are Noetherian. The result follows by Theorem 3.3.3. □

Note in particular that Corollary 3.3.4 applies in case M is a finitely generated projective right R-module or a right ideal of R (compare Example 3.1.4 and note that the ring R in Example 3.1.4 is not left Noetherian). Before giving an application of Corollary 3.3.4, we first require a result of Gentile and Levy.

Lemma 3.3.5. *Let R be a semiprime right and left Goldie ring and let M be a finitely generated nonsingular right R-module. Then M can be embedded in a finitely generated free right R-module.*

Proof. See [12, Proposition 6.19]. □

Corollary 3.3.6. *Let R be a semiprime right and left Noetherian ring and let M be a finitely generated nonsingular right R-module. Then M is Artinian if and only if every irreducible prime submodule of M is maximal.*

Proof. By Lemma 3.3.5, such a module can be embedded in a finitely generated free right R-module. The result follows by Corollary 3.3.4. □

Chapter 4

Right Fully Bounded Rings with Right Krull Dimension

In this chapter we shall be concerned with the class of rings which are right fully bounded and have right Krull dimension. We shall call such rings *right FBK* rings. This is a natural generalisation of right FBN rings (that is right fully bounded right Noetherian rings), which have played an important role in many areas, for example in the study of the Gabriel correspondence (Section 4.5), the bimodule condition and Krull symmetry (Section 4.6) and the Jacobson conjecture (Section 4.7). In general, many Noetherian results can be extended to the wider class of rings with Krull dimension and here we attempt to do the same; investigating how results on right FBN rings can be extended to right FBK rings. In particular, we investigate whether such right fully bounded rings with right Krull dimension satisfy the H-condition, giving a necessary and sufficient condition for them to do so and an example to show that this condition is not satisfied for all such rings.

4.1 Preliminaries

We begin with some preliminary results relating boundedness and singular modules. Recall Lemma 1.4.15, which says that if R is a semiprime right Goldie ring and A is a right R-module which is not singular, then A has a uniform submodule isomorphic to a right ideal of R.

Lemma 4.1.1. *Let R be a prime ring. Then R is right bounded if and only if R has no faithful finitely generated singular right modules.*

Proof. If R is not right bounded then R has an essential right ideal I which contains no nonzero ideals of R. Since $\mathrm{ann}_R(R/I)$ is an ideal of R contained in I, it must be zero. Thus R/I is a faithful right R-module. Clearly R/I is a finitely generated right R-module. Since I is an essential right ideal of R, it follows, by Lemma 2.1.3, that R/I is a singular right R-module.

Conversely, suppose that R is right bounded and let A be any finitely generated singular right R-module. Choose generators $a_1, \ldots, a_n$ for A (for some integer $n \geq 1$ and elements $a_i \in A$ $(1 \leq i \leq n)$). Then for each $1 \leq i \leq n$, a_i is annihilated by an essential right ideal I_i of R. The intersection $I = I_1 \cap \ldots \cap I_n$ is an essential right ideal of R, so, by assumption, there exists a nonzero ideal J of R such that $J \subseteq I$. Then $a_i J = 0$ for each $1 \leq i \leq n$ and, since J is an ideal of R, $AJ = 0$. Hence A is not faithful. □

Lemma 4.1.2. *Let R be a semiprime right Goldie ring and let U be a uniform right R-module. Then U is either singular or nonsingular.*

Proof. If U is not nonsingular then $Z(U) \neq 0$, in which case $Z(U)$ is an essential submodule of U, since U is uniform. Let $u \in U$. By Lemma 2.1.3, $(\mathrm{Z}(U) : u)$ is an essential right ideal of R, so, by Proposition 1.3.2, $ur \in \mathrm{Z}(U)$ for some regular element $r \in R$. Then, again by Proposition 1.3.2, $urs = 0$ for some regular element $s \in R$. Then rs is a regular element of R and since $rs \in \mathrm{ann}_R(u)$, it follows, once again by Proposition 1.3.2, that $\mathrm{ann}_R(u)$ is an essential right ideal of R, so $u \in \mathrm{Z}(U)$. Thus $\mathrm{Z}(U) = U$, so U is singular. □

Corollary 4.1.3. *Let R be a right bounded prime right Goldie ring and let A be a faithful finitely generated uniform right R-module. Then A is nonsingular.*

Proof. Since R is prime right bounded, A cannot be singular, by Lemma 4.1.1. Since R is prime right Goldie, it follows, by Lemma 4.1.2, that A is nonsingular. □

4.2 Right FBK Rings

We now consider right FBK rings and provide some simple generalisations of results on right FBN rings which extend easily to the right FBK case.

Definition. We shall call a ring R a *right FBK* ring if R is right fully bounded and has right Krull dimension.

Examples of (right) FBK rings include commutative rings with Krull dimension and, more generally, PI-rings with Krull dimension. The "matrix ring"

$$R = \begin{pmatrix} \mathbb{Z} & \mathbb{Z}(p^\infty) \\ 0 & \mathbb{Z} \end{pmatrix}$$

where p is a prime number, is a right (and left) FBK ring with right (and left) Krull dimension 1, but R is not right (or left) Noetherian (see Section 4.4).

If M is a uniform right module over a ring R, then we define the *assassinator* of M to be the set $\text{ass}(M) = \{r \in R | r \in \text{ann}_R(N)$ for some nonzero submodule N of $M\}$. Note that this is an ideal of R.

Our first result (taken from [13, Theorem 8.3]) is an important extension to rings with right Krull dimension of a result that can be proved relatively simply for right Noetherian rings. The result requires more work in the case of rings with right Krull dimension, although its proof uses the Noetherian-like properties of such rings.

Proposition 4.2.1. *Let R be a ring with right Krull dimension and let U be a uniform right R-module. Then*

(i) ass(U) is a prime ideal of R.

(ii) ass(U) = ass(U') = ann(U') for some nonzero submodule U' of U.

Proof. We use the following two crucial Noetherian-like properties of the ring R, both of which follow from the fact that R has right Krull dimension.

(1) For each proper ideal A of R there exists a finite set of prime ideals $P_1, \ldots, P_n$ of R for some integer $n \geq 1$, each containing A, such that $P_1 \cdots P_n \subseteq A$.

(2) R has the ascending chain condition on prime ideals.

(For proofs that (1) and (2) hold for a ring R with right Krull dimension, see Proposition 1.4.11 and Lemma 1.4.14 respectively.)

Consider the set

$S = \{P | P$ is a prime ideal of R such that $VP = 0$ for some nonzero submodule V of $U\}$.

By (1), some finite product of prime ideals of R is zero and it follows that the set S is non-empty. Hence, by (2), we may choose a prime ideal P of R maximal in S. Then $U'P = 0$ for some nonzero submodule U' of U. Clearly $P \subseteq \text{ann}_R(U')$. We show that

$P = \mathrm{ass}(U') = \mathrm{ann}_R(U')$. Since U is uniform, this is enough to prove the required result. For, clearly $P \subseteq \mathrm{ass}(U)$ and, if $r \in \mathrm{ass}(U)$, then $r \in \mathrm{ann}_R(V)$ for some nonzero submodule V of U. Since U is uniform, it follows that $V \cap U' \neq 0$, so $r \in \mathrm{ann}_R(V \cap U') = P$ and thus $P = \mathrm{ass}(U)$.

So, let U'' be a nonzero submodule of U'. Then $U''P = 0$, so that U'' can be considered as a right (R/P)-module and we can consider the ideal $\mathrm{ann}_{R/P}(U'')$ of R/P. The factor ring R/P inherits the two properties (1) and (2) from R and $U'' \neq 0$ implies that $\mathrm{ann}_{R/P}(U'') \neq R/P$, so, by (1), there exist prime ideals $\overline{P_1}, \ldots, \overline{P_n}$ of R/P for some integer $n \geq 1$ such that $\overline{P_1} \cdots \overline{P_n} \subseteq \mathrm{ann}_{R/P}(U'') \subseteq \overline{P_1} \cap \ldots \cap \overline{P_n}$. For each $1 \leq i \leq n$ $\overline{P_i}$ is of the form $\overline{P_i} = P_i/P$ for some prime ideal P_i of R such that $P \subseteq P_i$, so there exist prime ideals $P_1, \ldots, P_n$ of R such that $U''P_1 \cdots P_n = 0$ and $P \subseteq P_i$ for each $1 \leq i \leq n$. Thus, for some $1 \leq j \leq n$, P_j annihilates some nonzero submodule of U''. But $U'' \subseteq U' \subseteq U$ so this P_j annihilates a nonzero submodule of U and hence $P_j \in S$. Since $P \subseteq P_j$, it follows, by the maximal choice of P, that $P = P_j$. Thus $\overline{P_j} = \overline{0}$ and so $\mathrm{ann}_{R/P}(U'') = 0$. Hence $\mathrm{ann}_R(U'') = P$. It follows that $P = \mathrm{ass}(U') = \mathrm{ann}_R(U')$, as required. $\square$

In fact Proposition 4.2.1 holds for any ring which satisfies the two conditions (1) and (2) given in the above proof.

The next two results are simple generalisations of known results on right FBN rings. Both generalisations rely on the fact that a prime ring with right Krull dimension is prime right Goldie (see Proposition 1.4.9).

Proposition 4.2.2. *Let R be a right FBK ring. If P is a right primitive ideal of R (in particular, if P is a maximal ideal of R), then R/P is a simple Artinian ring.*

Proof. Without loss of generality, we may assume that $P = 0$ (since R/P is also a right FBK ring). Then R is a right primitive ring, so there exists a faithful simple right R-module, which may be written as R/M for some maximal right ideal M of R. As the right R-module R/M is faithful, M does not contain a nonzero ideal of R and hence M is not an essential right ideal of R, since R is prime right bounded. Therefore there exists a nonzero right ideal J of R such that $M \cap J = 0$. Then $J \cong (M + J)/M = R/M$ and so $J \subseteq \mathrm{Soc}(R_R)$. In particular, $\mathrm{Soc}(R_R)$ is nonzero.

Now, R is a prime ring with right Krull dimension, so, by Proposition 1.4.9, R is a prime right Goldie ring. As $\mathrm{Soc}(R_R) \neq 0$, it follows, by Lemma 2.2.22, that R is a simple Artinian ring. $\square$

Note, in particular, that it follows that every right primitive ideal in a right FBK ring is a maximal ideal.

Corollary 4.2.3. *Let R be a right FBK ring and let A be a simple right R-module. Then $ann_R(A)$ is a maximal ideal of R and $R/ann_R(A)$ is a simple Artinian ring. Hence, A is isomorphic to a right ideal of $R/ann_R(A)$ and $R/ann_R(A)$ is isomorphic to a finite direct sum of copies of A.*

The following result again generalises a result from right FBN rings to right FBK rings. Again we use the important result that a prime ring with right Krull dimension is prime right Goldie (see Proposition 1.4.9), along with the fact that modules with Krull dimension have finite uniform dimension (see Lemma 1.4.4).

Proposition 4.2.4. *Let R be a right FBK ring and let M be a nonzero right R-module. Then M has a nonzero submodule N such that $ann_R(N)$ is a prime ideal of R and N is isomorphic to a right ideal of $R/ann_R(N)$.*

Proof. Since M is nonzero, it has a nonzero finitely generated submodule, A say. Since A is finitely generated and R has right Krull dimension, A has Krull dimension. Therefore A has finite uniform dimension and thus has a uniform submodule, C say. Then C is a uniform submodule of M.

Let $P = \text{ass}(C)$. By Proposition 4.2.1, P is a prime ideal of R and $P = \text{ass}(C) = \text{ass}(D) = \text{ann}_R(D)$ for some nonzero submodule D of C. Since $P = \text{ann}_R(D)$, D can be considered as a right (R/P)-module. In fact, D is a nonzero fully faithful right (R/P)-module, since $P = \text{ass}(D)$.

Now, R/P is a prime right Goldie ring, as R/P is prime and has right Krull dimension. Let $\mathcal{C}(P)$ denote the set of elements $c \in R$ such that $c + P$ is a regular element of R/P. Suppose that $dc = 0$ for some $d \in D$ and $c \in \mathcal{C}(P)$. Then $d(cR + P) = 0$. But, by Proposition 1.3.2, $cR + P$ is an essential right ideal of R/P, so contains a nonzero ideal of R/P, since R/P is prime right bounded. Thus, there exists an ideal I of R such that $P \subsetneq I$ and $I \subseteq cR + P$. Then $dI = 0$ and, in fact, since I is a two-sided ideal of R, $(dR)I = 0$. If $d \neq 0$, then dR is a nonzero submodule of D and $I \subseteq \text{ann}_R(dR) \subseteq \text{ass}(D) = P$, a contradiction. So we must have $d = 0$. Hence D is a torsion-free right (R/P)-module. By Lemma 1.4.15, it follows that D has a submodule N isomorphic to a uniform right ideal of R/P. Moreover, since D is a fully-faithful right (R/P)-module, we have that $\text{ann}_R(N) = P$. $\square$

4.3 Right FBK Rings and the H-Condition

Recall that a ring R satisfies the H-condition if every finitely generated right R-module is finitely annihilated. A result of Cauchon [3, Théorème II 8] says that every right FBN ring has the H-condition (see Theorem 4.3.7). Although this result does not extend to right FBK rings, we are able to show that many aspects of the strong relationship which is found in right Noetherian rings between finite annihilation and the boundedness of the ring do also hold for rings with right Krull dimension. We begin by proving a general result on the relationship between finite annihilation and boundedness, before considering rings with right Krull dimension and investigating the H-condition for right FBK rings.

Proposition 4.3.1. *Let R be a ring with the H-condition. Then every factor ring of R is right bounded.*

Proof. It is easy to show that every factor ring of R also satisfies the H-condition, so it is enough to show that R is right bounded. This follows by Lemma 2.1.4. □

Before investigating the converse of Proposition 4.3.1 for rings with right Krull dimension, we outline some assumptions that we can make without loss of generality about the modules under consideration in this section.

Given a right module M over a right FBK ring R, if we want to show that M is finitely annihilated, then, by factoring out the ideal $\operatorname{ann}_R(M)$ of R, we may assume without loss of generality that M is a faithful right R-module. The following argument shows further that if M is finitely generated, then, without loss of generality, we may also assume that M is a uniform right R-module.

Lemma 4.3.2. *Let R be a ring with right Krull dimension and suppose that every finitely generated uniform right R-module is finitely annihilated. Then R satisfies the H-condition.*

Proof. Let M be a finitely generated right R-module. By Corollary 1.4.2, M has Krull dimension, so, by Lemma 1.4.4, M has finite uniform dimension. Therefore, as in the proof of Lemma 3.2.8, there exist a positive integer n and submodules L_i ($1 \leq i \leq n$) of M such that $0 = L_1 \cap \ldots \cap L_n$ and M/L_i is a uniform right R-module for each $1 \leq i \leq n$. By hypothesis, it follows that for each $1 \leq i \leq n$ there exists a finite subset $F_i \subseteq M$ such that $\operatorname{ann}_R(M/L_i) = \operatorname{ann}_R(\{x + L_i | x \in F_i\})$.

Now $\bigcup_{i=1}^n F_i$ is a finite subset of M and clearly $\operatorname{ann}_R(M) \subseteq \operatorname{ann}_R(\bigcup_{i=1}^n F_i)$. Let $r \in \operatorname{ann}_R(\bigcup_{i=1}^n F_i)$ and let $m \in M$. Let $1 \leq i \leq n$; then $xr = 0$ for all $x \in F_i$, so

$(x + L_i)r = xr + L_i = \overline{0}$ for all $x \in F_i$. Thus $r \in \mathrm{ann}_R(M/L_i)$ and so $mr \in L_i$. Hence $mr \in L_1 \cap \ldots \cap L_n = 0$, so $mr = 0$. Therefore $Mr = 0$, that is $r \in \mathrm{ann}_R(M)$. It follows that $\mathrm{ann}_R(M) = \mathrm{ann}_R(\bigcup_{i=1}^n F_i)$. Hence R satisfies the H-condition. □

Our next lemma applies to any ring. Note also that in the following three results we need not assume that the modules are finitely generated. However, when considering whether or not the ring R has the H-condition we will only apply them in the particular case of a finitely generated right R-module.

Lemma 4.3.3. *Let R be a ring and let M be a faithful right R-module. Suppose that there exists a submodule N of M such that,*

1. *$NP = 0$ for some prime ideal P of R,*
2. *N is a torsion-free right (R/P)-module,*
3. *R/P is a prime right Goldie ring.*

Suppose that $MA \subseteq N$ for some right ideal A of R such that A has finite uniform dimension as a right R-module. Then there exists a finite subset F of M such that $A \cap \mathrm{ann}_R(F) = 0$.

Proof. If $A = 0$ then the result is trivial, so suppose that $A \neq 0$. Now, $MA \subseteq N$ implies that $MAP = 0$, so that $AP = 0$, since M is a faithful right R-module. Hence A can be considered as a right (R/P)-module.

Since $A \neq 0$ and M is a faithful right R-module, $MA \neq 0$, so there exists an element $m_1 \in M$ such that $m_1 A \neq 0$. Let $A_1 = A \cap \mathrm{ann}_R(m_1)$, then $A_1 \subsetneqq A$. Now $A/A_1 \cong m_1 A \subseteq N$, so A/A_1 is a nonzero torsion-free right (R/P)-module.

If $A_1 = 0$ then the result holds with $F = \{m_1\}$. So suppose that $A_1 \neq 0$. Then as above there exists an element $m_2 \in M$ such that $m_2 A_1 \neq 0$. Let $A_2 = A_1 \cap \mathrm{ann}_R(m_2) = A \cap \mathrm{ann}_R(m_1, m_2)$. Then $A_2 \subsetneqq A_1$ and $A_1/A_2 \cong m_2 A_1 \subseteq MA \subseteq N$, so A_1/A_2 is a nonzero torsion-free right (R/P)-module. If $A_2 \neq 0$ then the process continues as above.

This process gives a strictly descending chain of submodules $A = A_0 \supset A_1 \supset A_2 \supset \cdots$, such that A_i/A_{i+1} is a nonzero torsion-free right (R/P)-module for each $i \geq 0$. It follows that for each $i \geq 0$, A_{i+1} is not an essential R-submodule of A_i. Hence there exists a nonzero R-submodule B_i of A_i such that $A_{i+1} \cap B_i = 0$. Thus $\mathrm{u}(A_i) \geq \mathrm{u}(A_{i+1} \oplus B_i) = \mathrm{u}(A_{i+1}) + \mathrm{u}(B_i) > \mathrm{u}(A_{i+1})$ and the uniform dimension is strictly decreasing at each step in the chain. But $\mathrm{u}(A) < \infty$, so the process must stop. Thus $A_n = 0$ for some $n \geq 1$ and then $A \cap \mathrm{ann}_R(m_1, \ldots, m_n) = 0$, so the result holds by taking $F = \{m_1, \ldots, m_n\}$. □

We now show that in the case of a faithful uniform right module M over a right FBK ring R, there is a nonzero submodule N of M satisfying the conditions of Lemma 4.3.3.

Lemma 4.3.4. *Let R be a right FBK ring and let M be a faithful uniform right R-module. Then there exists a nonzero submodule N of M such that*

1. *$NP = 0$ for some prime ideal P of R,*
2. *N is a torsion-free right (R/P)-module,*
3. *R/P is a prime right Goldie ring.*

Proof. Since M is a uniform right R-module, $P = \text{ass}(M)$ is a prime ideal of R by Proposition 4.2.1 and there exists a nonzero submodule N of M such that $P = \text{ass}(M) = \text{ass}(N) = \text{ann}_R(N)$.

Let $0 \neq u \in N$ and consider the submodule uR of N. Note that uR is a nonzero finitely generated uniform submodule of N and hence of M. Clearly $\text{ann}_R(uR) \subseteq \text{ass}(uR) \subseteq \text{ass}(M) = P$. But also $P = \text{ann}_R(N) \subseteq \text{ann}_R(uR)$. Therefore we have $P = \text{ass}(uR) = \text{ann}_R(uR)$. It follows that we may assume, without loss of generality, that N is a finitely generated (in fact cyclic) right R-module.

Since $P = \text{ann}_R(N)$, N can be considered as a right (R/P)-module and as such is finitely generated, faithful and uniform. Now R/P is a prime right bounded ring with right Krull dimension, so, by Proposition 1.4.9, R/P is a right bounded prime right Goldie ring. It follows, by Corollary 4.1.3, that N is a torsion-free right (R/P)-module.

Hence the conditions of Lemma 4.3.3 are satisfied by the nonzero right R-submodule N of M. □

Corollary 4.3.5. *Let R be a right FBK ring and let M be a faithful uniform right R-module. Then there exists a nonzero submodule N of M such that for a right ideal A of R, if $MA \subseteq N$, then $A \cap \text{ann}_R(F) = 0$ for some finite subset F of M.*

Proof. Since R has right Krull dimension, every right ideal of R has finite uniform dimension as a right R-module, so the result follows by Lemmas 4.3.3 and 4.3.4. □

At this point we briefly turn our attention to right Noetherian rings and, following a preliminary lemma, we are able to prove the previously mentioned result of Cauchon which shows that a right FBN ring necessarily satisfies the H-condition.

