Deformations, Extensions and Symmetries of Solutions to the WDVV equations

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Abstract

We investigate almost-dual-like solutions of the WDVV equations for which the metric, under the standard definition, is degenerate. Such solutions have previously been considered in [21] as complex Euclidean $\vee$-systems with zero canonical form but were not regarded as solutions since a non-degenerate metric is required for a solution. We have found that, in every case we considered, we can impose a metric and hence recover a solution. We also found that for the deformed $A_n(c)$ family (first appearing in [8]) with the choice of parameters that renders the metric singular we can also recover a solution. The generalised root system $A(n-1,n)$ (as it appears in our notation) has zero canonical form but we found that by restricting the covectors we can again recover a solution which we generalise to a family with $(n+1)$ parameters which we denote as P_n.

We next look at extended $\vee$-systems. These are root-systems which possess the small orbit property (as defined in [36]) which we then extend into a dimension perpendicular to the original system. We then impose the $\vee$-conditions onto these systems and obtain 1-parameter infinite families of $\vee$-systems. We also find that for the B_n family we can extend into two perpendicular directions.

We then go on to look at a generalisation of the Legendre transformations (which originally appeared in [13])which map solutions to WDVV to other solutions. We find that such transformations are generated not only by constant vector fields but by functional vector fields too and we find a very simple rule which such vector fields must obey. Finally

we link our work on extended $\vee$-systems and on generalised Legendre transformations to that on extended affine Weyl groups found in [16] and [17].

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Contents

Introduction

The origin of the WDVV equations was in 2D topological field theory at the end of the 1980s. They are systems of PDEs expressed in terms of the matrices of the third derivatives of a function F of n variables $(x_1, \ldots, x_n)$

$$F_i F_j^{-1} F_k = F_k F_j^{-1} F_i, \quad i, j, k = 1, \ldots, n,$$

where

$$(F_i)_{jk} = \frac{\partial^3 F}{\partial x_i \partial x_j \partial x_k}.$$

In the original formulation of the equations an additional requirement that there must exist a marked variable, let us call it x_1, such that the matrix F_1 has constant entries was made. An example of a solution (due to Kontsevich) for $n = 3$ with this additional assumption is

$$F(x_1, x_2, x_3) = \frac{1}{2} x_1 x_2^2 + \frac{1}{2} x_1^2 x_3 + \sum_{k=0}^{\infty} \frac{N_k}{3k-1} x_3^{3k-1} e^{k x_2},$$

where the recursion relation

$$\frac{N_k}{(3k-4)!} = \sum_{a+b=k} \frac{a^2 b(3b-1) b(2a-b)}{(3a-1)!(3b-1)!} N_a N_b,$$

guarantees that this is a solution to the WDVV equations. Here the numbers N_a, examples of Gromov-Witten invariants, give the number of rational curves of degree k passing through $3k-1$ points in complex projective space of dimension 2.

A few years later it was found that the prepotential of Seiberg-Witten theory satisfies a version of WDVV but without the assumption of a marked variable. An example of such a solution is

$$F = \sum_{\alpha \in \mathcal{R}} (\alpha \cdot z)^2 \log(\alpha \cdot z), \tag{1}$$

where $\mathcal{R}$ is any root system of a finite Coxeter group. Boris Dubrovin was able to unite these seemingly disconnected worlds of solutions via his concept of almost duality. This provided a systematic way of obtaining an 'almost dual' solution in which all of the variables are on an equal footing from a solution which had a marked variable. Going the other way, or reconstructing a solution, is fraught with much more difficulty. Dubrovin had formulated the WDVV equations in a geometric form as Frobenius manifolds and it was in this setting that almost duality was discovered.

At the end of the 1990s Veselov derived geometric conditions, called the $\vee$-conditions, which collections of covectors (called $\vee$-systems, the root systems in (1) for example) must satisfy in order to provide a solution to WDVV in the almost dual world. He also found both entirely new $\vee$-systems as well as deformations to some of the known root-system solutions which are not almost dual to any solution.

In this thesis, after two preliminary chapters outlining the background of the area, we discuss work in three distinct areas. The first, in Chapter 3, looks at recovering solutions in the almost-dual world when the canonical bilinear form defined for all $\vee$-systems (and which plays an essential role in providing a solution to the WDVV equations) is identically zero. Work had already been done in this direction in [4] where solutions were recovered for two $\vee$-systems whose canonical form, for certain values of their parameters, became identically zero. We show that the same can be done for various other $\vee$-systems and, most significantly, in Subsection 3.1.2 that we can recover a solution from the generalised root system $A(n-1,n)$ as defined in [36]. We have verified computationally that we can recover a solution for $n \leqslant 9$ and conjecture that we can for all n. We also conjecture that there exist generalised $\vee$-conditions that can be applied not only to $\vee$-systems with the canonical bilinear form but also to those for which we must 'impose' a metric. We also present some preliminary findings on similar work on polynomial solutions in Section 3.3.

In Chapter 4 we define extended $\vee$-systems. These utilise the small-orbit property of root systems defined in [36] to obtain an $(n+1)$-dimensional $\vee$-system from an n-dimensional one. Note that this does not give us previously unknown $\vee$-systems (they are subsets of the families of deformed $\vee$-systems found in [40]) but are significant in that, up to a Legendre transformation, they are almost-dual to Dubrovin and Zhang's extended affine Weyl solutions found in [16] (which we show in Chapter 5). In Section 4.3 we show that systems of B-type can be extended into 2 dimensions. This raises the question of whether the corresponding affine Weyl solutions can be extended into 2 dimensions also.

Chapter 5 builds on the work found in [9] and [30]. In both of these works what we call generalised Legendre fields were discussed. In the former they were thought of as maps between connections on F-manifolds (a generalisation of Frobenius manifolds) while in the latter they feature in the construction of an integrable hierarchy of hydrodynamic type, a generalisation of Dubrovin's principal hierarchy [13]. We show that we can generalise Dubrovin's Legendre transormations defined in [13] which were generated by only flat vector fields to those generated by any generalised Legendre field (see Propositions 5.3 and 5.14). In Section 5.3 we consider *twisted* Legendre transformations. These are the transformations induced between the almost-dual counterparts of a solution and its Legendre transformation. These had been considered in [9] and [33]. In the former *eventual identities*, vector fields which provide a new multiplication on an F-manifold were considered whereas we consider the special case of the Euler vector field as the eventual identity. We find the criterion for the twisted Legendre field to be flat also. In Section 5.4 we unite our work in Chapter 4 on extended $\vee$ systems and that on generalised Legendre transformations to show that extended $\vee$-systems are, up to a twisted Legendre transformation, almost-dual to Dubrovin and Zhang's extended affine Weyl solutions.

Finally, in Chapter 6 we consider possible avenues for further work.

Chapter 1

The WDVV Equations and Frobenius Manifolds

The subjects under consideration in this chapter are the Witten - Dijkgraaf - Verlinde - Verlinde (WDVV) equations which first appeared in the papers [11] and [42] in the context of two-dimensional topological field theory and their very closely related geometrical interpretation, Frobenius manifolds, which first appeared in [12].

1.1 The WDVV equations of associativity

The WDVV equations of associativity are an over-determined system of PDE which arise from the condition that the pair (F, η) where the function $F = F(t), \quad t = (t^1, \dots, t^n)$, is called the *prepotential* and η is a non-degenerate $n \times n$ matrix called the metric (which will be used, along with its inverse $\eta^{\alpha\beta} := (\eta_{\alpha\beta})^{-1}$ to lower and raise indices, respectively) define, via the functions (Einstein summation is assumed throughout this thesis)

$$c^{\gamma}_{\alpha\beta}(t) := \eta^{\gamma\varepsilon} c_{\varepsilon\alpha\beta}(t),$$

where $c_{\alpha\beta\gamma}$ is the $(0,3)$-tensor of third derivatives of F

$$c_{\alpha\beta\gamma}(t) := \frac{\partial^3 F(t)}{\partial t^\alpha \partial t^\beta \partial t^\gamma}, \tag{1.1}$$

a structure of an associative algebra $\mathcal{A}_t$ (which is also commutative by (1.1)) by

$$e_\alpha \cdot e_\beta = c^\gamma_{\alpha\beta}(t) e_\gamma,$$

for any t in the n-dimensional space with basis $e_1, \ldots, e_n$.

In other words we must have

$$(e_\alpha \cdot e_\beta) \cdot e_\gamma = e_\alpha \cdot (e_\beta \cdot e_\gamma),$$

$$c^\mu_{\alpha\beta}(t)[e_\mu \cdot e_\gamma] = c^\mu_{\beta\gamma}(t)[e_\alpha \cdot e_\mu],$$

$$[c^\mu_{\alpha\beta}(t) c^\lambda_{\mu\gamma}(t)] e_\lambda = [c^\mu_{\beta\gamma}(t) c^\lambda_{\alpha\mu}(t)] e_\lambda,$$

$$c_{\alpha\beta\nu}(t) \eta^{\nu\mu} c_{\mu\gamma\lambda}(t) = c_{\beta\gamma\nu}(t) \eta^{\nu\mu} c_{\alpha\mu\lambda}(t),$$

or

$$\frac{\partial^3 F(t)}{\partial t^\alpha \partial t^\beta \partial t^\nu} \eta^{\nu\mu} \frac{\partial^3 F(t)}{\partial t^\mu \partial t^\gamma \partial t^\lambda} = \frac{\partial^3 F(t)}{\partial t^\beta \partial t^\gamma \partial t^\nu} \eta^{\nu\mu} \frac{\partial^3 F(t)}{\partial t^\alpha \partial t^\mu \partial t^\lambda}, \tag{1.2}$$

for all $\alpha, \beta, \gamma, \lambda$ from 1 to n. It is this system which is called the *WDVV equations of associativity.*

As discussed in the introduction in some contexts we have the additional assumption that there exists a marked variable, t_1, such that the metric given by

$$\eta_{\alpha\beta} := c_{1\alpha\beta}, \tag{1.3}$$

is constant.

1.1.1 Quasihomogeneity and the Euler Vector Field

We will also often want to impose quasihomogeneity on F. A naïve definition of this is that we must have

$$F(\lambda^{d_1}t^1, \ldots, \lambda^{d_n}t^n) = \lambda^{d_F} F(t^1, \ldots, t^n), \tag{1.4}$$

for some numbers $d_1, \ldots, d_n, d_F$ (called the *degrees* of the t^i) for all $\lambda \neq 0$. Note that if all of the degrees are equal then this condition reduces to homogeneity.

By differentiating this equation with respect to λ and then setting $\lambda = 1$ we obtain

$$d_1 t_1 \frac{\partial F}{\partial t_1} + \ldots + d_n t_n \frac{\partial F}{\partial t_n} = d_F F(t_1, \ldots, t_n),$$

(which is a generalisation of Euler's theorem) and we can see that this may be written

$$\mathcal{L}_E F(t) = d_F F(t),$$

where

$$E = \sum_{i=1}^{n} d_i t^i \partial_i,$$

is called the *Euler vector field* and $\mathcal{L}_E$ is the Lie derivative along E (note that the shorthand ∂_α will refer to $\frac{\partial}{\partial t^\alpha}$ throughout this thesis). We may normalise the d_i so that $d_1 = 1$.

We can generalise the quasihomoegenity condition by considering the addition of a non-homogeneous quadratic function of $t_1, \ldots, t_n$ to F. This would render the $c_{\alpha\beta\gamma}(t)$, and hence the algebras $\mathcal{A}_t$, and also the metric, unchanged. The condition now reads

$$\mathcal{L}_E F(t) = d_F F(t) + A_{\alpha\beta} t^\alpha t^\beta + B_\alpha t^\alpha + C. \tag{1.5}$$

Furthermore, if we add a quadratic form to F

$$F'(t) = F(t) + A'_{\alpha\beta}t^\alpha t^\beta + B'_\alpha t^\alpha + C',$$

we have

$$\mathcal{L}_E F'(t) = \mathcal{L}_E F(t) + (d_\alpha + d_\beta)A'_{\alpha\beta}t^\alpha t^\beta + d_\alpha B'_\alpha t^\alpha, \tag{1.6}$$

and

$$d_F F'(t) = d_F(F(t) + A'_{\alpha\beta}t^\alpha t^\beta + B'_\alpha t^\alpha + C'), \tag{1.7}$$

so on substituting (1.5) and (1.7) into (1.6)

$$\begin{aligned}\mathcal{L}_E F'(t) = d_F F'(t) + (A_{\alpha\beta} - d_F A'_{\alpha\beta})t^\alpha t^\beta + (B_\alpha - d_F B'_\alpha)t^\alpha + (C - d_F C')\\ + (d_\alpha + d_\beta)A'_{\alpha\beta}t^\alpha t^\beta + d_\alpha B'_\alpha t^\alpha,\end{aligned} \tag{1.8}$$

$$\begin{aligned}= d_F F'(t) + [A_{\alpha\beta} + A'_{\alpha\beta}(d_\alpha + d_\beta - d_F)]t^\alpha t^\beta\\ + [B_\alpha + B'_\alpha(d_\alpha - d_F)]t^\alpha + (C - d_F C'),\end{aligned} \tag{1.9}$$

so we can recover

$$\mathcal{L}_E F'(t) = d_F F'(t),$$

as long as

$$d_F \neq d_\alpha + d_\beta, \quad d_F \neq d_\alpha, \quad d_F \neq 0,$$

for all α and β.

The notion of quasihomogeneity may be extended in another way (see [13]) to include instances where some of the $d_i = 0$. In that case we have

$$E = \sum_{i=1}^{n} d_i t^i \partial_i + \sum_{i|d_i=0} r^i \partial_i.$$

The Euler vector field is *conformal* (see [13]), in other words

$$\mathcal{L}_E \eta_{\alpha\beta} = D\eta_{\alpha\beta}, \tag{1.10}$$

and in our case $D = d_F - 1$. We can rewrite (1.10) as

$$E(\eta(\partial_\alpha, \partial_\beta)) - \eta([E, \partial_\alpha], \partial_\beta) - \eta(\partial_\alpha, [E, \partial_\beta]) = (d_F - 1)\eta(\partial_\alpha, \partial_\beta), \tag{1.11}$$

and since

$$[E, \partial_\alpha] = -d_\alpha \partial_\alpha,$$

we have

$$\eta(d_\alpha \partial_\alpha, \partial_\beta) + \eta(\partial_\alpha, d_\beta \partial_\beta) = (d_F - 1)\eta(\partial_\alpha, \partial_\beta)$$

that is

$$(d_\alpha + d_\beta - d_F + 1)\eta_{\alpha\beta} = 0.$$

So $\eta_{\alpha\beta} = 0$ unless $d_\alpha + d_\beta = d_F - 1$. This means that if we choose the d_i such that $d_\alpha + d_{n-\alpha+1} = d_F - 1$ the metric has the antidiagonal form

$$\eta_{\alpha\beta} = \delta_{\alpha+\beta,n+1}. \tag{1.12}$$

Integrating (1.3) yields the prepotential

$$F(t) = \frac{1}{2}(t^1)^2 t^n + \frac{1}{2} t^1 \sum_{\alpha=2}^{n-1} t^\alpha t^{n-\alpha+1} + f(t^2, \dots, t^n), \tag{1.13}$$

for some function $f(t^2, \ldots, t^n)$.

The only other possibility which gives a different prepotential is if $\eta_{11} \neq 0$. This is only possible if $d_F = 3d_1$ and the prepotential obtained is

$$F(t) = \frac{c}{6}(t^1)^3 + \frac{1}{2}t^1 \sum_{\alpha=1}^{n-1} t^\alpha t^{n-\alpha+1} + f(t^2, \ldots, t^n), \tag{1.14}$$

where c is a non-zero constant and $d_\alpha + d_{n-\alpha+1} = 2$.

1.1.2 Solutions for $n = 2$ with the assumption of a marked variable and quasihomogeneity

Since the algebras $\mathcal{A}_t$ are unital the WDVV equations of associativity are automatically satisfied. It is only the definition of the metric and quasihomogeneity which constrain the solutions. First consider the case $\eta_{11} = 0$. We only consider the case $d_1 \neq 0$ (so $d_1 = 1$). We also assume, for the moment, that $d_2 \neq 0$. From eqn. (1.13)

$$F(t^1, t^2) = \frac{1}{2}(t^1)^2 t^2 + f(t^2),$$

and eqn. (1.5) reads;

$$t^1 \frac{\partial F}{\partial t^1} + d_2 t^2 \frac{\partial F}{\partial t^2} = d_F F + \alpha(t^2)^2 + \beta t^2 + \gamma,$$

where α, β and γ are constants. Thus

$$(t^1)^2 t^2 + \frac{d_2}{2}(t^1)^2 t^2 = d_F (t^1)^2 t^2 \quad (\implies d_2 = d_F - 2),$$

and

$$d_2 t^2 f'(t^2) = d_F f(t^2) + \alpha(t^2)^2 + \beta t^2 + \gamma,$$

so

$$t^2 f'(t^2) - \frac{d_F}{d_F - 2} f(t^2) = \tilde{\alpha}(t^2)^2 + \tilde{\beta} t^2 + \tilde{\gamma},$$

putting $k = \frac{d_F}{d_F - 2}$, multiplying by $(t^2)^{-k-1}$ and rearranging gives

$$(t^2)^{-k} f(t^2) = \int \left(\frac{\tilde{\alpha}}{(t^2)^{k-1}} + \frac{\tilde{\beta}}{(t^2)^k} + \frac{\tilde{\gamma}}{(t^2)^{k+1}} \right) dt^2.$$

Integrating yields (ignoring quadratic terms)

$$F(t^1, t^2) = \frac{1}{2}(t^1)^2 t^2 + t_2^k, \quad d_F \neq 0, 2, 4, \tag{1.15}$$

$$F(t^1, t^2) = \frac{1}{2}(t^1)^2 t^2 + \mathrm{log} t^2, \quad d_F = 0, \tag{1.16}$$

$$F(t^1, t^2) = \frac{1}{2}(t^1)^2 t^2 + (t^2)^2 \mathrm{log} t^2, \quad d_F = 4. \tag{1.17}$$

In the case $d_2 = 0$ eqn. (1.5) reads;

$$t^1 \frac{\partial F}{\partial t^1} + r \frac{\partial F}{\partial t^2} = d_F F + \alpha (t^2)^2 + \beta t^2 + \gamma,$$

so

$$(t^1)^2 t^2 = \frac{d_F}{2} (t^1)^2 t^2 \quad (\implies d_F = 2),$$

and then

$$f'(t^2) - \frac{2}{r} f(t^2) = -\frac{r}{2}(t^1)^2 + \tilde{\alpha}(t^2)^2 + \tilde{\beta} t^2 + \tilde{\gamma}, \quad r \neq 0,$$

$$f(t^2) = \tilde{\alpha}(t^2)^2 + \tilde{\beta} t^2 + \tilde{\gamma}, \quad r = 0.$$

These give, again ignoring quadratic terms

$$F(t^1, t^2) = \frac{1}{2}(t^1)^2 t^2 + e^{\frac{2}{r} t^2}, \quad r \neq 0, \tag{1.18}$$

$$F(t^1, t^2) = \frac{1}{2}(t^1)^2 t^2, \quad r = 0. \tag{1.19}$$

In the case $\eta_{11} \neq 0$ eqn. (1.14) gives

$$F(t^1, t^2) = \frac{c}{6}(t^1)^3 + \frac{1}{2}(t^1)^2 t^2 + f(t^2),$$

and degrees $d_1 = d_2 = 1$. Substituting $F(t^1, t^2)$ into eqn. (1.5) gives

$$\frac{c}{2}(t^1)^3 + (t^1)^2 t^2 + \frac{1}{2}(t^1)^2 t^2 + t^2 f'(t^2) = d_F \left[\frac{c}{6}(t^1)^3 + \frac{1}{2}(t^1)^2 t^2 + f(t^2)\right] + \alpha(t^2)^2 + \beta t^2 + \gamma,$$

hence

$$\frac{c}{2}(t^1)^3 = \frac{c d_F}{6}(t^1)^3, \quad (\implies d_F = 3),$$

and hence

$$t^2 f'(t^2) - 3f(t^2) = \alpha(t^2)^2 + \beta t^2 + \gamma.$$

This equation can be solved to give

$$F(t^1, t^2) = \frac{c}{6}(t^1)^3 + \frac{1}{2}(t^1)^2 t^2 + (t^2)^3. \tag{1.20}$$

This completes the list of quasihomogeneous solutions to the WDVV equations of associativity in 2 dimensions.

1.2 Frobenius manifolds

Frobenius algebras began to be studied in the 1930s by Richard Brauer and his student Cecil James Nesbitt and were named by them in [7]. More recently interest in them has intensified due to their importance in *topological quantum field theories* (see [5]). It was shown in [10] that when these topological field theories are two-dimensional they are equivalent to Frobenius algebras. Frobenius manifolds, objects which possess the structure of a Frobenius algebra as outlined below, were introduced by Boris Dubrovin in [12]. They have a very intimate connection with solutions of the WDVV equations of associativity which are also quasihomogeneous.

Definition 1.21 *(Frobenius algebra) [13] An algebra $\mathcal{A}$ defined over a field k is said to be Frobenius if it associative, unital and is equipped with a non-degenerate bilinear form $\eta : \mathcal{A} \times \mathcal{A} \mapsto k$ which satisfies*

$$\eta(X \circ Y, Z) = \eta(X, Y \circ Z),$$

$\forall X, Y, Z \in \mathcal{A}$ where $\circ$ is the multiplication associated with $\mathcal{A}$.

We will be exclusively concerned with Frobenius algebras which are also commutative.

Definition 1.22 *(Frobenius manifold) A manifold $\mathcal{M}$ is called* Frobenius *if the tangent space $T_t\mathcal{M}$ at a point $t \in \mathcal{M}$ is a Frobenius algebra that varies smoothly with t and*

- *the invariant (with respect to $\circ$) inner product η is flat. This means that there must exist a set of coordinates (distinguished up to a Euclidean transformation), called the flat coordinates of η, in which the components of η are constants,*
- *the unity element, e, of $T_t\mathcal{M}$, is covariantly constant with respect to the Levi-Civita connection of η,*
- *the (0,4)-tensor $\nabla_W c(X, Y, Z)$ is totally symmetric $\forall W, X, Y, Z \in T_t\mathcal{M}$ where $c(X, Y, Z) = \eta(X \circ Y, Z)$,*
- *the Euler vector field, E, may be determined on $\mathcal{M}$ such that $\nabla(\nabla E) = 0$ (so E is linear in the flat coordinates) and $\mathcal{L}_E\eta = D\eta$, for a constant D, $\mathcal{L}_E\circ = \circ$ and $\mathcal{L}_E e = -e$.*

We now show that the definition of a Frobenius manifold is equivalent to the WDVV equations of associativity and quasihomogeneity.

Associativity of $\circ$ gives eqn. (1.2). In the flat coordinates covariant derivatives are just partial derivatives so we have $\partial_W c(X, Y, Z)$ is totally symmetric. Successive applications of the Poincaré lemma demonstrates the existence of the prepotential $F(t)$ that satisfies eqn. (1.1).

Since e is covariantly constant we can choose $e = \partial_1$ and we have

$$\eta(\partial_1 \circ X, Y) = \eta(X, Y) = c(\partial_1, X, Y),$$

which is eqn. (1.3).

Finally we need to show that $F(t)$ satisfies eqn.(1.5). From $\mathcal{L}_E e = -e$ we have that $[\partial_1, E] = \partial_1$, hence ∂_1 is an eigenvector of the operator $Q = \nabla E$ with eigenvalue 1 or $d_1 = 1$. From $\mathcal{L}_E \eta = D\eta$, we have that the constant matrix $\left(Q^\alpha_\beta\right)$ must satisfy $Q_{\alpha\beta} = D\eta_{\alpha\beta}$ for some constant D. Using $\mathcal{L}_E \eta = D\eta$ again along with $\mathcal{L}_E \circ = \circ$ gives

$$\mathcal{L}_E c_{\alpha\beta\gamma} = (1 + D)c_{\alpha\beta\gamma},$$

and using eqn. (1.1) this is

$$\partial_\alpha \partial_\beta \partial_\gamma [E^\varepsilon \partial_\varepsilon F - (1 + D)F] = 0,$$

which, on integrating, is eqn. (1.5).

1.3 The Dubrovin connection

In this section and the next we follow the approach found in [26]. Let ∇ be the Levi-Civita connection of the metric η.

Definition 1.23 *(Dubrovin connnection) The Dubrovin connection is the pencil of connections along a vector field X of another vector field Y*

$$ {}^{\lambda}\nabla_X(Y) := \nabla_X(Y) + \lambda X \circ Y. $$

for a constant λ.

Note ${}^{\lambda}\nabla$ is torsion-free. This follows from the torsion free property of the Levi-Civita connection and the commutativity of the multiplication.

Theorem 1.24 *For a manifold $\mathcal{M}$ equipped with metric η and multiplication $\circ$ there exists a prepotential F such that*

$$ c_{abd}(t) = \frac{\partial^3 F(t)}{\partial t^a \partial t^b \partial t^d}, $$

and $\circ$ is associative if and only if ${}^{\lambda}\nabla$ is flat.

Proof. Consider the curvature of ${}^{\lambda}\nabla$,

$$ R(X,Y)Z := ([{}^{\lambda}\nabla_X, {}^{\lambda}\nabla_Y] - {}^{\lambda}\nabla_{[X,Y]})Z. $$

We express it as

$$ R(X,Y)Z = \lambda^2 R_2(X,Y)Z + \lambda R_1(X,Y)Z, $$

(there is no constant term since ∇ is flat).

It is sufficient to work with flat vector fields ∂_a, ∂_b. From the λ-terms in

$$ [\nabla_{\partial a} + \lambda\partial_a\circ, \nabla_{\partial b} + \lambda\partial_b\circ]\partial_d = \big\{\nabla_{\partial_a}(\lambda\partial_b\circ) + \lambda\partial_a \circ (\nabla_{\partial_b}) + \lambda^2\partial_a \circ (\partial_b\circ) - \nabla_{\partial_b}(\lambda\partial_a\circ) - \lambda\partial_b \circ (\nabla_{\partial_a}) - \lambda^2\partial_b \circ (\partial_a\circ)\big\}\, \partial_d, \quad (1.25) $$

we see that $R_1 = 0$ if and only if

$$\partial_a c^e_{bd} = \partial_b c^e_{ad},$$

thus, by the Poincaré lemma there exists an F such that

$$c_{abd}(t) = \frac{\partial^3 F(t)}{\partial t^a \partial t^b \partial t^d}. \tag{1.26}$$

From the λ^2-terms in (1.25) we see that $R_2 = 0$ if and only if

$$\partial_a \circ (\partial_b \circ \partial_d) = \partial_b \circ (\partial_a \circ \partial_d),$$

in other words, if $\circ$ is associative and thus, with (1.26), we have the WDVV equations.$\square$

1.4 Semi-simple Frobenius manifolds

We will be primarily concerned with *semi-simple* Frobenius manifolds throughout this thesis. This means that the Frobenius algebra at a generic point t on the manifold is semi-simple.

Definition 1.27 *(Semi-simple algebra) An n-dimensional algebra $\mathcal{A}$ with multiplication $\circ$ is said to be semisimple if it is isomorphic, as a $\mathbb{C}$-algebra, to $\mathbb{C}^n$ with component-wise multiplication. This means that a basis, $e_1, \ldots, e_n$ of the algebra can be chosen with multiplication given by*

$$e_i \circ e_j = \delta^j_i e_i.$$

The coordinates on a Frobenius manifold on whose tangent space this multiplication holds, $u^1, \ldots, u^n$ are called *canonical*.

