Strategic Foundations of Oligopolies in General Equilibrium

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Abstract

In this thesis, I study the strategic foundations of oligopolies in general equilibrium by following the approach based on strategic market games. The thesis is organised as follows.

In Chapter 1, I first survey some of the main contributions on imperfect competition in production economies and the main problems which arise in this framework. I then focus on the literature on imperfect competition in exchange economies by considering the Cournot-Walras approach and strategic market games. I finally discuss the main contributions on the foundations of oligopolies.

In Chapter 2, I extend the non-cooperative analysis of oligopoly to exchange economies with infinitely many commodities and traders by using a strategic market game with trading posts. I prove the existence of a Cournot-Nash equilibrium with trade and show that the price vector and the allocation at the Cournot-Nash equilibrium converge to the Walras equilibrium when the number of traders increases. In a framework with infinitely many commodities, an oligopolist can be an "asymptotic oligopolist" if his market power is uniformly bounded away from zero on an infinite set of commodities, or an "asymptotic price-taker" if his market power converges to zero along the sequence of commodities. The former corresponds to the Cournotian idea of oligopolist. The latter describes an agent with a kind of mixed behaviour since his market power can be made arbitrary small by choosing an appropriate infinite set of commodities while it is greater than a positive constant on a finite set.

In Chapter 3, I further study oligopolies in economies with infinitely many commodities and traders. By using the strategic market game called "all for sale model", I prove the existence of an asymptotic price-taker. Heuristically, an asymptotic price-taker exists if at least one trader makes positive bids on an infinite number of commodities and in all markets the quantities of commodities exchanged are non-negligible.

In Chapter 4, I study if there is a non-empty intersection between the sets of Cournot-Nash and Walras allocations in mixed exchange economies, with oligopolists represented as atoms and small traders represented by a continuum. In a bilateral oligopoly setting, I show that a necessary and sufficient condition for a Cournot-Nash allocation to be a Walras allocation is that all atoms demand a null amount of one of the two commodities. I also provide four examples which show that this characterization holds non-vacuously.

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Preface

Nowadays real world economies are characterised by large firms which often spread their activities on many countries and it is hard to believe that they do not have market power. In such situation, it is not clear if markets are still an efficient way to allocate resources among agents. Crouch (2011) argues that a political debate that continues to be organised around market and state is missing the issues raised by the presence of big corporations. He also underlines that

> "The confrontation between the market and the state that seems to dominate political conflict in many societies conceals the existence of this third force, which is more potent than either and transforms the workings of both."

In this thesis, I analyse exchange economies in which the price-taking assumption is dropped and agents can influence the prices with their supply and demand decisions. To do so, I consider the literature on strategic market games which are models where traders can choose the amount of their initial endowments to put up in exchange for other commodities. In this framework, I study the conditions on the fundamentals of an economy under which market power arises endogenously in equilibrium, i.e., foundations of oligopolies.

The main contributions of this thesis are twofold. In Chapters 2 and 3, I focus on oligopolies in exchange economies with infinitely many commodities and traders to study agents with mixed behaviours having market power on some commodities while being competitive on others. Perhaps surprisingly, in such framework both agents who keep market power on an infinite set of commodities and agents whose market power vanishes along the sequence of commodities arise endogenously in equilibrium. This model is a starting point to analyse the differences between global and local oligopolists. In Chapter 4, I consider mixed exchange economies where oligopolists are represented as atoms and small traders are represented by a continuum. In this framework, I show the necessary and sufficient conditions under which the sets of Cournot-Nash and Walras allocations coincide. This result helps to understand when in a market with oligopolists a competitive outcome can arise endogenously in equilibrium.

Finally, in the rest of the thesis I use "we" instead of "I" because it is more formal and less personal. As proposed by Thomson (2011), in single author contributions "we" may mean the author and the reader.

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Declaration

I declare that, except where specific reference is mentioned about the contribution of others, this dissertation is a result of my own research work and has not been submitted for any other degree at the University of Glasgow or any other institution.

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Introduction

"Let me, to begin with, justify the choice, which may be surprising since the topic has been dormant for so long. First, I think that there is no microeconomic theory if it is not eventually cast in a general equilibrium framework. Of course, partial models give insights, and also serve as natural starting points. But, in its very essence, the economy is a system of interrelated markets where the influence of what happens at some point is unavoidably propagated throughout the system, via the chain of markets, through the possible substitutions and complementarities between goods that are due to tastes and technologies."

Gabszewicz (1999)

"Finally, it seems impossible to go on with analysing markets under the assumption of perfect competition. Direct observation of economic activity reveals that markets are the fields of "giants", operating simultaneously with a fringe of small competitors. Even partial analysis has taken this picture of the market when proposing oligopoly solutions to describe the outcomes of imperfectly competitive markets. Behind the demand function there is a myriad of "small" price-taking agents, while the supply side is occupied by few agents appearing as giants, contrasting with the dwarfs on the demand side."

Gabszewicz (2013)

1.1 Market power in general equilibrium theory

We have started this chapter with two quotes from Jean J. Gabszewicz as rhetorical devices to appeal for the studying of market power in general equilibrium models. We now briefly clarify why this area of research can provide a useful theoretical framework to address economic issues.

It is easy to recognise that many markets in real world economies are dominated by few giant firms which compete against each others. Even if it may be true that sometimes perfect competition arises despite the few number of agents, as in the Bertrand's model, usually imperfect competition prevails. Therefore, the first justification of studying market power is the necessity of realism in economic models. The hypothesis of price-taking behaviours, which is crucial in the theory of perfect competition, could hardly capture the features of markets in which agents can manipulate prices. Furthermore, imperfect competition may take away markets efficiency properties and then under such condition it is not clear if markets are still an efficient way to allocate commodities among agents. This clearly makes market power a fundamental matter to study.

The choice of using general equilibrium models is again justified by a realism requirement. Since multinational firms are active simultaneously in several markets and consumers' preferences are spread over many commodities, general equilibrium theory arises as a natural way to capture the complexity of this interrelated system. General equilibrium models can also be used to check the robustness and consistency of results obtained in partial analysis. Obviously, if we think about industrial organisation, this cannot be done systematically for all its rich research which aims to analyse in depth the industry level and not the whole economy. Finally, general equilibrium theory can provide a unified framework to study and compare different market structures. The study of monopoly, oligopoly, and monopolistic competition is usually conducted with different approaches within partial analysis and is based on *ad hoc* assumptions. This is further away from the comprehensive and coherent Walrasian system developed for perfect competitive economies.

In the next sections, we do not aim to review all the literature on general equilibrium with imperfect competition but we focus only on those contributions that provide an historical and theoretical framework for the next chapters of the thesis. The literature on imperfect competition in general equilibrium was masterly surveyed by Hart (1985), Gary-Bobo (1988), Bonanno (1990), and Gabszewicz (1999, 2002).

1.2 Imperfect competition: an historical overview

The first attempt to analyse analytically imperfect competition was made by Cournot (1838). He proposes a simply theoretical framework to study a market with oligopolistic firms and a large number of price taking consumers. He assumes that firms produce an homogeneous commodity and that consumers are represented by an inverse demand function. Given the demand function, each firm decides the quantity to produce in order to maximise its profit. The equilibrium concept introduced by Cournot, which is a particular case of the Nash equilibrium (see Nash (1951)), captures the strategic interactions among firms. In other words, each firm maximises its profit given the production levels of all other firms.

Many years later, Chamberlin (1933) developed a theoretical framework to study markets with differentiated commodities. He introduces the notion of monopolistic competition which describes an industry with differentiated commodities where each commodity is produced only by one firm whose strategic variable is the commodity's price. There are two types of monopolistic competition: small group and large group. The former considers an industry with few firms and then any change in price has an effect on the other few firms in the industry. Differently, the latter considers an industry with many firms and then "any adjustment of price [...] by a single producer spreads its influence over so many of his competitors that the impact felt by any one is negligible and does not lead him to any readjustment of his own situation" (Chamberlin (1933) p.83). This is the more studied case in the literature on monopolistic competition. By comparing the Cournot model and the large group case, we observe that in both of them firms can manipulate prices but while in Cournot there is strategic interaction among firms in the large group monopolistic competition firms ignore each others. Furthermore, both contributions are cast in partial equilibrium analysis which means that they do not consider the whole economy and the system is not closed because consumers do not formally belong to the model and they are just represented by an inverse demand function.

Negishi (1961) and Gabszewicz and Vial (1972) made one of the first attempts to extend to general equilibrium theory the model of imperfect competition introduced by Chamberlin (1933) and by Cournot (1838) respectively. We now briefly describe the main features of these two contributions (for a detailed comparison see Busetto (2005)).

Negishi (1961) proves the existence of an equilibrium in an economy with perfectly competitive consumer, some perfectly competitive firms and some monopolistically competitive firms. The assumptions on consumers and perfectly competitive firms are standard, the novelty is the assumption that each monopolistically competitive firm is characterized by a portfolio of commodities containing all the commodities on which the firm has market power. It is also assumed that portfolios are such that no firm controls all markets and none of the markets is dominated by two firms. In this way, monopoly and oligopoly are ruled out from the analysis. In this general equilibrium framework, monopolistic competition turns out to be an economic configuration in which monopolistically competitive firms have market power on some markets while behave competitively on all others and each market is possibly dominated by only one firm. In this model, production decisions are based on subjective (or perceived) demand curves which only show the prices conjectured by firms for different production levels and they may not coincide with the real market demands. This feature is a weakness of this approach because it introduces an element of arbitrariness in the model. Gary-Bobo (1989) showed that all feasible allocations such that the production of each firm is greater than zero and yields non-negative profits are equilibria in the sense defined by Negishi (1961). In other words, it is possible to find subjective demands which sustain each allocation as an equilibrium allocation. To overcome this indeterminacy problem, it is necessary to consider firms which face objective demands.

Gabszewicz and Vial (1972) in their attempt to extend the Cournot model to general equilibrium theory pioneered the objective demand approach, i.e., firms face objective demands generated by consumers utility maximisation problem. They define an economy with production where, as in the Cournot model, firms are assumed to be few whereas consumers are assumed to be many. Consumers, who behave competitively, provide firms with labour and other primary factors which are non-consumable and non-marketable. With these resources, firms, which have market power and behave as players of a non-cooperative game, produce consumption commodities. These commodities are then distributed to the consumers, who have provided primary factors, as real wages according to some preassigned shares. At the end of the production process, each consumer is thus endowed with his initial endowment plus the bundles of commodities which he receives as a shareholder of the firms. Exchange markets are then organized, where the consumers aim at improving their utility levels through trade. The rule of exchange consists in using a Walrasian price mechanism under which all markets clear. The equilibrium prices resulting from these exchanges allow firms to choose the production levels which maximise their profits given the production decisions of others.

Bonanno (1990) and Codognato (1994) summarised the main problems raised by the approach of Gabszewicz and Vial (1972). Here, we just focus on two of them, namely the existence of a Cournot-Walras equilibrium and the fact that profit maximisation may not be a rational objective for imperfectly competitive firms. Both these are crucial issues since they concern the consistency of the whole theory.

In the paper of Gabszewicz and Vial (1972), the existence of a Cournot-Walras equilibrium relies on two strong assumptions namely that the set of Walras equilibria resulting from the exchange of commodities among consumers contains a unique element and that firms' profit functions are quasi-concave. Clearly these two assumptions are not based on the fundamentals of the economy. Furthermore, the quasi-concavity of profit functions imposes restrictions on the shape of demand curves. The first assumption was actually relaxed by Roberts (1980) and Dierker and Grodal (1986). Differently, Roberts and Sonnenschein (1977) showed that even if we make standard assumptions on consumers and firms, profit functions need not to be quasi-concave. A possible way of avoiding this *ad hoc* assumption is to consider equilibria in mixed strategies. But the notion of mixed strategy is far from being clear in this framework and additionally Dieker and Grodal (1986) showed an example of non-existence even in mixed strategies.

Let us now discuss the problem of maximising profits in imperfect competitive economies. The following quotation from the referee's report of Gabszewicz and Vial (1972) raised the issue for the first time.

> "Consider a firm owned by many consumers, all of whom are identical. Given the strategies of the other firms in the economy, this firm chooses an output vector so as to maximize the wealth of each of its consumers. However, it is possible that this firm could choose a different strategy which would result in slightly lower wealth, but in a much lower price of some particular commodity which is greatly "desired" by the owners of the firm. Thus this alternative strategy might yield greater "real income" to the firms owners."

In other words, firms' owners are interested in monetary profits only in terms of their purchasing power and then they may not find profit maximisation as a rational objective if it leads to higher prices for the commodities they consume. One of the first attempts to overcome this difficulty was proposed by Mas-Colell

(1982) who considers a model where consumers with positive share ownership in firms consume only the numeraire commodity (which is endogenously determined). They can be seen as a class of capitalists who leaves in an outside world. A similar but probably more ingenious construction was proposed by Hart (1985). He considers a model with islands in which consumers consume only in the island where they live but get profits from other islands. In these two frameworks, profit maximisation is a consistent objective for firms because when they maximise profits they are also maximising the utility of their owners. These approaches can then reconcile the trade-off between firms' profit maximisation and shareholders' utility maximisation but at the price of moving the original Gabszewicz and Vial model toward partial equilibrium analysis as it was noticed by Mas-Colell (1982) and Hart (1985) themselves. Another way to circumvent the problem was proposed by Dierker and Grodal (1986) by considering an economy in which each firm is owned by only one consumer and the firm, instead of maximising profit, maximises the indirect utility function of its owner. However, this latter approach provides a well defined objective function for the firms only in the special case where each firm is owned by a single consumer. Dierker and Grodal (1999) and Dierker, Dierker, and Grodal (2001) proposed another objective for the firms based on the maximisation of shareholders' real wealth. A different and more drastic path was followed by Codognato and Gabszewicz (1991) and Gabszewicz and Michel (1997) which recasts the Cournot-Walras model in exchange economies. The advantage of analysing oligopoly in exchange economies is that it helps to study exchanges among traders who can influence prices avoiding all other problems which arise in production economies. We analyse in detail this approach in the next section.

1.3 Oligopolies in exchange economies

The study of oligopoly in exchange economies can be divided in two approaches: asymmetric oligopoly and symmetric oligopoly (see also Codognato (1988)). In the asymmetric oligopoly some consumers have market power while others act competitively. Such framework was introduced by Codognato and Gabszewicz (1991) and further studied by Gabszewicz and Michel (1997). In the symmetric oligopoly all consumers are treated symmetrically and all of them act strategically. This case was studied by using strategic market games which were introduced in the contributions of Shubik (1973), Shapley (1976), and Shapley and Shubik (1977).

We start by describing the first approach which is very close to the spirit of

Cournot-Walras in production economies. In order to clearly understand the institutional framework of Cournot-Walras in exchange economy, we first quote the description provided by Gabszewicz and Michel (1997) of how trade takes place in competitive economies.

> "In a competitive economy, exchange can be viewed as taking place in the following way. Each competitive trader comes with his initial endowment in each good to a central market-place, where the sum of these endowments for each good is supplied for trade. A price system is announced, which determines the income of each trader as the scalar product of this price vector by the vector of his initial endowments. Then each utility maximizing competitive traders buys back a bundle of the commodities, the value of which does not exceed his income. If the price system clears each market a competitive equilibrium obtains."

We now describe the institutional framework of Cournot-Walras in exchange economies. Consumers are divided in two groups: oligopolists and small traders. Each of them is characterised by an initial endowment which represents his wealth and by a utility function which describes his preferences. Additionally, each oligopolist is further characterised by a portfolio of commodities which contains the commodities on which he behaves strategically. Differently, small traders are assumed to behave competitively on all commodities. By analogy with the above story on perfect competition, a consumer behaves competitive on a commodity if he supplies the market-place with all his endowment of that commodity. Differently, a consumer has a strategic behaviour, if he supplies the market-place only with a restricted share of his endowment. Therefore, small traders send to the market-place all their endowments while oligopolists can restrict the supply of the commodities in the portfolios. The process of exchange can be seen as organised in two stages. First, each oligopolist chooses the amount of commodities he wants to sell keeping for later consumption the remaining share. In the second stage, each consumer demands the preferred commodity bundles, according to the Walrasian demands, and prices are computed in a way that all markets clear. In order to clarify the notion of Cournot-Walras equilibrium, we now show an example in a two-commodity exchange economy.

Example 1. Consider an exchange economy with the following set of consumers $\{1, 2, 3, 4\}$. The initial endowments and utility functions of traders are as follows. $w^i = (2, 0)$, $u^i(x^i) = \ln(1 + x_1^i) + x_2^i$, for $i = 1, 2$, and $w^i = (0, 5)$, $u^i(x^i) =$

$2x_1^i - \frac{1}{2}(x_1^i)^2 + x_2^i$, for $i = 3, 4$. We assume that consumers 1 and 2 are oligopolists while consumers 3 and 4 are small traders. Since we consider an exchange economy with only two commodities and corner endowments, to find the Cournot-Walras equilibrium we only need to calculate the Walrasian demands for the small traders and then find the best supply decisions of oligopolists. We solve the following maximisation problem to find the Walrasian demands of consumers 3 and 4.

$$\begin{aligned} \max_{x_1^i, x_2^i} \quad & 2x_1^i - \frac{1}{2}(x_1^i)^2 + x_2^i, \\ \text{subject to} \quad & p_1 x_1^i + p_2 x_2^i \leq p_2 5, \\ & x_1^i, x_2^i \geq 0, \end{aligned}$$

for $i = 3, 4$. It is straightforward to verify that the following Walrasian demands are the solutions of the maximisation problem above.

$$\begin{aligned} x_1^i &= 2 - \frac{p_1}{p_2}, \\ x_2^i &= \frac{5p_2^2 + p_1^2 - 2p_1 p_2}{p_2^2}, \end{aligned}$$

for $i = 3, 4$. Let q^1 and q^2 be the amounts of commodity 1 that consumers 1 and 2 send to the market-place respectively. We normalise to 1 the price of commodity 2 and p_1 is such that the market for commodity 1 clears, i.e., $x_1^3 + x_1^4 = q^1 + q^2$. Then, the price of commodity 1 is given by $p_1 = 2 - \frac{1}{2}(q^1 + q^2)$ and the commodity bundles of oligopolists are determined as follows. $x_1^i = w_1^i - q^i$ and $x_2^i = (2 - \frac{1}{2}(q^1 + q^2))q^i$, for $i = 1, 2$. We solve the following maximisation problem to find the optimal supply decision q^i.

$$\begin{aligned} \max_{q^i} \quad & \log(1 + 2 - q^i) + \Big(2 - \frac{1}{2}(q^1 + q^2)\Big)q^i, \\ \text{subject to} \quad & 0 \leq q^i \leq 2, \end{aligned}$$

for $i = 1, 2$. It is straightforward to verify that $q^1 = q^2 = 1$ is the solution of the maximisation problem. Hence, the price vector and the allocation at the Cournot-Walras equilibrium of this exchange economy are

$$\begin{aligned} (p_1, p_2) &= (1, 1), \\ (x_1^i, x_2^i) &= (1, 1), \text{ for } i = 1, 2, \\ (x_1^i, x_2^i) &= (1, 4), \text{ for } i = 3, 4. \end{aligned}$$

$\square$

The model proposed by Gabszewicz and Michel (1997) makes it possible to deal with Cournotian oligopoly in a general equilibrium framework while avoiding some of the problems raised by the Gabszewicz and Vial approach. However, the problem of existence of a Cournot-Walras equilibrium remains open. Bonnisseau and Florig (2003) provided an existence result which holds only for linear exchange economies. Another issue which arises in this framework was stressed by Okuno, Postelwaite, and Roberts (1980).

> "Traditional general equilibrium treatments of such situations [in which some but not all agents may have market power] have been deficient in that they have simply assumed a priori that certain agents behave as price takers while others act non-competitively, with no formal explanation being given as to why a particular agent should behave one way or the other."

Indeed, Gabszewicz and Michel (1997) do not provide a formal explanation on why some consumers are considered oligopolists while others are considered small traders. Obviously, a heuristic explanation is that consumers are oligopolists if they hold a commodity whose ownership is concentrated among few of them while small traders own commodities whose ownership is spread over many consumers. In any case, the example above shows that it is also possible to consider odd cases in which some consumers are assumed to be small traders even if there are only two of them.

Differently, symmetric oligopolies overcome the issue raised by Okuno et al. (1980) because in these models all traders behave strategically. This approach is usually based on strategic market games, which are games where all traders decide simultaneously how much of the commodities in their endowments put up in exchange for other commodities. There are many types of strategic market games (see Giraud (2003)) and here we focus on the trading post model and the window model which can be seen as different institutional mechanisms through which prices are determined.

In the former, trade is decentralised through a system of trading posts where commodities are exchanged. There are two variants of this model which were studied by Dubey and Shubik (1978) and Amir, Shubik, Sahi, and Yao (1990) respectively. In the first variant, there is one commodity which has the role of commodity money and it is used to buy all other commodities. So, each trader simultaneously puts up quantities of the commodities he holds in exchange for commodity money and quantities of commodity money in exchange for other commodities. In each trading post a commodity is exchanged for commodity money and the price is determined as the ratio of the two amounts exchanged. Differently, Amir et al. (1990) study a model in which there is a trading post for each pair of commodities, i.e., any commodity can be used to buy other commodities. Prices are then computed in each trading post as in the previous model but since there is a price for each pair of commodities, they are not necessarily consistent through pairs of markets in which the same commodity is exchanged.

In the latter, the window model, any commodity can be used to buy other commodities, as in Amir et al. (1990), but trade is centralised by a clearing house in which there is a “window” for each commodity. All bids to buy a commodity are aggregated in the window for that commodity and the clearing house carries out a massive coordinated computation in order to find consistent prices which clear all markets. This model was initially proposed informally by Lloyd Shapley and subsequently analysed in detail by Sahi and Yao (1989). In all these three contributions, Dubey and Shubik (1978), Amir et al. (1990), and Sahi and Yao (1989), the existence of a Cournot-Nash equilibrium was proved. We now consider an example of the strategic market game considered by Dubey and Shubik (1978) in which the Cournot-Nash equilibrium is used as equilibrium concept.

Example 2. Consider the exchange economy defined in Example 1 but in this case, since we deal with a strategic market game, all traders act strategically. Let commodity 2 be commodity money and we then normalize its price to 1. Since we consider an exchange economy with only two commodities and corner endowments, the only actions available to traders 1 and 2 are q^1 and q^2 respectively, which are the amounts of commodity 1 offered in exchange for commodity money. Analogously, the only actions available to traders 3 and 4 are b^3 and b^4 respectively, which are the amounts of commodity money offered for commodity 1. The price of commodity 1 is determined according the following rule

$$p_1 = \begin{cases} \frac{b^3+b^4}{q^1+q^2} & \text{if } q^1+q^2 \neq 0, \\ 0 & \text{if } q^1+q^2 = 0. \end{cases}$$

Traders' commodity bundles are determined as follows.

$$x_1^i = w_1^i - q^i,$$
$$x_2^i = p_1 q^i,$$

for $i = 1, 2$, and

$$x_1^i = \frac{b^i}{p_1},$$
$$x_2^i = w_2^i - b^i,$$

for $i = 3, 4$. We solve the following maximisation problems to find the Cournot-Nash equilibrium.

$$\begin{aligned} \max_{q^i} \quad & \ln(1 + 2 - q^i) + p_1 q^i, \\ \text{subject to} \quad & 0 \leq q^i \leq 2, \end{aligned}$$

for $i = 1, 2$, and

$$\begin{aligned} \max_{b^i} \quad & 2\frac{b^i}{p_1} - \frac{1}{2}\left(\frac{b^i}{p_1}\right)^2 + (5 - b^i), \\ \text{subject to} \quad & 0 \leq b^i \leq 5, \end{aligned}$$

for $i = 3, 4$. By combining the solutions of both maximisation problems, we obtain the following Cournot-Nash equilibrium.

$$(q^1, q^2, b^3, b^4) = \left(\frac{5 - \sqrt{17}}{2}, \frac{5 - \sqrt{17}}{2}, \frac{3\sqrt{17} - 11}{4}, \frac{3\sqrt{17} - 11}{4}\right).$$

Hence, the price vector and the allocation at the Cournot-Nash equilibrium of this exchange economy are

$$\begin{aligned} (p_1, p_2) &= \left(\frac{\sqrt{17} - 1}{4}, 1\right), \\ (x_1^i, x_2^i) &= \left(\frac{\sqrt{17} + 3}{4}, \frac{3\sqrt{17} - 11}{4}\right), \text{ for } i = 1, 2, \\ (x_1^i, x_2^i) &= \left(\frac{5 - \sqrt{17}}{4}, \frac{31 - 3\sqrt{17}}{4}\right), \text{ for } i = 3, 4. \end{aligned}$$

□

The Cournot-Walras approach in exchange economies and strategic market games have the common characteristic that consumers/traders supply the market with shares of their initial endowment in order to purchase other commodities and therefore both of them can be considered as Cournotian models. However, they differ in the institutional organisation of the exchange process which is purely Walrasian in Cournot-Walras whereas it is non-Walrasian in strategic market games. In the two examples above, the two exchange economies considered are the same but the allocations at the Cournot-Walras equilibrium and at the Cournot-Nash equilibrium are different because in Example 1 consumers 3 and 4 behave competitively while in Example 2 they behave strategically.

Since in strategic market games all traders are treated symmetrically, it is not necessary to make *ad hoc* assumptions on traders' behaviours. But this means that the problem of developing a model of asymmetric oligopoly in which the differences in traders' behaviours are not assumed a priori but are endogenously determined remains open. This is the so called problem of "foundations of asymmetric oligopoly" (hereafter simply "foundations of oligopoly"). We analyse this issue in the next section by continuing to focus on exchange economies.

1.4 Foundations of oligopolies

It is worth to stress that an issue similar to the foundations of oligopoly arises also in perfect competitive economies and it is called "foundations of perfect competition". The Walrasian analysis in the synthesis reached in the contributions of Arrow and Debreu (1954), Debreu (1959), and Arrow and Hahn (1971) crucially relies on the price-taking assumption. It is then important to study under which conditions on the fundamentals of an economy agents consider prices as given endogenously in equilibrium. We first recall briefly the main results obtained on the foundations of perfect competition and we then consider the foundations of oligopoly.

There are two approaches to provide a foundation to perfect competition: the cooperative approach and the non-cooperative one. The first is based on the notion of core which was first proposed by Edgeworth (1881) who was interested in showing how the presence of many agents would lead them to consider prices as given. Debreu and Scarf (1963) developed a general model of the Edgeworth idea. They provide a limit result showing that when an economy is replicated, the allocations in the core shrink to the Walras allocations. This result is heuristically based on the fact that when the number of trader increases traders' influence on prices decreases and at the limit totally disappears. Aumann (1964) established a result at the limit by considering an economy with a continuum of traders and by showing that the allocations in the core coincide with the Walras allocations. In this model, each trader is negligible and then he cannot influence the prices.

When the non-cooperative approach is used to give a foundation to the price-taking assumption, we provide a *strategic* foundation of perfect competition. Strategic market games were originally developed to study the strategic foundations of perfect competition because traders influence prices with their choices. Indeed, the papers of Dubey and Shubik (1978), Sahi and Yao (1989), and Amir et al.

(1990) showed that when the economy is replicated à la Debreu and Scarf, the price vector and the allocation, at the Cournot-Nash equilibrium, converge to the Walras equilibrium of the underlying exchange economy. By using strategic market games, it is also possible to establish results at the limit à la Aumann. Dubey and Shapley (1994) and Codognato and Ghosal (2000) extended to exchange economies with a continuum of traders the models of Dubey and Shubik (1978) and Sahi and Yao (1989) respectively. Mas-Colell (1980) surveyed the main contributions on the strategic foundations of perfect competition by considering also the approaches which do not rely on strategic market games (see Robert and Postlewaite (1974), Roberts (1980), Mas-Colell (1983), and Novshek and Sonnenschein (1983) among others).

The problem of providing a foundation to oligopoly can be essentially reduced in finding a mathematical framework in which endogenously in equilibrium some traders have market power and others behave competitively. In the cooperative approach, Drèze, Gabszewicz, and Gepts (1969) adopted a partial replication which leaves some traders as in the original economy while increasing the number of others. With this variation of the Debreu and Scarf replica, the set of traders becomes a mixture of "giants" and "dwarfs" where the former ones keep market power while the latter ones lose it. Differently, Gabszewicz and Mertens (1971) and Shitovitz (1973), by following the Aumann approach, considered the core of mixed exchange economies where oligopolists are represented as atoms, and small traders are represented by a continuum. An atom is a trader whose initial endowment is large compared to the total endowment of the economy while a trader in the continuum holds only a negligible part of it. This approach, based on mixed exchange economies, overcomes the problem raised by Okuno et al. (1980) as remarked in Shitovitz (1973).

> "The main point in our treatment is that the small and the large traders are not segregated into different groups a priori; they are treated on exactly the same basis. The distinctions we have found between them are an outcome of the analysis; they have not been artificially introduced in the beginning, as is the case in the classical approach."

In exchange situations with non-negligible and negligible traders, the core does not generally coincide with the Walras allocations because atoms can manipulate the coalition formation and the allocation of commodities at the core of an economy (see for instance Example 1 in Chapter 4). Nevertheless, Gabszewicz and Mertens

(1971) and Shitovitz (1973) established some equivalence theorems for mixed exchange economies, similar to the one proved by Aumann (1964) in continuum economies. In particular, Shitovitz's Theorem B shows that the presence of an arbitrary small (but not infinitesimal) oligopolist cancels out the market power of all oligopolists. Okuno et al. (1980) found this result so counterintuitive to call into question the use of the cooperative approach to study oligopoly in general equilibrium. In their contribution, they keep the framework of mixed exchange economy, which seems useful to study oligopoly because introduce an element of asymmetry among traders, and replace the core solution concept with a Cournot-Nash equilibrium of the strategic market game analysed by Dubey and Shubik (1978). In this non-cooperative framework, they show that, under the assumptions of Shitovitz's Theorem B, the Cournot-Nash allocations do not coincide with the Walras allocations. This result allows them to conclude that the non-cooperative approach is a useful one to study oligopoly in a general equilibrium framework as the small traders always have a negligible influence on prices, while the oligopolists keep their strategic power. Furthermore, price taking and price making behaviours are endogenously determined in equilibrium and they are not assumed a priori. However, Okuno et al. (1980) considered particular types of mixed exchange economies with only two commodities and interior initial endowments and they do not provide an existence result. Busetto, Codognato, and Ghosal (2011) generalized Okuno et alii's model to mixed exchange economies with a finite number of commodities by using the strategic market game analysed by Sahi and Yao (1989) and they proved for such model the existence of a Cournot-Nash equilibrium. Therefore, this contribution provides a strategic foundation of oligopoly since it proves the existence of a Cournot-Nash equilibrium in which endogenously traders on the continuum behave competitively while atoms keep market power.

This asymmetric oligopoly obtained by considering strategic market games in mixed exchange economies is similar to the asymmetric oligopoly considered in the Cournot-Walras approach in exchange economies. However, Codognato (1995) showed that in mixed exchange economies the set of Cournot-Nash allocations does not coincide with the set of Cournot-Walras allocations. This is due to the fact that the Cournot-Walras approach has an intrinsic two-stage structure –the supply decisions of oligopolists and the exchange of commodities among traders– whereas in strategic market games all traders choose their actions simultaneously. Consequently, the strategic foundation provided in Busetto et al. (2011) concerns a simultaneously oligopoly which is different from the case considered in the Cournot-

Walras approach. There is also a second reason for the non-equivalence between the two models which is more subtle. In the Cournot-Walras approach, each oligopolist is characterised by a mixed behaviour, since he behaves strategically in making his supply decisions on the commodities in his portfolio while acts competitively in demanding all commodities.[1] Differently, in strategic market games atoms behave strategically on all commodities.

The issue of providing a strategic foundations to the Cournot-Walras approach was studied by Busetto, Codognato, and Ghosal (2008). They first provide a new definition of Cournot-Walras equilibrium in which oligopolists behave strategically on all commodities in the economy. They then reformulate the window model as a two-stage game in which atoms move in the first stage and traders in the continuum move in the second stage. With this new definition of Cournot-Walras equilibrium and the two-stage game, they show that the set of Cournot-Walras equilibria coincide with a specific set of subgame perfect equilibria, which is called the set of Pseudo-Markov perfect equilibria. Unfortunately, the paper does not provide an existence result for a Pseudo-Markov perfect equilibrium, which is the Cournot-Walras equilibrium.

1.5 Outline of the thesis

This thesis studies oligopolies in two different kinds of exchange economies. In Chapters 2 and 3, we consider exchange economies with infinitely many commodities and traders while, in Chapter 4, we consider mixed exchange economies with two commodities. However, in all three chapters, the analysis is developed by using a strategic market game with trading posts where trade takes place by using commodity money.

In the previous sections, we have seen that both the contributions of Negishi (1961) and Gabszewicz and Michel (1997) consider agents with mixed behaviours: the monopolistically competitive firms and the oligopolists respectively. A critique similar to the one stressed by Okuno et al. (1980) can be raised for these two models: *no formal explanation is given as to why a particular trader should behave strategically on some commodities and competitively on others.* In Chapter 2 and 3, we address this issue by considering a strategic market game with a countable infinity of commodities and traders in order to see if in such framework traders

[1]In Example 1 it is not possible to see this feature because we consider a two-commodity exchange economy with corner endowments.

with mixed behaviours arise endogenously in equilibrium. Indeed, when a trader is active on an infinite set of commodities, his market power can vanish along the sequence of commodities or can remain non-negligible. To describe these new phenomena, we introduce the notion of "asymptotic oligopolist", which describes a trader whose market power is uniformly bounded away from zero on an infinite set of commodities, and the notion of "asymptotic price-taker", which describes a trader whose market power converges to zero along the sequence of commodities. It is the latter that represents a trader with a kind of mixed behaviour since his market power can be made arbitrary small by choosing an appropriate infinite set of commodities while it is greater than a positive constant on a finite set.