Lemma 4.3.6. *Let R be a right FBN ring, let M be a faithful uniform right R-module and let $P = ass(M)$. Then $l.ann_R(P)$ is an essential right ideal of R.*

Proof. It is clear that $\text{l.ann}_R(P)$ is a right ideal of R. Since R has finite right uniform dimension, it suffices to show that $\text{l.ann}_R(P)$ has nonzero intersection with each uniform right ideal of R. Let X be a uniform right ideal of R and let $P' = \text{ass}(X)$. By Proposition 4.2.1, there is a nonzero right ideal Y of R with $Y \subseteq X$ such that $P' = \text{ann}_R(Y)$. Because M is faithful we have $MY \neq 0$ and $\text{ann}_R(MY) = \text{ann}_R(Y) = P'$. Hence $P' = \text{ann}_R(MY) \subseteq \text{ass}(M) = P$. Again by Proposition 4.2.1, there is a nonzero submodule V of M such that $P = \text{ann}_R(V)$. Set $W = V \cap MY$. By Corollary 4.1.3, MY is nonsingular as a right (R/P')-module. But W is a nonzero submodule of MY and $WP = 0$. Therefore P/P' is not an essential right ideal of the prime ring R/P' and it follows that $P = P'$, since R/P' is prime. Hence $YP = 0$ and so $\text{l.ann}_R(P) \cap X \neq 0$. □

Theorem 4.3.7. *Let R be a right FBN ring. Then R has the H-condition.*

Proof. Since a right FBN ring is right FBK, it suffices, by Lemma 4.3.2 and by factoring out the right annihilator, to consider a finitely generated faithful uniform right R-module M and to show that M is finitely annihilated. Let $P = \text{ass}(M)$. Then, by Lemma 4.3.6, $A = \text{l.ann}_R(P)$ is an essential right ideal of R. Let $N = \{m \in M | mP = 0\}$. Then $\text{ann}_R(N) = P$, so, by Corollary 4.1.3, N is nonsingular as a right (R/P)-module. In fact, N satisfies the conditions of Lemma 4.3.3. Now $MAP = 0$, so $MA \subseteq N$. Then, by Lemma 4.3.3, $A \cap \text{ann}_R(F) = 0$ for some finite subset F of M and so $\text{ann}_R(F) = 0$. Since M is faithful this gives that $\text{ann}_R(M) = \text{ann}_R(F) = 0$, hence proving the required result. □

Note that to apply Corollary 4.1.3 we require that the submodule N be finitely generated, which follows since M is a finitely generated right module over a right Noetherian ring, so is itself Noetherian.

Recall that our original aim of this section was to investigate whether or not right FBK rings satisfy the H-condition. If the ring is both right bounded and right FBK then the following result shows that finitely generated faithful uniform modules are finitely annihilated, which then allows us to characterise which rings with right Krull dimension do satisfy the H-condition. Note that a right bounded ring is not necessarily right fully bounded (see Section 1.5 of the Introduction) and that a right fully bounded ring is not necessarily right bounded (see Example 4.4.1), so neither the boundedness nor the fully boundedness condition alone is sufficient.

Lemma 4.3.8. *Let R be a right bounded right FBK ring and let M be a finitely generated faithful uniform right R-module. Then M is finitely annihilated.*

Proof. Let N be a nonzero submodule of M satisfying the conditions of Corollary 4.3.5. We want to find an essential right ideal A of R such that $MA \subseteq N$. Then $A \cap \operatorname{ann}_R(F) = 0$ for some finite subset F of M and so $\operatorname{ann}_R(F) = 0$. Since M is faithful this gives that $\operatorname{ann}_R(M) = \operatorname{ann}_R(F) = 0$, hence proving the result.

Because M is finitely generated we have $M = m_1R + \cdots + m_kR$ for some positive integer $k \geq 1$ and elements $m_i \in M$ $(1 \leq i \leq k)$. Since M is uniform and N is a nonzero submodule of M, N is an essential submodule of M. Thus, by Lemma 2.1.3, for each $1 \leq i \leq k$ there exists an essential right ideal E_i of R such that $m_iE_i \subseteq N$. Let $E = E_1 \cap \ldots \cap E_k$, then E is an essential right ideal of R and $m_iE \subseteq N$ for all $1 \leq i \leq k$. Now R is right bounded, so E contains an ideal A of R which is essential as a right ideal of R. Then

$$MA = (m_1R + \cdots + m_kR)A \subseteq m_1A + \cdots + m_kA \subseteq m_1E + \cdots + m_kE \subseteq N$$

and the result holds. □

Here we are referring to a particular right R-module M, which we have assumed to be faithful, and hence, given our right FBK ring R, we need only the extra hypothesis that R is right bounded. In general we must take factor rings of R to ensure that all our modules are faithful and hence we must further suppose that every factor ring of R is right bounded in order to conclude that R satisfies the H-condition.

Proposition 4.3.9. *Let R be a ring with right Krull dimension such that every factor ring of R is right bounded. Then R has the H-condition.*

Proof. By Lemma 4.3.2, it suffices to show that every finitely generated uniform right R-module is finitely annihilated. So let M be a finitely generated uniform right R-module. By factoring out the right annihilator, $\operatorname{ann}_R(M)$, of M, we may consider M as a faithful right $(R/\operatorname{ann}_R(M))$-module. Since every factor ring of R is right bounded, $R/\operatorname{ann}_R(M)$ is a right bounded right FBK ring and the result follows by Lemma 4.3.8. □

The above argument, along with Proposition 4.3.1, also gives us the following necessary and sufficient condition for a ring with right Krull dimension to satisfy the H-condition.

Theorem 4.3.10. *Let R be a ring with right Krull dimension. Then R has the H-condition if and only if every homomorphic image of R is right bounded.*

Proof. Necessity holds by Proposition 4.3.1. Sufficiency follows by Proposition 4.3.9, since every factor ring of R is right bounded. $\square$

By Proposition 4.3.9 our original question as to whether right FBK rings satisfy the H-condition has become, "if R is a right FBK ring, then is every factor ring of R right bounded?" By Proposition 4.3.1 and Theorem 4.3.7 the answer is yes for right FBN rings. However, in the following section we give an example which shows that the answer can, in general, be no for right FBK rings.

4.4 Right FBK Rings without the H-Condition

We now give an example to show that a right FBK ring need not have the H-condition. We give details of a ring with right Krull dimension which is right fully bounded but is not itself right bounded. By Theorem 4.3.10, this shows that right FBK rings do not necessarily have the H-condition, so answering our original question in the negative.

Let p be a prime number, then $\mathbb{Z}(p^\infty)$ denotes the $\mathbb{Z}$-submodule of $\mathbb{Q}/\mathbb{Z}$ given by $\mathbb{Z}(p^\infty) = \{a/p^n + \mathbb{Z} | a \in \mathbb{Z}, n \geq 0\}$. The following ascending chain is a complete list of all $\mathbb{Z}$-submodules of $\mathbb{Z}(p^\infty)$,

$$0 \subset (1/p + \mathbb{Z})\mathbb{Z} \subset (1/p^2 + \mathbb{Z})\mathbb{Z} \subset \ldots \subseteq \cup_{n \geq 1}(1/p^n + \mathbb{Z})\mathbb{Z} = \mathbb{Z}(p^\infty),$$

and hence $\mathbb{Z}(p^\infty)$ is an Artinian but not Noetherian $\mathbb{Z}$-module, every proper submodule of which is cyclic.

Example 4.4.1. Let p be any prime number and let R be the "matrix ring" given by

$$R = \begin{pmatrix} \mathbb{Z} & \mathbb{Z}(p^\infty) \\ 0 & \mathbb{Z} \end{pmatrix}.$$

Then R is a right (and left) FBK ring but R is not right bounded.

Proof. In this proof we will use subscripts to denote the ring over which we are considering Krull dimension in each case. Since $\mathbb{Z}(p^\infty)$ is an Artinian $\mathbb{Z}$-module, we have $\mathrm{k}((\mathbb{Z}(p^\infty))_{\mathbb{Z}}) = 0$. Let $N = \begin{pmatrix} 0 & \mathbb{Z}(p^\infty) \\ 0 & 0 \end{pmatrix}$. Then N is an ideal of R such that $N^2 = 0$. Moreover, every right ideal of R contained in N is of the form $\begin{pmatrix} 0 & K \\ 0 & 0 \end{pmatrix}$ where K is a $\mathbb{Z}$-submodule of $\mathbb{Z}(p^\infty)$. Thus $\mathrm{k}(N_R) = \mathrm{k}((\mathbb{Z}(p^\infty))_{\mathbb{Z}}) = 0$. Next, there is a ring isomorphism between R/N and $\mathbb{Z} \oplus \mathbb{Z}$, so that $\mathrm{k}((R/N)_R) = \mathrm{k}((R/N)_{R/N}) = \mathrm{k}((\mathbb{Z} \oplus \mathbb{Z})_{\mathbb{Z}\oplus\mathbb{Z}}) = 1$. Thus R has right Krull dimension 1, since $\mathrm{k}(R_R) = \sup\{\mathrm{k}(N_R), \mathrm{k}((R/N)_R)\}$. Similarly R has left Krull dimension 1.

For any prime ideal P of R, we have that $N \subseteq P$, since $N^2 = 0 \subseteq P$. It follows that $R/P \cong ((R/N)/(P/N))$, so the ring R/P is commutative and hence bounded on both sides. Thus R is both right and left fully bounded. Therefore, R is both right and left FBK.

We now show that R is not right bounded. Let $E = \begin{pmatrix} 0 & \mathbb{Z}/p\mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix}$. Here we identify $\mathbb{Z}/p\mathbb{Z}$ with the $\mathbb{Z}$-submodule $((1/p) + \mathbb{Z})\mathbb{Z}$ of $\mathbb{Z}(p^\infty)$, since the two are isomorphic as $\mathbb{Z}$-modules. Let $\begin{pmatrix} a & x \\ 0 & b \end{pmatrix}$ be any nonzero element of R. If $a \neq 0$ then there exists an element $y \in \mathbb{Z}(p^\infty)$ such that $ay = 1+p\mathbb{Z}$ (choose an element $y'+\mathbb{Z} \in \mathbb{Z}(p^\infty)$ such that ay' and p have a greatest common divisor of 1, then there exist elements $c, d \in \mathbb{Z}$ such that $ay'c + pd = 1$ and we may take $y = y'c+\mathbb{Z}$). In this case, $\begin{pmatrix} a & x \\ 0 & b \end{pmatrix}\begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1+p\mathbb{Z} \\ 0 & 0 \end{pmatrix} \in E$. Suppose that $a = 0$. Now $\mathbb{Z}/p\mathbb{Z}$ is an essential $\mathbb{Z}$-submodule of $\mathbb{Z}(p^\infty)$ (in fact, every nonzero $\mathbb{Z}$-submodule of $\mathbb{Z}(p^\infty)$ contains $\mathbb{Z}/p\mathbb{Z}$), so $\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}$ is an essential $\mathbb{Z}$-submodule of $\mathbb{Z}(p^\infty) \oplus \mathbb{Z}$. It follows that $\begin{pmatrix} 0 & x \\ 0 & b \end{pmatrix} R \cap E \neq 0$. Thus E is an essential right ideal of R.

Now suppose that there exists an ideal A of R such that A is an essential right ideal of R and $A \subseteq E$. Then $\begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix} \in A$ for some $0 \neq c \in \mathbb{Z}$. It follows that $\begin{pmatrix} 0 & \mathbb{Z}(p^\infty) \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \mathbb{Z}(p^\infty) \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix} \subseteq A \subseteq E$. This is a contradiction, so there can be no such ideal $A \subseteq E$. Thus R is not right bounded. □

4.5 Right FBK Rings and the Gabriel Correspondence

There is a well-known correspondence between the prime ideals of a right Noetherian ring and the isomorphism classes of indecomposable injective right modules over the ring. In our notation a ring R is said to satisfy the *Gabriel correspondence* if the mapping given by $I \mapsto \mathrm{ass}(I)$ is a bijection between the isomorphism classes of indecomposable injective right R-modules and the prime ideals of R. It was shown by Gordon and Robson [13, Chapter 8] and independently by Krause [18, Theorem] and Lambek and Michler [20, Corollary 3.12], that the right Noetherian rings which satisfy the Gabriel correspondence are precisely the right FBN rings. In fact, in [13], Gordon and Robson show that right FBK rings satisfy the Gabriel correspondence and it is a version of their proof which we include here. We begin with a couple of preliminary lemmas concerning injective modules. Recall that a nonzero module is called *indecomposable* if it has no direct summands other than 0 and itself.

Lemma 4.5.1. *Let R be a ring and let E be an injective right R-module. Then E is indecomposable if and only if $E \neq 0$ and E is an injective hull of every nonzero submodule of itself.*

Proof. Suppose that E is indecomposable. Then $E \neq 0$ by definition. Let M be a nonzero submodule of E. Then $\mathrm{E}(M) \subseteq E$, so, by Lemma 2.2.17, $\mathrm{E}(M)$ is a direct summand of E. It follows that $\mathrm{E}(M) = E$.

Conversely suppose that E is nonzero and is an injective hull of every nonzero submodule of itself. Let E_1 and E_2 be submodules of E such that $E_1 \oplus E_2 = E$ and suppose that $E_1 \neq 0$. Then $E_1 \cap E_2 = 0$ and $E = \mathrm{E}(E_1)$. In particular, E is an essential extension of E_1 and hence $E_2 = 0$. Thus E is indecomposable. $\square$

Lemma 4.5.2. *Let R be a ring and let M be a right R-module. Then $\mathrm{E}(M)$ is an indecomposable right R-module if and only if M is uniform.*

Proof. Suppose that $\mathrm{E}(M)$ is an indecomposable right R-module. Let M_1 and M_2 be submodules of M such that $M_1 \cap M_2 = 0$ and suppose that $M_1 \neq 0$. Then $\mathrm{E}(M_1) \subseteq \mathrm{E}(M)$, so, by Lemma 2.2.17, $\mathrm{E}(M_1)$ is a direct summand of $\mathrm{E}(M)$. It follows that $\mathrm{E}(M_1) = \mathrm{E}(M)$. In particular, $\mathrm{E}(M)$ is an essential extension of M_1 and hence $M_2 = 0$. Thus M is uniform.

Conversely, suppose that M is uniform. Let E_1 and E_2 be submodules of the right R-module $\mathrm{E}(M)$ such that $E_1 \oplus E_2 = \mathrm{E}(M)$ and suppose that $E_1 \neq 0$. Then $E_1 \cap M \neq 0$ and $(E_1 \cap M) \cap (E_2 \cap M) = 0$. It follows that $E_2 \cap M = 0$ and hence $E_2 = 0$. Thus $\mathrm{E}(M)$ is an indecomposable right R-module. $\square$

Recall that a nonzero module M is called *k-critical* if M has Krull dimension and, for some ordinal $\alpha \geq 0$, $\mathrm{k}(M) = \alpha$ and $\mathrm{k}(M') < \alpha$ for each proper homomorphic image M' of M (in this case M may also be called α-k-critical). We now consider the injective hulls of k-critical modules.

Proposition 4.5.3. *Let R be a ring with right Krull dimension. Then*

(i) the indecomposable injective right R-modules are precisely the injective hulls of k-critical right R-modules.

(ii) two k-critical right R-modules have isomorphic injective hulls if and only if one contains an isomorphic copy of a nonzero submodule of the other.

Proof. (i) By Lemmas 3.2.1 and 3.2.7 (i), k-critical modules are uniform, so, by Lemma 4.5.2, their injective hulls are indecomposable. Conversely, if I is an indecomposable injective right R-module then, by Lemma 4.5.1, I is the injective hull of any of its nonzero submodules. Choose a nonzero cyclic submodule of I. This has Krull dimension, so, by Lemma 1.4.7, contains a k-critical submodule, C say. Then $I = \mathrm{E}(C)$.

(ii) Let C and D be k-critical right R-modules. Suppose that there exists a nonzero submodule C' of C isomorphic to a (necessarily nonzero) submodule D' of D. Then $\mathrm{E}(C)$ and $\mathrm{E}(D)$ are indecomposable by (i) so, by Lemma 4.5.1, $\mathrm{E}(C) = \mathrm{E}(C') \cong \mathrm{E}(D') = \mathrm{E}(D)$. Conversely suppose that there exists a right R-isomorphism $\phi : \mathrm{E}(C) \to \mathrm{E}(D)$. Consider the restriction of ϕ to the submodule $C' = \{c \in C | \phi(c) \in D\}$ of C. Now $C \neq 0$ implies that $\phi(C) \neq 0$, so $\phi(C') = \phi(C) \cap D \neq 0$, since $\mathrm{E}(D)$ is an essential extension of D. Thus $C' \neq 0$. But $\ker \phi|_{C'} = \ker \phi \cap C' = 0$, since ϕ is an isomorphism. Thus the nonzero submodules C' of C and $\phi(C')$ of D are isomorphic. $\square$

In order to prove our main theorem of this section, we will need the concept of critical dimension, which we define as follows. Let R be a ring with right Krull dimension and let I be an indecomposable injective right R-module. By Proposition 4.5.3 (i), $I = \mathrm{E}(C)$ for some k-critical right R-module C. In fact Proposition 4.5.3 (ii) shows that the ordinal $\mathrm{k}(C)$ depends only on the isomorphism class of $I = \mathrm{E}(C)$. We call this ordinal the *critical dimension* of I and denote it by $\mathrm{crdim}(I)$. Now C and hence also $I = \mathrm{E}(C)$ is uniform, so, by Proposition 4.2.1, $\mathrm{ass}(I) = P$ is a prime ideal of R and $P = \mathrm{ass}(D) = \mathrm{ann}_R(D)$ for some nonzero submodule D of I. Take $C' = C \cap D$. Then C' is a nonzero submodule of the k-critical right R-module C, so is itself k-critical. Further, $P = \mathrm{ann}_R(D) \subseteq \mathrm{ann}_R(C') \subseteq \mathrm{ass}(C') \subseteq \mathrm{ass}(D) = P$, since C' is a nonzero submodule of D, and it follows that $P = \mathrm{ass}(C') = \mathrm{ann}_R(C')$. Note also that, by Proposition 4.5.3 (i), I is indecomposable so, by Lemma 4.5.1, $I = E(C')$. Now C' is a right (R/P)-module with Krull dimension so $\mathrm{k}(C') \leq \mathrm{k}(R/P)$, by Lemma 1.4.6, and this gives us the inequality,

$$\mathrm{crdim}(I) = \mathrm{k}(C') \leq \mathrm{k}(R/\mathrm{ass}(I)).$$

For rings with Krull dimension, the Gabriel correspondence and boundedness are related to the equality of these two ordinals. Before investigating this relationship further we require some preliminary results concerning k-critical right ideals in semiprime rings with right Krull dimension. Note that, by Proposition 1.4.9, such rings are semiprime right Goldie.

Lemma 4.5.4. *A right ideal of a semiprime ring with right Krull dimension is k-critical if and only if it is uniform.*

Proof. By Lemmas 3.2.1 and 3.2.7, every k-critical module over any ring is uniform. For the converse, suppose that I is a uniform right ideal of a semiprime ring R with right Krull dimension. By definition I is nonzero. By Lemma 1.4.7, I contains a k-critical right ideal, C say. Then $C \neq 0$ and R is semiprime, so $C^2 \neq 0$. Therefore there exists an element $c \in C$ such that $cC \neq 0$ and hence $cI \neq 0$. Define a map $\varphi : I \to C$ by $\varphi(x) = cx$ for all $x \in I$, then φ is a right R-homomorphism. Suppose that $\ker \varphi \neq 0$. Let $0 \neq x \in I$, then, by Lemma 2.1.3, there exists an essential right ideal E of R such that $xE \subseteq \ker \varphi$. Then $\varphi(x)E = \varphi(xE) = 0$. But R is semiprime right Goldie, so is right nonsingular and thus $\varphi(x) = 0$. Therefore $\varphi(I) = 0$, that is $cI = 0$, which is a contradiction. Hence $\ker \varphi = 0$ so I is isomorphic to the submodule $\varphi(I)$ of C and it follows that I is k-critical. □

Lemma 4.5.5. *Let R be a semiprime ring with right Krull dimension. Then $k(R) = \sup\{k(C) | C$ is a k-critical right ideal of $R\}$.*

Proof. Since R is right Goldie there is an essential finite direct sum of uniform right ideals of R, say $E = U_1 \oplus \cdots \oplus U_n$ for some integer $n \geq 1$. By Lemma 4.5.4, each U_i is k-critical for $1 \leq i \leq n$. By Lemma 1.4.13, $\mathrm{k}(R/E) < \mathrm{k}(R)$ and so $\mathrm{k}(E) = \mathrm{k}(R)$, by Lemma 1.4.1. Thus $\mathrm{k}(R) = \mathrm{k}(U_i)$ for some $1 \leq i \leq n$ and the result follows. □

Corollary 4.5.6. *Let R be a prime ring with right Krull dimension and let C be a k-critical right ideal of R. Then $k(C) = k(R)$.*

Proof. Since R is a prime right Goldie ring, all uniform right ideals are subisomorphic, that is each uniform right ideal contains an isomorphic copy of each other uniform right ideal. Hence they all have the same Krull dimension, namely $\mathrm{k}(C)$. The result follows by Lemma 4.5.5. □

In fact, this result can be strengthened as follows.

Lemma 4.5.7. *Let R be a prime ring with right Krull dimension and let C be a k-critical right R-module. Then $k(C) = k(R)$ if and only if some nonzero submodule of C embeds in R.*

Proof. Suppose $\mathrm{k}(C) = \mathrm{k}(R)$. Let $E = \oplus_{i=1}^{n} C_i$ be an essential finite direct sum of critical right ideals of R. If $CE = 0$ then $\mathrm{Hom}(R/E, C) \neq 0$ so $\mathrm{k}(C) \leq \mathrm{k}(R/E)$. But this is a contradiction, since $\mathrm{k}(R/E) < \mathrm{k}(R) = \mathrm{k}(C)$. Hence $CE \neq 0$ and so $CC_i \neq 0$ for some $1 \leq i \leq n$. Thus there is a nonzero map $f : C_i \to C$, which must be a monomorphism or else $\mathrm{k}(C) = \mathrm{k}(f(C_i)) < \mathrm{k}(C_i) \leq \mathrm{k}(R) = \mathrm{k}(C)$, a contradiction. Then $f(C_i)$ is a nonzero submodule of C which embeds in R

Conversely, suppose that D embeds in R for some nonzero submodule D of C. Then $\mathrm{k}(D) = \mathrm{k}(C)$ and, by Corollary 4.5.6, $\mathrm{k}(D) = \mathrm{k}(R)$. □

We now return to our investigation of the Gabriel correspondence in right FBK rings.

