SUBGROUPS OF MAPPING CLASS GROUPS AND BRAID GROUPS

Alan McLeay

Submitted in fulfilment of the requirements for the Degree of Doctor of Philosophy

School of Mathematics and Statistics
College of Science and Engineering
University of Glasgow

May 2018

For my family, my friends, et ma française.

Abstract

This thesis studies the subgroup structure of mapping class groups. We use techniques that fall into two categories: analysing the group action on a family of simplicial complexes, and investigating regular, finite-sheeted covering spaces.

We use the first approach to prove that a wide class of normal subgroups of mapping class groups of punctured surfaces are geometric, that is, they have the extended mapping class group as their group of automorphisms, expanding on work of Brendle-Margalit. For example, we determine that every member of the Johnson filtration is geometric. By considering punctured spheres, we also establish the automorphism groups of many normal subgroups of the braid group.

The second approach is to relate subgroups of each of the mapping class groups associated to a covering space, namely, the liftable and symmetric mapping class groups. Given that the two surfaces have boundary, we consider covers in which either every mapping class lifts or every mapping class is fibre-preserving. We classify all covers that fall into one of these cases.

In Chapter 1 we recall some preliminaries before stating the main results of the thesis. We then extend Brendle-Margalit's definition of complexes of regions to surfaces with punctures. Chapter 2 proves that the automorphism group of a complex of regions is the extended mapping class group, resolving in part a metaconjecture of N. V. Ivanov. In Chapter 3 we construct a complex of regions associated to a general normal subgroup of a mapping class group of a surface with punctures. We then apply the main result of the previous chapter to establish that such a normal subgroup is geometric.

Finally, Chapter 4 presents joint work with Tyrone Ghaswala. We give a proof of the Birman-Hilden Theorem for surfaces with boundary and then prove the classifications of regular, finite-sheeted covering spaces of surfaces with boundary discussed above. We conclude by investigating an infinite family of branched covers of the disc. This family induces embeddings of the braid group into mapping class groups. We prove that each of these embeddings maps a standard generator of the braid group to a product of Dehn twists about curves forming a chain, providing an answer to a question of Wajnryb.

Contents

Acknowledgements

First and foremost I wish to thank my advisor Tara Brendle. I am indebted to her for introducing me to mapping class groups and so many other fascinating topics in mathematics. She has been extremely generous with her time, guidance, support, and perspective. For these reasons, and many others, I am very grateful.

I extend my thanks to mathematical mentors and collaborators: Javier Aramayona, Tyrone Ghaswala, Dan Margalit, and Shane Scott for providing many entertaining and stimulating conversations. They also provided places to sleep and food to eat. Special thanks go to my panel Chris Leininger and Vaibhav Gadre for reading through this thesis and providing valuable comments, and also to the convener of the committee Mischa Feigin .

I would like to thank my mother, father, brother, sister, and anyone else who considers me family. Their support over these three and a half years has been fantastic and, of course, expected.

I must also mention the lads, whose warmth, humour, and friendship I have found to be limitless. I hope that in acknowledging them in my thesis with these words I have expressed my gratitude, even if they never read them.

Last but in no way least, I thank Noémie for her love and support. I would not have started a PhD without her encouragement. I would not have continued it without her belief in me. No one has given more for this thesis to be completed, and no one will give more as a result. I hope to one day return her generosity.

Declaration

All work in this thesis was carried out by the author except Chapter 4 which is joint work with Tyrone Ghaswala of the University of Manitoba.