In Chapter 2, we extend the strategic market game developed by Dubey and Shubik (1978) to exchange economies with a double infinity of commodities and traders. The main results of the chapter are a theorem of existence of a Cournot-Nash equilibrium in which all commodities are exchanged and a theorem of convergence which shows that when the economy is replicated the price vector and the allocation, at the Cournot-Nash equilibrium, converge to the Walras equilibrium of the underlying exchange economy. Furthermore, we illustrate the notions of asymptotic oligopolist and asymptotic price-taker by means of examples. We also show the main difficulties which arise in proving the existence of asymptotic price-takers.

In Chapter 3, we address this last issue by considering a variation of the Dubey and Shubik game called "all for sale model". In this model, at the start of the game all traders are required to deposit all the commodities in their initial endowment, except commodity money, in the appropriate trading post in exchange for non-negotiable receipts. The fact that all commodities go into the trading posts at the beginning of the game simplifies the mathematical analysis of the model. In such framework, we provide the sufficient conditions on the fundamentals of an economy under which an asymptotic price-taker exists. Heuristically, an asymptotic price-taker exists if all markets are thick, i.e., the quantities of commodities exchanged in each trading post are non-negligible. Furthermore, we prove the existence of a Cournot-Nash equilibrium for the all for sale model with an infinite number of commodities and traders. It is worth to note that even if the games considered in the two chapters are similar the two proofs of existence require different assumptions and one is based on the Kakutani–Fan–Glicksberg Fixed Point Theorem while the other is based on the Brouwer–Schauder–Tychonoff Fixed Point Theorem (see Corollaries 17.55 and 17.56 in Aliprantis and Border (2006)).

In Chapter 4, which is the part devoted to mixed exchange economies, we study the strategic foundations of oligopoly in a bilateral oligopoly framework. In the previous section, we have seen that Gabszewicz and Mertens (1971) and Shitovitz (1973) were able to establish some equivalence results between the allocations in the core and the Walras allocations in mixed exchange economies. In this chapter, we raise the question whether, in the non-cooperative approach, an equivalence, or at least a non-empty intersection, between the sets of Cournot-Nash and Walras allocations may hold in mixed exchange economies. We answer to this question by showing that a necessary and sufficient condition for a Cournot–Nash allocation to be a Walras allocation is that all atoms demand a null amount of one of the two commodities. When our condition fails to hold, we also confirm, through some examples, the result obtained by Okuno et al. (1980): small traders always have a negligible influence on prices, while oligopolists keep their strategic power even when their behaviour turns out to be Walrasian in the cooperative approach. Furthermore, we discuss the threefold equivalence among the sets of core, Cournot-Nash, and Walras allocations with some examples. Unfortunately, our result depends on atoms' demand behaviours at a Cournot-Nash equilibrium and then further research should be devoted to find the conditions on the fundamentals of an economy, i.e., traders' size, endowments, and preferences, under which our theorem holds.

— 2 —

Non-Cooperative Oligopoly in Economies with Infinitely Many Commodities and Traders

2.1 Introduction

The celebrated works of Chamberlin (1933) and Robinson (1933) introduced the theory of monopolistic competition to overcome the cleavage between the complementary approaches of perfect competition and monopoly. Indeed both believed that the assumption of perfect competition was too restrictive and that the real world was characterised by a mixture of competitive behaviour and market power. The aim of this chapter is to develop a coherent and tractable theoretical model in which agents with some kind of mixed behaviour – market power on some markets and competitive behaviour on others – can arise endogenously in equilibrium without making *ad hoc* assumptions.

To consider traders who are "small" compared to the whole economy and may display the mixed behaviour, we extend the analysis of non-cooperative oligopoly to exchange economies with a countable infinity of commodities and traders. The infinity of commodities can be seen as a set of differentiated commodities by using the Hotelling line. Our analysis is based on the literature on strategic market games initiated by the seminal papers of Shubik (1973), Shapley (1976), and Shapley and Shubik (1977) to study exchange economies in which all traders can influence the

[0] Some of the materials of this chapter were published in Ghosal S., Tonin S. (2014), "Non-Cooperative Asymptotic Oligopoly in Economies with Infinitely Many Commodities", Discussion Paper 2014-15, Adam Smith Business School, University of Glasgow.

prices. In this games, when traders and commodities are finite, all traders turn out to have market power on all commodities. Dubey and Shapley (1994) and Codognato and Ghosal (2000) considered strategic market games with a continuum of traders to study agents that are negligible when compared to the whole economy and then behave competitively in equilibrium. Differently, in our framework with "small" agents, the market power of a trader active on an infinite set of commodities can vanish along the sequence of commodities or can remain non-negligible as in the finite case. To describe these phenomena, we introduce two new notions. We say that a trader is an "asymptotic oligopolist" if his market power is uniformly bounded away from zero on an infinite subset of commodities, otherwise, if his market power converges to zero along the sequence of commodities, we say that the trader is an "asymptotic price-taker". The former can be simply interpreted as the extension of the classical notion of oligopolist to infinite economies (see Cournot (1838) and Gabszewicz and Vial (1972) for the study of oligopolists in partial equilibrium and in general equilibrium respectively). The latter describes a trader with a kind of mixed behaviour since his market power can be made arbitrary small on an infinite set of commodities while it is greater than a positive constant only on a finite number of commodities.

In the previous literature on imperfect competition the mixed behaviour was usually obtained by considering agents characterised by portfolios of commodities which contain the commodities on which agents have market power. For instance, Negishi (1961) extended the theory of monopolistic competition of Chamberlin (1933) and Robinson (1933) from partial to general equilibrium. To do so, he considers monopolistically competitive firms characterized by portfolios of commodities containing all the commodities on which a firm has market power. He further assumes that portfolios are strict subsets of all commodities so that a monopolistically competitive firm has market power on a subset of commodities while it acts competitively on all others. In a similar vein, d'Aspremont, Dos Santos Ferreira, and Gérard-Varet (1997) considered exchange economies in which consumers are characterised by portfolios of commodities to introduce the "Cournotian Monopolistic Competition Equilibrium" which aims to generalize both the Cournot's solution and the monopolistic competition in partial equilibrium. Finally, Gabszewicz and Michel (1997) introduced the notion of portfolio in the Cournot-Walras approach in exchange economies initiated by Codognato and Gabszewicz (1991). In their model, some traders are defined oligopolists and each of them is characterized by a portfolio which is a subset of the commodities held by the oligopolist.

Therefore, each oligopolist has a mixed behaviour: strategic behaviour in supplying the commodities which belong to the portfolio and competitive behaviour in demanding all the commodities and in supplying the commodities which are not in the portfolio. A common feature of all these contributions is that the portfolio of commodities is a primitive of the model and then no formal explanation is given as to why a particular trader should behave strategically on some commodities and competitively on others. In contrast, by using strategic market games, traders' market power is not assumed but is endogenously determined in equilibrium.

Our contributions are as follows. We first define an exchange economy with a countable infinity of commodities and traders having a structure of multilateral oligopoly, that is an economy in which each trader holds only commodity money and one other commodity. We then extend the strategic market game analysed by Dubey and Shubik (1978) to this particular setting. This is a strategic market game with commodity money in which there is a trading post for each commodity where the commodity is exchanged for commodity money. The actions available to traders are offers, amounts of commodities put up in exchange for commodity money, and bids, amounts of commodity money given in exchange for other commodities. Since in this game a Cournot-Nash equilibrium with no trade always exists, we prove the existence of an "active" Cournot-Nash equilibrium at which all commodities are exchanged. After having defined the model and proved the existence result, we analyse traders' market power at a Cournot-Nash equilibrium by introducing the formal definitions of asymptotic oligopolist and asymptotic price-taker. We then consider some examples to illustrate these two notions and to show, heuristically, under which conditions an asymptotic oligopolist exists. Perhaps surprisingly, we construct an example where, even if the number of traders active in each trading post is not uniformly bounded from above, there are traders "big enough", in terms of initial endowment of commodity money, who are asymptotic oligopolists. In Example 4, we show why an asymptotic oligopolist and an asymptotic price-taker can be heuristically interpreted as global and local oligopolists respectively. Furthermore, this example clarifies why the definitions of asymptotic oligopolist and asymptotic price-taker are based on the notion of limit. As in the previous contributions on strategic market games (see Dubey and Shubik (1978), Sahi and Yao (1989), and Amir, Sahi, Shubik, and Yao (1990), among others), we prove that the price vector and the allocation, at the Cournot-Nash equilibrium, converge to the Walras equilibrium when the number of each type of trader tends to infinity. We finally consider the case of exchange economies with an infinite aggregate endowment of commodity money. In such setting, we prove the existence of an active Cournot-Nash equilibrium and we consider some examples with asymptotic oligopolists and asymptotic price-takers. Obviously, since there is an infinite amount of commodity money, a Walras equilibrium does not exist.

From a mathematical point of view, our approach relies on the literature on economies with infinitely many commodities and with a double infinity of commodities and traders initiated by Bewley (1972) and Balasko, Cass, and Shell (1980) respectively. Our existence result is different from the one provided by Dubey and Shubik (1978) because we prove the existence of a Cournot-Nash equilibrium at which all commodities are exchanged while they proved the existence of an "equilibrium point" that is a Cournot-Nash equilibrium in which some commodities are legitimately not exchanged (see Cordella and Gabszewicz (1998) and Busetto and Codognato (2006) for a detailed analysis on "legitimately inactive" trading posts). Furthermore, the proof of existence adapts the approach used by Bloch and Ferrer (2001) for the case of two commodities to a setting with an infinite set of commodities and it is based on the Brouwer–Schauder–Tychonoff Fixed Point Theorem and on the Generalized Kuhn-Tucker Theorem. This last theorem is systematically used to prove that each commodity is exchanged at the Cournot-Nash equilibrium. Since we deal with a framework with an infinite number of commodities, some non classical restrictions on marginal utilities are needed to ensure that the vector of prices lies in a compact set bounded away form zero.

The chapter is organised as follows. In Section 2.2, we introduce the mathematical model. In Section 2.3, we prove the existence theorem. In Section 2.4, we introduce the definitions of asymptotic oligopolist and asymptotic price-taker and we show the examples. In Section 2.5, we prove the convergence theorem. In Section 2.6, we consider exchange economies with an infinite aggregate endowment of commodity money. In Section 2.7, we draw some conclusions from our analysis. In the appendixes, we list some mathematical definitions and results, we relate our market power measure with the notions of marginal price and average price introduced by Okuno, Postlewaite, and Roberts (1980), and we compare our model with the strategic market game analysed by Dubey and Shubik (1978).

2.2 Mathematical model

In this section, we define an exchange economy with a countable infinity of commodities and traders and the strategic market game associated to it. Furthermore, we make the assumptions necessary to prove the existence of a Cournot-Nash equilibrium at which all commodities are exchanged.

Let T_t be a finite set with cardinality k strictly greater than 1. Elements of T_t are traders of type t. The set of traders is $I = \cup_{t=1}^{\infty} T_t$. The set of commodities is $J = \{0, 1, 2, \dots\}$. The consumption set is denoted by X. A commodity bundle x is a point in X with x_j the amount of commodity j. A trader i is characterised by an initial endowment, $w^i \in X$, and a utility function, $u^i : X \to \mathbb{R}$, which describes his preferences. Traders of the same type have the same initial endowment and utility function. The context should clarify whether the superscript refers to a trader type or to a trader. An exchange economy is then a set ${}_k\mathcal{E} = \{(u^i, w^i) : i \in I\}$, with k the number of traders of each type.

An allocation $\mathbf{x}$ is a specification of a commodity bundle x^i, for each $i \in I$, such that $\sum_{i \in I} x_j^i = \sum_{i \in I} w_j^i$, for each $j \in J$. A price vector is denoted by p. Given a price vector p, we define the budget set of a trader i to be $B^i(p) = \{x \in X : \sum_{j=0}^{\infty} p_j x_j^i \leq \sum_{j=0}^{\infty} p_j w_j^i\}$. A Walras equilibrium is a pair $(p, \mathbf{x})$ consisting of a price vector p and an allocation $\mathbf{x}$ such that x^i is maximal with respect to u^i in i's budget set, for each $i \in I$.

A commodity j is desired by a trader i if u^i is an increasing function of the variable x_j^i and $\lim_{x_j^i \to 0} \frac{\partial u^i}{\partial x_j^i} = \infty$, for any fixed choice of the other variables. The set of commodities desired by a trader i is denoted by L^i.

We make the following assumptions.

Assumption 1. Let σ be a positive constant. The initial endowment of a type t trader is such that $w_0^t > 0$, $w_t^t > \sigma$, and $w_j^t = 0$, for each $j \in J \setminus \{0, t\}$, for $t = 1, 2, \dots$.

Assumption 2. Let e be a positive constant such that $\sigma < e$. The aggregate initial endowment of each commodity is such that $\sum_{t=1}^{\infty} w_0^t < e$ and $\sum_{t=1}^{\infty} w_j^t = w_j^j < e$, for each $j \in J \setminus \{0\}$.

Assumption 3. The consumption set X is a subset of the space of non-negative bounded sequences ℓ_∞^+ endowed with the product topology, i.e., $X = \{x \in \ell_\infty^+ : \sup_j |x_j| \leq ke\}$.[1]

Assumption 4. The utility function of a type t trader is continuous, continuously

[1] All the mathematical definitions and results can be found in Appendix 2.A.

Frèchet differentiable,[2] non-decreasing, and concave, for $t = 1, 2, \ldots$. Moreover, let λ and f be two positive constants such that $\lambda < f$, the marginal utilities of a type t trader are such that (i) $\lambda \leq \frac{\partial u^t}{\partial x_0^t}(x^t) \leq f$ and $\lambda \leq \frac{\partial u^t}{\partial x_t^t}(x^t) \leq f$, for each $x^t \in X$; (ii) $\frac{\partial u^t}{\partial x_t^t}(x^t) \leq \frac{\partial u^t}{\partial x_0^t}(x^t)$, for each x^t such that $x_t^t = w_t^t$, for $t = 1, 2, \ldots$.

Assumption 5. A commodity j is desired by at least one type of trader, for each $j \in J \setminus \{0\}$.

The first assumption formalises the notion of multilateral oligopoly and the second ensures that the aggregate initial endowment of all commodities is uniformly bounded from above. The third assumption imposes restrictions on the consumption set which are standard in the literature on infinite economies. In the first part of Assumption 4, we made the classical restrictions on traders' preferences. We need the additional conditions (i) and (ii) on the marginal utilities of commodity money and the commodity held by the trader because the space of commodities is infinite. Condition (i) is needed to prove that the price vector lies in a compact set uniformly bounded away from zero. Condition (ii) guarantees that each trader puts up in exchange the commodity in his initial endowment at the Cournot-Nash equilibrium. Utility functions which satisfy these conditions are, for instance, linear in commodity money and separable respect to the commodity in the endowment of the trader, i.e., $u^t(x) = x_0^t + z^t(x_t^t) + v^t(x_1^t, \ldots, x_{t-1}^t, x_{t+1}^t, \ldots)$.[3] The last assumption is standard in the literature on strategic market games but our definition of desired commodity is stronger than the one of Dubey and Shubik (1978) because we impose restrictions on the limits of marginal utilities.

We now introduce the strategic market game ${}_k\Gamma$ associated with the exchange economy, with k the number of traders of each type.[4] In this game, each trader has two types of actions: the offer of the commodity in his initial endowment and the bids of commodity money on all other commodities. So, the strategy set of a trader i of type t is

$$S^i = \Big\{ s^i = (q_t^i, b_1^i, \ldots, b_{t-1}^i, b_{t+1}^i, \ldots) : 0 \leq q_t^i \leq w_t^i,\ b_j^i \geq 0, \text{ for } j \in J \setminus \{0, t\}, \text{ and } \sum_{j \neq 0, t} b_j^i \leq w_0^i \Big\},$$

[2] Differentiability should implicitly be understood to include the case of infinite partial derivatives along the boundary of the consumption set (for a discussion of this case, see, for instance, Kreps (2012), p. 58).

[3] Shubik and Yao (1989) made assumptions similar to ours to prove the existence of an equilibrium in an infinite-horizon strategic market game with commodity money.

[4] A similar game was introduced by Shubik (1973). In Appendix 2.C, we prove that in our setting the game ${}_k\Gamma$ is equivalent to the game analysed by Dubey and Shubik (1978) in terms of attainable commodity bundles at the Cournot-Nash equilibrium.

where q_t^i is the offer of commodity t that trader i puts up in exchange for commodity money and b_j^i is the bid of commodity money that he makes on commodity j. Without loss of generality, we make the following technical assumption on the strategy set.

Assumption 6. The set S^i is a subset of ℓ_∞^+ endowed with the product topology, for each $i \in I$, i.e., $S^i \subseteq \{s^i \in \ell_\infty^+ : \sup_j |s_j^i| \leq e\}$.

This assumption implies that S^i lies in a normed space and therefore in a Hausdorff space. Let $S = \prod_{i \in I} S^i$ and $S^{-z} = \prod_{i \in I \setminus \{z\}} S^i$. Let s and s^{-i} be elements of S and S^{-i} respectively.

In the game, there is a trading post for each commodity where its price is determined and the commodity is exchanged for commodity money. For each $s \in S$, the price vector $p(s)$ is such that

$$p_j(s) = \begin{cases} \frac{\bar{b}_j}{\bar{q}_j} & \text{if } \bar{q}_j \neq 0, \\ 0 & \text{if } \bar{q}_j = 0, \end{cases}$$

for each $j \in J \setminus \{0\}$, with $\bar{q}_j = \sum_{i \in T_j} q_j^i$ and $\bar{b}_j = \sum_{i \in I \setminus T_j} b_j^i$. By Assumption 2, the sums $\bar{q}_j$ and $\bar{b}_j$ are uniformly bounded from above, for each $j \in J \setminus \{0\}$. For each $s \in S$, the final holding $x^i(s)$ of a trader i of type t is such that

$$x_0^i(s) = w_0^i - \sum_{j \neq 0, t} b_j^i + q_t^i p_t(s), \tag{2.1}$$

$$x_t^i(s) = w_t^i - q_t^i,$$

$$x_j^i(s) = \begin{cases} \frac{b_j^i}{p_j(s)} & \text{if } p_j(s) \neq 0, \\ 0 & \text{if } p_j(s) = 0, \end{cases} \tag{2.2}$$

for each $j \in J \setminus \{0, t\}$.

The payoff function of a trader i, $\pi^i : S \to \mathbb{R}$, is such that $\pi^i(s) = u^i(x^i(s))$.

We now introduce the definitions of an active trading post, a best response correspondence, and a Cournot-Nash equilibrium.

Definition 1. A trading post for a commodity j is said to be active if $\bar{q}_j > 0$ and $\bar{b}_j > 0$, otherwise we say that the trading post is inactive.

Definition 2. The best response correspondence of a trader i is a correspondence $\phi^i : S^{-i} \to S^i$ such that

$$\phi^i(s^{-i}) \in \arg\max_{s^i \in S^i} \pi^i(s^i, s^{-i}),$$

for each $s^{-i} \in S^{-i}$.

Definition 3. An $\hat{s} \in S$ is a Cournot-Nash equilibrium of ${}_k\Gamma$ if $\hat{s}^i \in \phi^i(\hat{s}^{-i})$, for each $i \in I$.

We also define the following particular types of Cournot-Nash equilibria. An active Cournot-Nash equilibrium is a Cournot-Nash equilibrium such that all trading posts are active. A type-symmetric Cournot-Nash equilibrium is a Cournot-Nash equilibrium such that all traders of the same type play the same strategy. An interior Cournot-Nash equilibrium is a Cournot-Nash equilibrium such that $\sum_{j \neq 0,t} b^i_j < w^i_0$, for each $i \in I$.

2.3 Theorem of existence

In this section, we state and prove the theorem of existence of an active Cournot-Nash equilibrium for the game ${}_k\Gamma$. Before to do so, we introduce some additional notions and we prove some lemmas. Following Dubey and Shubik (1978), in order to prove the existence of a Cournot-Nash equilibrium, we introduce the perturbed strategic market game ${}_k\Gamma^\epsilon$, the function $x^i_0(x^i_1, x^i_2, \dots)$, and the set $Y^i(s^{-i}, \epsilon)$.[5] The perturbed strategic market game ${}_k\Gamma^\epsilon$ is a game defined as ${}_k\Gamma$ with the only exception that the price vector $p(s)$ becomes

$$p^\epsilon_j(s) = \frac{\bar{b}_j + \epsilon}{\bar{q}_j + \epsilon},$$

for each $j \in J \setminus \{0\}$, with $\epsilon > 0$. The interpretation is that an outside agency places a fixed bid of ϵ and a fixed offer of ϵ in each trading post. This does not change the strategy sets of traders, but does affect the prices, the final holdings, and the payoffs. Consider, without loss of generality, a trader i of type t and fix the strategies s^{-i} for all other traders. Let

$$x^i_0(x^i_1, x^i_2, \dots) = w^i_0 - \sum_{j \neq 0,t} \frac{(\bar{b}^i_j + \epsilon)x^i_j}{\bar{q}_j + \epsilon - x^i_j} + \frac{(\bar{b}_t + \epsilon)(w^i_t - x^i_t)}{\bar{q}^i_t + \epsilon + w^i_t - x^i_t}$$

and let

$$Y^i(s^{-i}, \epsilon) = \Big\{(x^i_1, x^i_2, \dots) \in X : x^i_t = w^i_t - q^i_t,\ x^i_j = b^i_j \frac{\bar{q}_j + \epsilon}{\bar{b}^i_j + b^i_j + \epsilon},$$
$$\text{for each } j \in J \setminus \{0, t\},\ \text{for each } s^i \in S^i\Big\},$$

with $\bar{q}^i_t = \bar{q}_t - q^i_t$ and $\bar{b}^i_j = \bar{b}_j - b^i_j$. The function $x^i_0(x^i_1, x^i_2, \dots)$ can be easily obtained by the function $x^i_0(s^i)$ in (2.1) by relabelling the variables. Furthermore,

[5] Dubey and Shubik (1978) denotes the set $Y^i(s^{-i}, \epsilon)$ with $D^i(Q, B, \epsilon)$.

it is straightforward to verify that this function is strictly concave since it is a sum of concave and strictly concave functions. In the next proposition, by following the proof in Appendix A of Dubey and Shubik (1978), we prove that $Y^i(s^{-i}, \epsilon)$ is a convex set.

Proposition 1. The set $Y^i(s^{-i}, \epsilon)$ is convex.

Proof. Consider, without loss of generality, a trader i of type t and fix the strategies s^{-i} for all other traders. Take two commodity bundles $x'^i, x''^i \in Y^i(s^{-i}, \epsilon)$ and consider $\tilde{x}^i = \alpha x'^i + (1-\alpha)x''^i$, with $\alpha \in (0,1)$. We want to show that $\tilde{x}^i \in Y^i(s^{-i}, \epsilon)$. Hence, there must exist a strategy $\tilde{s}^i \in S^i$ such that $x^i(\tilde{s}^i) = \tilde{x}^i$. Let $x'^i = x^i(s'^i)$ and $x''^i = x^i(s''^i)$. Consider first the commodity t. It is straightforward to verify that $\tilde{x}^i_t = x^i_t(\alpha q'^i_t + (1-\alpha)q''^i_t)$. Consider now a commodity $j \neq t$. By equation (2.2), the function $x^i_j(b^i_j)$ is concave in b^i_j, then

$$\tilde{x}^i_j = \alpha x'^i_j + (1-\alpha)x''^i_j = \alpha x^i_j(b'^i_j) + (1-\alpha)x^i_j(b''^i_j) \leq x^i_j(\alpha b'^i_j + (1-\alpha)b''^i_j) = x^{*i}_j.$$

By the intermediate value theorem and since setting $b'^i_j = 0$ and $b''^i_j = 0$ would make $x^{*i}_j = 0$, we may reduce b'^i_j and b''^i_j appropriately to get $\tilde{x}^i_j$, for each $j \in J \setminus \{0\}$. $\square$

In the next lemma, we prove the existence of a Cournot-Nash equilibrium in the perturbed game.

Lemma 1. Under Assumptions 1, 2, 3, 4, 5, and 6, for each $\epsilon > 0$, there exists a Cournot-Nash equilibrium for ${}_k\Gamma^\epsilon$.

Proof. Consider, without loss of generality, a trader i and fix the strategies s^{-i} for all other traders. In the perturbed game the payoff function π^i is continuous because it is a composition of continuous functions (see Theorem 17.23, p. 566 in Aliprantis and Border (2006), AB hereafter). By Tychonoff Theorem (see Theorem 2.61, p. 52 in AB), S^i is compact. By Weierstrass Theorem (see Corollary 2.35, p. 40 in AB), there exists a strategy $\hat{s}^i$ that maximises the payoff function. We then consider the best response correspondence $\phi^i : S^{-i} \to S^i$. Since S^i is a non-empty and compact Hausdorff space, by Berge Maximum Theorem (see Theorem 17.31, p. 570 in AB), ϕ^i is an upper hemicontinuous correspondence.

We show now that ϕ^i is a continuous function. Suppose that there are two feasible commodity bundles x'^i and x''^i that maximise the utility function. Consider the commodity bundle $\tilde{x}^i_i = \frac{1}{2}x'^i + \frac{1}{2}x''^i$. Since the utility function is concave, $u^i(\tilde{x}^i) \geq \frac{1}{2}u^i(x'^i) + \frac{1}{2}u^i(x''^i) = u^i(x'^i)$. Since $x^i_0(x^i_1, x^i_2, \dots)$ is strictly concave and

$Y^i(s^{-i}, \epsilon)$ is convex, there exists a $\gamma > 0$ such that $\tilde{x}^i + \gamma e_0$ is a feasible allocation[6]. But then, since the utility function is strictly increasing in x_0^i, $u^i(\tilde{x}^i + \gamma e_0) > u^i(x'^i)$, a contradiction. Therefore, there is only one commodity bundle that maximises the utility function and ϕ^i is a single valued correspondence. Hence, ϕ^i is a continuous function (see Lemma 17.6, p.559 in AB).

As we are looking for a fixed point in the strategy space S, let's consider $\phi^i : S \to S^i$. Let $\Phi : S \to S$ such that $\Phi(S) = \prod_{i \in I} \phi^i(S)$. The function Φ is continuous since it is a product of continuous functions (see Theorem 17.28, p. 568 in AB). By Tychonoff Theorem, S is compact. Moreover, S is a non-empty and convex Hausdorff space. Therefore, by Brouwer-Schauder-Tychonoff Theorem (see Corollary 17.56, p. 583 in AB), there exists a fixed point $\hat{s}$ of Φ, which is a Cournot-Nash equilibrium of the perturbed game ${}_k\Gamma^\epsilon$. □

In the next lemma, we prove that all traders make positive bids on the commodities which are desired at a Cournot-Nash equilibrium.

Lemma 2. At a Cournot-Nash equilibrium $\hat{s}$ of the perturbed game ${}_k\Gamma^\epsilon$, $\hat{b}_j^i > 0$, for each $j \in L^i$, for each $i \in I$.

Proof. Let $\hat{s}$ be a Cournot-Nash equilibrium of the perturbed game. Consider, without loss of generality, a trader i. First, suppose that $\hat{b}_l^i = 0$, for an $l \in L^i$, and $\sum_{j \neq 0,t} \hat{b}_j^i < w_0^i$. Consider a strategy s'^i such that $b_l'^i = \hat{b}_l^i + \gamma$, with γ sufficiently small, and all other actions equal to the actions of the original strategy $\hat{s}^i$. Since $\lim_{x_l^i \to 0} \frac{\partial u^i}{\partial x_l^i} = \infty$, $u^i(x^i(s'^i, \hat{s}^{-i})) > u^i(x^i(\hat{s}^i, \hat{s}^{-i}))$, a contradiction. Now, suppose that $\hat{b}_l^i = 0$, for an $l \in L^i$, and $\sum_{j \neq 0,t} \hat{b}_j^i = w_0^i$. Then, there exists a commodity m such that $\hat{b}_m^i > 0$. Consider a strategy s'^i such that $b_m'^i = \hat{b}_m^i - \gamma$, $b_l'^i = \hat{b}_l^i + \gamma$, with γ sufficiently small, and all other actions equal to the actions of the original strategy $\hat{s}^i$. Since $\lim_{x_l^i \to 0} \frac{\partial u^i}{\partial x_l^i} = \infty$, $u^i(x^i(s'^i, \hat{s}^{-i})) > u^i(x^i(\hat{s}^i, \hat{s}^{-i}))$, a contradiction. Hence, $\hat{b}_j^i > 0$, for each $j \in L^i$, for each $i \in I$. □

In the next lemma, we prove that all commodities are offered in the trading posts at a Cournot-Nash equilibrium.

Lemma 3. At a Cournot-Nash equilibrium $\hat{s}$ of the perturbed game ${}_k\Gamma^\epsilon$, $\bar{\hat{q}}_j > 0$, for each $j \in J \setminus \{0\}$.

Proof. Let $\hat{s}$ be a Cournot-Nash equilibrium of the perturbed game. Consider, without loss of generality, a trader i of type t. Then, $\hat{s}^i$ solves the following max-

[6] e_j is an infinite vector in ℓ_∞ whose jth component is 1, and all others are 0.

imisation problem

$$
\begin{aligned}
\max_{s^i} \quad & \pi^i(s^i, \hat{s}^{-i}), & & \\
\text{subject to} \quad & q_t^i \leq w_t^i, & (i) & \\
& \sum_{j \neq 0,t} b_j^i \leq w_0^i, & (ii) & \qquad (2.3)\\
& - q_t^i \leq 0, & (iii) & \\
& - b_j^i \leq 0, \text{ for each } j \in J \setminus \{0, t\}. & (iv) &
\end{aligned}
$$

The constraints can be written as a function $G : \ell_\infty \to Z$, with $Z \subset \ell_\infty$. It is straightforward to verify that Z contains a positive cone P with a non-empty interior and G is Fréchet differentiable. We now show that there exists an $h \in \ell_\infty$ such that $G(\hat{s}^i) + G'(\hat{s}^i)h < 0$, with G' the Fréchet derivative of G. In matrix form, it becomes

$$
\begin{bmatrix} \hat{q}_t^i - w_t^i \\ \sum_{j \neq t} \hat{b}_j^i - w_0^i \\ -\hat{q}_t^i \\ -\hat{b}_1^i \\ -\hat{b}_2^i \\ \cdots \end{bmatrix} + \begin{bmatrix} h_t \\ \sum_{j \neq t} h_j \\ -h_t \\ -h_1 \\ -h_2 \\ \cdots \end{bmatrix} < \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \cdots \end{bmatrix}.
$$

First, suppose that the constraints (i) and (ii) are not binding. Consider a vector h with $h_j = 0$, for each j such that $\hat{b}_j^i > 0$, and h_j positive and sufficiently small, for all other j. But then, $\hat{s}^i$ is a regular point. Now, suppose that the constraints (i) and (ii) are binding. Consider a vector h with h_j positive and sufficiently small, for each j such that $\hat{b}_j^i = 0$, and h_j negative and sufficiently small, for all other j. But then, $\hat{s}^i$ is a regular point. If either constraint (i) or (ii) is binding the previous argument leads, *mutatis mutandis*, to the same result. Hence, $\hat{s}^i$ is a regular point of the constrained set. Finally, since the game is perturbed and the utility function is continuously Fréchet differentiable, π^i is continuously Fréchet differentiable. Hence, we have proved that all the hypothesis of the Generalized Kuhn-Tucker Theorem are satisfied (see Appendix 2.A). Therefore, there exist non-negative multipliers $\hat{\lambda}_1^{i*}$ and $\hat{\mu}_t^{i*}$ such that

$$
\begin{aligned}
& \frac{\partial \pi^i}{\partial q_t^i}(\hat{s}^i, \hat{s}^{-i}) - \hat{\lambda}_1^{i*} + \hat{\mu}_t^{i*} = 0, \qquad (2.4)\\
& \hat{\lambda}_1^{i*}(\hat{q}_t^i - w_t^i) = 0, \\
& \hat{\mu}_t^{i*} \hat{q}_t^i = 0.
\end{aligned}
$$

Since the payoff function is defined as $\pi^i(s) = u^i(x^i(s))$, equation (2.4) becomes

$$\frac{\partial u^i}{\partial x_0^i}(x^i(\hat{s}))\frac{\bar{\hat{b}}_t + \epsilon}{\bar{\hat{q}}_t^i + \hat{q}_t^i + \epsilon}\left(1 - \frac{\hat{q}_t^i}{\bar{\hat{q}}_t^i + \hat{q}_t^i + \epsilon}\right) - \frac{\partial u^i}{\partial x_t^i}(x^i(\hat{s})) - \hat{\lambda}_1^{i*} + \hat{\mu}_t^{i*} = 0. \tag{2.5}$$

Suppose that $\bar{\hat{q}}_t = 0$. Then, $\hat{q}_t^i = 0$, for each trader i of type t, and $\hat{\lambda}_1^{i*} = 0$. Hence, the equation above becomes

$$\frac{\partial u^i}{\partial x_0^i}(x^i(\hat{s}))\frac{\bar{\hat{b}}_t + \epsilon}{\epsilon} - \frac{\partial u^i}{\partial x_t^i}(x^i(\hat{s})) + \hat{\mu}_t^{i*} = 0.$$

Since $\frac{\bar{\hat{b}}_t + \epsilon}{\epsilon} > 1$, by Lemma 2, and $\frac{\partial u^t}{\partial x_t^t}(x^t) \leq \frac{\partial u^t}{\partial x_0^t}(x^t)$ for each x^t such that $x_t^t = w_t^t$, by Assumption 4, then the left hand side of the equation is greater than zero, a contradiction. Hence, $\bar{\hat{q}}_j > 0$, for each $j \in J \setminus \{0\}$. □

By Lemmas 1, 2, and 3, there exists an active Cournot-Nash equilibrium in the perturbed game. In the next lemma, we prove that the price vector lies in a compact set bounded away from zero, for any ϵ, at an active Cournot-Nash equilibrium. Since the number of commodities is infinite, we could not apply the analogous lemma of Dubey and Shubik (1978).

Lemma 4. At an active Cournot-Nash equilibrium $\hat{s}$ of the perturbed game ${}_k\Gamma^\epsilon$, there exist two positive constants, independent from ϵ, C_j and D_j such that

$$C_j < p_j^\epsilon(\hat{s}) < D_j,$$

for each $j \in J \setminus \{0\}$. Moreover, C_j is uniformly bounded away from zero and D_j is uniformly bounded from above.

