A thesis submitted for the degree of Master of Science in the University of Glasgow

An investigation of acceleration waves in a perfectly conducting magnetohydrodynamic fluid

By

Abdullah A. Abdullah

The University of Glasgow

November, 1986

Preface

This dissertation is submitted to the University of Glasgow, in accordance with the requirements for the degree of Master of Science in Mathematics.

The work presented here has been carried out under the supervision of Dr. Kenneth A. Lindsay. I would like to express here my deepest gratitude to him for his guidance, constant interest and encouragement throughout the period of this research.

In addition, I wish to thank professor R. W. Ogden for providing me with every possible help in the department. I should also like to thank the Saudi goverment for financial support. Finally I am very grateful to my wife for her patience and encouragement.

Contents

	Page
- Introduction	1
- Chapter One	4
- Chapter Two	10
- Chapter Three	18
- Chapter four	24
- Chapter Five	29
- Chapter Six	42
- References	52

Introduction

Recently there has been interest in the mechanics of a perfectly conducting magnetohydrodynamic fluid due to their possible relevance to the behaviour of neutron stars. Roberts [1] has investigated the stability of a particular equilibrium configuration in a complex magnetic material such as a superconductor whose internal energy depends on the mean magnetic induction **B** in an arbitrary way. The relevance of this criterion to the configuration of a neutron star is discussed by Roberts [1] and by Muzikar & Pethick [2]. Straughan [3] developed the same criterion from a consideration of acceleration waves in this particular class of fluids.

In this context an acceleration wave is a propagating surface Σ across which the primitive quantities density ρ, velocity **V** and magnetic induction **B** are continuous, but their space and time derivatives are potentially discontinuous. Straughan showed that the propagation velocities, U, of such acceleration waves satisfy a sixth order polynomial which unexpectedly factorizes into a product of a quadratic and a quartic polynomial. Previous experience of acceleration waves eg. Lindsay & Straughan [4], [5] and Truesdell [6] would suggest that mechanical, thermal and magnetic waves might interact with each other, whereas this result of Straughan would indicate that perhapes in this material there are two distinct types of discontinuities present. Straughan showed that the stability criterion of Roberts was just the condition that the quartic polynomial have real, positive wave speeds.

The aim of this dissertation is firstly to explore the significance of this factorization and thereafter to derive amplitude equations for the development of the resulting discontinuities.

Elcrat [9] investigated the propagation of acceleration waves for the perfect classical fluid model in the case where the fluid ahead of the wavefront was moving but he did not consider any special flows. The acceleration wave analysis for this model is more subtle than that of the perfect fluid. Although the discontinuities in $[\dot{\mathbf{V}}],[\dot{\mathbf{B}}]$ are themselves vectors, for perfect fluids it often transpires that these discontinuities are in the direction of the normal, $\mathbf{n}$, to Σ so that, in effect, they act just like scalar discontinuities in the sense that we need only investigate the behaviour of $\mathbf{n}.[\dot{\mathbf{V}}]$ and $\mathbf{n}.[\dot{\mathbf{B}}]$. However in this case no such simplification materialises. It transpires that the quadratic polynomial gives rise to an Alfven wave in which density, the first derivatives of density, velocity and normal acceleration are continuous across Σ but the derivatives of the magnetic field are discontinuous across Σ. Specifically the discontinuity in $[\dot{\mathbf{B}}]$ is parallel to $\mathbf{B} \times \mathbf{n}$. The quartic polynomial gives rise to a fast and slow wave and corresponds to the situation in which all primitive quantities have discontinuous derivatives.

In the case of the quartic polynomial the amplitude equation is of Bernoulli type and has a closed form solution. In the Alfven wave situation, the corresponding amplitude equation is linear. Special attention is directed to the particular constitutive model in which η = constant and the amplitude equations are formulated for discontinuities propagating into a region at rest and at constant density. The solutions obtained in this case have the same form as the main problem, but are algebraically simpler.

In particular it is clear that under reasonable physical conditions the coefficient of c^2 is negative in the quartic polynomial case.

The problem of discussing the evolutionary behaviour of the amplitude of an acceleration wave requires the location of the singular surface to be determined for all times greater than the initial one. It is only when that is known that the amplitude equation can be studied with a view to determining the behaviour of the amplitude over the surface of the acceleration wave at any particular location and time. In order to obtain the required surface we use a " ray " method developed by Courant & Hilbert [7] and further developed by Varley & Cumberbatch [8], Elcrat [9], Seymour & Mortell [10], Whitham [11] and Wright [12].

Solutions are obtained for several initial profiles. Since the theory allows for variations in amplitude over the singular surface this generally results in the amplitude being a solution to a partial rather than an ordinary differential equation. However, the important feature of the ray method is that along the ray trajectories, the partial differential equation describing amplitude is essentially of ordinary type. in which the initial condition is dependent on surface variables. This introduces the possibility of the amplitude of finite sections of the wavefront becoming infinite in a finite time.

Initial surfaces which are plane, cylindrical and spherical are examined when the fluid velocity ahead of the wavefront is zero and the magnetic induction is either a constant or has a known value.

Chapter One

Electromagnetic Constitutive Laws.

The evolution of electromagnetic effects in a stationary material is governed by the Maxwell equations

$$\nabla \cdot \mathbf{D} = \rho \quad , \tag{1.1}$$

$$\nabla \cdot \mathbf{B} = 0 \quad , \tag{1.2}$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \quad , \tag{1.3}$$

$$\nabla \times \mathbf{H} - \frac{\partial D}{\partial t} = \mathbf{J} \tag{1.4}$$

where **D** is the electric displacement, ρ is the free charge density, **B** is the magnetic induction, **E** is the electric field intensity, **H** is the magnetic field intensity and **J** is the current density. In addition **B**, **H**, **D** and **E** are further connected by the constitutive relations,

$$\rho' = - \nabla \cdot \mathbf{P} \quad , \tag{1.5}$$

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \quad , \tag{1.6}$$

$$\mathbf{B} = \mathbf{H} + \mathbf{M} \tag{1.7}$$

where ρ' is the polarization charge density, ϵ_0 is the permittivity of free space, **P** is the polarization vector and **M** is the magnetization. When the material moves with non-relativistic velocity **V**, to first order in **V**, the local electric field $\mathbf{E}'$ is given by

$$\mathbf{E}' = \mathbf{E} + \mathbf{V} \times \mathbf{B} \tag{1.8}$$

and if we make the further constitutive assumption of an Ohmic material i.e. **J** is proportional to the local electric field intensity $\mathbf{E}'$, then

$$\mathbf{J} = \sigma(\mathbf{E} + \mathbf{V} \times \mathbf{B}) \tag{1.9}$$

where σ is known as the conductivity. In a perfectly conducting material non-zero local electric fields initiate infinite currents and so for a perfectly conducting medium if $\mathbf{J}$ is to be finite then $\mathbf{E}$, $\mathbf{V}$ and $\mathbf{B}$ must satisfy

$$\mathbf{E} = - \mathbf{V} \times \mathbf{B} \ . \tag{1.10}$$

Maxwell's third equation (1.3) and the field equation for a perfect conductor (1.10) can be used to compute the convected derivative of $\mathbf{B}$ in the following way :

$$\begin{aligned}
\dot{B}_i &= \frac{\partial B_i}{\partial t} + B_{i,r} V_r \\
&= - e_{ijk} E_{k,j} + B_{i,r} V_r \\
&= e_{ijk} (e_{kmn} V_m B_n)_{,j} + B_{i,r} V_r \\
&= (\delta_{im} \delta_{jn} - \delta_{in} \delta_{jm})(V_m B_{n,j} + V_{m,j} B_n) + B_{i,r} V_r .
\end{aligned}$$

$$\therefore \quad \dot{B}_i = V_i B_{j,j} + V_{i,j} B_j - V_j B_{i,j} - V_{j,j} B_i + V_r B_{i,r} . \tag{1.11}$$

Since $\mathbf{B}$ is solenoidal, equation (1.11) becomes

$$\dot{B}_i = V_{i,j} B_j - V_{j,j} B_i \ . \tag{1.12}$$

Stress Tensor

We wish to find the general form of the symmetric stress tensor σ_{ij} when it is a function of ρ, B_i and H_i and so with this end, we construct the form invariant scalar

$$\Phi = a_i b_j \sigma_{ij} (\rho, B_i, H_i) \quad . \tag{1.13}$$

We use a method which relies on ideas of Capelli [13] and which appears in Weyl [14].

Theorem

Suppose Φ is a function of the vectors $u^1, \ldots ,u^m$ and is form invariant under the group of orthogonal transformations then a complete table of invariants of the orthogonal group consists of

(I) All scalar products $\mathbf{u}^i \cdot \mathbf{v}^j \quad 1 \leqslant i,j \leqslant m$,

(II) All n×n determinants formed from any subset of the m vectors.

The invariants of type I are even and there are $m(m+1)/2$ distinct invariants whereas invariants of type II are odd and there are $\binom{m}{n}$ independent forms.

In our case $n = 3$ and so the version of the forementioned theorem appropriate to these circumstances is

Theorem

In three dimensional vector space, a complete table of typical basic invariants of the orthogonal group consists of

(i) All possible scalar products $\mathbf{u} \cdot \mathbf{v}$ (even invariants)

(ii) All possible vector triple products $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$ (odd invariants).

According to this theorem we may state that Φ is a function of the list of variables

$$a_i\, a_i,\ a_i\, b_i,\ a_i\, B_i,\ a_i\, H_i,\ b_j\, b_j,\ b_j\, B_j,\ b_j\, H_j,\ B_i\, B_i,\ B_i\, H_i,$$
$$(1.14)$$
$$H_i\, H_i,\ e_{ijk}\, a_i\, b_j\, B_k,\ e_{ijk}\, a_i\, b_j\, H_k,\ e_{ijk}\, a_i\, B_j\, H_k,\ e_{ijk}\, b_i\, B_j\, H_k\ .$$

Let us denote the invariants,

$$\rho,\ B_i\, B_i,\ B_i\, H_i,\ H_i\, H_i \quad \text{by the notation } \Sigma\ .$$

Since Φ is linear in $\mathbf{a}$, $\mathbf{b}$ then $a_i a_i$, $b_i b_i$ can be dropped from list (1.14) and hence

$$\begin{aligned}\Phi = {} & a_i\, b_i\, \Phi_1(\Sigma) + a_i\, B_i\, \Phi_2(b_j\, B_j,\ b_j\, H_j,\ \Sigma,\ e_{ijk}\, b_i\, B_j\, H_k) \\ & + a_i\, H_i\, \Phi_3(b_j\, B_j,\ b_j\, H_j,\ \Sigma,\ e_{ijk}\, b_i\, B_j\, H_k) \\ & + b_j\, B_j\, \Phi_4(\Sigma,\ e_{ijk}\, a_i\, B_j\, H_k) + b_j\, H_j\, \Phi_5(\Sigma,\ e_{ijk}\, a_i\, B_j\, H_k) \\ & + e_{ijk}\, a_i\, b_j\, B_k\, \Phi_6(\Sigma) + e_{ijk}\, a_i\, b_j\, H_k\, \Phi_7(\Sigma) \\ & + e_{ijk}\, a_i\, B_j\, H_k\, \Phi_8(\Sigma,\ e_{ijk}\, b_i\, B_j\, H_k)\ . \end{aligned} \qquad (1.15)$$

$$\therefore \quad \Phi = a_i\, b_i\, \Phi_1(\Sigma) + a_i\, B_i\, b_j\, B_j\, \Phi_{21}(\Sigma) + a_i\, B_i\, b_j\, H_j\, \Phi_{22}(\Sigma)$$

$$+ a_i\, B_i\, e_{jrs}\, b_j\, B_r\, H_s\, \Phi_{23}(\Sigma) + a_i\, H_i\, b_j\, B_j\, \Phi_{31}(\Sigma)$$

$$+ a_i\, H_i\, b_j\, H_j\, \Phi_{32}(\Sigma) + a_i\, H_i\, e_{jrs}\, b_j\, B_r\, H_s\, \Phi_{33}(\Sigma)$$

$$+ b_j\, B_j\, e_{irs}\, a_i\, B_r\, H_s\, \Phi_4(\Sigma) + b_j\, H_j\, e_{irs}\, a_i\, B_r\, H_s\, \Phi_5(\Sigma)$$

$$+ a_i\, b_j\, e_{ijk}\, B_k\, \Phi_6(\Sigma) + e_{ijk}\, a_i\, b_j\, H_k\, \Phi_7(\Sigma)$$

$$+ e_{irk}\, a_i\, B_r\, H_k\, e_{jst}\, b_j\, B_s\, H_t\, \Phi_8(\Sigma) \quad . \qquad (1.16)$$

$$\therefore \quad a_i\, b_j\, [\sigma_{ij} - \Phi_1(\Sigma)\, \delta_{ij} - B_i\, B_j\, \Phi_{21}(\Sigma) - B_i\, H_j\, \Phi_{22}(\Sigma)$$

$$- B_i\, e_{jrs}\, B_r\, H_s\, \Phi_{23}(\Sigma) - H_i\, B_j\, \Phi_{31}(\Sigma) - H_i\, H_j\, \Phi_{32}(\Sigma)$$

$$- H_i\, e_{jrs}\, B_r\, H_s\, \Phi_{33}(\Sigma) - e_{irs}\, B_r\, H_s\, B_j\, \Phi_4(\Sigma)$$

$$- e_{irs}\, B_r\, H_s\, H_j\, \Phi_5(\Sigma) - e_{ijk}\, B_k\, \Phi_6(\Sigma) - e_{ijk}\, H_k\, \Phi_7(\Sigma)$$

$$- e_{irk}\, e_{jst}\, B_r\, H_k\, B_s\, H_t\, \Phi_8(\Sigma)] = 0 \quad . \qquad (1.17)$$

Since a_i and b_j are arbitrary vectors then it follows that

$$\sigma_{ij} = \Phi_1\, \delta_{ij} + B_i\, B_j\, \Phi_{21} + B_i\, H_j\, \Phi_{22} + B_i\, e_{jrs}\, B_r\, H_s\, \Phi_{23}$$

$$+ H_i\, B_j\, \Phi_{31} + H_i\, H_j\, \Phi_{32} + H_i\, e_{jrs}\, B_r\, H_s\, \Phi_{33}$$

$$+ e_{irs}\, B_r\, H_s\, B_j\, \Phi_4 + e_{irs}\, B_r\, H_s\, H_j\, \Phi_5 + e_{ijk}\, B_k\, \Phi_6$$

$$+ e_{ijk}\, H_k\, \Phi_7 + e_{irk}\, e_{jst}\, B_r\, H_k\, B_s\, H_t\, \Phi_8 \qquad (1.18)$$

where the Φ's are functions of Σ only. If we now define

$C_i = e_{irs}\, B_r\, H_s$, then the symmetry of σ requires that

$$0 = \sigma_{ij} - \sigma_{ji} = (B_i\, H_j - B_j\, H_i)(\Phi_{22} - \Phi_{31})$$

$$+ (B_i\, C_j - B_j\, C_i)(\Phi_{23} - \Phi_4) + (H_i\, C_j - H_j\, C_i)(\Phi_{33} - \Phi_5)$$

$$+ 2\, e_{ijk}\, B_k\, \Phi_6 + 2\, e_{ijk}\, H_k\, \Phi_7 \quad . \qquad (1.19)$$