Theorem 1.28 *[26] Canonical coordinates on a semi-simple Frobenius manifold $(\mathcal{M}, \eta, \circ)$ always exist.*

Proof. Since $\mathcal{M}$ is Frobenius we have, by Theorem (1.24), that

$$[{}^{\lambda}\nabla_{e_i}, {}^{\lambda}\nabla_{e_j}](e_k) = {}^{\lambda}\nabla_{[e_i,e_j]}(e_k). \tag{1.29}$$

Also since $\mathcal{M}$ is associative and assuming η is flat (we will derive the conditions for this to be true) we need only consider the terms linear in λ in this equation. Define the Riemannian connection coefficients of η for the basis e_k:

$$\nabla_{e_i}(e_k) = \sum_q \mathit{\Gamma}^q_{ik} e_q. \tag{1.30}$$

The left side of (1.29) yields the λ-terms

$$\lambda e_i \circ \nabla_{e_j}(e_k) + \lambda \nabla_{e_i}(e_j \circ e_k) \quad - \quad (i \leftrightarrow j),$$

or

$$\lambda e_i \circ \sum_q \mathit{\Gamma}^q_{jk} e_q + \lambda \sum_q \delta^k_j \mathit{\Gamma}^q_{ik} e_q \quad - \quad (i \leftrightarrow j),$$

or

$$\lambda \sum_q (\delta^i_q \mathit{\Gamma}^q_{jk} + \delta^k_j \mathit{\Gamma}^q_{ik} - \delta^q_j \mathit{\Gamma}^q_{ik} - \delta^k_i \mathit{\Gamma}^q_{jk}) e_q. \tag{1.31}$$

Now introduce the obstructions to the commutativity of the e_i, the f^q_{ij} by

$$[e_i, e_j] = \sum_q f^q_{ij} e_q.$$

The right of (1.29) is

$${}^{\lambda}\nabla_{[e_i,e_j]}(e_k) = \nabla_{[e_i,e_j]}(e_k) + \lambda [e_i, e_j] \circ e_k,$$

and so yields λ-terms

$$\lambda[e_i, e_j] \circ e_k = \lambda \sum_q f_{ij}^q e_q \circ e_k$$

and this must equal 0 since the coefficient of e_k in (1.31) vanishes. Hence the $f_{ij}^q = 0$, the e_i pairwise commute and hence canonical coordinates exist ($e_i = \partial_{u^i}$). □

In fact the left side of (1.29) vanishes. We will use this fact to help us show when the metric in canonical coordinates is flat. First let us compute the metric in canonical coordinates. We have

$$\eta(\partial_i, \partial_j) = \eta(\partial_i \circ \partial_i, \partial_j) = \eta(\partial_i, \partial_i \circ \partial_j) = \eta(\partial_i, \partial_i \delta_i^j) = \delta_i^j \eta(\partial_i, \partial_i).$$

So the metric is diagonal. We will denote $\eta(\partial_i, \partial_i)$ by H_i^2 and introduce the *rotation coefficients*

$$\gamma_{ij} := \frac{\partial_i H_j}{H_i}.$$

Definition 1.32 *(Egoroff metric) A diagonal metric*

$$\eta = \sum \eta_{ii} (du^i)^2,$$

is said to be Egoroff if there exists a function $\Phi(u^1, \ldots, u^n)$ called the metric potential such that

$$\eta_{ii} = \frac{\partial \Phi}{\partial u^i}.$$

Lemma 1.33 *The rotation coefficients, γ_{ij}, are symmetric in i and j if and only if the metric is Egoroff.*

Proof. For an Egoroff metric

$$\gamma_{ij} = \frac{\partial_j \sqrt{\partial_i \Phi}}{\sqrt{\partial_j \Phi}} = \frac{1}{2} \frac{\partial_i \partial_j \Phi}{\sqrt{\partial_i \Phi} \sqrt{\partial_j \Phi}},$$

conversely, for γ_{ij} symmetric in i and j we have

$$\frac{\partial_j\sqrt{\eta_{ii}}}{\sqrt{\eta_{jj}}} = \frac{\partial_i\sqrt{\eta_{jj}}}{\sqrt{\eta_{ii}}},$$

$$\frac{1}{2}\frac{\partial_j\eta_{ii}}{\sqrt{\eta_{jj}\eta_{ii}}} = \frac{1}{2}\frac{\partial_i\eta_{jj}}{\sqrt{\eta_{jj}\eta_{ii}}},$$

$$\partial_j\eta_{ii} = \partial_i\eta_{jj},$$

as required. □

We will find the conditions on Φ such that the metric is flat in the next subsection but first we state and prove

Theorem 1.34 *Flatness of the metric on a semi-simple Frobenius manifold implies that it is Egoroff.*

Proof. For any metric $\eta = \sum \eta_{ij} du^i du^j$ the coefficients of the Levi-Civita connection are given by

$$\Gamma^k_{ij} = \sum_l \Gamma_{ijl}\eta^{lk},$$

where

$$\Gamma_{ijk} = \frac{1}{2}(\partial_i\eta_{jk} - \partial_k\eta_{ij} + \partial_j\eta_{ki}).$$

So the non-zero connection coefficients of $\eta = \sum H_i^2(du^i)^2$ are $(i \neq j)$:

$$\Gamma^i_{ii} = \frac{\partial_i H_i}{H_i} = \gamma_{ii}, \quad \Gamma^j_{ii} = -\frac{H_i}{H_j^2}\partial_j H_i = -\frac{H_i}{H_j}\gamma_{ji}, \quad \text{and} \quad \Gamma^i_{ij} = \Gamma^i_{ji} = \frac{\partial_j H_i}{H_i} = \frac{H_j}{H_i}\gamma_{ji}.$$

Hence by (1.30) we have

$$\nabla_i(\partial_i) = (\gamma_{ii})\partial_i - \left(\sum_{l\neq i}\frac{H_i}{H_l}\gamma_{li}\right)\partial_l, \tag{1.35}$$

and

$$\nabla_i(\partial_j) = \left(\frac{H_j}{H_i}\gamma_{ji}\right)\partial_i + \left(\frac{H_i}{H_j}\gamma_{ij}\right)\partial_j. \tag{1.36}$$

Now, the vanishing of the λ-terms on the left of (1.29) means that

$$\partial_i \circ \nabla_{\partial_j}(\partial_k) + \nabla_{\partial_i}(\partial_j \circ \partial_k) = \partial_j \circ \nabla_{\partial_i}(\partial_k) + \nabla_{\partial_j}(\partial_i \circ \partial_k).$$

This equation is trivially satisfied for $i = j$ and $i \neq j \neq k \neq i$ but on substituting (1.35) and (1.36) for the case $i \neq j = k$ gives

$$\begin{aligned}\partial_i \circ \left\{(\gamma_{jj})\partial_j - \left(\sum_{l\neq i}\frac{H_j}{H_l}\gamma_{lj}\right)\partial_l\right\}& \\ + \left(\frac{H_j}{H_i}\gamma_{ji}\right)\partial_i + \left(\frac{H_i}{H_j}\gamma_{ij}\right)\partial_j &= \partial_j \circ \left\{\left(\frac{H_j}{H_i}\gamma_{ji}\right)\partial_i + \left(\frac{H_i}{H_j}\gamma_{ij}\right)\partial_j\right\},\end{aligned}$$

$$\left(-\frac{H_j}{H_i}\gamma_{ij} + \frac{H_j}{H_i}\gamma_{ji}\right)\partial_i + \left(\frac{H_i}{H_j}\gamma_{ij}\right)\partial_j = \left(\frac{H_i}{H_j}\gamma_{ij}\right)\partial_j,$$

$$\gamma_{ij} = \gamma_{ji},$$

the result follows from Lemma (1.33) (the case of $j \neq i = k$ is identical). □

Proposition 1.37 *In canonical coordinates the unity field is*

$$e = \sum_i \partial_i, \tag{1.38}$$

and the Euler field is

$$E = \sum_i u^i \partial_i.$$

Proof. The first statement is easily seen from

$$\left(\sum_i \partial_i\right) \circ \partial_i = \partial_i \circ \partial_i = \partial_i.$$

The second statement follows from the requirements that $\mathcal{L}_E \circ = \circ$ and $\mathcal{L}_E \eta = D\eta$. The first of these is equivalent to

$$[E, X \circ Y] - [E, X] \circ Y - X \circ [E, Y] = X \circ Y,$$

for all vector fields X, Y. Evaluating this with $E = \sum E^i \partial_i$, $X = \partial_k$ and $Y = \partial_l$ gives

$$\partial_k E^i = \delta^i_k, \quad \text{so} \quad E^i = u^i + c^i,$$

where the c^i are constants. Similarly (1.11) with $X = \partial_i$ and $Y = \partial_j$ is equivalent to

$$E \partial_i \Phi = (D - 2) \partial_i \Phi.$$

Since $E \partial_i = \partial_i E - \partial_i$ this can be integrated to give

$$E \Phi = (D - 1)\Phi + \text{const},$$

and so the canonical coordinates can be normalised to give the result. $\square$

1.4.1 The Darboux-Egoroff equations

It remains to find the conditions the rotation coefficients must satisfy to ensure the metric is flat. These are called the Darboux-Egoroff equations. We will now derive them. Substituting (1.35) and (1.36) into

$$\nabla_i \nabla_j (\partial_k) = \nabla_j \nabla_i (\partial_k), \tag{1.39}$$

gives, for $i \neq j = k$

$$\nabla_i \left\{ \gamma_{jj}\partial_j - \left(\sum_{l\neq j} \frac{H_j}{H_l}\gamma_{lj} \right) \partial_l \right\} = \nabla_j \left\{ \left(\frac{H_j}{H_i}\gamma_{ji} \right) \partial_i + \left(\frac{H_i}{H_j}\gamma_{ij} \right) \partial_j \right\},$$

$$\begin{aligned}
&\partial_i(\gamma_{jj})\partial_j + \gamma_{jj} \left\{ \left(\frac{H_j}{H_i}\gamma_{ji} \right) \partial_i + \left(\frac{H_i}{H_j}\gamma_{ij} \right) \partial_j \right\} - \partial_i \left(\sum_{l\neq j} \frac{H_j}{H_l}\gamma_{lj} \right) \partial_l \\
&- \left(\frac{H_j}{H_i}\gamma_{ij} \right) \left\{ (\gamma_{ii})\partial_i - \left(\sum_{m\neq i} \frac{H_i}{H_m}\gamma_{mi} \right) \partial_m \right\} - \sum_{p\neq i,j} \left(\frac{H_j}{H_p}\gamma_{pj} \left\{ \left[\frac{H_p}{H_i}\gamma_{pi} \right] \partial_i + \left[\frac{H_i}{H_p}\gamma_{ip} \right] \partial_p \right\} \right) \\
&\qquad = \partial_j \left(\frac{H_j}{H_i}\gamma_{ji} \right) \partial_i + \left(\frac{H_j}{H_i}\gamma_{ji} \right) \left\{ \left(\frac{H_i}{H_j}\gamma_{ij} \right) \partial_j + \left(\frac{H_j}{H_i}\gamma_{ji} \right) \partial_i \right\} \\
&\qquad\quad + \partial_j \left(\frac{H_i}{H_j}\gamma_{ij} \right) \partial_j + \left(\frac{H_i}{H_j}\gamma_{ij} \right) \left\{ (\gamma_{jj})\, \partial_j - \left(\sum_{q\neq j} \frac{H_j}{H_q}\gamma_{qj} \right) \partial_q \right\}. \qquad (1.40)
\end{aligned}$$

Equating the coefficients of ∂_i,

$$\gamma_{jj}\frac{H_j}{H_i}\gamma_{ji} - \partial_i \left(\frac{H_j}{H_i}\gamma_{ij} \right) - \frac{H_j}{H_i}\gamma_{ij}\gamma_{ii} - \sum_{m\neq i,j} \left(\frac{H_j}{H_i}\gamma_{mj}\gamma_{mi} \right) = \partial_j \left(\frac{H_j}{H_i}\gamma_{ji} \right) + \left(\frac{H_j^2}{H_i^2}\gamma_{ji}^2 \right) - \gamma_{ij}^2,$$

performing the differentiation,

$$\begin{aligned}
&\frac{H_j}{H_i}\gamma_{jj}\gamma_{ji} - \frac{H_j}{H_i}\partial_i(\gamma_{ij}) - \gamma_{ij}\frac{H_i\partial_i H_j - H_j\partial_i H_i}{H_i^2} - \frac{H_j}{H_i}\gamma_{ij}\gamma_{ii} \\
&\qquad - \sum_{m\neq i,j} \left(\frac{H_j}{H_i}\gamma_{mj}\gamma_{mi} \right) = \frac{H_j}{H_i}\partial_i(\gamma_{ji}) + \gamma_{ji}\frac{H_i\partial_j H_j - H_j\partial_j H_i}{H_i^2} + \frac{H_j^2}{H_i^2}\gamma_{ji}^2 - \gamma_{ij}^2,
\end{aligned}$$

multiplying by $\frac{H_i}{H_j}$ and rearranging,

$$\begin{aligned}
&\partial_i(\gamma_{ij}) + \partial_j(\gamma_{ji}) + \sum_{m\neq i,j} (\gamma_{mj}\gamma_{mi}) = \gamma_{jj}\gamma_{ji} \\
&\qquad - \gamma_{ij} \left(\frac{H_i}{H_j}\gamma_{ij} - \gamma_{ii} \right) - \gamma_{ij}\gamma_{ii} - \gamma_{ji} \left(\gamma_{jj} - \frac{H_j}{H_i}\gamma_{ji} \right) - \frac{H_j}{H_i}\gamma_{ji}^2 + \frac{H_i}{H_j}\gamma_{ij}^2,
\end{aligned}$$

$$\partial_i(\gamma_{ij}) + \partial_j(\gamma_{ji}) + \sum_{m\neq i,j}(\gamma_{mj}\gamma_{mi}) = 0. \tag{1.41}$$

The coefficient of ∂_j in (1.40) is

$$\begin{aligned}
&\partial_i(\gamma_{jj}) - \partial_j\left(\frac{H_i}{H_j}\gamma_{ij}\right)\\
&\equiv \frac{\partial_i\partial_j H_j}{H_j} - \frac{\partial_i H_j \partial_j H_j}{H_j^2} - \gamma_{ij}\left(\frac{H_j\partial_j H_i - H_i\partial_j H_j}{H_j^2}\right) - \frac{H_i}{H_j}\left(\frac{H_i\partial_j\partial_i H_j - \partial_i H_j \partial_j H_i}{H_i^2}\right),\\
&\qquad = \frac{\partial_i\partial_j H_j}{H_j} - \frac{H_i}{H_j}\gamma_{ij}\gamma_{jj} - \gamma_{ij}\left(\gamma_{ji} - \frac{H_i}{H_j}\gamma_{jj}\right) - \frac{\partial_j\partial_i H_j}{H_j} + \gamma_{ij}\gamma_{ji} = 0.
\end{aligned}$$

Equating coefficients of ∂_l, $(l \neq i, j)$ in (1.40) gives

$$-\partial_i\left(\frac{H_j}{H_l}\gamma_{lj}\right) + \frac{H_j}{H_l}\gamma_{ij}\gamma_{li} - \frac{H_i H_j}{H_l^2}\gamma_{il}\gamma_{lj} = -\frac{H_i}{H_l}\gamma_{ij}\gamma_{lj},$$

$$-\gamma_{lj}\left(\frac{H_i}{H_l}\gamma_{ij} - \frac{H_j H_i}{H_l^2}\gamma_{il}\right) - \frac{H_j}{H_l}\partial_i(\gamma_{lj}) + \frac{H_j}{H_l}\gamma_{ij}\gamma_{li} - \frac{H_i H_j}{H_l^2}\gamma_{il}\gamma_{lj} = -\frac{H_i}{H_l}\gamma_{ij}\gamma_{lj},$$

$$\partial_i(\gamma_{lj}) = \gamma_{ij}\gamma_{li}. \tag{1.42}$$

Similar calculations show that (1.39) is satisfied for $i \neq j \neq k \neq i$ as long as (1.41) and (1.42) hold (there is nothing to prove for $i = j$).

Since the rotation coefficients are symmetric we can substitute (1.42) into (1.41) to obtain

$$\sum_m \partial_m \gamma_{ij} = 0,$$

or

$$e(\gamma_{ij}) = 0. \tag{1.43}$$

Corollary 1.44 *The rotation coefficients are quasihomogeneous with degree -1.*

Proof. Recall from Proposition (1.37) we had $\sum_k u^k \partial_i\partial_k\Phi = (D-2)\Phi_i$. Applying ∂_j gives

$\sum_k u^k \partial_j \partial_i \partial_k \Phi = (D-3)\Phi_{ij}$.These can be rewriiten as

$$\sum_k u^k \partial_k H_i^2 = (D-2)H_i^2,$$

or

$$\sum_k u^k \partial_k H_i = \frac{D-2}{2} H_i,$$

and

$$\sum_k u^k \partial_j \partial_k H_i^2 = (D-3)\partial_j H_i^2,$$

$$\sum_k u^k (2\partial_j H_i \partial_k H_i + 2H_i \partial_j \partial_k H_i) = (D-3)\partial_j H_i^2,$$

$$\frac{\partial_j H_i}{H_i} \sum_k u^k \partial_k H_i + \sum_k u^k \partial_j \partial_k H_i = (D-3)\partial_j H_i,$$

or

$$\sum_k u^k \partial_j \partial_k H_i = \left(\frac{D-4}{2}\right) \partial_j H_i.$$

Using these results we see that

$$E(\gamma_{ij}) = \sum_k u^k \partial_k (\gamma_{ij}) = \sum_k u^k \partial_k \left(\frac{\partial_i H_j}{H_i}\right),$$

$$= \sum_k u^k \left(\frac{H_i \partial_k \partial_i H_j - \partial_i H_j \partial_k H_i}{H_i^2}\right),$$

$$= \frac{\partial_j H_i}{H_i}\left(\frac{D-4}{2}\right) - \frac{\partial_j H_i}{H_i}\left(\frac{D-2}{2}\right),$$

$$E(\gamma_{ij}) = -\gamma_{ij}. \tag{1.45}$$

□

(1.42), (1.43) and (1.45) are the Darboux-Egoroff equations and their satisfaction is equivalent to the existence of a Frobenius structure.

1.5 The intersection form

Another metric exists on a Frobenius manifold called the *intersection form.* It is defined on the cotangent bundle T^*M by

$$g^*(dt^i, dt^j) := i_E(dt^i \circ dt^j),$$

where i_E means contraction of a 1-form with the Euler vector field, so

$$(dt^i, dt^j) = E^k c_k^{ij}.$$

Let us denote the intersection form by g^{ij}. Its matrix inverse (where it exists), g_{ij}, defines a metric on the *tangent* bundle, related to the original metric via

$$g(\partial_i, \partial_j) = \eta\left(E^{-1} \circ \partial_i, \partial_j\right), \tag{1.46}$$

and so is not defined where E is not invertible. (We will henceforth use the shorthand $g(\cdot,\cdot) := (\cdot,\cdot)$ and $\eta(\cdot,\cdot) :=< \cdot,\cdot >$).

To see this note that

$$(\partial_i, \partial_j) = c_{ij}^s \left\langle E^{-1}, \partial_s\right\rangle = c_{ij}^s \left(E^{-1}\right)^p < \partial_p, \partial_s >= c_{ij}^s \left(E^{-1}\right)^p \eta_{ps}$$

hence

$$\begin{aligned} g^{ia} g_{aj} &= E^k c_k^{ia} c_{aj}^s \left(E^{-1}\right)^p \eta_{ps}, \\ &= c_k^{ia} c_{ajp} E^k \left(E^{-1}\right)^p, \\ &= c_{aj}^i c_{kp}^a E^k \left(E^{-1}\right)^p, \end{aligned}$$

by associativity

$$= c^i_{aj} \left(\partial_k \circ \partial_p\right)^a E^k \left(E^{-1}\right)^p ,$$

$$= c^i_{aj} \left(E^k \partial_k \circ \left(E^{-1}\right)^p \partial_p\right)^a ,$$

$$= c^i_{aj} e^a = \left(e^a \partial_a \circ \partial_j\right)^i = c^i_{1j} = c_{1jb} \eta^{bi} = \eta_{jb} \eta^{bi} = \delta^i_j .$$

Definition 1.47 *(Flat pencil of metrics) Consider a manifold supplied with two non-proportional metrics on its cotangent bundle T^*M given in a coordinate system by their components g_1^{ij} and g_2^{ij} and with Christoffel symbols of their contravariant Levi-Civita connections $\mathit{\Gamma}^{ij}_{1k}$ and $\mathit{\Gamma}^{ij}_{2k}$. The two metrics are said to form a* flat pencil *if*

- *the metric*

$$g^{ij} := g_1^{ij} + \lambda g_2^{ij},$$

  is flat $\forall \lambda$ and

- *the contravariant Levi-Civita connection of this metric is given by*

$$\mathit{\Gamma}^{ij}_k = \mathit{\Gamma}^{ij}_{1k} + \lambda \mathit{\Gamma}^{ij}_{2k}.$$

We will show that $g^{\alpha\beta}$ and $\eta^{\alpha\beta}$ form a flat pencil but first let us recall some basic facts of differential geometry. The Levi-Civita connection of a metric $g^{\alpha\beta}$ is uniquely determined by

$$\nabla_k g^{ij} := \partial_k g^{ij} + \mathit{\Gamma}^i_{ks} g^{sj} + \mathit{\Gamma}^j_{ks} g^{is} = 0,$$

and

$$\mathit{\Gamma}^k_{ij} = \mathit{\Gamma}^k_{ji},$$

or, in terms of the contravariant components $\Gamma^{ij}_k := -g^{is}\Gamma^j_{sk}$

$$\partial_k g^{ij} = \Gamma^{ij}_k + \Gamma^{ji}_k, \tag{1.48}$$

and

$$g^{is}\Gamma^{jk}_s = g^{js}\Gamma^{ik}_s. \tag{1.49}$$

The Riemann curvature tensor in terms of the contravariant components is [14]

$$R^{ijk}_l := g^{is}g^{jt}R^k_{slt} = g^{is}\left(\partial_s\Gamma^{jk}_l - \partial_l\Gamma^{jk}_s\right) + \Gamma^{ij}_s\Gamma^{sk}_l - \Gamma^{ik}_s\Gamma^{sj}_l.$$

Lemma 1.50 *([13], Appendix D) If, for a flat metric in a coordinate system $x^1, \ldots, x^n$ both the components $g^{ij}(x)$ of the metric and $\Gamma^{ij}_k(x)$ of the corresponding contravariant Levi-Civita connection depend linearly on the coordinate x^1 then the metrics*

$$g^{ij}_1 := g^{ij} \quad \text{and} \quad g^{ij}_2 := \partial_1 g^{ij},$$

form a flat pencil, provided $\det\left(g^{ij}_2\right) \neq 0$. The corresponding Levi-Civita connections are

$$\Gamma^{ij}_{1k} := \Gamma^{ij}_k, \quad \Gamma^{ij}_{2k} := \partial_1\Gamma^{ij}_k.$$

Proof. The equations (1.48), (1.49) and the vanishing of the Riemann curvature tensor have constant coefficients. So the transformation

$$g^{ij}(x^1, \ldots, x^n) \mapsto g^{ij}(x^1 + \lambda, \ldots, x^n), \quad \Gamma^{ij}_k(x^1, \ldots, x^n) \mapsto \Gamma^{ij}_k(x^1 + \lambda, \ldots, x^n),$$

for an arbitrary λ maps the solutions of these equations to themselves. By the assumption we have

$$g^{ij}(x^1+\lambda,\ldots,x^n)=g_1^{ij}(x)+\lambda g_2^{ij}(x),\quad \mathit{\Gamma}_k^{ij}(x^1+\lambda,\ldots,x^n)=\mathit{\Gamma}_{1k}^{ij}+\lambda\mathit{\Gamma}_{2k}^{ij}(x).\qquad\square$$

Theorem 1.51 *$\eta^{\alpha\beta}$ and $g^{\alpha\beta}$ form a flat pencil of metrics*

Proof. We proceed as in [29]. This will be proved by showing that

$$P=g+\lambda\eta$$

satisfies the criteria of Lemma (1.50).

Firstly, to see that P is non-degenerate recall that for Euler vector fields with $d_1=1$ we can chose flat coordinates in such a way that

$$g^{\alpha\beta}(t)=E^1c_1^{\alpha\beta}+\sum_{i=2}^{n}E^ic_i^{\alpha\beta}=t^1\eta^{\alpha\beta}+\tilde{g}^{\alpha\beta}(t^2,\ldots,t^n).\tag{1.52}$$

So

$$g^{\alpha\beta}(t)+\lambda\eta^{\alpha\beta}=\eta^{\alpha\beta}(t^1+\lambda)+\tilde{g}^{\alpha\beta}(t^2,\ldots,t^n)\tag{1.53}$$

is non-degenerate $\forall\lambda$. Equation (1.52) also shows that g depends linearly on t^1. We also have that the contravariant Christoffel symbols of the intersection form are (see [15])

$$\frac{\partial}{\partial t^1}{}^g\mathit{\Gamma}_\gamma^{\alpha\beta}=\frac{\partial}{\partial t^1}\left\{\left(\frac{d-1}{2}\delta_\gamma^\varepsilon+(\nabla E)_\gamma^\varepsilon\right)c_\gamma^{\alpha\beta}\right\}=0,$$

since $\partial_1 c_\gamma^{\alpha\beta}=0$.

Finally we need to show that the curvature of P is zero. By (1.48) we have, in the flat coordinates of η

$$\frac{\partial}{\partial t^\gamma}P^{\alpha\beta}=\frac{\partial}{\partial t^\gamma}\left(g^{\alpha\beta}+\lambda\eta^{\alpha\beta}\right)=\frac{\partial}{\partial t^\gamma}g^{\alpha\beta}={}^g\mathit{\Gamma}_\gamma^{\alpha\beta}+{}^g\mathit{\Gamma}_\gamma^{\beta\alpha}.$$

(1.49) for P reads

$$(g^{\alpha\varepsilon}(t) + \lambda\eta^{\alpha\varepsilon})^P\Gamma^{\beta\kappa}_{\varepsilon} = (g^{\beta\varepsilon}(t) + \lambda\eta^{\beta\varepsilon})^P\Gamma^{\alpha\kappa}_{\varepsilon},$$

or, on using (1.53)

$$(\eta^{\alpha\varepsilon}(t^1+\lambda) + \tilde{g}^{\alpha\varepsilon}(t^2,\ldots,t^n))^P\Gamma^{\beta\kappa}_{\varepsilon} = (\eta^{\beta\varepsilon}(t^1+\lambda) + \tilde{g}^{\alpha\varepsilon}(t^2,\ldots,t^n))^P\Gamma^{\alpha\kappa}_{\varepsilon},$$

equating coefficients of t^1,

$$(\eta^{\alpha\varepsilon})^P\Gamma^{\beta\kappa}_{\varepsilon} = (\eta^{\beta\varepsilon})^P\Gamma^{\alpha\kappa}_{\varepsilon} \quad\Longrightarrow\quad (g^{\alpha\varepsilon})^P\Gamma^{\beta\kappa}_{\varepsilon} = (g^{\beta\varepsilon})^P\Gamma^{\alpha\kappa}_{\varepsilon}.$$

Hence

$${}^P\Gamma^{\alpha\beta}_{\gamma} = {}^g\Gamma^{\alpha\beta}_{\gamma} \quad\Longrightarrow\quad {}^PR^{\alpha}_{\beta\gamma\delta} = {}^gR^{\alpha}_{\beta\gamma\delta},$$

and ${}^gR^{\alpha}_{\beta\gamma\delta} = 0$. □

We can, with the additional conditions of quasihomogeneity and regularity, go the other way and show that a flat pencil of metrics gives rise to a Frobenius structure. We refer the reader to [14] for a comprehensive discussion.

1.6 The Landau-Ginsburg superpotential

For *any* semi-simple Frobenius manifold, M, one may construct a function of one variable (which may depend on $t = (t^1,\ldots,t^n)$), $\lambda(z;t)$, called the *Landau-Ginsburg superpotential*, in terms of which the Frobenius structure may be expressed thus [13]:

Theorem 1.54

$$
\begin{aligned}
< \partial_\alpha, \partial_\beta > &= -\sum \underset{d\lambda=0}{res} \left\{ \frac{\partial_\alpha \lambda(z) \partial_\beta \lambda(z)}{\lambda'(z)} \omega \right\}, \\
c(\partial_\alpha, \partial_\beta, \partial_\gamma) &= -\sum \underset{d\lambda=0}{res} \left\{ \frac{\partial_\alpha \lambda(z) \partial_\beta \lambda(z) \partial_\gamma \lambda(z)}{\lambda'(z)} \omega \right\}, \\
(\partial_\alpha, \partial_\beta) &= -\sum \underset{d\lambda=0}{res} \left\{ \frac{\partial_\alpha \log \lambda(z) \partial_\beta \log \lambda(z)}{(\log \lambda)'(z)} \omega \right\}, \\
c^*(\partial_\alpha, \partial_\beta, \partial_\gamma) &= -\sum \underset{d\lambda=0}{res} \left\{ \frac{\partial_\alpha \log \lambda(z) \partial_\beta \log \lambda(z) \partial_\gamma \log \lambda(z)}{(\log \lambda)'(z)} \omega \right\}.
\end{aligned}
$$

where $\partial_\alpha, \partial_\beta$ *and* ∂_γ *are arbitrary tangent vectors on* M *and* ω *is the* primary differential, *different choices of which lead to Frobenius manifolds related by* Legendre transformations *(see Section 5.4).*

The $c^*(\partial_\alpha, \partial_\beta, \partial_\gamma)$ refers to *almost dual* Frobenius manifolds whose multiplication is defined in terms of the intersection form (and which will be the subject of Chapter 3).

For the full details of why the above formulae hold we refer the reader to [13] and outline the reasoning here. The critical values of $\lambda(z, t)$ are the canonical coordinates on the semi-simple Frobenius manifold:

$$
\begin{aligned}
\lambda(q^i, t) &= u^i, \\
\left. \frac{d\lambda}{dz} \right|_{z=q^i} &= 0.
\end{aligned}
$$

Near a critical point λ must have an expansion

$$
\lambda = u_i - \frac{(z - q_i)^2}{2\eta_{ii}(u)} + O(z - q_i)^3.
$$

If we now consider

$$
-\sum \underset{d\lambda=0}{\mathrm{res}} \frac{\partial_i \lambda \partial_j \lambda}{d\lambda} dz,
$$

(where the choice of primary differential $\omega = dz$ has been made) we can see that the points at which $d\lambda = 0$ are the q_i. Also, at each $z = q_k$ we have

$$\partial_i \lambda \big|_{z=q_k} = \delta_{ik}.$$

So the residues are zero except when $i = j = k$. Thus

$$\begin{aligned} < \partial_i, \partial_i > &= -\operatorname*{res}_{z=q_i} \frac{1}{\lambda'}, \\ &= \operatorname*{res}_{z=q_i} \frac{1}{\frac{2(z-q_i)}{2\eta_{ii}} + O(z-q_i)^2}, \\ &= -\operatorname*{res}_{z=q_i} \frac{1}{z-q_i} \frac{1}{\frac{1}{\eta_{ii}} + O(z-q_i)}, \end{aligned}$$

$$\begin{aligned} &= \left. \frac{1}{\frac{1}{\eta_{ii}} + O(z-q_i)} \right|_{z=q_i}, \\ &= \eta_{ii}. \end{aligned}$$

So

$$< \partial_i, \partial_j >= \delta_{ij} \eta_{ii},$$

as required. The other three formulae are found similarly.

We are now in a position to present an important and well-known construction due to Dubrovin that gives Frobenius manifolds in all dimensions. It is intimately related to the family of Coxeter groups A_n, the details of which, along with all other finite Coxeter groups, will be explored in the next chapter.

Example 1.55 *Consider the n-dimensional manifold, M, identified with the space of polynomials*

$$\lambda(z) = z^{n+1} + a_1 z^{n-1} + a_2 z^{n-2} + \ldots + a_n, \tag{1.56}$$

with coordinate functions $a_1, \ldots, a_n$ *such that the derivative* $\dfrac{d\lambda}{dz}$ *has no repeated roots. The tangent plane to* M *at a point* $a = (a_1, \ldots, a_n)$ *consists of all polynomials of degree strictly less than* n*.*

M *is a semi-simple Frobenius manifold with canonical coordinates,* u_i*, given by the critical points of* $\lambda(z)$*:*

$$u_i = \lambda(\alpha_i) \quad where \quad \lambda'(\alpha_i) = 0, \quad for \quad i = 1, \ldots, n.$$

To see this first observe

$$\delta_i^j = \frac{\partial u_i}{\partial u_j} = \frac{\partial \lambda}{\partial u_j}(\alpha_i) + \lambda'(\alpha_i)\frac{\partial \alpha_i}{\partial u_j} = \frac{\partial \lambda}{\partial u_j}(\alpha_i) \quad since \quad \lambda'(\alpha_i) = 0.$$

This tells us that we must have

$$\frac{\partial \lambda}{\partial u_j} = \prod_{r \neq j} \frac{z - \alpha_r}{\alpha_j - \alpha_r}, \tag{1.57}$$

since both sides have the same degree, $n-1$*, and agree at* n*-points,* $\alpha_1, \ldots, \alpha_n$*.*

If we now define a metric (with reference to the start of this section) by

$$\eta_{ij} = -\underset{d\lambda=0}{res} \frac{\partial \lambda}{\partial u_i}\frac{\partial \lambda}{\partial u_j}\frac{dz}{\lambda'},$$

we see that $\dfrac{\partial \lambda}{\partial u_i}\dfrac{\partial \lambda}{\partial u_j}$ *is divsible by* $\lambda'(z) = (n+1)(z-\alpha_1)\ldots(z-\alpha_n)$ *for* $i \neq j$ *and so the* *residue vanishes. For $i = j$ we have*

$$\begin{aligned}
\eta_{ii} &= -\operatorname*{res}_{z=\alpha_i} \frac{\prod_{s\neq i} \frac{(z-\alpha_s)^2}{(\alpha_i-\alpha_s)^2}}{(n+1)\prod_l (z-\alpha_l)} dz, \\
&= -\operatorname*{res}_{z=\alpha_i} \frac{\prod_{s\neq i} \frac{(z-\alpha_s)}{(\alpha_i-\alpha_s)^2}}{(n+1)(z-\alpha_l)} dz, \\
&= -\frac{1}{n+1} \prod_{s\neq i} \frac{(z-\alpha_s)}{(\alpha_i-\alpha_s)^2}\bigg|_{z=\alpha_1}, \\
&= -\frac{1}{n+1} \prod_{s\neq i} \frac{1}{\alpha_i-\alpha_s},
\end{aligned}$$

and since

$$\lambda''(z) = (n+1)\sum_t \prod_{m\neq t} (z-\alpha_m),$$

we have

$$\eta_{ii} = -\frac{1}{\lambda''(\alpha_i)}.$$

To see that the metric is Egoroff consider an alternative, to (1.57), expression

$$\frac{\partial\lambda}{\partial u_j} = \sum_{r=1}^{n} \frac{\partial a_r}{\partial u_j} z^{n-r},$$

and equate coefficients of z^{n-1}:

$$\frac{\partial a_1}{\partial u_j} = \frac{1}{\prod\limits_{r\neq j} (\alpha_j - \alpha_r)}$$

or

$$\frac{\partial a_1}{\partial u_j} = -(n+1)\eta_{jj},$$

so the metric is Egoroff with potential

$$\Phi = -\frac{a_1}{n+1}.$$

Recall (1.38) that the unity field in canonical coordinates is $e = \sum_i \partial_i$ *so summing*

$$\delta_i^j = \sum_{r=1}^{n} \frac{\partial a_r}{\partial u^j} z^{n-r} \Bigg|_{z=\alpha_i},$$

tells us that $\sum_{r=1}^{n} e(a_r) z^{n-r}$ *is a polynomial of degree* $(n-1)$ *and has value 1 at* $z = \alpha_1, \ldots, \alpha_n$ *and so must be identically 1. So we have*

$$e = \frac{\partial}{\partial a_n}.$$

A similar calculation [26] gives the Euler vector field as

$$E = \frac{1}{n+1} \sum_r (r+1) a_r \frac{\partial}{\partial a_r}.$$

It remains to show that η *is a flat metric: introducing a new function* $\lambda(z) = w^{n+1}$, *inverting gives a Puiseaux series as* $z \to \infty$

$$z(w,t) = w + \frac{t_1}{w} + \frac{t_2}{w^2} + \ldots + \frac{t_n}{w^n} + \ldots,$$

and so

$$\eta\left(\frac{\partial\lambda}{\partial t_i}, \frac{\partial\lambda}{\partial t_j}\right) = -\mathop{res}_{z=\infty} \left(\frac{\partial\lambda}{\partial t_i}, \frac{\partial\lambda}{\partial t_j}\right) \frac{dz}{\lambda'(z)} = -\mathop{res}_{z=\infty} \frac{\lambda'(z)^2 dz}{w^{i+j}\lambda'(z)} = -\mathop{res}_{w=\infty} (n+1) w^{n-i-j} dw,$$

which gives constant coefficients [24].