Proposition 4.5.8. *Let R be a ring with right Krull dimension and let P be a prime ideal of R. Then there is a unique (up to isomorphism) indecomposable injective right R-module I such that $ass(I) = P$ and $crdim(I) = k(R/ass(I))$.*

Proof. Let U be a uniform right ideal of the prime right Goldie ring R/P. Then $\mathrm{ass}_{R/P}(U)$ is a prime ideal of R/P and $\mathrm{ass}_{R/P}(U) = \mathrm{ass}_{R/P}(U') = \mathrm{ann}_{R/P}(U')$ for some nonzero submodule U' of U. Since R/P is prime it follows that $\mathrm{ass}_{R/P}(U) = \mathrm{ann}_{R/P}(U) = 0$ and therefore $\mathrm{ass}_R(U) = P$. Also since U is a uniform right (R/P)-module, $\mathrm{E}(U_R)$ is an indecomposable injective right R-module. By Lemma 4.5.4, the notions of k-critical and uniform coincide for right ideals in the prime ring with right Krull dimension R/P, so U is a k-critical right ideal of R/P and hence, by Corollary 4.5.6, $\mathrm{crdim}(E(U_R)) = \mathrm{k}(U) = \mathrm{k}(R/P)$. Furthermore, $\mathrm{ass}_R(\mathrm{E}(U_R)) = \mathrm{ass}_R(U) = P$.

Now let I be any indecomposable injective right R-module with $\mathrm{ass}(I) = P$ and $\mathrm{crdim}(I) = \mathrm{k}(R/P)$. Then $I = \mathrm{E}(C)$ for some k-critical R-submodule C of I such that $\mathrm{ann}_R(C) = P$ and $\mathrm{k}(C) = \mathrm{k}(R/P)$. By Lemma 4.5.7, C contains a nonzero right (R/P)-module D which embeds in R/P and D is necessarily k-critical with $I = \mathrm{E}(C_R) = \mathrm{E}(D_R)$. Since the uniform right ideals of the prime right Goldie ring R/P are all subisomorphic, it follows that $I = \mathrm{E}(D_R) \cong \mathrm{E}(U_R)$. Therefore an indecomposable injective right R-module $I = \mathrm{E}(U_R)$ satisfying the conditions of the proposition exists and is unique up to isomorphism. □

This result shows that for a ring with right Krull dimension the Gabriel correspondence is equivalent to the condition that $\mathrm{crdim}(J) = \mathrm{k}(R/\mathrm{ass}(J))$ for each indecomposable injective right R-module J.

Before showing that right FBK rings satisfy the Gabriel correspondence, we require one further lemma.

Lemma 4.5.9. *Let R be a ring, let I be a two-sided ideal of R and let A be a right R-module such that $AI = 0$. Then $E(A_{R/I}) = \{e \in E(A_R) | eI = 0\}$.*

Proof. Let $E = \{e \in \mathrm{E}(A_R)|eI = 0\}$ and let X be any nonzero (R/I)-submodule of E (it is clear that E can be considered as a right (R/I)-module). Then X is also an R-submodule of E and hence of $\mathrm{E}(A_R)$ and thus $X \cap A \neq 0$. It follows that, as right (R/I)-modules, E is an essential extension of A.

We now show that E is an injective right (R/I)-module. Let $\overline{R} = R/I$, let $\overline{B}$ be a right ideal of $\overline{R}$ and let $\alpha : \overline{B} \to E$ be any right $\overline{R}$-homomorphism. Then $\overline{B} = B/I$ for some right ideal B of R such that $I \subseteq B$ and we can also consider α as a right R-homomorphism. Let $\pi : B \to B/I$ be the canonical projection, given by $\pi(b) = b + I$ for all $b \in B$. Then $\alpha\pi : B \to E \subseteq \mathrm{E}(A_R)$ is a right R-homomorphism, so can be extended to a right R-homomorphism $\varphi : R \to \mathrm{E}(A_R)$ such that $\varphi(b) = \alpha\pi(b) = \alpha(b + I)$ for all $b \in B$. But then $\varphi(r)i = \varphi(ri) = \alpha(ri + I) = \alpha(\overline{0}) = 0$ for all $r \in R$ and $i \in I$, since $I \subseteq B$ is a two-sided ideal of R, and it follows that φ is in fact a mapping $\varphi : R \to E$. Defining $\overline{\varphi} : R/I \to E$ by $\overline{\varphi}(r + I) = \varphi(r)$ for all $r \in R$, it is then easily checked that $\overline{\varphi}$ is a well-defined right (R/I)-homomorphism such that $\overline{\varphi}|_{\overline{B}} = \alpha$. Hence E is an injective right (R/I)-module. It follows that $E = \mathrm{E}(A_{R/I})$, as required. □

Theorem 4.5.10. *Let R be a right FBK ring. Then R satisfies the Gabriel correspondence; that is, the mapping $I \mapsto \mathrm{ass}(I)$ gives a bijection between the isomorphism classes of indecomposable injective right R-modules and prime ideals of R.*

Proof. Suppose that R does not satisfy the Gabriel correspondence. Then, by Proposition 4.5.8, there is an indecomposable injective right R-module J with $\mathrm{crdim}(J) < \mathrm{k}(R/\mathrm{ass}(J))$. Let $P = \mathrm{ass}(J)$. Then $J = \mathrm{E}(D)$ for some k-critical right R-module D such that $\mathrm{ann}_R(D) = P$. Let $J' = \{j \in J | jP = 0\}$. Then, by Lemma 4.5.9, $J' = \mathrm{E}(D_{R/P})$, so J' is an indecomposable injective right (R/P)-module. Since the hypothesis on R is inherited by factor rings, by factoring out the prime ideal $P = \mathrm{ass}(J)$ and replacing J by J', we may assume that R is a prime ring and that $\mathrm{ass}(J) = 0$ and $\mathrm{crdim}(J) < \mathrm{k}(R)$. Now choose a cyclic critical right R-module C such that $J = \mathrm{E}(C)$ and $\mathrm{ann}_R(C) = \mathrm{ass}(C) = \mathrm{ass}(J) = 0$. Then $C \cong R/I$ where I is a nonzero right ideal of R and so $\mathrm{ann}_R(R/I) = 0$. But $\mathrm{ann}_R(R/I) = 0$ is the largest ideal of R contained in I and hence I cannot be an essential right ideal of R, since, by hypothesis, R is prime right bounded. Thus $I \cap A = 0$ for some nonzero right ideal A of R. Then $A \cong (I \oplus A)/I \hookrightarrow R/I \cong C$, so A embeds in C. It follows that some nonzero submodule of C is isomorphic to A and hence embeds in R. Thus $\mathrm{k}(C) = \mathrm{k}(R)$, by Lemma 4.5.7. This contradicts the fact that $\mathrm{k}(C) = \mathrm{crdim}(\mathrm{E}(C)) = \mathrm{crdim}(J) < \mathrm{k}(R)$. Therefore R must satisfy the Gabriel correspondence. □

4.6 The Bimodule Condition and Krull Symmetry for Rings with Krull Dimension

Recall that if R and S are rings and ${}_SM_R$ is a left S-, right R-bimodule then we say that ${}_SM_R$ satisfies the *bimodule condition* if both ${}_SM$ and M_R have Krull dimension and $\mathrm{k}({}_SM) = \mathrm{k}(M_R)$.

It is shown in [25, Corollary 6.4.13] that if R and S are both FBN rings then every bimodule ${}_SM_R$ which is finitely generated on both sides satisfies the bimodule condition. This was originally proved by Jategaonkar in [14, Section 2]. Note that the bimodule condition and the related concept of strongly Krull symmetric bimodules are discussed in Section 3.2 of Chapter 3, where further examples of bimodules satisfying the bimodule condition are given.

Here we investigate the bimodule condition and find that for rings with Krull dimension it is closely related to the H-condition (and hence to boundedness properties of the ring). Whereas previously the H-condition always referred to the right-handed version, we will now need to specify which side the condition holds on. We will do this by saying that the ring R satisfies the H-condition on the right/left, or equivalently that R satisfies the right/left H-condition, as appropriate.

Lemma 4.6.1. *Let R be a ring and let M be a finitely annihilated right R-module with Krull dimension. Then $k(M) = k(R/ann_R(M))$.*

Proof. Since M is a finitely annihilated right R-module, $\mathrm{ann}_R(M) = \mathrm{ann}_R(m_1, \ldots, m_n)$ for some positive integer n and elements $m_i \in M$ $(1 \leq i \leq n)$. Then $R/\mathrm{ann}_R(M)$ embeds in M^n via the map $r + \mathrm{ann}_R(M) \mapsto (m_1r, \ldots, m_nr)$ for all $r \in R$ and hence has Krull dimension with $\mathrm{k}(R/\mathrm{ann}_R(M)) \leq \mathrm{k}(M)$. The result follows by Lemma 1.4.6 since M is a right $(R/\text{ann}_R(M))$-module with Krull dimension and the R- and $(R/\text{ann}_R(M))$-module structures of M coincide. □

For the next result we require the concept of the deviation of a partially ordered set, which we define as follows. If E is a partially ordered set we denote the partial ordering in the usual way using the symbol $\leq$. If $a, b \in E$ then we take $a < b$ to mean $a \leq b$ but $a \neq b$. If $a, b \in E$ then we denote by $[a, b]$ the subset of E consisting of all $x \in E$ satisfying $a \leq x \leq b$.

We define the notion of the *deviation* of a partially ordered set E, which we denote by $\text{dev}(E)$. The measure $\text{dev}(E)$ will be either an ordinal or one of the symbols $-\infty, +\infty$. The ordinals are ordered in the usual way and for any ordinal α we take $-\infty \leq \alpha \leq +\infty$. We determine by induction on the ordinal α the partially ordered sets E which have $\text{dev}(E) \leq \alpha$. We begin by defining $\text{dev}(E) = -\infty$ if E is a discrete partially ordered set (that is, if for $a, b \in E$ we have $a \leq b$ if and only if $a = b$). We define $\text{dev}(E) \leq 0$ if E is Artinian (that is, if every decreasing sequence in E terminates). Now suppose that $\alpha \geq 1$ is an ordinal and that we have determined all partially ordered sets F satisfying $\text{dev}(F) < \alpha$. We define $\text{dev}(E) \leq \alpha$ if for every decreasing sequence $a_1 > a_2 > a_3 > \cdots$ in E, $\text{dev}[a_{i+1}, a_i] < \alpha$ for all but finitely many $i = 1, 2, 3, \ldots$. Finally we define $\text{dev}(E) = +\infty$ if for all ordinals α, $\text{dev}(E) \nleq \alpha$. For example, $\text{dev}(E) = 0$ if and only if E is Artinian and non-discrete and, with the usual orderings, $\text{dev}(\mathbb{N}) = 0$, $\text{dev}(\mathbb{Z}) = 1$ and $\text{dev}(\mathbb{Q}) = +\infty$.

Given a module or bimodule M, $\mathcal{L}(M)$ will denote the lattice of submodules or sub-bimodules of M, ordered by inclusion. Note that the Krull dimension of a right module M over a ring R, $\text{k}(M)$, is equivalent to the deviation of $\mathcal{L}(M)$.

The following proposition is taken from [25, Proposition 6.4.13]. Note that throughout the proof all Krull dimensions will be on the right R-module side and so we will omit the R subscript.

Proposition 4.6.2. *Let R be a ring with the H-condition on the right, let S be any ring and let ${}_SM_R$ be a left S-, right R-bimodule such that M_R is Noetherian. Then dev$(\mathcal{L}({}_SM_R)) =$ k(M_R).*

Proof. Let $\mu(M)$ denote $\text{dev}(\mathcal{L}({}_SM_R))$. Since $\mathcal{L}({}_SM_R)$ is a sublattice of $\mathcal{L}(M_R)$ we have $\mu(M) \leq \text{k}(M)$. Suppose that the equality is not always true and choose M amongst the counterexamples to minimise $\mu(M)$. Say $\mu(M) = \beta$ for some ordinal β. Clearly $\beta \geq 0$ since equality holds trivially for zero modules. Now $\text{k}(M) > \beta$, so, by Lemma 1.4.8, there exists an R-submodule N of M with M/N a β-k-critical right R-module and in particular $\mathrm{k}(M/N) = \beta$. Let $A = \mathrm{ann}_R(M/N)$ and $M_1 = MA$, then M/N and M/M_1 are finitely annihilated right R-modules with Krull dimension and $A \subseteq \mathrm{ann}_R(M/M_1)$ and $M_1 \subseteq N$. Hence, by Lemmas 1.4.1 and 4.6.1,

$$\beta = \mathrm{k}(M/N) \leq \mathrm{k}(M/M_1) \leq \mathrm{k}(R/A) = \mathrm{k}(M/N) = \beta$$

and so $\mathrm{k}(M/M_1) = \beta$. By Lemma 1.4.1 again, it follows that $\mathrm{k}(M_1) > \beta$.

In fact, M_1 is a sub-bimodule of ${}_SM_R$ and hence M/M_1 is also a left S-, right R-bimodule. Now $\mu(M/M_1) \leq \mathrm{k}(M/M_1) = \beta$ and $\mu(M_1) \leq \mu(M) = \beta$. If $\mu(M/M_1) < \beta = \mathrm{k}(M/M_1)$ or $\mu(M_1) < \beta < \mathrm{k}(M_1)$ then the minimal choice of β is contradicted. Thus $\mu(M/M_1) = \mu(M_1) = \beta$. However, $\mu(M_1) < \mathrm{k}(M_1)$, so M_1 is another counterexample. Iteration gives us a chain $M = M_0 \supseteq M_1 \supseteq M_2 \supseteq \cdots$ of sub-bimodules of M with $\mu(M_n/M_{n+1}) = \beta$ for all $n \geq 0$. This contradicts the hypothesis that $\mu(M) = \beta$. □

Corollary 4.6.3. *Let R and S be rings with the H-condition on the right and left respectively and let ${}_SM_R$ be a left S-, right R-bimodule such that ${}_SM$ and M_R are both Noetherian. Then ${}_SM_R$ satisfies the bimodule condition, that is $\mathrm{k}({}_SM) = \mathrm{k}(M_R)$. If, further, ${}_SM$ and M_R are faithful then $\mathrm{k}({}_SS) = \mathrm{k}(R_R)$.*

Proof. That ${}_SM_R$ satisfies the bimodule condition follows from Proposition 4.6.2. If both ${}_SM$ and M_R are faithful then Lemma 4.6.1 shows that $\mathrm{k}({}_SS) = \mathrm{k}(R_R)$. □

One particular case of interest is the application of the above to rings.

Definition. A ring R is called *Krull symmetric* if R has left and right Krull dimension and $\mathrm{k}({}_RR) = \mathrm{k}(R_R)$.

Lenagan's Theorem (Proposition 2.2.19) shows that a Noetherian ring which is Artinian (that is has Krull dimension zero) on either side is Krull symmetric. The aforementioned result of Jategaonkar shows that FBN rings are Krull symmetric (Proposition 4.6.4). It is still an open question as to whether all Noetherian rings are Krull symmetric. For more information on the bimodule condition and Krull symmetric rings see [12, Appendix 9 p. 287].

Considering the ring as a bimodule over itself, our above work gives Jategaonkar's result that FBN rings are Krull symmetric (for note that, by Proposition 4.3.1 and Theorem 4.3.7, a Noetherian ring is fully bounded if and only if it has the H-condition).

Proposition 4.6.4. *Let R be a left and right Noetherian ring with left and right H-condition. Then R is Krull symmetric.*

Using a result of Gordon and Robson this can be pushed further to give an analogue for rings with Krull dimension of the fully bounded Noetherian result (see Theorem 5.3.4). We give details of this in Section 5.3 of Chapter 5.

4.7 The Jacobson Conjecture for Rings with Krull Dimension

We conclude this chapter by briefly considering the Jacobson conjecture for fully bounded rings with Krull dimension.

A ring R is said to satisfy the *Jacobson conjecture* if

$$\bigcap_{n\geq 0} \mathrm{J}^n(R) = 0,$$

where $\mathrm{J}(R)$ denotes the Jacobson radical of R (the intersection of the maximal right ideals of R, amongst many other characterisations (see Section 1.2 of the Introduction)).

It is well known that FBN rings satisfy the Jacobson conjecture [14, Theorem 3.7], but that one-sided FBN rings do not [5, Example 5.12]. In fact, [12, Theorem 8.12] shows that any left Noetherian right FBN ring satisfies the Jacobson conjecture. However, the following example shows that, even in the two-sided case, FBK rings do not necessarily satisfy the Jacobson conjecture.

Example 4.7.1. Let S be a discrete valuation ring (that is, a local principal ideal domain), let U be a simple S-module and let E be the injective hull of U_S. Consider the "matrix ring"

$$R = \left\{ \begin{pmatrix} a & e \\ 0 & a \end{pmatrix} | a \in S, e \in E \right\}.$$

Then R is a commutative ring. Let A be the ideal of R given by $A = \begin{pmatrix} 0 & E \\ 0 & 0 \end{pmatrix}$. Then $\mathrm{k}(A_R) = \mathrm{k}(E_S) = 0$, since $E = \mathrm{E}(U_S)$ is Artinian over the commutative Noetherian ring S (see [28, Theorem 4.30]). Also $R/A \cong S$, so $\mathrm{k}(R/A) = \mathrm{k}(S) = 1$, since S is a principal ideal domain. It follows that $\mathrm{k}(R) = 1$. Thus R is both right and left FBK. In fact, since R is commutative, every factor ring of R is both left and right bounded. However, if M is the unique maximal ideal of S then

$$\mathrm{J}(R) = \left\{ \begin{pmatrix} m & e \\ 0 & m \end{pmatrix} | m \in M, e \in E \right\}.$$

The commutative Noetherian ring S satisfies the Jacobson conjecture, so $\cap_{n\geq 0} M^n = 0$. Also E is injective over the domain S so is divisible and hence $EM = E$. It follows that

$$\bigcap_{n\geq 0} \mathrm{J}^n(R) = \begin{pmatrix} 0 & E \\ 0 & 0 \end{pmatrix} \neq 0,$$

so R does not satisfy the Jacobson conjecture.

Chapter 5

Krull Dimensions and their Duals

5.1 Introduction and Definitions

In previous chapters we introduced and made use of the concept of Krull dimension, detailing many of its basic properties in Section 1.4 of the Introduction. This particular dimension could more specifically be termed “module-theoretic Krull dimension” and there are in fact several related dimensions. In this chapter we consider a number of these various types of “Krull dimension” and investigate the relationships between them. We begin by defining the four main dimensions that we will be working with.

Let R be a ring and let M be a right R-module. The *Krull dimension* of M is defined in Section 1.4 of the Introduction and is denoted by $\mathrm{k}(M)$, if it exists. The dual Krull dimension of a module is defined similarly with ascending chains, as below.

Definition. Let R be a ring and let M be a right R-module. The *dual Krull dimension* of M_R, if it exists, is denoted by $\mathrm{k}^\circ(M_R)$ and defined as follows. $\mathrm{k}^\circ(M) = -1$ if and only if $M = 0$. If $\alpha \geq 0$ is an ordinal such that all modules with dual Krull dimension strictly less than α are known, then $\mathrm{k}^\circ(M) \leq \alpha$ if for every ascending chain $0 = M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots$ of submodules of M there is a positive integer n such that $\mathrm{k}^\circ(M_{i+1}/M_i) < \alpha$ for all $i \geq n$.

Note that $\mathrm{k}^\circ(M) = 0$ if and only if M_R is a nonzero Noetherian module. Further, results on deviations of partially ordered sets and their duals show that M has Krull dimension if and only if M has dual Krull dimension [25, Proposition 6.1.8].

The *right dual Krull dimension* of a ring R if it exists is the dual Krull dimension of the right R-module R_R, denoted by $\mathrm{k}^\circ(R)$.

Definition. Let R be a nonzero ring. Let $\mathrm{Spec}(R)$ denote the collection of prime ideals of

R. Let $\mathrm{Spec}_0(R)$ denote the collection of maximal ideals of R and for any ordinal $\alpha \geq 1$, let $\mathrm{Spec}_\alpha(R)$ denote the collection of prime ideals P of R such that all prime ideals of R properly containing P belong to $\bigcup_{0 \leq \beta < \alpha} \mathrm{Spec}_\beta(R)$. If there exists an ordinal $\alpha \geq 0$ such that $\mathrm{Spec}(R) = \mathrm{Spec}_\alpha(R)$ then we shall say that R has *classical Krull dimension* and the classical Krull dimension of R, denoted by $\mathrm{ck}(R)$, shall be the least ordinal $\gamma \geq 0$ such that $\mathrm{Spec}(R) = \mathrm{Spec}_\gamma(R)$. We shall define a ring to have classical Krull dimension -1 if and only if the ring is zero.

Definition. Let R be a ring and let M be a right R-module. The *dual classical Krull dimension* of M, if it exists, is denoted by $\mathrm{ck}^\circ(M)$ and defined as follows. $\mathrm{ck}^\circ(M) = -1$ if and only if $M = 0$. If $M \neq 0$ then $\mathrm{ck}^\circ(M)$ is the least number of generators as a two-sided ideal of a finitely generated proper two-sided ideal A of R such that $\mathrm{ann}_M(A)$ has finite (composition) length. The zero ideal is considered to have zero non-trivial generators so that $\mathrm{ck}^\circ(M) = 0$ if and only if M is nonzero and has finite (composition) length.

Note that dual classical Krull dimension takes only finite values and is otherwise undefined, unlike Krull dimension, dual Krull dimension and classical Krull dimension, which are defined for any ordinal values. For convenience sake however, we will sometimes consider a module without a given dimension to satisfy the condition that the module has that dimension with value infinity, with the convention that ∞ is "greater" than any finite or ordinal value. Note that only if there is possible ambiguity will we include subscripts to indicate the module structure under consideration, for example $\mathrm{k}(M_R)$.