Proof. Let $\hat{s}$ be an active Cournot-Nash equilibrium of the perturbed game. Without loss of generality, let $j = l$. We first establish the existence of C_l. Since the Cournot-Nash equilibrium is active, there exists a trader i of type l such that $\hat{q}_l^i > 0$. Then, a decrease γ in i's offer of commodity l is feasible, with $0 < \gamma \leq \hat{q}_l^i$, and has the following incremental effects on the final holding of trader i

$$\begin{aligned}
x_0^i(\hat{s}(\gamma)) - x_0^i(\hat{s}) &= (\hat{q}_l^i - \gamma)\frac{\bar{\hat{b}}_l + \epsilon}{\bar{\hat{q}}_l + \epsilon - \gamma} - \hat{q}_l^i\frac{\bar{\hat{b}}_l + \epsilon}{\bar{\hat{q}}_l + \epsilon}, \\
&= \frac{\bar{\hat{b}}_l + \epsilon}{\bar{\hat{q}}_l + \epsilon}\left((\hat{q}_l^i - \gamma)\frac{\bar{\hat{q}}_l + \epsilon}{\bar{\hat{q}}_l + \epsilon - \gamma} - \hat{q}_l^i\right) \geq -p_l^\epsilon(\hat{s})\gamma, \\
x_j^i(\hat{s}(\gamma)) - x_j^i(\hat{s}) &= 0, \text{ for each } j \in J \setminus \{0, l\}, \\
x_l^i(\hat{s}(\gamma)) - x_l^i(\hat{s}) &= \gamma.
\end{aligned}$$

The inequality in the preceding array follows from the fact that $\bar{\bar{q}}_l + \epsilon > \bar{\bar{q}}_l + \epsilon - \gamma$. Then, we obtain the following vector inequality

$$x^i(\hat{s}(\gamma)) \geq x^i(\hat{s}) - p_l^\epsilon(\hat{s})\gamma e_0 + \gamma e_l.$$

By using a linear approximation of the utility function around the point $x^i(\hat{s})$, we obtain

$$u^i(x^i(\hat{s}(\gamma))) - u^i(x^i(\hat{s})) \geq -\frac{\partial u^i}{\partial x_0^i}(x^i(\hat{s}))p_l^\epsilon(\hat{s})\gamma + \frac{\partial u^i}{\partial x_l^i}(x^i(\hat{s}))\gamma + O(\gamma^2).$$

Since $x^i(\hat{s})$ is an optimum point, the left hand side of the inequality is negative and then

$$p_l^\epsilon(\hat{s}) > \frac{\partial u^i}{\partial x_l^i}(x^i(\hat{s}))\bigg/\frac{\partial u^i}{\partial x_0^i}(x^i(\hat{s})) = C_l.$$

By Assumption 4(i), $C_l \geq \frac{\lambda}{f}$. Then, $C_j \geq \frac{\lambda}{f}$, for each $j \in J \setminus \{0\}$. Now, we establish the existence of D_l. Since there are at least two traders of each type, we consider a trader i of type l such that $\hat{q}_l^i \leq \frac{\bar{\hat{q}}_l}{2}$. We need to consider two cases. First, suppose that $\hat{q}_l^i < w_l^i$. Then, an increase γ in i's offer of commodity l is feasible, with $0 < \gamma < \min\{w_l^i - \hat{q}_l^i, \epsilon\}$, and has the following incremental effects on the final holding of trader i

$$\begin{aligned}
x_0^i(\hat{s}(\gamma)) - x_0^i(\hat{s}) &= (\hat{q}_l^i + \gamma)\frac{\bar{\hat{b}}_l + \epsilon}{\bar{\hat{q}}_l + \epsilon + \gamma} - \hat{q}_l^i\frac{\bar{\hat{b}}_l + \epsilon}{\bar{\hat{q}}_l + \epsilon}, \\
&= \frac{\bar{\hat{b}}_l + \epsilon}{\bar{\hat{q}}_l + \epsilon}\frac{\bar{\hat{q}}_l^i + \epsilon}{\bar{\hat{q}}_l^i + \hat{q}_l^i + \epsilon + \gamma}\gamma \geq \frac{1}{3}p_l^\epsilon(\hat{s})\gamma, \\
x_j^i(\hat{s}(\gamma)) - x_j^i(\hat{s}) &= 0, \text{ for each } j \in J \setminus \{0, l\}, \\
x_l^i(\hat{s}(\gamma)) - x_l^i(\hat{s}) &= -\gamma.
\end{aligned}$$

The inequality in the preceding array follows from the fact that $\hat{q}_l^i \leq \bar{\hat{q}}_l^i + \epsilon$ and $\gamma \leq \bar{\hat{q}}_l^i + \epsilon$. Then, we obtain the following vector inequality

$$x^i(\hat{s}(\gamma)) \geq x^i(\hat{s}) + \frac{1}{3}p_l^\epsilon(\hat{s})\gamma e_0 - \gamma e_l.$$

By using a linear approximation of the utility function around the point $x^i(\hat{s})$, we obtain

$$u^i(x^i(\hat{s}(\gamma))) - u^i(x^i(\hat{s})) \geq \frac{\partial u^i}{\partial x_0^i}(x^i(\hat{s}))\frac{1}{3}p_l^\epsilon(\hat{s})\gamma - \frac{\partial u^i}{\partial x_l^i}(x^i(\hat{s}))\gamma + O(\gamma^2).$$

Since $x^i(\hat{s})$ is an optimum point, the left hand side of the inequality is negative and then

$$p_l^\epsilon(\hat{s}) < 3\bigg(\frac{\partial u^i}{\partial x_l^i}(x^i(\hat{s}))\bigg/\frac{\partial u^i}{\partial x_0^i}(x^i(\hat{s}))\bigg) = D_l^1.$$

By Assumption 4(i), $D_l^1 \leq 3\frac{f}{\lambda}$. Now, suppose that $\hat{q}_l^i = w_l^i$. Then,

$$p_l^\epsilon(\hat{s}) < \frac{ke}{w_l^i} = D_l^2,$$

with ke the total endowment of commodity money in the economy. By Assumption 1, $D_l^2 \leq \frac{ke}{\sigma}$. Finally, we choose D_l such that $D_l = \max\{D_l^1, D_l^2\}$. Hence, D_j is uniformly bounded from above as D_j^1 and D_j^2 are uniformly bounded from above, for each $j \in J \setminus \{0\}$. □

In the next lemma, we prove that there exists a positive lower bound, independent from ϵ, for each bid made by a trader on a desired commodity at an active Cournot-Nash equilibrium.

Lemma 5. At an active Cournot-Nash equilibrium $\hat{s}$ of the perturbed game ${}_k\Gamma^\epsilon$, there exists a positive constant B_j^i, independent of ϵ, such that

$$0 < B_j^i \leq \hat{b}_j^i,$$

for each $j \in L^i$, for each $i \in I$.

Proof. Let $\hat{s}$ be an active Cournot-Nash equilibrium of the perturbed game. Consider, without loss of generality, a trader i of type t. Then, $\hat{s}^i$ solves the maximisation problem in (2.3). As in the proof of Lemma 3, all the hypothesis of the Generalized Kuhn-Tucker Theorem are satisfied and then there exist non-negative multipliers $\hat{\lambda}_2^{i*}$ and $\hat{\mu}_j^{i*}$, for each $j \in J \setminus \{0, t\}$, such that

$$\begin{aligned}
&\frac{\partial \pi^i}{\partial b_j^i}(\hat{s}^i, \hat{s}^{-i}) - \hat{\lambda}_2^{i*} + \hat{\mu}_j^{i*} = 0, \text{ for each } j \in J \setminus \{0, t\}, \qquad (2.6)\\
&\hat{\lambda}_2^{i*}\Big(\sum_{j \neq 0,t} \hat{b}_j^i - w_0^i\Big) = 0,\\
&\hat{\mu}_j^{i*}\hat{b}_j^i = 0, \text{ for each } j \in J \setminus \{0, t\}.
\end{aligned}$$

Consider, without loss of generality, a commodity $l \in L^i$. Since the payoff function is defined as $\pi^i(s) = u^i(x^i(s))$, equation (2.6) becomes

$$-\frac{\partial u^i}{\partial x_0^i}(x^i(\hat{s})) + \frac{\partial u^i}{\partial x_l^i}(x^i(\hat{s}))\frac{\bar{\hat{q}}_l + \epsilon}{\bar{\hat{b}}_l^i + \hat{b}_l^i + \epsilon}\left(1 - \frac{\hat{b}_l^i}{\bar{\hat{b}}_l^i + \hat{b}_l^i + \epsilon}\right) - \hat{\lambda}_2^{i*} + \hat{\mu}_j^{i*} = 0. \qquad (2.7)$$

By Lemma 4 and since $\hat{s}$ is an active Cournot-Nash equilibrium, we obtain the following inequality from the previous equation

$$-\frac{\partial u^i}{\partial x_0^i}(x^i(\hat{s})) + \frac{\partial u^i}{\partial x_l^i}(x^i(\hat{s}))\frac{1}{D_l}\left(\frac{\bar{\hat{b}}_l^i + \epsilon}{\bar{\hat{b}}_l^i + \hat{b}_l^i + \epsilon}\right) - \hat{\lambda}_2^{i*} \leq 0. \qquad (2.8)$$

Suppose that $\hat{b}_l^i \to 0$. Then, $\lim_{\hat{b}_l^i \to 0} \frac{\bar{\hat{b}}_l^i + \epsilon}{\bar{\hat{b}}_l^i + \hat{b}_l^i + \epsilon} = 1$ and $\lim_{\hat{b}_l^i \to 0} \frac{\partial u^i}{\partial x_l^i}(x^i(\hat{s})) = \infty$, as $l \in L^i$. But then, since $\frac{\partial u^i}{\partial x_0^i}(x^i)$ and D_l are bounded from above, the left hand side of the inequality is positive, a contradiction. Hence, there exists a positive lower bound B_j^i, independent of ϵ, at which the left hand side of equation (2.8) is equal to zero, for each $j \in L^i$, for each $i \in I$. □

We now state the existence theorem.

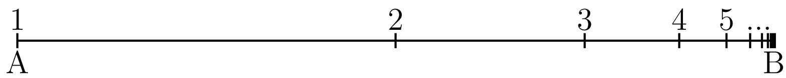
Theorem 1. Under Assumptions 1, 2, 3, 4, 5, and 6, there exists an active Cournot-Nash equilibrium for ${}_k\Gamma$.

Proof. Consider a sequence of $\{{}^g\epsilon\}_{g=1}^{\infty}$ converging to 0. By Lemmas 1, 2, and 3, in each perturbed game there exists an active Cournot-Nash equilibrium. Then, we can consider a sequence of active Cournot-Nash equilibria $\{{}^g\hat{s}\}_{g=1}^{\infty}$ associated to the sequence of ϵ. As proved before, S is compact and, by Lemma 4, $p^{\epsilon}({}^g\hat{s}) \in \prod_{j \neq 0}[C_j, D_j]$ with C_j uniformly bounded away from zero and D_j uniformly bounded from above, for each $j \in J \setminus \{0\}$. By Tychonoff Theorem, $\prod_{j \neq 0}[C_j, D_j]$ is compact. Then, we can pick a subsequence of $\{{}^g\hat{s}\}_{g=1}^{\infty}$ that converges to v such that $v \in S$ and $p(v) \in \prod_{j \neq 0}[C_j, D_j]$. Hence, v is a point of continuity of payoff functions and then it is a Cournot-Nash equilibrium, i.e., $\hat{v}$. It remains to prove that $\hat{v}$ is an active Cournot-Nash equilibrium. By Assumption 5 and Lemma 5, for each commodity $j \in J \setminus \{0\}$, $\bar{\hat{b}}_j \geq B_j^i > 0$, for a trader i such that $j \in L^i$. Suppose, without loss of generality, that there exists a commodity l such that $\bar{\hat{q}}_l = 0$. But then, $p_l(\hat{v}) \notin [C_l, D_l]$, a contradiction. Therefore, $\bar{\hat{q}}_j > 0$, for each $j \in J \setminus \{0\}$, and then $\hat{v}$ is an active Cournot-Nash equilibrium. □

2.4 Asymptotic oligopolies and asymptotic price takers

In this section, we analyse traders' market power at an active Cournot-Nash equilibrium of the game ${}_k\Gamma$. We study it by introducing the definitions of asymptotic oligopolist and asymptotic price-taker. These concepts allow us to extend the study of traders who can influence prices from finite to infinite economies. In fact, differently from the finite case, if a trader is active on an infinite number of commodities, he can keep market power on an infinite set of them or his market power can vanish along their sequence. We also provide some examples to illustrate these two notions and to show that the distinction is not trivial in our framework. Example 1 and 2 show economies with asymptotic oligopolists and asymptotic price-takers respectively. Example 3 shows that, perhaps surprisingly, even if the number of traders active in each trading post is not uniformly bounded from above, there are traders "big enough", in terms of initial endowment of commodity money, who are asymptotic oligopolists. This example also helps to understand heuristically under which conditions an asymptotic oligopolist exists. Finally, Example 4 shows why an asymptotic oligopolist can heuristically be seen as a global oligopolist and an asymptotic price-taker as a local one. We start our analysis by discussing a possible interpretation of the infinity of commodities and by defining an index to measure traders' market power.

It is straightforward to see that the mathematical structure of the exchange economy described in Section 2.2 is similar to an overlapping generation model (see Balasko, Cass, and Shell (1980), Wilson (1981), and Burke (1988), among others). As remarked in Geanakoplos and Polemarchakis (1991) "The countably infinite index of commodities need not to refer to calendar time. Location or any other characteristic suffice to give rise to economies analytically equivalent to economies of overlapping generations". Therefore, the infinite set of commodities can be interpreted as commodities in different locations or as a set of differentiated commodities respect to particular attribute. The next figure shows a possible arrangement of commodities on a Hotelling line. As the diagram makes clear,



commodity 1 is at the origin of the line, commodity 2 is in the middle of the line, commodity 3 is in the middle of the second half, and so on.

We measure traders' market power by using their market share in each trading post. Therefore, on the offers side, the market power of a trader i of type t on the commodity he holds can be measured by $q_t^i/\bar{q}_t$. The higher this ratio is, the higher is the market power of trader i. By Assumption 1, each commodity is offered by k traders and then each trader always keeps market power on the commodity held for any finite k. Similarly, on the bids side, the market power of a trader i of type t on a commodity j, with $j \neq t$, can be measured by the ratio $b_j^i/\bar{b}_j$. If $b_j^i = 0$, we say that trader i is a trivial price-taker on commodity j. Okuno et al. (1980) used a different way to measure traders' market power in strategic market games. They introduce the notions of marginal price and average price and they show that when the two prices are equal traders behave competitively. In Appendix 2.B, we show that our approach is equivalent to the one used by Okuno et al. (1980).

We now introduce the definitions of an asymptotic oligopolist and an asymptotic price-taker.

Definition 4. Consider an active Cournot-Nash equilibrium $\hat{s}$ in which there exists a trader i such that $\hat{b}^i_j > 0$ for an infinite number of commodities. We say that trader i is an asymptotic price-taker if $\lim_{j\to\infty} \hat{b}^i_j/\bar{\hat{b}}_j = 0$, otherwise trader i is an asymptotic oligopolist.

The key feature of an asymptotic oligopolist is that his market power is greater than a positive constant on an infinite subset of commodities. Differently, an asymptotic price-taker i is a trader such that, for any $\mu > 0$, there exists a $l \in J$ such that $b^i_j/\bar{b}_j < \mu$, for each $j > l$, i.e., his market power is smaller than μ on an infinite set of commodities, with μ arbitrary small. Therefore, an asymptotic price-taker is characterised by a mixed behaviour because his market power is greater than a positive constant on a finite set of commodities while it can be made arbitrary small on infinite sets of commodities.

To simplify computations, in all examples we consider logarithmic additive utility functions linear in commodity money and in the commodity held by the trader, i.e., $x_0+x_t+\sum_j \beta_j \ln x_j$.[7] The coefficient β_j converts the utilities associated with the consumption of commodities into commodity money. Furthermore, in the examples we consider only type-symmetric Cournot-Nash equilibria and then all superscripts denote types of traders.

In the first example, we show an exchange economy in which at the Cournot-Nash equilibrium type 1 traders are asymptotic oligopolists.

Example 1. Consider an exchange economy in which traders of type 1, 2, 3, and $t \geq 4$ have the following utility functions and initial endowments

$$
\begin{aligned}
u^1(x^1) &= x^1_0 + x^1_1 + \sum_{j=2}^{\infty} \delta^j \ln x^1_j & w^1 &= \left(\frac{1}{1-\delta}, 1, 0, \dots\right),\\
u^2(x^2) &= x^2_0 + \delta \ln x^2_1 + x^2_2 + \delta^3 \ln x^2_3 & w^2 &= \left(\frac{\delta^1}{1-\delta}, 0, 1, 0, \dots\right),\\
u^3(x^3) &= x^3_0 + x^3_3 + \delta^4 \ln x^3_4 & w^3 &= \left(\frac{\delta^2}{1-\delta}, 0, 0, 1, 0, \dots\right),\\
u^t(x^t) &= x^t_0 + x^t_t + \delta^{t+1} \ln x^t_{t+1} & w^t &= \left(\frac{\delta^{t-1}}{1-\delta}, 0, \dots, 0, 1, 0, \dots\right),
\end{aligned}
$$

[7] Logarithmic utility functions facilitate computations but are not continuous at the boundary. Therefore, they violate Assumption 4. This does not affect the current analysis but should be kept in mind.

with $\delta \in (0,1)$. The type-symmetric active Cournot-Nash equilibrium of the game ${}_k\Gamma$ associated to the exchange economy is

$$\begin{aligned}
\big(\hat{q}^1_1, \hat{b}^1_2, \hat{b}^1_3, \ldots, \hat{b}^1_j, \ldots\big) &= \big(\delta^1 G(1)^2, \delta^2 G(1), \delta^3 G(2), \ldots, \delta^j G(2), \ldots\big),\\
\big(\hat{q}^2_2, \hat{b}^2_1, \hat{b}^2_3, \hat{b}^2_4, \ldots, \hat{b}^2_j, \ldots\big) &= \big(\delta^2 G(1)^2, \delta^1 G(1), \delta^3 G(2), 0, \ldots, 0, \ldots\big),\\
\big(\hat{q}^3_3, \hat{b}^3_1, \hat{b}^3_2, \hat{b}^3_4, \hat{b}^3_5, \ldots, \hat{b}^3_j, \ldots\big) &= \big(2\delta^3 G(1)G(2), 0, 0, \delta^4 G(2), 0, \ldots, 0, \ldots\big),\\
\big(\hat{q}^t_t, \hat{b}^t_1, \ldots, \hat{b}^t_{t-1}, \hat{b}^t_{t+1}, \hat{b}^t_{t+2}, \ldots\big) &= \big(2\delta^t G(1)G(2), 0, \ldots, 0, \delta^{t+1} G(2), 0, \ldots\big),
\end{aligned}$$

for $t = 4, 5, \ldots$, with $G(y) = \big(1 - \frac{1}{yk}\big)$. At this Cournot-Nash equilibrium, type 1 traders are asymptotic oligopolists.

Proof. At the Cournot-Nash equilibrium, in a trading post for commodity $j \geq 3$ only types of traders 1 and $j-1$ are active and they both bid the same amount of commodity money. Therefore, the market power of type 1 traders, $\hat{b}^1_j/\bar{\hat{b}}_j$, is equal to $\frac{1}{2k}$, for $j \geq 3$. Hence, $\lim_{j\to\infty} \hat{b}^1_j/\bar{\hat{b}}_j = \frac{1}{2k}$. □

In this example, type 1 traders are asymptotic oligopolists because in equilibrium there are only two types of traders active in each trading post and all of them make the same bid. Heuristically, type 1 traders keep market power on all commodities because bids and markets' size shrink together along the sequence of commodities.

In the next example, we show an exchange economy in which at the Cournot-Nash equilibrium type 1 traders are asymptotic price-takers.

Example 2. Consider an exchange economy in which traders of type 1, 2, 3, and $t \geq 4$ have the following utility functions and initial endowments

$$\begin{aligned}
u^1(x^1) &= x^1_0 + x^1_1 + \sum_{j=2}^{\infty} \delta^j \ln x^1_j & w^1 &= \left(\frac{1}{1-\delta}, 1, 0, \ldots\right),\\
u^2(x^2) &= x^2_0 + \delta \ln x^2_1 + x^2_2 + \sum_{j=3}^{\infty} \delta^j \ln x^2_j & w^2 &= \left(\frac{\delta^1}{1-\delta}, 0, 1, 0, \ldots\right),\\
u^3(x^3) &= x^3_0 + x^3_3 + \sum_{j=4}^{\infty} \delta^j \ln x^3_j & w^3 &= \left(\frac{\delta^2}{1-\delta}, 0, 0, 1, 0, \ldots\right),\\
u^t(x^t) &= x^t_0 + x^t_t + \sum_{j=t+1}^{\infty} \delta^j \ln x^t_j & w^t &= \left(\frac{\delta^{t-1}}{1-\delta}, 0, \ldots, 0, 1, 0, \ldots\right),
\end{aligned}$$

with $\delta \in (0,1)$. The type-symmetric active Cournot-Nash equilibrium of the game ${}_k\Gamma$ associated to the exchange economy is

$$\begin{aligned}
\big(\hat{q}^1_1, \hat{b}^1_2, \hat{b}^1_3, \ldots, \hat{b}^1_j, \ldots\big) &= \big(\delta^1 G(1)^2, \delta^2 G(1), \delta^3 G(2), \ldots, \delta^j G(j-1), \ldots\big),\\
\big(\hat{q}^2_2, \hat{b}^2_1, \hat{b}^2_3, \ldots, \hat{b}^2_j, \ldots\big) &= \big(\delta^2 G(1)^2, \delta^1 G(1), \delta^3 G(2), \ldots, \delta^j G(j-1), \ldots\big),
\end{aligned}$$

$$\left(\hat{q}_3^3, \hat{b}_1^3, \hat{b}_2^3, \hat{b}_4^3, \ldots, \hat{b}_j^3, \ldots\right) = \left(2\delta^3 G(1)G(2), 0, 0, \delta^4 G(3), \ldots, \delta^j G(j-1), \ldots\right),$$
$$\left(\hat{q}_t^t, \hat{b}_1^t, \ldots, \hat{b}_{t-1}^t, \hat{b}_{t+1}^t, \ldots\right) = \left((t-1)\delta^t G(1)G(t-1), 0, \ldots, 0, \delta^{t+1} G(t), \ldots\right),$$

for $t = 4, 5, \ldots,$ with $G(y) = \left(1 - \frac{1}{yk}\right)$. At this Cournot-Nash equilibrium, all traders are asymptotic price-takers.

Proof. At the Cournot-Nash equilibrium, in a trading post for commodity $j \geq 3$ there are $j-1$ types of traders active and all of them bid the same amount of commodity money. Therefore, the market power of type t traders active on commodity j, $\hat{b}_j^t/\bar{\hat{b}}_j$, is equal to $\frac{1}{(j-1)k}$, for $j \geq 2$. Hence, $\lim_{j\to\infty} \hat{b}_j^t/\bar{\hat{b}}_j = 0$, for $t = 1, 2, \ldots$. □

In this example, in each trading post all traders types make the same bid and their number is strictly increasing. Therefore, the bids of each trader become negligible in comparison to the bids of all others along the sequence of commodities and then everyone is an asymptotic price-taker. A key difference with Example 1 is that the number of traders types active in each trading post is not uniformly bounded from above and this counteracts type 1 traders' market power. As remarked above, even if the market power of all traders vanishes along the sequence of commodities, each of them keeps market power on the commodity in the endowment since $q_t^t/\bar{q}_t = 1/k$, for $t = 1, 2, \ldots$. A different result is obtained if there exists a type of trader that places higher bids than other traders along the sequence of commodities.

In the next example, we show an exchange economy where there are asymptotic oligopolists even if the number of traders active in each trading post is not uniformly bounded from above.

Example 3. Consider an exchange economy in which traders of type 1, 2, 3, and $t \geq 4$ have the following utility functions and initial endowments

$$u^1(x^1) = x_0^1 + x_1^1 + \sum_{j=2}^{\infty} \frac{1}{j^2} \ln x_j^1, \qquad w^1 = \left(\frac{\pi^2}{6}, 1, 0, \ldots\right),$$
$$u^2(x^2) = x_0^2 + \delta \ln x_1^2 + x_2^2 + \sum_{j=3}^{\infty} \delta^j \ln x_j^2 \qquad w^2 = \left(\frac{\delta^1}{1-\delta}, 0, 1, 0, \ldots\right),$$
$$u^3(x^3) = x_0^3 + x_3^3 + \sum_{j=4}^{\infty} \delta^j \ln x_j^3 \qquad w^3 = \left(\frac{\delta^2}{1-\delta}, 0, 0, 1, 0, \ldots\right),$$
$$u^t(x^t) = x_0^t + x_t^t + \sum_{j=t+1}^{\infty} \delta^j \ln x_j^t \qquad w^t = \left(\frac{\delta^{t-1}}{1-\delta}, 0, \ldots, 0, 1, 0, \ldots\right).$$

For $k = 2$ and $\delta = 1/3$, the type-symmetric active Cournot-Nash equilibrium of the game ${}_k\Gamma$ associated to the exchange economy is

$$
\begin{aligned}
\left(\hat{q}_1^1, \hat{b}_2^1, \hat{b}_3^1, \ldots, \hat{b}_j^1, \ldots\right) &= \left(\frac{1}{12}, \frac{1}{8}, \frac{\sqrt{13}-1}{36}, \ldots, \frac{(2j+1)j^2 - 3^j - F(j)}{4j^2\left(j^2 - 3^j\right)}, \ldots\right),\\
\left(\hat{q}_2^2, \hat{b}_1^2, \hat{b}_3^2, \ldots, \hat{b}_j^2, \ldots\right) &= \left(\frac{1}{16}, \frac{1}{6}, \frac{7-\sqrt{13}}{108}, \ldots, \bar{\hat{b}}_j, \ldots\right),\\
\left(\hat{q}_3^3, \hat{b}_1^3, \hat{b}_2^3, \hat{b}_4^3, \ldots, \hat{b}_j^3, \ldots\right) &= \left(\frac{2+\sqrt{13}}{108}, 0, 0, \frac{227-\sqrt{4729}}{14040}, \ldots, \bar{\hat{b}}_j, \ldots\right),\\
\left(\hat{q}_t^t, \hat{b}_1^t, \ldots, \hat{b}_{t-1}^t, \hat{b}_{t+1}^t, \ldots\right) &= \left(\frac{(2j-5)j^2 + 3^j + F(j)}{3^j 8 j^2}, 0, \ldots, 0, \bar{\hat{b}}_j, \ldots\right),
\end{aligned}
$$

with $\bar{\hat{b}}_j = \frac{(5-2j)j^2 + 3^j(4j-7) - F(j)}{3^j 4(j-2)(3^j - j^2)}$ and $F(y) = \sqrt{(5-2y)^2 y^4 + 3^y 2y^2(6y-11) + 9^y}$, for $t = 4, 5, \ldots$. At this Cournot-Nash equilibrium, type 1 traders are asymptotic oligopolists.

Proof. At the Cournot-Nash equilibrium, the market power of type 1 traders, $\hat{b}_j^1/\bar{\hat{b}}_j$, is equal to $\frac{(2j-5)j^2 + 3^{j+1} - F(j)}{4(3^j - j^2)}$, for each $j \geq 3$. Hence, $\lim_{j\to\infty} \hat{b}_j^1/\bar{\hat{b}}_j = \frac{1}{2}$. $\square$

In this example, type 1 traders are again asymptotic oligopolists because in each trading post they place sufficiently higher bids and therefore their bids do not become negligible respect to other traders' bids. Heuristically, a trader, with a higher initial endowment of commodity money and a sequence of utility coefficients that converges to zero slowly enough, can be an asymptotic oligopolist even if the number of traders active in each trading post is not uniformly bounded from above. In Example 3, indeed, type 1 traders have an higher initial endowment of commodity money than in the previous examples and the sequence $\{\frac{1}{j^2}\}$ converges to zero slower that the sequence $\{\delta^j\}$.

In the next example, we show why the notions of asymptotic oligopolist and asymptotic price-taker are defined by using limits instead of limit points.

Example 4. Consider an exchange economy in which traders of type 1, 2, $s \geq 3$ odd, and $t \geq 4$ even have the following utility functions and initial endowments

$$
\begin{aligned}
u^1(x^1) &= x_0^1 + x_1^1 + \sum_{j=2}^{\infty} \delta^j \ln x_j^1 & w^1 &= \left(\frac{1}{1-\delta}, 1, 0, \ldots\right),\\
u^2(x^2) &= x_0^2 + \delta \ln x_1^2 + x_2^2 + \delta^3 \ln x_3^2 & w^2 &= \left(\frac{\delta^1}{1-\delta}, 0, 1, 0, \ldots\right),\\
u^s(x^s) &= x_0^s + x_s^s + \frac{1}{(s+1)^2} \ln x_{s+1}^s & w^s &= \left(\frac{1}{(s-1)^2}, 0, \ldots, 0, 1, 0, \ldots\right),
\end{aligned}
$$

$$u^t(x^t) = x_0^t + x_t^t + \delta^{t+1} \ln x_{t+1}^t \qquad w^t = \left(\frac{\delta^{t-1}}{1-\delta}, 0, \ldots, 0, 1, 0, \ldots\right).$$

For $k = 2$ and $\delta = 1/3$, the type-symmetric active Cournot-Nash equilibrium of the game ${}_k\Gamma$ associated to the exchange economy is

$$\begin{aligned}
(\hat{b}_1^1, \hat{b}_2^1, \hat{b}_3^1, \ldots, \hat{b}_j^1, \hat{b}_{j+1}^1, \ldots) &= \left(\frac{1}{12}, \frac{1}{18}, \frac{1}{36}, \ldots, \frac{6}{3^j 5 - j^2 + F(j)}, \frac{3}{3^{j+1}4}, \ldots\right), \\
(\hat{q}_2^2, \hat{b}_1^2, \hat{b}_3^2, \hat{b}_4^2, \ldots, \hat{b}_j^2, \ldots) &= \left(\frac{1}{36}, \frac{1}{6}, \frac{1}{36}, 0, \ldots, 0, \ldots\right), \\
(\hat{q}_s^s, \hat{b}_1^s, \ldots, \hat{b}_{s-1}^s, \hat{b}_{s+1}^s, \ldots) &= \left(\frac{3}{3^s 4}, 0, \ldots, 0, \frac{6}{5(s+1)^2 - 3^{s+1} + F(s+1)}, 0, \ldots\right), \\
(\hat{q}_t^t, \hat{b}_1^t, \ldots, \hat{b}_{t-1}^t, \hat{b}_{t+1}^t, \ldots) &= \left(\frac{t^2 + 3^t + F(t)}{3^t 8 t^2}, 0, \ldots, 0, \frac{3}{3^{t+1}4}, 0, \ldots\right),
\end{aligned}$$

for $s \geq 3$ odd and $t \geq 4$ even, with $F(y) = \sqrt{y^4 + 3^y 14 y^2 + 9^y}$. At this Cournot-Nash equilibrium, type 1 traders are asymptotic oligopolists.

Proof. At the Cournot-Nash equilibrium, the market power of type 1 traders, $\hat{b}_j^1/\bar{\hat{b}}_j$, is equal to $\frac{1}{4}$, for each $j \geq 3$ odd, and to $\frac{2j^2}{3j^2 + F(j) + 3^j}$, for each $j \geq 4$ even. Hence, the sequence of market power has two limit points $\frac{1}{4}$ and 0. □

In this example, the sequence of market power has two different limit points and even if one of them is zero, type 1 traders are asymptotic oligopolists because there is an infinite number of commodities on which their market power is greater than a positive constant. These traders can be seen as global oligopolists precisely because their market power is uniformly bounded away from zero on an infinite subset of commodities. In the definition of asymptotic price-takers, we have used the notion of limit to rule out such traders who have a market power uniformly bounded away from zero on odd commodities and which converges to zero on the even ones. By doing so, an asymptotic price-taker can be seen as a local oligopolist because his market power is greater than a positive constant on a finite set of commodities while it is vanishing on the tail of the sequence of commodities.

2.5 Convergence to the Walras equilibrium

In this section, we consider the relationship between the Walras equilibrium of the exchange economy and the Cournot-Nash equilibrium of the strategic market game. We show that if the number of traders of each type tends to infinity then the price vector and the allocation, at a Cournot-Nash equilibrium, converge to the Walras equilibrium of the underlying exchange economy.

Our framework is a particular case of the one considered by Wilson (1981) and it is straightforward to verify that under our assumptions a Walras equilibrium exists. This result relies crucially on the fact that there is a finite aggregate endowment of all commodities. We can then compare the Cournot-Nash equilibrium with the case in which all traders behave competitively, i.e., the Walras equilibrium. From the analysis of traders' market power in the previous section, it follows that the price vector and the allocation at a Cournot-Nash equilibrium do not coincide with the Walras equilibrium. In fact, even when the market power of asymptotic price-takers vanishes along the sequence of commodities, all traders continue to keep market power on the offers side.

Before to state the convergence theorem, we need to introduce some further notation, a definition, and a lemma. We denote by ${}_k\hat{s}$ a type symmetric Cournot-Nash equilibrium of the game ${}_k\Gamma$. To each ${}_k\hat{s}$ can be associated a vector ${}_k\tilde{s}$ which associates a strategy to each type of trader, i.e., ${}_k\tilde{s} \in \prod_{t=1}^{\infty} S^t$ and ${}_k\tilde{s}^t = {}_k\hat{s}^t$, for $t = 1, 2, \ldots$. We denote by $p({}_k\tilde{s})$ and ${}_h\mathbf{x}({}_k\tilde{s})$ a price vector and an allocation of an exchange economy ${}_h\mathcal{E}$ such that $p({}_k\tilde{s}) = p({}_k\hat{s})$ and $x^t({}_k\tilde{s}) = x^t({}_k\hat{s})$, for $t = 1, 2, \ldots$, and all h traders of type t have the same final holding $x^t({}_k\tilde{s})$. In the next definition, we introduce the notion of marginal price vector.[8]

Definition 5. Consider an active Cournot-Nash equilibrium $\hat{s}$ of ${}_k\Gamma$. The marginal price vector for a trader i of type t, $\bar{p}^i(\hat{s})$, is such that

$$\bar{p}^i_t(\hat{s}) = p_t(\hat{s})\left(1 - \frac{\hat{q}^i_t}{\bar{\hat{q}}_t}\right) \quad \text{and} \quad \bar{p}^i_j(\hat{s}) = p_j(\hat{s})\left(1 + \frac{\hat{b}^i_j}{\bar{\hat{b}}^i_j}\right), \text{ for each } j \in J \setminus \{0, t\}.$$

The following lemma is the analogous of Lemma 4 of Dubey and Shubik (1978) for a setting with infinitely many commodities.