We will return to this approach of calculating prepotentials from superpotentials in Chapter 3 when we consider almost-dual-like Frobenius manifolds and then again in Chapter 5 when we consider extended affine Weyl orbit spaces.

Chapter 2

Finite Coxeter Groups and Polynomial Frobenius Manifolds

This chapter is concerned with the class of solutions to the WDVV equations with the assumption that there exists a marked variable and which are quasihomogeneous and which have polynomial prepotentials. That these solutions are related to the finite Coxeter groups was first noted by V. I. Arnold [13] who observed, for $n = 3$, that the degrees of the polynomials were 1 more than the Coxeter numbers of the groups of symmetries of the Platonic solids (4 for the tetrahedron, 6 for the cube and 10 for the icosahedron).

We will see, via the Saito construction, how to obtain these polynomial prepotentials from a given Coxeter group but first we will discuss the classification of the finite Coxeter groups.

2.1 Finite Coxeter groups

In this section we summarise some of the key points from the first two chapters of [25]. Recall that we say a reflection, s_α, of $\mathbb{R}^n$ equipped with a positive definite symmetric bilinear form (μ, ν), in a vector $\alpha \in \mathbb{R}^n$ sends α to its negative whilst leaving the hyperplane

H_α orthogonal to α unchanged. For an arbitrary $\lambda \in \mathbb{R}^n$ we have

$$s_\alpha \lambda := \lambda - \frac{2(\lambda, \alpha)}{(\alpha, \alpha)} \alpha.$$

Definition 2.1 *(Root system) A root system, Φ is a finite set of non-zero vectors in $\mathbb{R}^n$ such that*

1. $\Phi \cap k\alpha = \{\alpha, -\alpha\} \quad \forall \alpha \in \Phi,\ k \in \mathbb{R},$
2. $s_\alpha \Phi = \Phi \quad \forall \alpha \in \Phi.$

We define its associated reflection group W as that generated by all reflections s_α, $\alpha \in \Phi$.

It turns out that the situation can be considerably simplified by considering only a system of *simple* roots, Δ. Such a system is a linearly independent subset of Φ such that all $\alpha \in \Phi$ is a linear combination of members of Δ with coefficients all of the same sign. In fact W is generated by reflections in the simple roots: if we denote the order of the product of reflections s_α and s_β by $m(\alpha, \beta)$ (so, for example $m(\alpha, \alpha) = 1$) we can say that W is generated by the set $S = \{s_\alpha, \alpha \in \Delta\}$ subject only to the relations

$$(s_\alpha s_\beta)^{m(\alpha,\beta)} = 1. \tag{2.2}$$

For a Coxeter group some of the $m(\alpha, \beta)$ can be infinite, [6], but all finite Coxeter groups have a presentation subject only to the relations (2.2). More precisely the pair (W, S) is called a Coxeter system. It turns out that finite Coxeter groups are precisely the finite reflection groups.

2.1.1 Coxeter graphs

We have seen that W is determined by the $m(\alpha, \beta)$. A way to encode this information is via a *Coxeter graph*. Such a graph has a set of vertices in one-to-one correspondence with

Δ and has vertices joined by an edge whenever $m(\alpha, \beta) \geqslant 3$ and labelled as such (but not when $m(\alpha, \beta) = 3$ since it occurs so frequently). For all vertices without an edge between them $m(\alpha, \beta) = 2$.

The Coxeter system (W, S) is *irreducible* if its Coxeter graph is connected. In other words, [6], S is non-empty and there exists no partition of S into two distinct subsets S' and S'' such that every element of S' commutes with every element of S''.

Theorem 2.3 *[6, 25] The graph of any irreducible finite Coxeter system* (W, S) *is isomorphic to one of the following;*

A_n	$\circ - \circ - \cdots - \circ - \circ$	$(n \geqslant 1)$
B_n	$\circ - \circ - \cdots - \circ \overset{4}{-} \circ$	$(n \geqslant 2)$
D_n	$\circ - \circ - \cdots - \overset{\circ \atop \vert}{\circ} - \circ$	$(n \geqslant 4)$
E_6	$\circ - \circ - \overset{\circ \atop \vert}{\circ} - \circ - \circ$	
E_7	$\circ - \circ - \overset{\circ \atop \vert}{\circ} - \circ - \circ - \circ$	
E_8	$\circ - \circ - \circ - \circ - \overset{\circ \atop \vert}{\circ} - \circ - \circ$	
F_4	$\circ - \circ \overset{4}{-} \circ - \circ$	
G_2	$\circ \overset{6}{-} \circ$	
H_3	$\circ \overset{5}{-} \circ - \circ$	
H_4	$\circ \overset{5}{-} \circ - \circ - \circ$	
$I_2(p)$	$\circ \overset{p}{-} \circ \quad (p = 5 \text{ or } p \geqslant 7)$	

2.1.2 Crystallographic reflection groups and root systems

Finite Coxeter groups can be subdivided according to whether they are crystallographic reflection groups or not. This is the same as saying whether the finite Coxeter group stabilizes a lattice $L \in \mathbb{R}^n$ (the $\mathbb{Z}$-span of a basis of $\mathbb{R}^n$) or not. Considering the fact that the atoms in a crystal occupy the nodes of a regular lattice indicates the origin of the name. It turns out that W is crystallographic if $m(\alpha, \beta) = 2, 3, 4, 6$ for all $\alpha \neq \beta$. We see immediately from Theorem (2.3) that the crystallographic reflection groups are the infinite families A_n, B_n and D_n and the exceptional groups E_6, E_7, E_8, F_4 and G_2. Crystallographic reflection groups are also called Weyl groups which explains the choice of W for a finite reflection froup in this chapter.

Definition 2.4 *A root system,* Φ *is said to be* crystallographic *if, in addition to the conditions of Definition 2.1 it also satisfies*

$$\frac{2(\alpha, \beta)}{(\beta, \beta)} \in \mathbb{Z} \textit{ for all } \alpha, \beta \in \Phi.$$

Crystallographic root systems are very closely related to crystallographic reflection groups but with a subtle difference: there are distinct crystallographic root systems B_n and C_n each having as Weyl group the group labelled above as B_n. Defining the coroots

$$\alpha^\vee := \frac{2\alpha}{(\alpha, \alpha)},$$

the set $\Phi^\vee$ of all coroots is also a crystallographic root system in $\mathbb{R}^n$ and, in most cases, is isomorphic to Φ. However, the root systems of B_n and C_n are dual to each other.

2.1.3 A description of the A_n, B_n and D_n families

We now present an explicit description of root systems of the infinite families of finite Coxeter groups. These will be studied in more detail in Chapter 3 when we examine $\vee$-systems.

- A_n

  W is the symmetric group S_{n+1}. This can be seen by considering a permutation acting on $\mathbb{R}^{n+1}$ by permuting the standard basis vectors $\varepsilon_1, \ldots, \varepsilon_n$. The transposition (ij) acts as a reflection, sending $\varepsilon_i - \varepsilon_j$ to its negative and fixing pointwise the orthogonal complement, those vectors in $\mathbb{R}^n$ having equal i^{th} and j^{th} components. Since S_{n+1} is generated by transpositions it is therefore a reflection group.

  S_{n+1} fixes pointwise the line spanned by $\varepsilon_1 + \ldots + \varepsilon_{n+1}$ and leaves stable the orthogonal complement, the hyperplane consisting of vectors whose coordinates add up to 0. In other words, S_{n+1} acts on $\mathbb{R}^n$ as a group generated by reflections.

  We define Φ to be the set of all vectors of squared length 2 in the intersection of this hyperplane with the standard lattice $\mathbb{Z}\varepsilon_1 + \ldots + \mathbb{Z}\varepsilon_{n+1}$. Hence Φ consists of the $n(n+1)$ vectors

$$\varepsilon_i - \varepsilon_j \quad (1 \leqslant i \neq j \leqslant n+1).$$

- B_n

  Other reflections can be defined by sign changes of the ε_i. These sign changes generate a group isomorphic to $(\mathbb{Z}/2\mathbb{Z})^n$ (normalised by S_n). W is the semi-direct product of this and S_n.

  Define Φ to be the set of all vectors in the standard lattice of squared length 1 or 2 and thus is comprised of the $2n$ short roots $\pm\varepsilon_i$ and the $2n(n-1)$ long roots $\pm\varepsilon_i \pm \varepsilon_j \quad (i < j)$.

- D_n

  W is the subgroup of the group of type B_n involving an even number of sign changes, the semidirect product of S_n and $(\mathbb{Z}/2\mathbb{Z})^{n-1}$.

  Define Φ as the set of vectors of squared length 2 in the standard lattice and thus is comprised of the $2n(n-1)$ roots $\pm\varepsilon_i \pm \varepsilon_j \quad (i < j)$.

For a similar description of the exceptional Coxeter groups see [6] or [25].

2.2 Polynomial Frobenius manifolds

We have already seen (Example 1.55) examples of Frobenius manifolds with polynomial Frobenius structures. That example found Frobenius manifolds corresponding to the Coxeter group A_n. In this section we will extend this idea to outline how to construct a Frobenius manifold from *any* finite Coxeter group. All such manifolds have polynomial Frobenius structures. In fact it was proved by Hertling in [23] that *all* semi-simple polynomial Frobenius manifolds arise from finite Coxeter groups.

2.2.1 Frobenius structures on Coxeter group orbit spaces

We now consider the action of a Coxeter group W on polynomial functions of the coordinates of the vector space V, $x^1, \ldots, x^n$ (which are defined by a choice of basis for V) on which W acts. We denote by $S(V)$ the ring of such polynomial functions and by $R = S(V)^W$ that subring of polynomials which is invariant under the action of W. It turns out that this subring has an interesting basis, determined by W.

Theorem 2.5 *(Chevalley [25]) Let R be the subalgebra of $\mathbb{R}[x_1, \ldots, x_n]$ consisting of W-invariant polynomials. Then R is generated as an $\mathbb{R}$-algebra by n homogeneous, algebraically independent elements $y^1, \ldots, y^n$ of positive degree, (together with 1). The degrees of these generators are uniquely determined by the Coxeter group W.*

Table 2.1 lists the orders of the invariant polynomials for all the finite Coxeter groups.

Coxeter Group	$d_n, \ldots, d_1$
A_n	2,3,...,n+1
B_n	2,4,6,...,2n
D_{2k}	$2, 4, \ldots, 2k-2, 2k, 2k, 2k+2 \ldots, 4k-2$
D_{2k+1}	$2, 4, \ldots, 2k, 2k+1, 2k+2, 2k+4, \ldots, 4k$
E_6	$2, 5, 6, 8, 9, 12$
E_7	$2, 6, 8, 10, 12, 14, 18$
E_8	$2, 8, 12, 14, 18, 20, 24, 30$
F_4	$2, 6, 8, 12$
G_2	$2, 6$
H_3	$2, 6, 10$
H_4	$2, 12, 20, 30$
$I_2(k)$	$2, k$

Table 2.1: The degrees of the invariant polynomials of the finite Coxeter groups

Example 2.6 *(Polynomial solutions for $n = 3$) Extending the calculations of subsection (1.1.2) to $n = 3$ requires us to consider associativity. We now look for solutions of the form*

$$F(t) = \frac{1}{2}(t^1)^2 t^3 + \frac{1}{2} t^1 (t^2)^2 + f(t^2, t^3).$$

Associativity imposes the single constraint on $f = f(x, y)$ (see [13])

$$f_{xxy}^2 = f_{yyy} + f_{xxx} f_{xyy}. \tag{2.7}$$

Recall that we normalise the degrees so $d_1 = 1$ and that $d_\alpha + d_{n-\alpha+1} = d_F - 1$. So this gives $d_2 = (d_F - 1)/2$ and $d_3 = d_F - 2$. Substituting these along with

$$f(x, y) = \sum_{p,q \in \mathbb{N}} a_{pq} x^p y^q$$

into (1.4) gives

$$\frac{1}{2}c^{d_F}[(t^1)^2t^3 + t^1(t^2)^2] + \sum_{p,q\in\mathbb{N}} a_{pq}c^{p(d_F-1)/2}(t^2)^p c^{q(d_F-2)}(t^3)^q =$$

$$c^{d_F}[\frac{1}{2}(t^1)^2t^3 + \frac{1}{2}t^1(t^2)^2 + \sum_{p,q\in\mathbb{N}} a_{pq}(t^2)^p(t^3)^q].$$

So we only have $a_{pq} \neq 0$ for

$$d_F\left(\frac{p}{2} + q - 1\right) = \frac{p}{2} + 2q.$$

We consider the case where the degrees are real positive numbers which means $d_F > 2$. So we require

$$2 < \frac{p+4q}{p+2q-2},$$

and since we are only interested in terms in which $p + q \geqslant 4$ (recalling that the flat coordinates are only defined up to linear transformations) this inequality reduces to

$$p < 4.$$

We therefore look for a function of the form

$$f = a_1x^3y^\alpha + a_2x^2y^\beta + a_3xy^\gamma + a_4y^\delta,$$

subject to the constraints $\alpha \geqslant 1$, $\beta \geqslant 2$, $\gamma \geqslant 3$ and $\delta \geqslant 4$. It turns out that the only 3-dimensional polynomial Frobenius manifolds are

$$\begin{aligned}
F_1(t) &= \frac{1}{2}(t^1)^2t^3 + \frac{1}{2}t^1(t^2)^2 + a_1^2(t^2)^2(t^3)^2 + \frac{4}{15}a_1^4(t^3)^5, \\
F_2(t) &= \frac{1}{2}(t^1)^2t^3 + \frac{1}{2}t^1(t^2)^2 + a_2(t^2)^3t^3 + 6a_2^2(t^2)^2(t^3)^3 + \frac{216}{35}a_2^4(t^3)^7, \\
F_3(t) &= \frac{1}{2}(t^1)^2t^3 + \frac{1}{2}t^1(t^2)^2 + a_3(t^2)^3(t^3)^2 + \frac{27}{15}a_3^2(t^2)^2(t^3)^5 + \frac{9}{11}a_3^4(t^3)^{11}.
\end{aligned} \tag{2.8}$$

for arbitrary constants a_1, a_2, a_3. We can eliminate the arbitrary constants in each of the above by putting

$$\begin{aligned}
t_1 &\rightarrow a_i^{\frac{2}{\deg(F_i)}} t_1, \\
t_2 &\rightarrow a_i^{-\frac{1}{\deg(F_i)}} t_2, \\
t_3 &\rightarrow a_i^{-\frac{4}{\deg(F_i)}} t_3.
\end{aligned}$$

These polynomials correspond to the Coxeter groups A_3, B_3 and H_3. This can be seen by reference to Table 2.1. For A_3 we see that t_3, t_2 and t_1 are invariant polynomials of degrees 2, 3 and 4 respectively. Hence each term of the first of the prepotentials (2.8) has degree 10. Similarly for B_3, t_3, t_2 and t_1 have orders 2, 4 and 6 and each term in the second prepotential has degree 14 and for H_3 the polynomials have degrees 2, 6 and 10 and each term in the third prepotential has degree 22.

Definition 2.9 *Writing $d_i :=$deg y^i we define*

$$h = d_1 > d_2 \geqslant \ldots \geqslant d_{n-1} > d_n = 2.$$

h is called the Coxeter number *of W.*

The degrees satisfy the duality condition

$$d_i + d_{n-i+1} = h + 2, \quad i = 1, \ldots, n.$$

Remark 2.10 *The coordinate y^1 and hence the vector field $\frac{\partial}{\partial y^1}$ are fixed up to a scalar multiple. Also, every Coxeter group has a degree 2 invariant polynomial which is a scalar multiple of the distance from the origin and can be chosen to be*

$$y^n = \frac{1}{2h} \sum_{i=1}^{n} (x^i)^2.$$

By a standard transformation of a $(0,2)$-tensor we have

$$g^{ij}(y) = \frac{\partial y^i}{\partial x^a} \frac{\partial y^j}{\partial x^a}, \tag{2.11}$$

and the corresponding contravariant Levi-Civita connection

$$\Gamma_k^{ij}(y) dy^k = \frac{\partial y^i}{\partial x^a} \frac{\partial^2 y^j}{\partial x^a \partial x^b} dx^b. \tag{2.12}$$

The *Saito metric*, introduced in [34], is given by

$$\eta^{ij}(y) := \partial_1 g^{ij}(y). \tag{2.13}$$

From Lemma (1.50) we see that if we can show that $g^{ij}(y)$ and $\Gamma_k^{ij}(y)$ depend at most linearly on y^1 and that $det(\eta^{ij}(y)) \neq 0$ then g^{ij} and η^{ij} form a flat pencil.

Proposition 2.14 *(4.1 in [13]) $g^{ij}(y)$ and $\Gamma_k^{ij}(y)$ depend linearly on y^1.*

Proof. From (2.11) and (2.12) we see that $g^{ij}(y)$ and $\Gamma_k^{ij}(y)$ are graded homogeneous polynomials of degrees

$$\deg g^{ij}(y) = d_i + d_j - 2, \tag{2.15}$$

$$\deg \Gamma^{ij}_k = d_i + d_j - d_k - 2.$$

Since $d_i + d_j \leqslant 2h = 2d_1$ these polynomials can be at most linear in y^1. $\square$

We now show the non-degenerateness of $\eta^{ij}(y)$ following [13].

Theorem 2.16 *The Saito metric has the form*

$$\eta^{ij} = 0, \quad i + j > n + 1,$$

with constant anti-diagonal elements

$$c_i := \eta^{i(n+1-i)}.$$

Also

$$c := det(\eta^{ij})$$

is a non-zero constant.

Proof. From (2.13) and (2.15) we have

$$\deg \eta^{ij}(y) = d_i + d_j - 2 - h.$$

So from the duality condition we have $\deg \eta^{i(n-i+1)} = 0$ and $\deg \eta^{ij} < 0$ for $i + j > n + 1$ and hence the first two assertions of the theorem.

To show the third we consider

$$D(y) := \det(g^{ij}(y)),$$

as a polynomial in y^1,

$$D(y) = c(y^1)^n + a_1(y^1)^{n-1} + \ldots + a_n,$$

where $a_1, \ldots, a_n$ are quasihomogeneous polynomials in $y^2, \ldots, y^n$ of degree $h, 2h, \ldots, nh$ respectively. Let λ be the eigenvector of a Coxeter transformation T with eigenvalue $e^{\frac{2\pi i}{h}}$ (see [6], Chap. V). Then

$$y^k(\lambda) = y^k(T\lambda) = y^k(e^{\frac{2\pi i}{h}}\lambda) = e^{\frac{2\pi i d_k}{h}} y^k(\lambda).$$

So we have

$$y^k(\lambda) = 0, \quad k = 2, \ldots, n,$$

but $D(\lambda) \neq 0$ (again see [6]) so we must have $c \neq 0$. $\square$

Further more, if the flat pencil is regular and quasihomogenous then there must be a unique Frobenius structure on the space of orbits of the Coxeter group (see [14])). A full discussion of this area is beyond the scope of this thesis so we quote a result from [13] and refer the reader there for a full treatment.

Theorem 2.17 *Let $t^1, \ldots, t^n$ be the flat coordinates of $\eta^{\alpha\beta}$ (called the Saito flat coordinates) on the space of orbits of a finite Coxeter group and*

$$\eta^{\alpha\beta} = \partial_1 (dt^\alpha, dt^\beta)^*,$$

be the corresponding constant Saito metric. Then there exists a quasihomogeneous polynomial $F(t)$ of degree $2h+2$ such that

$$(dt^\alpha, dt^\beta)^* = \frac{(d_\alpha + d_\beta - 2)}{h} \eta^{\alpha\lambda}\eta^{\beta\mu}\partial_\lambda\partial_\mu F(t). \tag{2.18}$$

$F(t)$ determines on the space of orbits a polynomial Frobenius structure with the structure constants

$$c_{\alpha\beta}^{\gamma} = \eta^{\gamma\varepsilon}\partial_\alpha\partial_\beta\partial_\varepsilon F(t),$$

the unity field

$$e = \partial_1,$$

the Euler vector field

$$E = \sum \left(1 - \frac{degt^\alpha}{h}\right) t^\alpha \partial_\alpha,$$

and the invariant inner product η.

Example 2.19 *(Dihedral groups $I_2(k)$) The Coxeter group $I_2(k)$ is the symmetry group of a regular k-gon in $\mathbb{R}^2$ centered at the origin and arranged symmetrically about the real axis. It is generated by the reflection $z \to \overline{z}$ and the rotation $z \to e^{2\pi i/k} z$. The basic invariant polynomials generating R are*

$$y^1 = z^k + \overline{z}^k,$$

and

$$y^2 = \frac{1}{2k} z\overline{z}.$$

the components of g^{ij} are given by

$$g^{11}(y) = (dy^1, dy^1) = 4\frac{\partial y^1}{\partial z}\frac{\partial y^1}{\partial \overline{z}} = 4k^2(z\overline{z})^{k-1} = (2k)^{k+1}(y^2)^{k-1},$$

$$g^{12}(y) = (dy^1, dy^2) = 2\left(\frac{\partial y^1}{\partial z}\frac{\partial y^2}{\partial \overline{z}} + \frac{\partial y^2}{\partial \overline{z}}\frac{\partial y^1}{\partial z}\right) = (z^k + \overline{z}^k) = y^1 \; (= g^{21}(y)),$$

$$g^{22} = 4\frac{\partial y^2}{\partial z}\frac{\partial y^2}{\partial \overline{z}} = \frac{2}{k} y^2.$$

Therefore the Saito metric is given by

$$\eta^{ij} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and so, in this instance, y^1 and y^2 coincide with the Saito flat coordinates. (2.18) gives

$$(2k)^{k+1}(y^2)^{k-1} = \frac{2k-2}{k}\partial_2\partial_2 F(t),$$

$$y^1 = \partial_1\partial_2 F(t),$$

and

$$y^2 = \partial_1\partial_1 F(t).$$

Thus on integrating we obtain

$$F = \frac{1}{2}(y^1)^2(y^2) + \frac{(2k)^{k+1}}{2(k^2-1)}(y^2)^{k+1}.$$

Note that this coincides (up to a linear transformation) with (1.15).

So we have two different methods of constructing polynomial Frobenius manifolds: via the residue calculations as seen in Example 1.55 and via the Saito construction utilising invariant theory as we have just seen. The former is computationally simpler.

Chapter 3

Almost Duality and $\vee$-systems

Dubrovin, in [15], introduced what are called *almost dual* Frobenius manifolds. These utilise the intersection form and a new multiplication on the tangent space of a Frobenius manifold to construct a new entity which satisfies all of the axioms of Frobenius manifolds except the covariant constancy of the unity. The new multiplication, $\star$ is defined by

$$u \star v := E^{-1} \circ u \circ v,$$

for $u, v \in T_tM$ and $t \in M \backslash \Sigma$ where Σ is called the discriminant and is where E is not invertible.

Clearly $\star$ is associative, commutative and has unity element E. Also, by (1.46)

$$(u \star v, w) = (E^{-1} \circ u \circ v, w) = \left\langle E^{-2} \circ u \circ v, w \right\rangle = \left\langle E^{-1} \circ u, E^{-1} \circ v \circ w \right\rangle = (u, v \star w),$$

in other words the matrix inverse of the intersection form is invariant with respect to $\star$.

It remains to prove that

$$\nabla^{\gamma} c_{\rho}^{\alpha\beta} = g^{\gamma\varepsilon} \partial_{\varepsilon} c_{\rho}^{\alpha\beta} - \mathit{\Gamma}_{\varepsilon}^{\gamma\alpha} c_{\rho}^{\varepsilon\beta} - \mathit{\Gamma}_{\varepsilon}^{\gamma\beta} c_{\rho}^{\alpha\varepsilon} + \mathit{\Gamma}_{\rho}^{\gamma\varepsilon} c_{\varepsilon}^{\alpha\beta}$$

is symmetric in α, β and γ, where ∇ is the Levi-Civita connection of the intersection form. We follow the proof in [15]. We can rewrite

$$g^{\gamma\varepsilon}\partial_\varepsilon c_\rho^{\alpha\beta} = g^{\gamma\varepsilon}\partial_\rho c_\varepsilon^{\alpha\beta} = \partial_\rho\left(c_\varepsilon^{\alpha\beta}g^{\varepsilon\gamma}\right) - c_\varepsilon^{\alpha\beta}\left(\Gamma_\rho^{\gamma\varepsilon} + \Gamma_\rho^{\varepsilon\gamma}\right),$$

to obtain

$$\nabla^\gamma c_\rho^{\alpha\beta} = \partial_\rho\left(c_\varepsilon^{\alpha\beta}g^{\varepsilon\gamma}\right) - \Gamma_\varepsilon^{\gamma\alpha}c_\rho^{\varepsilon\beta} - \Gamma_\varepsilon^{\gamma\beta}c_\rho^{\alpha\varepsilon} - c_\varepsilon^{\alpha\beta}\Gamma_\rho^{\varepsilon\gamma}.$$

Using the result (from [15]) that

$$\Gamma_\gamma^{\alpha\beta} = \left(\frac{1}{2} - \mathcal{V}\right)_\varepsilon^\beta c_\gamma^{\alpha\varepsilon},$$

where $\mathcal{V} := \frac{2-d}{2} - \nabla_\eta E$ we have

$$\nabla^\gamma c_\rho^{\alpha\beta} \;=\; \partial_\rho\left(c_\varepsilon^{\alpha\beta}g^{\varepsilon\gamma}\right) \;-\; \left(\frac{1}{2} - \mathcal{V}\right)_\lambda^\alpha c_\varepsilon^{\gamma\lambda}c_\rho^{\varepsilon\beta} \;-\; \left(\frac{1}{2} - \mathcal{V}\right)_\lambda^\beta c_\varepsilon^{\gamma\lambda}c_\rho^{\alpha\varepsilon} \;-\; \left(\frac{1}{2} - \mathcal{V}\right)_\lambda^\gamma c_\varepsilon^{\alpha\beta}c_\rho^{\varepsilon\lambda}.$$

By associativity

$$\nabla^\gamma c_\rho^{\alpha\beta} = \partial_\rho\left(c_\varepsilon^{\alpha\beta}g^{\varepsilon\gamma}\right) - \left\{\left(\frac{1}{2} - \mathcal{V}\right)_\lambda^\alpha c_\varepsilon^{\gamma\beta}c_\rho^{\varepsilon\lambda} + \left(\frac{1}{2} - \mathcal{V}\right)_\lambda^\beta c_\varepsilon^{\alpha\gamma}c_\rho^{\varepsilon\lambda} + \left(\frac{1}{2} - \mathcal{V}\right)_\lambda^\gamma c_\varepsilon^{\alpha\beta}c_\rho^{\varepsilon\lambda}\right\}.$$

Clearly the second term is symmetric in α, β and γ. Symmetry of the first term follows from

$$c_\varepsilon^{\alpha\beta}g^{\varepsilon\gamma} = i_E(c_\varepsilon^{\alpha\beta}dt^\varepsilon \circ dt^\gamma) = i_E(dt^\alpha \circ dt^\beta \circ dt^\gamma).$$

That E is also the Euler vector field of the new algebra structure follows from the formulae for the Lie derivatives

$$\mathcal{L}_E g^{\alpha\beta} = (d-1)g^\alpha\beta, \quad \mathcal{L}_E c_\gamma^{\alpha\beta} = (d-1)c_\gamma^{\alpha\beta}.$$

Proposition 3.1 *There exists a function $F^*(z)$ defined by*

$$\frac{\partial^3 F^*(z)}{\partial z^i \partial z^j \partial z^k} = G_{ia} G_{jb} \frac{\partial t^\gamma}{\partial z^k} \frac{\partial z^a}{\partial t^\alpha} \frac{\partial z^b}{\partial t^\beta} c_\gamma^{\alpha\beta}(t), \tag{3.2}$$

where $z^1, \ldots, z^n$ are flat coordinates of the intersection form, G_{ij} is the intersection form expressed in these coordinates, which satisfies the associativity equations

$$\frac{\partial^3 F^*}{\partial z^i \partial z^j \partial z^a} G^{ab} \frac{\partial^3 F^*}{\partial z^b \partial z^k \partial z^l} = \frac{\partial^3 F^*}{\partial z^l \partial z^j \partial z^a} G^{ab} \frac{\partial^3 F^*}{\partial z^b \partial z^k \partial z^i} \quad (i, j, k, l = 1, \ldots, n).$$

Proof. We have shown above that all of the axioms of a Frobenius manifold (except covariant constancy of the unity) are satisfied and hence such a prepotential must exist. Since $\star$ and $\circ$ are the same on T^*M we have

$$c_\gamma^{\alpha\beta}(t) = c_\gamma^{*\alpha\beta},$$

and the definition of F^* follows from a change of coordinates of this (2,1)-tensor and lowering the indices with G_{ij}. $\square$

3.1 $\vee$-systems

In the 1990s it was found that, in the context of Seiberg-Witten theory (see [1]), there existed solutions of what are known as the generalised WDVV equations. What distinguishes these equations from those discussed thus far is the absence of a need for one of the matrices of third derivatives of F to be independent of the coordinates of F and hence composed entirely of constant entries. It was found that such solutions take the form

$$F = \sum_{\alpha \in \mathcal{R}} (\alpha \cdot z)^2 \log(\alpha \cdot z), \tag{3.3}$$

where $\mathcal{R}$ is any finite Coxeter root system (see [27] and [28]). This class of solutions became ever broader as deformations of the A_n and B_n families found in [8], deformed root systems related to Lie superalgebras in [37] and restrictions of Coxeter systems along any parabolic subgroup in [22] were all found to yield prepotentials that solved the WDVV equations. These deformed solutions are no longer dual to a Frobenius manifold with covariantly constant unity and so are known as almost-dual-*like* solutions.

In [40] Veselov derived geometric conditions, the $\vee$-conditions, that any system of covectors must satisfy in order for (3.3) to solve the generalised WDVV equations. Such systems are called $\vee$-systems:

Definition 3.4 *($\vee$-system) Let V be a real vector space and $\mathcal{A}$ be a finite set of covectors spanning the dual space V^*. We associate the canonical form $G_{\mathcal{A}}$ on V:*

$$G_{\mathcal{A}}(x,y) = \sum_{\alpha\in\mathcal{A}} \alpha(x)\alpha(y), \tag{3.5}$$

where $x, y \in V$. This is a non-degenerate scalar product which establishes the isomorphism

$$\varphi_{\mathcal{A}} : V \to V^*,$$

and we denote

$$\varphi_{\mathcal{A}}^{-1}(\alpha) := \alpha^\vee.$$

$\mathcal{A}$ is called a $\vee$-system if, for any $\alpha \in \mathcal{A}$ (which we will refer to as 'the pivot'), and any 2-plane π containing α we have

$$\sum_{\beta\in\pi\cap\mathcal{A}} \beta(\alpha^\vee)\beta^\vee = \lambda\alpha^\vee, \tag{3.6}$$

for some constant $\lambda = \lambda(\alpha,\pi)$.

Remark 3.7 *When π contains 3 or more covectors λ must be the same no matter the choice of pivot, see [4].*

This was expanded to include complex vector spaces in [21]. For real vector spaces the only condition on the elements of $\mathcal{A}$ for the metric to be non-degenerate is that they span V^*. That the metric is non-degenerate is a stronger assumption for complex vector spaces, however. We have found examples of systems for which the metric under this definition is degenerate but which still solve the WDVV equations with another metric imposed on them. We detail these solutions in Subsection 3.1.3 and Section 3.2.

It will often be convenient to express covectors in the factorised form $\sqrt{h_\alpha}\alpha$, in which case we will refer to h_α as the *multiplicity* of the covector (since, by (3.3), this is how it appears in the prepotential). The multiplicity is not unique (obviously, any covector can be factorised in infinitely many ways) and does not contain any extra data. For instance, say (i, i, i) is a member of a $\vee$-system. The term that this would contribute to the prepotential is

$$(iz_1+iz_2+iz_3)^2\mathrm{Log}(iz_1+iz_2+iz_3) \simeq (-z_1^2-z_2^2-z_3^2-2z_1z_2-2z_1z_3-2z_2z_3)\mathrm{Log}(z_1+z_2+z_3)$$

(the prepotential is only defined up to quadratic terms) and we could say that the covector is $(1, 1, 1)$ with multiplicity -1. A negative multiplicity implies a complex $\vee$-system.