5.2 Classical Krull Dimension

Originally, the Krull dimension of a ring R was defined to be the supremum of the lengths of chains of prime ideals of R, being ∞ if no such supremum existed. This measure of dimension originated in the study of commutative Noetherian rings. An extension of this definition which included infinite ordinal values was introduced by Krause [17], thus allowing one to distinguish between various rings with infinite Krull dimension. It is Krause's definition we use above when referring to classical Krull dimension. Such dimensions are now generally referred to as the classical Krull dimension, in order to distinguish them from the more general definition of the Krull dimension of a module, which was introduced for finite ordinals by Rentschler and Gabriel [26] and extended to arbitrary ordinals again by Krause [17] and has been a widely studied subject since the

1970's (see [13], [25, Chapter 6] and [12, Chapter 13]). It is Krause's infinite ordinal definition that we use and term simply Krull dimension. In this section we consider the infinite ordinal definition of the classical Krull dimension and study its properties, not only out of independent interest, but also to aid later comparison with the general module-theoretic Krull dimension.

In general a ring may not have classical Krull dimension, however we have the following result.

Lemma 5.2.1. *A ring R has classical Krull dimension if and only if R satisfies the ascending chain condition on prime ideals.*

Proof. Suppose that a ring R has classical Krull dimension. Then $\mathrm{ck}(R) = \alpha$ for some ordinal α and so $\mathrm{Spec}(R) = \mathrm{Spec}_\alpha(R)$. Suppose, for contradiction, that there exists an infinite strictly ascending chain $P_1 \subset P_2 \subset \cdots$ of prime ideals of R. For each such chain $P_1 \in \mathrm{Spec}_\alpha(R)$. Choose β to be the least ordinal such that there exists such a strictly ascending chain of prime ideals with $P_1 \in \mathrm{Spec}_\beta(R)$. Then $P_2 \in \mathrm{Spec}_\gamma(R)$ for some $\gamma < \beta$, so the chain $P_2 \subset P_3 \subset \cdots$ contradicts the minimal choice of β. Hence there can be no such chain and R must satisfy the ascending chain condition on prime ideals.

Conversely, suppose that a ring R satisfies the ascending chain condition on prime ideals. Since the cardinality of the sets $\mathrm{Spec}_0(R) \subseteq \mathrm{Spec}_1(R) \subseteq \cdots$ is bounded, for example by $2^{|R|}$, this transfinite chain must terminate, so there exists an ordinal α such that $\mathrm{Spec}_\alpha(R) = \mathrm{Spec}_{\alpha+1}(R)$. If R does not have classical Krull dimension then $\mathrm{Spec}_\alpha(R) \neq \mathrm{Spec}(R)$, so we can choose a prime ideal P of R maximal in $\mathrm{Spec}(R) \setminus \mathrm{Spec}_\alpha(R)$. If Q is a prime ideal of R such that $P \subset Q$ then, by the maximal choice of P, we have $Q \in \mathrm{Spec}_\alpha(R)$. But then $P \in \mathrm{Spec}_{\alpha+1}(R) = \mathrm{Spec}_\alpha(R)$, a contradiction. Hence R must have classical Krull dimension. □

If K is a field then the polynomial ring in infinite indeterminates $R = K[x_1, x_2, x_3, \ldots] = \cup_{n \geq 1} K[x_1, \ldots, x_n]$ is an example of a ring which does not satisfy the ascending chain condition on prime ideals (since $Rx_1 \subseteq Rx_1 + Rx_2 \subseteq Rx_1 + Rx_2 + Rx_3 \subseteq \cdots$ is an infinite ascending chain of prime ideals of R) and hence does not have classical Krull dimension.

The definition of classical Krull dimension that we use extends the original definition of Krull dimension in terms of lengths of chains of prime ideals, allowing infinite ordinal values. We will call this original Krull dimension *f-Krull dimension*, since it takes only finite values (or infinity).

Definition. The *f-Krull dimension* of a ring R, denoted by $\mathrm{fk}(R)$, shall be defined to be the supremum of the lengths of chains of prime ideals of R, taking $\mathrm{fk}(R)$ to be ∞ if no such supremum exists.

The classical Krull dimension of R coincides with the f-Krull dimension of R provided it is finite, but replaces ∞ by an ordinal value unless R does not have classical Krull dimension.

Lemma 5.2.2. *A ring R has finite f-Krull dimension if and only if R has finite classical Krull dimension, in which case $fk(R) = ck(R)$.*

Proof. Let R be a ring with finite f-Krull dimension and let $\mathrm{fk}(R) = n$ for some positive integer n. Suppose that $\mathrm{Spec}_n(R) \neq \mathrm{Spec}(R)$. Then there exists a prime ideal $P_0 \in \mathrm{Spec}(R)$ of R such that $P_0 \notin \mathrm{Spec}_n(R)$. Hence there exists a prime ideal $P_1 \in \mathrm{Spec}(R)$ of R with $P_1 \supset P_0$ such that $P_1 \notin \mathrm{Spec}_{n-1}(R)$, and so there exists a prime ideal $P_2 \in \mathrm{Spec}(R)$ of R with $P_2 \supset P_1$ such that $P_2 \notin \mathrm{Spec}_{n-2}(R)$. Continuing in this way we obtain a chain of prime ideals of R,

$$P_{n+1} \supset P_n \supset \cdots \supset P_2 \supset P_1 \supset P_0$$

with $P_0 \notin \mathrm{Spec}_n(R)$, $P_1 \notin \mathrm{Spec}_{n-1}(R)$, $P_2 \notin \mathrm{Spec}_{n-2}(R)$, $\ldots$, $P_n \notin \mathrm{Spec}_0(R)$. This chain of prime ideals of R has length $n+1$, contradicting $\mathrm{fk}(R) = n$. Therefore we must have $\mathrm{Spec}_n(R) = \mathrm{Spec}(R)$ and so R has finite classical Krull dimension and $\mathrm{ck}(R) \leq n = \mathrm{fk}(R)$.

Conversely let R be a ring with finite classical Krull dimension and let $\mathrm{ck}(R) = m$ for some positive integer m. Suppose that there exists a chain of prime ideals of R,

$$P_{m+1} \supset P_m \supset P_{m-1} \supset \cdots \supset P_2 \supset P_1 \supset P_0$$

of length $m+1$. Then P_m is not a maximal prime, so $P_m \notin \mathrm{Spec}_0(R)$ and thus $\mathrm{Spec}_0(R) \neq \mathrm{Spec}(R)$. Now $P_m \supset P_{m-1}$ with $P_m \notin \mathrm{Spec}_0(R)$ so, by definition, $P_{m-1} \notin \mathrm{Spec}_1(R)$ and thus $\mathrm{Spec}_1(R) \neq \mathrm{Spec}(R)$. Continuing in this way we obtain $P_{m-2} \notin \mathrm{Spec}_2(R)$, $\ldots$, $P_1 \notin \mathrm{Spec}_{m-1}(R)$ and finally $P_0 \notin \mathrm{Spec}_m(R)$. Thus $\mathrm{Spec}_m(R) \neq \mathrm{Spec}(R)$, contradicting $\mathrm{ck}(R) = m$. Therefore there can exist no such chain of prime ideals of R of length $m+1$ and so R has finite f-Krull dimension and $\mathrm{fk}(R) \leq m = \mathrm{ck}(R)$. $\square$

We now consider various properties concerning the classical Krull dimension of a ring. The proofs of the first two lemmas are taken from [18, Lemma 1.3].

Lemma 5.2.3. *Let R be a ring, let I be an ideal of R and let $\alpha \geq 0$ be an ordinal.*

(i) If P is a prime ideal of R containing I, then $P \in Spec_\alpha(R)$ if and only if $P/I \in Spec_\alpha(R/I)$.

(ii) If R has classical Krull dimension, then so does the factor ring R/I and $ck(R/I) \leq ck(R)$.

Proof. Note that there is a one-to-one correspondence between prime ideals of R containing I and prime ideals of R/I, given by $P \leftrightarrow P/I$.

(i) Proceed by induction on α. The result is clear for $\alpha = 0$ in which case both P and P/I are maximal primes. Let $\alpha > 0$ and assume that the result holds for all ordinals $0 \leq \beta < \alpha$. Then, by definition, $P \in \mathrm{Spec}_\alpha(R)$ if and only if for every $Q \in \mathrm{Spec}(R)$ with $P \subset Q$, $Q \in \mathrm{Spec}_\beta(R)$ for some $\beta < \alpha$ and, by the induction hypothesis, this holds if and only if for every $Q/I \in \mathrm{Spec}(R/I)$ with $P/I \subset Q/I$, $Q/I \in \mathrm{Spec}_\beta(R/I)$ for some $\beta < \alpha$ which, again by definition, holds if and only if $P/I \in \mathrm{Spec}_\alpha(R/I)$.

(ii) If $\overline{P}$ is a prime ideal of R/I then $\overline{P} = P/I$ for some prime ideal P of R with $I \subseteq P$. Suppose that R has classical Krull dimension α. Then $P \in \mathrm{Spec}_\alpha(R)$, so $\overline{P} = P/I \in \mathrm{Spec}_\alpha(R/I)$ by (i). Hence $\mathrm{Spec}(R/I) = \mathrm{Spec}_\alpha(R/I)$, so R/I has classical Krull dimension and $\mathrm{ck}(R/I) \leq \alpha = \mathrm{ck}(R)$. □

Note that Lemma 5.2.3 (i) proves that if Q is a prime ideal of a ring R such that R/Q has classical Krull dimension and $\mathrm{ck}(R/Q) \leq \alpha$ for some ordinal $\alpha \geq 0$ then $Q \in \mathrm{Spec}_\alpha(R)$.

Lemma 5.2.4. *If R is a prime ring with classical Krull dimension and $P \neq 0$ is a prime ideal of R, then $ck(R/P) < ck(R)$.*

Proof. Note that R/P has classical Krull dimension by Lemma 5.2.3 (ii). Let $\mathrm{ck}(R) = \alpha$ for some ordinal $\alpha \geq 0$. Then 0 is a prime ideal of R, so $0 \in \mathrm{Spec}_\alpha(R)$ and thus $P \in \mathrm{Spec}_\beta(R)$ for some $\beta < \alpha$. If $\overline{Q} \in \mathrm{Spec}(R/P)$, then $\overline{Q} = Q/P$ for some prime ideal Q of R with $P \subseteq Q$. Then $Q \in \mathrm{Spec}_\gamma(R)$ for some $\gamma \leq \beta$ and hence $Q/P \in \mathrm{Spec}_\gamma(R/P)$, by Lemma 5.2.3 (i). Thus $\mathrm{Spec}(R/P) \subseteq \bigcup_{\gamma \leq \beta} \mathrm{Spec}_\gamma(R/P) = \mathrm{Spec}_\beta(R/P)$, which implies that $\mathrm{ck}(R/P) \leq \beta < \alpha = \mathrm{ck}(R)$. □

Our next lemma is taken from [18, Lemma 1.4].

Lemma 5.2.5. *Let R be a ring with classical Krull dimension $ck(R) \geq \alpha$ for some ordinal $\alpha \geq 0$. If $ck(R/I) < \alpha$ for every ideal $I \neq 0$ of R, then R is a prime ring with $ck(R) = \alpha$.*

Proof. Let P and Q be prime ideals of R with $P \subset Q$. Since $Q \neq 0$, $\text{ck}(R/Q) = \beta$ for some ordinal $\beta < \alpha$ and so we get $Q \in \text{Spec}_\beta(R)$, by the note after Lemma 5.2.3. Thus $P \in \text{Spec}_\alpha(R)$ for all $P \in \text{Spec}(R)$ and hence $\text{ck}(R) \leq \alpha$. Therefore $\text{ck}(R) = \alpha$.

Now assume that R is not prime and let A and B be nonzero ideals of R with $AB = 0$. Let $\beta = \max\{\text{ck}(R/A), \text{ck}(R/B)\}$ and let P be a prime ideal of R. Then $\beta < \alpha$ and we may assume that $A \subseteq P$. Then, by Lemma 5.2.3,

$$\text{ck}(R/P) = \text{ck}((R/A)/(P/A)) \leq \text{ck}(R/A) \leq \beta < \alpha,$$

so $P \in \text{Spec}_\beta(R)$. Thus $\text{Spec}(R) = \text{Spec}_\beta(R)$ with $\beta < \alpha$, which contradicts $\text{ck}(R) = \alpha$. □

Our final result of this section is taken from [18, Proposition 1.5]. Note that the ring R may be either left or right Noetherian and that, by Lemma 5.2.1, $\text{ck}(S)$ is defined for every epimorphic image S of such a ring R.

Proposition 5.2.6. *The following properties are equivalent for a one-sided Noetherian ring R.*

(i) R is a prime ring.

(ii) $ck(R/P) < ck(R)$ for every prime ideal $P \neq 0$ of R.

(iii) $ck(R/I) < ck(R)$ for every ideal $I \neq 0$ of R.

Proof. (i)⇒(ii) By Lemma 5.2.4.

(ii)⇒(iii) Let I be an ideal of R which is maximal with respect to the property that $\text{ck}(R/I) = \text{ck}(R) = \alpha$. If K/I is a nonzero ideal of R/I then $\text{ck}((R/I)/(K/I)) = \text{ck}(R/K) < \alpha = \text{ck}(R/I)$, by the maximality of I. It follows, by Lemma 5.2.5, that R/I is a prime ring, that is I is a prime ideal of R. Thus $I = 0$ by (ii).

(iii)⇒(i) By Lemma 5.2.5. □

5.3 Krull Dimension and Classical Krull Dimension

In this section we consider the relationship between classical Krull dimension and the general module-theoretic Krull dimension. Recall that the right Krull dimension of a ring R is defined to be the Krull dimension of the right R-module R, if it exists, and is denoted by $\text{k}(R)$.

In [26] Rentschler and Gabriel considered the case of finite-valued classical Krull dimension and showed that $\mathrm{ck}(R) \leq \mathrm{k}(R)$ for a right Noetherian ring R and stated (without proof) that equality holds if R is commutative Noetherian. In [17] Krause extended the definition of classical Krull dimension to infinite ordinals, as detailed in the previous section, and showed that, in general, $\mathrm{ck}(R) \leq \mathrm{k}(R)$ for every right Noetherian ring R, with equality if R is a right Noetherian right Matlis-ring (see [17] for definition). In [18] Krause then showed that if R is a right fully bounded ring with right Krull dimension then $\mathrm{ck}(R) \leq \mathrm{k}(R)$ and that equality holds if R is a right fully bounded right Noetherian ring. In [13] Gordon and Robson showed, independent of Krause, that, in fact, $\mathrm{ck}(R) = \mathrm{k}(R)$ for any right fully bounded ring R with right Krull dimension [13, Theorem 8.12]. It is Gordon and Robson's result which we detail in this section. We begin however, with the following example, which shows that a ring with classical Krull dimension need not have (right) Krull dimension.

Example 5.3.1. Let F be a field of nonzero characteristic p where p is a prime number and let G be the Prüfer p-group. Then the group algebra $R = F[G]$ has classical Krull dimension 0 but does not have Krull dimension.

Proof. Let the Abelian group G be generated by $x_1, x_2, \ldots$, where $x_1^p = 1$ and for all $i \geq 1$, $x_{i+1}^p = x_i$. Let $A = \omega G$ denote the augmentation ideal of the commutative ring R. For all $i \geq 1$, $(x_{i+1} - 1)^p = x_{i+1}^p - 1^p = x_i - 1$, since F has characteristic p. Thus $x_i - 1 \in A^p$ for all $i \geq 1$ and hence $A \subseteq A^p$. Therefore $A = A^2 = \cdots = A^p$ and A is an idempotent ideal.

Now let $0 \neq a \in A$. Then $a \in F[\langle x_n \rangle]$ for some $n \geq 1$. In fact $a \in \omega H$, where ωH is the augmentation ideal of the ring $S = F[H]$, where $H = \langle x_n \rangle$ is a finite cyclic group of order p^n. But $\omega H = S(x_n - 1)$ is a nilpotent ideal of S, since $(x_n - 1)^{p^n} = x_n^{p^n} - 1^{p^n} = 1 - 1 = 0$ so $(\omega H)^{p^n} = 0$, and hence a is nilpotent. Therefore A is a nil ideal of R. Now $R/A \cong F$, so A is a maximal ideal of R. Let P be a prime ideal of R. Then for each $a \in A$, $a^k = 0 \in P$ for some $k \geq 1$, so $a \in P$. Hence $A \subseteq P$ and so $A = P$ since A is maximal. Therefore A is the only prime ideal of R and hence $\mathrm{ck}(R) = 0$.

On the other hand, if R had Krull dimension then, by Proposition 1.4.10, the nil subring A of R would be nilpotent and, since A is idempotent, $A = 0$. This is a contradiction, so R does not have Krull dimension. □

Lemmas 1.4.14 and 5.2.1 show that a ring with right Krull dimension necessarily has classical Krull dimension. We are in fact able to show further that the classical Krull dimension is less than or equal to the Krull dimension.

Lemma 5.3.2. *Let R be a ring with right Krull dimension. Then R has classical Krull dimension and $ck(R) \leq k(R)$.*

Proof. The result is clear if $R = 0$, in which case $\mathrm{ck}(R) = \mathrm{k}(R) = -1$. So suppose that $\mathrm{k}(R) = \alpha$ for some ordinal $\alpha \geq 0$. If $\alpha = 0$ then R is right Artinian and hence, by Lemma 2.2.10, every prime ideal of R is maximal so that $\mathrm{ck}(R) = 0$.

Now suppose that $\alpha > 0$. Let P be any prime ideal of R. Then R/P is a prime ring with right Krull dimension so R/P is a prime right Goldie ring, by Proposition 1.4.9. Let Q be any prime ideal of R properly containing P. Then Q/P is an essential right ideal of R/P and hence, by Lemma 1.4.13,

$$\mathrm{k}(R/Q) = \mathrm{k}((R/P)/(Q/P)) < \mathrm{k}(R/P) \leq \mathrm{k}(R) = \alpha.$$

By induction on α, $\mathrm{ck}(R/Q) \leq \mathrm{k}(R/Q)$ and hence $Q \in \mathrm{Spec}_\beta(R)$ for some ordinal $0 \leq \beta < \alpha$, by the note after Lemma 5.2.3. Thus $P \in \mathrm{Spec}_\alpha(R)$. It follows that $\mathrm{Spec}_\alpha(R) = \mathrm{Spec}(R)$ and hence R has classical Krull dimension and $\mathrm{ck}(R) \leq \alpha = \mathrm{k}(R)$. □

Our main result of this section is taken from [13, Theorem 8.12] and shows that for right FBK rings the Krull dimension and the classical Krull dimension are in fact equal.

Proposition 5.3.3. *Let R be a right FBK ring. Then R has classical Krull dimension and $ck(R) = k(R)$.*

Proof. By Proposition 1.4.12, we may choose a prime ideal P of R with $\mathrm{k}(R) = \mathrm{k}(R/P)$. Suppose that $\mathrm{ck}(R/P) = \mathrm{k}(R/P)$, then, by Lemma 5.3.2,

$$\mathrm{k}(R/P) = \mathrm{ck}(R/P) \leq \mathrm{ck}(R) \leq \mathrm{k}(R) = \mathrm{k}(R/P)$$

and so $\mathrm{ck}(R) = \mathrm{k}(R)$. Therefore, without loss of generality, we may assume that R is a prime ring.

Let $\mathrm{k}(R) = \alpha$ for some ordinal $\alpha \geq -1$. The result is clear if α equals -1 or 0, so suppose that $\alpha \geq 1$. We claim that for any ordinal $\beta < \alpha$ there is a prime ideal Q of R such that $\beta \leq \mathrm{k}(R/Q) < \alpha$. By Lemma 1.4.13, there is certainly an essential right ideal E of R such that $\beta \leq \mathrm{k}(R/E) < \alpha$. But R is prime right bounded so E contains a nonzero ideal I of R and, since I is essential as a right ideal of R, $\beta \leq \mathrm{k}(R/E) \leq \mathrm{k}(R/I) < \alpha$. Choosing a prime ideal Q of R satisfying $\mathrm{k}(R/I) = \mathrm{k}(R/Q)$ (again by Proposition 1.4.12) then establishes the claim. Now, proceeding by induction on α, assume that the result is true for all ordinals strictly less than α. If $\mathrm{ck}(R) \neq \alpha$ then $\mathrm{ck}(R) < \alpha$, so there is a prime ideal Q of R such that $\mathrm{ck}(R) \leq \mathrm{k}(R/Q) < \alpha$. By induction hypothesis $\mathrm{k}(R/Q) = \mathrm{ck}(R/Q)$ and thus

$$\mathrm{ck}(R) \leq \mathrm{k}(R/Q) = \mathrm{ck}(R/Q) \leq \mathrm{ck}(R)$$

so $ck(R) = \mathrm{ck}(R/Q)$. This is impossible, by Lemma 5.2.4, since R is prime and Q is a nonzero prime ideal of R. Therefore $\mathrm{ck}(R) = \alpha = \mathrm{k}(R)$. □

In particular, a commutative ring R with Krull dimension has classical Krull dimension and satisfies $\mathrm{ck}(R) = \mathrm{k}(R)$.

We are now able to prove the following analogue for rings with Krull dimension of Jategaonkar's result that fully bounded Noetherian rings are Krull symmetric, as mentioned at the end of Section 4.6 of Chapter 4.

Theorem 5.3.4. *Let R be a FBK ring. Then R is Krull symmetric.*

Proof. Since classical Krull dimension is defined in terms of two-sided prime ideals, it is a symmetric concept. By hypothesis, R is both left and right FBK and hence, by Proposition 5.3.3, $\mathrm{k}({}_R R) = \mathrm{ck}(R) = \mathrm{k}(R_R)$. □

5.4 Krull Dimension and Dual Classical Krull Dimension

In this section we consider the relationship between Krull dimension and dual classical Krull dimension.