In view of the fact that $\mathbf{C} = \mathbf{B} \times \mathbf{H}$ then $H_i\, C_i = B_i\, C_i = 0$ and thus on contracting (1.19) with H_i and B_j we obtain respectively the identities

$$(\mathbf{H} \cdot \mathbf{B}\, H_j - B_j\, H^2)(\Phi_{22} - \Phi_{31}) + C_j\, [\mathbf{H} \cdot \mathbf{B}\, (\Phi_{23} - \Phi_4)$$

$$+ H^2\, (\Phi_{33} - \Phi_5) + 2\, \Phi_6] = 0$$

$$(1.20)$$

$$(\mathbf{B} \cdot \mathbf{H}\, B_i - H_i\, B^2)(\Phi_{22} - \Phi_{31}) - C_i\, [B^2\, (\Phi_{23} - \Phi_4)$$

$$+ \mathbf{B} \cdot \mathbf{H}\, (\Phi_{33} - \Phi_5) - 2\, \Phi_7] = 0 \ .$$

However since C is perpendicular to B and H then it is immediately obvious from (1.20) that

(i) $\Phi_{22} - \Phi_{31} = 0$ (1.21)

(ii) $H \cdot B\,(\Phi_{23} - \Phi_4) + H^2\,(\Phi_{33} - \Phi_5) + 2\,\Phi_6 = 0\,.$ (1.22)

(iii) $B \cdot H\,(\Phi_5 - \Phi_{33}) + B^2\,(\Phi_4 - \Phi_{23}) + 2\,\Phi_7 = 0\,.$ (1.23)

Equation (1.18) can now be simplified to

$$\sigma_{ij} = \Phi_1\,\delta_{ij} + B_i\,B_j\,\Phi_{21} + (B_i\,H_j + H_i\,B_j)\,\Phi_{22} + B_i\,C_j\,\Phi_{23} + H_i\,H_j\,\Phi_{32} + H_i\,C_j\,\Phi_{33} + C_i\,B_j\,\Phi_4 + C_i\,H_j\,\Phi_5 + e_{ijk}\,B_k\,\Phi_6 + e_{ijk}\,H_k\,\Phi_7 + C_i\,C_j\,\Phi_8\,. \quad (1.24)$$

After further algebra and the introduction of results (1.22) and (1.23) into (1.24) the stress tensor assumes here the canonical form

$$\sigma_{ij} = \Phi_1\,\delta_{ij} + \Phi_2\,B_i\,B_j + \Phi_3\,H_i\,H_j + \Phi_4\,C_i\,C_j + \Phi_5\,(B_i\,H_j + B_j\,H_i) + \Phi_6\,(B_i\,C_j + B_j\,C_i) + \Phi_7\,(H_j\,C_i + H_i\,C_j) \quad (1.25)$$

where Φ_i , $1 \leqslant i \leqslant 7$, are arbitrary functions of Σ which are related to the coefficients in the previous analysis but are not necessarily identical to them.

Special Case

In the important constitutive model in which the magnetisation **M** is parallel to **H** we have the constitutive relation $\mathbf{B} = \mu\mathbf{H}$ i.e. **B** and **H** are parallel.

In this case

(1) $|B| = \mu\,|H|$

(2) $\mathbf{C} = \mathbf{B} \times \mathbf{H} = 0$.

Thus equation (1.25) becomes

$$\sigma_{ij} = \Phi_1\,\delta_{ij} + \Phi_2\,B_i\,B_j + \frac{1}{\mu^2}\,\Phi_3\,B_i\,B_j + \frac{2}{\mu}\,\Phi_5\,B_i\,B_j \quad (1.26)$$

and the general form for the stress tensor in this case is

$$\sigma_{ij} = \Phi_1 \delta_{ij} + \Phi_2 B_i B_j \tag{1.27}$$

where Φ_1 and Φ_2 are functions of ρ, B and have been defined from (1.26) in the obvious way.

Chapter Two

The Mathematics of Singular Surfaces

Consider a regular surface $\Sigma(t)$ which is the common boundary of two regions R^+ and R^- in any real space. Let $\Phi(\mathbf{x},t)$ be a tensor-valued function which is continuous in the interiors of R^+ and R^- and which approches definite limit values Φ^+ and Φ^- as $\mathbf{x}$ approches a point $\mathbf{x}_0$ on $\Sigma(t)$ while remaining within R^+ and R^- respectively. When Φ is continuous across the surface these two values are identical. Otherwise there will be a jump across $\Sigma(t)$ at a surface point $\mathbf{x}_0$ given by

$$[\Phi] = \Phi^+ - \Phi^- \quad . \tag{2.1}$$

The quantity $[\Phi]$ is clearly a function of position and time t. When $[\Phi] \neq 0$, the surface $\Sigma(t)$ is said to be singular with respect to Φ.

In subsequent work, we often encounter [AB] where A and B can themselves be discontinuous. Elementary algebra verifies that

$$[AB] = A^+ \, [B] + B^+ \, [A] - [A] \, [B] \quad . \tag{2.2}$$

This form is particularly useful because A^+ and B^+ are determined from the region into which the discontinuity is propagating and thus they are known quantities.

The theory of singular surfaces can be constructed from Hadamard's lemma.

Hadamard's lemma

Let a tensor-valued function Φ be defined and continuously differentiable in regions R^+ and R^- which are separated by a smooth surface Σ and let Φ and $\Phi_{,i}$ tend to finite limits $(\Phi^+, (\Phi_{,i})^+)$ and $(\Phi^-, (\Phi_{,i})^-)$ as Σ is approched on paths interior to R^+ and R^- respectively. If $\mathbf{x} = \mathbf{x}(s)$ is a smooth curve on Σ and

Φ^+ and Φ^- are differentiable along this path, then

$$\frac{d\Phi^+}{ds} = (\Phi_{,i})^+ \frac{dx_i}{ds} \quad , \tag{2.3}$$

$$\frac{d\Phi^-}{ds} = (\Phi_{,i})^- \frac{dx_i}{ds} \quad . \tag{2.4}$$

In other words, the theorem of total differentiation holds true for the limiting values as Σ is approched from one side only.

The motion of surfaces

Consider a family of surfaces given by

$$\mathbf{x} = \mathbf{x}(t, \theta^\alpha) \tag{2.5}$$

where θ^α are any surface coordinates. This equation gives the location in $\mathbb{R}^3$ of the surface point θ^α at time t and thus it describes the motion of a surface. Specifically θ^α may not be material coordinates.

The velocity $\mathbf{u}$ of the surface point θ^α is defined by

$$\mathbf{u} = \left.\frac{\partial \mathbf{x}}{\partial t}\right|_{\theta^\alpha = \text{constant}.} \tag{2.6}$$

If θ^α is eliminated from (2.5), we can obtain a relation of the form

$$f(\mathbf{x}, t) = 0 \tag{2.7}$$

and on differentiation of (2.7) with respect to time at fixed θ^α,

$$\frac{\partial f}{\partial t} + \mathbf{u} \cdot \nabla f = 0 \quad . \tag{2.8}$$

Now $\mathbf{n} = \nabla f / |\nabla f|$ and so we may finally deduce that

$$\mathbf{u} \cdot \mathbf{n} = - \frac{\partial f}{\partial t} \Big/ |\nabla f| \ . \tag{2.9}$$

This relation effectively tells us that at any point $\mathbf{x}$ and time t, the normal velocity of the surface $u_n = \mathbf{u} \cdot \mathbf{n}$ is independent of the choice of surface coordinates.

Suppose we consider two parameterisations

$$\mathbf{x} = \mathbf{x}(\theta^\beta, t) \quad , \quad \mathbf{x} = \mathbf{x}(v^\alpha, t) \tag{2.10}$$

such that points of constant θ^β always move normal to Σ

$$\text{i.e.} \quad n_i \left.\frac{\partial x_i}{\partial t}\right|_{\theta^\beta} = u_n \ . \tag{2.11}$$

Clearly the coordinate representations θ^β and v^α are related by $v^\alpha = v^\alpha(\theta^\beta, t)$ and so

$$\left.\frac{\partial}{\partial t} x_i(v^\alpha, t)\right|_{\theta^\beta} = \left.\frac{\partial}{\partial t} x_i(v^\alpha(\theta^\beta, t), t)\right|_{\theta^\beta} = u_n\, n_i \tag{2.12}$$

$$\therefore \quad \left.\frac{\partial x_i}{\partial t}\right|_{\theta^\beta} + \frac{\partial x_i}{\partial v^\alpha} \left.\frac{\partial v^\alpha}{\partial t}\right|_{\theta^\beta} = u_n\, n_i \ . \tag{2.13}$$

Now define u^α, the coordinate drift velocity, to be $\left.\dfrac{\partial v^\alpha}{\partial t}\right._{\theta^\beta}$ and thus we obtain the result

$$\frac{\partial x_i}{\partial t} + u^\alpha\, x_{i,\alpha} = u_n\, n_i \ . \tag{2.14}$$

Suppose $F = F(v^\alpha, t)$ is defined on Σ where $v^\alpha = v^\alpha(\theta^\beta, t)$. We may compute the displacement derivative of F with respect to time, at fixed θ^β i.e. in a direction which is always normal to Σ in the following way :

$$\begin{aligned} \left.\frac{\delta F}{\delta t}\right|_{\theta^\beta} &= \left.\frac{\partial}{\partial t} F(v^\alpha(\theta^\beta, t), t)\right|_{\theta^\beta} \\ &= \frac{\partial F}{\partial t} + F_{,\alpha} \frac{\partial v^\alpha}{\partial t} \\ &= \frac{\partial F}{\partial t} + u^\alpha\, F_{,\alpha} \ . \end{aligned} \tag{2.15}$$

Now suppose that $\psi(\mathbf{x}, t)$ is a function defined on $\mathbb{R}^3$ and define $\psi(v^\alpha, t) = \psi(\mathbf{x}(v^\alpha, t), t)$ then the displacement derivative of ψ at fixed θ^β is defined by

$$\left.\frac{\delta \psi}{\delta t}\right|_{\theta^\beta} = \left.\frac{\partial \psi}{\partial t}\right|_{\mathbf{x}} + \psi_{,i} \left.\frac{\partial x_i}{\partial t}\right|_{\theta^\beta} \tag{2.16}$$

and if we use (2.11) then (2.16) becomes

$$\left.\frac{\delta \psi}{\delta t}\right|_{\theta^\beta} = \left.\frac{\partial \psi}{\partial t}\right|_{\mathbf{x}} + \psi_{,i}\, u_n\, n_i \ . \tag{2.17}$$

(see Truesdell & Toupin [15]).

Some formulae from the theory of surfaces

Here we list some results concerning surfaces embeded in three-dimensional Euclidean space. Let $\mathbf{x}$ be a set of coordinates describing $\mathbb{R}^3$ then any surface $\Sigma \subset \mathbb{R}^3$ may be parameterised in the form $\mathbf{x} = \mathbf{x}(u^\alpha, t)$ where α takes values 1 and 2. Since $ds^2 = dx_i\, dx_i$ then on Σ

$$ds^2 = x_{i,\alpha}\, x_{i,\beta}\, du^\alpha\, du^\beta\ . \tag{2.18}$$

We define $a_{\alpha\beta} = x_{i,\alpha}\, x_{i,\beta}$ so that on Σ

$$ds^2 = a_{\alpha\beta}\, du^\alpha\, du^\beta\ . \tag{2.19}$$

The quantities $a_{\alpha\beta}$ are the components of a symmetric covariant tensor called the metric tensor. For the surface Σ we must introduce another symmetric covariant tensor $b_{\alpha\beta}$ by the relations

$$\begin{aligned} x_{k,\alpha\beta} &= n_k\, b_{\alpha\beta}\ , \\ n_{k,\beta} &= -\, b_\beta{}^\alpha\, x_{k,\alpha} \end{aligned} \tag{2.20}$$

where , denotes covariant derivative with respect to θ^α. Equations (2.20) are often called the Gauss-Weingarten relations and the symmetric tensor $b_{\alpha\beta}$ is the curvature tensor for the surface Σ. If $\mathbf{n}$ is a unit normal to Σ then we may also prove that

$$a^{\alpha\beta}\, x_{i,\alpha}\, x_{j,\beta} = \delta_{ij} - n_i\, n_j \tag{2.21}$$

(see Eringen & Suhubi [16]) . In order to find the displacement derivative of the unit normal to the surface we first differentiate equation (2.11) with respect to x_α obtaining

$$u_{n,\alpha} = n_{i,\alpha}\, \frac{\partial x_i}{\partial t} + n_i\, \frac{\partial}{\partial t}(x_{i,\alpha})\ . \tag{2.22}$$

Since n_i and $x_{i,\alpha}$ are perpendicular then

$$x_{i,\alpha}\, \frac{\partial n_i}{\partial t} = -\, n_i\, \frac{\partial}{\partial t}(x_{i,\alpha}) \tag{2.23}$$

and consequently equation (2.22) becomes

$$u_{n,\alpha} = n_{i,\alpha} \frac{\partial x_i}{\partial t} - x_{i,\alpha} \frac{\partial n_i}{\partial t} \quad . \tag{2.24}$$

If we multiply both sides of equation (2.24) by $a^{\alpha\beta} x_{j,\beta}$ and use equations (2.20) and (2.21), we obtain

$$a^{\alpha\beta} x_{j,\beta} u_{n,\alpha} = (n_i n_j - \delta_{ij}) \frac{\partial n_i}{\partial t} - a^{\alpha\beta} b_\alpha{}^\gamma x_{j,\beta} x_{i,\gamma} \frac{\partial x_i}{\partial t} \quad . \tag{2.25}$$

Since $n_i \; \partial n_i/\partial t = 0$ then equation (2.25) becomes

$$- a^{\alpha\beta} x_{j,\beta} u_{n,\alpha} = \frac{\partial n_j}{\partial t} - b_\gamma{}^\beta u^\gamma x_{j,\beta} \tag{2.26}$$

where we have used the fact that $b_{\gamma\beta}$ is symmetric. In view of the Gauss-Weingarten relation $(2.20)_2$ and the definition of the displacement derivative, we may easily deduce that

$$\frac{\delta n_i}{\delta t} = - a^{\alpha\beta} x_{i,\beta} u_{n,\alpha} \; . \tag{2.27}$$

This equation relates the displacement derivative of the unit normal of the surface to the propagation velocity u_n.

Geometrical conditions of compatibility

Suppose that Φ^+, Φ^-, $(\Phi_{,i})^+$ and $(\Phi_{,i})^-$ are functions of surface coordinates in (2.3) and (2.4), then

$$\Phi_{,\alpha}{}^+ = \Phi_{,i}{}^+ x_{i,\alpha} \tag{2.28}$$

$$\Phi_{,\alpha}{}^- = \Phi_{,i}{}^- x_{i,\alpha} \quad . \tag{2.29}$$

From (2.28) and (2.29), we find that

$$[\Phi]_{,\alpha} = [\Phi_{,i}] x_{i,\alpha} \; . \tag{2.30}$$

If both sides of (2.30) are contracted with $a^{\alpha\beta} x_{j,\beta}$, then using (2.21), we find that

$$a^{\alpha\beta} x_{j,\beta} [\Phi]_{,\alpha} = (\delta_{ij} - n_i n_j) [\Phi_{,i}] \tag{2.31}$$

and thus we have obtained the first order compatibility condition.