3.1.1 The deformed A_n and B_n $\vee$-systems

In [32] the last of the residue formulae in Theorem 1.54 was used to derive the almost-dual-like prepotential corresponding to the superpotential

$$\lambda(z) = \prod_{i=0}^{n}(z - z^i)^{c_i}\Bigg|_{\sum\limits_{i=0}^{n} c_i z^i = 0} . \tag{3.8}$$

Note that when the $c_i = 1$ this is a factorisation of (1.56). (The restriction $\sum z^i = 0$ is equivalent to the coefficient of the z^n term being 0 in (1.56)).When at least 1 of the $c_1 \neq 1$, however, the construction in Example 1.55 no longer holds and we do not have a Frobenius manifold. Remarkably, though, we are still able to obtain almost-dual-like solutions with the z^i being the flat coordinates. We now summarise the calculations that appeared there.

The initial calculation involved another restriction on the values the c_i could take, namely the superpotential had the form

$$\lambda(z) = \prod_{i=0}^{m} (z - z^i)^{c_i} \Bigg|_{\sum\limits_{i=0}^{m} c_i z^i = 0}$$

and $\sum c_i = n + 1$. Residues were then calculated to obtain the following results.

Lemma 3.9 *For i, j, k distinct*

$$c^*_{ijk} = -\frac{c_i c_j c_k}{c_0^2} \sum_{r=1}^{m} \frac{c_r}{z^0 - z^r} - \frac{c_i c_j c_k}{c_0} \left(\frac{1}{z^0 - z^i} + \frac{1}{z^0 - z^j} + \frac{1}{z^0 - z^k} \right),$$

with precisely two of the indices identical we have

$$c^*_{iij} = -\frac{c_i^2 c_j}{c_0^2} \sum_{r=1}^{m} \frac{c_r}{z^0 - z^r} - \frac{c_i^2 c_j}{c_0} \left(\frac{2}{z^0 - z^i} + \frac{1}{z^0 - z^j} \right) + c_i c_j \left(\frac{1}{z^i - z^0} - \frac{1}{z^i - z^j} \right),$$

and with three identical indices

$$c^*_{iii} = -\frac{c_i^3}{c_0^3} \sum_{r=1}^{m} \frac{c_r}{z^0 - z^r} - \frac{3c_i^3}{c_0} \frac{1}{z^0 - z^i} + 3c_i^2 \frac{1}{z^i - z^0} + c_i \sum_{s \neq i} \frac{c_s}{z^i - z^s}.$$

These expressions may then be integrated to obtain the prepotential

$$F^* = \frac{1}{8}\sum_{r=0}^{m}\sum_{s\neq r} c_r c_s (z^r - z^s)^2 \log(z^r - z^s)^2 \Bigg|_{\sum\limits_{i=0}^{m} c_i z^i = 0} .$$

It was then shown that this result could be generalised to remove the restriction on the sum of the c_i resulting in the almost-dual-like prepotential

$$F^* = \frac{1}{8}\sum_{r=0}^{n}\sum_{s\neq r} c_r c_s (z^r - z^s)^2 \log(z^r - z^s)^2 \Bigg|_{\sum\limits_{i=0}^{n} c_i z^i = 0} .$$

From this prepotential we can extract the deformed A_n $\vee$-system found in [8]

$$\mathfrak{A}_n(c) = \begin{cases} \sqrt{c_j c_k}(e_j - e_k), & 1 \leqslant j < k \leqslant n-1, \\ \sqrt{c_j} e_j, & j = 1, \ldots, n-1, \end{cases} \tag{3.10}$$

where, without loss of generality, the restriction $z^n = 0$ has been applied as well as putting $c_n = 1$. However, in [8] the c_i are restricted to being positive but in [32] there is no such restriction to which the geometric interpretation that the negative c_i determine which Hurwitz space one is dealing with and the positive c_i determine which discriminant the solution comes from was given.

Similarly for the B_n superpotential

$$\lambda(z) = z^{2c_0} \prod_{i=1}^{N} \left(z^2 - (z^i)^2\right)^{c_i} , \tag{3.11}$$

one obtains the solution

$$F = \sum_{i=1}^{N} 2c_i(c_0 + c_i)(z^i)^2 \log z^i + \sum_{i\neq j} c_i c_j \left(z^i \pm z^j\right)^2 \log\left(z^i \pm z^j\right) ,$$

from which we can extract the $\vee$-system found in [8]

$$\mathfrak{B}_n(c) = \begin{cases} \sqrt{c_i c_j}(e_i \pm e_j), & 1 \leqslant i < j \leqslant n, \\ \sqrt{2c_i(c_i + c_0)}e_i, & i = 1, \dots, n. \end{cases} \tag{3.12}$$

In [8] it was required that the c_i are arbitrary real positive constants and $c_0 > -c_i \, \forall i$ but again in the superpotential approach no such stipulations are made.

3.1.2 Generalised root systems

There are certain special values of the deformation parameters in (3.10) and (3.12) which yield systems, called generalised root systems, that will be of particular interest to us in the next chapter and which Serganova in [36] defined thus:

Definition 3.13 *Let V be a finite-dimensional complex vector space with a non-degenerate bilinear form $(\,,\,)$. The finite set $\mathcal{U} \subset V \setminus \{0\}$ is called a generalized root system if the following conditions are fulfilled:*

1. *$\mathcal{U}$ spans V and $\mathcal{U} = -\mathcal{U}$;*

2. *if $\alpha, \beta \in \mathcal{U}$ and $(\alpha, \alpha) \neq 0$ then $2\frac{(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z}$ and $r_\alpha(\beta) = \beta - 2\frac{(\alpha,\beta)}{(\alpha,\alpha)}\alpha \in \mathcal{U}$;*

3. *if $\alpha \in \mathcal{U}$ and $(\alpha, \alpha) = 0$ then for any $\beta \in \mathcal{U}$ such that $(\alpha, \beta) \neq 0$ at least one of the vectors $\beta + \alpha$ or $\beta - \alpha$ belongs to $\mathcal{U}$.*

The classification of irreducible generalised root systems was given in [36] and coincides with the list of basic classical Lie superalgebras given in [41]. When all of the c_i in (3.12) and (3.10) are ± 1 these systems satisfy the above conditions. We now detail them.

- The series $A(m,n)$, $m \neq n-1$

  Consider the following data:

$$
\begin{aligned}
\mathcal{U} &= \{\alpha_{ij} := e_i - e_j \,, i \neq j \,, \quad i,j = 0\,,\dots\,,n+m\}|_{\sum \varepsilon_j z^j = 0} \,, \\
\varepsilon_i &= \begin{cases} +1 & \text{if } i = 0\,,\dots\,,m\,, \\ -1 & \text{if } i = m+1\,,\dots\,,n+m \end{cases} \,,
\end{aligned}
$$

$$
g = \sum_{i=0}^{m+n} \varepsilon_i (dz^i)^2 \Big|_{\sum \varepsilon_j z^j = 0} \,.
$$

  This constitutes a generalized root system (recall, elements of $\mathcal{U}$ are covectors and the metric g defines the vectors) and, as may be easily verified, a $\vee$-system with multiplicities $h_{\alpha_{ij}} = \varepsilon_i \varepsilon_j$. Note

$$
(\alpha_{ij}^\vee, \alpha_{ij}^\vee) = \varepsilon_i + \varepsilon_j \,,
$$

  so the squared lengths can be $+2\,, 0$ or -2, reflecting the split signature of the metric g . If $n=0$ the system reduces to the standard Coxeter root system for A_m .

- The series $B(m,n)$

  Consider the following data:

$$
\begin{aligned}
\mathcal{U} &= \{\pm e_i \pm e_j \,, i \neq j \,, i,j = 1\,,\dots\,,n+m\} \cup \{e_i \,, i = 1\,,\dots\,,n+m\} \,, \\
\varepsilon_i &= \begin{cases} +1 & \text{if } i = 1\,,\dots\,,m\,, \\ -1 & \text{if } i = m+1\,,\dots\,,n+m \end{cases} \,, \\
g &= \sum_{i=1}^{m+n} \varepsilon_i (dz^i)^2 \,.
\end{aligned}
$$

This constitutes a generalized root system (recall, elements of $\mathcal{U}$ are covectors and the metric g defines the vectors) and, as may be easily verified, a $\vee$-system with multiplicities $h_{\pm e_i \pm e_j} = h\varepsilon_i\varepsilon_j$ and $h_{\pm e_i} = 2\varepsilon_i(2\varepsilon_i + \gamma)$. Note

$$(\alpha_{ij}^{\vee}, \alpha_{ij}^{\vee}) = \varepsilon_i + \varepsilon_j \,, \quad (\alpha_i^{\vee}, \alpha_i^{\vee}) = \varepsilon_i \,,$$

(where $\alpha_{ij} = \pm e_i \pm e_j$ and $\alpha_i = \pm e_i$) so the squared lengths can be $+2\,, +1\,, 0\,, -1$ or -2 , reflecting the split signature of the metric g . If $n = 0$ the system reduces to the standard Coxeter root system for B_m .

- The case of $A(n-1, n)$

  When $m = n-1$ (and so equal numbers of the ε_i equal 1 and -1) the metric under the above definition (and which coincides with (3.5)) is singular and so the system is not a $\vee$-system. In fact, all of the covectors lie on the plane

$$\sum_{i=1}^{2n-1} z^i = 0,$$

  and so do not span V inevitably leading to a degenerate metric. However, we can still recover a solution to WDVV by restricting for a second time and so the metric is

$$g = \sum_{i=1}^{2n-1} \varepsilon_i (dz^i)^2 \Bigg|_{\sum_{j=1}^{2n-1} \varepsilon_j^j z = 0} \,,$$

  and we have a $(2n-2)$-dimensional system of of $n(2n-1)$ covectors.

Example 3.14 $A(2,3)$ *after the restrictions* $z^5 \to z^0 + z^1 + z^2 - z^3 - z^4$ *and* $z^4 \to$

$z^0+z^1+z^2-z^3$ have been applied becomes

$$A^{restr}(2,3)=\begin{cases} e_i-e_j, & 1\leqslant i<j\leqslant 3,\\ -e_1-e_2-e_3+2e_4, & \\ e_4, & \\ e_1+e_2+e_3-e_4, & \\ e_i-e_4, & i=1,2,3,\\ -e_i-e_j+e_4, & 1\leqslant i<j\leqslant 3,\\ e_i, & i=1,2,3 \end{cases}$$

with the multiplicity of the first 6 covectors being 1 and of the other 9 being -1. The metric is

$$G^{-1}=\begin{pmatrix} 0 & -1 & -1 & 1\\ -1 & 0 & -1 & 1\\ -1 & -1 & 0 & 1\\ 1 & 1 & 1 & -2 \end{pmatrix}.$$

In fact, we can generalise this system by introducing arbitrary constants c_i which take the rôle of the ε_i as well as featuring in the covectors themselves. The system is now

$$A^{restr}(n-1,n;c)=\begin{cases} \sqrt{-Sc_ic_j}(e_i-e_j), & 1\leqslant i<j\leqslant 2n-2,\\ \sqrt{\frac{-Sc_j}{c_{2n-1}}}(c_{2n-1}e_j+\sum c_ke_k), & j=1,\dots,2n-2,\\ S\sqrt{c_j}e_j, & j=1,\dots,2n-2,\\ \frac{S}{\sqrt{c_{2n-1}}}\sum\limits_{j=1}^{2n-2}c_je_j, & \end{cases}$$

where $S = \sum\limits_{i=1}^{2n-1} c_i$ and the metric is

$$g = \sum_{i=1}^{2n-2} [c_i(dz^i)^2] + \frac{1}{c_{2n-1}} \left(\sum_{i=1}^{2n-2} c_i dz^i \right),$$

which reduces to the above example for $n = 3$, $c_1 = c_2 = c_3 = 1$, $c_4 = c_5 = -1$. We have verified, computationally that this system solves the WDVV equations up to $n = 9$ and conjecture that it does so for all n. Note that the metric of this system by the usual definition (3.5) is identically zero.

See [37] for a comprehensive discussion of all of the generalised root systems appearing in [36] as well as a complete list of what are known as their *admissible deformations* (which also yield $\vee$-systems, see [21]).

3.1.3 Complex Euclidean $\vee$-systems

As we have already seen the above definition of a $\vee$-system excludes some of the generalised root systems ($A(n-1, n)$ as outlined above but also others, see [21]). In [21] this discrepancy was rectified by the introduction of *complex Euclidean $\vee$-systems* (which may or may not be $\vee$-systems):

Definition 3.15 *Let V be a complex Euclidean space, which is a complex vector space with a non-degenerate bilinear form B. Let $\mathcal{A}$ be a finite set of vectors in V. We say $\mathcal{A}$ is a* complex Euclidean $\vee$-*system if the canonical form (3.5) is proportional to B and any of its two-dimensional subsystems is either reducible or the restriction of (3.5) is proportional to the restriction of B on that plane.*

Crucially, the constant of proportionality is allowed to be 0 and so this definition includes systems with identically zero canonical form (although they can not be called $\vee$-systems, again see [21]).

However, as we have just seen, we *are* able to recover a solution to the WDVV equations by imposing a metric on such a system. Note that $A^{restr}(n-1,n;c)$ is *not*, in general, a complex Euclidean $\vee$-system: for arbitrary c_i two-dimensional subsystems of it do not have canonical forms (in their corresponding plane) proportional to the Euclidean bilinear form. We now generalise the infinite family of systems of covectors with zero canonical form $A^{restr}(n-1,n;c)$, show how to recover a solution to WDVV and will show, in Section 3.2, that we can recover solutions to WDVV from many other systems for which the canonical form is singular.

3.1.4 A further generalisation of $A^{restr}(n-1,n;c)$

The family $A^{restr}(n-1,n;c)$ just discussed can be generalised still further to include all dimensions (not just even ones) and, as we shall see, there exists an even more general system (which we call P_n) spanning an n-dimensional subspace of an $(n+1)$-dimensional space (from which $A^{restr}(n-1,n;c)$ can be recovered). Such a 'parent' system is, in fact, much easier to describe.

Definition 3.16 P_n $(n \geqslant 3)$ *is the* $(n+1)$*-parameter family of* $\frac{(n+1)(n+2)}{2}$ *covectors :*

$$\mathfrak{P}_n(c) = \begin{cases} e_j - e_k, & 1 \leqslant j < k \leqslant n+1, \\ (-S + c_j)e_j + \sum\limits_{m \neq j} c_m e_m, & j = 1, \dots, n+1, \end{cases}$$

where S is as defined above, the multiplicities of the $e_j - e_k$-type covectors are $-Sc_jc_k$ and those of the others are c_j. This system solves the WDVV equations with any metric of the form

$$g = \sum_{j=1}^{n+1} (c_i dz^i)^2 + B \left(\sum_{i=1}^{n+1} c_i dz^i \right)^2$$

with B an arbitrary constant.

We have verified computantionally that this system solves the WDVV equations up to $n = 7$ and again conjecture that it does so for all n. The canonical form for P_n is identically zero (this is clear since it does not span the space in which it is expressed) and it is not a complex Euclidean $\vee$-system for arbitrary c_i. We can obtain $A^{restr}(\frac{n}{2}, \frac{n}{2}+1; c)$ (and also precisely analogous odd-dimensional systems) by sending

$$z^{n+1} \to \frac{1}{c_{n+1}} \sum_{i=1}^{n} c_i z^i.$$

The reason that we say P_n is more general than $A^{restr}(n-1, n; c)$ is that this is not the only way we can obtain a system in an n-dimensional space from P_n. We can also send (for simplicity we consider the case where the $c_i = 1$),

$$z^{n+1} \to \frac{1-\sqrt{n+1}}{n} \sum_{j=1}^{n} z^j,$$

which gives

$$\mathfrak{P}_n^{restr} = \begin{cases} e_j - e_k, & 1 \leqslant j < k \leqslant n, \\ (\sqrt{n+1}-1)\sum_i e_i + n e_j, & j = 1, \ldots, n, \\ (n+1-\sqrt{n+1})\sum_i e_i - n(n+1)e_j, & j = 1, \ldots, n, \\ \sum_i e_i, & \end{cases}$$

with multiplicites for the the first two types of covector being $-n^2(n+1)$, 1 for the third and $n^2(n+1)$ for the last. This system solves the WDVV equations with the Euclidean metric.

An analogue to P_n with the 2-plane structure of B_{n+1} only exists for $n = 3, 4$:

$$\mathfrak{P}_n^{(B)} = \begin{cases} -ne_i + \sum\limits_{j \neq i} e_j, & i = 1, \dots, n+1, \\ (n+1)(e_i - e_j), & 1 \leqslant i < j \leqslant n+1, \\ (1-n)e_i + (1-n)e_j + 2 \sum\limits_{k \neq i,j} e_k, & 1 \leqslant i < j \leqslant n+1, \end{cases}$$

with metric

$$g = \sum_{j=1}^{n+1} (dz^i)^2 + B \left(\sum_{i=1}^{n+1} dz^i \right)^2$$

where B is an arbitrary constant and the multiplicites of the three types of covector are (in the above order) 1, m^2 and $\frac{4+m^2}{8}$ (where m is an arbitrary constant) for $n = 3$ and 5, 25 and 10 for $n = 4$. These turn out to be the $\vee$-systems $F_3(t)$ and (E_8, A_4) respectively (see [22]).

Since all the systems in this subsection have zero canonical form we must impose metrics on them to obtain a solution to the WDVV equations. The $\vee$-conditions do not apply to them since the 'natural' metric is central to the derivation of the $\vee$-conditions. We have shown that they solve the WDVV equations with the use of the computing package *Mathematica* but, of-course, this does not constitute a rigorous proof. Such a proof would require a new formulation of the $\vee$-conditions that included the case where a metric is imposed on a system of covectors.

Conjecture 3.17 *There exists a generalised version of the $\vee$-conditions which includes those cases where the canonical form is identically zero but a solution to the WDVV equations can be recovered by imposing a metric.*

We hope to prove this in future work.

3.2 More instances of imposing a metric to recover a solution to WDVV

We have already seen examples of complex Euclidean $\vee$-systems (Subsection 3.1.4) for which the canonical form (3.5) is zero (and hence are not $\vee$-systems) but for which we can impose a metric to recover a solution to WDVV.

There has already been work done in this direction in [4] where the equivalence of the $\vee$-conditions and a vector space V with non-degenerate bilinear form g and a collection of rank one endomorphisms $\{\rho_H := \alpha_H \otimes \alpha_H^\vee\}_{H\in\mathcal{H}}$ where the set $\mathcal{V} = \{\alpha\}_{\alpha\in\mathcal{V}}$ span V^* and $\mathcal{H} = \{H\}_{H\in\mathcal{H}}$ is a set of hyperplanes defined by $H = \mathrm{Ker}(\alpha)$ having the Kohno condition

$$[\sum_{H\in\mathcal{H},\, L\subset H} \rho_H,\, \rho_K] = 0,$$

for each ρ_K with $K \in L$ where L is a linear subspace of codimension 2 obtained by intersection of members of H, was shown. Also in that paper two examples of $\vee$-systems which have zero canonical form for certain values of their parameters were discussed ($D_3(t,s)$ with $s+t+1=0$ and $G_3(t)$ with $t=-1/2$) and it was shown that, at the singularity, they lose their unity field

$$\sum_{\alpha\in\mathcal{V}} \rho_{H_\alpha} = \mathrm{Id},$$

where H_α is the hyperplane defined by α.

Below we list several other instances for which we can recover a solution. It is natural to conjecture that we can recover a solution from all systems whose canonical form becomes singular for certain values of their parameters. Such singular counterparts do not appear to exist for the classical D_n family of $\vee$-systems or for the exceptional E_6, E_7 and E_8

$\vee$-systems. A possible direction of future work could be to derive the condition(s) a $\vee$-system must satisfy in order for it to have a corresponding singular solution or even to derive generalised $\vee$-conditions that are satisfied not only by the known $\vee$-systems but also by these systems which solve WDVV with an imposed metric. We now list the different cases.

- The case of the $B_n(c)$ family

  The metric of (3.12) is identically zero when $c_0 + \sum c_i = 0$ but we can recover a solution by dividing the metric by $c_0 + \sum c_i$, and then evaluating the inverse metric at $c_0 + \sum c_i = 0$. The relative squared-lengths of and angles between the covectors are just what we have in the non-singular case with this relation between the parameters.

- The case of $F_3(\frac{i}{\sqrt{2}})$ (see [35])

  With $t^2 = -\frac{1}{2}$ the first three covectors vanish and the metric is identically zero. We can recover a solution with the Euclidean metric, however.

- The case of $F_4(\frac{-i}{\sqrt{2}})$ (see [22])

  The system of 24 covectors

$$\widehat{\mathfrak{F}}_4 = \begin{cases} 2e_j, & 1 \leqslant j \leqslant 4, \\ i\sqrt{2}(e_j \pm e_k), & 1 \leqslant j < k \leqslant 4, \\ e_1 \pm e_2 \pm e_3 \pm e_4, & \end{cases}$$

  where the signs can be chosen arbitrarily solves the WDVV equations with the Euclidean metric. The relative lengths-squared of the covectors are 1 and -1 as opposed to 2 and 1 in the real case. The angles between the covectors are the same in both cases.

- The case of $AB(1,3)$

The $\vee$-system related to the Lie superalgebra $AB(1,3)$ (see [22]) consists of the 18 covectors

$$\mathfrak{AB}(1,3) = \begin{cases} ae_j, & 1 \leqslant j \leqslant 3, \\ be_4, & \\ (e_j \pm e_k), & 1 \leqslant j < k \leqslant 3, \\ c(e_1 \pm e_2 \pm e_3 \pm e_4), & \text{(signs chosen arbitrarily)}, \end{cases}$$

where

$$a^2 = 2(2c^2+1), \qquad b^2 = \frac{2c^2(2c^2-1)}{c^2+1}.$$

If we set $c^2 = -\frac{1}{2}$ the first three covectors vanish and the metric is identically zero. However, we can recover a solution with the metric

$$G = (dz^1)^2 + (dz^2)^2 + (dz^3)^2 - (dz^4)^2.$$

- The case of the $A_n(c)$

The $A_n(c)$ family's canonical form is singular when $\sum c_i + 1 = 0$. We can recover a solution, however, simply by dividing the metric by $\sum c_i + 1$, finding the inverse metric and then evaluating this at $\sum c_i + 1 = 0$. The relative squared-lengths of and angles between the covectors are just the same as in the non-singular case but with this relationship between the parameters substituted. There is an extra subtlety in this case to those previously encountered in that this inverse metric is actually singular itself.

3.3 A similar strategy applied to polynomial solutions

Inspired by the above solutions for which the conventional metric is singular we revisit the polynomial solutions of Example 2.6 and perform a similar operation: we look for prepotentials for which the metric defined in the usual way,

$$\eta_{\alpha\beta} = c_{1\alpha\beta}$$

is singular but when equipped with the antidiagonal metric (1.12) solves the WDVV equations. First consider prepotentials of the form

$$F = a_1 t_1^\alpha + a_2 t_2^\beta + a_3 t_3^\gamma + a_{12} t_1^\delta t_2^\varepsilon + a_{13} t_1^\zeta t_3^\eta + a_{23} t_2^\theta t_3^\lambda + a_{123} t_1^\mu t_2^\nu t_3^\rho,$$

where the coefficients and indices are arbitrary constants. We can make the metric singular by having F linear in t_1 or by putting the coefficients a_{12} and a_{123} or a_{13} and a_{123} equal to 0. By doing so we find three families of solutions to the WDVV equations (each equipped with the anti-diagonal metric) which are quasihomogeneous for certain values of their parameters but, since we have divorced the metric from the multiplication, no longer have a unity vector field and so do not have the structure of Frobenius manifolds.

Type I

$$F = a_1 t_1 t_3^\alpha + a_2 t_2 t_3^\beta + a_3 t_3^\gamma \qquad \alpha, \beta \geqslant 2, \qquad \gamma \geqslant 3.$$

This family is always quasihomogeneous.

Type II

$$F = a_1 t_1 t_3^{\alpha} + a_2 t_2 t_3^{\beta} + a_3 \frac{\alpha}{2} t_2^2 t_3^{\alpha-1} + a_4 t_3^{\gamma} \qquad \alpha, \beta \geqslant 2, \qquad \gamma \geqslant 3.$$

This family is quasihomogenous when $\gamma = 2\beta - \alpha + 1$.

Type III

$$F = a_1 t_1^{\alpha} + a_2 t_1 t_3^2 + a_3 t_3^{\gamma} + a_4 t_2^{\delta} \qquad \alpha, \gamma, \delta \geqslant 3.$$

This family is quasihomogeneous when $\alpha = \frac{\gamma}{\gamma-2}$.

It is interesting that, as in the non-singular case, we find that there are 3 different types of polynomial solutions (albeit multiparameter familes in the singular case) but an association with the three 3-dimensional Coxeter groups A_3, B_3 and H_3 (see Example (2.6)) is not immediately apparent.

Of course, it is not only polynomial solutions that can have a singular metric under the old definition, for instance the prepotential

$$F = t_1 e^{a_1 t_1 + a_2 t_2 - \frac{a_2^2}{a_1^2} t_3}, \tag{3.18}$$

solves WDVV with the antidiagonal metric.

We have also found instances of solutions where the metric is not flat, for example

$$F = a_1 t_1^3 + a_2 t_1 t_3^2 + a_3 t_3^{\alpha} + a_4 t_2^{\beta}, \qquad \alpha, \beta \geqslant 3,$$

$$\eta_{ab} = \partial_{t_a}\partial_{t_b}F = \begin{pmatrix} 3a_1t_1 & 0 & a_2t_3 \\ 0 & \frac{1}{2}(\beta-1)\beta a_4 t_2^{\beta-2} & 0 \\ a_2t_3 & 0 & a_2t_1 \end{pmatrix}.$$

A future direction of work would be to find what other functional forms which have singular metric under the old definition solve the WDVV equations with the anti-diagonal metric and to find the conditions that a pair (F, η) must possess in order to provide a solution.

Chapter 4

Extended $\vee$-systems

This chapter builds upon the superpotential approach found in Subsection 3.1.1. When the $c_i = 1$ for all i in (3.10) and (3.12) these reduce to the well-known A_n and B_n solutions, with configurations being the root vectors of these Coxeter groups. Introducing multiplicities, either in the zeros or poles of the superpotential - depending on the sign of the c_i - destroys this interpretation, and also introduces a split signature metric. From the analysis of the case when

$$\mathbf{c}^{ext} = \{c\,,\underbrace{1\,,\ldots\,,1}_{m}\,,\underbrace{-1\,,\ldots\,,-1}_{n}\}$$

it was found that the configuration could be interpreted as an extension into a perpendicular direction of the lower dimensional configuration defined by

$$\mathbf{c} = \{\underbrace{1\,,\ldots\,,1}_{m}\,,\underbrace{-1\,,\ldots\,,-1}_{n}\}\,.$$

The origin of this configuration - which defines a generalized root system - from a *rational* superpotential aids in the interpretation of its symmetries: the superpotential is invariant under interchange of zeros and of poles. Isotropic roots can be interpreted as an interchange of zeros and poles

The structure of the chapter is as follows. Firstly we construct extended $\vee$-systems. Starting with a $\vee$-system we extend the configuration into a one-dimension higher space by adding a one-dimensional orthogonal direction and adding certain special covectors to the original configuration. We then derive the (extra)-conditions required for this extended configuration to be a $\vee$-system. This construction utilizes the idea of a small-orbit, as introduced by Serganova [36]. Thus extended $\vee$-systems are $\vee$-systems, but in one dimension higher than the original system. We then consider the case of the generalised root systems.

We will return to these extended $\vee$-systems in the next chapter where we will perform Legendre transformations on them and see their connection to the extended affine Weyl group orbit spaces studied in [16] and [17].

Example 4.1 *Consider the following solution to the WDVV equations [33],*

$$F^{\star} = \frac{1}{4}\left(x^2\log x + y^2\log y - (x-y)^2\log(x-y)\right)$$

with $g = 2dx\,dy$. This solution is the almost dual to the Frobenius manifold defined by the prepotential

$$F = \frac{1}{2}t_1^2 t_2 + t_2^2\log t_2\,.$$

The configuration of vectors $\{\pm(1,0)\,,\pm(0,1)\,,\pm(1,-1)\}$ seems somewhat asymmetric.

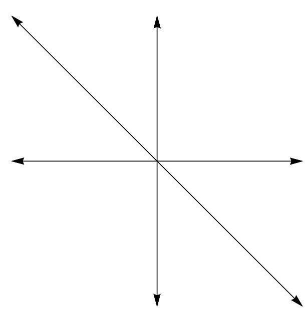

Figure 4.1: The geometry of the $\vee$-configuration

However this does not take into account the split signature of the metric. If one rotates the diagram and superimposes the light-cone, this iluminates the geometry of the configuration:

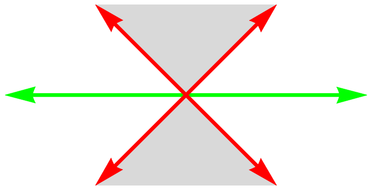

Figure 4.2: The geometry of the $\vee$-configuration, rotated with superimposed light cone

The vectors $\{\pm(1,0)\,,\pm(0,1)\}$ *are null and the vectors* $\pm(1,-1)$ *are spacelike.*

This example also motivates the main construction. We start with a configuration $\mathcal{U}=\{\pm(1,-1)\}$ spanning a space V (i.e. the configuration is the roots system for the Coxeter group A_1) and extend into a perpendicular direction defined by the normal vector $n=\frac{1}{2}(1,1)$. With this one obtains an extended space V^{ext}, and an extended space of configurations $\mathcal{U}^{ext}$ may be constructed by extending *certain* vectors into the perpendicular direction, so

$$\begin{aligned}(0,1) &= \frac{1}{2}(-1,1)+n\,,\\ (1,0) &= \frac{1}{2}(1,-1)+n\,.\end{aligned}$$

What makes the vectors $\pm\frac{1}{2}(1,-1)$ special is the following important property: their difference lies in $\mathcal{U}$. This is known as the small orbit property and its existence will be crucial to what follows.

Thus given a space V and configuration $\mathcal{U}$ we extend into a perpendicular space V^{ext} and form a new configuration $\mathcal{U}^{ext}$ by extending vectors in the small orbit of $\mathcal{U}$. Imposing the $\vee$-conditions on the extended configuration then constrains the various objects, notably the constants h_a associated to each covector.

Definition 4.2 *Let* $\mathcal{U}$ *be a* $\vee$*-system.*

(a) A small orbit ϑ_s of the $\vee$-system is a finite set of covectors such that

$$w_1 - w_2 \in \mathcal{U}$$

for all $w_1\,, w_2 \in \vartheta_s\,,$ with $w_1 \neq w_2$.

(b) An invariant small orbit is a particular case of a small orbit and consists of pairs (w, h_w), where w is a small orbit covector with associated multiplicity h_w, which satisfy the additional conditions:

(i) $\sum_{w\in\vartheta_s} h_w w(z)^2 = h_s(z,z)\,,$

(ii) $\sum_{w\in\vartheta_s} h_w w(z) = 0$

for all $z \in V$.

The first part of the definition is just an adaption of the concept of a small orbit for Weyl groups [36], and the adjective 'orbit' reflects this origin. In applications this set could be invariant under the action of the Weyl group (and hence a bone-fide orbit), but even if there is no such group we keep this adjective. In the second part of the definition, the adjective 'invariant' is used since, if $\mathcal{U}$ is a Coxeter configuration (i.e. the root system of a Coxeter group, with multiplicities equal on each orbit), the two conditions in part (b) are, by basic properties of invariant theory, automatically satisfied.