Lemma 6. At an interior type-symmetric active Cournot-Nash equilibrium ${}_k\hat{s}$ of the game ${}_k\Gamma$, a trader i of type t maximises his payoff at the fixed marginal price vector $\bar{p}^i({}_k\hat{s})$, for each $i \in I$.

Proof. Let ${}_k\hat{s}$ be an interior type-symmetric active Cournot-Nash equilibrium of the game ${}_k\Gamma$. Consider, without loss of generality, a trader i of type t and the following maximisation problem

$$\begin{aligned} &\max_{{}_k s^i} && \pi^i({}_k s^i, \bar{p}^i({}_k\hat{s})), \\ &\text{subject to} && q^i_t \leq w^i_t, \qquad (i) \end{aligned} \tag{2.9}$$

[8]This definition is equivalent to the one of Okuno et al. (1980) only if the Cournot-Nash equilibrium is such that $0 < \hat{q}^i_t < w^i_t$, $\hat{b}^i_j > 0$, for $j \in J \setminus \{0, t\}$, and $\sum_{j \neq 0,t} \hat{b}^i_j < w^i_0$, for each $i \in I$.

$$\sum_{j\neq 0,t} b^i_j \leq w^i_0, \qquad (ii)$$

$$-q^i_t \leq 0, \qquad (iii)$$

$$-b^i_j \leq 0, \text{ for each } j \in J \setminus \{0,t\}, \quad (iv)$$

with $\pi^i({}_k s^i, \bar{p}^i({}_k\hat{s}))$ a payoff function at which the price vector is fixed and equal to $\bar{p}^i({}_k\hat{s})$. Let $x^i({}_k s^i, \bar{p}^i({}_k\hat{s}))$ denote the commodity bundle of trader i when he plays ${}_k s^i$ and the price vector is fixed and equal to $\bar{p}^i({}_k\hat{s})$. As in the proof of Lemma 3, all the hypothesis of the Generalized Kuhn-Tucker Theorem are satisfied and then, if a ${}_k s^i$ solves the maximisation problem, there exist non-negative multipliers λ^{i*}_1, λ^{i*}_2 and μ^{i*}_j, for $j \in J \setminus \{0\}$, such that

$$\frac{\partial u^i}{\partial x^i_0}(x^i({}_k s^i, \bar{p}^i({}_k\hat{s})))\bar{p}^i_t({}_k\hat{s}) - \frac{\partial u^i}{\partial x^i_t}(x^i({}_k s^i, \bar{p}^i({}_k\hat{s}))) - \lambda^{i*}_1 + \mu^{i*}_t = 0, \qquad (2.10)$$

$$\lambda^{i*}_1(q^i_t - w^i_t) = 0,$$

$$\mu^{i*}_t q^i_t = 0,$$

$$-\frac{\partial u^i}{\partial x^i_0}(x^i({}_k s^i, \bar{p}^i({}_k\hat{s}))) + \frac{\partial u^i}{\partial x^i_j}(x^i({}_k s^i, \bar{p}^i({}_k\hat{s})))\frac{1}{\bar{p}^i_j({}_k\hat{s})} - \lambda^{i*}_2 + \mu^{i*}_j = 0, \qquad (2.11)$$

$$\lambda^{i*}_2\Big(\sum_{j\neq 0,t} b^i_j - w^i_0\Big) = 0,$$

$$\mu^{i*}_j b^i_j = 0, \text{ for each } j \in J \setminus \{0,t\}.$$

By using the definition of marginal price vector, it is straightforward to verify that equations (2.5) and (2.7) become (2.10) and (2.11) respectively. But then, ${}_k\hat{s}^i$, $\hat{\lambda}^{i*}_1$, $\hat{\lambda}^{i*}_2$, and $\hat{\mu}^{i*}_j$, for $j \in J \setminus \{0\}$, satisfy the first order conditions associated to the maximisation problem (2.9). Since the utility function is concave and prices are fixed, the payoff function $\pi^i({}_k s^i, \bar{p}^i({}_k\hat{s}))$ is concave. Hence, ${}_k\hat{s}^i$ is optimal for the maximisation problem (2.9).[9] $\square$

We now state and prove the convergence theorem.

Theorem 2. Consider a sequence of games $\{{}_k\Gamma\}^{\infty}_{k=2}$. Suppose that there exists a sequence of interior type-symmetric active Cournot-Nash equilibria, $\{{}_k\hat{s}\}^{\infty}_{k=2}$, such that the sequences $\{{}_k\tilde{s}\}^{\infty}_{k=2}$ and $\{p({}_k\tilde{s})\}^{\infty}_{k=2}$ converge to $\tilde{v}$ and to $p(\tilde{v})$, respectively. Then, the pair $(p(\tilde{v}), {}_h\mathbf{x}(\tilde{v}))$ is a Walras equilibrium of the exchange economy associated to the game ${}_h\Gamma$, for any h.

[9]This conclusion can be also obtained by Theorem 2 of Section 8.5 and Lemma 1 of Section 8.7 in Luenberger (1969).

Proof. Let $\{{}_k\Gamma\}_{k=2}^{\infty}$ be a sequence of games ${}_k\Gamma$. Assume that there exists a sequence of interior type-symmetric active Cournot-Nash equilibria $\{{}_k\hat{s}\}_{k=2}^{\infty}$ such that the sequences $\{{}_k\tilde{s}\}_{k=2}^{\infty}$ and $\{p({}_k\tilde{s})\}_{k=2}^{\infty}$ converge to $\tilde{v}$ and to $p(\tilde{v})$ respectively. Consider, without loss of generality, a trader i of type t. By Lemma 6, ${}_k\hat{s}^i$ solves the maximisation problem (2.9), for any k, and, since ${}_k\hat{s}^i$ is an interior type symmetric active Cournot-Nash equilibrium, the constraints (ii) and (iii) are not binding. Let $\bar{p}_0({}_k\hat{s}) = 1$, for any k.[10] It is straightforward to verify that $x^i({}_k\hat{s}^i, \bar{p}^i({}_k\hat{s}))$ belongs to the budget set at price $\bar{p}^i({}_k\hat{s})$, $B^i(\bar{p}^i({}_k\hat{s}))$, for any k. Now, suppose that there exists a commodity bundle $x'^i \in B^i(\bar{p}^i({}_k\hat{s}))$ such that $u^i(x'^i) > u^i(x^i({}_k\hat{s}^i, \bar{p}^i({}_k\hat{s})))$. Since the utility function is non-decreasing, $x'^i_j > x^i_j({}_k\hat{s}^i, \bar{p}^i({}_k\hat{s}))$, for at least one commodity j. But since $\sum_{j\neq 0,t} \hat{b}^i_j < w^i_0$ and $-\hat{q}^i_t < 0$, there exists a feasible strategy ${}_k s'^i \in S^i$ such that $x^i_j({}_k s'^i, \bar{p}^i({}_k\hat{s})) = x'^i_j$, a contradiction. Hence, the commodity bundle $x^i({}_k\hat{s}^i, \bar{p}^i({}_k\hat{s}))$ maximises the utility function on $B^i(\bar{p}^i({}_k\hat{s}))$, for each $i \in I$, for any k. Now, consider the sequence of marginal price vectors $\{\bar{p}^t({}_k\hat{s})\}_{k=2}^{\infty}$, for a representative trader of type t. By the assumptions of Theorem 2 and by the definition of marginal price vector, $\lim_{k\to\infty} \bar{p}^t_j({}_k\tilde{s}) = p_j(\tilde{v})$, for each $j \in J \setminus \{0\}$, for $t = 1, 2, \ldots$. Since $\hat{s}$ is a type symmetric Cournot-Nash equilibrium, D^2_j in Lemma 4 becomes $\frac{e}{\sigma}$. Then, C_j and D_j are independent from k, for each $j \in J \setminus \{0\}$. But then, by Lemma 4, $p(\tilde{v}) \in \prod_{j\neq 0}[C_j, D_j]$. Therefore, $\tilde{v}$ is a point of continuity of the payoff function and then the commodity bundle $x^t(\tilde{v})$ is optimal on $B^t(p(\tilde{v}))$, for $t = 1, 2, \ldots$. Hence, $(p(\tilde{v}), {}_h\mathbf{x}(\tilde{v}))$ is a Walras equilibrium for the exchange economy associated to ${}_h\Gamma$, for any h. $\square$

It is worth to note that Theorem 2 holds for all the examples in the previous section.

2.6 An infinity of commodity money

In all the previous sections, we have considered exchange economies with a finite aggregate endowment of commodity money. Since in the strategic market game commodity money is used as a medium of exchange, it is also useful to consider the case in which the aggregate initial endowment of commodity money is infinite. In this section, we first prove the existence of an active Cournot-Nash equilibrium for a strategic market game ${}_k\Gamma$ with an infinite amount of commodity money. We then show by two examples that also in such framework there are

[10] With a slight abuse of notation, $\bar{p}({}_k\hat{s})$ denotes also a marginal price vector in which the first element is $\bar{p}_0({}_k\hat{s})$.

both asymptotic oligopolists and asymptotic price-takers. We conclude the section by considering an interesting example in which, at the Cournot-Nash equilibrium, all traders consume all commodities and all of them are asymptotic price-takers. Finally, in this setting, the market clearing condition for commodity money does not hold and then a Walras equilibrium does not exist.

We make the following new assumptions.

Assumption 7. Let σ and e be positive constants such that $\sigma < e$. The initial endowment of a type t trader is such that $\sigma < w_0^t < e$, $\sigma < w_t^t < e$, and $w_j^t = 0$, for each $j \in J \setminus \{0, t\}$, for $t = 1, 2, \ldots$.

Assumption 8. Let a be a positive constant. A commodity j is desired by at least one type of trader and there are no more than a types of traders for which the utility function is increasing respect to the variable x_j, for each $j \in J \setminus \{0\}$.

Assumption 9. The consumption set X is a subset of the space of non-negative bounded sequences ℓ_∞^+ endowed with the product topology, i.e., $X = \{x \in \ell_\infty^+ : \sup_j |x_j| < ake\}$.

Assumption 7 formalises the notion of multilateral oligopoly and guarantees that the total endowment of each commodity except commodity money is uniformly bounded from above. Assumption 8 ensures that all commodities are consumed by a finite number of types of traders. We need a uniform upper bound on the number of traders who desire each commodity because the total endowment of commodity money is infinite. Assumption 9 imposes restrictions on the consumption set which are standard in the literature on infinite economies. The constant ake is the maximum amount of commodity money that a trader can receive in exchange for the commodity he holds.

We now state the existence theorem.

Theorem 3. Under Assumptions 4, 6, 7, 8, and 9, there exists an active Cournot-Nash equilibrium for ${}_k\Gamma$.

Proof. It is straightforward to verify that under Assumptions 4, 6, 7, 8, and 9, Lemmas 1, 2, 3, and 5 hold. Lemma 4 holds with D_l^2 equal $\frac{aek}{w_l^i}$. Therefore, by applying the same steps in the proof of Theorem 1, we can conclude that there exists an active Cournot-Nash equilibrium for a game ${}_k\Gamma$ with an infinite aggregate endowment of commodity money. $\square$

When there is an infinity of commodity money, each trader holds only a negligible quantity of the total amount of commodity money in the economy. In such setting, we can expect that no trader has enough money to be an asymptotic oligopolist. The following example shows that this conjecture is false.

Example 5. Consider an exchange economy in which traders of type 1, 2, 3, and $t \geq 4$ have the following utility functions and initial endowments

$$
\begin{aligned}
u^1(x^1) &= x_0^1 + x_1^1 + \sum_{j=2}^{\infty} \delta^j \ln x_j^1 & w^1 &= \left(\frac{1}{1-\delta}, 1, 0, \dots\right), \\
u^2(x^2) &= x_0^2 + \delta \ln x_1^2 + x_2^2 + \delta^3 \ln x_3^2 & w^2 &= (2, 0, 1, 0, \dots), \\
u^3(x^3) &= x_0^3 + x_3^3 + \delta^4 \ln x_4^3 & w^3 &= (2, 0, 0, 1, 0, \dots), \\
u^t(x^t) &= x_0^t + x_t^t + \delta^{t+1} \ln x_{t+1}^t & w^t &= (2, 0, \dots, 0, 1, 0, \dots),
\end{aligned}
$$

with $\delta \in (0,1)$. At the type-symmetric active Cournot-Nash equilibrium of the game ${}_k\Gamma$, type 1 traders are asymptotic oligopolists.

Proof. This exchange economy is identical to the one considered in Example 1 except for the fact that all types of traders $t \geq 2$ have an initial endowment of commodity money equal to 2. It is straightforward to verify that this does not affect the Cournot-Nash equilibrium which is the same of Example 1. Therefore, type 1 traders are asymptotic oligopolists. □

In the next example, we show an exchange economy in which at the Cournot-Nash equilibrium type 1 traders are asymptotic price-takers.

Example 6. Consider an exchange economy in which traders of type 1, 2, 3, and $t \geq 4$ have the following utility functions and initial endowments

$$
\begin{aligned}
u^1(x^1) &= x_0^1 + x_1^1 + \sum_{j=2}^{\infty} \delta^j \ln x_j^1 & w^1 &= \left(\frac{1}{1-\delta}, 1, 0, \dots\right), \\
u^2(x^2) &= x_0^2 + \ln x_1^2 + x_2^2 + \ln x_3^2 & w^2 &= \big(2, 0, 1, 0, \dots\big), \\
u^3(x^3) &= x_0^3 + x_3^3 + \ln x_4^3 & w^3 &= \big(2, 0, 0, 1, 0, \dots\big), \\
u^t(x^t) &= x_0^t + x_t^t + \ln x_{t+1}^t & w^t &= \big(2, 0, \dots, 0, 1, 0, \dots\big),
\end{aligned}
$$

with $\delta \in (0,1)$. For $k = 2$, the type-symmetric active Cournot-Nash equilibrium of the game ${}_k\Gamma$ associated to the exchange economy is

$$
\begin{aligned}
(\hat{q}_1^1, \hat{b}_2^1, \hat{b}_3^1, \dots, \hat{b}_j^1, \dots) &= \left(\frac{1}{4}, \frac{\delta^2}{2}, \frac{\delta^3(\delta^3 - 5 + F(3))}{4\delta^j - 4}, \dots, \frac{\delta^j(\delta^j - 5 + F(j))}{4\delta^j - 4}, \dots\right), \\
(\hat{q}_2^2, \hat{b}_1^2, \hat{b}_3^2, \hat{b}_4^2, \dots, \hat{b}_j^2, \dots) &= \left(\frac{\delta^2}{4}, \frac{1}{2}, \frac{1 - 5\delta^3 + F(3)}{4 - 4\delta^3}, 0, \dots, 0, \dots\right),
\end{aligned}
$$

$$(\hat{q}_3^3, \hat{b}_1^3, \hat{b}_2^3, \hat{b}_4^3, \ldots, \hat{b}_j^3, \ldots) = \left(\frac{\delta^3 + F(3) + 1}{4}, 0, 0, \frac{1 - 5\delta^4 + F(4)}{4 - 4\delta^4}, 0, \ldots, 0, \ldots\right),$$

$$(\hat{q}_t^t, \hat{b}_1^t, \ldots, \hat{b}_{t-1}^t, \hat{b}_{t+1}^t, \ldots) = \left(\frac{\delta^t + F(t) + 1}{4}, 0, \ldots, 0, \frac{1 - 5\delta^{t+1} + F(t+1)}{4 - 4\delta^{t+1}}, \ldots\right),$$

for $t = 4, 5, \ldots$, with $F(y) = \sqrt{(\delta^y + 14)\delta^y + 1}$. At this Cournot-Nash equilibrium, type 1 traders are asymptotic price-takers.

Proof. At the Cournot-Nash equilibrium, the market power of type 1 traders, $\hat{b}_j^1/\bar{\hat{b}}_j$, is equal to $\frac{3\delta^j - \sqrt{(\delta^j+14)\delta^j+1}+1}{4\delta^j - 4}$, for each $j \geq 3$. Hence, $\lim_{j\to\infty} \hat{b}_j^1/\bar{\hat{b}}_j = 0$. □

In the next example, we consider an exchange economy in which all traders desire all commodities and everyone is an asymptotic price-taker. Therefore, this example does not satisfy Assumption 8 but nevertheless there exists a Cournot-Nash equilibrium because traders' preferences have a particular pattern.

Example 7. Consider an exchange economy in which traders of type 1, 2, 3, and $t \geq 4$ have the following utility functions and initial endowments

$$u^1(x^1) = x_0^1 + x_1^1 + \sum_{j=2}^{\infty} \delta^j \ln x_j^1 \qquad w^1 = \left(\frac{1}{1-\delta}, 1, 0, \ldots\right),$$

$$u^2(x^2) = x_0^2 + x_2^2 + \delta^2 \ln x_1^2 + \sum_{j=3}^{\infty} \delta^{j-1} \ln x_j^2 \qquad w^2 = \left(\frac{1}{1-\delta}, 0, 1, 0, \ldots\right),$$

$$u^3(x^3) = x_0^3 + x_3^3 + \sum_{j=1}^{2} \delta^3 \ln x_j^3 + \sum_{j=4}^{\infty} \delta^{j-2} \ln x_j^3 \qquad w^3 = \left(\frac{1}{1-\delta}, 0, 0, 1, 0, \ldots\right),$$

$$u^t(x^t) = x_0^t + x_t^t + \sum_{j=1}^{t-1} \delta^t \ln x_j^t + \sum_{j=t+1}^{\infty} \delta^{j-t+1} \ln x_j^t \quad w^t = \left(\frac{1}{1-\delta}, 0, \ldots, 0, 1, 0, \ldots\right).$$

For $k = 2$ and $\delta = 1/2$, the type-symmetric active Cournot-Nash equilibrium of the game ${}_k\Gamma$ associated to the exchange economy is

$$(\hat{q}_1^1, \hat{b}_2^1, \hat{b}_3^1, \hat{b}_4^1, \ldots, \hat{b}_j^1, \ldots) = \left(\frac{m}{2}, \frac{\delta^2 m}{\delta^2 + m}, \frac{\delta^3 m}{\delta^3 + m}, \frac{\delta^4 m}{\delta^4 + m}, \ldots, \frac{\delta^j m}{\delta^j + m}, \ldots\right),$$

$$(\hat{q}_2^2, \hat{b}_1^2, \hat{b}_3^2, \hat{b}_4^2 \ldots, \hat{b}_j^2, \ldots) = \left(\frac{m}{2}, \frac{\delta^2 m}{\delta^2 + m}, \frac{\delta^2 m}{\delta^2 + m}, \frac{\delta^3 m}{\delta^3 + m} \ldots, \frac{\delta^{j-1} m}{\delta^{j-1} + m}, \ldots\right),$$

$$(\hat{q}_3^3, \hat{b}_1^3, \hat{b}_2^3, \hat{b}_4^3, \ldots, \hat{b}_t^3, \ldots) = \left(\frac{m}{2}, \frac{\delta^3 m}{\delta^3 + m}, \frac{\delta^3 m}{\delta^3 + m}, \frac{\delta^2 m}{\delta^2 + m}, \ldots, \frac{\delta^{j-2} m}{\delta^{j-2} + m}, \ldots\right),$$

$$(\hat{q}_t^t, \hat{b}_1^t, \ldots, \hat{b}_{t-1}^t, \hat{b}_{t+1}^t, \ldots) = \left(\frac{m}{2}, \frac{\delta^t m}{\delta^t + m}, \ldots, \frac{\delta^t m}{\delta^t + m}, \frac{\delta^2 m}{\delta^2 + m}, \frac{\delta^3 m}{\delta^3 + m}, \ldots\right),$$

for $t = 4, 5, \ldots$, with $m = \bar{\hat{b}}_j \approx 0.8415$, for each $j \in J\backslash\{0\}$.[11] At this Cournot-Nash equilibrium, all traders types are asymptotic price-takers.

[11] This approximated result was obtained with Mathematica.

Proof. At the Cournot-Nash equilibrium, given the particular structure of traders' preferences, it is straightforward to verify that the sum of bids is the same in each trading post and it is equal to m. Since $\lim_{j\to\infty} \hat{b}_j^t = 0$, for $t = 1, 2, \ldots$, $\lim_{j\to\infty} \hat{b}_j^t/\bar{\hat{b}}_j = 0$, for $t = 1, 2, \ldots$. □

2.7 Conclusion

In this chapter, we have extended the analysis of non-cooperative oligopoly to exchange economies with infinitely many commodities and traders. We have done so by considering the strategic market game analysed by Dubey and Shubik (1978) in a setting with a countable infinity of commodities and traders having the structure of multilateral oligopoly. In such framework, we have proved the existence of an active Cournot-Nash equilibrium and its convergence to the Walras equilibrium when the number of traders of each type tends to infinity.

To analyse traders' market power in infinite economies, we have introduced the notions of asymptotic oligopolist and asymptotic price-taker and we have showed, via a number of examples, that both cases arise endogenously in equilibrium in our framework. Therefore, the infinity of commodities and traders is not sufficient to guarantee that all traders are "small" and display a mixed behaviour, i.e., asymptotic price-takers. In the previous section, we have considered exchange economies with an infinite aggregate endowment of commodity money and by two examples we have shown that also in this setting both asymptotic oligopolists and asymptotic price-takers arise endogenously in equilibrium.

From the examples in Section 2.4, we can draw some conclusions about the problem of proving the existence of asymptotic oligopolists. In the first example, we have shown that some traders are asymptotic oligopolists because, heuristically, traders have similar preferences and only few of them are active in each trading post. In the second example, we have shown a case in which the number of traders active in each trading post is non uniformly bounded from above and all traders are asymptotic price-takers. In the third example, we have shown that, even if the number of traders active in each trading post is not uniformly bounded from above, some traders with a higher initial endowment of commodity money and with particular preferences can be asymptotic oligopolists. Therefore, whether or not a trader is an asymptotic oligopolist in equilibrium is sensitive to the initial endowment of commodity money and to the "size of the market" in each trading post, which depends on preferences throughout traders. We leave as an open problem for further research the determination of sufficient conditions to prove the existence of asymptotic oligopolists. In the next chapter, we prove the existence of asymptotic price-takers in a strategic market game called "all for sale model".

In Section 2.4, we have illustrated how our setting can be interpreted as a model of oligopoly with differentiated commodities by using the Hotelling line. Therefore, this framework may be useful to get some insights on competition policy issues.

2.A Mathematical appendix

In this appendix, we describe the mathematical notions that we have used in the chapter. The definitions and the theorems are based on Luenberger (1969) and the page number in brackets refers to it.

Definition (ℓ_∞ spaces)**.** The space ℓ_∞^+ consists of non-negative bounded sequences. The norm of an element $x = \{x_i\}$ in ℓ_∞^+ is defined as $\|x\|_\infty = \sup_i |x_i|$ (p. 29).

By assuming that the space ℓ_∞^+ is endowed with the product topology, we impose on ℓ_∞^+ the norm $\|x\|_\infty = \sup_i \|a_i x_i\|$ such that $\{a_i\}$ is a sequence of real number converging to zero (see Brown and Lewis (1981)).

Definition (Transformation T)**.** Let X and Y be linear vector spaces and let D be a subset of X. A rule which associates with every element $x \in D$ an element $y \in Y$ is said to be a transformation from X to Y with domain D. If y corresponds to x under T, we write $y = T(x)$ (p. 27).

Definition (Fréchet differentiable)**.** Let T be a transformation defined on an open domain D in a normed space X and having range in a normed space Y. If for fixed $x \in D$ and each $h \in X$ there exists $\delta T(x;h) \in Y$ which is linear and continuous with respect to h such that

$$\lim_{\|h\| \to 0} \frac{\|T(x+h) - T(x) - \delta T(x;h)\|}{\|h\|} = 0,$$

then T is said to be Fréchet differentiable at x and $\delta T(x;h)$ is said to be the Fréchet differential of T at x with incremental h (p. 172).

Definition (Continuously Fréchet differentiable)**.** The Fréchet differential of T at x with incremental h, $\delta T(x;h)$, can be written as $T'(x)h$ with T' the Fréchet derivative of T. If the correspondence $x \to T'(x)$ is continuous at the point x_0, we say that the Fréchet derivative of T is continuous at x_0. If the derivative of

T is continuous on some open sphere S, we say that T is continuously Fréchet differentiable on S (p. 175).

Definition (Normed dual)**.** Let X be a normed linear vector space. The space of all bounded linear function on X is called the normed dual of X and is denoted by X^* (p. 106).

Luenberger states the Regular Point definition (p. 248) and the Generalized Kuhn-Tucker Theorem (p. 249-250) for vector spaces. Since we deal with normed spaces, we state them for these particular spaces (see Example 1, p. 250).

Definition (Regular Point)**.** Let X be a normed vector space and let Z be a normed vector space with a closed positive cone P having non-empty interior. Let G be a mapping $G : X \to Z$ which is Fréchet differentiable. A point $x_0 \in X$ is said to be a regular point of the inequality $G(x) \leq 0$ if $G(x_0) \leq 0$ and there is an $h \in X$ such that $G(x_0) + G'(x_0) \cdot h < 0$.

Theorem (Generalized Kuhn-Tucker Theorem)**.** Let X be a normed vector space and Z be a normed vector space having a closed positive cone P. Assume that P contains an interior point. Let f be a Fréchet differentiable real-valued function on X and G a Fréchet differentiable mapping from X into Z. Suppose x_0 maximizes f subject to $G(x) \leq 0$ and that x_0 is a regular point of the inequality $G(x) \leq 0$. Then there is a $z_0^* \in Z^*$, $z_0^* \geq 0$ such that

$$f'(x_0) + z_0^* G'(x_0) = 0,$$
$$z_0^* \cdot G(x_0) = 0.$$

2.B Marginal and average prices

Okuno et al. (1980) introduce the notions of marginal price and average price to study traders' behaviours. They show that when the two prices are equal traders behave competitively. At a Cournot-Nash equilibrium $\hat{s}$, the average price vector is equal to $p(\hat{s})$ and traders' marginal price vectors are defined in Definition 5. The next proposition shows, in our framework of multilateral oligopoly, the relationship between the approach of Okuno et al. (1980) and traders' market share, which is used by us to measure traders' market power.

Proposition 2. Consider an active Cournot-Nash equilibrium $\hat{s}$ of ${}_k\Gamma$. For a trader i of type t, the marginal price vector is equal to the average price vector if and only if $\hat{q}_t^i/\bar{\hat{q}}_t = 0$ and $\hat{b}_j^i/\bar{\hat{b}}_j = 0$, for each $j \in J \setminus \{0, t\}$.

Proof. Let $\hat{s}$ be an active Cournot-Nash equilibrium of ${}_k\Gamma$. Consider a trader i of type t. First, assume that $\bar{p}^i(\hat{s}) = p(\hat{s})$. By Definition 5, $\hat{q}_t^i/\bar{\hat{q}}_t = 0$ and $\hat{b}_j^i/\bar{\hat{b}}_j^i = 0$, for each $j \in J \setminus \{0, t\}$. But then, $\hat{b}_j^i/\bar{\hat{b}}_j = 0$, for each $j \in J \setminus \{0, t\}$. Now, assume that $\hat{q}_t^i/\bar{\hat{q}}_t = 0$ and $\hat{b}_j^i/\bar{\hat{b}}_j = 0$, for each $j \in J \setminus \{0, t\}$. This implies that $\hat{b}_j^i = 0$ and then $\hat{b}_j^i/\bar{\hat{b}}_j^i = 0$, for each $j \in J \setminus \{0, t\}$. By Definition 5, $\bar{p}^i(\hat{s}) = p(\hat{s})$. □

2.C The game analysed by Dubey and Shubik (1978)

The game ${}_k\Gamma$ was introduced for exchange economies with a finite number of commodities by Shubik (1973). The strategy space of this game is different from the one considered by Dubey and Shubik (1978) where traders are allowed to sell and buy the same commodity. We introduce now the strategic market game analysed by Dubey and Shubik (1978) and we show how it is related to the game ${}_k\Gamma$. Let's call this game Λ. The strategy set of trader i is

$$Z^i = \Big\{ z^i = (q_1^i, b_1^i, q_2^i, b_2^i, \ldots, q_j^i, b_j^i, \ldots) : 0 \leq q_j^i \leq w_j^i,\ b_j^i \geq 0, \text{ for each } j \in J \setminus \{0\}, \text{ and } \sum_{j=1}^{\infty} b_j^i \leq w_0^i \Big\}.$$

Let $Z = \prod_{i \in I} Z^i$ and $Z^{-r} = \prod_{i \in I \setminus \{r\}} Z^i$. Let z and z^{-i} be elements of Z and Z^{-i} respectively. For each $z \in Z$, the price vector $p(z)$ is such that

$$p_j(z) = \begin{cases} \frac{\bar{b}_j}{\bar{q}_j} & \text{if } \bar{q}_j \neq 0, \\ 0 & \text{if } \bar{q}_j = 0, \end{cases}$$

for each $j \in J \setminus \{0\}$, with $\bar{q}_j = \sum_{i \in I} q_j^i$ and $\bar{b}_j = \sum_{i \in I} b_j^i$. For each $z \in Z$, the final holding $x^i(z)$ of a trader i is such that

$$x_0^i(z) = w_0^i - \sum_{j=1}^{\infty} b_j^i + \sum_{j=1}^{\infty} q_j^i p_j(z),$$

$$x_j^i(z) = \begin{cases} w_j^i - q_j^i + \frac{b_j^i}{p_j(z)} & \text{if } p_j(z) \neq 0, \\ 0 & \text{if } p_j(z) = 0, \end{cases}$$

for each $j \in J \setminus \{0\}$.

The payoff function of a trader i, $\pi^i : Z \to \mathbb{R}$, is such that $\pi^i(z) = u^i(x^i(z))$.

The following proposition establishes the relationship between the attainable allocations of the two games at a Cournot-Nash equilibrium.

Theorem 4. Consider an exchange economy ${}_k\mathcal{E}$ as defined in Section 2.2 which satisfies Assumption 1. An allocation $\mathbf{x}$ is attainable at a Cournot-Nash equilibrium $\hat{s}$ of the game ${}_k\Gamma$ if and only if the same allocation $\mathbf{x}$ is attainable at a Cournot-Nash equilibrium $\hat{z}$ of the game Λ.

Proof. Let ${}_k\mathcal{E}$ be an exchange economy which satisfies Assumption 1. First, assume that $\hat{s}$ is a Cournot-Nash equilibrium of the game ${}_k\Gamma$. Let $\hat{z}$ be a strategy profile such that, for a trader i of type t, $\hat{z}^i = (0, \hat{b}_1^i, 0, \hat{b}_2^i, \dots, \hat{q}_t^i, 0, \dots)$, for each $i \in I$. It is straightforward to verify that $\mathbf{x}(\hat{s})$ and $\mathbf{x}(\hat{z})$ are equal. Suppose that z is not a Cournot-Nash equilibrium for Λ. Then, there exists a trader i of type t that can increase his payoff by playing a strategy z'^i. The only action that can increase the trader's payoff is to increase the bid for commodity t, $b_t'^i$, because all other feasible deviations are also available in the game ${}_k\Gamma$. Then, $x_t^i(z'^i, \hat{z}^{-i}) > x_t^i(\hat{s})$. But, by decreasing $\hat{q}_t^i$, the commodity bundle $x^i(z'^i, z^{-i})$ is attainable also in the original game, a contradiction. Hence, $\hat{z}$ is a Cournot-Nash equilibrium for the game Λ. Now, assume that $\hat{z}$ is a Cournot-Nash equilibrium of the game Λ. Let $\hat{s}$ be a strategy profile such that, for a trader i of type t, $s^i = (\hat{q}_t^i - \frac{\hat{b}_t^i}{p_t(\hat{s})}, \hat{b}_1^i, \hat{b}_2^i, \dots, \hat{b}_{t-1}^i, \hat{b}_{t+1}^i, \dots)$, for each $i \in I$. It is straightforward to verify that $\mathbf{x}(\hat{z})$ and $\mathbf{x}(\hat{s})$ are equal. Suppose that $\hat{s}$ is not a Cournot-Nash equilibrium for ${}_k\Gamma$. Then, there exists a trader i of type t that can increase his payoff by playing a strategy s'^i. But, any possible deviation in ${}_k\Gamma$ is also available in Λ, a contradiction. Hence, $\hat{s}$ is a Cournot-Nash equilibrium for ${}_k\Gamma$. $\square$

As a corollary of this theorem, we can extend the existence theorem to the game Λ analysed by Dubey and Shubik (1978).

Corollary 1. In an exchange economy ${}_k\mathcal{E}$ as defined in Section 2.2, under Assumptions 1, 2, 3, 4, 5, and 6, there exists an active Cournot-Nash equilibrium for Λ.

Proof. Let ${}_k\mathcal{E}$ be an exchange economy which satisfies Assumptions 1, 2, 3, 4, 5, and 6. By Theorem 1, there exists an active Cournot-Nash equilibrium for the game ${}_k\Gamma$ associated to ${}_k\mathcal{E}$. Let $\mathbf{x}(\hat{s})$ be the allocation at the Cournot-Nash equilibrium $\hat{s}$. But then, by Theorem 4, there exists a Cournot-Nash equilibrium $\hat{z}$ for the game Λ such that $\mathbf{x}(\hat{z}) = \mathbf{x}(\hat{s})$. $\square$

— 3 —

Asymptotic Price-Takers in Economies with Infinitely Many Commodities and Traders

3.1 Introduction

In the previous chapter, we have introduced the notions of asymptotic oligopolist and asymptotic price-taker to study traders' market power in exchange economies with a countable infinity of commodities and traders. In this chapter, we focus on asymptotic price-takers and we study under which conditions on the fundamentals of an economy, i.e., initial endowments and preferences, an asymptotic price-taker exists.