Thus equation (2.31) becomes

$$[\Phi,_j] = [n_i\ \Phi,_i]\ n_j + a^{\alpha\beta}\ x_{j,\beta}\ [\Phi],_\alpha\ . \qquad (2.32)$$

If Φ is continuous across the surface Σ, equation (2.32) reduces to

$$[\Phi,_j] = [\Phi,_r\ n_r]\ n_j\ . \qquad (2.33)$$

If we now replace Φ by $\Phi,_i$ in equation (2.32), then it is clear that

$$[\Phi,_{ij}] = [n_r\ \Phi,_{ri}]\ n_j + a^{\alpha\beta}\ [\Phi,_i],_\alpha\ x_{j,\beta}\ . \qquad (2.34)$$

Further from (2.34) we may conclude that

$$[n_i\ \Phi,_{ij}] = [n_r\ n_i\ \Phi,_{ri}]\ n_j + a^{\alpha\beta}\ n_i\ [\Phi,_i],_\alpha\ x_{j,\beta} \qquad (2.35)$$

and thus

$$[\Phi,_{ij}] = [n_r\ n_s\ \Phi,_{rs}]\ n_i\ n_j + a^{\alpha\beta}\ n_j\ n_r\ [\Phi,_r],_\alpha\ x_{i,\beta}$$
$$+\ a^{\alpha\beta}\ [\Phi,_i],_\alpha\ x_{j,\beta}\ . \qquad (2.36)$$

In view of (2.32), $[\Phi,_{ij}]$ can finally be simplified to the form

$$[\Phi,_{ij}] = a^{\alpha\beta}\ (a^{\gamma\delta}\ b_{\alpha\gamma}\ [\Phi],_\delta + [n_k\ \Phi,_k],_\alpha)(n_j\ x_{i,\beta} + n_i\ x_{j,\beta})$$
$$-\ a^{\alpha\beta}\ a^{\delta\gamma}\ x_{i,\beta}\ x_{j,\delta}\ ([n_k\ \Phi,_k]\ b_{\alpha\gamma} - [\Phi],_{\alpha\gamma})$$
$$+\ [n_r\ n_s\ \Phi,_{rs}]\ n_i\ n_j\ . \qquad (2.37)$$

This is the second-order compatibility conditions. (see Truesdell & Toupin [15]) .

In particular if Φ is continuous across the surface then equation (2.37) reduces to

$$[\Phi,_{ij}] = [n_r\ n_s\ \Phi,_{rs}]\ n_i\ n_j - [n_k\ \Phi,_k]\ x_{j,\alpha}\ x_{i,\beta}\ b^{\alpha\beta}$$
$$+\ a^{\alpha\beta}\ [\Phi,_k\ n_k],_\alpha\ (n_j\ x_{i,\beta} + n_i\ x_{j,\beta})\ . \qquad (2.38)$$

<u>Kinematical conditions of compatibility</u>

Let us now evaluate $[\dot{\Phi}]$, where $\dot{\Phi}$ is the convected derivative of Φ given by

$$\dot{\Phi} = \frac{\partial\Phi}{\partial t} + \Phi,_j\ V_j\ . \qquad (2.39)$$

When we use equation (2.17), equation (2.39) becomes

$$\dot{\Phi} = \frac{\delta\Phi}{\delta t} - u_n\, n_j\, \Phi_{,j} + \Phi_{,j}\, V_j \quad . \tag{2.40}$$

If we take the jump of equation (2.40), we obtain

$$[\dot{\Phi}] = \frac{\delta}{\delta t}[\Phi] - (u_n\, n_j - V_j)\, [\Phi_{,j}] \ . \tag{2.41}$$

In particular when Φ is continuous across Σ, then using result (2.33) equation (2.41) becomes

$$[\dot{\Phi}] = -\, U\, [n_r\, \Phi_{,r}] \tag{2.42}$$

where

$$U = u_n - V_j\, n_j \ . \tag{2.43}$$

Now we wish to evaluate $[\ddot{\Phi}]$, $[\overline{\dot{\Phi}_{,j}}]$. From equation (2.41)

$$[\ddot{\Phi}] = \frac{\delta}{\delta t}[\dot{\Phi}] - (u_n\, n_j - V_j)\, [\dot{\Phi}_{,j}] \tag{2.44}$$

where

$$\begin{aligned} \dot{\Phi}_{,j} &= \left(\frac{\partial\Phi}{\partial t} + \Phi_{,k}\, V_k\right)_{,j} \\ &= \frac{\partial}{\partial t}(\Phi_{,j}) + \Phi_{,jk}\, V_k + \Phi_{,k}\, V_{k,j} \\ &= \overline{\dot{\Phi}_{,j}} + \Phi_{,k}\, V_{k,j} \ . \end{aligned} \tag{2.45}$$

Also, from equation (2.17)

$$[\overline{\dot{\Phi}_{,j}}] = \frac{\delta}{\delta t}\, [\Phi_{,j}] - (u_n\, n_k - V_k)\, [\Phi_{,jk}] \ . \tag{2.46}$$

In particular if Φ is continuous across Σ, then using results (2.38) and (2.43) equation (2.46) becomes

$$\begin{aligned} [\overline{\dot{\Phi}_{,j}}] = &-U\, n_j\, [n_r\, n_s\, \Phi_{,rs}] - a^{\alpha\beta}\, x_{j,\beta}\, n_s\, V_{s,\alpha}\, [n_r\, \Phi_{,r}] \\ &- a^{\alpha\beta}\, x_{j,\beta}\, (U\, [n_r\, \Phi_{,r}])_{,\alpha} + n_j\, L[n_r\, \Phi_{,r}] \end{aligned} \tag{2.47}$$

where

$$L(\ ) = \frac{\delta}{\delta t}(\ ) + V^{\alpha}(\ )_{,\alpha} \quad . \tag{2.48}$$

If we take the jump of equation (2.45) and use equation (2.47), we obtain

$$\begin{aligned}[\dot{\Phi}_{,j}] = &- U\, n_j\, [n_r\, n_s\, \Phi_{,rs}] - a^{\alpha\beta}\, x_{j,\beta}\, n_s\, V_{s,\alpha}\, [n_r\, \Phi_{,r}] \\ &- a^{\alpha\beta}\, x_{j,\beta}\, (U\, [n_r\, \Phi_{,r}])_{,\alpha} + n_j\, L[n_r\, \Phi_{,r}] + [\Phi_{,k}\, V_{k,j}] .\end{aligned} \tag{2.49}$$

When (2.49) is taken into (2.44) and (2.42) is used, we find that

$$\begin{aligned}[\ddot{\Phi}] = &- L(U\, [n_r\, \Phi_{,r}]) - U\, L[n_r\, \Phi_{,r}] - (u_n\, n_j - V_j)\, [\Phi_{,k}\, V_{k,j}] \\ &+ U^2\, [n_r\, n_s\, \Phi_{,rs}] - n_s\, V^{\alpha}\, V_{s,\alpha}\, [n_r\, \Phi_{,r}] \quad .\end{aligned} \tag{2.50}$$

Define

$$F = U\, [n_r\, n_s\, \Phi_{,rs}] - L[n_r\, \Phi_{,r}] \ . \tag{2.51}$$

Thus (2.50) becomes

$$\begin{aligned}[\ddot{\Phi}] = &- L(U\, [n_r\, \Phi_{,r}]) + U\, F - (u_n\, n_j - V_j)\, [\Phi_{,k}\, V_{k,j}] \\ &- n_s\, V^{\alpha}\, V_{s,\alpha}\, [n_r\, \Phi_{,r}] \ .\end{aligned} \tag{2.52}$$

When $[n_r\, n_s\, \Phi_{,rs}]$ is eliminated from (2.47), we obtain

$$\begin{aligned}\overline{[\dot{\Phi}_{,j}]} = &- a^{\alpha\beta}\, x_{j,\beta}\, (U\, [n_r\, \Phi_{,r}])_{,\alpha} - a^{\alpha\beta}\, x_{j,\beta}\, n_s\, V_{s,\alpha}\, [n_r\, \Phi_{,r}] \\ &- n_j\, F \ .\end{aligned} \tag{2.53}$$

Chapter Three

Propagation of mechanical and electromagnetic acceleration waves

The basic equations for a single phase perfectly conducting magnetohydrodynamic fluid are

$$\dot{\rho} + \rho V_{i,i} = 0 \, , \tag{3.1}$$

$$\dot{B}_i = V_{i,j} B_j - V_{j,j} B_i \, , \tag{3.2}$$

$$\rho \dot{V}_i = \rho f_i + \sigma_{ki,k} \, . \tag{3.3}$$

In these equations, a superposed dot denotes material time differentiation and ,j denotes partial differentiation with respect to x_j. Also ρ is the mass density at time t, V_i is the velocity field, f_i is the specific externally applied body force, B_i is the magnetic induction and σ_{ki} is the stress tensor. Suppose that the internal energy function ψ^* is a function of ρ and B_i. From Roberts [1] σ_{ki}, H_i and p are given by

$$\sigma_{ki} = -(p' + H \, . \, B) \, \delta_{ki} + H_k B_i \, , \tag{3.4}$$

$$H_i = \rho \frac{\partial \psi^*}{\partial B_i} \, , \tag{3.5}$$

$$p' = \rho^2 \frac{\partial \psi^*}{\partial \rho} \tag{3.6}$$

where **H** is the magnetic field and p' is pressure.

If $\psi = \psi^*(\rho, B_i)$ then it is easily shown that $\psi = \psi(\rho, B)$ where $B = \surd(B_i \, B_i)$. Thus from (3.5),

$$H_i = \eta \, B_i \tag{3.7}$$

where

$$\eta = \frac{\rho}{B} \frac{\partial \psi}{\partial B} \, . \tag{3.8}$$

Define $$P = p' + \eta \, B^2 \, . \tag{3.9}$$

$$\therefore \quad \sigma_{ki} = - P \, \delta_{ki} + \eta \, B_k \, B_i \, . \tag{3.10}$$

Equation (3.3) becomes

$$\rho \dot{V}_i = \rho f_i + \frac{\partial \sigma_{ki}}{\partial \rho} \rho_{,k} + \frac{\partial \sigma_{ki}}{\partial B_j} B_{j,k} \tag{3.11}$$

where

$$\begin{aligned} \frac{\partial \sigma_{ki}}{\partial \rho} &= - P_\rho \delta_{ik} + \eta_\rho B_i B_k , \\ \frac{\partial \sigma_{ki}}{\partial B_j} &= - \frac{P_B}{B} B_j \delta_{ik} + \frac{\eta_B}{B} B_i B_j B_k + \eta (B_i \delta_{jk} + B_k \delta_{ij}) \end{aligned} \tag{3.12}$$

Suppose the perfect fluid occupies a region A of Euclidean space for all time. Equations (3.1)–(3.11) are all then assumed to hold on $A \times (-\infty,\infty)$. Further we suppose that ρ, V_i, B_i and f_i are continuous functions of $\mathbf{x}$,t on $A \times (-\infty,\infty)$ and that there is a surface $\Sigma \times (-\infty,\infty)$, such that for each $(\mathbf{x},t) \in \Sigma \times (-\infty,\infty)$ a unit normal; $\mathbf{n}$, to Σ is defined at $\mathbf{x}$, and the speed of Σ at $(\mathbf{x},t)$ is u_n in the direction of $\mathbf{n}$. The quantities $\dot{V}_i$, $V_{i,j}$, $\dot{B}_i$, $B_{i,j}$, $\dot{\rho}$, $\rho_{,i}$ are assumed to be continuous functions of $\mathbf{x}$,t on $(A - \Sigma) \times (-\infty,\infty)$ but may have jump discontinuities across Σ. Such discontinuities are called acceleration waves.

From Hadamard's lemma and the assumed differentiability of ρ, V and B, it follows that (see Truesdell & Toupin [15] and Eringen & Suhubi [16]),

$$[B_{i,j}] = a_i n_j \quad , \tag{3.13}$$

$$[\rho_{,j}] = b n_j \quad , \tag{3.14}$$

$$[V_{i,j}] = c_i n_j \tag{3.15}$$

where

$$a_i = [n_j B_{i,j}] \quad , \tag{3.16}$$

$$b = [n_j \rho_{,j}] \quad , \tag{3.17}$$

$$c_i = [n_j V_{i,j}] \quad . \tag{3.18}$$

For later convenience we note that the vectors a_i and c_i can be rewritten in terms of normal and tangential components using the decomposition

$$a_i = a_n n_i + a^\alpha x_{i,\alpha}$$
$$c_i = c_n n_i + c^\alpha x_{i,\alpha} \tag{3.19}$$

where

$$a_n = a_i n_i$$
$$a^\alpha = a^{\alpha\beta} a_i x_{i,\beta}$$
$$c_n = c_i n_i \tag{3.20}$$
$$c^\alpha = a^{\alpha\beta} c_i x_{i,\beta}$$

Since **B** is solenoidal then from equation (3.13)

$$0 = [B_{i,i}] = a_i n_i \tag{3.21}$$

$$\text{i.e.} \quad a_n = 0 \, . \tag{3.22}$$

Hence for an acceleration wave, the magnetic discontinuity is always transverse.

On taking the jump of equations (3.1), (3.2) and (3.11) over Σ,

$$[\dot{\rho}] + \rho [V_{i,j}] = 0 \quad , \tag{3.23}$$

$$[\dot{B}_i] = [V_{i,j}] B_j - [V_{j,j}] B_i \quad , \tag{3.24}$$

$$\rho [\dot{V}_i] - [\rho_{,k}] \frac{\partial \sigma_{ki}}{\partial \rho} - \frac{\partial \sigma_{ki}}{\partial B_j} [B_{j,k}] = 0 \quad . \tag{3.25}$$

From chapter two we have already shown that

$$[\dot{B}_i] = - U a_i \quad , \tag{3.26}$$

$$[\dot{\rho}] = - U b \quad , \tag{3.27}$$

$$[\dot{V}_i] = - U c_i \quad . \tag{3.28}$$

When we take (3.13)-(3.15) into (3.23)-(3.25) using (3.26),(3.27) and (3.28), we obtain

$$- U b + \rho c_n = 0 \quad , \tag{3.29}$$

$$- U a_i = c_i B_n - c_n B_i \quad , \tag{3.30}$$

$$- \rho U c_i - \frac{\partial \sigma_{ki}}{\partial \rho} b n_k - \frac{\partial \sigma_{ki}}{\partial B_j} a_j n_k = 0 \ . \tag{3.31}$$

Let us initially observe that (3.30) indicates that $a_n = 0$ i.e. it is consistent with the fact that div $\mathbf{B} = 0$.

By elimination of c_i from equations (3.29),(3.30) and (3.31), it can be shown that,

$$a_j \left(B_n \frac{\partial \sigma_{ki}}{\partial B_j} n_k - \rho U^2 \delta_{ij}\right) + b \left(B_n n_k \frac{\partial \sigma_{ki}}{\partial \rho} + U^2 B_i\right) = 0 \ . \tag{3.32}$$

When equation (3.32) is contracted with n_i and B_i we obtain respectively

$$b \left(U^2 - P_\rho + \eta_\rho B_n^2\right) + \Omega \left(\frac{\eta_B}{B} B_n^2 - \frac{P_B}{B}\right) = 0 \tag{3.33}$$

$$b \left(U^2 B^2 - P_\rho B_n^2 + \eta_\rho B^2 B_n^2\right) + \Omega \left(\eta B_n^2 + \eta_B B_n^2 B - \rho U^2 - \frac{P_B}{B} B_n^2\right) = 0 \tag{3.34}$$

where

$$\Omega = a_j B_j \ . \tag{3.35}$$

Equations (3.33) and (3.34) are required to have b and Ω non-zero and so

$$\begin{vmatrix} U^2 - P_\rho + \eta_\rho B_n^2 & \dfrac{\eta_B}{B} B_n^2 - \dfrac{P_B}{B} \\ U^2 B^2 - P_\rho B_n^2 + \eta_\rho B^2 B_n^2 & \eta B_n^2 + \eta_B B_n^2 B - \rho U^2 - \dfrac{P_B}{B} B_n^2 \end{vmatrix} = 0 \tag{3.36}$$

From (3.36) we find a fast wave and a slow wave whose speeds are determined by the quartic

$$\rho U^4 + U^2 [(\rho \eta_\rho - \eta) B_n^2 - (\rho P_\rho + \frac{P_B}{B} B_\perp^2)]$$
$$+ B_n^2 [P_\rho (\eta + \frac{\eta_B}{B} B_\perp^2) - \eta_\rho (\eta B_n^2 + \frac{P_B}{B} B_\perp^2)] = 0. \qquad (3.37)$$

This is the same equation derived by Straughan [3], who showed that the condition necessary for the wavespeeds to be positive is just the stability condition of Roberts [1]. We shall assume that (3.37) yields two real wavespeeds.