Let us recall

Definition 4.3 *The set of fundamental weight vectors, w_i, for a Coxeter group W with simple roots $\alpha_i, \ldots, \alpha_n$ is given by*

$$2\frac{(w_i, \alpha_j)}{(\alpha_j, \alpha_j)} = \delta_i^j.$$

Small orbits for Coxeter configurations were introduced and classified by Serganova [36]:

Theorem 4.4 *Let ω_i be the fundamental weight vectors for a finite Coxeter group W. The small orbits of rank $\geqslant 2$ are given by:*

1. $A_n \, : \vartheta_s = W\omega_1$ or $W\omega_n$;
2. $B_n \, , BC_n \, , C_n(n \geqslant 4) \, : \vartheta_s = W\omega_1$;
3. $B_2 \, : \vartheta_s = W\omega_1$;
4. $B_3 \, : \vartheta_s = W\omega_1$ or $W\omega_3$;
5. $BC_2 \, : \vartheta_s = W\omega_1$ or $2W\omega_2$;
6. $C_3 \, , BC_3 \, : \vartheta_s = W\omega_1$;
7. $D_n(n \geqslant 3 \, , n \neq 4) \, : \vartheta_s = W\omega_1$;
8. $D_4 \, ; \vartheta = W\omega_1 \, , W\omega_3$ or $W\omega_4$;
9. $G_2 \, : \vartheta_s = W\omega_1$.

It is interesting to note that the exceptional Coxeter groups do not have any small orbits. In the following construction of extended $\vee$-systems the existence of a small orbit is shown to be sufficient but it is not necessary, see Example 4.16 where a variation of an extended $\vee$-system is constructed for the F_4 root system.

The main construction in the first part of this chapter may be explained by the following example.

Example 4.5 *We begin with the root system (automatically a $\vee$-system) for the Coxeter group A_2. This is shown in the left-hand diagram in Figure 4.3. The small-orbit is given,*

Figure 4.3: The roots of A_2 and the small orbit superimposed on the roots

by Theorem 4.4, by the orbit of a weight vector, and this orbit is shown, superimposed on the root system, in the right-hand diagram in Figure 4.3.

We now extend the configuration into a third dimension by adding a normal vector to the end of each small-orbit vector and adding its negative. For A_2 this is shown in Figure 4.4.

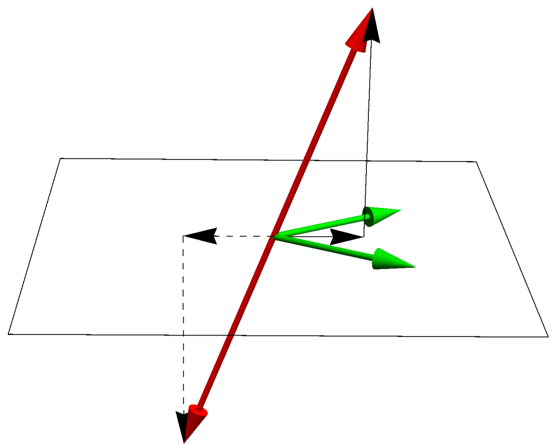

Figure 4.4: The (partial) construction of an extended $\vee$-system

Repeating the construction gives an extended configuration. For A_2 this is shown in Figure 4.5.

As in the previous example, the metric in the 3-dimensional space, and in particular its signature, depends on the geometry and multiplicities of these new extended vectors.

This extended configuration is not, at present, a $\vee$-system. One now imposes the $\vee$-

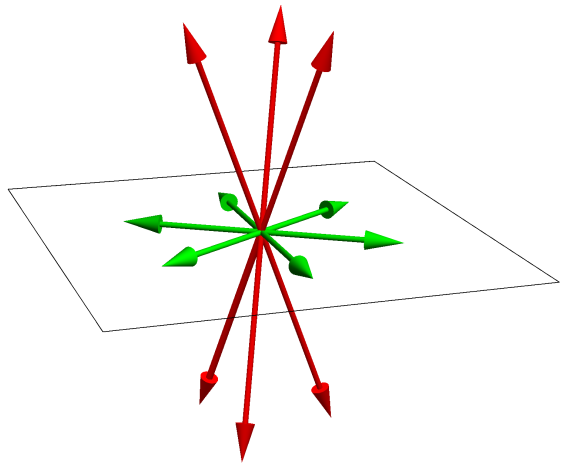

Figure 4.5: An extended $\vee$-system

conditions to obtain algebraic conditions which restricts the new data. It is at this stage that the small-orbit condition comes directly into play - it enables one to understand the 2-dimensional configurations on each two-plane Π. These algebraic conditions are given in Lemma 4.8 and Theorem 4.13.

4.1 Extended $\vee$-systems

4.1.1 Extended configurations

We begin by extending V by a 1-dimensional space $V^{\perp}$,

$$V^{ext} = V \oplus V^{\perp}$$

where $V^{\perp} = \text{span}\{n^{\vee}\}$ and V and $V^{\perp}$ are perpendicular subspaces of V^{ext}. The metric $(\,,\,)^{ext}$ on V^{ext} is determined by the original metric on V and the value $(n^{\vee}, n^{\vee})^{ext}$ which defines the perpendicular scale. Thus

$$(z_o + z^{\perp}, z_o + z^{\perp})^{ext} = (z_o, z_o) + (z^{\perp}, z^{\perp})^{ext}\,, \qquad z_o \in V\,, z^{\perp} \in V^{\perp}\,. \tag{4.6}$$

With this one can define the covector n. Once this extended space has been defined one can define the extended configuration $\mathcal{U}^{ext}$.

Definition 4.7 *Let $\mathcal{U}$ be a $\vee$-system with an invariant small orbit ϑ_s. The extended configurations $\mathcal{U}^{ext}$ are defined as:*

$$\mathcal{U}^{ext} = \mathcal{U} \cup \{\pm(w+n)\,, w \in \vartheta_s\} \cup \{\pm n\}\,.$$

We exclude reducible configurations, i.e. trivial extensions of the type $\mathcal{U} \cup \{\pm n\}$. The corresponding multiplicities for the new covectors will be denoted h_w (corresponding to the new covectors $\pm(w+n)$ and h_n (corresponding to the new covectors $\pm n$).

We now have two metrics on V^{ext}, the canonical metric given by (3.5) (now summed over the extended configuration) and the orthogonal decomposition given by (4.6). The following Lemma gives necessary and sufficient conditions for these to be equal.

Lemma 4.8 *Let $\mathcal{U}$ be a $\vee$-system with an invariant small orbit ϑ_s. The two metrics agree, i.e.*

$$h_{\mathcal{U}^{ext}}(x,y)^{ext} = \sum_{\alpha \in \mathcal{U}^{ext}} h_\alpha\, \alpha(x)\alpha(y)\,, \qquad x\,, y \in V^{ext}\,,$$

if and only if

$$h_{\mathcal{U}} + 2h_{\vartheta_s} = 2\{h_n + \sum_{w \in \vartheta_s} h_w\}(n^{\vee}, n^{\vee})\,.$$

With this, $h_{\mathcal{U}^{ext}} = h_{\mathcal{U}} + 2h_s$.

Proof. We prove this in the case $x = y$, the full result then follows from polarization.

Decomposing $z = z_o + z^\perp$ gives

$$\begin{aligned}
\sum_{\alpha\in\mathcal{U}^{ext}} h_\alpha \alpha(z)^2 &= \sum_{\alpha\in\mathcal{U}} h_\alpha\alpha(z)^2 + \sum_{w\in\vartheta_s}\left(h_w\left[(w+n)(z)\right]^2 + h_w\left[-(w+n)(z)\right]^2\right) \\
&\quad + h_n\left[n(z)\right]^2 + h_n\left[-n(z)\right]^2 \,, \\
&= \sum_{\alpha\in\mathcal{U}} h_\alpha\alpha(z_o)^2 + 2\sum_{w\in\vartheta_s} h_w\left[w(z_o)+n(z^\perp)\right]^2 + 2h_n n(z^\perp)^2 \,, \\
&= \left(h_{\mathcal{U}} + 2h_s\right)(z_o,z_o) + (2h_n + \sum_{w\in\vartheta_s} h_w) n(z^\perp)^2
\end{aligned}$$

where the invariant conditions have been used to derive the last line.

Since $\dim V^\perp = 1\,, z^\perp = \mu n^\vee$ for some scalar $\mu\,$, so

$$(n^\vee, z^\perp)^2 = (n^\vee, n^\vee)(z^\perp, z^\perp)\,.$$

Thus

$$\sum_{\alpha\in\mathcal{U}^{ext}} h_\alpha\alpha(z)^2 = \left(h_{\mathcal{U}}+2h_s\right)(z_o,z_o) + (2h_n + \sum_{w\in\vartheta_s} h_w)(n^\vee,n^\vee)(z^\perp,z^\perp)\,.$$

Since $(z,z) = (z_o,z_o) + (z^\perp, z^\perp)$ the result follows. □

Example 4.9 *Let $\mathcal{U} = \mathcal{R}_{A_n}$. We assume for now (these conditions will follow on the imposition of the $\vee$-conditions) that $h_n = 0\,, h_\alpha = 1$ for $\alpha\in\mathcal{R}_{A_n}$ (this fixes $h_{\mathcal{U}} = 2(n+1)$, the (dual) Coxeter number of A_n), and $h_w = \text{constant}$ for $w\in\vartheta_s$. We first find the constant h_s. From symmetry/invariant theory it follows that*

$$\begin{aligned}
\sum_{w\in\vartheta_s} h_w w(z_o)^2 &= h_s(z_o,z_o)\,, \\
\sum_{w\in\vartheta_s} h_w w(z_o) &= 0
\end{aligned}$$

for $z_o\in V$. Thus the invariant conditions are automatically satisfied. To find h_s we let

$z_o = \alpha$ and sum over $\alpha \in \mathcal{R}_{A_n}$. Thus

$$h_w \sum_{w\in\vartheta_s} \sum_{\alpha\in\mathcal{R}_{A_n}} (w,\alpha)^2 = h_s \sum_{\mathcal{R}_{A_n}} (\alpha,\alpha)\,.$$

Since $(\alpha,\alpha) = 2$ and $\sum_{\alpha\in\mathcal{R}_{A_n}} (w,\alpha)^2 = 2(n+1)(w,w)$, together with $\#\mathcal{R}_{A_n} = n(n+1)$, $\#\vartheta_s = n+1$ and $(w,w) = n/(n+1)$, it follows that $h_s = h_w$. The number of elements in ϑ_s follows from Serganova's classification and standard properties of weight vectors.

Having found h_s the result of Lemma 4.8 yields the condition

$$(n^\vee, n^\vee) = \frac{1}{h_w} + \frac{1}{1+n}\,.$$

Thus the construction gives a 1-parameter family of configurations, controlled by h_w (or alternatively, controlled by the perpendicular scale - the length $(n^\vee, n^\vee)$).

Example 4.1 falls into this class, with $n = 1, h_w = -1$ This gives $(n^\vee, n^\vee) = -\frac{1}{2}$ reflecting the split signature of the metric.

4.1.2 Imposition of the $\vee$-conditions

To impose the $\vee$-conditions on the extended configuration $\mathcal{U}^{ext}$ it is first necessary to classify the 2-dimensional arrangements. Since $\mathcal{U}$ is a priori a $\vee$-system one only has to understand plane arrangements through the origin and including some combination of the vectors $\pm(w+n)\,, \pm n$. It is here that the small orbit condition comes into play. Given $w_i\,, w_j \in \vartheta_s$, consider the plane containing the vectors $w_i + n\,, w_j + n$. Since

$$w_i - w_j \in \mathrm{span}\{w_i + n\,, w_j + n\}$$

it follows from the small orbit condition that the plane also contains an element $\alpha \in \mathcal{U}$. The set of planes for which one needs to impose the $\vee$ conditions depends on the geometric properties of the small orbit.

As is apparent from Figure 4.3, unlike roots, the negative of a small orbit vector may, or may not, be a small orbit vector. If it is not, then the pure normal vectors in the extended configuration must be absent.

Lemma 4.10 *Let* $w \in \vartheta_s$ *and suppose* $-w \notin \vartheta_s$. *Then*

$$\mathcal{U}^{ext} = \mathcal{U} \cup \{\pm(w+n)\,, w \in \vartheta_s\}\,.$$

Proof. Consider the intersection of the planes containing $\{\pm n\}$ with $\mathcal{U}$. Such configurations take the form

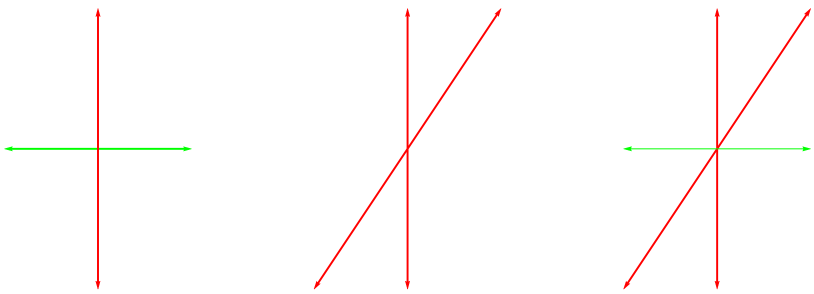

Figure 4.6: 2-plane configuration including the normal direction

where the vectors $\pm(w+n)$ and/or $\pm n$ may, or may not, be present in the configuration.

If $\pm(w+n)$ are not present the remaining vectors are perpendicular and the $\vee$-conditions are vacuous.

If $\pm(w+n)$ are present (so $h_w \neq 0$) the vectors $\pm\alpha$ may or may not be present. In either case the $\vee$-conditions imply that $h_w = 0$ and so such configurations cannot occur (it is here that the assumption $-w \notin \vartheta_s$ is used. Without this additional terms could appear).

Thus one arrives at a reducible configuration so by definition of $\mathcal{U}^{ext}$ (which excludes such reducible configurations),

$$\mathcal{U}^{ext} = \mathcal{U} \cup \{\pm(w+n)\,, w \in \vartheta_s\}\,. \qquad \square$$

Since the small orbit vectors of B_n have the property that $\pm w \in \vartheta_s$ and the small orbits vectors of A_n do not (as proved in [36]) we define the following:

Definition 4.11 *For all $w \in \vartheta_s$:*

(a) if $-w \notin \vartheta_s$ then $\mathcal{U}^{ext}$ is said to be of A-type;

(b) if $-w \in \vartheta_s$ then $\mathcal{U}^{ext}$ is said to be of B-type.

Example 4.12

- $(\mathcal{R}_{A_n})^{ext}$ *is of A-type;*
- $(\mathcal{R}_{B_n})^{ext}$ *is of B-type;*

In fact, with specific choices of normalizations;

$$\begin{aligned}(\mathcal{R}_{A_n})^{ext} &\cong \mathcal{R}_{A_{n+1}}\,,\\ (\mathcal{R}_{B_n})^{ext} &\cong \mathcal{R}_{B_{n+1}}\,,\end{aligned}$$

One now has to impose the $\vee$ conditions on the planes span$\{w_i + n\,, w_j + n\}$ and in particular on vectors in the intersection span$\{w_i + n\,, w_j + n\} \cap \mathcal{U}$. In general one can not say much about this intersection. It is at this stage that the small-orbit property comes into play again: it enables one to know what vectors are in this set. One obtains the 2-plane configurations shown in Figure 4.7 and in the following theorem the $\vee$-conditions are applied to these 2-plane configurations.

Note that, since we are considering collections of covectors with multiplicities, imposition of the $\vee$-conditions will, in general, yield 1 or more-parameter *families* of $\vee$-systems. Note also that small orbits are defined for collections of covectors without multiplicities and we merely use the small orbit property to identify the different 2-plane configurations whose constituent covectors, along with their multiplicites, we then apply the $\vee$-conditions to. Lastly note that the following theorem is applicable only to those $\vee$-systems which contain no other collinear covectors to α and $-\alpha$.

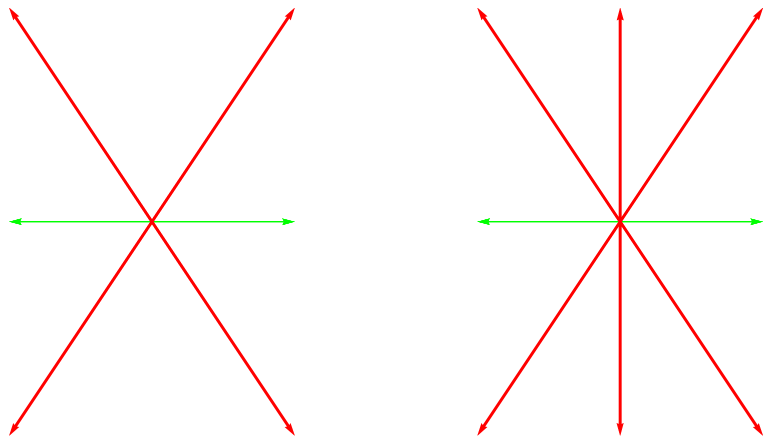

Figure 4.7: 2-plane configurations

Theorem 4.13 *The following constraints on the data* $\{h_{w_i}, h_n, n^\vee\}$ *are necessary and sufficient for the extended configurations* $\mathcal{U}^{ext}$ *to satisfy the* $\vee$*-conditions:*

$$
\begin{aligned}
h_{w_i}\left[(w_i^\vee, w_i^\vee) - (w_i^\vee, w_j^\vee)\right] &= h_{w_j}\left[(w_j^\vee, w_j^\vee) - (w_i^\vee, w_j^\vee)\right] \ , \\
h_{w_i}\left[(w_i^\vee, w_j^\vee) + (n^\vee, n^\vee)\right] &= h_{\alpha}\left[(w_j^\vee, w_j^\vee) - (w_i^\vee, w_j^\vee)\right] \ , \\
h_{w_j}\left[(w_i^\vee, w_j^\vee) + (n^\vee, n^\vee)\right] &= h_{\alpha}\left[(w_i^\vee, w_i^\vee) - (w_i^\vee, w_j^\vee)\right] \ ,
\end{aligned}
$$

where $\alpha = w_j - w_i$, *and (if the system is of B-type):*

$$h_n(n^\vee, n^\vee) = h_\alpha(w^\vee, w^\vee) + 2h_w \left\{(w^\vee, w^\vee) - (n^\vee, n^\vee)\right\} .$$

Also, if there exist roots α that are not of the form $\alpha = w_i - w_j$ then we must have $(\alpha^\vee, w_i^\vee) = 0$ (except, obviously, the case of $w_i = \alpha$).

Proof. Applying the $\vee$-condition (3.6) to a plane containing two extended small orbit covectors $(w_i + n)$ and $(w_j + n)$ and a root $\alpha = w_j - w_i$ (as on the left of Figure 4.7) we have, pivoting on $(w_i + n)$,

$$\begin{aligned} h_{w_i}[(w_i + n)(w_i^\vee + n^\vee)](w_i^\vee + n^\vee) + h_{w_j}[(w_j + n)(w_i^\vee + n^\vee)](w_j^\vee + n^\vee) \\ + h_\alpha[(w_j - w_i)(w_i^\vee + n^\vee)](w_j^\vee - w_i^\vee) = \lambda(w_i^\vee + n^\vee). \end{aligned}$$

Equating coefficients of $w_j^\vee$ gives

$$h_{w_j}[(w_j + n)(w_i^\vee + n^\vee)] + h_\alpha[(w_j - w_i)(w_i^\vee + n^\vee)] = 0,$$

or

$$h_{w_j}[(w_j^\vee, w_i^\vee) + (n^\vee, n^\vee)] = h_\alpha[(w_i^\vee, w_i^\vee) - (w_i^\vee, w_j^\vee)],$$

(the third of the above constraints) and equating coefficients of $n^\vee$

$$h_{w_i}[(w_i + n)(w_i^\vee + n^\vee)] + h_{w_j}[(w_j + n)(w_i^\vee + n^\vee)] = \lambda,$$

or

$$h_{w_i}[(w_i^\vee, w_i^\vee) + (n^\vee, n^\vee)] + h_{w_j}[(w_j^\vee, w_i^\vee) + (n^\vee, n^\vee)] = \lambda.$$

Applying the same condition but pivoting on $(w_j + n)$ gives

$$h_{w_i}[(w_i + n)(w_j^\vee + n^\vee)](w_i^\vee + n^\vee) + h_{w_j}[(w_j + n)(w_j^\vee + n^\vee)](w_j^\vee + n^\vee) \\ + h_\alpha[(w_j - w_i)(w_j^\vee + n^\vee)](w_j^\vee - w_i^\vee) = \lambda(w_j^\vee + n^\vee).$$

Equating coefficients of $w_j^\vee$ gives

$$h_{w_i}[(w_i + n)(w_j^\vee + n^\vee)] - h_\alpha[(w_j - w_i)(w_j^\vee + n^\vee)] = 0,$$

or

$$h_{w_i}[(w_i^\vee, w_j^\vee) + (n^\vee, n^\vee)] = h_\alpha[(w_j^\vee, w_j^\vee) - (w_i^\vee, w_j^\vee)],$$

(the second of the above constraints) and equating coefficients of $n^\vee$

$$h_{w_i}[(w_i + n)(w_j^\vee + n^\vee)] + h_{w_j}[(w_j + n)(w_j^\vee + n^\vee)] = \lambda,$$

or

$$h_{w_i}[(w_i^\vee, w_j^\vee) + (n^\vee, n^\vee)] + h_{w_j}[(w_j^\vee, w_j^\vee) + (n^\vee, n^\vee)] = \lambda,$$

and since λ is constant on a given plane we can equate this expression with that above to obtain

$$h_{w_i}[(w_i^\vee, w_i^\vee) + (n^\vee, n^\vee)] + h_{w_j}[(w_j^\vee, w_i^\vee) + (n^\vee, n^\vee)] \\ = h_{w_i}[(w_i^\vee, w_j^\vee) + (n^\vee, n^\vee)] + h_{w_j}[(w_j^\vee, w_j^\vee) + (n^\vee, n^\vee)],$$

or

$$h_{w_i}[(w_i^\vee, w_i^\vee) - (w_i^\vee, w_j^\vee)] = h_{w_j}[(w_j^\vee, w_j^\vee) - (w_i^\vee, w_j^\vee)],$$

(the first of the above constraints).

For systems of B-type we also have planes represented by the right of Figure 4.7 consisting of two extended small orbit covectors $(w+n)$ and $(-w+n)$, a root (which is also a small orbit covector) w and the pure normal covector n. Applying the $\vee$-condition, pivoting on w gives

$$h_\alpha w(w^\vee)w^\vee + h_n n(w^\vee)n^\vee + h_w(w+n)(w^\vee)(w^\vee+n^\vee) + h_w(-w+n)(w^\vee)(-w^\vee+n^\vee) = \lambda w^\vee.$$

Equating coefficients of $w^\vee$ yields

$$h_\alpha(w^\vee, w^\vee) + 2h_w(w^\vee, w^\vee) = \lambda.$$

Similarly pivoting on $n^\vee$ and equating coefficients of $n^\vee$ gives

$$h_n(n^\vee, n^\vee) + 2h_w(n^\vee, n^\vee) = \lambda.$$

Putting these two expressions for λ equal then gives the fourth constraint.

For the last condition consider a 2-plane spanned by an extended small orbit covector $\beta = w_i + n$ and a root α where $\alpha \neq w_i - w_j$ for any j. Then β and α are the only covectors in the 2-plane and so the $\vee$-condition reads (pivoting on β)

$$h_{w_i}\beta(\beta^\vee)\beta^\vee + h_\alpha\alpha(\beta^\vee)\alpha^\vee = \lambda\beta^\vee.$$

Equating coefficients of $\alpha^\vee$ gives $\alpha(\beta^\vee) = 0$ and thus $(\alpha^\vee, w_i^\vee) = 0$. $\square$

Note that this over-constrains the data $\{h_w, n^\vee\}$ but the constraints are completely determined by the geometry of small orbits.

Example 4.14 *Let $\mathcal{U} = \mathcal{R}_{A_n}$. Then from standard properties of weight vectors,*

$$(w_i^\vee, w_j^\vee) = \delta_{ij} - \frac{1}{n+1}\,.$$

With these one obtains $h_{w_i} = h_{w_j}$, i.e. all the h_{w_i} are constants, and, with the normalization $h_\alpha = 1\,, \alpha \in \mathcal{U}$, the condition

$$(n^\vee, n^\vee) = \frac{1}{h_w} + \frac{1}{1+n}\,.$$

Example 4.15 *Let $\mathcal{U} = \mathcal{R}_{G_2}$. We normalise these roots - generated by simple roots α and β by the conditions $(\alpha,\alpha) = 2\,, (\alpha,\beta) = -3\,, (\beta,\beta) = 6$. From (3.5) one finds*

$$6h_s + 18h_l = h_{\mathcal{U}}\,,$$

where h_s and h_l are the multiplicities of the short and long roots. From [36] the small orbit is the A_2-subsystem generated by the set $\{\pm\alpha\,, \pm(\alpha+\beta)\,, \pm(2\alpha+\beta)\}$ and using this one finds

$$h_{\vartheta_s} = 6h_w\,.$$

Lemma 4.8 implies

$$3h_s + 9h_l + 6h_w = (h_n + 6h_w)\ (n^\vee, n^\vee)\,.$$

Theorem 4.13 then implies the equation

$$\begin{aligned} -3h_l + h_n\left\{-1 + (n^\vee, n^\vee)\right\} &= 0\,, \\ -2h_s - 4h_w + \left\{h_n + 2h_w\right\}(n^\vee, n^\vee) &= 0\,. \end{aligned}$$

Assuming $h_w \neq 0$ (otherwise the construction collapses) one can solve equations to obtain *the extended configuration data in terms of the original* $\vee$*-data* $\{h_s\,, h_l\}$:

$$\begin{aligned} h_w &= \frac{1}{2}(h_s - 3h_l)\,, \\ h_n &= \frac{3(h_s - 3h_l)^2}{h_s + 3h_l}\,, \\ (n^\vee, n^\vee) &= \frac{h_s + 3h_l}{h_s - 3h_l}\,. \end{aligned}$$

Note that one requires a slight constraint on the original data: $h_s \neq \pm 3h_l$ *. This extended* $\vee$*-system coincides, after some linear algebra and redefinitions, to the system* $G_3(t)$ *presented in [21].*

Example 4.16 *Let* $\mathcal{U} = \mathcal{R}_{F_4}$ *. Consider the subset of* $\mathcal{U}$ *,* $e_i \pm e_j$*,* $1 \leqslant i < j \leqslant 4$*. This subset has the property that* either *the difference* or *the sum of any two members is proportional to a root. It turns out that we can extend this subset into a perpendicular dimension just as we did for the above small orbits and, after applying the* $\vee$*-conditions to find the multiplicities we obtain the* $\vee$*-system*

$$\mathfrak{F}_4{}^{(ext)} = \begin{cases} e_i, & i = 1, \dots 4, \\ e_i \pm e_j, & 1 \leqslant i < j \leqslant 4 \\ \frac{1}{2}(e_1 \pm e_2 \pm e_3 \pm e_4), & \\ \frac{1}{\sqrt{2}}(e_i \pm e_j \pm e_5), & 1 \leqslant i < j \leqslant 4 \\ \sqrt{3}\, e_5, & \end{cases}$$

where the signs are chosen arbitrarily. This is not a new $\vee$*-system, however, it is the system* (E_8, A_1^3) *found in [22].*

4.2 Extensions of generalized root systems

Returning to the generalised root systems defined in the last chapter, the small orbit for the series $A(m,n)$ is defined as [36]:

$$\vartheta = \left\{ w_i := e_i + \frac{1}{n-m+1} \sum_{r=0}^{m+n} \varepsilon_r e_r \right\}$$

(so trivially, $\alpha_{ij} = w_i - w_j$). That this is the only small orbit (up to sign) was proved in [36].

Applying Theorem 4.13 gives the multiplicities $h_{w_i} = \varepsilon_i h_w$ for some h_w , and with this the invariant conditions imply that $h_\vartheta = h_w$. Finally one obtains the condition

$$\frac{1}{h_w} + \frac{1}{m+1-n} = (n^\vee, n^\vee)\,.$$

This then gives an extended $\vee$-system $\mathcal{U}^{ext}$, with one free parameter: h_w fixes the perpendicular scale, or vice-versa.

For the series $B(m,n)$ the small orbit is defined as [36]

$$\vartheta = \{\pm e_i\}$$

That this is the only small orbit (up to sign) was proved in [36].

Applying Lemma 4.8 and Theorem 4.13 gives the multiplicities $h_{w_i} = \varepsilon_i h_w$ and the conditions

$$\begin{aligned} h_w &= \frac{h}{(n^\vee, n^\vee)}\,, \\ h_n &= \frac{2h}{(n^\vee, n^\vee)} + \frac{h\gamma}{(n^\vee, n^\vee)}\,. \end{aligned}$$

This then gives an extended $\vee$-system $\mathcal{U}^{ext}$, with one free parameter: the perpendicular scale.

4.2.1 Symmetries of the extended configurations

Consider again the extended configuration shown in Figure 4.5. The original configuration $\mathcal{U} = R_{A_2}$ is by definition invariant under reflections - the group A_2 - as are the small orbit vectors (which form an irregular orbit). The reflections generated by these roots can be extended to the whole space (acting trivially in the perpendicular direction). Thus the whole 3-dimensional configuration is invariant under A_2 but not, in general, under reflections generated by the new roots $\pm(w+n)$. Clearly this idea generalises - any symmetry is inherited by the extended configuration.

However, we now have to consider the generalized root system of $A(m,n)$ which has roots of positive, negative and zero length. On writing $\alpha_{rs} = w_r - w_s$ the non-isotropic roots define reflections in the hyperplanes $\alpha_{ij}(z) = 0$ and the whole configuration $\mathcal{U}^{ext}$ is invariant under reflections in the original roots.

More explicitly, if α_{ij} is a non-isotropic root then it is easy to show, using the small-orbit property, that

$$r_{\alpha_{ij}}(w_k) = \begin{cases} w_k & \text{if } i,j,k, \text{ distinct,} \\ w_i & \text{if } j=k\,, \\ w_j & \text{if } i=k\,, \end{cases}$$

and the reflection $r_{\alpha_{ij}}(\alpha_{rs})$ can then be found using linearity. However, as an abstract group, one may define a transformation $r_{\alpha_{ij}}$ for all roots (including isotropic roots) by the above formulae. While $\left(r_{\alpha_{ij}}\right)^2 = id$, the lengths of the covectors are not preserved under the action of $r_{\alpha_{ij}}$. For example, since

$$(w_i^\vee, w_j^\vee) = \varepsilon_i \delta_{ij} + \frac{1}{n-(m+1)}$$

a 'reflection' in an isotropic root will change the length of the roots. With this interpretation/definition of $r_{\alpha_{ij}}$ the whole extended configuration $\mathcal{U}^{ext}$ is invariant under the above action of $r_{\alpha_{ij}}$.

If one extends the action of $r_{\alpha_{ij}}$ to the multiplicities ε_i via the formula

$$r_{\alpha_{ij}}(\varepsilon_k) = \begin{cases} \varepsilon_k & \text{if } i, j, k, \text{ distinct,} \\ \varepsilon_i & \text{if } j = k\,, \\ \varepsilon_j & \text{if } i = k\,, \end{cases}$$

then the plane $\sum \varepsilon_i z^i = 0$ is invariant under the action of $r_{\alpha_{ij}}$ (which now interchanges *both* z^i with z^j and ε_i with ε_j).

A model for this comes from the space of rational functions

$$\lambda\left(\{z^k\}, \{\varepsilon_k\} \,:\, z\right) = \prod_{k=0}^{m+n} (z - z^k)^{\varepsilon_k} \Big|_{\sum \varepsilon_j z^j = 0} \,.$$

So for all roots α_{ij},

$$\lambda\left(\{r_{\alpha_{ij}} z^i\}, \{r_{\alpha_{ij}} \varepsilon_i\} \,:\, z\right) = \lambda\left(\{z^i\}, \{\varepsilon_i\} \,:\, z\right)\,.$$

If $n = 0$ then one recovers the standard invariant polynomial model for A_m

$$\lambda(z) = \prod_{i=0}^{m} (z - z^i) \Big|_{\sum z^r = 0}$$

which is invariant under the interchange of zeros. Repeating the argument for $B(m, n)$ results in the rational function

$$\lambda\left(\{z^k\}, \{\varepsilon_k\} \,:\, z\right) = \prod_{k=1}^{m+n} \left(z^2 - (z^k)^2\right)^{\varepsilon_k}\,.$$

If $n = 0$ then one recovers the standard invariant polynomial model for B_m .