List of Symbols

Σ	a surface
$\Sigma_{g,n}^m$	a surface of genus g with n punctures and m boundary components
$\mathrm{Mod}(\Sigma)$	the mapping class group of Σ
$\mathrm{Mod}(\Sigma, \mathcal{B})$	the subgroup of $\mathrm{Mod}(\Sigma)$ whose elements preserve the set of points $\mathcal{B}$
$\mathrm{Mod}^{\pm}(\Sigma)$	the extended mapping class group of Σ
$\mathrm{Mod}^{\pm}(\Sigma, \mathcal{B})$	the subgroup of $\mathrm{Mod}^{\pm}(\Sigma)$ whose elements preserve the set of points $\mathcal{B}$
$p : \widetilde{\Sigma} \to \Sigma$	a regular, finite-sheeted, possibly branched covering space
$\mathrm{LMod}(\Sigma, \mathcal{B})$	the liftable mapping class group
$\mathrm{SMod}(\widetilde{\Sigma})$	the symmetric mapping class group
$\mathbf{c}$	an essential simple closed curve
$i(\mathbf{a}, \mathbf{b})$	the geometric intersection number of the curves $\mathbf{a}$ and $\mathbf{b}$
$T_{\mathcal{C}}$	a chain twist defined by $T_{\mathbf{c}_1} T_{\mathbf{c}_2} \ldots T_{\mathbf{c}_k}$ where $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_k\}$ is a chain
$\mathcal{C}(\Sigma)$	the curve complex
$\mathcal{R}(\Sigma)$	the set of $\mathrm{Mod}^{\pm}(\Sigma)$-orbits of regions of Σ
η_A	the natural homomorphism defined by the action of $\mathrm{Mod}^{\pm}(\Sigma)$ on $\mathcal{C}_A(\Sigma)$
$\mathcal{C}_A(\Sigma)$	the complex of regions associated to the subset $A \subseteq \mathcal{R}(\Sigma)$
$\mathcal{S}$	the set of separating curves in a surface
$\mathcal{S}(A)$	the subset of $\mathcal{S}$ whose elements separate representatives of $A \subseteq \mathcal{R}(\Sigma)$
$\mathcal{S}(A; k, l)$	the (k, l)-extended set anchored by A
$\mathrm{PAut}(\mathcal{G})$	the pure automorphism group of the fundamental groupoid $\mathcal{G}$
$\mathrm{LAut}_{\mathcal{H}}(\mathcal{G})$	the liftable automorphism group of the groupoid $\mathcal{G}$ where $\mathcal{H} < \mathcal{G}$ is normal
$\mathrm{SAut}(\mathcal{K})$	the symmetric automorphism group

Chapter 1

Introduction

The *mapping class group* $\mathrm{Mod}(\Sigma)$ is the group of symmetries of an oriented surface Σ. In more formal language it is the group of isotopy classes of orientation-preserving self-homeomorphisms of the surface, relative to the boundary.

Mapping class groups are ubiquitous in mathematics. They naturally arise in 4-manifold and 3-manifold topology, through the study of Lefschetz fibrations and Heegaard splittings. Moreover, the mapping class group of a surface has a natural action on the Teichmüller space of the surface. The quotient by this action is the moduli space of Riemann surfaces and hence mapping class groups play a key role in algebraic geometry. Mapping class groups can also be thought of as generalisations of braid groups, arithmetic groups, and automorphism groups of free groups.

Broadly speaking, this thesis studies aspects of the subgroup structure of mapping class groups. In particular, we focus on surfaces that are not closed, that is, they may have punctures, or boundary, or both. We do this in two ways. The first is by investigating the action of the mapping class group on a very general class of simplicial complexes. This approach is inspired by the seminal paper of Ivanov [37] and the recent work of Brendle-Margalit [13]. The second method is by leveraging covering spaces of surfaces. In this case we will make use of the well-known Birman-Hilden Theorem [7].

In Chapter 2 we prove Theorem 1.4.2, a result which partially resolves a metaconjecture of Ivanov. In Chapter 3 we use Theorem 1.4.2 to prove that a each member of a wide class of normal subgroups of the mapping class is geometric, that is, their automorphism groups are the *extended mapping class group.* Finally in Chapter 4 we determine the relative size of two natural subgroups arising from regular, finite-sheeted, possibly branched covering spaces of surfaces with boundary. We end the final chapter by investigating a family of injective homomorphisms from the braid group into mapping class groups.

1.1 Preliminaries

We begin with an overview of mapping class groups, including some basic results and definitions before highlighting the main theorems of this thesis in the next section.

1.1.1 Mapping class groups

Throughout this thesis we will use the notation Σ for an oriented surface, that is, a topological space whose interior locally resembles $\mathbb{R}^2$ with some fixed orientation. A surface Σ is of *finite type* if its fundamental group is finitely generated; otherwise it is said to be of *infinite type*. From now on we will assume that every surface is of finite type unless stated otherwise. Examples of surfaces of finite type include spheres, annuli, and tori. When denoting a specific surface we may use the notation $\Sigma^m_{g,n}$ for a surface homeomorphic to the complement of n singular points and m open discs in a closed surface of genus g. We say that $\Sigma^m_{g,n}$ has m *boundary components* and n *punctures*. When the surface has no boundary components we omit the superscript and when the surface has no punctures we usually omit the second subscript.