<u>Case (1)</u>

<u>U satisfies equation (3.36)</u>

In this case we find a fast wave and a slow wave whose speeds are determined by the quartic equation (3.37). Here all the primitive quantities have discontinuous derivatives. Also b and Ω are linearly related.

<u>Case (2)</u>

<u>U does not satisfy equation (3.36)</u>

In this case

(i) $b = 0$, i.e. first derivatives of density are continuous across the surface Σ.

(ii) $\Omega = 0$, i.e. **a** is normal to the plane containing **n** and **B**.

(iii) $c_n = 0$, i.e. the normal acceleration components are continuous across Σ.

The corresponding discontinuity equations are

$$c_n = 0 \quad , \qquad (3.38)$$

$$B_n c_i + U a_i = 0 \quad , \qquad (3.39)$$

$$- \rho U c_i - a_j n_k \frac{\partial \sigma_{ki}}{\partial B_j} = 0 \quad . \qquad (3.40)$$

From (3.39) and (3.40), we obtain

$$U^2 = \frac{\eta}{\rho} B_n^2 \ . \tag{3.41}$$

This result was also obtained by Straughan [3] and determines the velocity of Alfven waves.

Chapter Four

Ray Equations

Suppose that the initial position of the acceleration wavefront is

$$x_i = x_i(0,\theta^\alpha) \tag{4.1}$$

where θ^α are any acceptable pair of surface coordinates. We want to determine the location at time t of Σ, where Σ is a wavefront advancing with a known normal speed u_n and initially starting on the surface (4.1).

Theorem

The motion of any point $x_i(t,\theta^\alpha)$ on the singular surface is a solution of the ordinary differential equations

$$\frac{dx_i}{dt} = \frac{\partial u_n}{\partial n_i} \tag{4.2}$$

$$\frac{dn_i}{dt} = (n_i\, n_j - \delta_{ij})\,\frac{\partial u_n}{\partial x_j} \tag{4.3}$$

satisfying the initial conditions

$$x_i = x_i(0,\theta^\alpha) \quad , \quad n_i = n_i(0,\theta^\alpha) \tag{4.4}$$

where $n_i(t,\theta^\alpha)$ is the normal to the singular surface at $x_i(t,\theta^\alpha)$ and $n_i(0,\theta^\alpha)$ is the normal to the initial surface at $x_i(0,\theta^\alpha)$ (see Varley & Cumberbatch [8]).

Proof

To establish equations (4.2) and (4.3) let $\psi(\mathbf{x},t)$ be any continuous function whose first order partial derivatives are continuous and such that

$$\psi(\mathbf{x},t) = 0 \tag{4.5}$$

is the acceleration wavefront under consideration and $\partial\psi/\partial t \neq 0$ in some neighbourhood of $\psi = 0$. Suppose that $\mathbf{n}(\mathbf{x},t)$ is the unit normal to the surface $\psi(\mathbf{x},t)$ = constant, in its direction of propagation, and u_n is its speed.

If we differentiate equation (4.5) with respect to time, we obtain

$$\frac{\partial\psi}{\partial t} + \frac{\partial\psi}{\partial x_i}\frac{\partial x_i}{\partial t} = 0 \quad . \tag{4.6}$$

From equation (2.11), it follows that at fixed θ^α,

$$\frac{\partial\psi}{\partial t} + \frac{\partial\psi}{\partial x_i} u_n \, n_i = 0 \tag{4.7}$$

thus

$$\psi_t + |\nabla\psi| \, u_n = 0 \tag{4.8}$$

$$\text{i.e.} \quad u_n = - \psi_t \,/\, |\nabla\psi| \ . \tag{4.9}$$

Now $n_i = \psi_{,i} \,/\, |\nabla\psi|$ and if we use equation (4.9), this become

$$\frac{n_i}{u_n} = - \frac{\psi_{,i}}{\psi_t} \quad . \tag{4.10}$$

Now suppose that

$$\psi = \psi(\mathbf{x},t) = \text{constant} \tag{4.11}$$

where $t = t(\mathbf{x})$, then if we differentiate (4.11) with respect to time we obtain

$$\psi_{,i} + \psi_t \frac{\partial t}{\partial x_i} = 0 \ . \tag{4.12}$$

Thus on using (4.12), equation (4.10) yields

$$\frac{n_i}{u_n} = \frac{\partial t}{\partial x_i} \ . \tag{4.13}$$

Let us suppose that $u_n = u_n(\mathbf{n}, \mathbf{x}, t)$ has the property that for any scalar C

$$C \, u_n = u_n(C \, \mathbf{n}, \mathbf{x}, t) \tag{4.14}$$

i.e. u_n is homogenous of degree one in $\mathbf{n}$. In particular if $C = u_n^{-1}$ then

$$1 = u_n(\frac{\mathbf{n}}{u_n}, \mathbf{x}, t) \ . \tag{4.15}$$

After differentiating (4.15) with respect to x_i and using result (4.13),

$$\frac{\partial u_n}{\partial\left(\frac{n_j}{u_n}\right)} \frac{\partial\left(\frac{n_j}{u_n}\right)}{\partial x_i} + \frac{\partial u_n}{\partial x_i} + \frac{n_i}{u_n} \frac{\partial u_n}{\partial t} = 0 \quad . \tag{4.16}$$

From (4.13)

$$\frac{\partial^2 t}{\partial x_i \partial x_j} = \frac{\partial}{\partial x_i}\left(\frac{n_j}{u_n}\right) = \frac{\partial}{\partial x_j}\left(\frac{n_i}{u_n}\right) . \tag{4.17}$$

Thus along any curve $\mathbf{x} = \mathbf{x}(s)$ given by

$$\frac{dx_i}{ds} = \frac{\partial u_n}{\partial\left(\frac{n_i}{u_n}\right)} \tag{4.18}$$

we can show from (4.16)-(4.18) that

$$\frac{d}{ds}\left(\frac{n_i}{u_n}\right) = -\left(\frac{\partial u_n}{\partial x_i} + \frac{n_i}{u_n} \frac{\partial u_n}{\partial t}\right) \tag{4.19}$$

along such a curve $t = t(s)$ varies so that

$$\frac{dt}{ds} = \frac{\partial t}{\partial x_i} \frac{dx_i}{ds} = \frac{n_i}{u_n} \frac{\partial u_n}{\partial\left(\frac{n_i}{u_n}\right)} = 1 \ . \tag{4.20}$$

In terms of $u_n(\mathbf{n}, \mathbf{x}, t)$ equations (4.18) and (4.20) imply that

$$\frac{dx_i}{dt} = \frac{\partial u_n}{\partial n_i} \tag{4.21}$$

which is the first of the required equations .

Further from (4.19)

$$u_n \frac{d}{dt}\left(\frac{n_i}{u_n}\right) = -\left(\frac{\partial u_n}{\partial x_i} + \frac{n_i}{u_n} \frac{\partial u_n}{\partial t}\right) \tag{4.22}$$

and hence it follows that

$$\frac{dn_i}{dt} = \frac{n_i}{u_n}\left(\frac{du_n}{dt} - \frac{\partial u_n}{dt}\right) - \frac{\partial u_n}{\partial x_i} \quad . \tag{4.23}$$

However, $\mathbf{n}$ is a unit vector and so from (4.23) we may conclude that

$$\frac{du_n}{dt} = \frac{\partial u_n}{\partial t} + u_n n_i \frac{\partial u_n}{\partial x_i} \tag{4.24}$$

from which we may eventually deduce the second of our required equations, namely

$$\frac{dn_i}{dt} = (n_i\, n_j - \delta_{ij})\ \frac{\partial u_n}{\partial x_j} \qquad (4.25)$$

Special Case

Ray Equations For $u_n = V_n + U(\mathbf{x}, B_n)$

Here we suppose that $u_n = V_i\, n_i + U(\mathbf{x}, B_i\, n_i)$. In order to make u_n homogeneous of degree one as is required by the above theorem, we rewrite u_n in the form

$$u_n = V_i\, n_i + U(x, \frac{B_i\, n_i}{\sqrt{(n_r\, n_r)}})\sqrt{(n_r\, n_r)}\ . \qquad (4.26)$$

When equation (4.26) for u_n is differentiated with respect to n_i, we produce

$$\frac{\partial u_n}{\partial n_i} = V_i + U(\frac{1}{2}\ \frac{2n_i}{\sqrt{(n_r\, n_r)}}) + $$

$$+ \sqrt{(n_r\, n_r)}\ \frac{\partial U}{\partial B_n}\ \frac{\partial}{\partial n_i}(\frac{B_i\, n_i}{\sqrt{(n_r\, n_r)}})$$

$$= V_i + U\, n_i + \frac{\partial U}{\partial B_n}\ [\ \frac{B_i}{\sqrt{(n_r\, n_r)}} - \frac{B_n\, n_i}{(\sqrt{(n_r\, n_r)})^3}]$$

$$= V_i + U\, n_i + \frac{\partial U}{\partial B_n}\ [B_i - B_n\, n_i]\ .$$

However $B_i - B_n\, n_i = B^\alpha\, x_{i,\alpha}$ and so the first ray equation in this case finally becomes

$$\dot{x}_i = V_i + U\, n_i + \frac{\partial U}{\partial B_n}\ B^\alpha\, x_{i,\alpha}\ . \qquad (4.27)$$

Differentiation along rays

For any function $\Phi(\mathbf{x}, t)$

$$\frac{d\Phi}{dt} = \frac{\partial \Phi}{\partial t} + \frac{\partial \Phi}{\partial x_i}\ \frac{dx_i}{dt}\ . \qquad (4.28)$$

When the value of $\dot{x}_i$ is substituted in (4.28), we obtain

$$\frac{d\Phi}{dt} = \frac{\partial \Phi}{\partial t} + \Phi_{,i}\ (\ V_i + U\, n_i + \frac{\partial U}{\partial B_n}\ B^\alpha\, x_{i,\alpha})\ . \qquad (4.29)$$

If we introduce the definition

$$L(\Phi) = \frac{\delta\Phi}{\delta t} + V^{\alpha}\,\Phi_{,\alpha} \tag{4.30}$$

then equation (4.29) becomes

$$\frac{d\Phi}{dt} = L(\Phi) + \frac{\partial U}{\partial B_n}\,B^{\alpha}\,\Phi_{,\alpha} \tag{4.31}$$

Ray equations for Alfven waves

In this case $U^2 = \dfrac{\eta}{\rho}\,B_n{}^2$ and hence the ray equations become

$$\dot{x}_i = \frac{\partial u_n}{\partial n_i} = V_i + \surd(\eta/\rho)\,B_i \tag{4.32}$$

$$\dot{n}_i = (n_i\,n_j - \delta_{ij})\,n_r\,(V_r + \surd(\eta/\rho)\,B_r)_{,j}\ . \tag{4.33}$$

Also if Φ is any function defined along the ray then we may show that

$$\frac{d\Phi}{dt} = L(\Phi) + \surd(\eta/\rho)\,B^{\alpha}\,\Phi_{,\alpha}\ . \tag{4.34}$$

Chapter Five

The amplitude equations

In this chapter we derive the amplitude equations of the electromagnetic acceleration wave. When equations (3.1), (3.2) and (3.11) are differentiated materially with respect to time, we obtain

$$\ddot{\rho} + \dot{\rho}\, V_{i,i} + \rho\, \dot{\overline{V_{i,i}}} = 0 \quad , \qquad (5.1)$$

$$\ddot{B}_i = V_{i,j}\, \dot{B}_j + \dot{\overline{V_{i,j}}}\, B_j - \dot{\overline{V_{j,j}}}\, B_i - V_{j,j}\, \dot{B}_i \quad , \qquad (5.2)$$

$$\rho\, \ddot{V}_i + \dot{\rho}\, \dot{V}_i - \dot{\rho}\, f_i - \rho\, \dot{f}_i - \frac{\partial \sigma_{ki}}{\partial \rho}\, \dot{\overline{\rho_{,k}}} - \rho_{,k}\, \dot{\rho}\, \frac{\partial^2 \sigma_{ki}}{\partial \rho^2}$$

$$- \rho_{,k}\, \dot{B}_j\, \frac{\partial^2 \sigma_{ki}}{\partial B_j \partial \rho} - \frac{\partial \sigma_{ki}}{\partial B_j}\, \dot{\overline{B_{j,k}}} - \frac{\partial^2 \sigma_{ki}}{\partial \rho \partial B_j}\, \dot{\rho}\, B_{j,k}$$

$$- \frac{\partial^2 \sigma_{ki}}{\partial B_j \partial B_r}\, \dot{B}_r\, B_{j,k} = 0 \quad . \qquad (5.3)$$

If we take jumps of equations (5.1), (5.2) and (5.3) then we produce

$$[\ddot{\rho}] + [\dot{\rho}\, V_{i,i}] + \rho\, [\dot{\overline{V_{i,i}}}] = 0 \quad , \qquad (5.4)$$

$$[\ddot{B}_i] = [V_{i,j}\, \dot{B}_j] + B_j\, [\dot{\overline{V_{i,j}}}] - B_i\, [\dot{\overline{V_{j,j}}}] - [V_{j,j}\, \dot{B}_i] \qquad (5.5)$$

$$\rho\, [\ddot{V}_i] + [\dot{\rho}\, \dot{V}_i] - f_i\, [\dot{\rho}] - \frac{\partial \sigma_{ki}}{\partial \rho}\, [\dot{\overline{\rho_{,k}}}] - [\rho_{,k}\, \dot{\rho}]\, \frac{\partial^2 \sigma_{ki}}{\partial \rho^2}$$

$$- [\rho_{,k}\, \dot{B}_j]\, \frac{\partial^2 \sigma_{ki}}{\partial B_j \partial \rho} - \frac{\partial \sigma_{ki}}{\partial B_j}\, [\dot{\overline{B_{j,k}}}] - [\dot{\rho}\, B_{j,k}]\, \frac{\partial^2 \sigma_{ki}}{\partial \rho \partial B_j}$$

$$- \frac{\partial^2 \sigma_{ki}}{\partial B_j \partial B_r}\, [\dot{B}_r\, B_{j,k}] = 0 \quad . \qquad (5.6)$$

From chapter two we have already shown that

$$[\ddot{B}_i] = -\,L(U\, a_i) + U\, F_i + U\, a_i\, c_n - U\, c_k\, (B_{i,k})^+ \,, \tag{5.7}$$

$$[\overline{\dot{B}_{i,j}}] = -\,F_i\, n_j - a^{\alpha\beta}\,(U\, a_i)_{,\beta}\, x_{j,\alpha} - a_i\, n_k\,(V_{k,j})^+ \,, \tag{5.8}$$

$$[\ddot{\rho}] = U\, E - L(U\, b) + U\, b\, c_n - U\, c_s\,(\rho_{,s})^+ \quad , \tag{5.9}$$

$$[\overline{\dot{\rho}_{,k}}] = -\,E\, n_k - b\, n_s\,(V_{s,k})^+ - a^{\alpha\beta}\,(U\, b)_{,\beta}\, x_{k,\alpha} \quad , \tag{5.10}$$

$$[\ddot{V}_i] = U\, S_i - L(U\, c_i) + U\, c_n\, c_i - U\, c_s\,(V_{i,s})^+ \quad , \tag{5.11}$$

$$[\overline{\dot{V}_{i,j}}] = -\,S_i\, n_j - a^{\alpha\beta}\,(U\, c_i)_{,\beta}\, x_{j,\alpha} - c_i\, n_s\,(V_{s,j})^+ \,. \tag{5.12}$$