Lemma 5.4.1. *Let R be a right Artinian ring and let M be a right R-module with Krull dimension. Then M has finite length.*

Proof. Let J denote the radical of R (see Corollary 1.2.2). Then, by Proposition 1.2.1, J is nilpotent, so $J^k = 0$ for some integer $k \geq 1$. Consider the chain of submodules of M,

$$M = MJ^0 \supseteq MJ \supseteq MJ^2 \supseteq \cdots \supseteq MJ^{k-1} \supseteq MJ^k = 0.$$

Let $1 \leq i \leq k$ and consider the factor module MJ^{i-1}/MJ^i. This quotient can be considered as a right (R/J)-module and as such is semisimple, since R/J is a semiprime Artinian ring. Since MJ^{i-1}/MJ^i has finite Goldie dimension (it inherits Krull dimension from M) it must be a finite sum of simple modules and hence has finite length, both as a right

(R/J)- and as a right R-module. Applying this to each factor in the chain it follows that M has finite length as a right R-module. □

Note that this lemma in fact holds regardless of whether the ring is right or left Artinian and also independently regardless of whether the module structure is on the right or left.

Lemma 5.4.2. *Let R be a ring such that $k(R) = ck(R)$. Then R is right Artinian if and only if every prime ideal of R is maximal.*

Proof. This follows since R is right Artinian if and only if $\mathrm{k}(R) = 0$, in which case $\mathrm{ck}(R) = 0$, and this holds if and only if every prime ideal of R is maximal. □

Note that, by Proposition 5.3.3, $\mathrm{k}(R) = \mathrm{ck}(R)$ for a right fully bounded ring R with right Krull dimension. In particular, a corollary of Lemma 5.4.2 is therefore the well-known result that a commutative Noetherian ring is Artinian if and only if every prime ideal is maximal (see Corollary 2.2.11). If A is a finitely generated ideal of a ring R then we will denote the minimum number of generators of A by $\mathrm{g}(A)$, taking $\mathrm{g}(0) = 0$.

Theorem 5.4.3. *Let R be a right fully bounded ring with right Krull dimension and let M be a right R-module with Krull dimension. Then M has dual classical Krull dimension and $ck^\circ(M) \leq k(R)$.*

Proof. We proceed by induction on the right Krull dimension of the ring R, $\mathrm{k}(R)$. If $\mathrm{k}(R) = 0$ then R is right Artinian and, since M has Krull dimension, M has finite length, by Lemma 5.4.1. Thus $\mathrm{ck}^\circ(M) = 0 = \mathrm{k}(R)$ (in fact, in this case, $\mathrm{ck}^\circ(M) = \mathrm{k}^\circ(M) = \mathrm{ck}(R) = \mathrm{k}(R) = 0$).

Now suppose that $\mathrm{k}(R) \geq 1$. Let $P_1, \ldots, P_m$ be the distinct minimal prime ideals of R (see Proposition 1.4.11). Since, by Proposition 5.3.3, $\mathrm{ck}(R) = \mathrm{k}(R) \geq 1$ there exists a prime ideal P of R which is not minimal. Then, for all $1 \leq i \leq m$, $P \cap (\cap_{j \neq i} P_j) \nsubseteq P_i$ so there exists an element $c_i \in P \cap (\cap_{j \neq i} P_j)$ such that $c_i \notin P_i$. Putting $a_1 = c_1 + \cdots + c_m$ gives an element $a_1 \in P$ such that $a_1 \notin P_i$ for all $1 \leq i \leq m$. Let $A_1 =< a_1 >$ denote the ideal of R generated by a_1. Consider the quotient ring R/A_1. By Proposition 1.4.12, there exists a prime ideal Q of R with $A_1 \subseteq Q$ such that $\mathrm{k}(R/A_1) = \mathrm{k}(R/Q)$ and, by the choice of a_1, there exists a minimal prime ideal P_j of R (for some $1 \leq j \leq m$) such that $P_j \subsetneqq Q$. Then $\mathrm{k}(R/A_1) = \mathrm{k}(R/Q) < \mathrm{k}(R/P_j) \leq \mathrm{k}(R)$, so $\mathrm{k}(R/A_1) < \mathrm{k}(R)$. Now R/A_1 is a right FBK ring and $N = \mathrm{ann}_M(A_1)$ is a right (R/A_1)-module with Krull dimension, so, by induction hypothesis, N has dual classical Krull dimension and $\mathrm{ck}^\circ(N) \leq \mathrm{k}(R/A_1) < \mathrm{k}(R)$.

Therefore there exists a finitely generated proper ideal A/A_1 of R/A_1 (for some ideal A of R with $A_1 \subseteq A$) such that $\mathrm{ann}_N(A/A_1)$ has finite length and $\mathrm{g}(A/A_1) = \mathrm{ck}^\circ(N) < \mathrm{k}(R)$. But then A is a finitely generated proper ideal of R with $\mathrm{ann}_M(A) = \mathrm{ann}_N(A/A_1)$ and $\mathrm{g}(A) \leq \mathrm{g}(A/A_1) + 1 \leq \mathrm{k}(R)$. It follows that M has dual classical Krull dimension and $\mathrm{ck}^\circ(M) \leq \mathrm{k}(R)$. □

Note that Theorem 5.4.3 is not true if R is a simple ring with right Krull dimension which is not right Artinian (for example $R = A_1 = \mathbb{C}[x, y]$ where $xy - yx = 1$), since in this case R_R does not have dual classical Krull dimension.

5.5 Dual Classical Krull Dimension

Over the course of the next few sections we investigate the relationship between dual Krull dimension and dual classical Krull dimension, proving that for Artinian modules over certain rings these dimensions are equal. We begin our study by looking at some of the basic properties of the dual classical Krull dimension of a module. For the most part we will be considering commutative rings.

We begin by noting that it is clear that if R is a ring, M is a right R-module with dual classical Krull dimension and N is a submodule of M, then N has dual classical Krull dimension and $\mathrm{ck}^\circ(N) \leq \mathrm{ck}^\circ(M)$. Note that for ease we use the convention that a module M without dual classical Krull dimension satisfies $\mathrm{ck}^\circ(M) = \infty$ with all such infinities being equal to each other and greater than any ordinal value.

Lemma 5.5.1. *Let R be a ring, let M be a right R-module and let N be a proper submodule of M such that N has finite length. Then $ck^\circ(M) \leq ck^\circ(M/N)$.*

Proof. There is nothing to prove if M/N does not have dual classical Krull dimension, so suppose that $\mathrm{ck}^\circ(M/N) = n$ for some integer $n \geq 0$. Then there exists a finitely generated proper ideal A of R with $\mathrm{g}(A) = n$ such that $\mathrm{ann}_{M/N}(A)$ has finite length. Now

$$\mathrm{ann}_M(A)/(N \cap \mathrm{ann}_M(A)) \cong (\mathrm{ann}_M(A) + N)/N \subseteq \mathrm{ann}_{M/N}(A),$$

so $\mathrm{ann}_M(A)/(N \cap \mathrm{ann}_M(A))$ has finite length. Since N has finite length, it follows that $\mathrm{ann}_M(A)$ has finite length. Thus $\mathrm{ck}^\circ(M) \leq n = \mathrm{ck}^\circ(M/N)$. □

Lemma 5.5.2. *Let R be a commutative ring, let M be an R-module and let N be a submodule of M such that N has finite length. Then $ck^\circ(M/N) \leq ck^\circ(M)$.*

Proof. There is nothing to prove if M does not have dual classical Krull dimension or $M = 0$, so suppose that $\text{ck}^\circ(M) = n$ for some integer $n \geq 0$. Then there exists a finitely generated proper ideal A of R with $\text{g}(A) = n$ such that $\text{ann}_M(A)$ has finite length. Now $\text{ann}_{M/N}(A) = \{m + N | mA \subseteq N\} = T/N$ where T is the submodule $T = \{m \in M | mA \subseteq N\}$ of M and it suffices to prove that T has finite length. Write $A = a_1R + \cdots + a_nR$ for some $a_1, \ldots, a_n \in A$. Define a map $\varphi : T \to N^n$ by $\varphi(t) = (ta_1, \ldots, ta_n)$ for all $t \in T$. Now, as R-modules $T/\ker\varphi$ embeds in N^n and hence $T/\ker\varphi$ has finite length. Since $\ker\varphi = \text{ann}_M(A)$ has finite length, it follows that T has finite length, as required. □

Corollary 5.5.3. *Let R be a commutative ring, let M be an R-module and let N be a proper submodule of M such that N has finite length. Then $ck^\circ(M) = ck^\circ(M/N)$.*

Proof. This follows from Lemmas 5.5.1 and 5.5.2. □

Note that the above result means that either both M and M/N have dual classical Krull dimension and their values are equal, or neither M nor M/N has dual classical Krull dimension.

Lemma 5.5.4. *Let R be a commutative ring, let M be an R-module and let A be a finitely generated ideal of R such that $ann_M(A)$ has finite length. Then $ann_M(A^s)$ has finite length for any integer $s \geq 1$.*

Proof. Let $Y_s = \text{ann}_M(A^s)$ for each integer $s \geq 1$. Then $Y_1 \subseteq Y_2 \subseteq \cdots$ and $Y_sA \subseteq Y_{s-1}$ for all $s \geq 2$. We show that Y_s has finite length by induction on s. By hypothesis the result is true for $s = 1$, so suppose that $s \geq 2$. Then $Y_s/Y_{s-1} = \text{ann}_{M/Y_{s-1}}(A)$. As in the proof of Lemma 5.5.2, $\text{ann}_{M/Y_{s-1}}(A)$ has finite length since both $\text{ann}_M(A)$ and, by induction hypothesis, Y_{s-1} do. Thus Y_s/Y_{s-1} has finite length. Again by induction hypothesis, Y_{s-1} has finite length and it follows that Y_s has finite length. □

Lemma 5.5.5. *Let R be a ring, let M be an R-module and let $n \geq 1$ be an integer. Then $ck^\circ(M^n) = ck^\circ(M)$.*

Proof. It is clear that $\text{ck}^\circ(M) \leq \text{ck}^\circ(M^n)$, since M is isomorphic to a submodule of M^n. For the converse suppose that $\text{ck}^\circ(M) = m$ for some integer m (since the result is clear if M does not have dual classical Krull dimension). Then there exists a finitely generated proper ideal A of R with $\text{g}(A) = m$ such that $\text{ann}_M(A)$ has finite length. Now $\text{ann}_{M^n}(A) = \oplus_{i=1}^n \text{ann}_M(A)$, so $\text{ann}_{M^n}(A)$ has finite length. Thus M^n has dual classical Krull dimension and $\text{ck}^\circ(M^n) \leq \text{g}(A) = m = \text{ck}^\circ(M)$, as required. □

Lemma 5.5.6. *Let R be a commutative ring, let A be an ideal of R and let M be an Artinian R-module. Then there exists a finitely generated ideal B of R with $B \subseteq A$ such that $ann_M(B^n) = ann_M(A^n)$ for all integers $n \geq 0$.*

Proof. Consider the set F of submodules of M of the form $\text{ann}_M(A')$, where A' is a finitely generated ideal of R contained in A. Since F is non-empty (it certainly contains $M = \text{ann}_M(0)$) and M is Artinian, F contains a minimal element, $\text{ann}_M(B)$ say, where $B \subseteq A$ is a finitely generated ideal of R. Clearly $\text{ann}_M(A) \subseteq \text{ann}_M(B)$. Let $a \in A$. Then $\text{ann}_M(B+aR) \subseteq \text{ann}_M(B)$ and $B+aR$ is a finitely generated ideal of R with $B+aR \subseteq A$, so $\text{ann}_M(B+aR) = \text{ann}_M(B)$ by the minimal choice of $\text{ann}_M(B)$. Thus $\text{ann}_M(B).a = 0$ for all $a \in A$ and hence $\text{ann}_M(B) = \text{ann}_M(A)$. It follows by a simple induction argument (using the commutativity of the ring R) that $\text{ann}_M(B^n) = \text{ann}_M(A^n)$ for all integers $n \geq 0$. □

A ring is called *quasi-local* if it has a unique maximal ideal (note that a ring is called *local* if it is quasi-local and Noetherian). Quasi-local commutative rings will play an important role in our study of the relationship between dual Krull dimension and dual classical Krull dimension.

Proposition 5.5.7. *Let R be a quasi-local commutative ring with unique maximal ideal J and let M be an Artinian R-module. Then M has (finite) dual classical Krull dimension.*

Proof. By Lemma 5.5.6, there is a finitely generated ideal B of R with $B \subseteq J$ such that $\text{ann}_M(B) = \text{ann}_M(J)$. Note in particular that B is a proper ideal of R. Now $\text{ann}_M(J) = \text{Soc}(M)$ and, since M is Artinian, $\text{Soc}(M)$ has finite length. Thus $\text{ann}_M(B)$ has finite length and the result follows by the definition of dual classical Krull dimension. □

Let R be a commutative ring and let M be an R-module. If S is a non-empty subset of R then $\text{rad}(S)$ is defined to be the set $\text{rad}(S) = \{r \in R | r^k \in S \text{ for some integer } k \geq 1\}$. The module M is called *coprimary* if $M = Mr$ for all $r \in R$ such that $r \notin \text{rad}(\text{ann}_R(M))$. If M is a coprimary R-module then it can be shown that $P = \text{rad}(\text{ann}_R(M))$ is a prime ideal of R and we will say that M is a *P-coprimary* module and that P is the prime ideal *associated with M*. By [16, Theorem 1], for any commutative ring R every Artinian R-module is expressible as the sum of a finite number of coprimary R-modules. In fact, every Artinian R-module has a *normal coprimary decomposition*, meaning that the prime ideals associated with the decomposition are distinct and the decomposition is irredundant in the sense that no member of the decomposition may be removed and still have the sum equal M. For further details on coprimary modules and coprimary decomposition see [16].

Lemma 5.5.8. *Let R be a commutative ring, let A be a finitely generated ideal of R and let M be a nonzero R-module with normal coprimary decomposition $M = N_1 + \cdots + N_k$ for some integer $k \geq 1$ and P_i-coprimary modules N_i $(1 \leq i \leq k)$. Then the following statements are equivalent.*

(i) $MA = M$.

(ii) $A \nsubseteq P_i$ *for all* $1 \leq i \leq k$.

(iii) $M = Ma$ *for some element* $a \in A$.

Proof. See [16, Proposition 6]. □

The following lemma will be used later when we consider the relationship between dual Krull dimension and dual classical Krull dimension.

Lemma 5.5.9. *Let R be a quasi-local commutative ring with unique maximal ideal J and let M be an Artinian R-module. If $ck^\circ(M) > 0$ then there exists a submodule M' of M with $ck^\circ(M') = ck^\circ(M)$ and an element $x \in J$ satisfying $M'x = M'$.*

Proof. By Proposition 5.5.7, M has finite dual classical Krull dimension, so let $\mathrm{ck}^\circ(M) = n$ for some integer $n \geq 1$ and suppose that $A = \sum_{i=1}^{n} a_i R$ is a proper ideal of R such that $\mathrm{ann}_M(A)$ has finite length. Then, by Lemma 5.5.4, $\mathrm{ann}_M(A^s)$ has finite length for any integer $s \geq 1$. Now M is Artinian so there exists an integer $t \geq 1$ with $MA^t = MA^{t+1}$. Put $B = A^t$ and $M' = MB$. Now B is finitely generated, by $b_1, \ldots, b_m \in B$ say for some integer $m \geq 1$. Define a map $\phi : M \to \oplus_{i=1}^{m} M'$ by $\phi(x) = (xb_1, \ldots, xb_m)$ for all $x \in M$. Then $\ker \phi = \mathrm{ann}_M(B) = \mathrm{ann}_M(A^t)$ has finite length and $\ker \phi \neq M$ since $\mathrm{ck}^\circ(M) > 0$. Hence $\mathrm{ck}^\circ(M) = \mathrm{ck}^\circ(M/\ker \phi) = \mathrm{ck}^\circ(\phi(M))$, by Corollary 5.5.3. Also $\mathrm{ck}^\circ(\oplus_{i=1}^{m} M') = \mathrm{ck}^\circ(M')$, by Lemma 5.5.5. Thus

$$\mathrm{ck}^\circ(M') \leq \mathrm{ck}^\circ(M) = \mathrm{ck}^\circ(\phi(M)) \leq \mathrm{ck}^\circ(\oplus_{i=1}^{m} M') = \mathrm{ck}^\circ(M').$$

Therefore the submodule $M' = MB = MA^t$ satisfies $\mathrm{ck}^\circ(M') = \mathrm{ck}^\circ(M)$. Further, $M' = M'A$ and hence, by the above discussion of coprimary modules and Lemma 5.5.8, there exists an element $x \in J$ such that $M'x = M'$. □

5.6 Polynomial Functions

In order to prove that for Artinian modules over certain families of rings dual Krull dimension and dual classical Krull dimension are equal, we will require the concept of polynomial functions, which we introduce and study some basic properties of in this section.

Definition. Let G be an Abelian group. A map $f : \mathbb{Z} \to G$ is called a *polynomial function* if there exist integers $d, n_0 \geq 0$ and elements $g_0, \ldots, g_d \in G$ such that for all $n \geq n_0$,

$$f(n) = \sum_{i=0}^{d} \binom{n+i}{i} g_i.$$

If $g_d \neq 0$ then we say that f is of *degree* d. If $g_i = 0$ for all i (that is $f(n) = 0$ for all sufficiently large n) then we say that f is of degree -1. The degree of a polynomial function f will be denoted by $\mathrm{d}(f)$

Our first lemma shows that polynomial functions and their degrees are well defined.

Lemma 5.6.1. *Let G be an Abelian group and let $f : \mathbb{Z} \to G$ be a polynomial function. Then the representation and degree of f as a polynomial function are unique.*

Proof. Suppose that

$$f(n) = \sum_{i=0}^{d} \binom{n+i}{i} g_i = \sum_{i=0}^{d'} \binom{n+i}{i} g'_i \tag{5.1}$$

for all $n \geq n_0$, for some integers $d, d', n_0 \geq 0$ and elements $g_0, \ldots, g_d, g'_0, \ldots, g'_{d'} \in G$. This equation gives a polynomial in n with coefficients in $\mathbb{Q}G$ which is zero for all sufficiently large n. Since such a polynomial can only have a finite number of distinct roots it must be identically zero. Therefore the coefficients in equation (5.1) must be equal, so $d = d'$ and $g_i = g'_i$ for all $i \geq 0$. It follows that the representation and degree of f as a polynomial function are well defined. □

Let G be an Abelian group. With any mapping $f : \mathbb{Z} \to G$ there is associated a mapping $\triangle f : \mathbb{Z} \to G$ given by

$$\triangle f(n) = f(n) - f(n-1).$$

Lemma 5.6.2. *Let G be an Abelian group, let $f : \mathbb{Z} \to G$ be a map and let $d \geq 0$ be an integer. Then f is a polynomial function of degree d if and only if $\triangle f$ is a polynomial function of degree $d - 1$.*

Proof. The result is clear if $d = 0$, since f is a polynomial function of degree 0 if and only if f is eventually constant, which happens if and only if $\triangle f$ is eventually zero, that is if and only if $\triangle f$ is a polynomial function of degree -1. So suppose that f is a polynomial function of degree $d \geq 1$. Then there exist an integer $n_0 \geq 0$ and elements $g_0, \ldots, g_d \in G$ such that $f(n) = \sum_{i=0}^{d} \begin{pmatrix} n+i \\ i \end{pmatrix} g_i$ for all $n \geq n_0$. Then for all $n \geq n_0$,

$$\begin{aligned}
\triangle f(n) &= f(n) - f(n-1) \\
&= \sum_{i=0}^{d} \begin{pmatrix} n+i \\ i \end{pmatrix} g_i - \sum_{i=0}^{d} \begin{pmatrix} n-1+i \\ i \end{pmatrix} g_i \\
&= \sum_{i=0}^{d} \left(\begin{pmatrix} n+i \\ i \end{pmatrix} - \begin{pmatrix} n-1+i \\ i \end{pmatrix} \right) g_i \\
&= \sum_{i=1}^{d} \begin{pmatrix} n+i-1 \\ i-1 \end{pmatrix} g_i \\
&= \sum_{j=0}^{d-1} \begin{pmatrix} n+j \\ j \end{pmatrix} g'_j
\end{aligned}$$

where $j = i - 1$ and $g'_j = g_{j+1}$ for $0 \leq j \leq d-1$. Thus $\triangle f$ is a polynomial function of degree $d - 1$.

Conversely, suppose that $\triangle f$ is a polynomial function of degree $d - 1$, with $d \geq 1$. There exist an integer $n_1 \geq 0$ and elements $h_0, \ldots, h_{d-1} \in G$ such that $\triangle f(n) = \sum_{i=0}^{d-1} \begin{pmatrix} n+i \\ i \end{pmatrix} h_i$ for all $n \geq n_1$. Define $g(n) = f(n) - \sum_{i=0}^{d-1} \begin{pmatrix} n+i+1 \\ i+1 \end{pmatrix} h_i$. Then for all $n \geq n_1$,

$$\begin{aligned}
\triangle g(n) &= g(n) - g(n-1) \\
&= \triangle f(n) - \sum_{i=0}^{d-1} \left(\begin{pmatrix} n+i+1 \\ i+1 \end{pmatrix} - \begin{pmatrix} n+i \\ i+1 \end{pmatrix} \right) h_i \\
&= \triangle f(n) - \sum_{i=0}^{d-1} \begin{pmatrix} n+i \\ i \end{pmatrix} h_i \\
&= 0.
\end{aligned}$$

Therefore g must be eventually constant, that is $g(n) = g_0$ for all $n \geq n_1$ for some $g_0 \in G$.