Curves

A *simple closed curve* on a surface Σ is an embedding of the circle $\mathbf{c} : S^1 \to \Sigma$. We will usually use the term *curve* when this is unambiguous. We call a curve *essential* if it is not isotopic to a point or a boundary component. Note then that curves that are isotopic to punctures are not essential. Abusing notation, we will write $\mathbf{c}$ for the image of $\mathbf{c}$ in Σ.

For two curves $\mathbf{c}_1$ and $\mathbf{c}_2$ we write $i(\mathbf{c}_1, \mathbf{c}_2) = |\mathbf{c}_1 \cap \mathbf{c}_2|$. We usually consider pairs of curves with minimal intersection with respect to their isotopy class. We therefore use the same notation for the minimal intersection of two isotopy classes of curves, that is, $i([\mathbf{c}_1], [\mathbf{c}_2]) = |\mathbf{c}_1 \cap \mathbf{c}_2|$, where $\mathbf{c}_1$ and $\mathbf{c}_2$ are representative curves with minimal intersection. Throughout this thesis we will often refer to both a curve and its isotopy class by the same name.

We say that a set of curves $\{\mathbf{c}_i\}$ *fills* the surface Σ if when we cut Σ along each of the $\mathbf{c}_i$ we get a collection of discs with zero or one punctures, and annuli that share a boundary component with Σ. Equivalently, the set of curves $\{\mathbf{c}_i\}$ fills Σ if there exist no essential curves disjoint from each $\mathbf{c}_i$. For any surface Σ (other than a disc, a punctured disc, an annulus, or a pair of pants) it is a fact that there exists a pair curves that fills Σ [27]. Indeed, there exist infinitely many such pairs.

Mapping class groups

Let Σ be a surface let and $\mathrm{Homeo}^+(\Sigma)$ be the group of self-homeomorphisms that preserve the orientation of the surface Σ. We define $\mathrm{Homeo}^+(\Sigma, \partial\Sigma)$ to be the subgroup consisting of all homeomorphisms that preserve the boundary pointwise. Note here that the '+' is superfluous, as all homeomorphisms that fix the boundary pointwise are necessarily orientation-preserving. For two homeomorphisms $f_1, f_2 \in \mathrm{Homeo}^+(\Sigma)$ we write $f_1 \sim f_2$ when f_1 and f_2 are isotopic. Isotopy is an equivalence relation and so we

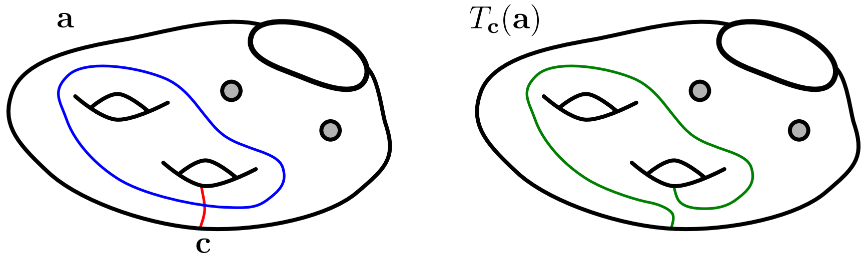

Figure 1.1: The Dehn twist of the curve $\mathbf{a}$ about the curve $\mathbf{c}$, written $T_{\mathbf{c}}(\mathbf{a})$.

define the *mapping class group* of the surface as follows;

$$\mathrm{Mod}(\Sigma) := \mathrm{Homeo}^+(\Sigma, \partial\Sigma)/\sim .$$

We sometimes consider a fixed set of *marked points* $\mathcal{B}$ in a surface Σ. In this case we define the subgroup $\mathrm{Mod}(\Sigma, \mathcal{B}) < \mathrm{Mod}(\Sigma)$ consisting of all elements represented by homeomorphisms that fix the marked points setwise. It is true that if $|\mathcal{B}| = n$ then $\mathrm{Mod}(\Sigma_g^m, \mathcal{B}) \cong \mathrm{Mod}(\Sigma_{g,n}^m)$ and the justification is left to the reader.