With the aid of these expressions, the jump equations (5.4), (5.5) and (5.6) become

$$U\, E - \rho\, S_n + X = 0 \tag{5.13}$$

$$U\, F_i + B_n\, S_i - S_n\, B_i + Y_i = 0 \tag{5.14}$$

$$\rho\, U\, S_i + E\, n_k\, \frac{\partial \sigma_{ki}}{\partial \rho} + F_j\, n_k\, \frac{\partial \sigma_{ki}}{\partial B_j} + Z_i = 0 \tag{5.15}$$

where

$$X = -\,L(U\, b) + 2\, U\, b\, c_n - U\, c_s\,(\rho_{,s})^+ - \rho\, c_i\, n_s\,(V_{s,i})^+$$
$$- 2\, U\, b\,(V_{i,i})^+ - \rho\, a^{\alpha\beta}\,(U\, c_i)_{,\beta}\, x_{i,\alpha} \quad , \tag{5.16}$$

$$Y_i = -\,L(U\, a_i) + 2\, c_n\, U\, a_i - B_i\, c_j\, n_s\,(V_{s,j})^+ - U\, c_s\,(B_{i,s})^+$$
$$+ U\, a_j\,(V_{i,j})^+ - 2\, U\, a_i\,(V_{j,j})^+ + c_n\, B_j\,(V_{i,j})^+$$
$$- B_i\, a^{\alpha\beta}\,(U\, c_j)_{,\beta}\, x_{j,\alpha} + B^{\beta}\,(U\, c_i)_{,\beta} \quad , \tag{5.17}$$

$$
\begin{aligned}
Z_i = & - \rho\, L(U\, c_i) + a^{\alpha\beta}\, (U\, b)_{,\beta}\, x_{k,\alpha} \frac{\partial \sigma_{ki}}{\partial \rho} - U\, b^2\, n_k \frac{\partial^2 \sigma_{ki}}{\partial \rho^2} \\
& - 2\, U\, b\, a_j\, n_k \frac{\partial^2 \sigma_{ki}}{\partial \rho \partial B_j} + a^{\alpha\beta}\, (U\, a_j)_{,\beta}\, x_{k,\alpha} \frac{\partial \sigma_{ki}}{\partial B_j} \\
& - \frac{1}{\rho}\, U\, b\, (B_{j,k})^+ \frac{\partial \sigma_{ki}}{\partial B_j} + U\, a_r\, (B_{j,k})^+ \frac{\partial^2 \sigma_{ki}}{\partial B_j \partial B_r} \\
& - U\, a_r\, a_j\, n_k \frac{\partial^2 \sigma_{ki}}{\partial B_j \partial B_r} + U\, b\, (B_{j,k})^+ \frac{\partial^2 \sigma_{ki}}{\partial \rho \partial B_j} \\
& - (\rho_{,k})^+ \left(\frac{1}{\rho}\, U\, b \frac{\partial \sigma_{ki}}{\partial \rho} - U\, b \frac{\partial^2 \sigma_{ki}}{\partial \rho^2} - U\, a_j \frac{\partial^2 \sigma_{ki}}{\partial B_j \partial \rho} \right) \\
& + (V_{s,s})^+ \left(\rho\, b\, n_k \frac{\partial^2 \sigma_{ki}}{\partial \rho^2} + b\, n_k\, B_j \frac{\partial^2 \sigma_{ki}}{\partial B_j \partial \rho} + \rho\, U\, c_i \right. \\
& \left. + a_j\, n_k\, B_r \frac{\partial^2 \sigma_{ki}}{\partial B_j \partial B_r} + \rho\, a_j\, n_k \frac{\partial^2 \sigma_{ki}}{\partial \rho \partial B_i} \right) - \rho\, U\, c_s\, (V_{i,s})^+ \\
& + b\, n_s \frac{\partial \sigma_{ki}}{\partial \rho} (V_{s,k})^+ - b\, n_k\, B_s\, (V_{j,s})^+ \frac{\partial^2 \sigma_{ki}}{\partial B_j \partial \rho} \\
& + a_j\, n_r\, (V_{r,k})^+ \frac{\partial \sigma_{ki}}{\partial B_j} - a_j\, n_k\, B_s\, (V_{r,s})^+ \frac{\partial^2 \sigma_{ki}}{\partial B_j \partial B_r} \; .
\end{aligned}
\tag{5.18}
$$

Equations (5.13), (5.14) and (5.15) are three differential equations to be solved for the six unknown quantities b, a_i and c_i. To do this, however, we first need to remove the terms involving E, F_i and S_i. It can be shown with the aid of (3.12) that

$$
\begin{aligned}
& E\, (U^2 + \eta_\rho\, B_n^2 - P_\rho)\, B_i + E\, P_\rho\, B^\alpha\, x_{i,\alpha} + F_i\, (\eta\, B_n^2 - \rho\, U^2) \\
& + F_n\, B_n\, \eta\, B_i + B_n\, \Omega^*\, \left(\eta_B \frac{B_n\, B_i}{B} - n_i \frac{P_B}{B} \right) = Q_i
\end{aligned}
\tag{5.19}
$$

where

$$Q_i = B_n\, Z_i - \rho\, U\, Y_i + U\, X\, B_i \tag{5.20}$$

$$\Omega^* = F_i\, B_i \; . \tag{5.21}$$

Case (1)

Here U satisfies equation (3.36).

Define

$$\zeta = U^2 - P_\rho + \eta_\rho B_n^2 \quad , \tag{5.22}$$

$$\xi = \rho U^2 - \eta B_n^2 - \frac{P_B}{B} B_\perp^2 \quad , \tag{5.23}$$

$$\mu = \frac{1}{B} (\eta_B B_n^2 - P_B) \, . \tag{5.24}$$

From (3.37) it is clear that

$$\zeta \, \xi + \mu \, P_\rho \, B_\perp^2 = 0 \, . \tag{5.25}$$

When equation (5.19) is contracted with n_i and B_i we obtain respectively

$$\zeta \, E + \mu \, \Omega^* = \frac{1}{B_n} [\, Q_n + F_n \, (\rho \, U^2 - 2 \, \eta \, B_n^2)] \quad , \tag{5.26}$$

$$E \, P_\rho \, B_\perp^2 - \xi \, \Omega^* = - \frac{Q_n}{B_n} B_\perp^2 + Q^\alpha B_\alpha + F_n \frac{B^2}{B_n} (\eta \, B_n^2 - \rho \, U^2) \, . \tag{5.27}$$

If we eliminate Ω^* from (5.26) and (5.27) and use equation (5.25), we find that

$$\begin{aligned} - B_\perp^2 \, (U^2 + \eta_\rho \, B_n^2)[Q_n - F_n \, (2 \, \eta \, B_n^2 - \rho \, U^2)] & \\ + \zeta \, B_n \, [Q^\alpha B_\alpha + B_n \, F_n \, (\eta \, B_n^2 - \eta \, B_\perp^2 - \rho \, U^2)] = 0. & \end{aligned} \tag{5.28}$$

This equation describes the amplitude of the acceleration wave.

It is clear that the amplitude equation contains b, a_i and c_i and their derivatives. However we can eliminate both a_i and c_i to obtain a partial differential equation for b alone as follows :-

(i) From equation (3.29) we see that $c_n = Ub/\rho$.

(ii) Since $a_n = 0$, equation (3.30) and (3.31) have form

$$U \, a_\alpha + B_n \, c_\alpha = B_\alpha \frac{U \, b}{\rho} \quad , \tag{5.29}$$

$$\rho \, U \, c^\beta \, x_{i,\beta} + n_k \, a^\beta \, x_{j,\beta} \frac{\partial \sigma_{ki}}{\partial B_j} = - \rho \, U \, c_n \, n_i - b \, n_k \frac{\partial \sigma_{ki}}{\partial \rho} \, . \tag{5.30}$$

When equation (5.30) is multiplied by $a^{\alpha\gamma}\, x_{i,\gamma}$; we obtain

$$\rho\ U\ c^{\alpha} + n_k\ a^{\beta}\ a^{\alpha\gamma}\ x_{j,\beta}\ x_{i,\gamma}\ \frac{\partial\sigma_{ki}}{\partial B_j} = -\ b\ n_k\ a^{\alpha\gamma}\ x_{i,\gamma}\ \frac{\partial\sigma_{ki}}{\partial\rho}\quad . \tag{5.31}$$

However,

$$n_k\ a^{\beta}\ a^{\alpha\gamma}\ x_{j,\beta}\ x_{i,\gamma}\ \frac{\partial\sigma_{ki}}{\partial B_j} = B_n\ a^{\beta}\ B_{\beta}\ B^{\alpha}\ \frac{\eta_B}{B} + \eta\ B_n\ a^{\alpha} \tag{5.32}$$

thus equation (5.31) becomes

$$\rho\ U\ c^{\alpha} + B_n\ a^{\beta}\ B_{\beta}\ B^{\alpha}\ \frac{\eta_B}{B} + \eta\ B_n\ a^{\alpha} = -\ b\ n_k\ a^{\alpha\gamma}\ x_{i,\gamma}\ \frac{\partial\sigma_{ki}}{\partial\rho}\quad . \tag{5.33}$$

Now (5.29) and (5.33) represent four equations in the four unknowns c_1, c_2, a_1 and a_2. Thus if the determinant of equations (5.29) and (5.33) is non-zero then we must solve these equations for a_1, a_2, c_1 and c_2 in terms of b. The required determinant is

$$\begin{vmatrix} U & 0 & B_n & 0 \\ 0 & U & 0 & B_n \\ \eta\ B_n + B_n\ B_1 B^1\ \dfrac{\eta_B}{B} & B_n\ B^1\ B_2\ \dfrac{\eta_B}{B} & \rho\ U & 0 \\ B_n\ B_1\ B^2\ \dfrac{\eta_B}{B} & \eta\ B_n + B_n\ B_2 B^2\ \dfrac{\eta_B}{B} & 0 & \rho\ U \end{vmatrix} \tag{5.34}$$

and after further algebra, this determinant has value

$$(\rho\ U^2 - \eta\ B_n{}^2)(\rho\ U^2 - \eta\ B_n{}^2 - \frac{\eta_B}{B}\ B_n{}^2\ B_{\alpha}\ B^{\alpha})\quad . \tag{5.35}$$

We require expression (5.35) to be non-zero. Since we are dealing with case (1) in which $\rho\ U^2 - \eta\ B_n{}^2 \neq 0$ then we require to show that

$$U^2 = \frac{B_n{}^2}{\rho}\ (\eta + \frac{\eta_B}{B}\ B_1{}^2) \tag{5.36}$$

cannot be a solution of equation (3.37). Suppose the contrary then

$$\frac{B_n^2\ B_\perp^2\ \eta_B}{B\ \rho}\ [\frac{\eta_B}{B}\ B_n^2\ B_\perp^2 + (\rho\ \eta_\rho + \eta)\ B_n^2 - \rho\ P_\rho - \frac{P_B}{B}\ B_\perp^2]$$

$$+ \frac{B_n^2}{\rho\ B}\ B_\perp^2\ \frac{\partial(P,\rho\eta)}{\partial(\rho,B)} \equiv 0$$

$$\therefore\quad \frac{B_n^2\ B_\perp^2}{B\ \rho}\ [\frac{\eta_B^2\ B_n^2\ B_\perp^2}{B} + \eta_B\ B_n^2\ (\rho\eta)_\rho - \rho\ P_\rho\ \eta_B - \eta_B\ B_\perp^2\ \frac{P_B}{B}$$

$$+ P_\rho\ (\rho\eta)_B - (\rho\eta)_\rho\ P_B] \equiv 0$$

$$\therefore\quad \frac{B_n^2\ B_\perp^2}{\rho\ B^2}\ [\eta_B\ B_\perp^2 + B\ (\rho\eta)_\rho][\eta_B\ B_n^2 - P_B] \equiv 0$$

$$\therefore\quad -\frac{B_n^2\ B_\perp^2}{\rho\ B^2}\ [\eta_B\ B_\perp^2 + B\ (\rho\eta)_\rho]^2 \equiv 0 \qquad (5.37)$$

where we have used the fact that $\eta_B\ B_n^2 - P_B = -\ [\eta_B\ B_\perp^2 + B\ (\rho\eta)_\rho]$ We may show that we cannot find $\eta(\rho,B)$ such that p^*, p^*_ρ are always non-negative and for which $\eta_B\ B_\perp^2 + B\ (\rho\eta)_\rho \equiv 0$ and hence we may conclude that determinant (5.34) is non-singular.

We can determine the form of the amplitude equation (5.28), but instead we shall determine the coefficients of $L(b)$, $b_{,\beta}$, b^2, $b\ b_\alpha^{\ \alpha}$, $b\ (V_{s,s})^+$, $b\ (V_{r,s})^+$, $b\ (\rho_{,k})^+$ and b.