Hence, for all $n \geq n_1$,

$$\begin{aligned} f(n) &= g(n) + \sum_{i=0}^{d-1} \binom{n+i+1}{i+1} h_i \\ &= g_0 + \sum_{j=1}^{d} \binom{n+j}{j} h_{j-1} \\ &= \sum_{j=0}^{d} \binom{n+j}{j} g_j \end{aligned}$$

where $j = i+1$ and $g_j = h_{j-1}$ for $1 \leq j \leq d$. Thus f is a polynomial function of degree d. $\square$

Corollary 5.6.3. *Let G be an Abelian group, let $d \geq -1$ be an integer and let $f_1 : \mathbb{Z} \to G$ and $f_2 : \mathbb{Z} \to G$ be functions such that $f_1(n) = f_2(n+1)$ for all $n \geq n_0$ for some integer $n_0 \geq 0$. Then f_1 is a polynomial function of degree d if and only if f_2 is a polynomial function of degree d.*

Proof. Proceed by induction on d. The result is clear for $d = -1$, since both f_1 and f_2 are zero for large n if either is a polynomial function of degree -1. So suppose that $d \geq 0$ and that the result is true for degrees strictly less than d. Now $\triangle f_1(n) = f_1(n) - f_1(n-1) = f_2(n+1) - f_2(n) = \triangle f_2(n+1)$ for all $n \geq n_0$. By Lemma 5.6.2, f_1 is a polynomial function of degree d if and only if $\triangle f_1$ is a polynomial of degree $d-1$ and, by the induction hypothesis, this holds if and only if $\triangle f_2$ is a polynomial function of degree $d-1$, which, again by Lemma 5.6.2, holds if and only if f_2 is a polynomial function of degree d. $\square$

5.7 Graded Modules, Chain Conditions and Polynomial Functions

In this section we detail some results concerning chain conditions on graded modules and use these to deduce that certain functions are polynomial functions. Though relatively technical, these results will be used in later sections when considering the equality of dual Krull dimension and dual classical Krull dimension and a rigorous exposition and proof of them is worthwhile. Before proving our first main result of this section (Proposition 5.7.3) we require a couple of preliminary lemmas.

Note that throughout this section all rings will be commutative. For a ring R, we will often use $\mathcal{M}$ to denote a Serre subcategory of the category of R-modules; that is $\mathcal{M}$ has the property that for an exact sequence of R-modules

$$0 \to M' \to M \to M'' \to 0$$

$M \in \mathcal{M}$ if and only if $M', M'' \in \mathcal{M}$. For example this holds if $\mathcal{M}$ is the entire category of R-modules or if $\mathcal{M}$ is the set of Noetherian R-modules, Artinian R-modules or R-modules with finite length.

Proposition 5.7.1. *Let R be a commutative ring, let M and N be R-modules and let $\mathcal{M}$ be a Serre subcategory of the category of R-modules. Let $s \geq 1$ be an integer and let $f_i : M \to N$ $(1 \leq i \leq s)$ be R-homomorphisms.*

(i) If $M \in \mathcal{M}$ and $N = \sum_{i=1}^{s} \mathrm{im} f_i$ then $N \in \mathcal{M}$.

(ii) If $N \in \mathcal{M}$ and $0 = \cap_{i=1}^{s} \ker f_i$ then $M \in \mathcal{M}$.

Proof. (i) Note first that $M \in \mathcal{M}$ implies that $M^k \in \mathcal{M}$ for all integers $k \geq 1$, by induction on k. For, suppose that $M^k \in \mathcal{M}$ for some integer $k \geq 1$. Then the exact sequence

$$0 \to M^k \to M^{k+1} \to M \to 0$$

with the standard mappings gives that $M^{k+1} \in \mathcal{M}$. Now let the map $f : M^s \to N$ be defined by $f(m_1, \ldots, m_s) = f_1(m_1) + \cdots + f_s(m_s)$ for all $m_i \in M$ $(1 \leq i \leq s)$. Then

$$0 \to \ker f \hookrightarrow M^s \xrightarrow{f} N \to 0$$

is an exact sequence. By the above argument $M^s \in \mathcal{M}$ and hence $N \in \mathcal{M}$.

(ii) Define a map $g : M \to N^s$ by $g(m) = (f_1(m), \ldots, f_s(m))$ for all $m \in M$. Then g is a monomorphism and

$$0 \to M \xrightarrow{g} N^s \to N^s/\mathrm{im} g \to 0$$

is an exact sequence. By the above argument $N^s \in \mathcal{M}$ and hence $M \in \mathcal{M}$. □

Now let R be a commutative ring and let s and l be integers with $s \geq 1$. Let $R[x_1, \ldots, x_s]$ denote the polynomial ring over R in commuting indeterminates $x_1, \ldots, x_s$. Consider a graded $R[x_1, \ldots, x_s]$-module $M = \oplus_{n=-\infty}^{l} M_n$ (note that for all $-\infty < n < l$, if $x \in \{x_1, \ldots, x_s\}$ and $m \in M_n$ then $mx \in M_{n+1}$, with $mx = 0$ if $n = l$). For integers $t \geq 0$ put

$$N_t = \mathrm{ann}_M(x_s^{t+1} R)/\mathrm{ann}_M(x_s^t R).$$

Then N_t is also a graded $R[x_1, \ldots, x_s]$-module. But N_t is annihilated by x_s, so may be regarded as an $R[x_1, \ldots, x_{s-1}]$-module. For $t \geq 1$ let $\alpha_t : N_t \to N_{t-1}$ denote the mapping given by

$$\alpha_t(a + \mathrm{ann}_M(x_s^t R)) = ax_s + \mathrm{ann}_M(x_s^{t-1} R)$$

for all $a \in \mathrm{ann}_M(x_s^{t+1} R)$. Then α_t is a graded $R[x_1, \ldots, x_{s-1}]$-monomorphism. For $t \geq 0$ let $\beta_t : N_t \to N_0$ be the composite mapping $\alpha_1 \cdots \alpha_t$. Then β_t is also a graded $R[x_1, \ldots, x_{s-1}]$-monomorphism. Finally, for each $t \geq 0$ and each submodule A of M let A_t be the submodule of N_t given by,

$$\begin{aligned} A_t &= [(A \cap \mathrm{ann}_M(x_s^{t+1} R)) + \mathrm{ann}_M(x_s^t R)]/\mathrm{ann}_M(x_s^t R) \\ &= [(A + \mathrm{ann}_M(x_s^t R)) \cap \mathrm{ann}_M(x_s^{t+1} R)]/\mathrm{ann}_M(x_s^t R). \end{aligned}$$

Lemma 5.7.2. *Let R be a commutative ring, let s and l be integers with $s \geq 1$ and let $M = \oplus_{n=-\infty}^{l} M_n$ be a graded $R[x_1, \ldots, x_s]$-module. Let A and B be submodules of M and let A_t, B_t, α_t and β_t be defined as above.*

(i) If $A \subseteq B$ then $A_t \subseteq B_t$ for all $t \geq 0$.

(ii) If $A \subseteq B$ and $A_t = B_t$ for all $t \geq 0$ then $A = B$.

(iii) $A_0, \beta_1(A_1), \beta_2(A_2), \ldots$ is a decreasing sequence of $R[x_1, \ldots, x_{s-1}]$-submodules of N_0.

Proof. (i) Clear.

(ii) Suppose that $A \subsetneqq B$. Consider an element $b \in B \setminus A$, where $b = \sum_{n=k}^{l} b_n$ for some $k \leq l$, where each b_n is a homogeneous element of degree n (that is, an element of M_n). Then $b \neq 0$ but $bx_s^{l-k+1} = 0$. Let $t \geq 0$ be such that $b \in \mathrm{ann}_M(x_s^{t+1} R) \setminus \mathrm{ann}_M(x_s^t R)$. Choose $b \in B \setminus A$ such that the corresponding t is minimal. Then the image of b in B_t is not in A_t, for otherwise there exists $b' \in A \cap \mathrm{ann}_M(x_s^{t+1} R)$ such that $b - b' \in \mathrm{ann}_M(x_s^t R)$, which, by the minimality of t, would imply that $b - b' \in A$, contradicting $b \in B \setminus A$. Therefore $A = B$, as required.

(iii) We have $Ax_s \subseteq A$, so that, by (i), $(Ax_s)_t \subseteq A_t$ for $t \geq 0$. Now $\alpha_t(A_t) = (Ax_s)_{t-1}$, so $\alpha_t(A_t) \subseteq A_{t-1}$ for $t \geq 1$. Applying β_{t-1} gives that $\beta_t(A_t) \subseteq \beta_{t-1}(A_{t-1})$ for $t \geq 1$. $\square$

We are now able to prove our first main result of this section.

Proposition 5.7.3. *Let R be a commutative ring, let $s \geq 0$ be an integer and let $M = \oplus_{n=-\infty}^{\infty} M_n$ be a graded $R[x_1, \ldots, x_s]$-module.*

(i) M is a Noetherian $R[x_1, \ldots, x_s]$-module if and only if there exist integers k and l such that

(a) $M_n = 0$ for $n < k$.

(b) $M_{n+1} = \sum_{i=1}^{s} M_n x_i$ for $n \geq l$.

(c) M_n is a Noetherian R-module for $k \leq n \leq l$.

(ii) M is an Artinian $R[x_1, \ldots, x_s]$-module if and only if there exist integers k and l such that

(a) $M_n = 0$ for $n > l$.

(b) $ann_{M_n}(\sum_{i=1}^{s} x_i R) = 0$ for $n < k$.

(c) M_n is an Artinian R-module for $k \leq n \leq l$.

Proof. We prove the result in case (ii); in case (i) the proof of the necessity of conditions (a),(b) and (c) is similar, while sufficiency follows from Hilbert's basis theorem, since (a) and (b) show that M is a finite sum of submodules $M_n R[x_1, \ldots, x_s]$ $(k \leq n \leq l)$ and each of these summands is a Noetherian $R[x_1, \ldots, x_s]$-module by Hilbert's basis theorem and (c).

(ii)($\Rightarrow$) Suppose that M is an Artinian $R[x_1, \ldots, x_s]$-module. Consider the following chains of $R[x_1, \ldots, x_s]$-submodules of M,

$$\cdots \supseteq \sum_{n=-1}^{\infty} M_n \supseteq \sum_{n=0}^{\infty} M_n \supseteq \sum_{n=1}^{\infty} M_n \supseteq \cdots \tag{5.2}$$

$$\cdots \supseteq \sum_{-\infty}^{n=1} \operatorname{ann}_{M_n}(\sum_{i=1}^{s} x_i R) \supseteq \sum_{-\infty}^{n=0} \operatorname{ann}_{M_n}(\sum_{i=1}^{s} x_i R) \supseteq \sum_{-\infty}^{n=-1} \operatorname{ann}_{M_n}(\sum_{i=1}^{s} x_i R) \supseteq \cdots \tag{5.3}$$

and for each n,

$$L_0 R[x_1, \ldots, x_s] \supseteq L_1 R[x_1, \ldots, x_s] \supseteq L_2 R[x_1, \ldots, x_s] \supseteq \cdots \tag{5.4}$$

where $L_0 \supseteq L_1 \supseteq L_2 \supseteq \cdots$ is a descending chain of R-submodules of M_n. The termination of these chains imply (a), (b) and (c) respectively.

($\Leftarrow$) The converse is proved by induction on s. Suppose that $s = 0$. In this case the empty sum in (b) is zero, so (b) gives that $M_n = \operatorname{ann}_{M_n}(0) = 0$ for $n < k$. Thus (a), (b) and (c) together give that M is a finite sum of Artinian R-modules, so is itself an Artinian R-module and hence is an Artinian $R[x_1, \ldots, x_s]$-module. Now assume that $s \geq 1$ and that conditions (a), (b) and (c) hold. By induction hypothesis, assume that for graded

$R[x_1, \ldots, x_{s-1}]$-modules conditions (a), (b) and (c) are sufficient to ensure that the module is Artinian. We want to show that M is Artinian.

We begin by showing that for $t \geq 0$ the modules N_t, as defined above, satisfy (a), (b) and (c) and so are Artinian. For each n consider the homomorphisms $f_i^n : M_n \to M_{n+1}$ given by multiplication by x_i for $1 \leq i \leq s$ and let $\mathcal{M}$ be the set of Artinian R-modules. Applying Proposition 5.7.1 (ii) it follows, by downward induction on n, using conditions (b) and (c), that M_n is an Artinian R-module for all $n \leq l$. Now, the homogeneous part of N_t of degree n, denoted by $(N_t)_n$, is a subfactor of M_n, so $(N_t)_n = 0$ for $n > l$ and $(N_t)_n$ is an Artinian R-module for $n \leq l$. Consider $\overline{a} \in \operatorname{ann}_{N_t}(\sum_{i=1}^{s-1} x_i R)$ and let a be a representative of $\overline{a}$ in $\operatorname{ann}_M(x_s^{t+1} R)$. Then $ax_i \in \operatorname{ann}_M(x_s^t R)$ for all $1 \leq i \leq s$. Hence $ax_s^t \in \operatorname{ann}_M(\sum_{i=1}^{s} x_i R) \subseteq \sum_{n=k}^{l} M_n$ by condition (b) and so $a \in \sum_{n=k-t}^{l-t} M_n + \operatorname{ann}_M(x_s^t R)$. It follows that $\overline{a} \in \sum_{n=k-t}^{l-t} (N_t)_n$. Thus $\operatorname{ann}_{(N_t)_n}(\sum_{i=1}^{s-1} x_i R) = 0$ for $n < k - t$ and so N_t satisfies conditions (a), (b) and (c). By induction hypothesis N_t is an Artinian $R[x_1, \ldots, x_{s-1}]$-module for all $t \geq 0$.

Now consider a descending chain

$$A_0 \supseteq A_1 \supseteq A_2 \supseteq \cdots \tag{5.5}$$

of $R[x_1, \ldots, x_s]$-submodules of M. With each A_i associate $R[x_1, \ldots, x_{s-1}]$-submodules A_{it} of N_t for each $t \geq 0$, as defined above. By Lemma 5.7.2 (i), $A_{0t} \supseteq A_{1t} \supseteq A_{2t} \supseteq \cdots$ for all $t \geq 0$, so for each t there exists an integer i_t and an $R[x_1, \ldots, x_{s-1}]$-submodule Q_t of N_t such that $A_{it} = Q_t$ for all $i \geq i_t$. By Lemma 5.7.2 (iii), for $i \geq \max(i_t, i_{t+1})$

$$\beta_t(Q_t) = \beta_t(A_{it}) \supseteq \beta_{t+1}(A_{it+1}) = \beta_{t+1}(Q_{t+1})$$

so $Q_0 = \beta_0(Q_0) \supseteq \beta_1(Q_1) \supseteq \beta_2(Q_2) \supseteq \cdots$ is a descending chain of $R[x_1, \ldots, x_{s-1}]$-submodules of N_0. Thus there exists an integer $T \geq 0$ such that $\beta_t(Q_t) = \beta_{t+1}(Q_{t+1})$ for all $t \geq T$. Put $I = \max(i_0, \ldots, i_T)$, then

$$A_{it} = A_{i+1t} (= Q_t) \text{ for all } i \geq I \text{ and } 0 \leq t \leq T.$$

We show that

$$A_{it} = A_{i+1t} (= Q_t) \text{ for all } i \geq I \text{ and } t \geq 0$$

so that, by Lemma 5.7.2 (ii), $A_i = A_{i+1}$ for all $i \geq I$, that is, the original descending chain (5.5) stops and M is Artinian. We prove this final step by induction on $t \geq T$, noting first that the result is true for $t = T$. Suppose that $t \geq T$ and $i \geq I$, then

$$\beta_t(A_{it}) \supseteq \beta_{t+1}(A_{it+1}) \supseteq \beta_{t+1}(Q_{t+1}) = \beta_t(Q_t) = \beta_t(A_{it}),$$

by induction hypothesis. Thus $\beta_{t+1}(A_{it+1}) = \beta_{t+1}(Q_{t+1})$ for all $i \geq I$ and, as β_{t+1} is a monomorphism, $A_{it+1} = Q_{t+1} (= A_{i+1t+1})$ for all $i \geq I$, as required. $\square$

Now let R be a commutative ring and let $\mathcal{N}$ be a set of R-modules such that if A and B are elements of $\mathcal{N}$ and $\alpha : A \to B$ is an R-homomorphism then $\ker \alpha$ is in $\mathcal{N}$. Note that a Serre subcategory of the category of R-modules satisfies this condition. For each integer $s \geq 0$ let $\mathcal{N}_s$ (respectively $\mathcal{N}'_s$) denote the set of graded $R[x_1, \ldots, x_s]$-modules $M = \oplus_{n=-\infty}^{\infty} M_n$ such that $M_n \in \mathcal{N}$ for all n and

(a) when $s = 0$, there exists an integer n_0 such that $M_n = 0$ for all $n > n_0$ (respectively $n < n_0$).

(b) when $s > 0$, both the kernel and cokernel of the map $X_s : M \to M$ given by multiplication by x_s are in $\mathcal{N}_{s-1}$ (respectively $\mathcal{N}'_{s-1}$).

Further, let G be an Abelian group and let $\mathcal{L} : \mathcal{N} \to G$ be a mapping such that whenever $0 \to A' \to A \to A'' \to 0$ is an exact sequence of R-homomorphisms between elements of $\mathcal{N}$ then $\mathcal{L}(A) = \mathcal{L}(A') + \mathcal{L}(A'')$.

Proposition 5.7.4. *Let R be a commutative ring, let $s \geq 0$ be an integer and let $M = \oplus_{n=-\infty}^{\infty} M_n$ be a graded $R[x_1, \ldots, x_s]$-module. Let $\mathcal{N}, \mathcal{N}_s, \mathcal{N}'_s$, G and $\mathcal{L}$ be defined as above.*

(i) If $M \in \mathcal{N}_s$ then the mapping $f_M : \mathbb{Z} \to G$ given by $f_M(n) = \mathcal{L}(M_n)$ is a polynomial function of degree at most $s - 1$.

(ii) If $M \in \mathcal{N}'_s$ then the mapping $f'_M : \mathbb{Z} \to G$ given by $f'_M(n) = \mathcal{L}(M_{-n})$ is a polynomial function of degree at most $s - 1$.

Proof. We prove (i) by induction on $s \geq 0$; (ii) follows similarly. Suppose that $s = 0$. Then $M \in \mathcal{N}_0$ gives that there exists an integer n_0 such that $M_n = 0$ for all $n > n_0$. The exact sequence property of $\mathcal{L}$ gives that $\mathcal{L}(0) = 0$. Thus $f_M(n) = \mathcal{L}(M_n) = 0$ for all $n > n_0$ and f_M is a polynomial function of degree -1.

Now consider the inductive step when $s > 0$. As above let $X_s : M \to M$ denote the map given by multiplication by x_s. The exact sequence

$$0 \to \operatorname{ann}_M(x_s R) \to M \xrightarrow{X_s} M \to M/Mx_s \to 0$$

gives two exact sequences

$$0 \to \operatorname{ann}_{M_n}(x_s R) \to M_n \to M_n x_s \to 0 \tag{5.6}$$

and (noting that $M_n x_s \subseteq M_{n+1}$)

$$0 \to M_n x_s \to M_{n+1} \to M_{n+1}/M_n x_s \to 0 \tag{5.7}$$

where the map $M_n \to M_n x_s$ in (5.6) is given by multiplication by x_s. Now, $M/Mx_s = \oplus_{n=-\infty}^{\infty} M_n/M_{n-1}x_s$ and $\mathrm{ann}_M(x_sR) = \oplus_{n=-\infty}^{\infty} \mathrm{ann}_{M_n}(x_sR)$ are graded $R[x_1, \ldots, x_{s-1}]$-modules. By hypothesis, $M \in \mathcal{N}_s$, so the kernel and cokernel of $X_s : M \to M$ are in $\mathcal{N}_{s-1}$, that is $\mathrm{ann}_M(Mx_s)$ and M/Mx_s are in $\mathcal{N}_{s-1}$. By induction hypothesis, the mappings $f_{\mathrm{ann}_M(x_sR)} : \mathbb{Z} \to G$ and $f_{M/Mx_s} : \mathbb{Z} \to G$ given by $f_{\mathrm{ann}_M(x_sR)}(n) = \mathcal{L}(\mathrm{ann}_{M_n}(x_sR))$ and $f_{M/Mx_s}(n) = \mathcal{L}(M_n/M_{n-1}x_s)$ respectively are polynomial functions of degree at most $s-2$. Now, equations (5.6) and (5.7) give that

$$\mathcal{L}(M_n) = \mathcal{L}(\mathrm{ann}_{M_n}(x_sR)) + \mathcal{L}(M_n x_s)$$

and

$$\mathcal{L}(M_{n+1}) = \mathcal{L}(M_n x_s) + \mathcal{L}(M_{n+1}/M_n x_s)$$

and hence

$$\mathcal{L}(M_{n+1}) - \mathcal{L}(M_n) = \mathcal{L}(M_{n+1}/M_n x_s) - \mathcal{L}(\mathrm{ann}_{M_n}(x_sR)).$$

This says that

$$\triangle f_M(n+1) = f_M(n+1) - f_M(n) = f_{M/Mx_s}(n+1) - f_{\mathrm{ann}_M(x_sR)}(n).$$

Let $g : \mathbb{Z} \to G$ be the mapping given by $g(n) = f_{M/Mx_s}(n+1)$, then Corollary 5.6.3 says that g is a polynomial function of degree at most $s-2$. Let $h : \mathbb{Z} \to G$ be the mapping given by $h(n) = \triangle f_M(n+1)$. Then $h(n) = g(n) - f_{\mathrm{ann}_M(x_sR)}(n)$ and, as a difference of two polynomial functions of degree at most $s-2$, h is a polynomial function of degree at most $s-2$. By Corollary 5.6.3 again, this gives that $\triangle f_M$ is a polynomial function of degree at most $s-2$. It follows, by Lemma 5.6.2, that f_M is a polynomial function of degree at most $s-1$. □

Our next result is an extension of Proposition 5.7.3.