For surfaces without boundary we define the *extended* mapping class group $\mathrm{Mod}^{\pm}(\Sigma)$ to be the group consisting of the equivalence classes of all self-homeomorphisms of the surface Σ, including the orientation-reversing ones. It can be seen that $\mathrm{Mod}(\Sigma)$ is an index two subgroup of $\mathrm{Mod}^{\pm}(\Sigma)$.

We will now give some examples of homeomorphisms and mapping classes.

Dehn twists

Let $\Sigma_0^2 = S^1 \times [0, 1]$ be an annulus with some fixed orientation. We define a homeomorphism of Σ_0^2 as follows;

$$\begin{aligned} T : \Sigma_0^2 &\to \Sigma_0^2 \\ (\theta, t) &\mapsto (\theta + 2t\pi, t). \end{aligned}$$

The homeomorphism T fixes the boundary pointwise and so T is orientation-preserving. It follows then that T represents an element of $\mathrm{Mod}(\Sigma_0^2)$. In fact, this element generates the group $\mathrm{Mod}(\Sigma_0^2) \cong \mathbb{Z}$, see Farb-Margalit [27, Proposition 2.4].

Now, every surface Σ contains a simple closed curve $\mathbf{c}$. Writing $A_{\mathbf{c}}$ for an annular neighbourhood of $\mathbf{c}$ we can define an orientation-preserving homeomorphism $f : \Sigma_0^2 \to A_{\mathbf{c}}$. The *Dehn twist about* $\mathbf{c}$ is defined as follows;

$$T_{\mathbf{c}}(x) = \begin{cases} f \circ T \circ f^{-1}(x) \text{ if } x \in A_{\mathbf{c}} \\ x \text{ otherwise.} \end{cases}$$

If Σ is homeomorphic a to disc or a punctured disc then any such Dehn twist is isotopic to the identity homeomorphism. If Σ is any other surface and $\mathbf{c}$ is essential, or isotopic to a boundary component, then $T_{\mathbf{c}}$ represents a non-trivial element of $\mathrm{Mod}(\Sigma)$.

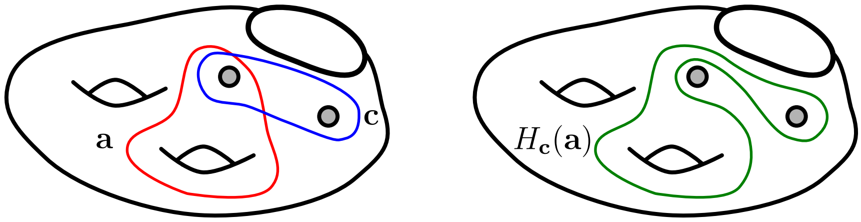

Figure 1.2: The Half twist of the curve **a** about the curve **c**, written $H_{\mathbf{c}}(\mathbf{a})$.

From now on we will refer to both the mapping class and the homeomorphism as a *Dehn twist* and write $T_{\mathbf{c}} \in \mathrm{Mod}(\Sigma)$. We note that two isotopic curves admit Dehn twists that are equal as mapping classes. Note that there is a choice between "twisting left" and "twisting right". We will use the convention that $T_{\mathbf{c}}$ is a left twist and $T_{\mathbf{c}}^{-1}$ is a right twist.

Dehn twists are fundamental in the study of mapping class groups. This is partly because, in some sense, they are among the "smallest elements" as they are defined on annular subregions. The main reason they are of such importance however, is that they generate any mapping class group of a closed surface Σ_g. Moreover, there exists a finite generating set of Dehn twists for $\mathrm{Mod}(\Sigma_g)$ consisting of $2g+1$ elements as shown by Humphries [34]. In fact, the mapping class group of any surface without punctures is finitely generated by Dehn twists. If Σ is a surface with punctures however, we need another type of mapping class to generate $\mathrm{Mod}(\Sigma)$.

Half twists

Let $\mathbf{D}_2 \cong \Sigma_{0,2}^1$ be a disc with two punctures. As with the case of the annulus given above, the mapping class group $\mathrm{Mod}(\mathbf{D}_2) \cong \mathbb{Z}$ is generated by a single element. We can think about a representative homeomorphism of this generator as a *half Dehn twist*, or simply a *half twist*. Heuristically, we cut a line between the two punctures, resulting in a surface whose interior is homeomorphic to the interior of an annulus. We then perform a half Dehn twist to this annulus and glue back along the line. This preserves the boundary component of $\mathbf{D}_2$ and it "swaps the punctures".