Coefficient of $L(b)$ in amplitude equation is

$$-\ 2\ U^2\ B_n\ B_\perp^2\ \frac{P_\rho}{\xi}\ (\xi + \rho\ \zeta)\ . \qquad (5.38)$$

Coefficient of $b_{,\beta}$ in amplitude equation is

$$-\ B_\perp^2\ (U^2 + \eta_\rho\ B_n^2)\ U\ B^\beta\ \frac{P_\rho}{\xi}\ [\frac{\xi}{P_\rho}\ (U^2 - \eta_\rho\ B_n^2) + \eta\ B_n^2 - \rho\ U^2$$

$$-\frac{\eta_B}{B}\ B_n^2\ B_\perp^2] + \zeta\ U\ B_n\ B^\beta\ \frac{P_\rho}{\xi}\ [2\ \rho\ U^2\ B_n - 2\ \eta\ B_n^3 - \eta\ B_n\ B_\perp^2$$

$$-\frac{\eta_B}{B}\ B_n\ B_\perp^4 + \frac{B_\perp^2}{B_n}\ \{\frac{\xi}{P_\rho}\ (U^2 - \eta_\rho\ B_n^2) - \rho\ U^2\}]. \qquad (5.39)$$

After a long calculation it can be shown that

$$\frac{\text{Coefficients of } b_{,\beta}}{\text{Coefficents of } L(b)} = \frac{\partial U}{\partial B_n} B^\beta . \tag{5.40}$$

If we use equation (4.31), then the amplitude equation has form

$$\frac{db}{dt} + A_1 (V_{s,s})^+ b + A_{rs} (V_{r,s})^+ b + A_3 b_\alpha{}^\alpha b + A_k (\rho_{,k})^+ b$$
$$+ A_5 b + A_6 b^2 = 0 \tag{5.41}$$

where

$$K A_1 = [(\zeta B_n B^\alpha x_{i,\alpha} - B_\perp{}^2 n_i (U^2 + \eta_\rho B_n{}^2)] [2 U^2 B_i$$
$$- \rho B_n n_k \frac{\partial^2 \sigma_{ki}}{\partial \rho^2} - B_n n_k B_j \frac{\partial^2 \sigma_{ki}}{\partial \rho \partial B_j} - \rho B_n n_k B^\alpha x_{j,\alpha} \frac{P_\rho}{\xi} \frac{\partial^2 \sigma_{ki}}{\partial \rho \partial B_j}$$
$$- \rho U^2 B_n \{\frac{(\xi - \rho P_\rho) B_i}{\rho B_n \xi} + \frac{P_\rho n_i}{\xi}\} - 2 \rho U^2 \frac{P_\rho}{\xi} B^\alpha x_{i,\alpha}$$
$$- B_n n_k B_r B^\alpha x_{j,\alpha} \frac{P_\rho}{\xi} \frac{\partial^2 \sigma_{ki}}{\partial B_j \partial B_r}] \quad , \tag{5.42}$$

$$K A_{rs} = [\zeta B_n B^\alpha x_{i,\alpha} - B_\perp{}^2 n_i (U^2 + \eta_\rho B_n{}^2)] [\rho U^2 B^\alpha \frac{P_\rho}{\xi} x_{s,\alpha} \delta_{ir}$$
$$+ U^2 B_s \delta_{ir} + B_n n_k B_s \frac{\partial^2 \sigma_{ki}}{\partial \rho \partial B_r} + B_n n_k B_s B^\alpha x_{j,\alpha} \frac{P_\rho}{\xi} \frac{\partial^2 \sigma_{ki}}{\partial B_j \partial B_r}$$
$$- B_n n_r \frac{\partial \sigma_{si}}{\partial \rho} + \rho B_n U^2 \{\frac{(\xi - \rho P_\rho)}{\rho B_n \xi} B_s + \frac{P_\rho n_s}{\xi}\} \delta_{ir}$$
$$- B_n n_r B^\alpha x_{j,\alpha} \frac{P_\rho}{\xi} \frac{\partial \sigma_{si}}{\partial B_j}] - [B_\perp{}^2 (U^2 + \eta_\rho B_n{}^2)$$
$$(2 \eta B_n{}^2 - \rho U^2) + \zeta B_n{}^2 (\eta B_n{}^2 - \eta B_\perp{}^2$$
$$- \rho U^2)] (B^\alpha x_{s,\alpha} n_r \frac{P_\rho}{\xi}) \quad , \tag{5.43}$$

$$K A_3 = (B_n n_j B^\beta x_{k,\beta} U \frac{P_\rho}{\xi} \frac{\partial \sigma_{ki}}{\partial B_j}) [B_\perp{}^2 n_i (U^2 + \eta_\rho B_n{}^2)$$
$$- \zeta B_n B^\alpha x_{i,\alpha}] \quad , \tag{5.44}$$

$$K\,A_k = [\zeta\,B_n\,B^\alpha\,x_{i,\alpha} - B_\perp{}^2\,n_i\,(U^2 + \eta_\rho\,B_n{}^2)]\,[\frac{U\,B_n}{\rho}\,\frac{\partial\sigma_{ki}}{\partial\rho}$$

$$+ U^3\,B_i\,\{\frac{(\xi - \rho\,P\rho)}{\rho\,\xi\,B_n}\,B_k + \frac{P\rho\,n_k}{\xi}\} - U\,B_n\,B^\alpha\,x_{j,\alpha}\,\frac{P_\rho}{\xi}\,\frac{\partial^2\sigma_{ki}}{\partial\rho\partial B_j}$$

$$- U\,B_n\,\frac{\partial^2\sigma_{ki}}{\partial\rho^2}] \quad , \tag{5.45}$$

$$K\,A_5 = [\zeta\,B_n\,B^\alpha\,x_{i,\alpha} - B_\perp{}^2\,n_i\,(U^2 + \eta\rho\,B_n{}^2)]\,[\rho\,B_n\,L[U^2\,\{\frac{P_\rho\,n_i}{\xi}$$

$$+ \frac{(\xi - \rho\,P_\rho)}{\rho\,\xi\,B_n}\,B_i\}] - B_n\,a^{\alpha\beta}\,x_{k,\alpha}\,(U\,\frac{P_\rho}{\xi}\,B^\gamma)_{,\beta}\,x_{j,\gamma}\,\frac{\partial\sigma_{ki}}{\partial B_j}$$

$$- B_n\,a^{\alpha\beta}\,U_{,\beta}\,x_{k,\alpha}\,\frac{\partial\sigma_{ki}}{\partial\rho} - U\,B_n\,B^\alpha\,x_{r,\alpha}\,(B_{j,k})^+\,\frac{P_\rho}{\xi}\,\frac{\partial^2\sigma_{ki}}{\partial B_j\partial B_r}$$

$$+ \frac{1}{\rho}\,U\,B_n\,(B_{j,k})^+\,\frac{\partial\sigma_{ki}}{\partial B_j} + \rho\,U\,B^\beta\,\{U^2\,\frac{(\xi - \rho\,P_\rho)}{\rho\,B_n\,\xi}\,B_i\,\frac{P_\rho n_i}{\xi}\}_{,\beta}$$

$$- U\,B_n\,(B_{j,k})^+\,\frac{\partial^2\sigma_{ki}}{\partial\rho\partial B_j} - \rho\,U\,L(\frac{U\,P_\rho}{\xi}\,B^\alpha\,x_{i,\alpha}) + U\,B_i\,L(U)$$

$$- \rho\,U^3\,(B_{i,s})^+\,\{\frac{(\xi - \rho\,P\rho)}{\rho\,B_n\,\xi}\,B_s + \frac{P\rho\,n_s}{\xi}\}]$$

$$+ [B_\perp{}^2\,(U^2 + \eta_\rho\,B_n{}^2)(2\,\eta\,B_n{}^2 - \rho\,U^2) + \zeta\,B_n{}^2\,(\eta\,B_n{}^2$$

$$- \eta\,B_\perp{}^2 - \rho\,U^2)]\,[U\,n_i\,(B_{i,s})^+\,\{\frac{(\xi - \rho\,P_\rho)}{\rho\,B_n\,\xi}\,B_s + \frac{P_\rho\,n_s}{\xi}\}$$

$$- \frac{B^\beta}{U}\,(\frac{U^2}{\rho})_{,\beta} + B^\beta\,n_{i,\beta}\,U\,\{\frac{(\xi - \rho\,P\rho)}{\rho\,B_n\,\xi}\,B_i + \frac{P\rho\,n_i}{\xi}\}$$

$$+ \frac{B_n}{U}\,a^{\alpha\beta}\,\{U^2\,\frac{(\xi - \rho\,P\rho)}{\rho\,B_n\,\xi}\,B_\alpha\}_{,\beta} - B^\alpha\,x_{i,\alpha}\,L(n_i)\,\frac{P_\rho}{\xi}] \quad , \tag{5.46}$$

$$K\,A_6 = [\zeta\,B_n\,B^\alpha\,x_{i,\alpha} - B_\perp{}^2\,n_i\,(U^2 + \eta_\rho\,B_n{}^2)]\,[U\,B_n\,n_k\,\frac{\partial^2\sigma_{ki}}{\partial\rho^2}$$

$$+ 2\,U\,B_n\,n_k\,B^\alpha\,x_{j,\alpha}\,\frac{P\rho}{\xi}\,\frac{\partial^2\sigma_{ki}}{\partial\rho\partial B_j} - \frac{2}{\rho}\,U^3\,B_i + 2\,U^3\,B^\alpha\,x_{i,\alpha}\,\frac{P_\rho}{\xi}$$

$$+ U\,B_n\,n_k\,B^\alpha\,x_{r,\alpha}\,B^\beta\,x_{j,\beta}\,\frac{P\rho^2}{\xi^2}\,\frac{\partial^2\sigma_{ki}}{\partial B_j\partial B_r}\,] \tag{5.47}$$

and where

$$\kappa = -\,2\,U^2\,B_n\,B_\perp{}^2\,\frac{P_\rho}{\xi}\,(\xi + \rho\,\zeta)\,. \tag{5.48}$$

Thus the solution of the amplitude equation (5.41) has form

$$b = \frac{b_0\,\exp(-\int_0^t \alpha\,dt)}{1 + b_0\int_0^t A_6\,\exp(-\int_0^s \alpha\,ds)\,dt} \tag{5.49}$$

where

$$\alpha = A_1\,(V_{s,s})^+ + A_{rs}\,(V_{r,s})^+ + A_3\,b_\alpha{}^\alpha + A_k\,(\rho_{,k})^+ + A_5$$

<u>Case (2)</u>

Here we are considering the situation of Alfven waves in which case

$$b = \Omega = 0 \qquad , \qquad U^2 = \frac{\eta}{\rho}\,B_n{}^2\,.$$

Thus in this case equation (5.19) becomes

$$E\,(U^2 + \eta_\rho\,B_n{}^2 - P_\rho)\,B_i + E\,P_\rho\,B^\alpha\,x_{i,\alpha} + \eta\,F_n\,B_n\,B_i$$
$$+\,\Omega^*\,B_i\,(\frac{\eta_B}{B}\,B_n{}^2 - \frac{P_B}{B}) + \frac{P_B}{B}\,\Omega^*\,B^\alpha\,x_{i,\alpha} = Q_i\,. \tag{5.50}$$

When equation (5.50) is contracted with n_i, we obtain

$$Q_n = (E\,\zeta + \mu\,\Omega^* + \eta\,F_n\,B_n{}^2)\,B_n. \tag{5.51}$$

If ζ is now eliminated between equations (5.50) and (5.51) then

$$Q_i = Q_n\,\frac{B_i}{B_n} + (E\,P_\rho + \Omega^*\,\frac{P_B}{B})\,B^\alpha\,x_{i,\alpha} \tag{5.52}$$

from which we can readily deduce that

$$Q^\alpha = \frac{Q^\beta\,B_\beta\,B^\alpha}{B_\perp{}^2}\,. \tag{5.53}$$

i.e. Q^α and B^α are parallel. This equation describes the amplitude of the Alfven wave.

When (5.16), (5.17) and (5.18) is taken into (5.20), we obtain

$$Q_i = \rho\, B_n\, L(U\, c_i) - \rho\, U\, L(a_i) - B_n\, a^{\alpha\beta}\, x_{k,\alpha}\, (U\, a_j)_{,\beta} \frac{\partial \sigma_{ki}}{\partial B_j}$$

$$- \rho\, U^2\, c_s\, (B_{i,s})^+ + B_n\, U\, a_r \frac{\partial^2 \sigma_{ki}}{\partial B_j \partial B_r} (a_j\, n_k - (B_{j,k})^+)$$

$$+ \rho\, U\, B^\beta\, (U\, c_i)_{,\beta} + (\rho_{,k})^+ [U^2\, B_i\, c_k - B_n\, U\, a_j \frac{\partial^2 \sigma_{ki}}{\partial \rho \partial B_j}]$$

$$- (V_{s,s})^+ [B_n\, a_j\, n_k\, B_r \frac{\partial^2 \sigma_{ki}}{\partial B_j \partial B_r} + \rho\, B_n\, n_k\, a_j \frac{\partial^2 \sigma_{ki}}{\partial \rho \partial B_j}$$

$$+ \rho\, B_n\, U\, c_i + 2\, \rho\, U^2\, a_i] + (V_{r,k})^+ [\rho\, U\, B_n\, c_k\, \delta_{ir}$$

$$- B_n\, a_j\, n_r \frac{\partial \sigma_{ki}}{\partial B_j} + B_n\, a_j\, n_s\, B_k \frac{\partial^2 \sigma_{si}}{\partial B_j \partial B_r} + \rho\, U^2\, a_k\, \delta_{ir}]. \tag{5.54}$$

It is clear that the amplitude equation (5.53) contains a_i and c_i and their derivatives. However c_i can be eliminated to obtain a partial differential equation for a_α alone. Thus the amplitude equation has form

$$\frac{d|\tilde{a}|}{dt} - \frac{U\, |\tilde{a}|}{B_n\, B_\perp^2}\, e_{irk}\, e_{jpq}\, n_r\, n_p\, B_k\, B_q\, (B_{i,j})^+ + \frac{\eta_B\, U\, B_j\, B_k\, |\tilde{a}|\, B_{j,k}^+}{2\, B\, \eta\, B_n}$$

$$+ \frac{U\, B_k\, |\tilde{a}|\, \eta_\rho\, (\rho_{,k})^+}{2\, \eta\, B_n} + \frac{|\tilde{a}|}{2\, \eta} (V_{s,s})^+ (2\, \eta + B\, \eta_B + \rho\, \eta_\rho)$$

$$- \frac{|\tilde{a}|}{2\, B\, \eta} B_r\, B_k\, \eta_B\, (V_{r,k})^+ = 0 \tag{5.55}$$

where

$$\tilde{a} = |a| \sqrt{(\eta\, U\, B_n / \rho)}$$

and

$$|a| = a\, B_\perp\, .$$

Thus the solution of the amplitude equation (5.55) has form

$$|\tilde{a}| = |\tilde{a}_0| \exp(\textstyle\int_0^t \alpha\, dt) \tag{5.56}$$

where the form of α is obvious from (5.55) .

Special Case η = constant

In this event we can see from (3.8) that ψ must have form

$$\psi = \frac{\eta\ B^2}{2\ \rho} + f(\rho) \quad . \tag{5.57}$$

If equation (5.57) is differentiated with respect to ρ, we obtain

$$\psi_\rho = -\frac{B^2}{2\ \rho^2}\ \eta + f'(\rho) \quad . \tag{5.58}$$

From equations (3.6), (3.9) and (5.58) we find that

$$P = \frac{\eta\ B^2}{2} + f'(\rho)\ \rho^2 \quad . \tag{5.59}$$

Thus

$$P_\rho = (\rho^2\ f_\rho)_\rho \tag{5.60}$$

which is a function of ρ only and

$$P_B = \eta\ B \quad . \tag{5.61}$$

We shall consider the situation in which the fluid ahead of the wavefront is at rest and at constant density.

Case(1)

Here U satisfies the quartic equation (3.37) which in this case has form

$$\rho\ U^4 - U^2\ (\rho\ P_\rho + \eta\ B^2) + \eta\ P_\rho\ B_n{}^2 = 0 \tag{5.62}$$

and the amplitude equation has form

$$\frac{dc}{dt} + \alpha\ c + \beta\ c^2 = 0 \tag{5.63}$$

where

$$\beta = -\frac{U}{2}\ \gamma[\frac{\rho U^2 - \eta\ B_n{}^2}{\rho U^4 - \eta\ P_\rho\ B_n{}^2}]\ [\frac{2\rho U^2\ (U^2 - P_\rho)}{\rho U^4 - \eta\ P_\rho\ B_n{}^2} + \rho\ U\ U_\rho + 2] \quad ,$$

$$\alpha = \frac{1}{2\,U\,B_n}\,L(U\,B_n) - L(B_i)\,\frac{U^2\,(\rho\,U^2\,n_i - \eta\,B_n\,B_i)}{2\,B_n\,(\rho\,U^4 - \eta\,P_\rho\,B_n^2)} + \frac{B^\alpha\,U_{,\alpha}}{2\,B_n}$$

$$+ (B_{i,j}) + \frac{U^3\,(\rho\,U^2\,n_i - \eta\,B_n\,B_i)\,(\rho\,U^2\,n_j - \eta\,B_n\,B_j)}{(\rho\,U^2 - \eta\,B_n^2)\,(\rho\,U^4 - \eta\,P_\rho\,B_n^2)}$$

$$- \frac{B^\alpha\,(\rho\,U^4 + \eta\,P_\rho\,B_n^2)}{4\,U\,B_n\,(\rho\,U^2 - \eta\,B_n^2)}\left[\frac{U^2\,(\rho\,U^2 - \eta\,B_n^2)}{\rho\,U^4 - \eta\,P_\rho\,Bn^2}\right]_{,\alpha}$$

$$- \frac{\rho\,U^3\,(\rho\,U^2 - \eta\,B_n^2)}{2\,B_n\,(\rho\,U^4 - \eta\,P_\rho\,Bn^2)}\left(\frac{U^2\,B^\alpha}{\rho\,U^2 - \eta\,B_n^2}\right)_{,\alpha}$$

$$- \frac{\rho\,\eta\,b_{\alpha\beta}\,B^\alpha\,B^\beta\,U^5}{2\,(\rho\,U^2 - \eta\,B_n^2)\,(\rho\,U^4 - \eta\,P_\rho\,B_n^2)}\ .$$

In the expression for β we have already observed that

$$\frac{\rho\,U^2 - \eta\,B_n^2}{\rho\,U^4 - \eta\,P_\rho\,B_n^2} = \frac{U^2\,B_\perp^2}{U^4\,B_\perp^2 + B_n^2\,(U^2 - P_\rho)^2} > 0\ .$$

Further,

$$\frac{2\,\rho\,U^2\,(U^2 - P_\rho)}{\rho\,U^4 - \eta\,P_\rho\,B_n^2} = \frac{2\,\rho\,U^2\,\eta\,B_\perp^2}{\eta^2\,B_n^2\,B_\perp^2 + (\rho\,U^2 - \eta\,B_n^2)^2} > 0$$

and since we would anticipate that U_ρ is also positive, then we expect β to be negative for any real material.