Proposition 5.7.5. *Let R be a commutative ring, let $s \geq 0$ be an integer and let $M = \oplus_{n=-\infty}^{\infty} M_n$ be a graded $R[x_1, \ldots, x_s]$-module. Let $\mathcal{M}$ be a Serre subcategory of the category of R-modules and let $\mathcal{M}_s$ and $\mathcal{M}'_s$ be defined as above.*

(i) M is a Noetherian member of $\mathcal{M}_s$ if and only if there exist integers k and l such that

(a) $M_n = 0$ *for* $n < k$.

(b) $M_{n+1} = \sum_{i=1}^{s} M_n x_i$ *for* $n \geq l$.

(c) M_n *is a Noetherian member of* $\mathcal{M}$ *for* $k \leq n \leq l$.

(ii) M *is an Artinian member of* $\mathcal{M}'_s$ *if and only if there exist integers* k *and* l *such that*

(a) $M_n = 0$ *for* $n > l$.

(b) $ann_{M_n}(\sum_{i=1}^{s} x_i R) = 0$ *for* $n < k$.

(c) M_n *is an Artinian member of* $\mathcal{M}$ *for* $k \leq n \leq l$.

Proof. We prove (ii) only; (i) follows similarly. If M is an Artinian member of $\mathcal{M}'_s$ then (a) and (b) follow by Proposition 5.7.3 (ii) and M_n is Artinian for $k \leq n \leq l$. By definition of $\mathcal{M}'_s$, $M_n \in \mathcal{M}$ for all n and (c) follows.

The converse is proved by induction on $s \geq 0$. If $s = 0$ then (b) gives that $M_n = 0$ for $n < k$. By (c), M_n is an Artinian member of $\mathcal{M}$ for $k \leq n \leq l$ and, by (a), $M_n = 0$ for $n > l$. Hence $M = M_k \oplus \cdots \oplus M_l$ is an Artinian member of $\mathcal{M}'_0$. Suppose now that $s > 0$ and that (ii) (a),(b) and (c) hold. By Proposition 5.7.3 (ii), M is an Artinian $R[x_1, \ldots, x_s]$-module. It remains to show that $M \in \mathcal{M}'_s$. Consider the homomorphisms $f_i : M_n \to M_{n+1}$ given by multiplication by x_i for $i = 1, \ldots, s$. By (b), $\cap_{i=1}^{s} \ker f_i = \cap_{i=1}^{s} \mathrm{ann}_{M_n}(x_i R) = \mathrm{ann}_{M_n}(\sum_{i=1}^{s} x_i R) = 0$ for $n < k$, so (c) and Proposition 5.7.1 (ii) give that $M_n \in \mathcal{M}$ for all $n \leq l$, by downward induction. It remains to show that the kernel, $\mathrm{ann}_M(x_s R)$, and cokernel, M/Mx_s, of the map $X_s : M \to M$ given by multiplication by x_s are in $\mathcal{M}'_{s-1}$. This is done by showing that $\mathrm{ann}_M(x_s R)$ and M/Mx_s both satisfy (ii) (a),(b) and (c) with s replaced by $s-1$ and then applying the induction hypothesis. Now, M is an Artinian $R[x_1, \ldots, x_s]$-module and it follows that $\mathrm{ann}_M(x_s R)$ and M/Mx_s are both Artinian $R[x_1, \ldots, x_{s-1}]$-modules, so satisfy (a) and (b), by Proposition 5.7.3 (ii). Further, their homogeneous parts of degree n are respectively a submodule and a factor module of M_n, which is an Artinian member of the Serre subcategory $\mathcal{M}$, so they are both Artinian members of $\mathcal{M}$. □

We can now prove our main result of this section, which is used later when we consider the relationship between dual Krull dimension and dual classical Krull dimension.

Proposition 5.7.6. *Let* R *be a commutative ring, let* M *be an Artinian* R*-module and let* A *be an ideal of* R*. Suppose that* $ann_M(A)$ *is contained in a Serre subcategory* $\mathcal{M}$ *of* *the category of R-modules. Then the R-submodule $ann_M(A^n)$ is contained in $\mathcal{M}$ for all $n \geq 0$. Moreover, if G is an Abelian group and $\mathcal{L} : \mathcal{M} \to G$ is a map such that whenever $0 \to N' \to N \to N'' \to 0$ is an exact sequence of R-modules in $\mathcal{M}$ $\mathcal{L}(N) = \mathcal{L}(N') + \mathcal{L}(N'')$, then the mapping $f : \mathbb{Z} \to G$ given by $f(n) = \mathcal{L}(ann_M(A^n))$ for $n \geq 0$ and $f(n) = 0$ for $n \leq -1$ is a polynomial function.*

Proof. By Lemma 5.5.6, we may assume that A is finitely generated, so suppose that $A = \sum_{i=1}^{s} a_i R$ for some integer $s \geq 1$ and elements $a_i \in A$ $(1 \leq i \leq s)$. Define $\overline{M}_n = \mathrm{ann}_M(A^{-n})/\mathrm{ann}_M(A^{-n-1})$ for $n \leq -1$ and $\overline{M}_n = 0$ for $n \geq 0$ and put $\overline{M} = \oplus_{n=-\infty}^{\infty} \overline{M}_n$. For $\overline{m} \in \overline{M}_{n-1}$ and indeterminates x_i $(1 \leq i \leq s)$, define products $\overline{m}x_i$ by

$$(m + \mathrm{ann}_M(A^{-n}))x_i = ma_i + \mathrm{ann}_M(A^{-n-1})$$

for $n \leq -1$ and $\overline{m}x_i = ma_i = 0$ for $n = 0$, where m is a representative of $\overline{m}$. This makes $\overline{M}$ into a graded $R[x_1, \ldots, x_s]$-module. Then

(a) $\overline{M}_n = 0$ for $n > -1$

(b) $\mathrm{ann}_{\overline{M}_n}(\sum_{i=1}^{s} x_i R) = 0$ for $n < -1$

(c) $\overline{M}_{-1} = \mathrm{ann}_M(A)$ is an Artinian element of $\mathcal{M}$.

(a) and (c) are clear. For (b),

$$\begin{aligned}
\mathrm{ann}_{\overline{M}_n}(\sum_{i=1}^{s} x_i R) &= \{m + \mathrm{ann}_M(A^{-n-1}) | (m + \mathrm{ann}_M(A^{-n-1}))x_i = \overline{0} \text{ for all } i = 1, \ldots, s\} \\
&= \{m + \mathrm{ann}_M(A^{-n-1}) | ma_i \in \mathrm{ann}_M(A^{-n-2}) \text{ for all } i = 1, \ldots, s\} \\
&= \{m + \mathrm{ann}_M(A^{-n-1}) | mA \subseteq \mathrm{ann}_M(A^{-n-2})\} \\
&= \{m + \mathrm{ann}_M(A^{-n-1}) | m \in \mathrm{ann}_M(A^{-n-1})\} \\
&= 0
\end{aligned}$$

By Proposition 5.7.5 (ii), it follows that $\overline{M}$ is an Artinian element of $\mathcal{M}'_s$. In particular, $\overline{M}_n \in \mathcal{M}$ for all n. By Proposition 5.7.4 (ii), the mapping $f' : \mathbb{Z} \to G$ given by $f'(n) = \mathcal{L}(\overline{M}_{-n})$ is a polynomial function of degree at most $s - 1$.

Now consider the exact sequences

$$0 \to \mathrm{ann}_M(A^{n-1}) \to \mathrm{ann}_M(A^n) \to \overline{M}_{-n} \to 0$$

for $n \geq 1$. Since $\mathrm{ann}_M(A) \in \mathcal{M}$ and $\overline{M}_{-n} \in \mathcal{M}$ for all n, induction shows that $\mathrm{ann}_M(A^n) \in \mathcal{M}$ for all $n \geq 1$. These exact sequences also show that $\triangle f(n) = f(n) - f(n-1) =$

$\mathcal{L}(\mathrm{ann}_M(A^n)) - \mathcal{L}(\mathrm{ann}_M(A^{n-1})) = \mathcal{L}(\overline{M}_{-n}) = f'(n)$ for all $n \geq 1$. Clearly $\triangle f(n) = 0 = f'(n)$ for all $n \leq 0$. By Lemma 5.6.2, it follows that f is a polynomial function of degree at most s, proving the result. □

Note that the proof of this proposition shows that (under the conditions detailed above) if A is a finitely generated ideal with s generators then the map $f : \mathbb{Z} \to G$ given by $f(n) = \mathcal{L}(\mathrm{ann}_M(A^n))$ for $n \geq 0$ and $f(n) = 0$ for $n \leq -1$ is a polynomial function of degree at most s.

We finish this section with a number of other technical results that follow from Proposition 5.7.3 and which will also be needed in our consideration of the relationship between dual Krull dimension and dual classical Krull dimension. Our first result is a dual analogue of the Artin-Rees Lemma (see [25, Lemma 4.1.10]). If S is a subset of a (not necessarily commutative) ring R and K is a subset of a (right) R-module M then $(K :_M S)$ will denote the set $(K :_M S) = \{m \in M | mS \subseteq K\}$. Note that $(0 :_M S) = \mathrm{ann}_M(S)$.

Proposition 5.7.7. *Let R be a commutative ring, let M be an Artinian R-module and let N be a submodule of M. For any ideal A of R there exists an integer $r \geq 0$ such that*

$$N + ann_M(A^n) = ((N + ann_M(A^r)) :_M A^{n-r})$$

for all $n \geq r$.

Proof. Let A be an ideal of R. Suppose first that the result holds for finitely generated ideals. By Lemma 5.5.6, there is a finitely generated ideal B of R such that $B \subseteq A$ and $\mathrm{ann}_M(A^n) = \mathrm{ann}_M(B^n)$ for all $n \geq 0$. Then there exists an integer $r \geq 0$ such that

$$\begin{aligned} N + \mathrm{ann}_M(A^n) = N + \mathrm{ann}_M(B^n) &= ((N + \mathrm{ann}_M(B^r)) :_M B^{n-r}) \\ &\supseteq ((N + \mathrm{ann}_M(A^r)) :_M A^{n-r}) \supseteq N + \mathrm{ann}_M(A^n) \end{aligned}$$

for all $n \geq r$. Hence $N + \mathrm{ann}_M(A^n) = ((N + \mathrm{ann}_M(A^r)) :_M A^{n-r})$ for all $n \geq r$. Thus we may assume that A is finitely generated, so suppose that $A = \sum_{i=1}^{s} a_i R$ for some integer $s \geq 1$ and elements $a_i \in A$ $(1 \leq i \leq s)$. Put $\overline{M}_{-n} = M/\mathrm{ann}_M(A^n)$ for $n \geq 0$ and $\overline{M}_n = 0$ for $n > 0$ and put $\overline{M} = \oplus_{n=-\infty}^{\infty} \overline{M}_n$. For indeterminates $x_1, \ldots, x_s$, the R-module $\overline{M}$ can be made into a graded $R[x_1, \ldots, x_s]$-module by putting $(m + \mathrm{ann}_M(A^{-n}))x_i = ma_i + \mathrm{ann}_M(A^{-n-1})$ for $n < 0$. Then

(a) $\overline{M}_n = 0$ for $n > 0$

(b) $\mathrm{ann}_{\overline{M}_n}(\sum_{i=1}^{s} x_i R) = 0$ for $n < 0$

(c) $\overline{M}_0 = M$ is an Artinian R-module.

where (a) and (c) are clear and (b) follows as in the proof of Proposition 5.7.6. Hence, by Proposition 5.7.3 (ii), $\overline{M}$ is an Artinian $R[x_1, \ldots, x_s]$-module.

Now, for $i \geq 0$ consider

$$N_i = \oplus_{n=-i}^{0}((N + \mathrm{ann}_M(A^{-n}))/\mathrm{ann}_M(A^{-n})) + \oplus_{n=-\infty}^{-i-1}(((N + \mathrm{ann}_M(A^{i})) :_M A^{-n-i})/\mathrm{ann}_M(A^{-n})).$$

Then $N_0 \supseteq N_1 \supseteq \cdots$ is a descending chain of $R[x_1, \ldots, x_s]$-submodules of $\overline{M}$. Since $\overline{M}$ is Artinian there exists an integer $r \geq 0$ such that $N_n = N_r$ for all $n \geq r$. Equating the homogeneous parts of degree n in N_n and N_r gives

$$(N + \mathrm{ann}_M(A^n))/\mathrm{ann}_M(A^n) = ((N + \mathrm{ann}_M(A^r)) :_M A^{n-r})/\mathrm{ann}_M(A^n)$$

for all $n \geq r$. Hence $N + \mathrm{ann}_M(A^n) = ((N + \mathrm{ann}_M(A^r)) :_M A^{n-r})$ for all $n \geq r$, as required. $\square$

Our next result provides an analogue of Krull's Intersection Theorem and is referred to by Kirby as the Union Theorem [15, Proposition 4].

Proposition 5.7.8. *Let R be a commutative ring, let A be an ideal of R and let M be an Artinian R-module. Then a submodule N of M contains $\bigcup_{n=0}^{\infty} \mathrm{ann}_M(A^n)$ if and only if $N = (N :_M A)$.*

Proof. Suppose that N is a submodule of M such that $N = (N :_M A)$. Then $(N :_M A) = (N :_M A^2) = (N :_M A^3) = \cdots$, so $\mathrm{ann}_M(A^n) \subseteq (N :_M A^n) = (N :_M A) = N$ for all $n \geq 0$. Thus $\cup_{n=0}^{\infty}\mathrm{ann}_M(A^n) \subseteq N$.

Conversely, suppose that N is a submodule of M such that $\cup_{n=0}^{\infty}\mathrm{ann}_M(A^n) \subseteq N$. By Proposition 5.7.7, there exists an integer $r \geq 0$ such that $N + \mathrm{ann}_M(A^n) = ((N + \mathrm{ann}_M(A^r)) :_M A^{n-r})$ for all $n \geq r$. Then $N = (N :_M A^{n-r})$ for all $n \geq r$ and, in particular, putting $n = r + 1$ gives $N = (N :_M A)$. $\square$

In the next proposition, recall that $\mathrm{J}(R)$ denotes the Jacobson radical of a ring R (see Section 1.2 of the Introduction).

Proposition 5.7.9. *Let R be a commutative ring and let A be an ideal of R. Then $A \subseteq \mathrm{J}(R)$ if and only if $M = \cup_{n=0}^{\infty} \mathrm{ann}_M(A^n)$ for all Artinian R-modules M.*

Proof. Suppose that $M = \cup_{n=0}^{\infty}\mathrm{ann}_M(A^n)$ for all Artinian R-modules M. Let I be a maximal ideal of R. Then R/I is a field, so is an Artinian R-module and hence $R/I = \cup_{n=0}^{\infty}\mathrm{ann}_{R/I}(A^n)$. Thus, for some $n \geq 0$ the unit of R/I is in $\mathrm{ann}_{R/I}(A^n)$, so $A^n \subseteq I$ and hence $A \subseteq I$, since I is maximal and hence prime. This is true for any maximal ideal I of R and therefore $A \subseteq \mathrm{J}(R)$.

Conversely, suppose that $A \subseteq \mathrm{J}(R)$ and let M be an Artinian R-module. Consider an element $b \in M$. Then $\cup_{n=0}^{\infty}\mathrm{ann}_M(A^n) \subseteq bA + \cup_{n=0}^{\infty}\mathrm{ann}_M(A^n)$, so Proposition 5.7.8 gives that $b \in ((bA + \cup_{n=0}^{\infty}\mathrm{ann}_M(A^n)) :_M A) = bA + \cup_{n=0}^{\infty}\mathrm{ann}_M(A^n)$. Thus, $b(1-a) \in \cup_{n=0}^{\infty}\mathrm{ann}_M(A^n)$ for some element $a \in A$. Now $a \in A \subseteq \mathrm{J}(R)$, so $1-a$ is a unit of R and thus $b \in \cup_{n=0}^{\infty}\mathrm{ann}_M(A^n)$. This is true for all $b \in M$ and hence $M = \cup_{n=0}^{\infty}\mathrm{ann}_M(A^n)$. □

A corollary of Proposition 5.7.9 provides the following analogue of Nakayama's Lemma.

Corollary 5.7.10. *Let R be a commutative ring and let A be an ideal of R with $A \subseteq J(R)$. If M is an Artinian R-module such that $ann_M(A) = 0$ then $M = 0$.*

Proof. $\mathrm{ann}_M(A) = 0$ implies that $\mathrm{ann}_M(A^n) = 0$ for all integers n and hence, by Proposition 5.7.9, $M = \cup_{n=0}^{\infty}\mathrm{ann}_M(A^n) = 0$. □

5.8 Dual Krull Dimension and Dual Classical Krull Dimension for Artinian Modules over Quasi-Local Commutative Rings

In this section, using the work of the previous sections concerning polynomial functions, we detail a result of R. N. Roberts [27] and D. Kirby [15], which says that if R is a quasi-local commutative ring and M is an Artinian R-module then M has dual Krull dimension and dual classical Krull dimension and $\mathrm{k}^{\circ}(M) = \mathrm{ck}^{\circ}(M)$.

Proposition 5.8.1. *Let R be a quasi-local commutative ring with unique maximal ideal J, let M be an Artinian R-module and let $r \geq 0$ be an integer. Suppose that M satisfies $k^{\circ}(M) \leq r$. Then $ck^{\circ}(M) \leq r$.*

Proof. Proceed by induction on r. If $M = 0$ then the result is trivial, since in this case $\mathrm{k}^{\circ}(M) = \mathrm{ck}^{\circ}(M) = -1$. If $\mathrm{k}^{\circ}(M) = 0$ then M has finite length and so $\mathrm{ck}^{\circ}(M) = 0$. Therefore the result holds in the case $r = 0$. Suppose now that $r = k$ for some integer $k > 0$ and that the result holds for all integers $0 \leq r < k$. We may assume that $\mathrm{k}^{\circ}(M) > 0$

and so $\mathrm{ck}^\circ(M) > 0$. Hence, by Lemma 5.5.9, we may assume that there exists an element $x \in J$ such that $Mx = M$. Consider the ascending chain

$$0 \subseteq \mathrm{ann}_M(x) \subseteq \mathrm{ann}_M(x^2) \subseteq \cdots$$

of submodules of M. Since $\mathrm{k}^\circ(M) \leq k$ there exists an integer $n \geq 0$ such that

$$\mathrm{k}^\circ(\mathrm{ann}_M(x^{m+1})/\mathrm{ann}_M(x^m)) \leq k-1$$

for all $m \geq n$. In particular

$$\mathrm{k}^\circ(\mathrm{ann}_M(x^{n+1})/\mathrm{ann}_M(x^n)) \leq k-1.$$

Since $Mx = M$, the map $f : \mathrm{ann}_M(x^{n+1})/\mathrm{ann}_M(x^n) \to \mathrm{ann}_M(x)$ induced by multiplication by x^n is an isomorphism. Hence $\mathrm{k}^\circ(\mathrm{ann}_M(x)) \leq k-1$. By induction hypothesis, it follows that $\mathrm{ck}^\circ(\mathrm{ann}_M(x)) \leq k-1$. Thus there exists a proper ideal A of R with $\mathrm{g}(A) \leq k-1$ such that $\mathrm{ann}_{\mathrm{ann}_M(x)}(A)$ has finite length. Now $\mathrm{ann}_{\mathrm{ann}_M(x)}(A) = \mathrm{ann}_M(A) \cap \mathrm{ann}_M(x) = \mathrm{ann}_M(A + xR)$, so $\mathrm{ann}_M(A+xR)$ has finite length. Further, $A + xR$ is a proper ideal of R (since $A + xR \subseteq J$) with $\mathrm{g}(A + xR) \leq (k-1) + 1 = k$. Thus $\mathrm{ck}^\circ(M) \leq k$. The result follows by induction. □

If N is any module with finite length then we will denote this length by $\mathcal{L}(N)$. Note that this gives a function $\mathcal{L}$ from the Serre subcategory of modules of finite length over a ring to the integers $\mathbb{Z}$ and that this function is additive in the sense that whenever $0 \to A' \to A \to A'' \to 0$ is an exact sequence of homomorphisms between such modules $\mathcal{L}(A) = \mathcal{L}(A') + \mathcal{L}(A'')$.

Let R be a quasi-local commutative ring with unique maximal ideal J and let M be an Artinian R-module. Then $\mathrm{ann}_M(J) = \mathrm{Soc}(M)$ has finite length. By Lemmas 5.5.4 and 5.5.6, $\mathrm{ann}_M(J^n)$ has finite length for all $n \geq 0$. Let the function $f_M : \mathbb{Z} \to \mathbb{Z}$ be given by $f_M(n) = \mathcal{L}(\mathrm{ann}_M(J^n))$ for $n \geq 0$ and $f_M(n) = 0$ for $n \leq -1$. Then, by Proposition 5.7.6, f_M is a polynomial function. We will call f_M the *Hilbert polynomial* of M and the degree of f_M will be denoted by $\mathrm{d}(f_M)$ or by $\mathrm{d}(M)$.

Lemma 5.8.2. *Let R be a quasi-local commutative ring with unique maximal ideal J. Let $\mathcal{L}$ be the function from the Serre subcategory of R-modules of finite length to the integers $\mathbb{Z}$ that gives the length of a module. Let M be an Artinian R-module and let $f_M : \mathbb{Z} \to \mathbb{Z}$ be the function defined by $f_M(n) = \mathcal{L}(ann_M(J^n))$ for $n \geq 0$ and $f_M(n) = 0$ for $n \leq -1$. Let A be an ideal of R such that $ann_M(A)$ has finite length and let $g_M : \mathbb{Z} \to \mathbb{Z}$ be the* *function defined by* $g_M(n) = \mathcal{L}(ann_M(A^n))$ *for* $n \geq 0$ *and* $g_M(n) = 0$ *for* $n \leq -1$. *Then* g_M *is a polynomial function and* $d(g_M) = d(f_M)$.