Let $\mathbf{c}$ be a curve bounding a disc $\mathbf{D}_{\mathbf{c}}$ with two punctures in Σ. Let $f : \mathbf{D}_2 \to \mathbf{D}_{\mathbf{c}}$ be an orientation-preserving homeomorphism. Writing H for the homeomorphism described above, we define the *half twist about* $\mathbf{c}$ as follows;

$$H_{\mathbf{c}}(x) = \begin{cases} f \circ H \circ f^{-1}(x) \text{ if } x \in \mathbf{D}_{\mathbf{c}} \\ x \text{ otherwise.} \end{cases}$$

From now on we refer to the mapping class containing such a homeomorphism as a *half twist* and we denote it by $H_{\mathbf{c}} \in \mathrm{Mod}(\Sigma)$. A picture of a half twist is shown in Figure 1.2. Now, half twists generate mapping class groups of punctured discs and punctured spheres. Furthermore, any mapping class group is generated by a set of finitely many Dehn twists and finitely many half twists [27, Corollary 4.15]. As noted previously,

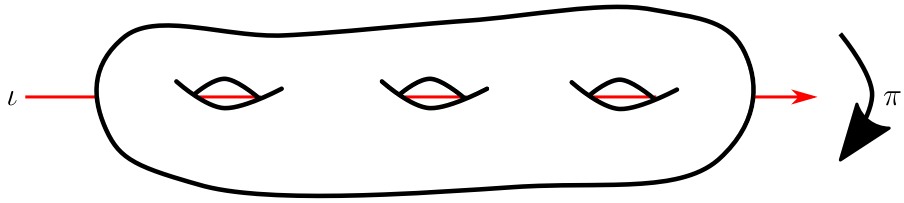

Figure 1.3: The hyperelliptic involution $\iota : \Sigma_3 \to \Sigma_3$. There are 8 fixed points indicated by where the red "kebab skewer" intersects Σ_3.

the groups $\mathrm{Mod}(\Sigma_{g,n}^m)$ and $\mathrm{Mod}(\Sigma_g^m, \mathcal{B})$ are isomorphic when $|\mathcal{B}| = n$. It is easy to see that the construction of half twists can be generalised to mapping classes that "swap two marked points". We also refer to such mapping classes as half twists and it will be clear from context to which one is being referred.

So far, we have seen that $\mathrm{Mod}(\Sigma)$ is finitely generated. In fact, the extended mapping class group $\mathrm{Mod}^{\pm}(\Sigma)$ is also finitely generated as we only require one additional orientation-reversing mapping class.

1.1.2 Classification and supports

So far we have seen two mapping classes that generate all others. The definitions in this section amount to a statement of the Nielsen-Thurston classification of all possible elements of $\mathrm{Mod}(\Sigma_{g,n})$.

Supports

The *support* R of a homeomorphism $f \in \mathrm{Homeo}^+(\Sigma, \partial\Sigma)$ is the minimal subsurface of Σ with essential boundary components such that f restricted to the complement of R is the identity homeomorphism. Here, we mean minimal with respect to subsurface containment. Similarly we say that R is a *support* of the mapping class $[f] \in \mathrm{Mod}(\Sigma)$. The support of a mapping class is unique up to homotopy equivalence of subsurfaces [8].

Periodic mapping classes

We call $[f]$ a *periodic* element of $\mathrm{Mod}(\Sigma)$ if there exists an integer k such that $[f]^k = \mathrm{id}$. In other words, $[f]$ generates a finite cyclic subgroup. Note that the support of any periodic mapping class is the entire surface Σ. This implies that the mapping class group of a surface with boundary cannot contain non-trivial periodic elements. This fact will be explored in greater depths in Chapter 4. An example of a periodic mapping class we will visit multiple times in Chapters 1 and 4 is the *hyperelliptic involution* which we denote $\iota \in \mathrm{Mod}(\Sigma_{g,n})$.