The solution of the amplitude equation (5.63) will be

$$c = \frac{c_0\,\exp(-\int_0^t \alpha\,dt)}{1 + c_0\int_0^t \beta\,\exp(-\int_0^s \alpha\,ds)\,dt} \qquad (5.64)$$

(i) If $c_0 < 0$, then since $\beta < 0$

$$1 + c_0\int_0^t \beta\,\exp(-\int_0^s \alpha\,ds)\,dt$$

is always positive and hence c is never infinite. In fact c is always negative. Also $c_0 < 0$ implies that $b < 0$, i.e. ρ is decreasing which means that we are moving from a low pressure region to a region of constant pressure.

(ii) If $c_0 > 0$, then $1 + c_0\int_0^t \beta\,\exp(-\int_0^s \alpha\,ds)\,dt$ may or may not be zero depending perhapes on the size of c_0

case(2)

Here we are considering the Alfven wave situation in which case

$$b = \Omega = 0 \qquad , \qquad \rho\, U^2 = \eta\, B_n^{\,2} \,.$$

The amplitude equation in this case has form

$$\frac{d(U\ |a|)}{dt} = \frac{\sqrt{(\eta/\rho)}}{B_\perp^{\,2}}\, U\ |a|\ e_{irs}\ e_{jpq}\ n_r\ n_p\ B_s\ B_q\ (B_{i,j})^+ \qquad (5.65)$$

and the solution will be

$$U\ |a| = U_0\ |a_0|\ \exp\!\left(\int_0^t \frac{\sqrt{(\eta/\rho)}}{B_\perp^{\,2}}\ e_{irs}\ e_{jpq}\ n_r\ n_p\ B_s\ B_q\ (B_{i,j})^+ \right). \qquad (5.66)$$

Suppose that $\mathbf{A} = \mathbf{n} \times \mathbf{B}$, then (5.65) becomes

$$U\ |a| = U_0\ |a_0|\ \exp\!\left(\int_0^t \frac{\sqrt{(\eta/\rho)}}{B_\perp^{\,2}}\ A_i\ A_j\ (B_{i,j})^+ \right) . \qquad (5.67)$$

If $\mathbf{B} = f(\rho)\ (-y,\ x,\ 0)$ and $\mathbf{n} = (n_1,\ n_2,\ 0)$, then $A_1 = A_2 = 0$ and

$$U\ |a| = U_0\ |a_0| \qquad (5.68)$$

which means that the jump in acceleration is constant.

Chapter Six

Examples

Propagation of a wavefront into a region at rest and at constant density.

Here the ray equations has the form

$$\dot{x}_i = U\, n_i + \frac{\partial U}{\partial B_n} B^\alpha x_{i,\alpha}$$
$$\dot{n}_i = (n_i\, n_j - \delta_{ij}) \frac{\partial U}{\partial x_j} \qquad (6.1)$$

where

$$U = U(B, B_n) \ .$$

Firstly we discuss the propagation of a wavefront into a region of constant magnetic field. We may take $B = (B,0,0)$ without loss of generality. From $(6.1)_2$ we may deduce that n_i = constant on a ray and thus B_n is also constant on the ray. Further $\partial U/\partial B_n$ and $B^\alpha x_{i,\alpha} = B_i - B_n n_i$ are constants. Thus the ray equation $(6.1)_1$ may be integrated to obtain

$$x_i = \{U\, n_i + \frac{\partial U}{\partial B_n}(B_i - B_n\, n_i)\}\, t + x_i(0,\theta^\alpha). \qquad (6.2)$$

Specifically, if U is homogeneous of degree one in B_n then $B_n\, \partial U/\partial B_n = U$ and therefore the surface described by (6.2) assumes the simplified form

$$x_i = \frac{\partial U}{\partial B_n} B_i\, t + x_i(0,\theta^\alpha) \quad . \qquad (6.3)$$

The Alfven situation is a special case of (6.2) where U is linear in B_n and $\partial U/\partial B_n = \surd(\eta/\rho)$.

(a) Propagation of a plane wave with normal direction **n**

Here we suppose that the initial profile is a plane wave with equation $\mathbf{X} \cdot \mathbf{n} = \alpha$ then from (6.2) we have

$$\mathbf{x} \cdot \mathbf{n} = \alpha + U\,t \,. \qquad (6.4)$$

i.e. the final surface is a plane wave parallel to the original surface and propagating with speed $U(B,B_n)$.

(b) Propagation of a cylindrical wave

In this section we suppose that the initial wavefront is the surface of the cylinder

$$\mathbf{x}(0,\theta^\alpha) = (R\cos\theta, R\sin\theta, Z) \,.$$

The unit normal to this cylinder is $\mathbf{n}(0,\theta^\alpha) = (\cos\theta,\sin\theta,0)$ and thus the location of the singular surface at time t is readily seen from (6.2) to be

$$\mathbf{x} = (U - B_n \frac{\partial U}{\partial B_n})\, t\, (\cos\theta,\sin\theta,0) + \frac{\partial U}{\partial B_n}\, t\, (B,0,0) + (R\cos\theta, R\sin\theta, Z) \,. \qquad (6.5)$$

Equation (6.5) has component form given by

$$x = (U - B_n \frac{\partial U}{\partial B_n})\, t \cos\theta + t\, B \frac{\partial U}{\partial B_n} + R\cos\theta \,,$$

$$y = (U - B_n \frac{\partial U}{\partial B_n})\, t \sin\theta + R\sin\theta \,, \qquad (6.6)$$

$$z = Z \,.$$

In order to find the cartesian equation of the singular surface at time $t > 0$, it is necessary to eliminate θ from (6.6) bearing in mind that $B_n = B\cos\theta$.

(c) Propagation of a spherical wave

Here we suppose that the initial wavefront is the surface of the sphere

$$x(0,\theta^{\alpha}) = (R \sin\theta \cos\phi, R \sin\theta \sin\phi, R \cos\theta). \quad (6.7)$$

The unit normal to this sphere is

$$n = (\sin\theta \cos\phi, \sin\theta \sin\phi, \cos\theta)$$

and thus the location of the singular surface at time t can be shown to be

$$\begin{aligned} x = {} & (U - B_n \frac{\partial U}{\partial B_n})\, t\, (\sin\theta \cos\phi, \sin\theta \sin\phi, \cos\theta) \\ & + \frac{\partial U}{\partial B_n}\, t\, (B,0,0) + (R \sin\theta \cos\phi, R \sin\theta \sin\phi, R \cos\theta). \end{aligned} \quad (6.8)$$

In component form

$$\begin{aligned} x &= (U - B_n \frac{\partial U}{\partial B_n})\, t \sin\theta \cos\phi + \frac{\partial U}{\partial B_n}\, t\, B + R \sin\theta \cos\phi \quad , \\ y &= (U - B_n \frac{\partial U}{\partial B_n})\, t \sin\theta \sin\phi + R \sin\theta \sin\phi \quad , \\ z &= (U - B_n \frac{\partial U}{\partial B_n})\, t \cos\theta + R \cos\theta \end{aligned} \quad (6.9)$$

where $B_n = B \sin\theta \cos\phi$.

Alfven wave propagation

Here the ray equation has the form described by (6.3) where U is linear in B_n and $\partial U/\partial B_n = \sqrt{(\eta/\rho)}$.

Propagation of a cylindrical and spherical wave

For the cylindrical initial surface we have

$$\begin{aligned} x &= \sqrt{(\eta/\rho)}\, B\, t + R \cos\theta \quad , \\ y &= R \sin\theta \quad , \\ z &= Z \quad . \end{aligned} \quad (6.10)$$

If θ is partially eliminated between $(6.10)_{1,2}$ then

$$(x - B\ t\ \surd(\eta/\rho))^2 + y^2 = R^2 \tag{6.11}$$

and in the special case in which $\eta \equiv$ constant then we have a cylinder radius R, centre $(B\ t\ \surd(\eta/\rho),0)$.

Similarly for the initial spherical profile (6.7) we may show that

$$\begin{aligned} x &= \surd(\eta/\rho)\ B\ t + R \sin\theta \cos\phi \quad , \\ y &= R \sin\theta \sin\phi \quad , \\ z &= R \cos\theta \quad . \end{aligned} \tag{6.12}$$

If θ is partially eliminated between $(6.12)_{1,2}$ then

$$(x - \surd(\eta/\rho)\ B\ t)^2 + y^2 + z^2 = R^2 \ . \tag{6.13}$$

and in the special case in which $\eta \equiv$ constant then we have a sphere radius R, centre $(B\ t\ \surd(\eta/\rho),0,0)$.

<u>Cylindrical magnetic fields</u>

Let us assume that the magnetic field has form

$$\mathbf{B} = f(\rho)\ (-y,x,0) \quad , \quad \rho = \surd(x^2 + y^2) \tag{6.14}$$

which, we observe in passing, satisfies div $\mathbf{B} = 0$.

If we define $g(\rho) = \surd(\eta/\rho)$, then equation $(6.1)_1$ becomes

$$\dot{x}_i = g(\rho)\ B_i \ . \tag{6.15}$$

In component form

$$\begin{aligned} \dot{x} &= -\ h(\rho)\ y \quad , \\ \dot{y} &= h(\rho)\ x \quad , \\ \dot{z} &= 0 \end{aligned} \tag{6.16}$$

where $h(\rho) = f(\rho)\ g(\rho)$.

From $(6.16)_{1,2}$ we find that

$$x\,\dot{x} + y\,\dot{y} = 0 \tag{6.17}$$

which implies that

$$x^2 + y^2 = \rho^2 = \rho_0{}^2 \tag{6.18}$$

$$\text{i.e.} \quad \rho = \rho_0(\theta) \ . \tag{6.19}$$

Also

$$\frac{d}{dt}\left(\frac{y}{x}\right) = \sec^2(\theta)\,\frac{d\theta}{dt} = \frac{\dot{y}}{x} - \frac{y\,\dot{x}}{x^2} = h\,\sec^2(\theta) \tag{6.20}$$

and so

$$\dot{\theta} = h \ . \tag{6.21}$$

From equation (6.19) and (6.21) we may deduce that

$$\begin{aligned} x &= \rho_0(\theta)\,\cos(\theta + h(\rho_0(\theta))\,t) \\ y &= \rho_0(\theta)\,\sin(\theta + h(\rho_0(\theta))\,t) \ . \end{aligned} \tag{6.22}$$

In figures 1 and 2, the locus (6.22) is sketched when $f(\rho) = 1$ and for initial profiles

$$(1) \quad \rho_0 = 2\cos\theta \quad , \qquad (2) \quad \rho_0 = 2 + \cos\theta \ .$$

In figures 3 and 4, we sketch (6.22) when $f(\rho) = 1/\rho^2$ and the initial profiles are

$$(3) \quad \rho_0 = -\sec\theta \quad , \quad \pi/2 < \theta < 3\pi/2 \quad ,$$

$$(4) \quad \rho_0 = \cos\theta + \sin\theta + \sqrt{[(\cos\theta + \sin\theta)^2 + 14]} \ .$$

<u>A conservation property of Alfven waves</u>

Suppose that η = constant , $\mathbf{B} = (B_x, B_y, 0)$. If A is the closed area bounded by a wavefront then

$$\text{Area} = \int_{\partial A} x\,\frac{dy}{d\theta}\,d\theta \ . \tag{6.23}$$

If (6.23) is differentiated with respect to t, then we obtain

$$\frac{d}{dt}(\text{Area}) = \int_{\partial A} \left(\dot{x}\,\frac{dy}{d\theta} + x\,\frac{d\dot{y}}{d\theta}\right)d\theta \ . \tag{6.24}$$

On using (6.15) equation (6.24) becomes

$$\frac{d}{dt}\,(\text{Area}) = \int_{\partial A} g\, B_x \frac{dy}{d\theta} + x\,\frac{d(g\ B_y)}{d\theta}\, d\theta \quad . \tag{6.25}$$

After further algebra and the use of Green's theorm, we obtain

$$\frac{d}{dt}\,(\text{Area}) = g \iint_A \left(\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y}\right) dx\ dy \tag{6.26}$$

and since **B** is solenoidal equation (6.26) becomes

$$\frac{d}{dt}\,(\text{Area}) = 0 \tag{6.27}$$

i.e. The area enclosed by the Alfven wave is conserved although the shape may vary.