As described in the previous chapter, an asymptotic price-taker is a trader who makes positive bids on an infinite set of commodities and whose market power converges to zero along the sequence of commodities. This trader exhibits a kind of mixed behaviour since his market power can be made arbitrary small on an infinite set of commodities while it is greater than a positive constant only on a finite number of them. In the previous literature on imperfect competition, the mixed behaviour was usually obtained by assuming that agents are characterised by portfolios of commodities that contain the commodities on which agents have market power. Negishi (1961) considered monopolistically competitive firms which have market power on the commodities in their portfolios while they behave competitively on all other commodities. In this way, he extended the theory of monopolistic competition of Chamberlin (1933) and Robinson (1933) from partial to general equilibrium. Gabszewicz and Michel (1997) and d'Aspremont, Dos Santos Ferreira, and Gérard-Varet (1997) considered exchange economies in which consumers are characterized by portfolios of commodities to study the Cournot-Walras equilibrium and the Cournotian Monopolistic Competition equilibrium respectively. In all these contributions, traders' behaviours on each commodity are assumed a priori without giving any formal explanation as to why a particular trader should behave strategically on some commodities and competitively on others. In contrast, by using strategic market games, we provide the sufficient conditions on the fundamentals of an economy under which asymptotic price-takers, who are traders with a kind of mixed behaviour, arise endogenously in equilibrium.

Traders' market power is measured by traders' market share as in Chapter 2, i.e., the market power of trader i on commodity j is $b_j^i/\bar{b}_j$. From the examples in Section 2.4, it is clear that the sequence of sums of bids, $\{\bar{b}_j\}$, converges always to zero when there is a finite aggregate endowment of commodity money. Therefore, in such framework, the existence of an asymptotic price-taker depends crucially on the convergence rates to zero of the sequence of trader's bids and the sequence of sums of bids. Unfortunately, it turns out that it is a difficult task to obtain specific convergence rates by making assumptions on the fundamentals of an economy. In this chapter, we overcome these difficulties in the following way. We consider exchange economies with an infinite aggregate endowment of commodity money, as in Section 2.6, and we further simplify our analysis by considering a variation of the strategic market game analysed by Dubey and Shubik (1978), namely the "all for sale model", which was introduced by Shapley (1976) and Shapley and Shubik (1977). In this model, at the start of the game all traders are required to deposit all their commodities, except commodity money, in the appropriate trading post in exchange for non-negotiable receipts. These receipts will be redeemed after trade for the amount of commodity money obtained from the sale of each trader's commodities. Hence, the only actions available to traders are bids, amounts of commodity money offered in exchange for other commodities.

In such framework, we prove the existence of an active Cournot-Nash equilibrium in which the sum of bids in each trading post is uniformly bounded away from zero. Consequently, the proof of existence of an asymptotic price-taker reduces to show that there exists a trader who is active on an infinite number of commodities. Indeed, if the sums of bids are uniformly bounded away from zero then the sequence of market power, $\{b_j^i/\bar{b}_j\}$, converges to zero as the sequence of bids of a trader always converges to zero. As remarked above, to prove the existence of asymptotic price-takers it is crucial to assume that the aggregate endowment of commodity money is infinite. Unfortunately, this rules out the existence of a Walras equilibrium, but it helps to clarify the connections between the fundamentals of an economy and the mixed behaviour. The infinity of commodity money can be interpreted as a limiting case where the total endowment of commodity money is much greater than traders' initial endowments.

Our contributions are as follows. We first prove the existence of an active Cournot-Nash equilibrium in which the price vector lies in a compact set uniformly bounded away from zero. Under the assumptions of the model, this implies that the sums of bids along trading posts are uniformly bounded away from zero at a Cournot-Nash equilibrium. The fact that all commodities go into the trading posts at the beginning of the game means that one side of the market is always active and this simplifies the proof of existence. Furthermore, the assumptions needed to prove the existence are somehow less restrictive than the ones made in the previous chapter. For instance, it is not necessary to assume the structure of multilateral oligopoly but some restrictions on traders' marginal utilities are still necessary.[1] We finally prove the existence of an asymptotic price-taker. Heuristically, a trader active on an infinite number of commodities is an asymptotic price-taker if all markets are thick, i.e., the quantities of commodity money and commodities in all trading posts are non-negligible. The results of the two theorems are discussed with some examples which clarify the role of the assumptions and why it is crucial for the existence of asymptotic price-takers that prices are uniformly bounded away from zero. We conclude the chapter by comparing our results with the contributions on monopolistic competition of Dixit and Stiglitz (1977), Hart (1985), and Pascoa (1993).

The chapter is organised as follows. In Section 3.2, we introduce the mathematical model. In Section 3.3, we prove the existence theorem. In Section 3.4, we prove the existence of an asymptotic price-taker and we discuss it by means of examples. In Section 3.5, we draw some conclusions from our analysis and we compare our results with some previous contributions on monopolistic competition.

3.2 Mathematical model

The set of commodities is $J = \{0, 1, 2, \dots\}$. Let I be a countable set which denotes the set of traders. This set I is partitioned in two sets T and H. The

[1] In finite economies, the existence of a Cournot-Nash equilibrium in the all for sale model was proved by Dubey and Shubik (1977) in a framework with exogenous uncertainty.

consumption set is denoted by X. A commodity bundle x is a point in X with x_j the amount of commodity j. A trader i is characterized by an initial endowment, $w^i \in X$, which represents his wealth, and a utility function, $u^i : X \to \mathbb{R}$, which describes his preferences. An exchange economy is then a set $\mathcal{E} = \{(u^i, w^i) : i \in I\}$.

A commodity j is desired by a trader i if u^i is an increasing function of the variable x^i_j, for any fixed choice of the other variables. The set of commodities desired by a trader i is denoted by L^i.

We make the following assumptions.

Assumption 1. Let σ, e, and g be positive constants such that $\sigma < e$. The initial endowment of commodity money of a trader i is such that $\sigma < w^i_0 < e$, for each $i \in I$. The total endowment of a commodity j is such that $\sigma < \sum_i w^i_j < e$, for each $j \in J \setminus \{0\}$. Furthermore, a trader i is endowed with less than g different commodities, for each $i \in I$.

Assumption 2. Let f and n be two positive constants. For each trader $i \in T$, (i) the utility function, u^i, is continuously Frèchet differentiable, non-decreasing, and concave; (ii) $\sharp L^i \leq n$, i.e, a trader i desires less than n commodities; (iii)$\frac{\partial u^i}{\partial x^i_j}(x^i) < f$, for each $x^i \in X$, for each $j \in J$.

Assumption 3. For each trader $i \in H$, the utility function, u^i, is continuous, non-decreasing, and concave.

Assumption 4. Let a and λ be positive constants such that $\lambda < f$. For each commodity $j \in J \setminus \{0\}$, (i) there are less than a traders that desire commodity j; (ii) there exist two traders $s, t \in T$ such that $\frac{\partial u^s}{\partial x^s_j}(x^s), \frac{\partial u^t}{\partial x^t_j}(x^t) > \lambda$, for each $x^s, x^t \in X$.

Assumption 5. The consumption set X is a subset of the space of non-negative bounded sequences ℓ^+_∞ endowed with the product topology, i.e., $X = \{x \in \ell^+_\infty : \sup_j |x_j| \leq aeg\}$.

The first assumption guarantees that the aggregate endowment of all commodities except commodity money is uniformly bounded away from zero and from above. Differently, the aggregate endowment of commodity money is infinite and this is crucial to prove the existence of an asymptotic price-taker. The last part of Assumption 1 implies that traders' wealth is finite. Assumptions 2(i) and 2(ii) are classical restrictions on traders' preferences in infinite economies. Assumption

2(iii) is trivially true when the number of commodities is finite.[2] Assumptions 2 and 3 imply that traders in the set T desire only a finite number of commodities and traders in the set H can desire an infinite number of commodities. The first part of Assumption 4 imposes an upper bound a on the number of traders who desire each commodity. This guarantees that the sum of bids in each trading post is finite. The second part implies that for each commodity there are at least two traders whose utility functions are strictly increasing respect to it. Finally, the last assumption is common in economies with infinitely many commodities. The constant aeg is the maximum amount of commodity money that a trader can receive in exchange for the commodities he holds.

We now introduce the strategic market game Γ associated with the exchange economy $\mathcal{E}$. For each commodity $j \in J \setminus \{0\}$, there is a trading post where commodity j is exchanged for commodity money 0. At the beginning of the game all traders are required to deposit all of their commodities, except commodity money, in the appropriate trading post. They receive non-negotiable receipts which will be redeemed after trade for the amount of commodity money obtained from the sale of each trader's commodities. The strategy set of a trader i is

$$S^i = \Big\{ s^i = (b_1^i, b_2^i, b_3^i, \dots) : b_j^i \geq 0, \text{ for each } j \in J \setminus \{0\}, \text{ and } \sum_j b_j^i \leq w_0^i \Big\},$$

where b_j^i is the amount of commodity money that trader i bids on commodity j. Without loss of generality, we make the following technical assumption

Assumption 6. The set S^i is a subset of ℓ_∞^+ endowed with the product topology, for each $i \in I$, i.e., $S^i \subseteq \{s^i \in \ell_\infty^+ : \sup |s_j^i| \leq e\}$.

This assumption implies that S^i lies in a normed space and therefore in a Hausdorff space.

Let $S = \prod_{i \in I} S^i$ and $S^{-z} = \prod_{i \in I \setminus \{z\}} S^i$. Let s and s^{-i} be elements of S and S^{-i} respectively. For each $s \in S$, the price vector $p(s)$ is such that

$$p_j(s) = \frac{\bar{b}_j}{\bar{w}_j},$$

for each $j \in J \setminus \{0\}$, with $\bar{b}_j = \sum_{i \in I} b_j^i$ and $\bar{w}_j = \sum_{i \in I} w_j^i$. By Assumption 1 and 4(ii), the sums $\bar{b}_j$ and $\bar{w}_j$ are finite, for each $j \in J \setminus \{0\}$. For each $s \in S$, the final

[2] The classical definition of differentiability excludes the case of infinite partial derivatives along the boundary of the consumption set (see for instance Amir et al. (1990)).

holding $x^i(s)$ of a trader i is such that

$$x_0^i(s) = w_0^i - \sum_{j=1}^{\infty} b_j^i + \sum_{j=1}^{\infty} w_j^i p_j(s), \tag{3.1}$$

$$x_j^i(s) = \begin{cases} \frac{b_j^i}{p_j(s)} & \text{if } p_j(s) \neq 0, \\ 0 & \text{if } p_j(s) = 0, \end{cases} \tag{3.2}$$

for each $j \in J \setminus \{0\}$.

The payoff function of a trader i, $\pi^i : S \to \mathbb{R}$, is such that $\pi^i(s) = u^i(x^i(s))$.

We now introduce the definitions of an active trading post, a best response correspondence, and a Cournot-Nash equilibrium.

Definition 6. A trading post for a commodity j is said to be active if $\bar{w}_j > 0$ and $\bar{b}_j > 0$, otherwise we say that the trading post is inactive.

Definition 1. The best response correspondence of a trader i is a correspondence $\phi^i : S^{-i} \to S^i$ such that

$$\phi^i(s^{-i}) \in \arg\max_{s^i \in S^i} \pi^i(s^i, s^{-i}),$$

for each $s^{-i} \in S^{-i}$.

Definition 2. An $\hat{s} \in S$ is a Cournot-Nash equilibrium of Γ, if $\hat{s}^i \in \phi^i(\hat{s}^{-i})$, for each $i \in I$.

Furthermore, we say that a Cournot-Nash equilibrium is active if all trading posts are active.

3.3 Theorem of existence

In this section, we state and prove the theorem of existence of an active Cournot-Nash equilibrium for the game Γ such that the price vector lies in a compact set uniformly bounded away from zero. Before to do so, we define the perturbed strategic market game and we prove two lemmas. The perturbed strategic market game Γ^ϵ is a game defined as Γ with the only exception that the price becomes

$$p_j^\epsilon(s) = \frac{\bar{b}_j + \epsilon}{\bar{w}_j},$$

for each $j \in J \setminus \{0\}$, with $\epsilon > 0$. The interpretation is that an outside agency places fixed bids of ϵ in all trading posts. This does not change the strategy sets of traders, but does affect the prices, the final holdings, and the payoffs. In the next lemma we prove the existence of a Cournot-Nash equilibrium in the perturbed game.

Lemma 1. Under Assumptions 1, 2, 3, 4, 5, and 6, for each $\epsilon > 0$, there exists a Cournot-Nash equilibrium for Γ^ϵ.

Proof. Consider, without loss of generality, a trader i and fix the strategies s^{-i} for all other traders. In the perturbed game the payoff function π^i is continuous because it is a composition of continuous functions (see Theorem 17.23, p. 566 in Aliprantis and Border (2006), AB hereafter). By Tychonoff Theorem (see Theorem 2.61, p. 52 in AB), S^i is compact. By Weierstrass Theorem (see Corollary 2.35, p. 40 in AB), there exists a strategy $\hat{s}^i$ that maximises the payoff function. We can then consider the best response correspondence $\phi^i : S^{-i} \to S^i$. Since S^i is a non-empty and compact Hausdorff space, by Berge Maximum Theorem (see Theorem 17.31, p. 570 in AB), ϕ^i is an upper hemicontinuous correspondence.

We show now that ϕ^i has convex and closed-valued. Suppose that there are two feasible strategies s'^i and s''^i which belong to $\phi^i(s^{-i})$. We need to prove that $\tilde{s}^i = \alpha s'^i + (1-\alpha)s''^i$, with $\alpha \in (0,1)$, belongs to $\phi^i(s^{-i})$. Let $x'^i = x^i(s'^i)$ and $x''^i = x^i(s''^i)$. Since the utility function is concave, also the commodity bundle $\tilde{x}^i = \alpha x'^i + (1-\alpha)x''^i$ maximises the utility function. By equations (3.1) and (3.2), $x^i_0(s^i)$ is linear in s^i and $x^i_j(s^i)$ is concave respect to s^i, for each $j \in J \setminus \{0\}$. Therefore,

$$\tilde{x}^i = \alpha x^i(s'^i) + (1-\alpha)x^i(s''^i) \leq x^i(\alpha s'^i + (1-\alpha)s''^i).$$

But then $\tilde{x}^i \leq x^i(\tilde{s})$. Hence, $\tilde{s}$ belongs to $\phi^i(s^{-i})$ and then $\phi^i(s^{-i})$ has convex-valued. By the continuity of the payoff function π^i, it follows that ϕ^i has closed-valued.

Since S^i is closed and ϕ^i is upper hemicontinuous and closed-valued, ϕ^i has a closed graph by the Closed Graph Theorem (see Theorem 17.11, p. 561 in AB). As we are looking for a fixed point in strategy space S, let's consider $\phi^i : S \to S^i$. Let $\Phi : S \to S$ such that $\Phi(S) = \prod_{i \in I} \phi^i(S)$. Since $\Phi(S)$ is a product of correspondences with closed graph and non-empty convex values, it is straightforward to verify that the correspondence Φ has closed graph and non-empty convex values. Moreover, S is a non-empty compact convex subset of a locally convex Hausdorff space. Therefore, by Kakutani-Fan-Glicksberg Theorem (see Corollary 17.55, p. 583 in AB), there exists a fixed point $\hat{s}$ of Φ which is a Cournot-Nash equilibrium of the perturbed game Γ^ϵ. $\square$

In the next lemma, we prove that the price vector lies in a compact set bounded away from zero, for any ϵ, at a Cournot-Nash equilibrium.

Lemma 2. At a Cournot-Nash equilibrium $\hat{s}$ of the perturbed game Γ^ϵ, there exist two positive constants, independent from ϵ, C_j and D_j such that

$$C_j < p_j^\epsilon(\hat{s}) < D_j,$$

for each $j \in J \setminus \{0\}$. Moreover, C_j is uniformly bounded away from zero and D_j is uniformly bounded from above.

Proof. Let $\hat{s}$ be a Cournot-Nash equilibrium of the perturbed game. Without loss of generality, let $j = l$. First, we establish the existence of D_l. It is straightforward to verify that the highest possible price is $p_l = \frac{ae}{\sigma}$. Hence, $D_j = \frac{ae}{\sigma}$, for each $j \in J\setminus\{0\}$. Now, we establish the existence of C_l. By Assumption 4(ii), there are at least two traders belonging to the set T such that $\frac{\partial u^i}{\partial x_l^i}(x^i) > \lambda$. Between them, consider the trader i such that $\hat{b}_l^i \leq \frac{\bar{\hat{b}}_l}{2}$. We consider two cases. First, suppose that $\sum_{j\neq 0}\hat{b}_j^i < w_0^i$. An increase γ in i's bid for l is feasible, with $0 < \gamma \leq \min\{w_0^i - \sum_{j\neq 0}\hat{b}_j^i, \epsilon\}$, and has the following incremental effects on the final holding of trader i

$$\begin{aligned}
x_0^i(\hat{s}(\gamma)) - x_0^i(\hat{s}) =& -\gamma + w_l^i \frac{\bar{\hat{b}}_l + \epsilon + \gamma}{\bar{w}_l} - w_l^i \frac{\bar{\hat{b}} + \epsilon}{\bar{w}_l}, \\
=& -\gamma + \frac{w_l^i}{\bar{w}_l}\gamma \geq -\gamma \\
x_j^i(\hat{s}(\gamma)) - x_j^i(\hat{s}) =& 0, \text{ for each } j \in J \setminus \{0, l\}, \\
x_l^i(\hat{s}(\gamma)) - x_l^i(\hat{s}) =& (\hat{b}_l^i + \gamma)\frac{\bar{w}_l}{\bar{\hat{b}}_l + \epsilon + \gamma} - \hat{b}_l^i \frac{\bar{w}_l}{\bar{\hat{b}}_l + \epsilon}, \\
=& \frac{\bar{w}_l}{\bar{\hat{b}}_l + \epsilon} \frac{\bar{\hat{b}}_l^i + \epsilon}{\bar{\hat{b}}_l^i + \hat{b}_l^i + \epsilon + \gamma}\gamma \geq \frac{1}{3p_l^\epsilon(\hat{s})}\gamma,
\end{aligned}$$

with $\bar{\hat{b}}_j^i = \bar{\hat{b}}_j - \hat{b}_j^i$. The inequality in the preceding array follows from the fact that $\hat{b}_l^i \leq \bar{\hat{b}}_l^i + \epsilon$ and $\gamma \leq \bar{\hat{b}}_l^i + \epsilon$. Then, we obtain the following vector inequality[3]

$$x^i(\hat{s}(\gamma)) \geq x^i(\hat{s}) - \gamma e_0 + \frac{1}{3p_l^\epsilon(\hat{s})}\gamma e_l.$$

By using a linear approximation of the utility function around the point $x^i(\hat{s})$, we obtain

$$u^i(x^i(\hat{s}(\gamma))) - u^i(x^i(\hat{s})) \geq -\frac{\partial u^i}{\partial x_0^i}(x^i(\hat{s}))\gamma + \frac{\partial u^i}{\partial x_l^i}(x^i(\hat{s}))\frac{1}{3p_l^\epsilon(\hat{s})}\gamma + O(\gamma^2).$$

[3] e_j is an infinite vector in ℓ_∞ whose jth component is 1, and all others are 0.

Since $x^i(\hat{s})$ is an optimum point, the left hand side of the equation is negative and then

$$p_l^\epsilon(\hat{s}) > \frac{1}{3}\left(\frac{\partial u^i}{\partial x_l^i}(x^i(\hat{s}))\Big/\frac{\partial u^i}{\partial x_0^i}(x^i(\hat{s}))\right) = C_l^1.$$

By Assumption 2(iii) and since $\frac{\partial u^i}{\partial x_l^i}(x^i) > \lambda$, $C_l^1 \geq \frac{\lambda}{f}$. Now, suppose $\sum_{j\neq 0} \hat{b}_j^i = w_0^i$. By Assumption 2(ii), there exists a bid $\hat{b}_h^i$ such that $\hat{b}_h^i > \frac{w_0^i}{n}$. If $h = l$, then

$$p_l^\epsilon(\hat{s}) = p_h^\epsilon(\hat{s}) \geq \frac{w_0^i}{n\bar{w}_l} = C_l^2.$$

By Assumption 1, $C_l^2 \geq \frac{\sigma}{ne}$. If $h \neq l$, then trader i can decrease $\hat{b}_h^i$ by a small γ, with $0 < \gamma < \hat{b}_h^i$, and increase $\hat{b}_l^i$ by the same amount, with the following incremental effects on the final holding of trader i

$$\begin{aligned}
x_0^i(\hat{s}(\gamma)) - x_0^i(\hat{s}) &= w_l^i\frac{\bar{b}_l + \gamma}{\bar{w}_l} + w_h^i\frac{\bar{b}_h - \gamma}{\bar{w}_h} - w_l^i\frac{\bar{b}_l}{\bar{w}_l} - w_h^i\frac{\bar{b}_h}{\bar{w}_h} \\
&= \frac{w_l^i}{\bar{w}_l}\gamma - \frac{w_h^i}{\bar{w}_h}\gamma \geq -\frac{w_h^i}{\bar{w}_h}\gamma \geq -\gamma \\
x_j^i(\hat{s}(\gamma)) - x_j^i(\hat{s}) &= 0, \text{ for each } j \in J \setminus \{0, l, h\}, \\
x_l^i(\hat{s}(\gamma)) - x_l^i(\hat{s}) &\geq \frac{1}{3p_l^\epsilon(\hat{s})}\gamma, \\
x_h^i(\hat{s}(\gamma)) - x_h^i(\hat{s}) &= (\hat{b}_h^i - \gamma)\frac{\bar{w}_h}{\bar{\hat{b}}_h + \epsilon - \gamma} - \hat{b}_h^i\frac{\bar{w}_h}{\bar{\hat{b}}_h + \epsilon}, \\
&= -\frac{\bar{\hat{b}}_h^i + \epsilon}{\bar{\hat{b}}_h + \epsilon - \gamma}\frac{1}{p_h^\epsilon(\hat{s})}\gamma \geq -\frac{1}{p_h^\epsilon(\hat{s})}\gamma \geq -\frac{n\bar{w}_h}{w_0^i}\gamma.
\end{aligned}$$

The inequality in the preceding array follows from the fact that $\bar{\hat{b}}_h + \epsilon - \gamma \geq \bar{\hat{b}}_h^i + \epsilon$. Then, we obtain the following vector inequality

$$x^i(\hat{s}(\gamma)) \geq x^i(\hat{s}) - \gamma e_0 + \frac{1}{3p_l^\epsilon(\hat{s})}\gamma e_l - \frac{n\bar{w}_h}{w_0^i}\gamma e_h.$$

By using a linear approximation of the utility function around the point $x^i(\hat{s})$, we obtain

$$\begin{aligned}
u^i(x^i(\hat{s}(\gamma))) - u^i(x^i(\hat{s})) \geq -\frac{\partial u^i}{\partial x_0^i}(x^i(\hat{s}))\gamma + \frac{\partial u^i}{\partial x_l^i}(x^i(\hat{s}))\frac{1}{3p_l^\epsilon(\hat{s})}\gamma \\
-\frac{\partial u^i}{\partial x_h^i}(x^i(\hat{s}))\frac{n\bar{w}_h}{w_0^i}\gamma + O(\gamma^2).
\end{aligned}$$

Since $x^i(\hat{s})$ is an optimum point, the left hand side of the equation is negative and then

$$p_l^\epsilon(\hat{s}) > \frac{1}{3}\frac{\partial u^i}{\partial x_l^i}(x^i(\hat{s}))\Big/\left(\frac{\partial u^i}{\partial x_0^i}(x^i(\hat{s})) + \frac{n\bar{w}_h}{w_0^i}\frac{\partial u^i}{\partial x_h^i}(x^i(\hat{s}))\right) = C_l^3.$$

By Assumptions 1, 2, and since $\frac{\partial u^i}{\partial x_l^i}(x^i) > \lambda$, $C_l^3 \geq \frac{1}{3}\frac{\sigma}{\sigma+ne}\frac{\lambda}{f}$. Finally, we choose C_l such that $C_l = \min\{C_l^1, C_l^2, C_l^3, \}$. Since C_j^1, C_j^2, and C_j^3 are uniformly bounded away from zero, C_j is uniformly bounded above from zero, for each $j \in J \setminus \{0\}$. $\square$

We now state the existence theorem.

Theorem 1. Under Assumptions 1, 2, 3, 4, 5, and 6, the game Γ has an active Cournot-Nash equilibrium $\hat{s}$ such that the vector $p(\hat{s})$ is uniformly bounded away from zero.

Proof. Consider a sequence of $\{{}^g\epsilon\}_{g=1}^{\infty}$ converging to 0. By Lemma 1, in each perturbed game there exists a Cournot-Nash equilibrium. Then, we can consider the sequence of Cournot-Nash equilibria, $\{{}^g\hat{s}\}_{g=1}^{\infty}$, associated to the sequence of ϵ. As proved before, S is compact and, by Lemma 2, $p^{\epsilon}({}^g\hat{s}) \in \prod_{j\neq 0}[C_j, D_j]$ with C_j uniformly bounded away from zero and D_j uniformly bounded from above, for each $j \in J \setminus \{0\}$. By Tychonoff Theorem, $\prod_{j\neq 0}[C_j, D_j]$ is compact. Then, we can pick a subsequence of $\{{}^g\hat{s}\}_{g=1}^{\infty}$ that converges to v such that $v \in S$ and $p(v) \in \prod_{j\neq 0}[C_j, D_j]$. Therefore, v is a point of continuity of payoff functions and then v is a Cournot-Nash equilibrium with prices uniformly bounded away from zero. It remains to prove that $\hat{v}$ is an active Cournot-Nash equilibrium. By Assumption 1, $\bar{w}_j \geq \sigma$, for each $j \in J \setminus \{0\}$. Suppose, without loss of generality, that there exists a commodity l such that $\bar{\hat{b}}_l = 0$. But then, $p_l(\hat{v}) \notin [C_l, D_l]$, a contradiction. Therefore, $\bar{\hat{b}}_j > 0$, for each $j \in J \setminus \{0\}$, and then $\hat{v}$ is an active Cournot-Nash equilibrium. $\square$

We have shown the existence of an active Cournot–Nash equilibrium with prices bounded away from zero under the assumptions that there is a set of traders who desire a finite number of commodities (Assumption 2(i)) and that marginal utilities are uniformly bounded from above (Assumption 2(ii)). These assumptions are crucial to prove that prices are uniformly bounded away from zero. The next two examples clarify the role of these assumptions and show why prices converge to zero when the two assumptions do not hold. In order to simplify the computations, in both examples the set of traders H is empty and traders' utility functions are neither differentiable nor continuous at the boundary of the consumption set. This does not affect the current analysis but should be kept in mind. In the first example, we consider an exchange economy in which the number of commodities desired by each trader is finite but it is not uniformly bounded from above.

Example 1. Consider an exchange economy having as set of traders $I = T = \{1, 1', 2, 2', 3, 3', \dots\}$. Trader t and t' have the same initial endowment and utility function. Traders 1, 2, 3, 4, and $t \geq 5$ have the following utility functions and initial endowments

$$
\begin{aligned}
u^1(x^1) &= x_0^1 + \ln x_1^1 & w^1 &= (1, 1, 0, \dots),\\
u^2(x^2) &= x_0^2 + \ln x_2^2 + \ln x_3^2 & w^2 &= (1, 0, 1, 0, \dots),\\
u^3(x^3) &= x_0^3 + \sum_{j=4}^{6} \ln x_j^3 & w^3 &= (1, 0, 0, 1, 0, \dots),\\
u^4(x^4) &= x_0^4 + \sum_{j=7}^{10} \ln x_j^4 & w^4 &= (1, 0, 0, 0, 1, 0, \dots),\\
u^t(x^t) &= x_0^t + \sum_{j=1+\sum_{i=1}^{t-1} i}^{t+\sum_{i=1}^{t-1} i} \ln x_j^t & w^t &= (1, 0, \dots, 0, 1, 0, \dots).
\end{aligned}
$$

The Cournot-Nash equilibrium of the game Γ associated to the exchange economy is

$$
\begin{aligned}
(\hat{b}_1^1, \hat{b}_2^1, \hat{b}_3^1, \hat{b}_4^1, \dots, \hat{b}_j^1, \dots) &= \left(\frac{1}{2}, 0, 0, 0, \dots, 0, \dots\right),\\
(\hat{b}_1^2, \hat{b}_2^2, \hat{b}_3^2, \hat{b}_4^2, \dots, \hat{b}_j^2, \dots) &= \left(0, \frac{1}{2}, \frac{1}{2}, 0, \dots, 0, \dots\right),\\
(\hat{b}_1^3, \hat{b}_2^3, \hat{b}_3^3, \hat{b}_4^3, \hat{b}_5^3, \hat{b}_6^3, \hat{b}_7^3, \dots, \hat{b}_j^3, \dots) &= \left(0, 0, 0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, \dots, 0, \dots\right),\\
(\hat{b}_1^4, \dots, \hat{b}_6^4, \hat{b}_7^4, \hat{b}_8^4, \hat{b}_9^4, \hat{b}_{10}^4, \hat{b}_{11}^4, \dots, \hat{b}_j^4, \dots) &= \left(0, \dots, 0, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, 0, \dots, 0, \dots\right),\\
(\hat{b}_1^t, \dots, \hat{b}^t_{1+\sum_{i=1}^{t-1} i}, \dots, \hat{b}^t_{t+\sum_{i=1}^{t-1} i}, \dots) &= \left(0, \dots, 0, \frac{1}{t}, \dots, \frac{1}{t}, 0, \dots\right).
\end{aligned}
$$

for $t = 5, 6, \dots$. At this Cournot-Nash equilibrium the price vector $p(\hat{s})$ is not uniformly bounded away from zero.

Proof. At the Cournot-Nash equilibrium, in each trading post there are two traders active t and t' and both of them make the same bid $\frac{1}{t}$. Since $\lim_{j\to\infty} \bar{\hat{b}}_j = 0$ and $\bar{w}_j = 2$, for each $j \in J \setminus \{0\}$, $\lim_{j\to\infty} \hat{p}_j(s) = 0$. Hence, $p(\hat{s})$ is not uniformly bounded away from zero. $\square$

In the next example, we consider an exchange economy in which the marginal utilities of the desired commodities are not uniformly bounded from above when these commodities are consumed in positive quantities.

Example 2. Consider an exchange economy having as set of traders $I = T = \{1, 1', 2, 2', 3, 3', \dots\}$. Trader t and t' have the same initial endowment and utility function. Traders 1, 2, 3, 4, $s \geq 5$ odd, and $t \geq 6$ even have the following utility functions and initial endowments

$$
\begin{aligned}
&u^1(x^1) = x_0^1 && w^1 = (1, 1, 0, \dots),\\
&u^2(x^2) = x_0^2 + 2\ln x_1^2 && w^2 = (1, 0, 1, 0, \dots),\\
&u^3(x^3) = x_0^3 && w^3 = (1, 0, 0, 1, 0, \dots),\\
&u^4(x^4) = x_0^4 + \ln x_2^4 + 4\ln x_3^4 && w^4 = (1, 0, 0, 0, 1, 0, \dots),\\
&u^s(x^s) = x_0^s && w^s = (1, 0, \dots, 0, 1, 0, \dots),\\
&u^t(x^t) = x_0^t + \ln x_{t-2}^t + t\ln x_{t-1}^t && w^t = (1, 0, \dots, 0, 1, 0, \dots).
\end{aligned}
$$

The Cournot-Nash equilibrium of the game Γ associated to the exchange economy is

$$
\begin{aligned}
(\hat{b}_1^s, \hat{b}_2^s, \dots, \hat{b}_j^s, \dots) &= (0, 0, \dots, 0, \dots),\\
(\hat{b}_1^2, \hat{b}_2^2, \hat{b}_3^2, \dots, \hat{b}_j^2, \dots) &= (1, 0, 0, \dots, 0, \dots),\\
(\hat{b}_1^4, \hat{b}_2^4, \hat{b}_3^4, \hat{b}_4^4, \dots, \hat{b}_j^4, \dots) &= \left(0, \frac{1}{4}, \frac{3}{4}, 0, \dots, 0, \dots\right),\\
(\hat{b}_1^t, \dots, \hat{b}_{t-2}^t, \hat{b}_{t-1}^t, \hat{b}_t^t, \dots) &= \left(0, \dots, 0, \frac{1}{t}, \frac{t-1}{t}, 0, \dots\right),
\end{aligned}
$$

for $s \geq 1$ odd and $t \geq 6$ even. At this Cournot-Nash equilibrium the price vector $p(\hat{s})$ is not uniformly bounded away from zero.

Proof. At the Cournot-Nash equilibrium, in each trading post there are two traders active t and t' and both of them make the same bid. The price of a commodity j odd is $p_j(\hat{s}) = \frac{j}{j+1}$ and the price of a commodity j even is $p_j(\hat{s}) = \frac{1}{j+2}$. Then, the subsequence of odd commodities' prices converges to 1 while the subsequence of even commodities' prices converges to 0. Hence, $p(\hat{s})$ is not uniformly bounded away from zero. □

3.4 Asymptotic price-takers

We start this section by recalling the definitions of market power, asymptotic price-taker, and asymptotic oligopolist from the previous chapter.

In the game Γ, the market power of a trader i on commodity j can be measured by the ratio $b_j^i/\bar{b}_j$. The higher this ratio is, the higher is the market power of trader i on commodity j. If $b_j^i = 0$, we say that trader i is a trivial price-taker on commodity j.

Definition 3. Consider an active Cournot-Nash equilibrium $\hat{s}$ in which there exists a trader h such that $\hat{b}_j^h > 0$ for an infinite number of commodities. We say that trader h is an asymptotic price-taker if $\lim_{j\to\infty} \hat{b}_j^h / \bar{\hat{b}}_j = 0$, otherwise we say that trader h is an asymptotic oligopolist.

The key features of an asymptotic price-taker are that he consumes an infinite number of commodities and his market power converges to zero along the sequence of commodities.

In the next theorem, we prove the existence of an asymptotic price-taker.