4.3 Extending into two dimensions

We have found that this technique of extending the small orbit vectors into a perpendicular dimension can be, for B-type configurations, generalised to extending them into two perpendicular dimensions. We now extend V by a 2-dimensional space $V^{\perp}$ which is spanned by perpendicular vectors $n^{\vee}$ and $m^{\vee}$ and our configurations are defined thus

Definition 4.17 *For a B-type $\vee$-system $\mathcal{U}$ with invariant small orbit ϑ_s the two dimensional extended configurations are given by*

$$\mathcal{U}^{ext} = \mathcal{U} \cup \{\pm(w \pm n \pm m)\,, w \in \vartheta_s\} \cup \{\pm n\} \ \cup \{\pm m\} \cup \{\pm n \pm m\}.$$

Such as system can be easily seen to be equivalent to (3.12) with all the c_i equal for $i > 1$:

Theorem 4.18 *The system of n^2 covectors*

$$B_n^{2D}(c,k) = \begin{cases} \sqrt{c}(e_i), & 1 \leqslant i \leqslant 2, \\ \sqrt{\frac{2(k+4)}{c}} e_i, & 3 \leqslant i \leqslant n+2, \\ \sqrt{\frac{c+k}{2}}(e_1 \pm e_2), & \\ \frac{2}{\sqrt{c}}(e_i \pm e_j), & 3 \leqslant i < j \leqslant n+2, \\ \pm e_1 \pm e_2 \pm e_i, & 3 \leqslant i \leqslant n+2 \quad (0\, or 1 minus\ sign) \end{cases} \tag{4.19}$$

where c and k are arbitrary real positive constants is a $\vee$-system for $n \geqslant 2$.

Proof. By making the linear transformation $f_1 = e_1 + e_2$, $f_2 = e - 1 - e_2$ and putting $c = 4c_1^2$, $k = 4c_0c_1$ we have (3.12) with the c_i equal for $i > 1$. □

Chapter 5

Generalised Legendre-type Transformations

Legendre-type transformations were defined in [13]. They are a map from one solution of the WDVV equations to another generated by the vector field $\partial_{t^\kappa},\ \kappa = 1, \ldots, n$ via $\partial_{t^\alpha} = \partial_{t^\kappa} \circ \partial_{\widehat{t}^\alpha}$ where $\widehat{t}^\alpha$ are the new coordinates. We have found that the generating vector field of these transformations need not be constant but needs to satisfy certain conditions which we will prove.

In this chapter we will first review the Legendre transformations as discussed in [13] and then go on to show how they can be generalised to include those generated by functional vector fields. We then find exactly what generalised Legendre fields there are for the 2-dimensional Frobenius manifolds of Chapter 1 and that they exist for the A_3 Frobenius manifold.

Next, we perform a Legendre transformation on an extended $\vee$-system. Such Legendre transformations are symmetries of the WDVV-equations, and hence map solutions to solutions [13]. To perform such a transformation requires the choice of a direction, and for extended $\vee$-systems there is a natural choice of direction, namely the newly introduced orthogonal direction perpendicular to the original space. Using this direction for the Legendre transformation maps rational solutions, i.e. those of the form (3.3) to trigonometric systems, that is, to solutions of the form

$$F = \text{cubic } + \sum_{\alpha \in \mathcal{U}} h_\alpha Li_3(e^{\alpha(z)}) \,, \tag{5.1}$$

where Li_3 is the tri-logarithm function

$$Li_3 := \sum_{r=1}^{\infty} \frac{z^r}{r^3}.$$

A separate theory of trigonometric $\vee$-systems has been developed by M.Feigin [20].

Finally, we make the connection between extended $\vee$-systems and the almost-dual Frobenius manifolds for the extended affine Weyl group orbit spaces as constructed and studied in [16] and [17]. In particular the following is proved:

Theorem 5.2 *Let W be a finite irreducible classical Coxeter group of rank N and let $\widetilde{W}$ be the extended affine Weyl group of W with arbitrary marked node. Then up to a Legendre transformation, the almost dual prepotentials of the classical extended affine Weyl group orbit spaces $\mathbb{C}^{N+1}/\widetilde{W}$ are, for specific values of the free data, the extended $\vee$-systems of the $\vee$-system R_W .*

Proposition 5.3 (Legendre-type transformations) *Given a prepotential F expressed in coordinates t^α and equipped with metric $\eta_{\alpha\beta}$ which satisfies the WDVV equations a Legendre transformation yields another solution with new prepotential $\widehat{F}$ given by*

$$\frac{\partial^2 \widehat{F}}{\partial \widehat{t^\alpha} \partial \widehat{t^\beta}} = \frac{\partial^2 F}{\partial t^\alpha \partial t^\beta},$$

and new co-ordinates given by

$$\widehat{t_\alpha} = \partial_\alpha \partial_\kappa F(t),$$

whilst the new metric $<\cdot,\cdot>_\kappa$, *related to the old via*

$$< \partial_a, \partial_b >_\kappa = < \partial_\kappa \circ \partial_\kappa, \partial_a \circ \partial_b >,$$

remains invariant.

Proof. The metric is invariant since, by the Frobenius property

$$\begin{aligned} < \partial_\alpha, \partial_\beta >_\kappa &= < \partial_\kappa \circ \partial_\alpha, \partial_\kappa \circ \partial_\beta >, \\ &= < \partial_\alpha, \partial_\beta > . \end{aligned}$$

Rewriting

$$\partial_\kappa \circ \partial_{\widehat{\alpha}} = \frac{\partial t^\sigma}{\partial \widehat{t}^\alpha} c^\mu_{\kappa\sigma} \frac{\partial}{\partial t^\mu},$$

but

$$\partial_\kappa \circ \partial_{\widehat{\alpha}} = \frac{\partial}{\partial t^\alpha} = \delta^\mu_\alpha \frac{\partial}{\partial t^\mu},$$

so we have

$$\frac{\partial t^\sigma}{\partial \widehat{t}^\alpha} c^\mu_{\kappa\sigma} = \delta^\mu_\alpha.$$

On multiplying by $\frac{\partial \widehat{t}^\alpha}{\partial t^\nu}$

$$\frac{\partial \widehat{t}^\alpha}{\partial t^\nu} \frac{\partial t^\sigma}{\partial \widehat{t}^\alpha} c^\mu_{\kappa\sigma} = \frac{\partial \widehat{t}^\mu}{\partial t^\nu},$$

but

$$\frac{\partial \widehat{t}^\alpha}{\partial t^\nu} \frac{\partial t^\sigma}{\partial \widehat{t}^\alpha} = \delta^\sigma_\nu,$$

so

$$c^\mu_{\kappa\nu} = \frac{\partial \widehat{t}^\mu}{\partial t^\nu},$$

or

$$c_{\kappa\nu\mu} = \frac{\partial \widehat{t}_\mu}{\partial t^\nu},$$

(since $\eta_{\mu\alpha} = \widehat{\eta}_{\mu\alpha}$), integrating with respect to t^ν gives

$$\widehat{t}_\mu = \frac{\partial^2 F}{\partial t^\kappa \partial t^\mu},$$

up to a constant which may be ignored.

For the remaining part of the proposition observe that

$$\begin{aligned} < \partial_\alpha \circ \partial_\beta, \partial_\gamma > &= c_{\alpha\beta\gamma}, \\ < \partial_{\widehat{\alpha}} \circ \partial_{\widehat{\beta}}, \partial_{\widehat{\gamma}} >_\kappa &= c_{\widehat{\alpha}\widehat{\beta}\widehat{\gamma}}. \end{aligned}$$

Now $\frac{\partial}{\partial \widehat{t}^\sigma} < \partial_{\widehat{\alpha}} \circ \partial_{\widehat{\beta}}, \partial_{\widehat{\gamma}} >_\kappa$ is totally symmetric in α, β, γ and σ. So, by the Poincaré lemma $\exists \widehat{F}$ such that

$$\frac{\partial^3 \widehat{F}}{\partial_{\widehat{\alpha}} \partial_{\widehat{\beta}} \partial_{\widehat{\gamma}}} =< \partial_{\widehat{\alpha}} \circ \partial_{\widehat{\beta}}, \partial_{\widehat{\gamma}} >_\kappa .$$

So

$$\begin{aligned} \frac{\partial^3 \widehat{F}}{\partial_{\widehat{\alpha}} \partial_{\widehat{\beta}} \partial_{\widehat{\gamma}}} &= < \partial_\alpha \circ \partial_\beta, \partial_{\widehat{\gamma}} >, \\ &= \frac{\partial t^\mu}{\partial \widehat{t}^\gamma} < \partial_\alpha \circ \partial_\beta, \partial_\mu >, \\ &= \frac{\partial t^\mu}{\partial \widehat{t}^\gamma} \frac{\partial}{\partial t^\mu} F_{\alpha\beta}, \\ &= \frac{\partial}{\partial \widehat{t}^\gamma} F_{\alpha\beta}, \\ \frac{\partial^2 \widehat{F}}{\partial_{\widehat{\alpha}} \partial_{\widehat{\beta}}} &= \frac{\partial^2 F}{\partial_\alpha \partial_\beta}. \end{aligned}$$

□

Example 5.4 (B.1 in [13]) *Take the prepotential*

$$F = \frac{1}{2}(t^1)^2 t^2 + e^{t^2}.$$

Performing S_2 yields new coordinates

$$\widehat{t}_1 = t^1,$$

$$\widehat{t}_2 = e^{t^2},$$

or, on raising the indices

$$\widehat{t}^1 = e^{t^2},$$

$$\widehat{t}^2 = t^1.$$

The second derivatives of the new prepotential are

$$\begin{aligned} \widehat{F}_{\widehat{1}\widehat{1}} &= t^2 &= \log \widehat{t}^1 \,, \\ \widehat{F}_{\widehat{1}\widehat{2}} &= t^1 &= \widehat{t}^2 \,, \\ \widehat{F}_{\widehat{2}\widehat{2}} &= e^{t^2} &= \widehat{t}^1 \,, \end{aligned}$$

which, on integrating gives the new prepotential,

$$\widehat{F} = \frac{1}{2}(\widehat{t}^2)^2 \widehat{t}^1 + \frac{1}{2}(\widehat{t}^1)^2 \left(\log \widehat{t}^1 - \frac{3}{2} \right).$$

5.1 Generalised Legendre-type transformations

Legendre-type transformations as defined above are restricted to those transformations generated by the constant vector fields ∂_κ. The new metric is given by $\eta(X,Y) = \tilde{\eta}(\partial_\kappa \circ X, \partial_\kappa \circ Y)$. We have found that the notion of a Legendre-type transformation may be broadened to include transformations which are generated by functional vector fields [9]. Note that there is no requirement to impose a homogeneity condition on the generalised Legendre field (see Example 5.18).

Definition 5.5 *Suppose $\tilde{\eta}$ is a metric and $\tilde{\nabla}$ its Levi-Civita connection. Define*

$$\eta(X,Y) = \tilde{\eta}(\partial \circ X, \partial \circ Y),$$

and

$$\nabla_X Y = \partial^{-1} \circ \tilde{\nabla}_X (\partial \circ Y).$$

Definition 5.6 *(generalised Legendre field) A vector field ∂ on a Frobenius manifold $(\mathcal{M}, \eta, \nabla, \circ, e, E)$ (see Definition 1.22) is said to be a generalised Legendre field if*

$$Y \circ \tilde{\nabla}_X \partial = X \circ \tilde{\nabla}_Y \partial,$$

for all $X, Y \in T\mathcal{M}$.

Such vector fields have been studied before: in [9] where their ability to provide a map between torsion-free metric connections was investigated and here we extend those ideas to show that generalised Legendre fields generate transformations between solutions of the WDVV equations and, in Proposition 5.14, we give a coordinate description of the transformation between flat coordinate systems which is a direct generalisation of the original transformation a defined in [13] ; and in [30] where they appeared in the theory of hydrodynamics systems associated to F-manifolds as generators of commuting flows

$$u_t = \partial \circ u_X,$$

which generalises the principal hierarchy defined by Dubrovin in the case of Frobenius manifolds [13]. The role of ∂ here is somewhat different: it plays a role in defining a symmetry between *different* principal hierarchies. The use of conservation laws to define new sets of variables for hydrodynamic systems is well established and the theory developed here may be seen as a generalisation of this idea (in the sense that the hierachies as defined above have an underlying connection, and hence one can use ∂ to provide a map between such connections). Further, since the Legendre condition comes from the preservation of the torsion-fee condition, the theory could be developed to more general situations where one has torsion-free, but not metric, connections [3].

Definition 5.7 *(generalised Legendre transformation) A generalised Legendre transformation is a map, generated by a generalised Legendre field, from a Frobenius manifold with metric, connection and multiplication* $(\tilde{\eta}, \tilde{\nabla}, \circ)$ *to one with similar structures* $(\eta, \nabla, \circ)$ *(as defined above).*

The following four propositions show that such a transformation yields a solution to the WDVV equations.

Proposition 5.8 ∇ *is the Levi-Civita connection of* η *if and only if* ∂ *is a generalised Legendre field.*

Proof. By definition

$$\begin{aligned}
(\nabla_X \eta)(Y, Z) &= X[\eta(Y, Z)] - \eta(\nabla_X Y, Z) - \eta(Y, \nabla_X Z), \\
&= X[\tilde{\eta}(\partial \circ Y, \partial \circ Z)] - \tilde{\eta}(\partial \circ \partial^{-1} \circ \nabla_X Y, \partial \circ Z) - \tilde{\eta}(\partial \circ Y, \partial \circ \partial^{-1} \circ \nabla_X Z), \\
&= X[\tilde{\eta}(\partial \circ Y, \partial \circ Z)] - \tilde{\eta}(\tilde{\nabla}_X(\partial \circ Y), \partial \circ Z) - \tilde{\eta}(\partial \circ Y, \tilde{\nabla}_X(\partial \circ Z)), \\
&= (\tilde{\nabla}_X \tilde{\eta})(\partial \circ Y, \partial \circ Z).
\end{aligned}$$

So it follows that ∇ is a metric connection of η. Also by definition

$$\begin{aligned} T^{\nabla} &= \nabla_X Y - \nabla_y X - [X,Y], \\ &= \partial^{-1} \circ \tilde{\nabla}_X(\partial \circ Y) - \partial^{-1} \circ \tilde{\nabla}_Y(\partial \circ X) - [X,Y]. \end{aligned}$$

By assumption $(\tilde{\nabla}\circ)$ is totally symmetric so

$$\tilde{\nabla}_X(Y \circ Z) - (\tilde{\nabla}_X Y) \circ Z - (\tilde{\nabla}_X Z) \circ Y = \tilde{\nabla}_Y(X \circ Z) - (\tilde{\nabla}_Y X) \circ Z - (\tilde{\nabla}_Y Z) \circ X,$$

or (on putting $Z = \partial$)

$$\tilde{\nabla}_X(\partial \circ Y) - \tilde{\nabla}_Y(\partial \circ X) = (\tilde{\nabla}_X Y) \circ \partial + (\tilde{\nabla}_X \partial) \circ Y - (\tilde{\nabla}_Y X) \circ \partial - (\tilde{\nabla}_Y \partial) \circ X.$$

So we have

$$\begin{aligned} T^{\nabla}(X,Y) &= \partial^{-1} \circ \left\{ (\tilde{\nabla}_X Y) \circ \partial + (\tilde{\nabla}_X \partial) \circ Y - (\tilde{\nabla}_Y X) \circ \partial - (\tilde{\nabla}_Y \partial) \circ X \right\} - [X,Y], \\ &= \left\{ \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X,Y] \right\} + \partial^{-1} \circ \left\{ (\tilde{\nabla}_X \partial) \circ Y - (\tilde{\nabla}_Y \partial) \circ X \right\}, \\ &= T^{\tilde{\nabla}}(X,Y) + \partial^{-1} \left\{ (\tilde{\nabla}_X \partial) \circ Y - (\tilde{\nabla}_Y \partial) \circ X \right\}, \end{aligned}$$

so ∇ is torsion-free if and only if $(\tilde{\nabla}_X \partial) \circ Y = (\tilde{\nabla}_Y \partial) \circ X$. $\square$

Proposition 5.9 *$\tilde{\eta}$ is a flat metric if and only if η is a flat metric.*

Proof. The Riemann curvature tensor of η is

$$\begin{aligned}
R(X,Y)Z &= \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, \\
&= \partial^{-1}\circ\tilde{\nabla}_X(\partial\circ\nabla_Y Z) - \partial^{-1}\circ\tilde{\nabla}_Y(\partial\circ\nabla_X Z) - \partial^{-1}\circ\tilde{\nabla}_{[X,Y]}(\partial\circ Z), \\
&= \partial^{-1}\circ\tilde{\nabla}_X\left\{\tilde{\nabla}_Y(\partial\circ Z)\right\} - \partial^{-1}\circ\tilde{\nabla}_Y\left\{\tilde{\nabla}_X(\partial\circ Z)\right\} - \partial^{-1}\circ\tilde{\nabla}_{[X,Y]}(\partial\circ Z), \\
&= \partial^{-1}\circ\left\{\tilde{\nabla}_X\tilde{\nabla}_Y - \tilde{\nabla}_Y\tilde{\nabla}_X - \tilde{\nabla}_{[X,Y]}\right\}(\partial\circ Z), \\
&= \partial^{-1}\circ\tilde{R}(X,Y)(\partial\circ Z).
\end{aligned}$$

□

Proposition 5.10 *$\nabla\circ$ is totally symmetric if and only if $\tilde{\nabla}\circ$ is totally symmetric and ∂ is a generalised Legendre field.*

Proof. By definition

$$\begin{aligned}
(\nabla_X\circ)(Y,Z) - (\nabla_Y\circ)(X,Z) = \nabla_X(Y\circ Z) - Y\circ(\nabla_X Z) - Z\circ(\nabla_X Y) \\
+ \nabla_Y(X\circ Z) - X\circ(\nabla_Y Z) - Z\circ(\nabla_Y X),
\end{aligned}$$

$$\begin{aligned}
= \partial^{-1}\circ\tilde{\nabla}_X(\partial\circ Y\circ Z) - Y\circ\left\{\partial^{-1}\circ\tilde{\nabla}_X(\partial\circ Z)\right\} - Z\circ\left\{\partial^{-1}\circ\tilde{\nabla}_X(\partial\circ Y)\right\} + \\
\partial^{-1}\circ\tilde{\nabla}_Y(\partial\circ X\circ Z) - X\circ\left\{\partial^{-1}\circ\tilde{\nabla}_Y(\partial\circ Z)\right\} - Z\circ\left\{\partial^{-1}\circ\tilde{\nabla}_Y(\partial\circ X)\right\},
\end{aligned}$$

$$\begin{aligned}
= \partial^{-1}\{\tilde{\nabla}_X(\partial\circ Y\circ Z) - Y\circ\tilde{\nabla}_X(\partial\circ Z) - Z\circ\tilde{\nabla}_X(\partial\circ Y) \\
- \tilde{\nabla}_Y(\partial\circ X\circ Z) + X\circ\tilde{\nabla}_Y(\partial\circ Z) + Z\circ\tilde{\nabla}_Y(\partial\circ X)\}. \quad (5.11)
\end{aligned}$$

Also, by assumption

$$(\tilde{\nabla}_x\circ)(Y,\partial\circ Z) - (\tilde{\nabla}_Y\circ)(X,\partial\circ Z) = 0,$$

so similarly

$$0 = \tilde{\nabla}_X(\partial \circ Y \circ Z) - Y \circ \tilde{\nabla}_X(\partial \circ Z) - \partial \circ Z \circ \tilde{\nabla}_X(\partial \circ Y) \\ - \tilde{\nabla}_Y(\partial \circ X \circ Z) + X \circ \tilde{\nabla}_Y(\partial \circ Z) + \partial \circ Z \circ \tilde{\nabla}_Y(\partial \circ X). \quad (5.12)$$

(5.11)+$\partial^{-1}\circ$(5.12) yields

$$\begin{aligned}(\nabla_X \circ)(Y, Z) - (\nabla_Y \circ)(X, Z) &= \partial^{-1} \circ \{\partial \circ Z \circ \tilde{\nabla}_X Y - \partial \circ Z \circ \tilde{\nabla}_Y X - \\ &\quad Z \circ \tilde{\nabla}_X(\partial \circ Y) + Z \circ \tilde{\nabla}_Y(\partial \circ X)\}, \\ &= Z \circ (\tilde{\nabla}_X Y - \tilde{\nabla}_Y X) - Z \circ \partial^{-1} \circ \{\tilde{\nabla}_X(\partial \circ Y) - \tilde{\nabla}_Y(\partial \circ X)\}, \\ &= Z \circ [X, Y] - Z \circ [\nabla_X Y - \nabla_Y X],\end{aligned}$$

(since $T^{\nabla} = T^{\tilde{\nabla}} = 0$)

$$= Z \circ [X, Y] - Z \circ [X, Y] = 0,$$

(since ∂ is a generalised Legendre field). □

Corollary 5.13 *From Proposition 5.9 we have that if* ${}^{\lambda}\nabla_X Y := \nabla_X Y + \lambda X \circ Y$ *and* ${}^{\lambda}\tilde{\nabla}_X Y := \tilde{\nabla}_X Y + \lambda X \circ Y$ *then.*

$$R^{{}^{\lambda}\nabla}(X, Y)Z = \partial^{-1} \circ R^{{}^{\lambda}\tilde{\nabla}}(X, Y)(\partial \circ Z).$$

Since the flatness of ${}^{\lambda}\nabla$ is equivalent to having a solution to WDVV we have a new solution but only defined implicitly at this stage. We now go on to give an explicit coordinate dependent realisation of this abstract result. The following proposition states how a generalised Legendre-type transformation generated via $\partial \circ \partial_{\hat{t}^\alpha} = \partial_{t^\alpha}$ may be perfomed.

Proposition 5.14 (Generalised Legendre-type transformations) *Given a solution* *to the WDVV equations as in Proposition (5.3) and a generalised Legendre field ∂ a generalised Legendre-type transformation yields a new solution where the new prepotential and new metric transform precisely analogously to those in a Legendre-type transformation. The new co-ordinates are given by*

$$\frac{\partial \widehat{t}_\alpha}{\partial t^\beta} = \partial^\gamma c_{\alpha\beta\gamma}. \tag{5.15}$$

Proof. Whereas we had

$$\frac{\partial t^\sigma}{\partial \widehat{t}^\alpha} c^\mu_{\kappa\sigma} = \delta^\mu_\alpha,$$

in Proposition (5.3) we now have

$$\partial^\beta \frac{\partial t^\sigma}{\partial \widehat{t}^\alpha} c^\mu_{\beta\sigma} = \delta^\mu_\alpha.$$

On multiplying by $\frac{\partial \widehat{t}^\alpha}{\partial t^\nu}$

$$\partial^\beta \frac{\partial \widehat{t}^\alpha}{\partial t^\nu} \frac{\partial t^\sigma}{\partial \widehat{t}^\alpha} c^\mu_{\beta\sigma} = \frac{\partial \widehat{t}^\mu}{\partial t^\nu},$$

but

$$\frac{\partial \widehat{t}^\alpha}{\partial t^\nu} \frac{\partial t^\sigma}{\partial \widehat{t}^\alpha} = \delta^\sigma_\nu,$$

so

$$\partial^\beta c^\mu_{\beta\nu} = \frac{\partial \widehat{t}^\mu}{\partial t^\nu},$$

or

$$\partial^\beta c_{\beta\nu\mu} = \frac{\partial \widehat{t}_\mu}{\partial t^\nu},$$

(since $\eta_{\mu\alpha} = \widehat{\eta}_{\mu\alpha}$). Using the symmetry of $c_{\beta\mu\nu}$ it follows that

$$\frac{\partial \widehat{t}_\mu}{\partial t^\nu} = \frac{\partial \widehat{t}_\nu}{\partial t^\mu},$$

and hence $\widehat{t_\mu} = \partial_{t^\mu} h$ for some locally defined function h. □

5.1.1 Infinite families of generalised Legendre fields

The geometry of the deformed flat connection encodes a canonical class of Legendre fields. Expanding a flat section ${}^{(\lambda)}\nabla_X s = 0$ as a power series

$$s = \sum_{n=0}^{\infty} \lambda^n \partial_{(n)}$$

and equating coefficients gives

$$\begin{aligned} \nabla_X \partial_{(0)} &= 0\,, & (5.16)\\ \nabla_X \partial_{(n)} &= X \circ \partial_{(n-1)}\,. & (5.17) \end{aligned}$$

Thus each of the fields $\partial_{(n)}$ are Legendre fields. Conversely, starting from a flat vector field $\partial_{(0)}$ one may recursively construction the flat section, with each $\partial_{(n)}$ being a Legendre field. If $\partial_{(0)} = \frac{\partial}{\partial t^\kappa}$ for some κ one obtains an infinite family of Legendre fields labelled by (n,κ). This is similar to the procedure, outlined in [30], of defining recursively the higher flows of an integrable hierarchy of systems of hydrodynamic type from the primary flows defined using a basis of flat vector fields.

Note that, when written in coordinate form (5.17) takes the form

$$\frac{\partial}{\partial t^\alpha} \partial^\beta_{(n,\kappa)} = c^\beta_{\alpha\sigma} \partial^\sigma_{(n-1,\kappa)}\,.$$

Furthermore, it was shown in [13] that the vector field may be written in terms of (scalar) Hamiltonian densities

$$\partial_{(n,\kappa)} = \eta^{\alpha\beta} \frac{\partial h_{(n,\kappa)}}{\partial t^\alpha} \frac{\partial}{\partial t^\beta}\,.$$

With this the coordinate transformation takes a simple form:

$$\begin{aligned}\frac{\partial \tilde{t}^\alpha}{\partial t^\beta} &= \partial^\sigma_{(n,\kappa)} c^\alpha_{\sigma\beta}\,, \\ &= \frac{\partial}{\partial t^\beta} \partial^\alpha_{(n+1,\kappa)}\end{aligned}$$

and hence $\tilde{t}^\alpha = \partial^\alpha_{(n+1,\kappa)}$ or $\tilde{t}_\alpha = \dfrac{\partial h_{(n+1,\kappa)}}{\partial t^\alpha}$ (on lowering an index with the metric η).

Example 5.18 *Returning to the two-dimensional example with prepotential*

$$F = \frac{1}{2} x^2 y + e^y,$$

and Euler vector field

$$E = x\partial_x + 2\partial_y,$$

we now look for Legendre fields of the form

$$\partial_L = a(x,y)\partial_x + b(x,y)\partial_y.$$

If we now apply the condition $(\tilde{\nabla}_x \partial_L) \circ \partial_y = (\tilde{\nabla}_y \partial_L) \circ \partial_x$ *we have, on equating coefficients and recalling that* ∂_x *is the identity and* $\partial_y \circ \partial_y = c^\alpha_{yy}\partial_\alpha = e^y \partial_x$*,*

$$\partial_y : \quad \partial_x a = \partial_y b, \tag{5.19a}$$

$$\partial_x : \quad e^y \partial_x b = \partial_y a. \tag{5.19b}$$

If we additionally impose homogeneity $\mathcal{L}_E \partial_L = \mu \partial_L$ *(equivalent to* $[E, \partial_L] = \mu\partial_L$*) we* *obtain, by equating coefficients*

$$\partial_x : \qquad x\frac{\partial a}{\partial x} + 2\frac{\partial a}{\partial y} - a = \mu a,$$
$$\partial_y : \qquad x\frac{\partial b}{\partial x} + 2\frac{\partial b}{\partial y} = \mu b.$$

Substituting trial functions $a(x,y) = x^{\mu+1}A\left(\frac{e^y}{x^2}\right)$, $b(x,y) = x^{\mu}B\left(\frac{e^y}{x^2}\right)$ *into (5.19) and putting* $z = \frac{e^y}{x^2}$ *yields*

$$1) \qquad (\mu+1)A(z) - 2zA'(z) = zB'(z),$$
$$2) \qquad \mu B(z) - 2zB'(z) = A'(z).$$

Differentiating both of these equations and eliminating $B(z)$ *gives*

$$z(1-4z)A''(z) + 2(2\mu-1)zA'(z) - \mu(\mu+1)A(z) = 0,$$

putting $w = 4z$ *we obtain*

$$w(1-w)A''(w) + \frac{2\mu-1}{2}wA'(w) - \frac{\mu(\mu+1)}{4}A(w) = 0,$$

and see that this is of the form of the hypergeometric differential equation

$$w(1-w)A''(w) + [c-(a+b+1)w]A'(w) - abA(w) = 0,$$

with $a = -\frac{\mu}{2}, b = -\frac{\mu+1}{2}$ *and* $c = 0$ *which has singular points at* $w = 0, 1$ *and* ∞*. The solutions to this differential equation are constructed from the hypergeometric function*

$${}_2F_1(a,b;c;w) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n}\frac{w^n}{n!},$$

where the rising Pochhammer symbol

$$(q)_n = \begin{cases} 1, & n = 0 \\ q(q+1)\cdots(q+n-1), & n > 0 \end{cases}.$$

There exists two linearly independent solutions around each of the singular points [2]:

$$\begin{aligned}
A_{1(0)}(w) &= w\,{}_2F_1(a+1, b+1; 2; w), \\
A_{2(0)}(w) &= w\,{}_2F_1(a+1, b+1; 2; w)\ln w + \\
&\quad w \sum_{n=1}^{\infty} \left\{ w^n \frac{(a+1)_n (b+1)_n}{(2)_n n!} \sum_{s=1}^{n} \left[\frac{1}{s+a} + \frac{1}{s+b} - \frac{1}{s+1} - \frac{1}{s} \right] \right\} + \frac{1}{ab}, \\
A_{1(1)}(w) &= {}_2F_1(a, b; 1+a+b-c; 1-w), \\
A_{2(1)}(w) &= (1-w)^{c-a-b}\,{}_2F_1(c-a, c-b; 1+c-a-b; 1-w), \\
A_{1(\infty)}(w) &= w^{-a}\,{}_2F_1(a, 1+a-c; 1+a-b; w^{-1}), \\
A_{2(\infty)}(w) &= w^{-b}\,{}_2F_1(b, 1+b-c; 1+b-a; w^{-1}).
\end{aligned}$$

(Note that if any of $a, b, c-a$ or $c-b$ is an integer one or more of the above hypergeometric series terminates and the solution is of the form $w^{\alpha}(1-w)^{\beta} p_n(w)$ where $p_n(w)$ is a polynomial of order n. This is called the degenerate case of the hypergeometric differential equation, see [19] for full details.)

Similarly for the hypergeometric differential equation for $B(w)$ we have $a = -\frac{\mu}{2}, b = \frac{1-\mu}{2}$ and $c = 1$. Due to the value of c the solutions around $w = 0$ are

$$\begin{aligned}
B_{1(0)}(w) &= {}_2F_1(a, b; 1; w), \\
B_{2(0)}(w) &= {}_2F_1(a, b; 1; w)\ln w + \\
&\quad \sum_{n=1}^{\infty} \left\{ w^n \frac{(a)_n (b)_n}{(n!)^2} \sum_{s=0}^{n-1} \left[\frac{1}{s+a} + \frac{1}{s+b} - \frac{2}{s+1} \right] \right\},
\end{aligned}$$

the solutions around $w = 1$ and $w = \infty$ take the same form as those for $A(w)$.

For particular values of μ these solutions reduce to much simpler expressions. For instance when $\mu = -1$

$$A(z) = \frac{C_1}{2\sqrt{1-4z}} + C_2,$$

$$B(z) = \frac{-C_1}{\sqrt{1-4z}},$$

and when $\mu = 0$

$$A(z) = C_1\sqrt{1-4z},$$

$$B(z) = -2C_1\mathrm{ArcTan}\sqrt{1-4z} + C_2.$$

We now summarise similar results for the other two-dimensional prepotentials (see subsection (1.1.2)):

- $F(x,y) = \frac{1}{2}x^2y$

  The treatment for this prepotential is considerably simpler than that above and does not involve the hypergeometric differential equation. We find

$$\begin{aligned} a(x,y) &= C_1x^{\mu+1}, \\ b(x,y) &= C_1(\mu+1)y + C_2. \end{aligned}$$

- $F(x,y) = \frac{1}{2}x^2y + y^2\mathrm{log}y$

  The independent variable in the hypergeometric differential equations for both $A(w)$ and $B(w)$ is $w = \frac{y}{8x^2}$. For $A(w)$ we have $a = -\frac{\mu}{4}, b = -\frac{1+\mu}{2}$ and $c = 1$ (and so solutions of the same form as for $B(w)$ above), and for $B(w)$, $a = \frac{1}{8}(-6 - 3\mu + \sqrt{(2-7\mu)(2+\mu)})$, $b = \frac{1}{8}(-6-3\mu-\sqrt{(2-7\mu)(2+\mu)})$ and $c = 0$ (and so solutions of the same form as for those of $A(w)$ above).