Proof. Since $\mathrm{ann}_M(A)$ has finite length, g_M is a polynomial function, by Proposition 5.7.6, and further $\mathrm{ann}_M(A)J^k = 0$ for some integer $k \geq 1$. Thus $\mathrm{ann}_M(J) \subseteq \mathrm{ann}_M(A) \subseteq \mathrm{ann}_M(J^k)$, since $A \subseteq J$, and so, by a simple induction argument, $\mathrm{ann}_M(J^n) \subseteq \mathrm{ann}_M(A^n) \subseteq \mathrm{ann}_M(J^{nk})$ for all integers $n \geq 1$. It follows that $f_M(n) \leq g_M(n) \leq f_M(nk)$ for all $n \geq 1$ and hence $\mathrm{d}(f_M) = \mathrm{d}(g_M)$. $\square$

Proposition 5.8.3. *Let R be a quasi-local commutative ring, let M be an Artinian R-module and let f_M be the Hilbert polynomial of M with degree $d(M)$, defined as above. Then $d(M) \leq ck^\circ(M)$*

Proof. By Proposition 5.5.7, M has dual classical Krull dimension, so let A be a finitely generated proper ideal of R such that A has $\mathrm{ck}^\circ(M)$ generators (that is $\mathrm{g}(A) = \mathrm{ck}^\circ(M)$) and $\mathrm{ann}_M(A)$ has finite length. Let g_M be the polynomial function associated with A, as in Lemma 5.8.2. Then, by Lemma 5.8.2, $\mathrm{d}(M) = \mathrm{d}(f_M) = \mathrm{d}(g_M)$. Now, by Proposition 5.7.6, $\mathrm{d}(g_M) \leq \mathrm{g}(A) = \mathrm{ck}^\circ(M)$ and the result follows. $\square$

Proposition 5.8.4. *Let R be a quasi-local commutative ring and let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of Artinian R-modules. Let f_M, $f_{M'}$ and $f_{M''}$ be the Hilbert polynomials of M, M' and M'' respectively. Put $d = d(M) = d(f_M)$. Then $f_{M''}$ has degree at most d and the coefficient of n^k in the polynomial $f_M - f_{M'}$ is equal to the coefficient of n^k in $f_{M''}$ for all integers $k \geq d$.*

Proof. We may assume that M' is a submodule of M and that $M'' = M/M'$. Let J be the unique maximal ideal of R. By the additivity of the length function we have that

$$\begin{aligned}\mathcal{L}(\mathrm{ann}_M(J^n)) - \mathcal{L}(\mathrm{ann}_{M'}(J^n)) &= \mathcal{L}(\mathrm{ann}_M(J^n)/\mathrm{ann}_{M'}(J^n)) \\ &= \mathcal{L}(\mathrm{ann}_M(J^n)/(\mathrm{ann}_M(J^n) \cap M')) \\ &= \mathcal{L}((\mathrm{ann}_M(J^n) + M')/M').\end{aligned}$$

By Proposition 5.7.7, there exists an integer $r \geq 0$ such that

$$\mathrm{ann}_M(J^n) + M' = (M' + \mathrm{ann}_M(J^r) :_M J^{n-r})$$

for all integers $n \geq r$. Thus

$$(M' :_M J^{n-r}) \subseteq (M' + \mathrm{ann}_M(J^r) :_M J^{n-r}) = \mathrm{ann}_M(J^n) + M'$$

for all $n \geq r$ and hence

$$\mathcal{L}((M' :_M J^{n-r})/M') \leq \mathcal{L}((\mathrm{ann}_M(J^n) + M')/M') \leq \mathcal{L}((M' :_M J^n)/M')$$

for all $n \geq r$. It follows that

$$f_{M''}(n-r) \leq f_M(n) - f_{M'}(n) \leq f_{M''}(n) \tag{5.8}$$

for all sufficiently large n. Now, $f_M - f_{M'}$ is a polynomial of degree at most d and so $f_{M''}$ is a polynomial of degree at most d. Thus the coefficients in (5.8) are equal (and zero) when $k > d$. Dividing by n^d and allowing n to tend to infinity establishes the result for $k = d$. □

We are now able to prove the equality of dual Krull dimension and dual classical Krull dimension in the case of Artinian modules over quasi-local commutative rings.

Theorem 5.8.5. *Let R be a quasi-local commutative ring and let M be an Artinian R-module. Then $k^\circ(M) = ck^\circ(M) = d(M)$.*

Proof. We may assume that $M \neq 0$, else the result is trivial. By Propositions 5.8.1 and 5.8.3, it remains to prove that if $r \geq 0$ is an integer and $\mathrm{d}(M) \leq r$ then $\mathrm{k}^\circ(M) \leq r$. This is proved by induction on r. Suppose that $r = 0$. Let J be the unique maximal ideal of R. Let $\mathcal{L}$ be the function from the Serre subcategory of R-modules of finite length to the integers $\mathbb{Z}$ that gives the length of a module. Then $\mathcal{L}(\mathrm{ann}_M(J^n))$ eventually becomes constant as n increases and so there exists an integer N such that $\mathrm{ann}_M(J^s) = \mathrm{ann}_M(J^N)$ for all $s \geq N$. Thus, by Proposition 5.7.9, $M = \cup_{i=0}^{\infty}\mathrm{ann}_M(J^i) = \mathrm{ann}_M(J^N)$. Therefore M is annihilated by J^N and so has finite length. Thus $\mathrm{k}^\circ(M) = 0$, as required.

Suppose now that $r = k$ for some integer $k > 0$ and that the result holds for all integers $0 \leq r < k$. Suppose that $\mathrm{d}(M) = k$. Consider an ascending chain $M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots$ of submodules of M. For each integer $s \geq 0$ let g_s denote the Hilbert polynomial of the factor module M_{s+1}/M_s. By Proposition 5.8.4, if $s \geq 0$ then the coefficient of n^k in the polynomial $f_{M_{s+1}} - f_{M_0}$ is equal to the coefficient of n^k in the polynomial $g_s + g_{s-1} + \cdots + g_0$. It follows that g_t has degree k for only a finite number of values of t. Therefore there exists an integer $\beta \geq 0$ such that $\mathrm{d}(M_{t+1}/M_t) \leq k-1$ for all $t \geq \beta$. By induction hypothesis, $\mathrm{k}^\circ(M_{t+1}/M_t) \leq k-1$ for all $t \geq \beta$. Hence $\mathrm{k}^\circ(M) \leq k$. The result follows by induction. □

5.9 Dual Krull Dimension and Dual Classical Krull Dimension for Artinian Modules over Commutative Rings

In Section 5.8 we detailed a result of R. N. Roberts [27] and D. Kirby [15], which says that if R is a quasi-local commutative ring and M is an Artinian R-module then M has dual Krull dimension and dual classical Krull dimension and $\mathrm{k}^{\circ}(M) = \mathrm{ck}^{\circ}(M)$. In this section we, at least partially, extend this result to Artinian modules over arbitrary commutative rings.

We begin with some notation and a simple lemma. Let R be a commutative ring and let M be an R-module. For an ideal P of R define

$$\begin{aligned} \mathrm{M}(P) &= \{m \in M | mP^k = 0 \text{ for some integer } k \geq 1\} \\ &= \bigcup_{k \geq 1} \mathrm{ann}_M(P^k). \end{aligned}$$

Lemma 5.9.1. *Let R be a commutative ring and let M be a finitely generated Artinian R-module. Then M has finite length.*

Proof. It suffices to show that M is a Noetherian R-module. By factoring out the annihilator of M from the ring R we may assume that M is a faithful R-module. Since R is commutative and M is finitely generated, it follows that M is finitely annihilated. Thus R embeds in a finite direct sum of copies of M. Therefore R is Artinian and hence Noetherian. Since M is finitely generated, it follows that M is Noetherian. □

Lemma 5.9.2. *Let R be a commutative ring and let M be an Artinian R-module. Then $M = \sum_P M(P)$, where the sum is over the maximal ideals P of R.*

Proof. Let $0 \neq m \in M$. By Lemma 5.9.1, mR has finite length, so

$$mR = X_0 \supset X_1 \supset X_2 \supset \cdots \supset X_n = 0$$

for some integer $n \geq 0$ and submodules X_i of mR $(0 \leq i \leq n)$, where X_{i-1}/X_i is a simple R-module for each $1 \leq i \leq n$. Let $Q_i = \mathrm{ann}_R(X_{i-1}/X_i)$ for each $1 \leq i \leq n$, then $Q_1, \ldots, Q_n$ are maximal ideals of R and $mQ_1 \cdots Q_n = 0$. Therefore, there are integers $t \geq 1$ and $s_i \geq 1$ $(1 \leq i \leq t)$ and distinct maximal ideals $P_1, \ldots, P_t$ of R such that $mP_1^{s_1} \cdots P_t^{s_t} = 0$. Now,

$$P_2^{s_2} \cdots P_t^{s_t} + P_1^{s_1} P_3^{s_3} \cdots P_t^{s_t} + \cdots + P_1^{s_1} \cdots P_{t-1}^{s_{t-1}} = R,$$

so

$$\begin{aligned} m &\in m(P_2^{s_2}\cdots P_t^{s_t} + P_1^{s_1}P_3^{s_3}\cdots P_t^{s_t} + \cdots + P_1^{s_1}\cdots P_{t-1}^{s_{t-1}}) \\ &\subseteq mP_2^{s_2}\cdots P_t^{s_t} + mP_1^{s_1}P_3^{s_3}\cdots P_t^{s_t} + \cdots + mP_1^{s_1}\cdots P_{t-1}^{s_{t-1}} \\ &\subseteq \mathrm{M}(P_1) + \mathrm{M}(P_2) + \cdots + \mathrm{M}(P_t). \end{aligned}$$

Thus $M \subseteq \sum_P \mathrm{M}(P)$, where the sum is over the maximal ideals P of R, and the result follows. □

Lemma 5.9.3. *Let R be a commutative ring and let M be an Artinian R-module. Then $\sum_P M(P)$, where the sum is over the maximal ideals P of R, is a direct sum.*

Proof. Suppose that $m_1 + \cdots + m_h = 0$ for some integer $h \geq 1$ and $m_i \in M(Q_i)$ where Q_i is a maximal ideal of R $(1 \leq i \leq h)$. Then there exists some integer $g \geq 1$ such that $m_i Q_i^g = 0$ for all $1 \leq i \leq h$. Now, $m_1(Q_2^g \cdots Q_h^g) = (-m_2 - \cdots - m_h)(Q_2^g \cdots Q_h^g) = 0$, so $m_1(Q_1^g + (Q_2^g \cdots Q_h^g)) = 0$. But $Q_1^g + (Q_2^g \cdots Q_h^g) = R$ and it follows that $m_1 = 0$. Similarly $m_i = 0$ for all $1 \leq i \leq h$. Therefore the sum $\sum_P \mathrm{M}(P)$ over the maximal ideals P of R is a direct sum. □

Corollary 5.9.4. *Let R be a commutative ring and let M be an Artinian R-module. Then $M = M(P_1) \oplus \cdots \oplus M(P_n)$ for some integer $n \geq 1$ and distinct maximal ideals P_i of R $(1 \leq i \leq n)$.*

Proof. Since M is Artinian, the direct sum $M = \oplus_P \mathrm{M}(P)$ where P ranges over the maximal ideals of R (see Lemmas 5.9.2 and 5.9.3) must be finite. □

If R is a commutative ring and P is a maximal ideal of R then we will denote the localisation of R at P by R_P. Then R_P is a commutative quasi-local ring with unique maximal ideal $R_P P$ (see [28, Section 5.1] for further details on localisation).

Proposition 5.9.5. *Let R be a commutative ring and let M be an Artinian R-module. Then there exists an integer $n \geq 1$ and distinct maximal ideals P_i of R $(1 \leq i \leq n)$ such that*

$$M = M(P_1) \oplus \cdots \oplus M(P_n).$$

Further, for each $1 \leq i \leq n$ $M(P_i)$ is an Artinian module over the quasi-local commutative ring R_{P_i} and $k^\circ(M(P_i)_R) = k^\circ(M(P_i)_{R_{P_i}})$ and $ck^\circ(M(P_i)_R) = ck^\circ(M(P_i)_{R_{P_i}})$.

Proof. Let P be a maximal ideal of R, let $x \in \mathrm{M}(P)$ and let $r \in R$ and $t \in R$ with $t \notin P$. Then there exists an integer $n \geq 1$ such that $xP^n = 0$. Now $R = tR + P^n$, so $1 = ts + w$ for some $s \in R$ and $w \in P^n$. Thus $x = x(ts + w) = xts + xw = xts$. For an element $u \in R$ let $\overline{u}$ denote the corresponding element of R_P. Then define $x(\overline{r}/\overline{t}) = xrs$ where r, s and t are as above. It can be checked that this multiplication is well defined and makes $\mathrm{M}(P)$ into an R_P-module and further, that the R- and R_P-module structures of $\mathrm{M}(P)$ coincide. See [28, Section 5.1] for further details. The result follows. □

Theorem 5.9.6. *Let R be a commutative ring and let M be an Artinian R-module. Then $k^\circ(M) \leq ck^\circ(M)$.*

Proof. By Proposition 5.9.5, there exists an integer $n \geq 1$ and maximal ideals P_i of R $(1 \leq i \leq n)$ such that

$$M = \mathrm{M}(P_1) \oplus \cdots \oplus \mathrm{M}(P_n)$$

and further, for each $1 \leq i \leq n$ $\mathrm{M}(P_i)$ is an Artinian module over the quasi-local commutative ring R_{P_i} such that $\mathrm{k}^\circ(\mathrm{M}(P_i)_R) = \mathrm{k}^\circ(M(P_i)_{R_{P_i}})$ and $\mathrm{ck}^\circ(\mathrm{M}(P_i)_R) = \mathrm{ck}^\circ(M(P_i)_{R_{P_i}})$. Then, by Theorem 5.8.5,

$$\begin{aligned}\mathrm{k}^\circ(M) = \sup_{1\leq i\leq n} \mathrm{k}^\circ(\mathrm{M}(P_i)_R) &= \sup_{1\leq i\leq n} \mathrm{k}^\circ(\mathrm{M}(P_i)_{R_{P_i}}) \\ &= \sup_{1\leq i\leq n} \mathrm{ck}^\circ(\mathrm{M}(P_i)_{R_{P_i}}) = \sup_{1\leq i\leq n} \mathrm{ck}^\circ(\mathrm{M}(P_i)_R) \leq \mathrm{ck}^\circ(M).\end{aligned}$$

□

Note that Theorem 5.9.6 shows that if R is a commutative ring and M is an Artinian R-module then M has finite dual Krull dimension.

5.10 Krull Dimension and Dual Krull Dimension

Following on from the results of the previous sections relating the various forms of Krull dimension, we hoped to show that if R is a commutative ring with Krull dimension and M is an R-module with Krull dimension, then $\mathrm{k}^\circ(M) \leq \mathrm{k}(R)$. Unfortunately, we have thus far been unable to prove this conjectured result. In this section we provide a proof in the case M is Artinian. We begin by stating our conjecture.

Conjecture 5.10.1. *Let R be a commutative ring with Krull dimension and let M be an R-module with Krull dimension. Then M has dual Krull dimension and $k^\circ(M) \leq k(R)$.*

Our first approach is to use the result detailed in Section 5.9 which says that if R is a commutative ring and M is an Artinian R-module then M has dual Krull dimension and dual classical Krull dimension and $\mathrm{k}^{\circ}(M) \leq \mathrm{ck}^{\circ}(M)$. Then, since commutative rings are fully bounded, the results of Section 5.4 apply, so $\mathrm{k}^{\circ}(M) \leq \mathrm{ck}^{\circ}(M) \leq \mathrm{k}(R)$.

Theorem 5.10.2. *Let R be a commutative ring with Krull dimension and let M be an Artinian R-module. Then $k^{\circ}(M) \leq k(R)$.*

Proof. By Theorem 5.9.6, $\mathrm{k}^{\circ}(M) \leq \mathrm{ck}^{\circ}(M)$. Now, since commutative rings are fully bounded, Theorem 5.4.3 applies and so $\mathrm{k}^{\circ}(M) \leq \mathrm{ck}^{\circ}(M) \leq \mathrm{k}(R)$. □

This proves the desired result in case M has Krull dimension zero, however it is not clear how this method could be extended to modules of arbitrary Krull dimension.

An alternative approach is to use a result of Lemonnier [21, Corollaire 4.5] which shows that if R is a commutative Noetherian ring and M is an R-module with Krull dimension then $\mathrm{k}^{\circ}(M) \leq \mathrm{k}(R)$. This gives us the start of an argument to prove our conjecture by induction on $\mathrm{k}(R)$, since $\mathrm{k}(R) = 0$ means that R is Artinian and hence Noetherian. However, despite many reductions and preliminary results, we are unable to proceed any further than the case $\mathrm{k}(R) = 1$.

"THIS IS NOT AN EXIT"

References

[1] F. W. Anderson and K. R. Fuller, *Rings and categories of modules*, Volume 13 of Graduate Texts in Mathematics. Second Edition, Springer-Verlag, New York, 1992.

[2] J. A. Beachy, *On quasi-Artinian rings*, J. London Math. Soc. (2) 3 (1971) 449–452.

[3] G. Cauchon, *Les T-anneaux, la condition (H) de Gabriel et ses consequences*, Comm. Algebra 4 (1) (1976) 11–50.

[4] L. Chambless, *N-dimension and N-critical modules. Application to Artinian modules* Communications in algebra 8 (16) (1980) 1561–1592.

[5] A. W. Chatters and C. R. Hajarnavis, *Rings with chain conditions*, Volume 44 of Research Notes in Mathematics. Pitman (Advanced Publishing Program), Boston, Mass., 1980.

[6] B. E. Ellis, *American Psycho*, Picador, London, 1991.

[7] C. Faith, *Modules finite over endomorphism rings*, in Lectures on Rings and Modules (Tulane Univ. Ring and Operator Theory Year, 1970–1971, Vol. I) (eds A. Dold and B. Eckmann), Lecture Notes in Mathematics 246. Springer-Verlag, Berlin, 1972, pp. 145–189.

[8] P. Gabriel, *Des catégories abéliennes*, Bull. Soc. Math. France 90 (1962) 323–448.

[9] A. Ghorbani, *Further characterizations of Artinian rings*, Acta. Math. Hungar. 102 (1–2) (2004) 85–89.

[10] J. S. Golan, *Localization of noncommutative rings*, No. 30 of Pure and Applied Mathematics. Marcel Dekker, Inc. New York, 1975.

[11] A. W. Goldie, *Torsion-free modules and rings*, Journal of Algebra 1 (1964) 268–287.

[12] K. R. Goodearl and R. B. Warfield, Jr., *An introduction to noncommutative Noetherian rings*, London Mathematical Society Student Texts Volume 16. Cambridge University Press, Cambridge, 1989.

[13] R. Gordon and J. C. Robson, *Krull dimension*, No. 133 of Memoirs of the American Mathematical Society. American Mathematical Society, Providence, R.I., 1973.

[14] A. V. Jategaonkar, *Jacobson's Conjecture and Modules over fully bounded Noetherian rings*, J. Algebra 30 (1974) 103–121.

[15] D. Kirby, *Artinian Modules and Hilbert Polynomials*, Quart. J. Math. Oxford (2) 24 (1973) 47–57.

[16] D. Kirby, *Coprimary Decomposition of Artinian Modules*, J. London Math. Soc. (2), 6 (1973) 571–576.

[17] G. Krause, *On the Krull-Dimension of Left Noetherian Left Matlis-Rings*, Math. Zeit. 118 (1970) 207–214.

[18] G. Krause, *On fully left bounded left Noetherian rings*, J. Algebra 23 (1972) 88–99.

[19] J. Lambek, *Lectures on Rings and Modules*, Blaisdell Publishing Co., Waltham, 1966.

[20] J. Lambek and G. Michler, *The torsion theory at a prime ideal of a right Noetherian ring*, J. Algebra 25 (1973) 364–389.

[21] B. Lemonnier, *Dimension de Krull et codeviation. Application au theoreme d'Eakin*, Communications in Algebra 6 (16) (1978) 1647–1665.

[22] T. H. Lenagan, *Artinian ideals in Noetherian rings*, Proc. Amer. Math. Soc. 51 (1975) 499–500.

[23] V. T. Markov, *PI-rings and semiprime rings having faithful modules with Krull dimension*, Algebra and Logic 36 (5) (1997) 330–335.

[24] R. L. McCasland and P. F. Smith, *Prime submodules of Noetherian modules*, Rocky Mountain J. Math. 23 (3) (1993) 1041–1062.

[25] J. C. McConnell and J. C. Robson, *Noncommutative Noetherian rings.* John Wiley & Sons, Chichester, 1987.

[26] R. Rentschler and P. Gabriel, *Sur la dimension des anneaux et ensembles ordonnés*, C. R. Acad. Sci. Paris 265 (1967) 712–715.

[27] R. N. Roberts, *Krull dimension for Artinian modules over quasi local commutative rings*, Quart. J. Math. Oxford (3) 26 (1975) 269–273.

[28] D. W. Sharpe and P. Vámos, *Injective modules*, No. 62 of Cambridge Tracts in Mathematics and Mathematical Physics. Cambridge University Press, London, 1972.

[29] P. F. Smith, *Radical submodules and uniform dimension of modules*, Turkish J. Math. 28 (2004) 255–270.

[30] P. F. Smith and A. R. Woodward, *Artinian rings and the H-condition*, Bull. London Math. Soc. 38 (2006) 571–574.

[31] P. F. Smith and A. R. Woodward, *On finitely annihilated modules and Artinian rings*, Mediterr. J. Math. 3 (2006) 301–311.

[32] P. F. Smith and A. R. Woodward, *Krull dimension of bimodules*, Journal of Algebra 310 (2007) 405–412.

[33] B. Stenström, *Rings of quotients*. Springer-Verlag, New York, 1975.

[34] P. Vámos, *The dual of the notion of "finitely generated"*, J. London Math. Soc. 43 (1968) 643–646.

[35] O. Zariski and P. Samuel, *Commutative algebra Vol. 1*, Volume 28 of Graduate Texts in Mathematics. Springer-Verlag, New York, 1975.