Theorem 2. Let $\mathcal{E}$ be an exchange economy which satisfies Assumptions 1, 2, 3, 4, 5, and 6. If there exists a trader $h \in H$ whose utility function is such that $\lim_{x_j^h \to 0} \frac{\partial u^h}{\partial x_j^h} = \infty$, for an infinite number of commodities, then h is an asymptotic price-taker.

Proof. By Theorem 1, under Assumptions 1, 2, 3, 4, 5, and 6, there exists an active Cournot-Nash equilibrium $\hat{s}$ of Γ such that the vector $p(\hat{s})$ is uniformly bounded away from zero. Let $h \in H$ be a trader whose utility function is such that $\lim_{x_j^h \to 0} \frac{\partial u^h}{\partial x_j^h} = \infty$, for an infinite number of commodities. First, suppose that $\hat{b}_l^h = 0$, for a commodity l such that $\lim_{x_l^h \to 0} \frac{\partial u^h}{\partial x_l^h} = \infty$, and $\sum_{j\neq 0} \hat{b}_j^h < w_0^h$. Consider a strategy s'^h such that $b_l'^h = \hat{b}_l^h + \gamma$, with γ sufficiently small, and all other actions equal to the actions of the original strategy $\hat{s}^h$. Since $\lim_{x_l^h \to 0} \frac{\partial u^h}{\partial x_l^h} = \infty$, $u^h(x^h(s'^h, \hat{s}^{-h})) > u^h(x^h(\hat{s}^h, \hat{s}^{-h}))$, a contradiction. Now, suppose that $\hat{b}_l^h = 0$, for a commodity l such that $\lim_{x_l^h \to 0} \frac{\partial u^h}{\partial x_l^h} = \infty$, and $\sum_{j\neq 0} \hat{b}_j^h = w_0^h$. Then, there exists a commodity m such that $\hat{b}_m^h > 0$. Consider a strategy s'^h such that $b_m'^h = \hat{b}_m^h - \gamma$, $b_l'^h = \hat{b}_l^h + \gamma$, with γ sufficiently small, and all other actions equal to the actions of the original strategy $\hat{s}^h$. Since $\lim_{x_l^h \to 0} \frac{\partial u^h}{\partial x_l^h} = \infty$, $u^h(x^h(s'^h, \hat{s}^{-h})) > u^h(x^h(\hat{s}^h, \hat{s}^{-h}))$, a contradiction. Hence, $\hat{b}_j^h > 0$, for each commodity j such that $\lim_{x_j^h \to 0} \frac{\partial u^h}{\partial x_j^h} = \infty$. Therefore, trader h makes positive bids on an infinite number of commodities and his sequence of bids, $\{\hat{b}_j^h\}$, converges to zero because his endowment of commodity money is finite. Since $p(\hat{s})$ is bounded away from zero and the aggregate endowment of each commodity is uniformly bounded away from zero by Assumption 1, the sequence of sums of bids, $\{\bar{\hat{b}}_j\}$, is uniformly bounded away from zero. But then, $\lim_{j\to\infty} \hat{b}_j^h / \bar{\hat{b}}_j = 0$. Hence, trader h is an asymptotic price-taker. $\square$

In the subsequent examples, we show that a price vector uniformly bounded away from zero is crucial to prove the existence of asymptotic price-takers. The next example shows that if prices converge to zero along the sequence of commodities then a trader who consumes and infinite number of commodities can be an asymptotic oligopolist.

Example 3. Consider an exchange economy having as set of traders $I = \{1, 2, 3, \ldots\}$. Traders 1, 2, 3, and $t \geq 4$ have the following utility functions and initial endowments

$$
\begin{aligned}
u^1(x^1) &= x_0^1 + \sum_{j=1}^{\infty} \frac{1}{j^2} \ln x_j^1 & w^1 &= (1, 1, 0, \ldots),\\
u^2(x^2) &= x_0^2 + \ln x_1^2 & w^2 &= (1, 0, 1, 0, \ldots),\\
u^3(x^3) &= x_0^3 + \frac{1}{4} \ln x_2^3 & w^3 &= (1, 0, 0, 1, 0, \ldots),\\
u^t(x^t) &= x_0^t + \frac{1}{(t-1)^2} \ln x_{t-1}^t & w^t &= \big(1, 0, \ldots, 0, 1, 0, \ldots\big).
\end{aligned}
$$

The Cournot-Nash equilibrium of the game Γ associated to the exchange economy is

$$
\begin{aligned}
(\hat{b}_1^1, \hat{b}_2^1, \hat{b}_3^1, \ldots, \hat{b}_j^1, \ldots) &= \left(\frac{1}{2}, \frac{1}{8}, \frac{1}{18}, \ldots, \frac{1}{2j^2}, \ldots\right),\\
(\hat{b}_1^2, \hat{b}_2^2, \hat{b}_3^2, \ldots, \hat{b}_j^2, \ldots) &= \left(\frac{1}{2}, 0, 0, \ldots, 0, \ldots\right),\\
(\hat{b}_1^3, \hat{b}_2^3, \hat{b}_3^3, \ldots, \hat{b}_j^3, \ldots) &= \left(0, \frac{1}{8}, 0, \ldots, 0, \ldots\right),\\
(\hat{b}_1^t, \ldots, \hat{b}_{t-1}^t, \hat{b}_t^t, \hat{b}_{t+1}^t, \ldots) &= \left(0, \ldots, 0, \frac{1}{2(j-1)^2}, 0, \ldots\right),
\end{aligned}
$$

for $t = 4, 5, \ldots$. At this Cournot-Nash equilibrium trader 1 is an asymptotic oligopolist.

Proof. At the Cournot-Nash equilibrium, in a trading post for commodity j only traders 1 and $j+1$ are active and they both make the same bid $\frac{1}{2j^2}$. Therefore, the market power of trader 1, $\hat{b}_j^1/\bar{\hat{b}}_j$, is equal to $\frac{1}{2}$, for $j \geq 1$. Hence, $\lim_{j\to\infty} \hat{b}_j^1/\bar{\hat{b}}_j = \frac{1}{2}$. □

The next example shows that there are exchange economies with asymptotic price-takers even when prices converge to zero along the sequence of commodities. In other words, a price vector uniformly bounded away from zero is a sufficient condition for the existence of asymptotic price takers but it is not a necessary condition.

Example 4. Consider an exchange economy having as set of traders $I = \{1, 2, 3, \dots\}$. Traders 1, 2, 3, and $t \geq 4$ have the following utility functions and initial endowments

$$\begin{aligned}
u^1(x^1) &= x_0^1 + \sum_{j=1}^{\infty} \frac{1}{j^2} \ln x_j^1 & w^1 &= (1, 1, 0, \dots),\\
u^2(x^2) &= x_0^2 + \ln x_1^2 & w^2 &= (1, 0, 1, 0, \dots),\\
u^3(x^3) &= x_0^3 + \ln x_2^3 & w^3 &= (1, 0, 0, 1, 0, \dots),\\
u^t(x^t) &= x_0^t + \ln x_{t-1}^t & w^t &= \big(1, 0, \dots, 0, 1, 0, \dots\big).
\end{aligned}$$

The Cournot-Nash equilibrium of the game Γ associated to the exchange economy is

$$\begin{aligned}
(\hat{b}_1^1, \hat{b}_2^1, \hat{b}_3^1, \dots, \hat{b}_j^1, \dots) &= \left(\frac{1}{2}, \frac{1}{6}, \frac{1}{12}, \dots, \frac{1}{j + j^2}, \dots\right),\\
(\hat{b}_1^2, \hat{b}_2^2, \hat{b}_3^2, \dots, \hat{b}_j^2, \dots) &= \left(\frac{1}{2}, 0, 0, \dots, 0, \dots\right),\\
(\hat{b}_1^3, \hat{b}_2^3, \hat{b}_3^3, \dots, \hat{b}_j^2, \dots) &= \left(0, \frac{1}{3}, 0, \dots, 0, \dots\right),\\
(\hat{b}_1^t, \dots, \hat{b}_{t-2}^t, \hat{b}_{t-1}^t, \hat{b}_t^t, \dots) &= \left(0, \dots, 0, \frac{1}{t}, 0, \dots\right),
\end{aligned}$$

for $t = 4, 5, \dots$. At this Cournot-Nash equilibrium trader 1 is an asymptotic price-taker.

Proof. At the Cournot-Nash equilibrium, in a trading post for commodity j only traders 1 and $j + 1$ are active. The market power of trader 1, $\hat{b}_j^1/\bar{\hat{b}}_j$, is equal to $\frac{1}{1+j}$, for each $j \geq 2$. Hence, $\lim_{j\to\infty} \hat{b}_j^1/\bar{\hat{b}}_j = 0$. □

The condition that the price vector is uniformly bounded away from zero and Assumption 1 on initial endowments imply that the sums of bids and offers along the sequence of trading posts are uniformly bounded away from zero. This can be interpreted as a case in which all markets are "thick". Gretsky and Ostroy (1984) introduced the notions of "thick" and "thin" to refer to markets with many and few traders respectively. Similarly, Shubik (1973) call them "broad" and "thin" markets. Differently, in our framework thick means that the quantities of commodities and commodity money along the sequence of trading posts are uniformly bounded away from zero. In Examples 3 and 4, the sums of bids converge to zero along the sequence of trading posts and then markets are not thick.

Heuristically, Theorem 2, Examples 3, and Example 4 imply that if all markets are thick then all traders active on an infinite number of commodities are asymptotic price-takers. Differently, if markets are not thick the existence of an asymptotic price-taker depends on the pattern of preferences among traders.

3.5 Discussion of the model

The papers of Dixit and Stiglitz (1977), Hart (1985), and Pascoa (1993) developed different models to study monopolistic competition in the spirit of Chamberlin in production economies. All these contributions are characterised by firms which have market power on some commodities and consider as given the prices of all others. This is a crucial feature of Chamberlinian monopolistic competition and it is similar to the mixed behaviour obtained by using the portfolio of commodities. In the paper of Dixit and Stiglitz (1977) and Hart (1985), the mixed behaviour does not arise endogenously in equilibrium but it is assumed and justified by saying that firms can ignore their impact on others because their number is very large, i.e., firms are small compared to the whole economy. Differently, in Pascoa (1993) the mixed behaviour arises endogenously because the set of firms is an atomless continuum and then each firm is negligible. Unfortunately, it seems to be no hope to prove the existence of a Cournot-Nash equilibrium in such framework and then Pascoa (1993) proved the existence of an approximate equilibrium.

In the framework of this chapter, the mixed behaviour of an asymptotic price-taker is characterised by an approximate competitive behaviour since traders' market power is never zero but it can be made arbitrary small on infinite set of commodities. To obtain a clear endogenous split between strategic and competitive behaviours, it seems necessary to deal with models with a continuum of traders and a continuum of commodities. In the literature on the core, the equivalence and non-equivalence results were already extended to economies with a continuum of traders and commodities by Mas-Colell (1975) and Ostroy and Zame (1994) respectively. Nevertheless, in non-cooperative game theory many non existence results arise in games with a continuum of players and an infinite dimensional space of strategies as shown by Kahn, Rath, and Sun (1997) and Sun and Zhang (2015).

3.6 Conclusion

In this chapter, we have extended the study of mixed behaviour to Cournotian games by focusing on the bids side. The mixed behaviour is formalised by the notion of asymptotic price-taker. The idea that a trader is small compared to the whole economy is captured by considering economies with infinitely many commodities and traders and by assuming that the aggregate endowment of commodity money is infinite. In such framework, each trader holds only a negligible part of the total endowment of commodity money. In Theorem 1, we have proved the existence of a Cournot-Nash equilibrium for the all for sale model with an infinite number of commodities and traders. The difficulties which characterise the setting with a continuum of traders and strategies are overcome with our approach based on countable infinities. In Theorem 2, we provide the sufficient conditions on the fundamentals of an economy under which an asymptotic price-taker exists. Surprisingly, Example 3 shows that the infinities of commodities, traders, and endowment of commodity money are not sufficient to obtain the mixed behaviour which characterises asymptotic price-takers but additional restrictions on preferences are necessary.

— 4 —

Atomic Cournotian Traders May Be Walrasian

4.1 Introduction

In his celebrated paper, Aumann (1964) proved that, in exchange economies with a continuum of traders, the core coincides with the set of Walras allocations. He also suggested

> "Of course, to the extent that individual consumers or merchants are in fact not anonymous (think of General Motors), the continuous model is inappropriate, and our results do not apply to such a situation. But, in that case, perfect competition does not obtain either. In many real markets the competition is indeed far from perfect; such markets are probably best represented by a mixed model, in which some of the traders are points in a continuum, and others are individually significant."

Some years later, by following Aumann's suggestion, Gabszewicz and Mertens (1971) and Shitovitz (1973) introduced the mixed exchange economy model, i.e., an exchange economy with oligopolists, represented as atoms, and small traders, represented by an atomless continuum, in order to analyse oligopoly in a general equilibrium framework. Gabszewicz and Mertens (1971) showed that, if atoms are not "too" big, the core still coincides with the set of Walras allocations whereas Shitovitz (1973), in his Theorem B, proved that this result also holds if the atoms

[0]Some of the materials of this chapter were published in Codognato G., Ghosal S., Tonin S. (2015), "Atomic Cournotian traders may be Walrasian", *Journal of Economic Theory*, **159**, 1-14.

are of the same type, i.e., have the same initial endowments and preferences. It is worthy to note that Gabszewicz and Mertens were not satisfied of their result since they find an “extravagant condition” which divides exchange economies into two main categories: the one in which there is an equivalence result and the interesting one.

Analogously, Okuno et al. (1980) considered the result obtained by Shitovitz (1973) so counterintuitive to call into question the use of the core as the solution concept to study oligopoly in general equilibrium.[1] This led them to replace the core with the Cournot-Nash equilibrium of a model of simultaneous, non-cooperative exchange between oligopolists and small traders as the appropriate solution for the analysis of oligopoly in general equilibrium. The model of non-cooperative exchange they used belongs to a line of research initiated by Shubik (1973), Shapley (1976), and Shapley and Shubik (1977) (see Giraud (2003) for a survey of this literature). In particular, they considered a mixed exchange economy with two commodities which are both held by all traders. Furthermore, they assumed that no trader is allowed to be both buyer and seller of any commodity. In this framework, they showed that, if there are two atoms of the same type who demand, at a Cournot-Nash equilibrium, a positive amount of the two commodities, then the corresponding Cournot-Nash allocation is not a Walras allocation. Therefore, under the assumptions of Shitovitz's Theorem B, demanding a non-null amount of the two commodities by all the atoms is a sufficient condition for a Cournot-Nash allocation not to be a Walras allocation. This proposition allowed Okuno et al. (1980) to conclude that the non-cooperative model they considered is a useful one to study oligopoly in a general equilibrium framework as the small traders always have a negligible influence on prices, while oligopolists keep their strategic power even when their behaviour turns out to be Walrasian in the cooperative framework considered by Shitovitz (1973).

In this paper, we raise the question whether, in mixed exchange economies, an equivalence, or at least a non-empty intersection, between the sets of Walras and Cournot-Nash allocations may hold. In order to further simplify our analysis, we consider the model of bilateral oligopoly introduced by Gabszewicz and Michel (1997) and further analysed by Bloch and Ghosal (1997), Bloch and Ferrer (2001), Dickson and Hartley (2008), Amir and Bloch (2009), among others. By using this model, we still remain in a two-commodity setting but we assume that each trader

[1] Okuno et al. (1980) did not quote the result obtained by Gabszewicz and Mertens (1971). Nevertheless, their argument also holds, *mutatis mutandis*, for this result.

holds only one of the two commodities whose aggregate amount is strictly positive in the economy. We shall use a mixed bilateral oligopoly version of the strategic market game with commodity money analysed by Dubey and Shubik (1978) and by Dubey and Shapley (1994). In particular, Dubey and Shapley (1994) considered an economy with an atomless continuum of traders and proved that in such framework the sets of Walras and Cournot-Nash allocations coincide, thereby providing a non-cooperative version of Aumann's theorem. However, since in this strategic market game trade takes place by using commodity money, liquidity problems can arise and even if all traders are negligible there can be a non-equivalence between Cournot-Nash and Walras allocations. For this reason, the equivalence result is obtained under more restrictive assumptions than the ones made Aumann (1964). However, in exchange economies with two commodities, liquidity problems do not arise and the coincidence between the sets of Cournot-Nash and Walras allocations can be proved under more general conditions.

We first show, through some examples, that the threefold equivalence among the sets of Walras, core, and Cournot-Nash allocations may not hold, in mixed exchange economies with a bilateral oligopoly configuration, even under the assumptions made by Gabszewicz and Mertens (1971) and Shitovitz (1973). These examples confirm the result obtained by Okuno et al. (1980). We then answer our main question by proving a theorem which states that demanding a null amount of one of the two commodities by all the atoms is a necessary and sufficient condition for a Cournot-Nash allocation to be a Walras allocation. We also provide four examples which show that this characterisation theorem is non-vacuous.

Our result depends only on atoms' demand behaviour at a Cournot-Nash equilibrium. This opens the door to a research on the conditions on the fundamentals of an economy, i.e., traders' size, initial endowments, and preferences, under which our theorem holds. We start an investigation in this direction by providing two necessary conditions, expressed in terms of bounds on atoms' marginal rates of substitution, for our result to hold when atoms' preferences are represented by additively separable utility functions and by quasi linear utility functions respectively. Finally, we show that, in the mixed bilateral oligopoly framework, our main theorem can be extended to the model of non-cooperative exchange with complete markets proposed by Amir, Sahi, Shubik, and Yao (1990).

The paper is organised as follows. In Section 4.2, we introduce the mathematical model. In Section 4.3, we state the main equivalence theorems. In Section 4.4, we provide some examples and we state and prove our main theorem. In Section 4.5, we show two necessary conditions on the fundamentals of an economy under which our result holds. In Section 4.6, we extend our main result to the model analysed by Amir et al. (1990). In Section 4.7, we draw some conclusions from our analysis.

4.2 Mathematical model

We consider an exchange economy with oligopolists, represented as atoms, and small traders, represented by an atomless part. The space of traders is denoted by the measure space $(T, \mathcal{T}, \mu)$, where T is the set of traders, $\mathcal{T}$ is the σ-algebra of all μ-measurable subsets of T, and μ is a real valued, non-negative, countably additive measure defined on $\mathcal{T}$. We assume that $(T, \mathcal{T}, \mu)$ is finite, i.e., $\mu(T) < \infty$. This implies that the measure space $(T, \mathcal{T}, \mu)$ contains at most countably many atoms. Let T_0 denote the atomless part of T. A null set of traders is a set of measure 0. Null sets of traders are systematically ignored throughout the paper. Thus, a statement asserted for "each" trader in a certain set is to be understood to hold for all such traders except possibly for a null set of traders. A coalition is a non-null element of $\mathcal{T}$. The word "integrable" is to be understood in the sense of Lebesgue.

In the exchange economy, there are two different commodities. A commodity bundle is a point in $\mathbb{R}^2_+$. An assignment of commodity bundles to traders is an integrable function $\mathbf{x} : T \to \mathbb{R}^2_+$. There is a fixed initial assignment $\mathbf{w}$, satisfying the following assumption.

Assumption 1. There is a coalition S such that $\mathbf{w}_1(t) > 0$, $\mathbf{w}_2(t) = 0$, for each $t \in S$, and $\mathbf{w}_1(t) = 0$, $\mathbf{w}_2(t) > 0$, for each $t \in S^c$.

The preferences of each trader $t \in T$ are described by a utility function $u_t : \mathbb{R}^2_+ \to \mathbb{R}$, satisfying the following assumptions.

Assumption 2. $u^t : \mathbb{R}^2_+ \to \mathbb{R}$ is continuous, strongly monotone, and quasi-concave, for each $t \in T$.

Let $\mathcal{B}(\mathbb{R}^2_+)$ denote the Borel σ-algebra of $\mathbb{R}^2_+$. Moreover, let $\mathcal{T} \bigotimes \mathcal{B}$ denote the σ-algebra generated by the sets $E \times F$ such that $E \in \mathcal{T}$ and $F \in \mathcal{B}$.

Assumption 3. $u : T \times \mathbb{R}^2_+ \to \mathbb{R}$, given by $u(t, x) = u^t(x)$, for each $t \in T$ and for each $x \in \mathbb{R}^2_+$, is $\mathcal{T} \bigotimes \mathcal{B}$-measurable.

An allocation is an assignment $\mathbf{x}$ for which $\int_T \mathbf{x}(t)\, d\mu = \int_T \mathbf{w}(t)\, d\mu$. An allocation $\mathbf{y}$ dominates an allocation $\mathbf{x}$ via a coalition S if $u^t(\mathbf{y}(t)) \geq u^t(\mathbf{x}(t))$, for each $t \in S$, $u^t(\mathbf{y}(t)) > u^t(\mathbf{x}(t))$, for a non-null subset of traders t in S, and $\int_S \mathbf{y}(t)d\mu = \int_S \mathbf{w}(t)d\mu$. The core is the set of all allocations which are not dominated via any coalition. A Walras equilibrium is a pair $(p^*, \mathbf{x}^*)$, consisting of a price vector p^* and an allocation $\mathbf{x}^*$, such that $p^*\mathbf{x}^*(t) = p^*\mathbf{w}(t)$ and $u^t(\mathbf{x}^*(t)) \geq u^t(y)$, for each $y \in \{x \in \mathbb{R}^2_+ : p^*x = p^*\mathbf{w}(t)\}$, for each $t \in T$.

A price vector is a non-null vector $p \in \mathbb{R}^2_+$. A Walras allocation is an allocation $\mathbf{x}^*$ for which there exists a price vector p^* such that the pair $(p^*, \mathbf{x}^*)$ is a Walras equilibrium.

We now introduce the strategic market game associated to the exchange economy. We consider a two-commodity version of the game analysed by Dubey and Shubik (1978) and adapted for mixed exchange economies. To do so, we follow the contribution of Busetto, Codognato, and Ghosal (2011) on a strategic market game in mixed exchange economies. Since this is a game in which trade takes place by using commodity money, we assume, without loss of generality, that commodity 2 has this role. A strategy correspondence is a correspondence $\mathbf{S} : T \to \mathcal{P}(\mathbb{R}^2_+)$ such that, for each $t \in T$, $\mathbf{S}(t) = \{(q, b) \in \mathbb{R}^2_+ : q \leq \mathbf{w}_1(t) \text{ and } b \leq \mathbf{w}_2(t)\}$, where q is the amount of commodity 1 puts up in exchange for commodity money and b is the bid of commodity money offered for commodity 1. A strategy selection is an integrable function $\mathbf{s} : T \to \mathbb{R}_+$, such that $\mathbf{s}(t) \in \mathbf{S}(t)$, for each $t \in T$. We denote by $\mathbf{s} \setminus s(t)$ the strategy selection obtained from $\mathbf{s}$ by replacing $\mathbf{s}(t)$ with $s(t) \in \mathbf{S}(t)$.

Since commodity 2 has the role of money, we normalize its price to one, i.e., $p_2 = 1$. Therefore, given a strategy selection $\mathbf{s}$, the price of commodity 1 is determined as follows

$$p_1 = \begin{cases} \frac{\bar{\mathbf{b}}}{\bar{\mathbf{q}}} & \text{if } \bar{\mathbf{q}} \neq 0, \\ 0 & \text{if } \bar{\mathbf{q}} = 0, \end{cases} \tag{4.1}$$

with $\bar{\mathbf{q}} = \int_T \mathbf{q}(t)d\mu$ and $\bar{\mathbf{b}} = \int_T \mathbf{b}(t)d\mu$. We denote by $p(\mathbf{s})$ a function which associates with each strategy selection $\mathbf{s}$ the price vector $(p_1, 1)$ satisfying (4.1).

Given a strategy selection $\mathbf{s}$ and a price vector p, consider the assignment determined as follows

$$\mathbf{x}_1(t, \mathbf{s}(t), p) = \begin{cases} \mathbf{w}_1(t) - \mathbf{q}(t) + \frac{\mathbf{b}(t)}{p_1} & \text{if } p_1 \neq 0, \\ \mathbf{w}_1(t) - \mathbf{q}(t) & \text{if } p_1 = 0, \end{cases}$$

$$\mathbf{x}_2(t, \mathbf{s}(t), p) = \mathbf{w}_2(t) - \mathbf{b}(t) + \mathbf{q}(t)p_1,$$

for each $t \in T$. Given a strategy selection $\mathbf{s}$ and the function $p(\mathbf{s})$, the traders' final holdings are determined according with this rule and consequently expressed by the assignment

$$\mathbf{x}(t) = \mathbf{x}(t, \mathbf{s}(t), p(\mathbf{s})),$$

for each $t \in T$.[2] It is straightforward to show that this assignment is an allocation.

We are now able to define a notion of Cournot-Nash equilibrium for this reformulation of the Dubey and Shubik model.

Definition 1. A strategy selection $\hat{\mathbf{s}}$ is a Cournot-Nash equilibrium if

$$u^t(\mathbf{x}(t, \hat{\mathbf{s}}(t), p(\hat{\mathbf{s}}))) \geq u^t(\mathbf{x}(t, \hat{\mathbf{s}} \setminus s(t), p(\hat{\mathbf{s}} \setminus s(t)))),$$

for each $s(t) \in \mathbf{S}(t)$ and for each $t \in T$.

It is straightforward to verify that a strategy selection $\mathbf{s}$ such that $(\mathbf{q}(t), \mathbf{b}(t)) = (0, 0)$ for each $t \in T$ is always a Cournot-Nash equilibrium. We call it trivial Cournot-Nash equilibrium since there is no trade. Differently, if a Cournot-Nash equilibrium $\hat{\mathbf{s}}$ is such that $\bar{\hat{\mathbf{q}}} > 0$ and $\bar{\hat{\mathbf{b}}} > 0$, we say that $\hat{\mathbf{s}}$ is an active Cournot-Nash equilibrium. Since we want to establish equivalence results, in the rest of the paper the trivial Cournot-Nash equilibrium is systematically ignored and we refer to active Cournot-Nash equilibria simply as Cournot-Nash equilibria.

A Cournot-Nash allocation is an allocation $\hat{\mathbf{x}}$ such that $\hat{\mathbf{x}}(t) = \mathbf{x}(t, \hat{\mathbf{s}}(t), p(\hat{\mathbf{s}}))$, for each $t \in T$, where $\hat{\mathbf{s}}$ is a Cournot-Nash equilibrium.

4.3 Theorems of equivalence

In this section, we first state the equivalence result between the core and the set of Walras allocations established by Aumann (1964) in atomless economies and we prove the analogous theorem between the sets of Cournot-Nash and Walras allocations. We then recall the equivalence results between the core and the set of Walras allocations obtained by Gabszewicz and Mertens (1971) and Shitovitz (1973) in mixed exchange economies.

Theorem 1. Under Assumptions 1, 2, and 3, if $T = T_0$, then the core coincides with the set of Walras allocations.

Proof. See the proof of the main theorem in Aumann (1964). □

The equivalence theorem between Cournot-Nash and Walras allocations could not be established in such generality because it holds only for active Cournot-Nash equilibria. Indeed, when the Walras allocation is interior for a non-null subset of

[2] In order to save in notation, with some abuse, we denote by $\mathbf{x}$ both the function $\mathbf{x}(t)$ and the function $\mathbf{x}(t, \mathbf{s}(t), p(\mathbf{s}))$.

traders, there exists an allocation associated to the trivial Cournot-Nash equilibrium which breaks the equivalence result. But by excluding the trivial Cournot-Nash equilibrium, we do not have an equivalence when a Walras allocation is equal to the initial assignments.

Theorem 2. Let $\mathbf{x}$ be an allocation such that $\mathbf{x}(t) \gg 0$ for a non-null subset of traders $t \in T$. Under Assumptions 1, 2, and 3, if $T = T_0$, the allocation $\mathbf{x}$ is a Cournot-Nash allocation if and only if it is a Walras allocation.

Proof. Let $\mathbf{x}$ be a Cournot-Nash allocation. Then, there exists a strategy selection $\hat{\mathbf{s}}$, which is a Cournot-Nash equilibrium, such that $\mathbf{x}(t) = \hat{\mathbf{x}}(t) = \mathbf{x}(t, \hat{\mathbf{s}}(t), p(\hat{\mathbf{s}}))$, for each $t \in T$. We now show that $\hat{\mathbf{x}}$ is also a Walras allocation. Let $\hat{p} = p(\hat{\mathbf{s}})$. Since $\hat{\mathbf{x}}(t) \gg 0$ for a non-null subset of traders $t \in T$, $\hat{p} \gg 0$. We have that $\hat{p}\hat{\mathbf{x}}(t) = \hat{p}\mathbf{w}(t)$, for each $t \in T$, since $\hat{p}_2 = 1$ and

$$\begin{aligned}\hat{p}_1\hat{\mathbf{x}}_1(t) + \hat{p}_2\hat{\mathbf{x}}_2(t) = \hat{p}_1\left(\mathbf{w}_1(t) - \hat{\mathbf{q}}(t) + \frac{\hat{\mathbf{b}}(t)}{\hat{p}_1}\right) + \\ \hat{p}_2\big(\mathbf{w}_2(t) - \hat{\mathbf{b}}(t) + \hat{\mathbf{q}}(t)\hat{p}_1\big) = \hat{p}_1\mathbf{w}_1(t) + \hat{p}_2\mathbf{w}_2(t),\end{aligned} \tag{4.2}$$

for each $t \in T$. Suppose that there exist a trader $t \in T$ and a commodity bundle $\tilde{x}$ such that $u^t(\tilde{x}) > u^t(\hat{\mathbf{x}}(t))$ and $\tilde{x} \in \{x \in \mathbb{R}^2_+ : \hat{p}x = \hat{p}\mathbf{w}(t)\}$. Since the utility function is strongly monotone, by Assumption 2, $\tilde{x}_1 > \hat{\mathbf{x}}_1(t)$ or $\tilde{x}_2 > \hat{\mathbf{x}}_2(t)$. First, assume that the trader t is such that $\mathbf{w}_1(t) > 0$ and $\mathbf{w}_2(t) = 0$. Consider the case where $\tilde{x}_1 > \hat{\mathbf{x}}_1(t)$. If $\hat{\mathbf{s}}(t)$ is such that $\hat{\mathbf{q}}(t) = 0$ then $\hat{p}\tilde{x} \neq \hat{p}\mathbf{w}(t)$, a contradiction. If $\hat{\mathbf{s}}(t)$ is such that $\hat{\mathbf{q}}(t) > 0$, then there exists a strategy $\tilde{s}(t) \in \mathbf{S}(t)$ such that $\tilde{x} = \mathbf{x}(t, \hat{\mathbf{s}} \setminus \tilde{s}(t), p(\hat{\mathbf{s}}))$. Note that $p(\hat{\mathbf{s}}) = p(\hat{\mathbf{s}} \setminus \tilde{s}(t))$ as $\hat{p}_1 = \frac{\int_T \hat{\mathbf{b}}(t) d\mu}{\int_T \hat{\mathbf{q}}(t) d\mu}$ and $\hat{p}_2 = 1$. But then,

$$u^t(\mathbf{x}(t, \hat{\mathbf{s}} \setminus \tilde{s}(t), p(\hat{\mathbf{s}}))) > u^t(\mathbf{x}(t, \hat{\mathbf{s}}(t), p(\hat{\mathbf{s}}))),$$

a contradiction. Consider the case where $\tilde{x}_2 > \hat{\mathbf{x}}_2(t)$. If $\hat{\mathbf{s}}(t)$ is such that $\hat{\mathbf{q}}(t) = \mathbf{w}_2(t)$, then $\hat{p}\tilde{x} \neq \hat{p}\mathbf{w}(t)$, a contradiction. If $\hat{\mathbf{s}}(t)$ is such that $\hat{\mathbf{q}}(t) < \mathbf{w}_2(t)$, then there exists a strategy $\tilde{s}(t) \in \mathbf{S}(t)$ such that $\tilde{x} = \mathbf{x}(t, \hat{\mathbf{s}} \setminus \tilde{s}(t), p(\hat{\mathbf{s}}))$. But then,

$$u^t(\mathbf{x}(t, \hat{\mathbf{s}} \setminus \tilde{s}(t), p(\hat{\mathbf{s}}))) > u^t(\mathbf{x}(t, \hat{\mathbf{s}}(t), p(\hat{\mathbf{s}}))),$$

a contradiction. Now, assume that the trader t is such that $\mathbf{w}_1(t) = 0$ and $\mathbf{w}_2(t) > 0$. Then, the previous argument leads, *mutatis mutandis*, to the same kinds of contradictions. Therefore, $(\hat{p}, \hat{\mathbf{x}})$ is a Walras equilibrium. Now, let $\mathbf{x}$ be a Walras allocation. Then, there exist a price p^* and an assignment $\mathbf{x}^* = \mathbf{x}$ such that the pair $(p^*, \mathbf{x}^*)$ is a Walras equilibrium. We now show that $\mathbf{x}^*$ is also a Cournot-Nash allocation. It is straightforward to verify that in two-commodity exchange economies there exists always a strategy selection $\mathbf{s}^*$ such that $\mathbf{x}^*(t) = \mathbf{x}(t, \mathbf{s}^*(t), p(\mathbf{s}^*))$, for each $t \in T$. Since utility functions are strongly monotone, by Assumption 2, $p^* \gg 0$. Since, given a trader $t \in T$, $p(\mathbf{s}^*)\mathbf{x}^*(t) = p(\mathbf{s}^*)\mathbf{w}(t)$ by equation (4.2) and p^* is the unique price vector such that $p^*\mathbf{x}^*(t) = p^*\mathbf{w}(t)$, $p^* = p(\mathbf{s}^*)$. Suppose that $\mathbf{s}^*$ is not a Cournot-Nash equilibrium. Then, there exist a trader $t \in T$ and a strategy $\tilde{s} \in \mathbf{S}(t)$ such that

$$u^t(\mathbf{x}(t, \mathbf{s}^* \setminus \tilde{s}(t), p(\mathbf{s}^*))) > u^t(\mathbf{x}(t, \mathbf{s}^*(t), p(\mathbf{s}^*))).$$

Note that $p(\mathbf{s}^*) = p(\mathbf{s}^* \setminus \tilde{s}(t))$ as $p_1^* = \frac{\int_T \mathbf{b}^*(t) d\mu}{\int_T \mathbf{q}^*(t) d\mu}$. But then, there exists $\tilde{x} = \mathbf{x}(t, \mathbf{s}^* \setminus \tilde{s}(t), p(\mathbf{s}^*))$ such that $u^t(\tilde{x}) > u^t(\mathbf{x}^*(t))$ and $p^*\tilde{x} = p^*\mathbf{w}(t)$, by equation (4.2), a contradiction. Therefore, $\mathbf{s}^*$ is a Cournot-Nash equilibrium. □

Gabszewicz and Mertens (1971) and Shitovitz (1973) showed that an equivalence between the core and the set of Walras allocations may hold even when the space of traders contains atoms. In order to state their two main theorems, we need to introduce some further notation and definitions. Two traders $\tau, \rho \in T$ are said to be of the same type if $\mathbf{w}(\tau) = \mathbf{w}(\rho)$ and $u^\tau = u^\rho$. Let $A = \{A_1, A_2, \ldots, A_k, \ldots\}$ be a partition of the set of atoms $T \setminus T_0$ such that A_k contains all the atoms who are of the same type as an atom $\tau_k \in A_k$, for each $k = 1, \ldots, |A|$, where $|A|$ denotes the cardinality of the partition A. Moreover, let T_k be the set of the traders $t \in T$ who are of the same type as the atoms in A_k, for each $k = 1, \ldots, |A|$. Given a set T_k, denote by τ_{hk} the h-th atom belonging to the set T_k, for each $h = 1, \ldots, |A_k|$, where $|A_k|$ denotes the cardinality of the set A_k. We can now state the two theorems.