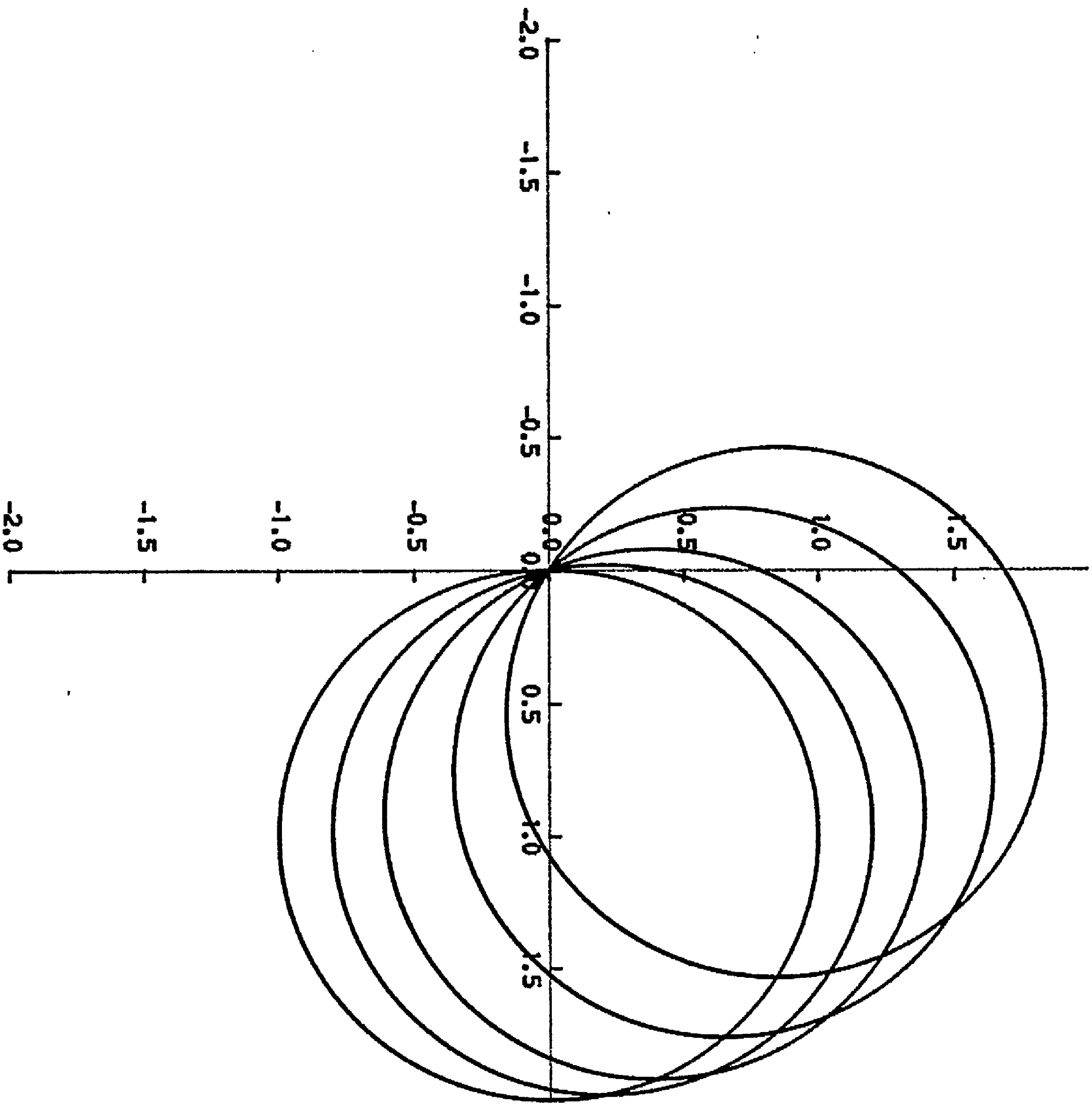

Figure 1

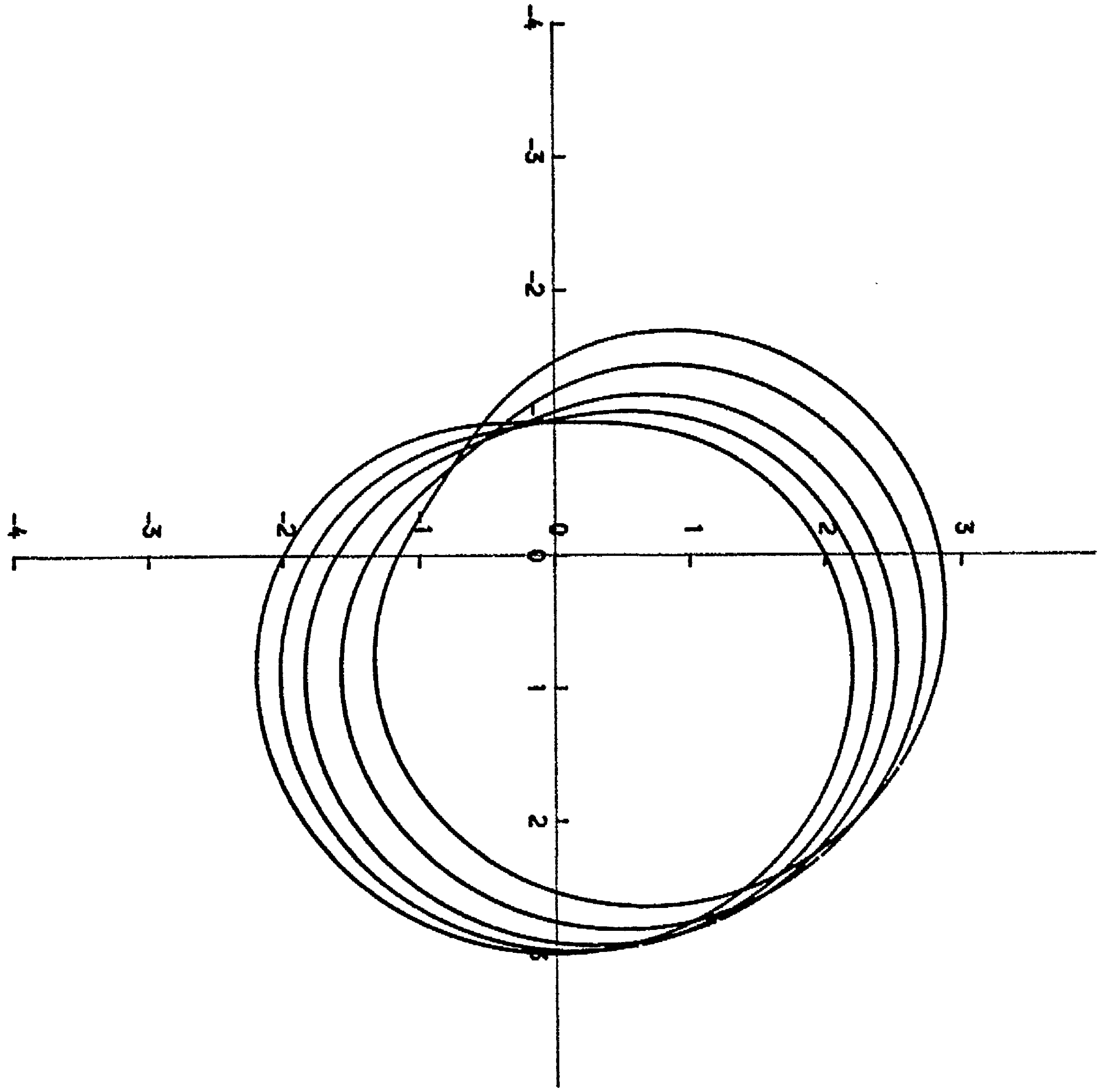

Figure 2

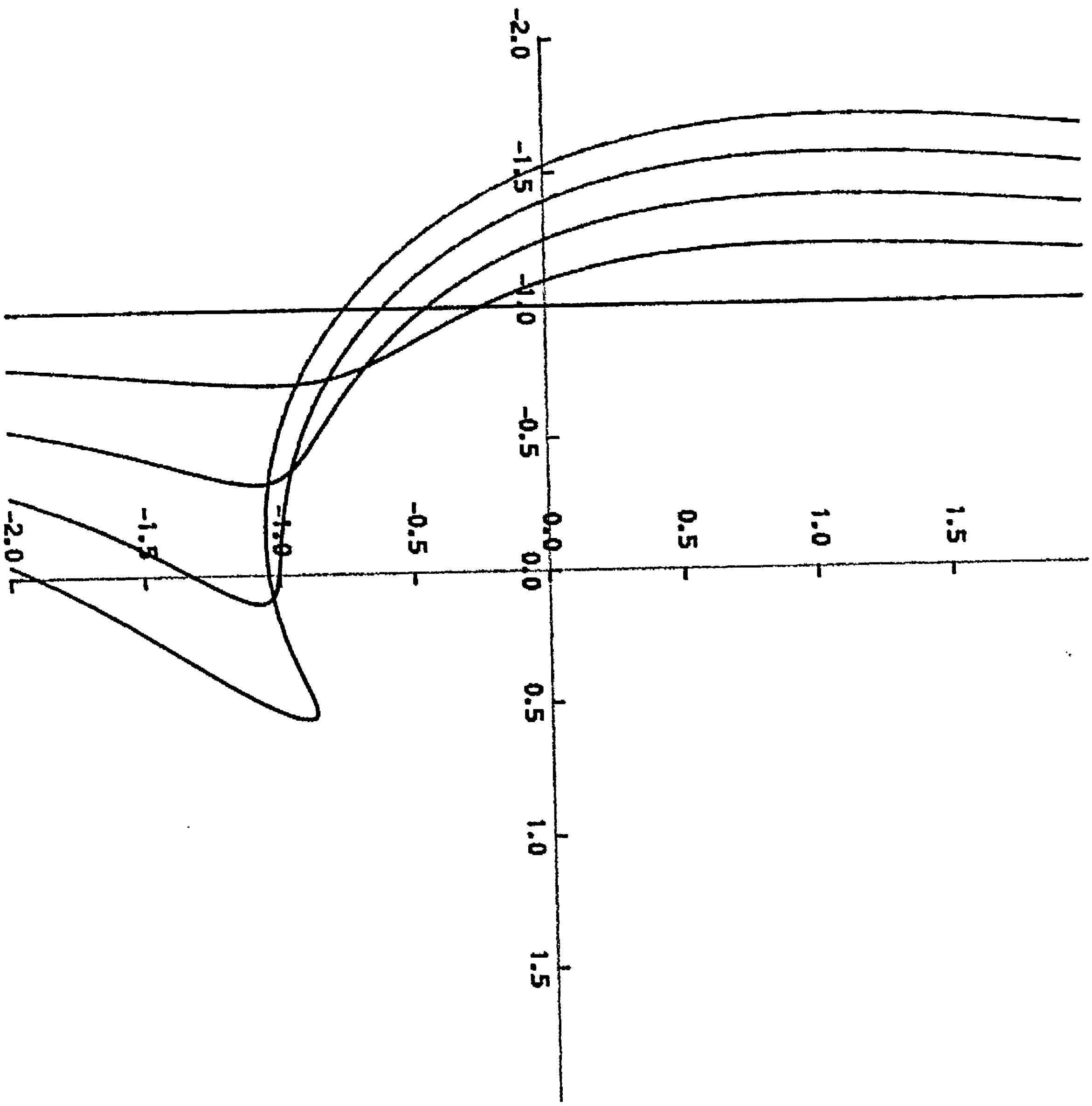

Figure 3

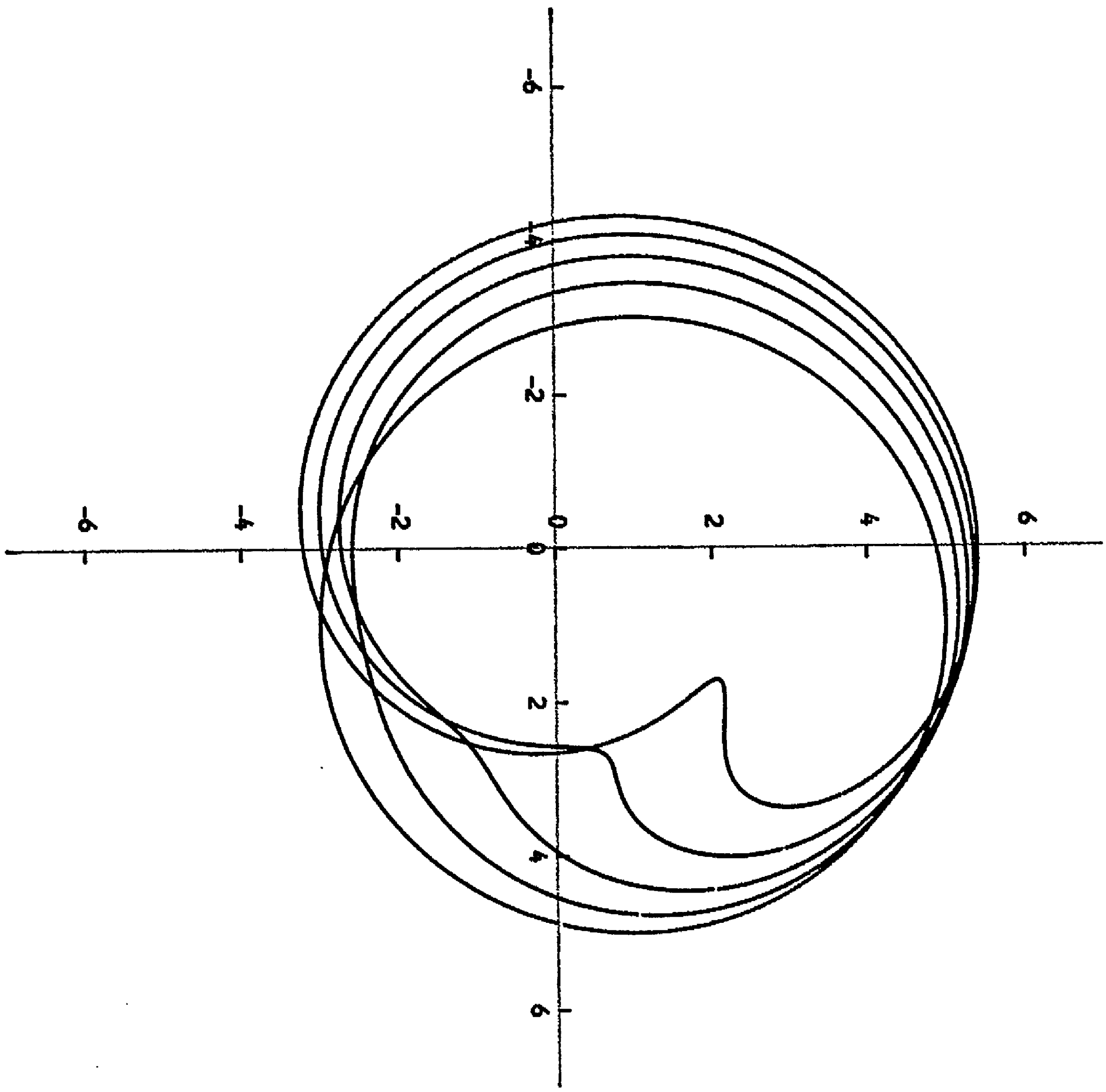

Figure 4

References

[1] Roberts, P. H. Equilibria and stability of a fluid type II superconductor. Q. J. Mech. Appl. Math. 34 (1981), 327–343.

[2] Muzikar, P. and C. J. Pethick. Flux bunching in type II superconductors. Phys. Rev. B24 (1981), 2533–2539.

[3] Straughan, B. Nonlinear waves in general magnetic fluid. J. Appl. Math. Phys. (ZAMP) 37 (1986) 274–279.

[4] Lindsay,K. A. and B. Straughan. Acceleration waves and second sound in a perfect fluid. Arch. Rational Mech. Anal. 68 (1978), 53–87.

[5] Lindsay, K. A. and B. Straughan. Propagation of mechanical and temperature acceleration waves in thermoelastic materials. J. Appl. Math. Phys. (ZAMP) 30 (1979), 477–490.

[6] Truesdell, C. J. Rational Mech. Anal. 2 (1953), 655–656.

[7] Courant, R. and D. Hilbert. Methods of mathematical physics Vol. 2 (Interscience publication, New York _ London, 1962).

[8] Varley, E. and E. Cumberbatch, J. Inst. Maths. Applics. 1 (1965), 101–112.

[9] Elcrat, A. R. Int. J. Engng. Sci. 15 (1977), 29–34.

[10] Seymour, B. R. and M. P. Mortell. Nonlinear geometrical acoustics, article in, " Mechanics Today ", Vol. 2. (Pergamon 1975), 251–312.

[11] Witham, G. B. Linear and nonlinear waves, (Wiley – Intercience publication, New York – London – Sydney – Toranto, 1974).

[12] Wright, T. W. Arch. Rational Mech. Anal. 50 (1973), 237–277

[13] Capelli, A. Math. Ann. 29, 1887, pp 331.

[14] Weyl, H. The classical group, 1946.

[15] Truesdell, C. and R. Toupin. The classical field theories, article in, " Handbuch der physik ", Vol. III/1 (Spring - Verlag, Berlin - Gottingon - Heidelberg, 1960).

[16] Eringen, A. and E. Suhubi. Elastodynamics. Vol. 1, 1974.