- $F(x,y) = \frac{1}{2}x^2y + y^k$

  For $A(w)$ *we have* $w = \frac{8k(2-k)}{k-1}\frac{y^{k-1}}{x^2} + 1$,
  $a = \frac{1}{4}\left\{(2\mu-3)(k-1) - \sqrt{\frac{k[35+(3-2\mu)^2(k^2-3k)+8\mu(\mu-4)]-25}{k-1}}\right\}$,
  $b = \frac{1}{4}\left\{(2\mu-3)(k-1) + \sqrt{\frac{k[35+(3-2\mu)^2(k^2-3k)+8\mu(\mu-4)]-25}{k-1}}\right\}$,
  and $c = k(\mu - \frac{3}{2}) - \mu + \frac{1}{2}$. *When* c *is not an integer the solutions around* $z = 0$ *take the form*

$$\begin{aligned} A_{1(0)}(w) &= {}_2F_1(a,b;c;w), \\ A_{2(0)}(w) &= z^{1-c}\,{}_2F_1(a-c+1, b-c+1; 2-c; w). \end{aligned}$$

  When $c = 0$ *or* 1 *the solutions take the form discussed above and similar expressions for other integral values of* c *(see [2]).*

  For $B(w)$ *we have* $w = \frac{8(2-k)}{k-1}\frac{y^{k-1}}{x^2} + 1$,
  $a = \frac{1}{4}\left\{k(3-2\mu) - \frac{6k-2}{k-1} - \sqrt{\frac{k^3(3-2\mu)^2-4+4k[14+\mu(\mu-7)]-k^2[8\mu(\mu-5)+45]}{k-1}}\right\}$,
  $b = \frac{1}{4}\left\{k(3-2\mu) - \frac{6k-2}{k-1} + \sqrt{\frac{k^3(3-2\mu)^2-4+4k[14+\mu(\mu-7)]-k^2[8\mu(\mu-5)+45]}{k-1}}\right\}$,
  and $c = k\mu + \frac{k(3k-7)}{2(1-k)}$ *and the solutions depend on the integrality of* c, *precisely as for* $A(w)$.

- $F = \frac{1}{2}x^2y + \log y$

  For $A(w)$ *we have* $w = -\frac{24}{13+12\mu}x^2y + 1$,
  $a = \frac{1}{4}\left(-\mu - \sqrt{-\mu(4+3\mu)}\right)$,
  $b = \frac{1}{4}\left(-\mu + \sqrt{-\mu(4+3\mu)}\right)$,
  $c = \frac{2-\mu}{2} + \frac{1}{13-12\mu}$,
  and the solutions depend on the integrality of c *as in the previous example.*

For $B(w)$ we have $w = \frac{24}{13+12\mu} z$ and solutions of the form

$$B(w) = \frac{(1-w)^{\frac{\mu-2}{26+24\mu}}}{w} {}_2F_1(a,b;c;w),$$

where

$$\begin{aligned}
c &= 12(\mu+1)(2-\mu), \\
a &= 18 - 6\mu^2 - \sqrt{348 + \mu(169 + 6\mu(13 + \mu(25 + 6\mu)))}, \\
b &= 18 - 6\mu^2 + \sqrt{348 + \mu(169 + 6\mu(13 + \mu(25 + 6\mu)))},
\end{aligned}$$

and so the solutions depend on the integrality of c as in the above examples.

5.1.2 Generalised Legendre fields for A_3

The prepotential of the Frobenius manifold associated with the Coxeter group A_3 is

$$F = \frac{1}{2}x^2 z + \frac{1}{2}xy^2 + y^2 z^2 + \frac{4}{15}z^5,$$

and has Euler vector field

$$E = x\frac{\partial}{\partial x} + \frac{3}{4}y\frac{\partial}{\partial y} + \frac{1}{2}z\frac{\partial}{\partial z}.$$

We now look look for Legendre fields of the form

$$\partial_L = a(x,y,z)\partial_x + b(x,y,z)\partial_y + c(x,y,z)\partial_z.$$

Applying homogeneity, $[E, \partial_L] = \mu \partial_L$ we obtain

$$x\frac{\partial a}{\partial x} + \frac{3}{4}y\frac{\partial a}{\partial y} + \frac{1}{2}z\frac{\partial a}{\partial z} - a \quad = \mu a, \tag{5.20}$$

$$x\frac{\partial b}{\partial x} + \frac{3}{4}y\frac{\partial b}{\partial y} + \frac{1}{2}z\frac{\partial b}{\partial z} - \frac{3}{4}b \quad = \mu b, \tag{5.21}$$

$$x\frac{\partial c}{\partial x} + \frac{3}{4}y\frac{\partial c}{\partial y} + \frac{1}{2}z\frac{\partial c}{\partial z} - \frac{1}{2}c \quad = \mu c. \tag{5.22}$$

Applying the condition

$$\partial_\beta \circ \nabla_\alpha \partial_L = \partial_\alpha \circ \nabla_\beta \partial_L,$$

for $\alpha, \beta = x, y, z \quad (\alpha \neq \beta)$, yields nine relations by equating coefficients of $\partial_x x, \partial_y$ and ∂_z which can be reduced to six linearly independent ones:

$$\frac{\partial b}{\partial z} = 4y\frac{\partial c}{\partial z} + 4z\frac{\partial c}{\partial y}, \quad \frac{\partial b}{\partial y} = \frac{\partial c}{\partial z} + 4z\frac{\partial c}{\partial x}, \quad \frac{\partial b}{\partial x} = \frac{\partial c}{\partial y}$$

$$\frac{\partial a}{\partial z} = 4y\frac{\partial c}{\partial y} + 16z^2\frac{\partial c}{\partial x}, \quad \frac{\partial a}{\partial y} = 4y\frac{\partial c}{\partial x} + 4z\frac{\partial c}{\partial y}, \quad \frac{\partial a}{\partial x} = \frac{\partial c}{\partial z}.$$

Introducing the variables $w = \frac{y}{x^{3/4}}, v = \frac{z}{x^{1/2}}$ the homogeneity relations are satisfied by

$$a = x^{1+\mu}h(v,w), \quad b = x^{3/4+\mu}g(v,w) \quad \text{and} \quad c = x^{1/2+\mu}f(v,w),$$

where f, g, h are functions to be found. The six relations then become

$$\frac{\partial g}{\partial v} = 4(1/2+\mu)wf - 2wv\frac{\partial f}{\partial v} + \frac{\partial f}{\partial w}(4v - 3w^2),$$

$$\frac{\partial g}{\partial w} = 4(1/2+\mu)vf - 3wv\frac{\partial f}{\partial w} + \frac{\partial f}{\partial v}(1 - 2v^2),$$

$$(3/4+\mu)g - 3/4w\frac{\partial g}{\partial w} - 1/2v\frac{\partial g}{\partial v} = \frac{\partial f}{\partial w},$$

$$\frac{\partial h}{\partial v} = 16v^2(1/2+\mu)f - 8v^3\frac{\partial f}{\partial v} + \frac{\partial f}{\partial w}(4w - 12v^2w),$$

$$\frac{\partial h}{\partial w} = 4w(1/2+\mu)f - 2wv\frac{\partial f}{\partial v} + \frac{\partial f}{\partial w}(4v-3w^2),$$

$$(1+\mu)h - 3/4w\frac{\partial h}{\partial w} - 1/2v\frac{\partial h}{\partial v} = \frac{\partial f}{\partial v}.$$

These equations can be rearranged to obtain

$$\begin{aligned}
\frac{\partial g}{\partial v} &= \frac{4\{4\tilde{f}wC_3 + \tilde{g}C_2 - \tilde{h}wC_1\}}{A}, \\
\frac{\partial g}{\partial w} &= \frac{8\tilde{f}C_2 - \tilde{g}w(C_1+4C_6) + 8\tilde{h}C_5}{A}, \\
\frac{\partial f}{\partial v} &= \frac{8\tilde{f}C_4 - \tilde{g}wC_1 - 4\tilde{h}w(8v^2+4-15vw^2)}{A}, \\
\frac{\partial f}{\partial w} &= \frac{2\{-2\tilde{f}wC_1 + \tilde{g}C_5 - \tilde{h}C_6\}}{A}, \\
\frac{\partial h}{\partial v} &= \frac{8\{-\tilde{f}[v^2(3(C_2-C_4)+4v(8v-9w^2))+9w^4] + \tilde{g}(C_3-2) + 2\tilde{h}C_4\}}{A}, \\
\frac{\partial h}{\partial w} &= \frac{4\{4\tilde{f}w(C_3-2) + \tilde{g}C_2 - \tilde{h}wC_1\}}{A},
\end{aligned}$$

where $C_1 = 24v^3 + 9w^2 - 20v$, $\quad C_2 = 3w^2 + 4v(4v^4 - vw^2 - 1)$, $\quad C_3 = 3v(2v - 4v^3 + w^2)$,

$C_4 = 8v^3 + 16v^5 + 3w^2 - 19v^2w^2$, $\quad C_5 = 8v^4 + 3vw^2 - 2$, $\quad C_6 = 10v^2 - 3$,

$A = 16(2v^2-1)(2v^2+1)^2 + 16vw(9-14v^2) - 27w^4$,

and $\tilde{f} = (1+2\mu f)$, $\quad \tilde{g} = g(3+4\mu)$, $\quad \tilde{h} = h(1+\mu)$.

The system is now in the form

$$\frac{\partial u}{\partial x} = \psi^u_x(v,w,f,g,h), \quad x = v,w, \quad u = f,g,h,$$

and lengthy calculations show that

$$\frac{\partial \psi^\alpha_v}{\partial w} - \frac{\partial \psi^\alpha_w}{\partial v} + \sum_{u=f,g,h}\left(\frac{\partial \psi^\alpha_v}{\partial u}\psi^u_w - \frac{\partial \psi^\alpha_w}{\partial u}\psi^u_v\right) = 0,$$

for each of $\alpha = f,g,h$ and hence satisfies the conditions of

Theorem 5.23 *(Frobenius [18]) The necessary and sufficient conditions for the unique solution $u^\alpha = u^\alpha(x)$ to the system*

$$\frac{\partial u^\rho}{\partial x^i} = \psi_i^\rho(x,u), \quad i = 1, \dots, n, \quad \rho = 1, \dots, N,$$

such that $u(x_0) = u_0$ to exist for any initial data $(u_0, x_0) \in \mathbb{R}^{n+N}$ is that the relations

$$\frac{\partial \psi_i^\alpha}{\partial x^j} - \frac{\partial \psi_j^\alpha}{\partial x^i} + \sum_\beta \left(\frac{\partial \psi_i^\alpha}{\partial u^\beta} \psi_j^\beta - \frac{\partial \psi_j^\alpha}{\partial u^\beta} \psi_i^\beta \right) = 0, \quad i,j = 1, \dots, n \quad \alpha, \beta = 1, \dots, N$$

hold.

Hence the equations generated by the generalised Legendre condition are consistent and so generalised Legendre fields for A_3 exist.

Legendre transformations are maps between solutions of the WDVV equations and so may be applied to standard solutions as well as almost-dual-like solutions.

5.2 Legendre transformations and trigonometric $\vee$-systems

In this section we apply such a Legendre transformation to the solution of the WDVV equations given by extended $\vee$-systems, as constructed in the last chapter. Such systems have a distinguished vector that may be used to define the Legendre transformation, namely the vector in the perpendicular, or extended, direction $\partial = \frac{\partial}{\partial z^\perp}$.

Since $\mathcal{U}^{ext} = \mathcal{U} \cup \mathcal{U}'$, the new variables are given purely in terms of the small-orbit data. In particular:

$$\widehat{z}_\alpha = \frac{\partial^2}{\partial z^\alpha \partial z^\perp} \left\{ \sum_{\beta \in \mathcal{U}'} h_\beta \beta(z)^2 \log \beta(z) \right\} \tag{5.24}$$

since terms involving $\alpha(z)$ with $\alpha \in \mathcal{U}$ are $z^\perp$ independent. The difficulty in applying such a transformation is computational: one has to invert the above change of variables. In the following example we invert these equations for the case when $\mathcal{U} = \mathcal{R}_{A_n}$. The procedure is very general and may easily be applied to other systems.

Example 5.25 *We study the case when* $\mathcal{U} = \mathcal{R}_{A_n}$, *when the the original system is the set of roots of the* A_n *Coxeter group. To do this we utilisze the fact that for* A_n *we have* $\#\vartheta_s = n+1$ *so we can use* n *of them as a a basis for* V. *We label these covectors* $w_i\,, i=0\,,\ldots\,,n$ *with* $w_0 = -\sum_{i=1}^n w_i$.

Using this basis we define (recall $z = z_o + z^{\perp}$*)*

$$\begin{aligned} z_i &= w_i(z)\,, \quad i=0\,,\ldots\,,n \\ z_\perp &= n(z)\,. \end{aligned}$$

Note that $\sum_{i=0}^n z_i = 0$. *With this, the change of variables given by (5.24) reduces to*

$$\begin{aligned} \widehat{z}_i &= \frac{\partial^2}{\partial z_i \partial z_\perp}\left\{ \sum_{r=0}^n 2h_\beta (z_i + z_\perp)^2 \log(z_i + z_\perp)\right\}\,, \\ \widehat{z}_\perp &= \frac{\partial^2}{\partial (z_\perp)^2}\left\{ \sum_{r=0}^n 2h_\beta (z_i + z_\perp)^2 \log(z_i + z_\perp)\right\}\,. \end{aligned}$$

On absorbing constants, or, using the quadratic freedom in the definition of F *one obtains the simple system*

$$\begin{aligned} \widehat{z}_i &= 4h_\beta \left\{\log(z_i + z_\perp) - \log(z_0 + z_\perp)\right\}\,, \\ \widehat{z}_\perp &= 4h_\beta \sum_{r=0}^n \log(z_i + z_\perp) \end{aligned}$$

which is straightforward to invert. This yields

$$\begin{aligned} z_i + z_\perp &= e^{\frac{1}{4h_\beta(n+1)}(\widehat{z}_0+\widehat{z}_\perp)} \,.\, e^{\frac{1}{4h_\beta}\widehat{z}_i}\,, \qquad i = 1\,,\ldots\,,n\,, \\ z_0 + z_\perp &= e^{\frac{1}{4h_\beta(n+1)}(\widehat{z}_0+\widehat{z}_\perp)}\,. \end{aligned}$$

One could go further and solve these, but this is not actually required. All that is required are the terms $\alpha(z)$ and $\beta(z)$. In fact, the above formula are precisely the $\beta(z)$-terms, and to find the $\alpha(z)$-terms one can again use the small-orbit property and write each α as the difference of two small-orbit covectors. Thus if $\alpha = w_i - w_j$ then

$$\begin{aligned} \alpha(z) &= (w_i - w_j)(z)\,, \\ &= \begin{cases} z_i - z_j & \text{if } i\,, j \neq 0\,, \\ z_0 - z_j & \text{if } i = 0\,, j \neq 0\,. \end{cases} \\ &= \begin{cases} e^{\frac{1}{4h_\beta(n+1)}(\widehat{z}_0+\widehat{z}_\perp)}\left(e^{\frac{1}{4h_\beta}\widehat{z}_i} - e^{\frac{1}{4h_\beta}\widehat{z}_j}\right)\,, & \text{if } i\,, j \neq 0\,, \\ e^{\frac{1}{4h_\beta(n+1)}(\widehat{z}_0+\widehat{z}_\perp)}\left(1 - e^{\frac{1}{4h_\beta}\widehat{z}_i}\right)\,, & \text{if } i = 0\,, j \neq 0\,. \end{cases} \end{aligned}$$

To complete the Legendre transformation one has to integrate the equations to find $\widehat{F}$ Using these formulae one finds, schematically, that

$$\frac{\partial^2 \widehat{F}}{\partial \widehat{z}_i \partial \widehat{z}_j} = \text{linear term } + \sum_\sigma \text{terms involving } \left\{\log\left[1 - e^{\sigma(\widehat{z})}\right]\right\}$$

with similar formulae for the $\widehat{z}_\perp$-derivatives. It is important to note that $\widehat{z}_\perp$ only occurs in the linear terms in these expressions. Integrating yields a solution of the WDVV equation *of trigonometric type,*

$$\widehat{F} = \text{cubic term } + \sum_{\sigma \in \widehat{\mathcal{U}}} c_\sigma Li_3 \left(e^{\sigma(z)} \right)$$

where, again, $\widehat{z}_\perp$ only occurs in the cubic term.

Explicit examples of this kind may be found in [20, 33].

Example 5.26 *Applying this Legendre transformation to Example 4.1 yields the prepotential [33]*

$$\widehat{F} = \frac{1}{24} z_\perp^3 - \frac{1}{8} z_\perp z^2 + \frac{1}{2} \left\{ Li_3 \left(e^z \right) + Li_3 \left(e^{-z} \right) \right\} .$$

This is the almost-dual prepotential associated to the potential

$$F = \frac{1}{2} t_1^2 t_2 + e^{t_2} ,$$

which is construction from the extended affine Weyl group $A_1^{(1)}$.

The above example shows a connection between extended $\vee$-systems and extended affine Weyl groups. This link - via a Legendre transformation - will form the subject of the next section.

5.3 Twisted Legendre-type transformations

As we saw in Chapter 3 given a Frobenius manifold F we have an almost-dual manifold $F^\star$. We may also apply a Legendre transformation to F to get a new manifold $\widehat{F}$ and apply almost duality to that. Schematically we have

$$\begin{array}{ccccc} & (F; \tilde{\eta}, \tilde{\nabla}) & \xrightarrow{S_\kappa} & (\widehat{F}; \eta, \nabla) \\ \text{almost duality} & \downarrow & & \downarrow \\ & (F^\star; \widehat{g}, \widehat{\nabla}) & \xrightarrow{\widehat{S}_\kappa} & (\widehat{F}^\star; \check{g}, \check{\nabla}) \end{array}$$

where η and $\tilde{\eta}$, and ∇ and $\tilde{\nabla}$ are related as in Definition (5.5) and $\widehat{S}_\kappa$ is the *twisted* Legendre transformation constructed in [33] and generated by the vector field

$$\widehat{\partial} = E \circ \partial \,,$$

i.e the generating field ∂ twisted by multiplication by the Euler vector field. $\widehat{S}_\kappa$ is a generalised Legendre transformation generated by $\widehat{\partial}$ (see Theorem 5.29). Generally $\widehat{\partial}$ is not a flat vector field but we find the condition for it to be so in this section.

From [23] (Theorem 9.4(a),(e)) $\widehat{\nabla}$ is related to $\tilde{\nabla}$ by

$$\widehat{\nabla}_X(Y) = E \circ \tilde{\nabla}(E^{-1} \circ Y) - \tilde{\nabla}_{E^{-1} \circ Y}(E) \circ X + \frac{1}{2}(D+1) X \circ Y \circ E^{-1}, \tag{5.27}$$

where D is the constant given by

$$L_E(\tilde{g}) = D\tilde{g},$$

so we have

$$\check{\nabla}_X(Y) = E \circ \nabla_X(E^{-1} \circ Y) - \nabla_{E^{-1} \circ Y}(E) \circ X + \frac{1}{2}(\check{D}+1) X \circ Y \circ E^{-1},$$

$$= E \circ \left\{ \partial^{-1} \circ \tilde{\nabla}_X(\partial \circ E^{-1} \circ Y) \right\} - \left\{ \partial^{-1} \circ \tilde{\nabla}_{E^{-1} \circ Y}(\partial \circ E) \right\} \circ X + \frac{1}{2}(\check{D}+1) X \circ Y \circ E^{-1},$$

$$= \partial^{-1} \circ \left\{ \widehat{\nabla}_X(\partial \circ Y) + \tilde{\nabla}_{E^{-1} \circ \partial \circ Y}(E) \circ X - \frac{1}{2}(D+1) X \circ Y \circ E^{-1} \right\}$$
$$- \partial^{-1} \circ \tilde{\nabla}_{E^{-1} \circ Y}(\partial \circ E) \circ X + \frac{1}{2}(\check{D}+1) X \circ Y \circ E^{-1},$$

(on substituting (5.27) for the first term)

$$= \partial^{-1} \circ \widehat{\nabla}_X(\partial \circ Y) + \partial^{-1} \left\{ \tilde{\nabla}_{E^{-1} \circ \partial \circ Y}(E) - \tilde{\nabla}_{E^{-1} \circ Y}(\partial \circ E) \right\} \circ X + \frac{1}{2}(\check{D} - D) X \circ Y \circ E^{-1}.$$

In [9] it was shown that

$$\tilde{\nabla}_Y(u \circ \mathcal{E}^{-1}) = \tilde{\nabla}_{u \circ Y}(\mathcal{E}^{-1}) + U \circ Y,$$

with

$$U = [\mathcal{E}, u] + [e, \mathcal{E}] \circ u.$$

So with $\mathcal{E} \to E^{-1}, u \to \partial$ and $Y \to E^{-1} \circ Y$ we have

$$\tilde{\nabla}_{E^{-1} \circ Y}(\partial \circ E) = \tilde{\nabla}_{E^{-1} \circ \partial \circ Y}(E) + U \circ E^{-1} \circ Y,$$

with

$$U = [E, \partial] + [e, E] \circ \partial,$$

thus

$$\check{\nabla}_X(Y) = \partial^{-1} \circ \widehat{\nabla}_X(\partial \circ Y) + \partial^{-1} \circ \left\{ -U \circ E^{-1} \circ Y \right\} \circ X + \frac{1}{2}(\check{D} - D) X \circ Y \circ E^{-1}.$$

To find U recall that

$$\begin{aligned} E &= \sum_i d_i t^i \partial_i + \sum_{a: d_a = 0} r_a \partial_a, \\ e &= \partial_1, \\ \partial &= \partial_\kappa, \end{aligned}$$

thus $[e, E] = e$, $[E, \partial] = -d_\kappa \partial$ and so $U = (1 - d_\kappa)\partial$. Also recall $D = 2 - d$ and $\check{D} = 2 - \check{d}$. So

$$\check{\nabla}_X(Y) = \partial^{-1} \circ \widehat{\nabla}_X(\partial \circ Y) + \partial^{-1} \circ \left\{ -(1 - d_\kappa)\partial \circ E^{-1} \circ Y \right\} \circ X + \frac{1}{2}(d - \check{d}) X \circ Y \circ E^{-1},$$

$$= \partial^{-1} \circ \widehat{\nabla}_X(\partial \circ Y) + \left[\frac{1}{2}(d - \check{d}) - (1 - d_\kappa)\right] X \circ Y \circ E^{-1}.$$

From [13], Appendix B we have that

$$d = -2\mu_1,$$

so

$$\check{d} = -2\mu_\kappa$$

and

$$d_\kappa = 1 - q_\kappa = 1 - (\mu_\kappa - \mu_1).$$

Therefore

$$\frac{1}{2}(d - \check{d}) - (1 - d_\kappa) = \frac{1}{2}(-2\mu_1 + 2\mu_\kappa) - 1 + 1 - (\mu_\kappa - \mu_1) = 0,$$

hence

$$\check{\nabla}_X(Y) = \partial^{-1} \circ \widehat{\nabla}_X(\partial \circ Y).$$

Hence if we know flat vector fields for $\widehat{\nabla}$ then we can construct flat vector fields for $\check{\nabla}$.

Moreover we can find out when $\widehat{\partial} = E \circ \partial$ is flat for $\widehat{\nabla}$ from (5.27)

$$\begin{aligned}
\widehat{\nabla}_X \widehat{\partial} &= E \circ \tilde{\nabla}_X(E^{-1} \circ \widehat{\partial}) - \tilde{\nabla}_{E^{-1} \circ \widehat{\partial}}(E) \circ X + \frac{1}{2}(D+1) X \circ \widehat{\partial} \circ E^{-1}, \\
&= E \circ \tilde{\nabla}_X(\partial) - \tilde{\nabla}_\partial(E) \circ X + \frac{1}{2}(D+1) X \circ \partial,
\end{aligned}$$

the first term on the right is 0 since ∂ is flat for $\tilde{\nabla}$ and since $\tilde{\nabla}$ is torsion-free

$$\begin{aligned}
\tilde{\nabla}_\partial E - \tilde{\nabla}_E \partial - [\partial, E] &= 0, \\
\tilde{\nabla}_\partial E &= [\partial, E],
\end{aligned}$$

again since ∂ is flat for $\tilde{\nabla}$. So

$$\widehat{\nabla}_X \widehat{\partial} = \left\{ [E, \partial] + \frac{1}{2}(3-d)\partial \right\} \circ X,$$

and since

$$[E, \partial] = \sum_i [d_i t^i \partial_i, \partial_\kappa] = -d_\kappa \partial,$$

we have that

$$\widehat{\nabla}_X \widehat{\partial} = \left[-d_\kappa + \frac{1}{2}(3-d) \right] \partial \circ X.$$

On substituting

$$\begin{aligned} d_\kappa &= 1 - q_\kappa = 1 - (\mu_\kappa - \mu_1), \\ d &= -2\mu_1, \end{aligned}$$

we see that

$$\widehat{\nabla}_X \widehat{\partial} = \left(\mu_\kappa + \frac{1}{2} \right) \partial \circ X, \tag{5.28}$$

and so $\widehat{\partial}$ is flat for $\widehat{\nabla}$ when $\mu_\kappa = -\frac{1}{2}$ and there is a twisted Legendre transformation that is actually a standard Legendre transformation when $-\frac{1}{2}$ is in the spectrum of the Frobenius manifold.

Theorem 5.29 *$\widehat{\partial}$ is a generalised Legendre field for $\widehat{\nabla}$.*

Proof. From (5.28) we have

$$Y \star \widehat{\nabla}_X \widehat{\partial} = \left(\mu_\kappa + \frac{1}{2} \right) \partial \circ E^{-1} \circ X \circ Y.$$

The right side of this equation is symmetric in X and Y so we have

$$Y \star \widehat{\nabla}_X \widehat{\partial} = X \star \widehat{\nabla}_Y \widehat{\partial}.$$

□

Example 5.30 *Consider the prepotential*

$$F = \frac{1}{4} t_2^2 t_3 + \frac{1}{2} t_1 t_3^2 + \frac{1}{2} t_1^2 \log t_2,$$

with Euler vector field

$$E = 2t_1 \partial_{t_1} + \frac{3}{2} t_2 \partial_{t_2} + t_3 \partial_{t_3}.$$

This has $\mu_1 = -\frac{1}{2}$. We will return to this Frobenius manifold in Example 5.35.

5.4 Extended $\vee$-systems and almost duality for extended affine Weyl orbit spaces

One of the main problems in the theory of almost-dual *type* solutions to the WDVV equations (this class including $\vee$-systems) is to indentfy those which are precisely the almost-dual prepotentials to some Frobenius manifold. In [15] a reconstruction theorem was proved, but the theorem is extremely difficult to use in practice. It relies on solving a linear Lax pair, and finding a vector field (to play the rôle of e) which acts on this solution in a specific way. In this section we bypass this reconstruction theorem and prove the following.

Theorem 5.31 *Let W be a finite irreducible classical Coxeter group of rank N and let $\widetilde{W}$ be the extended affine Weyl group of W with arbitrary marked node. Then up to a Legendre transformation, the almost dual prepotentials of the classical extended affine Weyl group orbit spaces $\mathbb{C}^{N+1}/\widetilde{W}$ are, for specific values of the free data, the extended $\vee$-systems of the $\vee$-system R_W. In particular:*

- *The almost dual prepotential corresponding to the orbit space*

$$\mathbb{C}^{l+1}/\widetilde{W}^{(k)}(A_l)$$

is the Legendre transformation, along the extended direction, of the extended $\vee$*-system with:*

  (i) original $\vee$*-system*

$$\begin{aligned} \mathcal{U} &= R_{A_l}\,, \\ h_\alpha &= 1 \qquad \forall \alpha \in \mathcal{U}\,; \end{aligned}$$

  (ii) data for extension (of A-type):

$$\begin{aligned} \vartheta_s &= \{w(\omega_1)\,|w \in W(A_l)\}\,, \\ h_w &= -(l+1-k) \qquad \forall w \in \mathcal{U}^{ext}\,. \end{aligned}$$

  (iii) superpotential data: The superpotential for the extended $\vee$*-system is given by (3.8) with*

$$\mathbf{c}^{ext} = \{-(l+1-k)\,,\underbrace{1\,,\ldots\,,1}_{l+1}\}\,.$$

- *The almost dual prepotential corresponding to the orbit space*

$$\mathbb{C}^{l+1}/\widetilde{W}^{(k)}(C_l)$$

with flat structure defined by the constant m*, where* $0 \leqslant m \leqslant l-k$*, is the Legendre transformation, along the extended direction, of the extended* $\vee$*-system with (where* $s = -2(l-(k+m))$ *,:*

(i) original $\vee$-system

$$\begin{aligned} \mathcal{U} &= R_{B_l}\,, \\ h_{\alpha_{short}} &= 1+s \\ h_{\alpha_{long}} &= 1 \end{aligned}$$

for all $\alpha_{short/long} \in \mathcal{U}$;

(ii) data for extension (of B-type):

$$\begin{aligned} \vartheta_s &= \{w(\omega_1)\,|w \in W(B_l)\}\,, \\ h_w &= -2k \qquad \forall w \in \mathcal{U}^{ext}\,, \\ h_n &= -2k(s+2k)\,. \end{aligned}$$

(iii) superpotential data: The superpotential for the extended $\vee$-system is given by (3.11) with

$$\mathbf{c}^{ext} = \{-2k\,, \underbrace{1\,, \ldots\,, 1}_{l}\}\,.$$

The proof will utilize a Hurwitz space construction. For extended affine Weyl groups of type A this construction was given in [16]. For types $B\,, C\,, D$ this was given in [17].

Hurwitz spaces are moduli spaces of pairs $(\mathcal{C}, \lambda)$, where $\mathcal{C}$ is a Riemann surface of degree g and λ is a meromorphic function on $\mathcal{C}$ of degree N. It was shown in [13] that such spaces may be endowed with the structure of a Frobenius manifold. The $g = 0$ case is particularly simple - meromorphic functions from the Riemann sphere to itself are just given by rational functions. It is into this category of Frobenius manifolds that the examples constructed above fall.

More specifically, the Hurwitz space $H_{g,N}(k_1\,,\dots\,,k_l)$ is the space of equivalence classes $[\lambda : \mathcal{C} \to \mathbb{P}^1]$ of N-fold branched covers[1] with:

- M simple ramification points $P_1, \dots, P_M \in \mathcal{L}$ with distinct *finite* images $l_1, \dots, l_M \in \mathbb{C} \subset \mathbb{P}^1$;
- the preimage $\lambda^{-1}(\infty)$ consists of l points: $\lambda^{-1}(\infty) = \{\infty_1, \dots, \infty_l\}$, and the ramification index of the map p at the point ∞_j is k_j $(1 \leqslant k_j \leqslant N)$.

The Riemann-Hurwitz formula implies that the dimension of this space is $M = 2g + l + N - 2$. One has also the equality $k_1 + \cdots + k_l = N$. For $g > 0$ one has to introduce a covering space, but this is unnecessary in the $g = 0$ case that will be considered here.

In this construction there is a certain ambiguity; one has to choose a so-called primary differential (also known as a primitive form). Different choices produce different solutions to the WDVV equations, but such solutions are related by Legendre transformation S_κ. The Hurwitz data $\{\lambda, \omega\}$ from which one constructs a solution $F_{\{\lambda,\omega\}}$ consists of the map λ (also known as the superpotential) and a particular primary differential ω [13]. Thus, again schematically, one has:

$$F_{\{\lambda,\omega\}} \overset{S_\kappa}{\longleftrightarrow} \widehat{F}_{\{\lambda,\widehat{\omega}\}}$$

(note the map λ does not change, though it might undergo a coordinate transformation). The metrics $<,>$, $(,)$ and multiplications $\circ\,,\star$ are determined by Theorem 1.54.

We divide the proof into the A case and the $B\,,C\,,D$ cases.