Theorem 3. Under Assumptions 1, 2, and 3, if, either $|A| = 1$ and $\sum_{h=1}^{|A_1|} \frac{\mu(\tau_{h1})}{\mu(T_1)} < 1$, or, $|A| > 1$ and $\sum_{k=1}^{|A|} \sum_{h=1}^{|A_k|} \frac{\mu(\tau_{hk})}{\mu(T_k)} \leq 1$, then the core coincides with the set of Walras allocations.

Proof. See the proof of the theorem in Gabszewicz and Mertens (1971). □

Theorem 4. Under Assumptions 1, 2, and 3, if $|A| = 1$ and $|A_1| \geq 2$, then the core coincides with the set of Walras allocations.

Proof. See the proof of Theorem B in Shitovitz (1973). □

Okuno et al. (1980) already showed that the equivalence stated by Theorem 4 (Shitovitz's Theorem B) does not extend to the set of Cournot-Nash allocations. In the next section, we further investigate the relation between the core and the sets of Walras and Cournot-Nash allocations.

4.4 Some examples and two theorems

In this section, we provide some examples to extend Okuno et alii's results to mixed exchange economies with corner endowments. We then show some examples in which the set of Cournot-Nash allocations coincides with the Walras allocations which, it turn, coincides with the core. Moreover, we answer our main question by providing two theorems. The first theorem shows sufficient conditions under which the sets of Cournot-Nash and Walras allocations are not equivalent. The second theorem establishes a necessary and sufficient condition for a Cournot-Nash allocation to be a Walras allocation.

We start our analysis by considering Example 1 in Shitovitz (1973) where the market of commodity 2 is monopolistic. The example shows that Theorems 3 and 4 cannot be extended to this case as $|A| = 1$, $|A_1| = 1$, and $\frac{\mu(\tau_{11})}{\mu(T_1)} = 1$. Moreover, in this market configuration, the sets of Walras and Cournot-Nash allocations are disjoint as there is no Cournot-Nash equilibrium (there exists only the trivial Cournot-Nash equilibrium).

Example 1. Consider the following specification of the exchange economy satisfying Assumptions 1, 2, and 3. $T_0 = [0, 1]$, $A_1 = \{2\}$, T_0 is taken with Lebesgue measure, $\mu(2) = 1$, $\mathbf{w}(t) = (4, 0)$, $u^t(x) = \sqrt{x_1} + \sqrt{x_2}$, for each $t \in T_0$, $\mathbf{w}(2) = (0, 4)$, $u^2(x) = \sqrt{x_1} + \sqrt{x_2}$. Then, there is an allocation in the core, which is not a Walras allocation, and there is no Cournot-Nash allocation.

Proof. The unique Walras equilibrium is the pair $(p^*, \mathbf{x}^*)$, where $(p_1^*, p_2^*) = (1, 1)$, $(\mathbf{x}_1^*(t), \mathbf{x}_2^*(t)) = (2, 2)$, for each $t \in T_0$, $(\mathbf{x}_1^*(2), \mathbf{x}_2^*(2)) = (2, 2)$. As shown by Shitovitz (1973), the allocation $\tilde{\mathbf{x}}$ such that $(\tilde{\mathbf{x}}_1(t), \tilde{\mathbf{x}}_2(t)) = (1, 1)$, for each $t \in T_0$, $(\tilde{\mathbf{x}}_1(2), \tilde{\mathbf{x}}_2(2)) = (3, 3)$ is in the core but it is not a Walras allocation. Suppose that there is a Cournot-Nash allocation $\hat{\mathbf{x}}$. Then, there is a strategy selection $\hat{\mathbf{s}}$ which is a Cournot-Nash equilibrium and which is such that $\hat{\mathbf{x}}(t) = \mathbf{x}(t, \hat{\mathbf{s}}(t), p(\hat{\mathbf{s}}))$, for each $t \in T$. In particular, $\mathbf{x}(2, \hat{\mathbf{s}}(2), p(\hat{\mathbf{s}})) = (\bar{\hat{\mathbf{q}}}, 4 - \hat{\mathbf{b}}(2))$. Let $s'(2)$ be a strategy such that $0 < b'(2) < \hat{\mathbf{b}}(2)$. Then,

$$u^2(\mathbf{x}(2, \hat{\mathbf{s}} \setminus s'(2), p(\hat{\mathbf{s}} \setminus s'(2)))) > u^2(\mathbf{x}(2, \hat{\mathbf{s}}(2), p(\hat{\mathbf{s}}))),$$

as $\mathbf{x}(2, \hat{\mathbf{s}} \setminus s'(2), p(\hat{\mathbf{s}} \setminus s'(2))) = (\bar{\hat{\mathbf{q}}}, 4 - b'(2))$ and u^2 is strongly monotone, a contradiction. Then, there is no Cournot-Nash allocation. □

In the following example, all traders have the same utility function as in Example 1 but a competitive fringe competes with the monopolist in the market for commodity 2. The core coincides with the set of Walras allocations as the assumptions of Theorem 3 are satisfied but no Cournot-Nash allocation is in the core.

Example 2. Consider the following specification of the exchange economy satisfying Assumptions 1, 2, and 3. $T_0 = [0,2]$, $A_1 = \{3\}$, T_0 is taken with Lebesgue measure, $\mu(3) = 1$, $\mathbf{w}(t) = (4,0)$, $u^t(x) = \sqrt{x_1} + \sqrt{x_2}$, for each $t \in [0,1]$, $\mathbf{w}(t) = (0,4)$, $u^t(x) = \sqrt{x_1} + \sqrt{x_2}$, for each $t \in [1,2]$, $\mathbf{w}(3) = (0,4)$, $u^3(x) = \sqrt{x_1} + \sqrt{x_2}$. Then, there is a unique allocation in the core which is also the unique Walras allocation but which is not a Cournot-Nash allocation.

Proof. The unique Walras equilibrium is the pair $(p^*, \mathbf{x}^*)$, where $(p_1^*, p_2^*) = (\sqrt{2}, 1)$, $(\mathbf{x}_1^*(t), \mathbf{x}_2^*(t)) = (4\sqrt{2}-4, 8\sqrt{2}-8)$, for each $t \in [0,1]$, $(\mathbf{x}_1^*(t), \mathbf{x}_2^*(t)) = (4-2\sqrt{2}, 8-4\sqrt{2})$, for each $t \in [1,2]$, $(\mathbf{x}_1^*(3), \mathbf{x}_2^*(3)) = (4-2\sqrt{2}, 8-4\sqrt{2})$. Then, by Theorem 3, the unique Walras allocation is also the unique allocation in the core as $|A| = 1$, $|A_1| = 1$, and $\frac{\mu(\tau_{11})}{\mu(T_1)} < 1$. Suppose that $\mathbf{x}^*$ is also a Cournot-Nash allocation. Then, there is a strategy selection $\mathbf{s}^*$ which is a Cournot-Nash equilibrium and which is such that $\mathbf{x}^*(t) = \mathbf{x}(t, \mathbf{s}^*(t), p(\mathbf{s}^*))$, for each $t \in T$. But then, $\mathbf{s}^*$ must be such that $\mathbf{q}^*(t) = 8 - 4\sqrt{2}$, for each $t \in [0,1]$, $\mathbf{b}^*(t) = 4\sqrt{2} - 4$, for each $t \in [1,2]$, $\mathbf{b}^*(3) = 4\sqrt{2} - 4$. However, the unique Cournot-Nash equilibrium is the strategy selection $\hat{\mathbf{s}}$ where $(\hat{\mathbf{q}}(t), \hat{\mathbf{b}}(t)) = (2.221, 0)$, for each $t \in [0,1]$, $(\hat{\mathbf{q}}(t), \hat{\mathbf{b}}(t)) = (0, 1.779)$, for each $t \in [1,2]$, $(\hat{\mathbf{q}}(3), \hat{\mathbf{b}}(3)) = (0, 0.993)$.[3] It is straightforward to see that $\mathbf{s}^* \neq \hat{\mathbf{s}}$. Then, the unique Walras allocation is not a Cournot-Nash allocation. □

In the following example, all traders have the same utility function as in Example 1 but there are two oligopolists of the same type in the market for commodity 2. The core coincides with the set of Walras allocations as the assumptions of Theorem 4 are satisfied but no Cournot-Nash allocation is in the core.

Example 3. Consider the following specification of the exchange economy satisfying Assumptions 1, 2, and 3. $T_0 = [0,1]$, $A_1 = \{2,3\}$, T_0 is taken with Lebesgue measure, $\mu(2) = \mu(3) = 1$, $\mathbf{w}(t) = (4,0)$, $u^t(x) = \sqrt{x_1} + \sqrt{x_2}$, for each $t \in T_0$, $\mathbf{w}(2) = \mathbf{w}(3) = (0,4)$, $u^2(x) = u^3(x) = \sqrt{x_1} + \sqrt{x_2}$. Then, there is a unique allocation in the core which is also the unique Walras allocation but which is not a Cournot-Nash allocation.

[3] These results are obtained with Mathematica. Since the exact strategies are complicate formulas, we have written the approximate ones. The actions $\hat{\mathbf{q}}(t)$, for each $t \in [0,1]$, $\hat{\mathbf{b}}(t)$, for each $t \in [1,2]$, and $\hat{\mathbf{b}}(3)$ are the roots of the following polynomials $x^6 - 64 + x^5 + 640x^4 - 3200x^3 + 8960x^2 - 13312x + 8192$, $x^6 + 40x^5 - 400x^4 + 1920x^3 - 5120x^2 + 7168x - 4096$, and $x^6 + 20x^5 + 160x^4 + 640x^3 + 1280x^2 + 2048x - 4096$ respectively.

Proof. The unique Walras equilibrium is the pair $(p^*, \mathbf{x}^*)$, where $(p_1^*, p_2^*) = (\sqrt{2}, 1)$, $(\mathbf{x}_1^*(t), \mathbf{x}_2^*(t)) = (4\sqrt{2}-4, 8\sqrt{2}-8)$, for each $t \in T_0$, $(\mathbf{x}_1^*(2), \mathbf{x}_2^*(2)) = (\mathbf{x}_1^*(3), \mathbf{x}_2^*(3)) = (4 - 2\sqrt{2}, 8 - 4\sqrt{2})$. Then, by Theorem 4, the unique Walras allocation is also the unique allocation in the core as $|A| = 1$ and $|A_1| = 2$. Suppose that $\mathbf{x}^*$ is also a Cournot-Nash allocation. Then, there is a strategy selection $\mathbf{s}^*$ which is a Cournot-Nash equilibrium and which is such that $\mathbf{x}^*(t) = \mathbf{x}(t, \mathbf{s}^*(t), p(\mathbf{s}^*))$, for each $t \in T$. But then, $\mathbf{s}^*$ must be such that $\mathbf{q}^*(t) = 8 - 4\sqrt{2}$, for each $t \in [0, 1]$, $\mathbf{b}^*(2) = \mathbf{b}^*(3) = 4\sqrt{2} - 4$. However, the unique Cournot-Nash equilibrium is the strategy selection $\hat{\mathbf{s}}$ where $(\hat{\mathbf{q}}(t), \hat{\mathbf{b}}(t)) = (1.903, 0)$, for each $t \in [0, 1]$, $(\hat{\mathbf{q}}(2), \hat{\mathbf{b}}(2)) = (\hat{\mathbf{q}}(3), \hat{\mathbf{b}}(3)) = (0, 0.864)$.[4] It is straightforward to see that $\mathbf{s}^* \neq \hat{\mathbf{s}}$. Then, the unique Walras allocation is not a Cournot-Nash allocation. □

In Examples 2 and 3, there are atoms who demand a strictly positive amount of both commodities at a Walras equilibrium and the sets of Walras and Cournot-Nash allocations are disjoint. The following theorem generalises these examples providing a sufficient condition for a Walras allocation not to be a Cournot-Nash allocation. In order to state the theorem, we need a further assumption on traders' utility functions.

Assumption 4. $u^t : \mathbb{R}^2_+ \to \mathbb{R}$ is differentiable, for each $t \in T \setminus T_0$.[5]

Theorem 5. Under Assumptions 1, 2, 3, and 4, if the pair $(p^*, \mathbf{x}^*)$ is a Walras equilibrium such that $\mathbf{x}^*(\tau) \gg 0$, for an atom $\tau \in T \setminus T_0$, then $\mathbf{x}^*$ is not a Cournot-Nash allocation.

Proof. Assume that the pair $(p^*, \mathbf{x}^*)$ is a Walras equilibrium such that $\mathbf{x}^*(\tau) \gg 0$, for an atom $\tau \in T \setminus T_0$. Moreover, suppose that $\mathbf{x}^*$ is a Cournot-Nash allocation. Then, there is a strategy selection $\mathbf{s}^*$ such that $\mathbf{x}^*(t) = \mathbf{x}(t, \mathbf{s}^*(t), p(\mathbf{s}^*))$, for each $t \in T$, where $\mathbf{s}^*$ is a Cournot-Nash equilibrium. Since, given a trader $t \in T$, $p(\mathbf{s}^*)\mathbf{x}^*(t) = p(\mathbf{s}^*)\mathbf{w}(t)$ and p^* is the unique price vector such that $p^*\mathbf{x}^*(t) = p^*\mathbf{w}(t)$, $p^* = p(\mathbf{s}^*)$. First, consider an atom $\tau \in T \setminus T_0$ such that $\mathbf{w}_1(\tau) > 0$ and $\mathbf{w}_2(\tau) = 0$. At a Cournot-Nash equilibrium, for the atom τ, the marginal rate of substitution must be equal to the marginal rate at which he can trade off commodity 1 for commodity 2 (see Okuno et al. (1980)). Moreover, at a Walras equilibrium, the

[4] The actions $\hat{\mathbf{q}}(t)$, for each $t \in [0, 1]$, and $\hat{\mathbf{b}}(2) = \hat{\mathbf{b}}(3)$ are the roots of the following polynomials $3x^3 - 4x^2 + 64x - 128$ and $6x^3 + 7x^2 + 8x - 16$ respectively.

[5] In this assumption, differentiability should be implicitly understood to include the case of infinite partial derivatives along the boundary of the consumption set (for a discussion of this case, see, for instance, Kreps (2012), p. 58).

marginal rate of substitution must be equal to the relative price of commodity 1 in terms of commodity 2, where $p_2^* = 1$. These conditions are expressed by the following equations

$$\frac{dx_2}{dx_1} = -p_1^* \frac{\bar{\mathbf{q}}^* - \mathbf{q}^*(\tau)\mu(\tau)}{\bar{\mathbf{q}}^*} = -p_1^*.$$

Then, we must have $\mathbf{q}^*(\tau) = 0$. But then, $(\mathbf{x}_1^*(\tau), \mathbf{x}_2^*(\tau)) = (\mathbf{w}_1(\tau), 0)$, a contradiction. Hence, $\mathbf{x}^*$ is not a Cournot-Nash allocation. Now, consider an atom $\tau \in T \backslash T_0$ and assume that $\mathbf{w}_1(\tau) = 0$ and $\mathbf{w}_2(\tau) > 0$. At a Cournot-Nash equilibrium, for the atom τ, the marginal rate of substitution must be equal to the marginal rate at which he can trade off commodity 1 for commodity 2 (see Okuno et al. (1980)). Moreover, at a Walras equilibrium, the marginal rate of substitution must be equal to the relative price of commodity 1 in terms of commodity 2, where $p_2^* = 1$. These two conditions are expressed by the following equations

$$\frac{dx_2}{dx_1} = -p_1^* \frac{\bar{\mathbf{b}}^*}{\bar{\mathbf{b}}^* - \mathbf{b}^*(\tau)\mu(\tau)} = -p_1^*.$$

Then, we must have $\mathbf{b}^*(\tau) = 0$. But then, $(\mathbf{x}_1^*(\tau), \mathbf{x}_2^*(\tau)) = (0, \mathbf{w}_2(\tau))$, a contradiction. Hence, $\mathbf{x}^*$ is not a Cournot-Nash allocation. □

The following example differs from Example 2 only in that the monopolist and the competitive fringe have quasi-linear utility functions. It shows that, under the assumptions of Theorem 3, the converse of Theorem 5 does not hold. At the unique Walras equilibrium, both the monopolist and the competitive fringe demand a null amount of commodity 2 and this unique Walras allocation is also the unique allocation in the core but it is not a Cournot-Nash allocation.

Example 4. Consider the following specification of the exchange economy satisfying Assumptions 1, 2, and 3. $T_0 = [0, 2]$, $A_1 = \{3\}$, T_0 is taken with Lebesgue measure, $\mu(3) = 1$, $\mathbf{w}(t) = (4, 0)$, $u^t(x) = \sqrt{x_1} + \sqrt{x_2}$, for each $t \in [0, 1]$, $\mathbf{w}(t) = (0, 4)$, $u^t(x) = \sqrt{x_1} + \frac{1}{10}x_2$, for each $t \in [1, 2]$, $\mathbf{w}(3) = (0, 4)$, $u^3(x) = \sqrt{x_1} + \frac{1}{10}x_2$. Then, there is a unique allocation in the core which is also the unique Walras allocation but which is not a Cournot-Nash allocation.

Proof. The unique Walras equilibrium is the pair $(p^*, \mathbf{x}^*)$, where $(p_1^*, p_2^*) = (\sqrt{3} + 1, 1)$, $(\mathbf{x}_1^*(t), \mathbf{x}_2^*(t)) = (8 - 4\sqrt{3}, 8)$, for each $t \in [0, 1]$, $(\mathbf{x}_1^*(t), \mathbf{x}_2^*(t)) = (2\sqrt{3} - 2, 0)$, for each $t \in [1, 2]$, $(\mathbf{x}_1^*(3), \mathbf{x}_2^*(3)) = (2\sqrt{3} - 2, 0)$. Then, by Theorem 3, the unique Walras allocation is also the unique allocation in the core as $|A| = 1$, $|A_1| = 1$, and $\frac{\mu(\tau_{11})}{\mu(T_1)} < 1$. Suppose that $\mathbf{x}^*$ is also a Cournot-Nash allocation. Then, there is a strategy selection $\mathbf{s}^*$ which is a Cournot-Nash equilibrium and which is such that $\mathbf{x}^*(t) = \mathbf{x}(t, \mathbf{s}^*(t), p(\mathbf{s}^*))$, for each $t \in T$. But then, $\mathbf{s}^*$ must be such that $\mathbf{q}^*(t) = 2\sqrt{21} - 6$, for each $t \in [0,1]$, $\mathbf{b}^*(t) = 4$, for each $t \in [1,2]$, $\mathbf{b}^*(3) = 4$. However, the unique Cournot-Nash equilibrium is the strategy selection $\hat{\mathbf{s}}$ where $(\hat{\mathbf{q}}(t), \hat{\mathbf{b}}(t)) = (2.857, 0)$, for each $t \in [0,1]$, $(\hat{\mathbf{q}}(t), \hat{\mathbf{b}}(t)) = (0, 4)$, for each $t \in [1,2]$, $(\hat{\mathbf{q}}(3), \hat{\mathbf{b}}(3)) = (0, 3.140)$.[6] It is straightforward to see that $\mathbf{s}^* \neq \hat{\mathbf{s}}$. Then, the unique Walras allocation is not a Cournot-Nash allocation. $\square$

The following example differs from Example 3 only in that the two oligopolists have quasi-linear utility functions. It shows that, under the assumptions of Theorem 4, the converse of Theorem 5 does not hold. At the unique Walras equilibrium, the two oligopolists demand a null amount of commodity 2 and this unique Walras allocation is also the unique allocation in the core but it is not a Cournot-Nash allocation.

Example 5. Consider the following specification of the exchange economy satisfying Assumptions 1, 2, and 3. $T_0 = [0,1]$, $A_1 = \{2,3\}$, T_0 is taken with Lebesgue measure, $\mu(2) = \mu(3) = 1$, $\mathbf{w}(t) = (4,0)$, $u^t(x) = \sqrt{x_1} + \sqrt{x_2}$, for each $t \in T_0$, $\mathbf{w}(2) = \mathbf{w}(3) = (0,4)$, $u^2(x) = u^3(x) = \sqrt{x_1} + \frac{1}{10}x_2$. Then, there is a unique allocation in the core which is also the unique Walras allocation but which is not a Cournot-Nash allocation.

Proof. The unique Walras equilibrium is the pair $(p^*, \mathbf{x}^*)$, where $(p_1^*, p_2^*) = (\sqrt{3} + 1, 1)$, $(\mathbf{x}_1^*(t), \mathbf{x}_2^*(t)) = (8 - 4\sqrt{3}, 8)$, for each $t \in T_0$, $(\mathbf{x}_1^*(2), \mathbf{x}_2^*(2)) = (\mathbf{x}_1^*(3), \mathbf{x}_2^*(3)) = (2\sqrt{3} - 2, 0)$. Then, by Theorem 4, the unique Walras allocation is also the unique allocation in the core as $|A| = 1$ and $|A_1| = 2$. Suppose that $\mathbf{x}^*$ is also a Cournot-Nash allocation. Then, there is a strategy selection $\mathbf{s}^*$ which is a Cournot-Nash equilibrium and which is such that $\mathbf{x}^*(t) = \mathbf{x}(t, \mathbf{b}^*(t), p(\mathbf{b}^*))$, for each $t \in T$. But then, $\mathbf{s}^*$ must be such that $\mathbf{q}^*(t) = 2\sqrt{21} - 6$, for each $t \in T_0$, $\mathbf{b}^*(2) = \mathbf{b}^*(3) = 4$. However, the unique Cournot-Nash equilibrium is the strategy selection $\hat{\mathbf{s}}$ where $(\hat{\mathbf{q}}(t), \hat{\mathbf{b}}(t)) = (2.857, 0)$, for each $t \in [0,1]$, $(\hat{\mathbf{q}}(2), \hat{\mathbf{b}}(2)) = (\hat{\mathbf{q}}(3), \hat{\mathbf{b}}(3)) = (0, 3.140)$.[7] It is straightforward to see that $\mathbf{s}^* \neq \hat{\mathbf{s}}$. Then, the unique Walras allocation is not a Cournot-Nash allocation. $\square$

We now address the question whether, in mixed exchange economies, an equivalence, or at least a non-empty intersection, between the sets of Walras and Cournot-

[6] The actions $\hat{\mathbf{q}}(t)$, for each $t \in [0,1]$, and $\hat{\mathbf{b}}(3)$ are the roots of the following polynomials $x^7 + 4x^6 - 16x^5 - 400x^4 + 6400x^3 - 38400x^2 + 102400x - 102400$ and $x^7 + 20x^6 + 160x^5 + 1040x^4 + 6080x^3 + 20224x^2 + 25600x - 640000$.

[7] The actions $\hat{\mathbf{q}}(t)$, for each $t \in [0,1]$, and $\hat{\mathbf{b}}(2) = \hat{\mathbf{b}}(3)$ are the roots of the following polynomials $2x^3 - 25x^2 + 200x - 400$ and $8x^3 + 50x^2 - 625$.

Nash allocations may hold. The following example differs from Example 4 only for the lower "weight" of commodity 2 for traders who have quasi-linear utility functions. At the unique Walras equilibrium, both the monopolist and the competitive fringe demand a null amount of commodity 2 and this unique Walras allocation is also the unique allocation in the core and the unique Cournot-Nash allocation.

Example 6. Consider the following specification of the exchange economy satisfying Assumptions 1, 2, and 3. $T_0 = [0, 2]$, $A_1 = \{3\}$, T_0 is taken with Lebesgue measure, $\mu(3) = 1$, $\mathbf{w}(t) = (4, 0)$, $u^t(x) = \sqrt{x_1} + \sqrt{x_2}$, for each $t \in [0, 1]$, $\mathbf{w}(t) = (0, 4)$, $u^t(x) = \sqrt{x_1} + \frac{1}{30}x_2$, for each $t \in [1, 2]$, $\mathbf{w}(3) = (0, 4)$, $u^3(x) = \sqrt{x_1} + \frac{1}{30}x_2$. Then, there is a unique allocation in the core which is also the unique Walras allocation and the unique Cournot-Nash allocation.

Proof. The unique Walras equilibrium is the pair $(p^*, \mathbf{x}^*)$, where $(p_1^*, p_2^*) = (\sqrt{3} + 1, 1)$, $(\mathbf{x}_1^*(t), \mathbf{x}_2^*(t)) = (8 - 4\sqrt{3}, 8)$, for each $t \in [0, 1]$, $(\mathbf{x}_1^*(t), \mathbf{x}_2^*(t)) = (2\sqrt{3} - 2, 0)$, for each $t \in [1, 2]$, $(\mathbf{x}_1^*(3), \mathbf{x}_2^*(3)) = (2\sqrt{3} - 2, 0)$. Then, by Theorem 3, the unique Walras allocation is also the unique allocation in the core as $|A| = 1$, $|A_1| = 1$, and $\frac{\mu(\tau_{11})}{\mu(T_1)} < 1$. The unique Cournot-Nash equilibrium is the strategy selection $\hat{\mathbf{s}}$ where $(\hat{\mathbf{q}}(t), \hat{\mathbf{b}}(t)) = (4\sqrt{3} - 4, 0)$, for each $t \in [0, 1]$, $(\hat{\mathbf{q}}(t), \hat{\mathbf{b}}(t)) = (0, 4)$, for each $t \in [1, 2]$, $(\hat{\mathbf{q}}(3), \hat{\mathbf{b}}(3)) = (0, 4)$. But then, $\mathbf{x}^*(t) = \mathbf{x}(t, \hat{\mathbf{s}}(t), p(\hat{\mathbf{s}}))$, for each $t \in T$. Hence, the unique Walras allocation is also the unique Cournot-Nash allocation. $\square$

The following example differs from Example 5 only for the lower "weight" of commodity 2 for traders who have quasi-linear utility functions. At the unique Walras equilibrium, the two oligopolists demand a null amount of commodity 2 and this unique Walras allocation is also the unique allocation in the core and the unique Cournot-Nash allocation.

Example 7. Consider the following specification of the exchange economy satisfying Assumptions 1, 2, and 3. $T_0 = [0, 1]$, $A_1 = \{2, 3\}$, T_0 is taken with Lebesgue measure, $\mu(2) = \mu(3) = 1$, $\mathbf{w}(t) = (4, 0)$, $u^t(x) = \sqrt{x_1} + \sqrt{x_2}$, for each $t \in T_0$, $\mathbf{w}(2) = \mathbf{w}(3) = (0, 4)$, $u^2(x) = u^3(x) = \sqrt{x_1} + \frac{1}{30}x_2$. Then, there is a unique allocation in the core which is also the unique Walras allocation and the unique Cournot-Nash allocation.

Proof. The unique Walras equilibrium is the pair $(p^*, \mathbf{x}^*)$, where $(p_1^*, p_2^*) = (\sqrt{3} + 1, 1)$, $(\mathbf{x}_1^*(t), \mathbf{x}_2^*(t)) = (8 - 4\sqrt{3}, 8)$, for each $t \in T_0$, $(\mathbf{x}_1^*(2), \mathbf{x}_2^*(2)) = (\mathbf{x}_1^*(3), \mathbf{x}_2^*(3)) = (2\sqrt{3} - 2, 0)$. Then, by Theorem 4, the unique Walras allocation is also the unique allocation in the core as $|A| = 1$ and $|A_1| = 2$. The unique Cournot-Nash equilibrium is the strategy selection $\hat{\mathbf{s}}$ where $(\hat{\mathbf{q}}(t), \hat{\mathbf{b}}(t)) = (4\sqrt{3}-4, 0)$, for each $t \in [0, 1]$, and $(\hat{\mathbf{q}}(2), \hat{\mathbf{b}}(2)) = (\hat{\mathbf{q}}(3), \hat{\mathbf{b}}(3)) = (0, 4)$. But then, $\mathbf{x}^*(t) = \mathbf{x}(t, \hat{\mathbf{s}}(t), p(\hat{\mathbf{s}}))$, for each $t \in T$. Hence, the unique Walras allocation is also the unique Cournot-Nash allocation. □

Examples 6 and 7 differ from Examples 4 and 5 as, in the latter, all atoms who hold commodity 2 demand a null amount of this commodity at a Walras equilibrium but not at a Cournot-Nash equilibrium whereas, in the former, they also demand a null amount of commodity 2 at a Cournot-Nash equilibrium. The following theorem generalises Examples 6 and 7 as it shows that demanding a null amount of one of the two commodities by all the atoms is a necessary and sufficient condition for a Cournot-Nash allocation to be a Walras allocation.

Theorem 6. Under Assumptions 1, 2, 3, and 4, let $\hat{\mathbf{s}}$ be a Cournot-Nash equilibrium and let $\hat{p} = p(\hat{\mathbf{s}})$ and $\hat{\mathbf{x}}(t) = \mathbf{x}(t, \hat{\mathbf{s}}(t), p(\hat{\mathbf{s}}))$, for each $t \in T$. Then, the pair $(\hat{p}, \hat{\mathbf{x}})$ is a Walras equilibrium if and only if $\hat{\mathbf{x}}_1(t) = 0$ or $\hat{\mathbf{x}}_2(t) = 0$, for each $t \in T \setminus T_0$.

Proof. Let $\hat{\mathbf{s}}$ be a Cournot-Nash equilibrium and let $\hat{\mathbf{x}}(t) = \mathbf{x}(t, \hat{\mathbf{s}}(t), p(\hat{\mathbf{s}}))$, for each $t \in T$, and $\hat{p} = p(\hat{\mathbf{s}})$. Suppose that the pair $(\hat{p}, \hat{\mathbf{x}})$ is a Walras equilibrium. Moreover, suppose that $\hat{\mathbf{x}}(\tau) \gg 0$, for an atom $\tau \in T \setminus T_0$. Then, $\hat{\mathbf{x}}$ is not a Cournot-Nash allocation, by Theorem 5, a contradiction. Hence, $\hat{\mathbf{x}}_1(t) = 0$ or $\hat{\mathbf{x}}_2(t) = 0$, for each $t \in T \setminus T_0$. Conversely, suppose that $\hat{\mathbf{x}}_1(t) = 0$ or $\hat{\mathbf{x}}_2(t) = 0$, for each $t \in T \setminus T_0$. Consider an atom $\tau \in T \setminus T_0$. First, assume that $\mathbf{w}_1(\tau) = 0$ and $\mathbf{w}_2(\tau) > 0$. Consider the case where $\hat{\mathbf{x}}_1(\tau) = 0$. Then, $\hat{\mathbf{b}}(\tau) = 0$ and $\hat{\mathbf{x}}(\tau) = (0, \mathbf{w}_2(\tau))$. We have that $\hat{p}\hat{\mathbf{x}}(\tau) = \hat{p}\mathbf{w}(\tau)$ since

$$\hat{p}_1\hat{\mathbf{x}}_1(\tau) + \hat{p}_2\hat{\mathbf{x}}_2(\tau) = \hat{p}_1 0 + \hat{p}_2(\mathbf{w}_2(\tau) - 0) = \hat{p}_2\mathbf{w}_2(\tau).$$

Let $\hat{x}_2(x_1)$ be a function such that $u^\tau(x_1, x_2(x_1)) \equiv u^\tau(\hat{\mathbf{x}}(\tau))$, for each $0 \leq x_1 \leq \frac{\mathbf{w}_2(\tau)}{\hat{p}_1}$. We have that

$$\frac{\partial u^\tau(\hat{\mathbf{x}}(\tau))}{\partial x_1} \frac{\bar{\hat{\mathbf{b}}} - \hat{\mathbf{b}}(\tau)\mu(\tau)}{\bar{\hat{\mathbf{b}}}} \frac{1}{\hat{p}_1} - \frac{\partial u^\tau(\hat{\mathbf{x}}(\tau))}{\partial x_2} \leq 0$$

as $\hat{\mathbf{b}}(\tau) = 0$. Then,

$$\frac{\partial u^\tau(\hat{\mathbf{x}}(\tau))}{\partial x_1} \frac{1}{\hat{p}_1} - \frac{\partial u^\tau(\hat{\mathbf{x}}(\tau))}{\partial x_2} \leq 0$$

as $\frac{\bar{\hat{\mathbf{b}}} - 0}{\bar{\hat{\mathbf{b}}}} = 1$. But then, $\frac{d\hat{x}_2(0)}{dx_1} \geq -\hat{p}_1$. Consider the case where $\frac{d\hat{x}_2(0)}{dx_1} = -\hat{p}_1$. Then, $u^\tau(\hat{\mathbf{x}}(\tau)) \geq u^\tau(y)$, for each $y \in \{x \in \mathbb{R}^2_+ : \hat{p}x = \hat{p}\mathbf{w}(\tau)\}$, as u^τ is quasi-concave, by

Assumption 2. Consider now the case where $\frac{d\hat{x}_2(0)}{dx_1} > -\hat{p}_1$. Then, $\frac{d\hat{x}_2(x_1)}{dx_1} > -\hat{p}_1$, for each $0 \leq x_1 \leq \frac{\mathbf{w}_2(\tau)}{\hat{p}_1}$, as u^τ is quasi-concave, by Assumption 2. Suppose that there exists a commodity bundle $\tilde{x} \in \{x \in \mathbb{R}^2_+ : \hat{p}x = \hat{p}\mathbf{w}(\tau)\}$ such that $u^\tau(\tilde{x}) > u^\tau(\hat{\mathbf{x}}(\tau))$. Then, $\tilde{x}_2 > \hat{x}_2(\tilde{x}_1)$ as u^τ is strongly monotone, by Assumption 2. But then, by the Mean Value Theorem, there exists some $\bar{x}_1$ such that $0 < \bar{x}_1 < \tilde{x}_1$ and such that

$$\frac{d\hat{x}_2(\bar{x}_1)}{dx_1} = \frac{\hat{x}_2(0) - \hat{x}_2(\tilde{x}_1)}{0 - \tilde{x}_1} < -\hat{p}_1,$$

a contradiction. Therefore, $u^\tau(\hat{\mathbf{x}}(\tau)) \geq u^\tau(y)$ for each $y \in \{x \in \mathbb{R}^2_+ : \hat{p}x = \hat{p}\mathbf{w}(\tau)\}$. Consider now the case where $\hat{\mathbf{x}}_2(\tau) = 0$. Then, $\hat{\mathbf{b}}(\tau) = \mathbf{w}_2(\tau)$ and $\hat{\mathbf{x}}(\tau) = (\frac{\mathbf{w}_2(\tau)}{\hat{p}_1}, 0)$. We have that $\hat{p}\hat{\mathbf{x}}(\tau) = \hat{p}\mathbf{w}(\tau)$ since

$$\hat{p}_1\hat{\mathbf{x}}_1(\tau) + \hat{p}_2\hat{\mathbf{x}}_2(\tau) = \hat{p}_1\frac{\mathbf{w}_2(\tau)}{\hat{p}_1} + \hat{p}_2(\mathbf{w}_2(\tau) - \mathbf{w}_2(\tau)) = \hat{p}_2\mathbf{w}_2(\tau).$$

Let $\hat{x}_2(x_1)$ be a function such that $u^\tau(x_1, \hat{x}_2(x_1)) \equiv u^\tau(\hat{\mathbf{x}}(\tau))$, for each $0 \leq x_1 \leq \frac{\mathbf{w}_2(\tau)}{\hat{p}_1}$. We have that

$$\frac{\partial u^\tau(\hat{\mathbf{x}}(\tau))}{\partial x_1}\frac{\bar{\hat{\mathbf{b}}} - \hat{\mathbf{b}}(\tau)\mu(\tau)}{\bar{\hat{\mathbf{b}}}}\frac{1}{\hat{p}_1} - \frac{\partial u^\tau(\hat{\mathbf{x}}(\tau))}{\partial x_2} \geq 0$$

as $\hat{\mathbf{b}}(\tau) = \mathbf{w}_2(\tau)$. Then,

$$\frac{\partial u^\tau(\hat{\mathbf{x}}(\tau))}{\partial x_1}\frac{1}{\hat{p}_1} - \frac{\partial u^\tau(\hat{\mathbf{x}}(\tau))}{\partial x_2} > 0$$

as $\frac{\bar{\hat{\mathbf{b}}} - \mathbf{w}_2(\tau)\mu(\tau)}{\bar{\hat{\mathbf{b}}}} < 1$. But then, $\frac{d\hat{x}_2(x_1)}{dx_1} < -\hat{p}_1$, for each $0 \leq x_1 \leq \frac{\mathbf{w}_2(\tau)}{\hat{p}_1}$, as u^τ is quasi-concave, by Assumption 2. Suppose that there exists a commodity bundle $\tilde{x} \in \{x \in \mathbb{R}^2_+ : \hat{p}x = \hat{p}\mathbf{w}(\tau)\}$ such that $u^\tau(\tilde{x}) > u^\tau(\hat{\mathbf{x}}(\tau))$. Then, $\tilde{x}_2 > \hat{x}_2(\tilde{x}_1)$ as u^τ is strongly monotone, by Assumption 2. But then, by the Mean Value Theorem, there exists some $\bar{x}_1$ such that $\tilde{x}_1 < \bar{x}_1 < \frac{\mathbf{w}_2(\tau)}{\hat{p}_1}$ and such that

$$\frac{d\hat{x}_2(\bar{x}_1)}{dx_1} = \frac{\hat{x}_2(\tilde{x}_1) - \hat{x}_2(\frac{\mathbf{w}_2(\tau)}{\hat{p}_1})}{\tilde{x}_1 - \frac{\mathbf{w}_2(\tau)}{\hat{p}_1}} > -\hat{p}_1,$$

a contradiction. Therefore, $u^\tau(\hat{\mathbf{x}}(\tau)) \geq u^\tau(y)$, for each $y \in \{x \in \mathbb{R}^2_+ : \hat{p}x = \hat{p}\mathbf{w}(\tau)\}$. Now, assume that $\mathbf{w}_1(\tau) > 0$ and $\mathbf{w}_2(\tau) = 0$. Then, the previous argument leads, *mutatis mutandis*, to the same kind of contradictions. We then conclude that $\hat{p}\hat{\mathbf{x}}(t) = \hat{p}\mathbf{w}(t)$ and $u^t(\hat{\mathbf{x}}(t)) \geq u^t(y)$ for each $y \in \{x \in \mathbb{R}^2_+ : \hat{p}x = \hat{p}\mathbf{w}(t)\}$, for each $t \in T \setminus T_0$. Finally, it is straightforward to show (see the proof of Theorem 2) that $\hat{p}\hat{\mathbf{x}}(t) = \hat{p}\mathbf{w}(t)$ and $u^t(\hat{\mathbf{x}}(t)) \geq u^t(y)$, for each $y \in \{x \in \mathbb{R}^2_+ : \hat{p}x = \hat{p}\mathbf{w}(t)\}$, for each $t \in T_0$. Hence, the pair $(\hat{p}, \hat{\mathbf{x}})$ is a Walras equilibrium. $\square$

Examples 6 and 7 show that Theorem 6 is non-vacuous when atoms demand, at a Cournot-Nash equilibrium, a null amount of the commodity they hold. The following two examples show that it is also non-vacuous when atoms demand, at a Cournot-Nash equilibrium, a null amount of the commodity they do not hold.

The structure of the following example differs from that of Example 6 for a further competitive fringe which holds commodity 2 and is not of the same type as the monopolist. At the unique Walras equilibrium, both the monopolist and the competitive fringe with traders of the same type as the monopolist demand a null amount of commodity 1 and this unique Walras allocation is also the unique allocation in the core and the unique Cournot-Nash allocation.

Example 8. Consider the following specification of the exchange economy satisfying Assumptions 1, 2, and 3. $T_0 = [0, 3]$, $A_1 = \{4\}$, T_0 is taken with Lebesgue measure, $\mu(4) = 1$, $\mathbf{w}(t) = (4, 0)$, $u^t(x) = \sqrt{x_1} + \sqrt{x_2}$, for each $t \in [0, 1]$, $\mathbf{w}(t) = (0, 4)$, $u^t(x) = \sqrt{x_1} + \sqrt{x_2}$, for each $t \in [1, 2]$, $\mathbf{w}(t) = (0, 4)$, $u^t(x) = \frac{1}{4}x_1 + \sqrt{x_2}$, for each $t \in [2, 3]$, $\mathbf{w}(4) = (0, 4)$, $u^4(x) = \frac{1}{4}x_1 + \sqrt{x_2}$. Then, there is a unique allocation in the core which is also the unique Walras allocation and the unique Cournot-Nash allocation.

Proof. The unique Walras equilibrium is the pair $(p^*, \mathbf{x}^*)$, where $(p_1^*, p_2^*) = (1, 1)$, $(\mathbf{x}_1^*(t), \mathbf{x}_2^*(t)) = (2, 2)$, for each $t \in [0, 1]$, $(\mathbf{x}_1^*(t), \mathbf{x}_2^*(t)) = (2, 2)$, for each $t \in [1, 2]$, $(\mathbf{x}_1^*(t), \mathbf{x}_2^*(t)) = (0, 4)$, for each $t \in [2, 3]$, $(\mathbf{x}_1^*(4), \mathbf{x}_2^*(4)) = (0, 4)$. Then, by Theorem 3, the unique Walras allocation is also the unique allocation in the core as $|A| = 1$, $|A_1| = 1$, and $\frac{\mu(\tau_{11})}{\mu(T_1)} < 1$. The unique Cournot-Nash equilibrium is the strategy selection $\hat{\mathbf{s}}$ where $(\hat{\mathbf{q}}(t), \hat{\mathbf{b}}(t)) = (2, 0)$, for each $t \in [0, 1]$, $(\hat{\mathbf{q}}(t), \hat{\mathbf{b}}(t)) = (0, 2)$, for each $t \in [1, 2]$, $(\hat{\mathbf{q}}(t), \hat{\mathbf{b}}(t)) = (0, 0)$, for each $t \in [2, 3]$, and $(\hat{\mathbf{q}}(4), \hat{\mathbf{b}}(4)) = (0, 0)$. But then $\mathbf{x}^*(t) = \mathbf{x}(t, \hat{\mathbf{s}}(t), p(\hat{\mathbf{s}}))$, for each $t \in T$. Hence, the unique Walras allocation is also the unique Cournot-Nash allocation. □

The structure of the following example differs from that of Example 7 for a further competitive fringe which holds commodity 2 and is not of the same type as the two oligopolists. At the unique Walras equilibrium, the two oligopolists demand a null amount of commodity 1 and this unique Walras allocation is also the unique allocation in the core and the unique Cournot-Nash allocation.

Example 9. . Consider the following specification of the exchange economy satisfying Assumptions 1, 2, and 3. $T_0 = [0, 2]$, $A_1 = \{3, 4\}$, T_0 is taken with Lebesgue measure, $\mu(3) = \mu(4) = 1$, $\mathbf{w}(t) = (4, 0)$, $u^t(x) = \sqrt{x_1} + \sqrt{x_2}$, for each $t \in [0, 1]$,

$\mathbf{w}(t) = (0,4)$, $u^t(x) = \sqrt{x_1} + \sqrt{x_2}$, for each $t \in [1,2]$, $\mathbf{w}(3) = \mathbf{w}(4) = (0,4)$, $u^3(x) = u^4(x) = \frac{1}{4}x_1 + \sqrt{x_2}$. Then, there is a unique allocation in the core which is also the unique Walras allocation and the unique Cournot-Nash allocation.

Proof. The unique Walras equilibrium is the pair $(p^*, \mathbf{x}^*)$, where $(p_1^*, p_2^*) = (1,1)$, $(\mathbf{x}_1^*(t), \mathbf{x}_2^*(t)) = (2,2)$, for each $t \in [0,1]$, $(\mathbf{x}_1^*(t), \mathbf{x}_2^*(t)) = (2,2)$, for each $t \in [1,2]$, $(\mathbf{x}_1^*(3), \mathbf{x}_2^*(3)) = (\mathbf{x}_1^*(4), \mathbf{x}_2^*(4)) = (0,4)$. Then, by Theorem 4, the unique Walras allocation is also the unique allocation in the core as $|A| = 1$ and $|A_1| = 2$. The unique Cournot-Nash equilibrium is the strategy selection $\hat{\mathbf{s}}$ where $(\hat{\mathbf{q}}(t), \hat{\mathbf{b}}(t)) = (2,0)$, for each $t \in [0,1]$, $(\hat{\mathbf{q}}(t), \hat{\mathbf{b}}(t)) = (0,2)$, for each $t \in [1,2]$, $(\hat{\mathbf{q}}(3), \hat{\mathbf{b}}(3)) = (\hat{\mathbf{q}}(4), \hat{\mathbf{b}}(4)) = (0,0)$. But then $\mathbf{x}^*(t) = \mathbf{x}(t, \hat{\mathbf{s}}(t), p(\hat{\mathbf{s}}))$, for each $t \in T$. Hence, the unique Walras allocation is also the unique Cournot-Nash allocation. □

4.5 Necessary conditions for an equivalence

In all the examples of the previous section, preferences are represented by additively separable utility functions, i.e., utility functions of the form $u(x) = v_1(x_1) + v_2(x_2)$, for each $x \in \mathbb{R}^2_+$. In this section, we first provide a necessary condition for Theorem 6 to hold when atoms' preferences are of this kind.

Proposition 1. Under Assumptions 1, 2, 3, and 4, let $\hat{\mathbf{s}}$ be a Cournot-Nash equilibrium and let $\hat{\mathbf{x}}(t) = \mathbf{x}(t, \hat{\mathbf{s}}(t), p(\hat{\mathbf{s}}))$, for each $t \in T$. Then, for each $t \in T \setminus T_0$ such that $u^t(x) = v_1^t(x_1) + v_2^t(x_2)$, $\hat{\mathbf{x}}_1(t) = 0$ only if $-\frac{\partial u^t(0,x_2)}{\partial x_1} / \frac{\partial u^t(0,x_2)}{\partial x_2} > -\infty$, for each $x_2 \in \mathbb{R}_+$, and $\hat{\mathbf{x}}_2(t) = 0$ only if $-\frac{\partial u^t(x_1,0)}{\partial x_1} / \frac{\partial u^t(x_1,0)}{\partial x_2} < 0$, for each $x_1 \in \mathbb{R}_+$.

Proof. Let $\hat{\mathbf{s}}$ be a Cournot-Nash equilibrium and let $\hat{\mathbf{x}}(t) = \mathbf{x}(t, \hat{\mathbf{s}}(t), p(\hat{\mathbf{s}}))$, for each $t \in T$. Moreover, let $\hat{p} = p(\hat{\mathbf{s}})$. Consider an atom $\tau \in T \setminus T_0$ such that $u^\tau(x) = v_1^\tau(x_1) + v_2^\tau(x_2)$. Suppose that $\hat{\mathbf{x}}_1(\tau) = 0$. By the same argument used in the proof of Theorem 6, it follows that

$$-\frac{\frac{\partial u^\tau(\hat{\mathbf{x}}(\tau))}{\partial x_1}}{\frac{\partial u^\tau(\hat{\mathbf{x}}(\tau))}{\partial x_2}} \geq -\hat{p}_1.$$

Then,

$$-\frac{\frac{\partial u^\tau(0,\hat{\mathbf{x}}_2(\tau))}{\partial x_1}}{\frac{\partial u^\tau(0,\hat{\mathbf{x}}_2(\tau))}{\partial x_2}} > -\infty.$$

But then, $\frac{\partial u^\tau(0,\hat{\mathbf{x}}_2(\tau))}{\partial x_1} = \frac{\partial v_1^\tau(0)}{\partial x_1} = \frac{\partial u^\tau(0,x_2)}{\partial x_1} < +\infty$, for each $x_2 \in \mathbb{R}_+$. Moreover, $\frac{\partial u^\tau(0,x_2)}{\partial x_2} > 0$, for each $x_2 \in \mathbb{R}_+$, as u^τ is strongly monotone, by Assumption 2.

Therefore, $-\frac{\partial u^\tau(0,x_2)}{\partial x_1}/\frac{\partial u^\tau(0,x_2)}{\partial x_2} > -\infty$, for each $x_2 \in \mathbb{R}_+$. Suppose that $\hat{\mathbf{x}}_2(\tau) = 0$. By the same argument used in the proof of Theorem 6, it follows that

$$-\frac{\frac{\partial u^\tau(\hat{\mathbf{x}}(\tau))}{\partial x_1}}{\frac{\partial u^\tau(\hat{\mathbf{x}}(\tau))}{\partial x_2}} < -\hat{p}_1.$$

Then,

$$-\frac{\frac{\partial u^\tau(\hat{\mathbf{x}}_1(\tau),0)}{\partial x_1}}{\frac{\partial u^\tau(\hat{\mathbf{x}}_1(\tau),0)}{\partial x_2}} < 0.$$

But then, $\frac{\partial u^\tau(\hat{\mathbf{x}}_1(\tau),0)}{\partial x_2} = \frac{\partial v_2^\tau(0)}{\partial x_2} = \frac{\partial u^\tau(x_1,0)}{\partial x_2} < +\infty$, for each $x_1 \in \mathbb{R}_+$. Moreover, $\frac{\partial u^\tau(x_1,0)}{\partial x_1}$ > 0, for each $x_1 \in \mathbb{R}_+$, as u^τ is strongly monotone, by Assumption 2. Therefore, $-\frac{\partial u^\tau(x_1,0)}{\partial x_1}/\frac{\partial u^\tau(x_1,0)}{\partial x_2} < 0$, for each $x_1 \in \mathbb{R}_+$. Hence, for each $t \in T \setminus T_0$ such that $u^t(x) = v_1^t(x_1) + v_2^t(x_2)$, $\hat{\mathbf{x}}_1(t) = 0$ only if $-\frac{\partial u^t(0,x_2)}{\partial x_1}/\frac{\partial u^t(0,x_2)}{\partial x_2} > -\infty$, for each $x_2 \in R_+$, and $\hat{\mathbf{x}}_2(t) = 0$ only if $-\frac{\partial u^t(x_1,0)}{\partial x_1}/\frac{\partial u^t(x_1,0)}{\partial x_2} < 0$, for each $x_1 \in R_+$. □

In Examples 4, 5, 6, 7, 8, and 9, atoms' preferences are represented by quasi-linear utility functions in commodity 2, i.e., utility functions of the form $u(x) = v(x_1) + k\, x_2$, for each $x \in \mathbb{R}^2_+$. We finally provide a necessary condition for Theorem 6 to hold when atoms' preferences are of this kind and atoms' initial endowment are such that $\mathbf{w}_1(t) = 0$ and $\mathbf{w}_2(t) > 0$.

Proposition 2. Under Assumptions 1, 2, 3, and 4, let $\hat{\mathbf{s}}$ be a Cournot-Nash equilibrium and let $\hat{\mathbf{x}}(t) = \mathbf{x}(t, \hat{\mathbf{s}}(t), p(\hat{\mathbf{s}}))$, for each $t \in T$. Then, for each $t \in T \setminus T_0$ such that $u^t(x) = v^t(x_1) + k\, x_2$, $\mathbf{w}_1(t) = 0$, and $\mathbf{w}_2(t) > 0$, $\hat{\mathbf{x}}_2(t) = 0$ only if $-\frac{\partial u^t(x_1,0)}{\partial x_1}/\frac{\partial u^t(x_1,0)}{\partial x_2} < -\frac{\mathbf{w}_2(t)}{\bar{\mathbf{w}}_1}$, for each $x_1 \in \mathbb{R}_+$, with $\bar{\mathbf{w}}_1 = \int_T \mathbf{w}_1(t) d\mu$.

Proof. Let $\hat{\mathbf{s}}$ be a Cournot-Nash equilibrium and let $\hat{\mathbf{x}}(t) = \mathbf{x}(t, \hat{\mathbf{s}}(t), p(\hat{\mathbf{s}}))$, for each $t \in T$. Moreover, let $\hat{p} = p(\hat{\mathbf{s}})$. Consider an atom $\tau \in T \setminus T_0$ such that $u^\tau(x) = v^\tau(x_1) + k\, x_2$, $\mathbf{w}_1(\tau) = 0$, and $\mathbf{w}_2(\tau) > 0$. Suppose $\hat{\mathbf{x}}_2(\tau) = 0$. Then $\hat{\mathbf{b}}(\tau) = \mathbf{w}_2(\tau)$ and $\hat{\mathbf{x}}(\tau) = (\frac{\mathbf{w}_2(\tau)}{p_1}, 0)$. By the same argument used in the proof of Theorem 6, it follows that

$$-\frac{\frac{\partial u^\tau(\hat{\mathbf{x}}(\tau))}{\partial x_1}}{\frac{\partial u^\tau(\hat{\mathbf{x}}(\tau))}{\partial x_2}} \leq -\hat{p}_1.$$

Since $\hat{p}_1 = \frac{\bar{\mathbf{b}}}{\bar{\mathbf{q}}}$, $\hat{p}_1 \geq \frac{\mathbf{w}_2(\tau)}{\bar{\mathbf{w}}_1}$. Let $\hat{x}_2(x_1)$ be a function such that $u^\tau(x_1, x_2(x_1)) \equiv u^\tau(\hat{\mathbf{x}}(\tau))$, for each $0 \leq x_1 \leq \frac{\mathbf{w}_2(\tau)}{\hat{p}_1}$. Then, $\frac{d\hat{x}_2(x_1)}{dx_1} < -\frac{\mathbf{w}_2(\tau)}{\bar{\mathbf{w}}_1}$, for each $0 \leq x_1 \leq \frac{\mathbf{w}_2(\tau)}{\hat{p}_1}$, as u^τ is quasi-concave, by Assumption 2. But then

$$-\frac{\frac{\partial u^\tau(x_1,0)}{\partial x_1}}{\frac{\partial u^\tau(x_1,0)}{\partial x_2}} < -\frac{\mathbf{w}_2(\tau)}{\bar{\mathbf{w}}_1},$$

for each $x_1 \in R_+$, as u^τ is quasi-linear in x_2. Hence, for each $t \in T \setminus T_0$ such that $u^t(x) = v_1^t(x_1) + k\,x_2$, $\hat{\mathbf{x}}_2(t) = 0$ only if $-\frac{\partial u^t(x_1,0)}{\partial x_1} / \frac{\partial u^t(x_1,0)}{\partial x_2} < -\frac{\mathbf{w}_2(t)}{\bar{\mathbf{w}}_1}$, for each $x_1 \in R_+$. □

4.6 Discussion of the model

Here, we address the question whether, in the mixed bilateral oligopoly framework considered so far, the result of Theorem 6 also holds for the model of non-cooperative exchange considered by Amir et al. (1990). Dubey and Shubik (1978) distinguished between commodity money and all other commodities. Then, there is one market (trading post) for each commodity where commodity money can be exchanged directly for one of the other commodities. The direct exchange between any other two commodities is ruled out. Differently Amir et al. (1990) (Model 2 hereafter) analysed a model where markets are complete, i.e., all commodities can be used for trade. In this model, there is a market for each pair of commodities with the price in a market being the ratio of the total amount of bids in each of the two commodities which are exchanged in that market.

In general, with more than two commodities, the sets of Cournot-Nash allocations of the two models differ as, in Model 1, only commodity money can be used for trade whereas, in Model 2, all commodities can be used for trade. Furthermore, since in this model prices are determined for each pairs of commodities, they are not necessarily consistent through pairs of markets in which the same commodity is exchanged.

We now introduce Model 2. A strategy correspondence is a correspondence $\mathbf{B} : T \to \mathcal{P}(\mathbb{R}^4_+)$ such that, for each $t \in T$, $\mathbf{B}(t) = \{b \in \mathbb{R}^4_+ : \sum_{j=1}^{2} b_{ij} \leq \mathbf{w}_i(t),\ i = 1, 2\}$, where b_{ij} represents the amount of commodity i that trader t offers in exchange for commodity j. A strategy selection is an integrable function $\mathbf{b} : T \to \mathbb{R}^4_+$, such that, for each $t \in T$, $\mathbf{b}(t) \in \mathbf{B}(t)$. Given a strategy selection $\mathbf{b}$, we define the aggregate matrix $\bar{\mathbf{B}} = (\int_T \mathbf{b}_{ij}(t) d\mu)$. Moreover, we denote by $\mathbf{b} \setminus b(t)$ the strategy selection obtained from $\mathbf{b}$ by replacing $\mathbf{b}(t)$ with $b(t) \in \mathbf{B}(t)$.[8]

Given a strategy selection $\mathbf{b}$, the 2×2 matrix P is said to be the price matrix generated by $\mathbf{b}$ if

$$p_{ij} = \begin{cases} \frac{\bar{\mathbf{b}}_{ij}}{\bar{\mathbf{b}}_{ji}} & \text{if } \bar{\mathbf{b}}_{ji} \neq 0, \\ 0 & \text{if } \bar{\mathbf{b}}_{ji} = 0, \end{cases}$$

[8] In order to save in notation, with some abuse, we denote by $\mathbf{b}(t)$ both a strategy selection of a trader t in Model 2 and an action of a trader t in Model 1. The context should clarify whether $\mathbf{b}(t)$ is a trader's strategy selection or a trader's action.

with $i, j = 1, 2$. We denote by $P(\mathbf{b})$ a function which associates with each strategy selection $\mathbf{b}$ the price matrix P generated by $\mathbf{b}$. Given a strategy selection $\mathbf{b}$ and a price matrix P, consider the assignment determined as follows

$$\mathbf{x}_j(t, \mathbf{b}(t), p) = \mathbf{w}_j(t) - \sum_{i=1}^{2} \mathbf{b}_{ji}(t) + \sum_{i=1}^{2} \mathbf{b}_{ij}(t) p_{ij},$$

for $j = 1, 2$, for each $t \in T$. Given a strategy selection $\mathbf{b}$ and the function $P(\mathbf{b})$, the traders' final holdings are determined according with this rule and consequently expressed by the assignment

$$\mathbf{x}(t) = \mathbf{x}(t, \mathbf{b}(t), P(\mathbf{b})),$$

for each $t \in T$.[9] It is straightforward to show that this assignment is an allocation.

We are now able to define a notion of Cournot-Nash equilibrium for Model 2.

Definition 2. A strategy selection $\tilde{\mathbf{b}}$ such that $\bar{\tilde{\mathbf{b}}}_{12} > 0$ and $\bar{\tilde{\mathbf{b}}}_{21} > 0$ is a Cournot-Nash equilibrium if

$$u_t(\mathbf{x}(t, \tilde{\mathbf{b}}(t), P(\tilde{\mathbf{b}}))) \geq u_t(\mathbf{x}(t, \tilde{\mathbf{b}} \setminus b(t), P(\tilde{\mathbf{b}} \setminus b(t)))),$$

for each $b(t) \in \mathbf{B}(t)$ and for each $t \in T$.[10]

A Cournot-Nash allocation of Model 2 is an allocation $\tilde{\mathbf{x}}$ such that $\tilde{\mathbf{x}}(t) = \mathbf{x}(t, \tilde{\mathbf{b}}(t), P(\tilde{\mathbf{b}}))$, for each $t \in T$, where $\tilde{\mathbf{b}}$ is a Cournot-Nash equilibrium of Model 2.

The following theorem shows an equivalence between the sets of Cournot-Nash allocations of Model 1 and Model 2.

Theorem 7. Under Assumptions 1, 2, and 3, the sets of Cournot-Nash allocations of Model 1 and Model 2 coincide.

Proof. Let $\hat{\mathbf{x}}$ be a Cournot-Nash allocation of Model 1. Then, there is a strategy selection $\hat{\mathbf{s}}$ which is a Cournot-Nash equilibrium of Model 1 and is such that $\hat{\mathbf{x}}(t) = \mathbf{x}(t, \hat{\mathbf{s}}(t), p(\hat{\mathbf{s}}))$, for each $t \in T$. Consider the strategy selection $\hat{\mathbf{b}}$ of Model 2 such that $\hat{\mathbf{b}}_{11}(t) = \hat{\mathbf{b}}_{22}(t) = 0$, $\hat{\mathbf{b}}_{12}(t) = \hat{\mathbf{q}}(t)$, and $\hat{\mathbf{b}}_{21}(t) = \hat{\mathbf{b}}(t)$. It is straightforward to show that $p_{12} = \frac{1}{p_1}$ and $p_{21} = p_1$. Then $\hat{\mathbf{x}}(t) = \mathbf{x}(t, \hat{\mathbf{b}}(t), P(\hat{\mathbf{b}}))$, for each $t \in T$.

[9] In order to save in notation, with some abuse, we denote by $\mathbf{x}$ both the function $\mathbf{x}(t)$ and the function $\mathbf{x}(t, \mathbf{b}(t), P(\mathbf{b}))$.

[10] Note that this definition of a Cournot-Nash equilibrium refers only to equilibria at which the markets for commodities 1 and 2 are active.

Suppose that $\hat{\mathbf{x}}$ is not a Cournot-Nash allocation of Model 2. Then, there exist a trader $t \in T$ and a strategy $b(t) \in \mathbf{B}(t)$ such that

$$u^t(\mathbf{x}(t, \hat{\mathbf{b}} \setminus b(t), P(\hat{\mathbf{b}} \setminus b(t)))) > u^t(\mathbf{x}(t, \hat{\mathbf{b}}(t), P(\hat{\mathbf{b}}))).$$

Consider a strategy $s(t)$ such that $q(t) = b_{12}(t)$ and $b(t) = b_{21}(t)$. Let $p_2 = 1$ and it is straightforward to show that $p_1 = p_{21}$. Then, $\mathbf{x}(t, \hat{\mathbf{s}} \setminus s(t), p(\hat{\mathbf{s}} \setminus s(t))) = \mathbf{x}(t, \hat{\mathbf{b}} \setminus b(t), P(\hat{\mathbf{b}} \setminus b(t)))$. But then,

$$u^t(\mathbf{x}(t, \hat{\mathbf{s}} \setminus s(t), p(\hat{\mathbf{s}} \setminus s(t)))) > u^t(\mathbf{x}(t, \hat{\mathbf{s}}(t), p(\hat{\mathbf{s}}))),$$

a contradiction. Therefore, $\hat{\mathbf{x}}$ is a Cournot-Nash allocation of Model 2. Let $\tilde{\mathbf{x}}$ be a Cournot-Nash allocation of Model 2. Suppose that $\tilde{\mathbf{x}}$ is not a Cournot-Nash allocation of Model 1. Then, the previous argument leads, *mutatis mutandis*, to the same kind of contradictions. Therefore, $\tilde{\mathbf{x}}$ is a Cournot-Nash allocation of Model 1. Hence, the sets of Cournot-Nash allocations of Model 1 and Model 2 coincide. □

The following corollary shows that Theorem 6 holds, *mutatis mutandis*, for Model 2.

Corollary 1. Under Assumptions 1, 2, 3, and 4, let $\tilde{\mathbf{b}}$ be a Cournot-Nash equilibrium of Model 2 and let $\tilde{p} = (\frac{\tilde{\bar{\mathbf{b}}}_{21}}{\tilde{\bar{\mathbf{b}}}_{12}}, 1)$ and $\tilde{\mathbf{x}}(t) = \mathbf{x}(t, \tilde{\mathbf{b}}(t), P(\tilde{\mathbf{b}}))$, for each $t \in T$. Then, the pair $(\tilde{p}, \tilde{\mathbf{x}})$ is a Walras equilibrium if and only if $\tilde{\mathbf{x}}_1(t) = 0$ or $\tilde{\mathbf{x}}_2(t) = 0$, for each $t \in T \setminus T_0$.

Proof. Let $\tilde{\mathbf{b}}$ be a Cournot-Nash equilibrium of Model 2 and let $\tilde{p} = (\frac{\tilde{\bar{\mathbf{b}}}_{21}}{\tilde{\bar{\mathbf{b}}}_{12}}, 1)$ and $\tilde{\mathbf{x}}(t) = \mathbf{x}(t, \tilde{\mathbf{b}}(t), P(\tilde{\mathbf{b}}))$, for each $t \in T$. By Theorem 7, there exists a Cournot-Nash equilibrium of Model 1, $\tilde{\mathbf{s}}$, such that $\tilde{\mathbf{q}}(t) = \tilde{\mathbf{b}}_{12}(t)$ and $\tilde{\mathbf{b}}(t) = \tilde{\mathbf{b}}_{21}(t)$, for each $t \in T$, for which $\tilde{\mathbf{x}}(t) = \mathbf{x}(t, \tilde{\mathbf{s}}(t), p(\tilde{\mathbf{s}}))$ and $\tilde{p} = p(\tilde{\mathbf{s}}) = (\frac{\tilde{\bar{\mathbf{b}}}_{21}}{\tilde{\bar{\mathbf{b}}}_{12}}, 1)$. Then, $\tilde{\mathbf{x}}(t)$ is a Cournot-Nash allocation for Model 1. Hence, by Theorem 6, the pair $(\tilde{p}, \tilde{\mathbf{x}})$ is a Walras equilibrium if and only if $\tilde{\mathbf{x}}_1(t) = 0$ or $\tilde{\mathbf{x}}_2(t) = 0$, for each $t \in T \setminus T_0$. □

4.7 Conclusion

In this paper, we have reconsidered, in the framework of bilateral oligopoly, the problem raised by Okuno et al. (1980) about the non-cooperative foundation of oligopolistic behaviour in general equilibrium. We can now summarize the implications of the previous analysis. The condition which requires that the atoms are not "too" big, introduced by Gabszewicz and Mertens (1971), is not necessary for the equivalence between the core and the set of Walras allocations, as shown by Theorem 4, but it is sufficient for this equivalence, by Theorem 3; moreover, it is neither necessary nor sufficient for a non-empty intersection between the sets of Walras and Cournot-Nash allocations as shown, respectively, by Examples 7 and 4. The condition which requires that there are only atoms of the same type, introduced by Shitovitz (1973), is not necessary for the equivalence between the core and the set of Walras allocations, as shown by Theorem 3, but it is sufficient for this equivalence, by Theorem 4; moreover, it is neither necessary nor sufficient for a non-empty intersection between the sets of Walras and Cournot-Nash allocations as shown, respectively, by Examples 6 and 5. Theorem 6 states that the condition which characterises the non-empty intersection of the sets of Walras and Cournot-Nash allocations requires that each atom demands a null amount of one commodity. Moreover, Examples 6, 7, 8, and 9 show that this characterisation condition is non-vacuous. Propositions 1 and 2 provide a rationale for these examples by exhibiting necessary conditions, expressed in terms of bounds on atoms' marginal rates of substitution, for Theorem 6 to hold when atoms' preferences are represented by additively separable utility functions and quasi-linear utility function respectively. We leave as an open problem for further research the generalisation of this proposition, namely, the determination of more general assumptions on traders' size, endowments, and preferences under which our characterisation condition holds. This analysis could help to understand more deeply which are the differences between atoms' Walrasian behaviour in a cooperative and in a non-cooperative framework. We have also proved that Theorem 6 can be extended, in bilateral oligopoly, to the model of non-cooperative exchange introduced by Amir at al. (1990). Some further research should also be devoted to the possibility of generalising the results achieved in this paper to an exchange economy with more than two commodities.

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