5.4.1 Extended Affine Weyl orbit spaces of type A

In [16] it was shown, given an extended affine group $\widetilde{W}^{(k)}(A_l)$, that the orbit space $\mathbb{C}^{l+1}/\widetilde{W}^{(k)}(A_l)$ maybe endowed with the structure of a Frobenius manifold. Additionally, it was shown that this space is isomorphic to the space of trigonometric polynomials

[1] Dubrovin uses the different notation $H_{g,k_1-1\,,\dots\,,k_l-1}$.

of bi-degree $(k, l+1-k)$, namely, functions of the form

$$\lambda(\tilde{z}) = e^{ik\tilde{z}} + a_1 e^{i(k-1)\tilde{z}} + \cdots + a_{l+1} e^{-i(l+1-k)\tilde{z}} \tag{5.32}$$

with the Frobenius manifold structures being given by Theorem 1.54 with the choice of primary differential $\omega = d\tilde{z}$.

This space is related, via a Legendre transformation, to the Hurwitz space $\mathcal{M}_{k,l+1-k}$ of rational functions of the form

$$\lambda(z) = z^k + \alpha_1 z^{k-2} + \ldots + \frac{t^\circ}{z - z^\circ} + \ldots + \frac{\alpha_{l+1}}{(z - z^\circ)^{l+1-k}} \tag{5.33}$$

with primary differential $\omega = dz$. The coefficient t° turns out, by evaluating certain residues, to be a flat coordinate and hence a change in primary differential is given by

$$d\tilde{z} = \{\partial_{t^\circ} \lambda(z)\}\, dz$$

so $\tilde{z} = \log(z - z^\circ)$, and this induces the Legendre transformation between the two Frobenius manifolds, i.e. this change of primary differential induces a change of variable that maps (5.32) to (5.33). Thus the Frobenius manifold structures on $\mathbb{C}^{l+1}/\widetilde{W}^{(k)}(A_l)$ and $\mathcal{M}_{k,l+1-k}$ are related by a Legendre transformation.

Rewriting the rational function (5.33) in this form

$$\lambda(z) = \left. \frac{\prod_{i=1}^{l+1}(z - z_i)}{(z - z^\circ)^{l+1-k}} \right|_{\sum_{i=1}^{l+1} z_i - (l+1-k) z^\circ = 0}$$

thus gives a $\vee$-system which is of extended type, i.e. an extension of the A_l $\vee$-system constructed above, with the zeros of the superpotential being flat coordinates for the metric g. This is a special case, with

$$\mathbf{k} = \{-(l+1-k)\,,\underbrace{1\,,\ldots\,,1}_{l+1}\}$$

of the general superpotential (3.8).

At this stage one could directly calculate the almost-dual prepotential from the trigonometric superpotential, and the result is a solution which is derived from a trigonometric $\vee$-system [33]. However, we will take a different approach and perform the twisted Legendre transformation.

$$\begin{tikzcd}
\mathcal{M}_{k,l+1-k} \arrow[r, leftrightarrow, "\text{Legendre transformation}"] \arrow[d] & \mathbb{C}^{l+1}/\widetilde{W}^{(k)}(A_l) \arrow[d] \\
\left\{\begin{array}{c}\text{almost dual}\\ \text{prepotential of}\\ \text{rational type}\end{array}\right\} \arrow[r, leftrightarrow, "\text{twisted Legendre transformation}"] & \left\{\begin{array}{c}\text{almost dual}\\ \text{prepotential of}\\ \text{trigonometric type}\end{array}\right\}
\end{tikzcd}$$

It turns out that, with the choice $\partial = \partial_{t^\circ}$ the twisted Legendre field is actually constant.

Lemma 5.34 *The twisted Legendre field $E \circ \frac{\partial}{\partial t^\circ}$ is constant in the $\{z^i\}$-variables. Moreover, the field is perpendicular to the space*

$$TV = \text{span}\left\{\frac{\partial}{\partial w^i}\,, i = 1\,,\ldots\,,l\right\}\,,$$

where the variables z^i and w^i are related by the linear change of variables

$$\begin{aligned} w^i &= z^i - \frac{1}{l+1}\sum_{j=1}^{l+1} z^j\,, \qquad j=1\,,\ldots\,,l+1\,,\\ w^{\perp} &= \frac{1}{l+1-k}\sum_{j=1}^{l+1} z^j\,, \end{aligned}$$

together with the constraint $\sum_{i=1}^{l+1} w^i = 0\,.$

Proof. We calculate

$$g\left(E\circ\frac{\partial}{\partial t^{\circ}},\frac{\partial}{\partial z^i}\right)$$

Using the above formulae,

$$\begin{aligned} g\left(E\circ\frac{\partial}{\partial t^{\circ}},\frac{\partial}{\partial z^i}\right) &= \eta\left(\frac{\partial}{\partial t^{\circ}},\frac{\partial}{\partial z^i}\right)\,,\\ &= \sum \mathrm{res}_{d\lambda=0}\left\{\frac{\frac{\partial\lambda}{\partial t^{\circ}}\frac{\partial\lambda}{\partial z^i}}{\lambda'}dz\right\}\,,\\ &= \sum \mathrm{res}_{d\lambda=0}\left\{\frac{1}{(z-z^{\circ})}\frac{\partial\lambda}{\partial z^i}\frac{\lambda}{\lambda'}dz\right\}\,. \end{aligned}$$

A standard residue calculation gives

$$g\left(E\circ\frac{\partial}{\partial t^{\circ}},\frac{\partial}{\partial z^i}\right) = -\frac{1}{l+1-k}\,.$$

In these coordinates,

$$g = \sum_{i=1}^{l+1}(dz^i)^2 - \frac{1}{l+1-k}\left(\sum_{j=1}^{l+1} dz^j\right)^2\,,$$

and hence

$$E \circ \frac{\partial}{\partial t^\circ} = \frac{1}{k} \sum_{j=1}^{l+1} \frac{\partial}{\partial z^j}\,.$$

In the w^i variables,

$$E \circ \frac{\partial}{\partial t^\circ} = \frac{l+1}{k(l+i-k)} \frac{\partial}{\partial w^\perp}\,.$$

and

$$g = \sum_{i=1}^{l} (dw^i)^2 \bigg|_{\sum w^i = 0} - \frac{k(l+1-k)}{l+1} (dw^\perp)^2$$

and hence the twisted Legendre field is perpendicular to the space TV. □

The following example illustrates Theorem 5.31 and also points to the above theory being more widely applicable than we have thus far shown. It shows that, indeed, the Legendre transformation along the extended direction (z_3 in the notation of the example) of the prepotential of the extended A_2 $\vee$-system is almost dual to the prepotential of the orbit space $\mathbb{C}^3/\tilde{W}^{(1)}(A_2)$ and that the Legendre field ($\partial_{\widehat{t_1}}$) generating the Legendre transformation from the prepotential of the Hurwitz space $\mathcal{M}_{1,2}$ to that of $\mathbb{C}^3/\tilde{W}^{(1)}(A_2)$ induces a flat twisted Legendre field to generate the (hence standard, not generalised) Legendre transformation from the extended A_2 $\vee$-system (see the end of the example). It also shows, however, that the same is true when we perform a different Legendre transformation on the prepotential of $\mathbb{C}^3/\tilde{W}^{(1)}(A_2)$ and for which the twisted Legendre field is not flat. We hope to find which Legendre transformations of extended affine Weyl orbit spaces give prepotentials which have extended $\vee$-systems as their almost-dual in future work.

Example 5.35 *The extended A_2 Frobenius manifold found in [16] has prepotential*

$$F = \frac{1}{2} t_1^2 t_3 + \frac{1}{4} t_1 t_2^2 + t_2 e^{t_3} - \frac{1}{96} t_2^4,$$

and Euler vector field

$$E = t_1\partial_1 + \frac{1}{2}t_2\partial_2 + \frac{3}{2}\partial_3.$$

The flat coordinates, z_i, of the intersection form are given, implicitly, by

$$\begin{aligned} t_1 &= e^{\frac{2}{3}z_3}(e^{z_1} + e^{-z_2} + e^{z_2 - z_1}), \\ t_2 &= e^{\frac{1}{3}z_3}(e^{z_2} + e^{-z_1} + e^{z_1 - z_2}), \\ t_3 &= z_3. \end{aligned}$$

We can thus use (3.2) to find the almost dual prepotential:

$$\begin{aligned} F^* = {} & \frac{1}{2}\left[Li_3(e^{2z_2 - z_1}) + Li_3(e^{z_1 - 2z_2})\right] + \frac{1}{2}\left[Li_3(e^{z_2 - 2z_1}) + Li_3(e^{2z_1 - z_2})\right] \\ & + \frac{1}{2}\left[Li_3(e^{z_1 + z_2}) + Li_3(e^{-z_1 - z_2})\right] + \frac{1}{4}z_1 z_2(z_1 - z_2) - \frac{2}{3}(z_1^2 - z_1 z_2 + z_2^2) + \frac{2}{27}z_3^3, \end{aligned}$$

and can perform generalised Legendre transformations generated by the vector fields

$$\widehat{\partial}_{t_i} = E \circ \partial_{t_i} \quad i = 2, 3.$$

Calculating these vector fields gives

$$\begin{aligned} \widehat{\partial}_2 &= \frac{3}{2}e^{t_3}\partial_1 + \left(t_1 - \frac{t_2^2}{4}\right)\partial_2 + \frac{1}{4}t_2\partial_3, \\ \widehat{\partial}_3 &= 2t_2 e^{t_3}\partial_1 + 3e^{t_3}\partial_2 + t_1\partial_3, \end{aligned}$$

or in the z-coordinates

$$\begin{aligned} \widehat{\partial}_2 = \frac{e^{\frac{z_3}{3}}}{2}\Bigg[& \frac{1}{3}\left(-e^{z_1 - z_2} + 2e^{-z_1} - e^{z_2}\right)\partial_{z_1} + \frac{1}{3}\left(e^{z_1 - z_2} + e^{-z_1} - 2e^{z_2}\right)\partial_{z_2} + \\ & \frac{1}{2}\left(e^{z_1 - z_2} + e^{-z_1} + e^{z_2}\right)\partial_{z_3}\Bigg] \end{aligned}$$

$$\widehat{\partial}_3 = e^{\frac{2z_3}{3}} \left[\frac{1}{3}\left(e^{-z_2} + e^{z_2 - z_1} - 2e^{z_1}\right)\partial_{z_1} + \frac{1}{3}\left(2e^{-z_2} - e^{z_2 - z_1} - e^{z_1}\right)\partial_{z_2} + \left(e^{-z_2} + e^{z_2 - z_1} + e^{z_1}\right)\partial_{z_3}\right].$$

We can now perform the transformations as outlined in Proposition 5.14.

Transformation $\widehat{\partial}_2$

From

$$\frac{\partial \widehat{z}_\alpha}{\partial z^\beta} = \partial^\gamma c_{\alpha\beta\gamma},$$

we obtain new coordinates (indices are written subscript for typographical clarity)

$$\begin{aligned}
\widehat{z}_1 &= e^{\frac{1}{3}z_3}(e^{z_1 - z_2} - 2e^{-z_1} + e^{z_2}),\\
\widehat{z}_2 &= -e^{\frac{1}{3}z_3}(e^{z_1 - z_2} + e^{-z_1} - 2e^{z_2}),\\
\widehat{z}_3 &= -\frac{3}{2}e^{\frac{1}{3}z_3}(e^{z_1 - z_2} + e^{-z_1} + e^{z_2}).
\end{aligned}$$

which can be inverted

$$\begin{aligned}
z_1 &= \log\left(\frac{(3\widehat{z}_2 - 2\widehat{z}_3)(3\widehat{z}_1 - 3\widehat{z}_2 - 2\widehat{z}_3)}{(3\widehat{z}_1 + 2\widehat{z}_3)^2}\right)^{\frac{1}{3}},\\
z_2 &= \log\left(\frac{-(3\widehat{z}_2 - 2\widehat{z}_3)^2}{(3\widehat{z}_1 + 2\widehat{z}_3)(3\widehat{z}_1 - 3\widehat{z}_2 - 2\widehat{z}_3)}\right)^{\frac{1}{3}},\\
z_3 &= \log\left(-\frac{1}{9^3}(3\widehat{z}_2 - 2\widehat{z}_3)(3\widehat{z}_1 - 3\widehat{z}_2 - 2\widehat{z}_3)(3\widehat{z}_1 + 2\widehat{z}_3)\right).
\end{aligned}$$

Then from

$$\frac{\partial^2 \widehat{F}^*}{\partial \widehat{z}_a \partial \widehat{z}_b} = \frac{\partial^2 F^*}{\partial z_a \partial z_b}$$

we derive the prepotential

$$\widehat{F}^*_{(2)} = \sum_{\alpha\in\mathcal{R}} (\alpha\cdot z)^2 \log(\alpha\cdot z) + \sum_{\beta\in\mathcal{W}_2} (\beta\cdot z)^2 \log(\beta\cdot z)$$

where (after the linear transformation $z_2 \to -z_2$)

$$\mathcal{R} = \begin{cases} (1,2,0) \\ (1,-1,0) \\ (2,1,0) \end{cases},$$

the roots of the A_2 system and

$$\mathcal{W}_2 = \begin{cases} \pm(1,0,2/3) \\ \pm(0,1,2/3) \\ \pm(-1,-1,2/3) \end{cases},$$

one of the small orbits of the A_2 system plus $2/3$ in the perpendicular direction, and the other small orbit minus $2/3$ in the perpendicular direction.

Transformation $\widehat{\partial}_3$

Similarly we find

$$\begin{aligned}
\widehat{z}_1 &= e^{\frac{2}{3}z_3}(e^{z_2-z_1} - 2e^{z_1} + e^{-z_2}), \\
\widehat{z}_2 &= e^{\frac{2}{3}z_3}(e^{z_2-z_1} + e^{z_1} - 2e^{-z_2}), \\
\widehat{z}_3 &= -3e^{\frac{2}{3}z_3}(e^{z_2-z_1} + e^{z_1} + e^{-z_2}),
\end{aligned}$$

inverting,

$$
\begin{aligned}
z_1 &= \log\left(\frac{(3\widehat{z}_1-\widehat{z}_3)^2}{(3\widehat{z}_2+2\widehat{z}_3)(3\widehat{z}_1-3\widehat{z}_2+\widehat{z}_3)}\right)^{\frac{1}{3}},\\
z_2 &= \log\left(\frac{-(3\widehat{z}_1-\widehat{z}_3)(3\widehat{z}_1-3\widehat{z}_2+\widehat{z}_3)}{(3\widehat{z}_1+\widehat{z}_3)^2}\right)^{\frac{1}{3}},\\
z_3 &= \log\left(-\frac{1}{9^3}(3\widehat{z}_1-\widehat{z}_3)(3\widehat{z}_1-3\widehat{z}_2+\widehat{z}_3)(3\widehat{z}_1+\widehat{z}_3)\right)^{\frac{1}{2}}.
\end{aligned}
$$

and the prepotential is

$$
\widehat{F}^*_{(3)} = \sum_{\alpha\in\mathcal{R}}(\alpha\cdot z)^2\log(\alpha\cdot z) - 2\sum_{\beta\in\mathcal{W}_\ni}(\beta\cdot z)^2\log(\beta\cdot z)
$$

where $\mathcal{R}$ is as above but now

$$
\mathcal{W}_3 = \begin{cases} \pm(1,0,-1/3) \\ \pm(0,1,-1/3) \\ \pm(-1,-1,-1/3) \end{cases},
$$

the small orbits plus or minus $1/3$ *in the perpendicular direction.*

We can perform the standard Legendre transformations S_2 *and* S_3 *to obtain, respectively, the prepotentials*

$$
\begin{aligned}
\widehat{F}_{(2)} &= \frac{1}{12}\widehat{t}_2^3 + \widehat{t}_1\widehat{t}_2\widehat{t}_3 + \frac{1}{2}\widehat{t}_1^2\log\widehat{t}_1 + \frac{1}{3}\widehat{t}_1\widehat{t}_3^3,\\
\widehat{F}_{(3)} &= \frac{1}{4}\widehat{t}_2^2\widehat{t}_3 + \frac{1}{2}\widehat{t}_1\widehat{t}_3^2 + \frac{1}{2}\widehat{t}_1^2\log\widehat{t}_2.
\end{aligned}
$$

Schematically, then, we have

$$\begin{array}{ccccc} \widehat{F}_{(3)} & \xleftarrow{S_3} & F & \xrightarrow{S_2} & \widehat{F}_{(2)} \\ \downarrow & & \downarrow & & \downarrow \\ \widehat{F}^*_{(3)} & \xleftarrow{\widehat{S}_3} & F^* & \xrightarrow{\widehat{S}_2} & \widehat{F}^*_{(2)} \end{array}$$

By composing the three relevant coordinate transformations we find, for S_2*,*

$$\begin{aligned} \widehat{t}_1 &= -\frac{1}{9^3}(3\widehat{z}_1 + 2\widehat{z}_3)(3\widehat{z}_1 - 3\widehat{z}_2 - 2\widehat{z}_3)(3\widehat{z}_2 - 2\widehat{z}_3), \\ \widehat{t}_2 &= \frac{1}{27}[\widehat{z}_3^2 - 3(\widehat{z}_1^2 - \widehat{z}_1\widehat{z}_2 + \widehat{z}_2^2)], \\ \widehat{t}_3 &= -\frac{1}{3}\widehat{z}_3, \end{aligned}$$

and can verify, with (3.2), that $\widehat{F}^*_{(2)}$ *is indeed almost dual to* $\widehat{F}_{(2)}$*. Similarly for* S_3*,*

$$\begin{aligned} \widehat{t}_1 &= \frac{1}{27}[\widehat{z}_3^2 - 3(\widehat{z}_1^2 - \widehat{z}_1\widehat{z}_2 + \widehat{z}_2^2)], \\ \widehat{t}_2 &= \frac{2}{27}\sqrt{(3\widehat{z}_1 - \widehat{z}_3)(3\widehat{z}_1 - 3\widehat{z}_2 + \widehat{z}_3)(3\widehat{z}_2 + \widehat{z}_3)}, \\ \widehat{t}_3 &= -\frac{1}{3}\widehat{z}_3, \end{aligned}$$

and again can check that $\widehat{F}^*_{(3)}$ *is almost dual to* $\widehat{F}_{(3)}$*. Also, performing the Legendre transformation* S_1 *on* $\widehat{F}_{(3)}$ *takes us back to* F *and* $\mu_1 = -1/2$*. Writing* E *in the* $\widehat{t}-$*coordinates we have*

$$E = 2\widehat{t}_1\partial_{\widehat{t}_1} + \frac{3}{2}\widehat{t}_2\partial_{\widehat{t}_2} + \widehat{t}_3\partial_{\widehat{t}_3},$$

and calculating the generating field, $E \circ \widehat{t}_1$*, for the generalised Legendre transform of* $\widehat{F}^*_{(3)}$ *gives*

$$E \circ \partial_{\widehat{t}_1} = \widehat{t}_3\partial_{\widehat{t}_1} + \frac{\widehat{t}_1}{\widehat{t}_2}\partial_{\widehat{t}_2} + \frac{3}{2}\partial_{\widehat{t}_3},$$

which, in the $\widehat{z}-$coordinates is

$$E \circ \partial_{\widehat{t}_1} = -\frac{9}{2}\partial_{\widehat{z}_3},$$

so is flat and thus generates a 'standard' Legendre transformation.

5.4.2 Extended Affine Weyl orbit spaces of type $B\,,C\,,D$

In a similar way, given an extended affine Weyl group of type $B\,,C\,,D$ there exists Frobenius manifold structures on the corresponding orbit space. In [16] this was constructed for a specific choice of marked node and in [17] this construction was generalized to the case of an arbitrary marked node. Thus given a extended affine Weyl goup of C-type, $\widetilde{W}^{(k)}(C_l)$ one can construct a Frobenius manifold structure on the orbit space $\mathbb{C}^{l+1}/\widetilde{W}^{(k)}(C_l)$. However, unlike the A-case, there is an additional freedom in the choice of flat structure on the orbit space, and this freedom is defined in terms of an additional integer $0 \leqslant m \leqslant l-k$. Thus the Frobenius manifold structure on the orbit space - defined by the pair (k,l) - depends on the triple (k,l,m). This Frobenius manifold will be denoted $\mathcal{M}_{k,m}(C_l)$.

This construction also covers the orbit spaces, and their Frobenius manifold structures, for the extended affine Weyl groups $\widetilde{W}^{(k)}(B_l)$ and $\widetilde{W}^{(k)}(D_l)$. The ring of invariant polynomials (freely generated by an appropriate Chevalley-type theorem) for these groups may be obtained from those constructed for the group $\widetilde{W}^{(k)}(C_l)$ by simple changes of variable, and this leads to isomorphic Frobenius manifolds. thus it suffices to study the orbit space $\mathbb{C}^{l+1}/\widetilde{W}^{(k)}(C_l)$ with the Frobenius manifold structure $\mathcal{M}_{k,m}(C_l)$.

Furthermore, it was shown that $\mathcal{M}_{k,m}(C_l)$ coincides with the Frobenius manifold structure on the space of cosine-Laurent series of tri-degree $(2k,2m,2l)$, namely functions of the form

$$\lambda(\tilde{z}) = \frac{1}{(\cos^2\tilde{z}-1)^m}\sum_{j=0}^{l} a_j \cos^{2(k+m-j)}(\tilde{z})\,, \tag{5.36}$$

with the choice of primary differential $\omega = d\tilde{z}$.

This space is related, via a Legendre transformation, to a space of $\mathbb{Z}_2$-invariant rational functions of the form

$$\lambda(z) = z^{2m} + \alpha_{m-1} z^{2(m-1)} + \ldots + \alpha_0 + \sum_{r=1}^{l-(k+m)} \frac{\beta_r}{z^{2r}} + \sum_{s=1}^{k} \frac{\gamma_s}{(z^2 - (z^\circ)^2)^s} \tag{5.37}$$

with primary differential $\omega = dz$. We denote this space $\mathcal{M}^{\mathbb{Z}_2}_{m,l-(k+m),k}$.

The coefficient t°, defined by the term

$$\lambda(z) = \ldots + \frac{z^\circ t^\circ}{(z^2 - (z^\circ)^2)} + \ldots$$

turns out, by evaluating certain residues, to be a flat coordinate and hence a change in primary differential is given by

$$d\tilde{z} = \{\partial_{t^\circ} \lambda(z)\}\, dz\,.$$

so (up to an overall constant that may be ignored) $z = i z^\circ \cot \tilde{z}$, and this induces the Legendre transformation between the two Frobenius manifolds, i.e. this change of primary differential induces a change of variable that maps (5.36) to (5.37). Thus the Frobenius manifold structures on $\mathcal{M}_{k,l}(C_l)$ and the $\mathbb{Z}_2$-graded Hurwitz space $\mathcal{M}^{\mathbb{Z}_2}_{m,l-(k+m),k}$ are related by a Legendre transformation.

As in the A-case, the twisted Legendre transformation between the corresponding almost dual manifolds turns out to a normal Legendre transformation. The extended-$\vee$-systems may be easily calculated by writing the superpotential in the form

$$\lambda(z) = \frac{\prod_{i=1}^{l} \left(z^2 - (z^i)^2\right)}{z^{2(l-(k+m))} \left(z^2 - (z^\circ)^2\right)^{2k}}\,.$$

Unlike the A-case the z_o variable is not constrained: the flat coordinates for the metric are $z^i\,, i = 0\,, \ldots\,, l\,$. This is a special case, with $s = -2\,(l - (k+m))$ and

$$\mathbf{k} = \{-2k\,,\underbrace{1\,,\ldots\,,1}_{l}\}$$

of the general superpotential (3.11).

Lemma 5.38 *The twisted Legendre field $E \circ \frac{\partial}{\partial t^\circ}$ is constant in the $\{z^i\}$-variables. Moreover, the field is perpendicular to the space*

$$TV = \operatorname{span}\left\{\frac{\partial}{\partial z^i}\,, i = 1\,, \ldots\,, l\right\}\,.$$

Proof. Since

$$4k\frac{\partial \lambda}{\partial t^\circ} = \frac{\partial \log \lambda}{\partial z^o}$$

it immediately follows that, for all $i = 0\,, \ldots\,, l\,$:

$$g\left(E \circ \frac{\partial}{\partial t^\circ}, \frac{\partial}{\partial z^i}\right) = g\left(\frac{1}{4k}\frac{\partial}{\partial z^\circ}, \frac{\partial}{\partial z^i}\right)\,.$$

Hence

$$E \circ \frac{\partial}{\partial t^\circ} = \frac{1}{4k}\frac{\partial}{\partial z^\circ}\,.$$

In these variables

$$g = \sum_{i=1}^{l}(dz^i)^2 - 2k(dz^\circ)^2\,,$$

□

and hence it follows that the twisted Legendre field is perpendicular to the space $TV\,$.

We end by noting two things:

(1) To make the connection with Section 4.1

$$
\begin{aligned}
TV \text{ is spanned by } &\left\{\frac{\partial}{\partial w^i} \text{ for } A \text{ type}\right\}, \left\{\frac{\partial}{\partial z^i} \text{ for } B, C, D \text{ type}\right\} \\
TV^{\perp} \text{ is spanned by } &\left\{\frac{\partial}{\partial w^{\perp}} \text{ for } A \text{ type}\right\}, \left\{\frac{\partial}{\partial z^{\perp}} \text{ for } B, C, D \text{ type}\right\}
\end{aligned}
$$

So the twisted Legendre field $E \circ \frac{\partial}{\partial t^{\circ}}$ is perpendicular to the space TV (the constraint in the A-case, $\sum_{i=1}^{l+1} w^i = 0$, is just the manifestation of the standard representation of the A_l roots system as a hyperplane in $\mathbb{R}^{l+1}$.

(2) Since $\tilde{\partial}$ is constant in these (flat)-coordinates, the twisted Legendre transformation is actually a normal Legendre transformation, and such a normal Legendre transformation has already been performed in Section 5.2

The choice of original Legendre field $\partial = \partial_{t^{\circ}}$ was very special - others choices would have resulted in a non-constant twisted Legendre field. As was shown in Section 5.3 this special property comes from the fact that $\mu = -\frac{1}{2}$ lies in the spectrum of the underlying Frobenius manifold [31].

Example 5.39 *We have also found trigonometric $\vee$-systems related to those B_n systems extended into 2 dimensions (see (4.19)), for instance*

$$
\begin{aligned}
F = {} & 512Li_3(e^{\frac{z_1}{4}}) + 512Li_3(e^{\frac{z_2}{4}}) - 80Li_3(e^{\frac{z_1}{2}}) - 80Li_3(e^{\frac{z_2}{2}}) \\
& + 32Li_3(e^{\frac{z_1-z_2}{4}}) + 32Li_3(e^{\frac{z_2-z_1}{4}}) + 64Li_3(e^{\frac{z_1+z_2}{4}}) + \frac{1}{4}(z_1^2 z_2 + z_1 z_2^2) - \frac{1}{12}(z_1^3 + z_2^3) \\
& \pm \frac{1}{4\sqrt{2}}(z_3^2 z_4 + z_3 z_4^2) \pm \frac{1}{12\sqrt{2}}(z_3^3 + z_4^3) \pm \frac{1}{2\sqrt{2}}(z_3 + z_4)(z_1^2 + z_2^2).
\end{aligned}
$$

The existence of such prepotentials immediately raises the question of what (if anything) they are almost-dual to. It is logical to conjecture that the construction of the extended affine Weyl groups in [16] can be similarly extended.

Chapter 6

Conclusions

An obvious question that arises from our work on almost-dual-like solutions that have singular canonical form under the usual definition is whether there exists a generalised version of the $\vee$-conditions which are satisfied by both $\vee$-systems and these new solutions (both complex Euclidean $\vee$-systems with zero canonical form and other systems with simgular metric). Since the canonical form plays a central rôle in derivation of the $\vee$-conditions an entirely new approach would be required in the derivation of the generalised version.

The classification of $\vee$-systems is still open (even in dimension and 3 and restricted to the real case, see [35]) and we have shown that there exists new solutions whose canonical form is identically zero but for which we can recover a solution by imposing a metric. We have computationally found that these solve the WDVV equations for certain values of n and an important future direction would be to prove, or otherwise, that they do for all n A deeper understanding of these new solutions may therefore shed light on the 'standard' $\vee$-systems themselves.

We also saw that our approach of investigating what happens to solutions when the conventional metric becomes singular is not restricted to the logarithmic solutions (1). We found new, multi-parameter polynomial solutions in dimension 3 and a new functional form of solutions (3.18) simply by considering functions that has identically zero metric under the old definition. Not all such functions give a solution, however, so an important question to ask is what are the other conditions a function must satisfy in order to provide a solution. Also how, if at all, are these solutions related to the logarithmic solutions with zero canonical form.

As remarked earlier, and as is apparent from Figure 4.4, an extended $\vee$-system, based, say, on the root system R_W of a classical Weyl group W of rank n, is invariant under the action of W . But on performing a Legendre transformation one obtains configuration invariant under an *extended* affine Weyl group of rank $n+1$. It is therefore natural to ask what is the origin of this extra symmetry that does not appear in the extended $\vee$-system. The answer lies in the precise nature of the Legendre transformation $\widehat{z} \leftrightarrow z$. For example, in the A_n example the perpendicular direction $z_\perp$ is invariant under the affine translation

$$\widehat{z}_\perp \mapsto \widehat{z}_\perp + 8\pi i h_\beta (n+1)$$

since the transformation is exponential. Thus in $\widehat{z}$-space one has a symmetry that is not apparent in the original z-space. Together with the original action of W, one thus obtains an extended affine group action.

The construction in this thesis is dependent on the existence of a small orbit and for exceptional Coxeter groups such orbits do not exist. However the construction in [16], coupled with almost-duality, guarantees a trigonometric $\vee$-system for such exceptional cases (for a specific marked node in [16], and conjecturally for an arbitrary marked node). Whether such systems are the Legendre-transformed versions of some extended rational $\vee$-system is unknown, though it is natural to conjecture that they are. More generally, a natural question to ask is whether there is some direct map between rational $\vee$-systems and trigonometric $\vee$-systems, and if not, to find under what conditions it does exist.

Further examples too would be of interest. there has been recent work on the classification of $\vee$-systems [3, 35] and it would be interesting to see if extended versions of these system exist. One could ask, for example, how the matroid for the extended systems can be constructed from the matroid of the original system. The complex-reflection/Shephard group examples recently constructed in [3] would be a good place to start: these already have interesting symmetry groups automatically build into their construction.

The small orbit property also provides an explanation of the ad-hoc construction of elliptic $\vee$-systems [39] and elliptic solutions to the WDVV equations. These solutions have, as their leading term, a function that by itself is a solution to the WDVV equations of the form (3.3), but an irregular orbit had to be added, but which irregular orbit was unclear. It turns out that the irregular orbit are precisely small orbits. One observation coming from these results is that, for Coxeter groups, the existence of an irreducible quartic invariant polynomial is equivalent to the existence of a small orbit. Whether this is significant or just accidental is unclear.

The WDVV-equations and the rational solutions come from the commutativity, or zero-curvature relations, for the deformed connection

$$\nabla_a = \partial_a + \kappa \sum_{\alpha \in \mathcal{U}} \frac{(\alpha, a)}{(\alpha, z)} \alpha^\vee \otimes \alpha \, .$$

The construction in this chapter can also be thought of in terms of extending such a connection into a dimension higher. The geometry of such a construction also deserves to be studied. Such questions also appear in the Hurwitz space description. For example, consider the A_n-example and its extension. This construction corresponds to adding an extra term to the superpotential:

$$\left. \prod_{i=0}^{n} (z - z^i) \right|_{\sum_{i=1}^{n} z^i = 0} \mapsto \left. \frac{\prod_{i=0}^{n} (z - z_i)}{(z - z^\circ)^k} \right|_{\sum_{i=0}^{n} z_i - n z^\circ = 0}$$

and the geometry of Hurwitz theory requires that $k \in \pm\mathbb{N}$. Algebraically this restriction is not required in the $\vee$-system. It is here that the sign of k effects the geometry (but not the algebra). If k is positive this generates the extended affine Weyl orbit space and a solution that is almost dual to the corresponding Frobenius manifold. If k is negative the superpotential no longer has a pole, but a multiple root. This corresponds to the induced Frobenius structures on discriminant surfaces with a larger manifold [38]. That such induced structures on discriminant generate solutions to the WDVV equations of the form (3.3) was proved in [22].

Finally, root systems and $\vee$-systems appear in many other places in mathematical physics: in the theory of Calogero-Moser and Schrödinger operators, for example [37], and there are rational and trigonometric versions of both of these. Whether these are connected by a suitable Legendre transformation is unknown.

All these questions require further work.

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