Consider a closed surface Σ_g embedded in $\mathbb{R}^3$ such that there exists a straight line that intersects Σ_g at $2g+2$ points. Rotation by π (or 180°) about this axis defines the homeomorphism ι, see Figure 1.3. It is clear that this element is periodic and that it generates the cyclic subgroup $\langle \iota \rangle \cong \mathbb{Z}/2\mathbb{Z}$.

Reducible mapping classes

A *reducible* element is one that fixes an isotopy class of a multicurve (a disjoint set of essential simple closed curves). An example of a reducible mapping class is a Dehn twist. Note that in the case where the surface is a torus with no punctures, the only isotopy class of curves fixed by $T_{\mathbf{c}}$ is $\mathbf{c}$ itself. Unlike periodic mapping classes, there are no restrictions on the supports of reducible mapping classes. For example, the hyperelliptic involution ι is also reducible in that it fixes a set of so-called *symmetric* curves. As we have seen the support of ι is the entire surface.

Pseudo-Anosov mapping classes

A *pseudo-Anosov* mapping class is one which is neither periodic, nor reducible. This definition, while concise, does not do justice to the deep structure behind pseudo-Anosovs. Informally these mapping classes stretch and squeeze the surface along two *transverse measured foliations*. We will visit this structure in Chapter 3 and use it to prove the main result of this thesis.

1.2 Normal subgroups

In this section we consider possibly punctured surfaces without boundary. We say that a normal subgroup N of $\mathrm{Mod}(\Sigma)$ is *geometric* if its automorphism group is the extended mapping class group $\mathrm{Mod}^{\pm}(\Sigma)$. In his seminal paper, Ivanov showed that the mapping class group of a surface of genus at least three, or of genus two with punctures, is itself geometric [37]. The analogous result was given by Korkmaz for surfaces of genus one and spheres with punctures [45]. The proofs of these results use the action of $\mathrm{Mod}^{\pm}(\Sigma)$ on the *curve complex*, a simplicial complex related to Σ that we will define in Section 1.4.1. Ivanov's result, and proof, acted as a springboard for a series of related results; see Bavard-Dowdall-Rafi [3], Brendle-Margalit [12], Bridson-Pettet-Souto [16], Irmak [35], and Kida [43] among many others.

The Torelli group and the Johnson filtration

There is a natural action of $\mathrm{Mod}(\Sigma)$ on the first homology group of the surface $H_1(\Sigma)$. The associated representation is known as the *symplectic representation*. This representation has a large kernel $\mathcal{I}(\Sigma)$ known as the *Torelli group* and has been an object of great interest in the study of mapping class groups, see Johnson [39] [40], Kasahara [41], Mess [56], and Putman [59] [61], to name only a few. It was shown by Farb-Ivanov for a closed surface Σ of genus at least 5 that $\mathcal{I}(\Sigma)$ is geometric [26].

The Torelli group can be generalised as follows. Write Γ_0 for the fundamental group of the surface Σ and define $\Gamma_k := [\Gamma_0, \Gamma_{k-1}]$ to be the k^{th} term in the lower central series of Γ_0. We now define a sequence of groups $\{\mathcal{N}_k(\Sigma)\}$ to be the kernels of the group homomorphisms

$$\mathrm{Mod}(\Sigma) \to \mathrm{Out}(\Gamma_0/\Gamma_k).$$

It was shown by Bass-Lubotzky that this is a filtration, that is, an infinite sequence of nested groups with trivial intersection [2]. The first term in this sequence is the Torelli group and the second term is named the *Johnson kernel* $\mathcal{J}(\Sigma)$. We call the entire sequence the *Johnson filtration.* It was shown by Brendle-Margalit for closed surfaces of genus at least 4 that $\mathcal{J}(\Sigma)$ is geometric [12]. A proof has also recently been given by Brendle-Margalit, for closed surfaces of genus at least 7, that $\mathcal{N}_k(\Sigma)$ is geometric for all k [13]. This result was originally announced by Bridson-Pettet-Souto [16].

In fact the work of Brendle-Margalit goes much further. They prove that for closed surfaces, each member of a wide class normal subgroups is geometric. This class includes (but goes well beyond) all the examples given above, provided the genus of the surface is high enough.

1.2.1 Statement of the theorem

Theorem 1.2.1 of this thesis extends the result of Bredle-Margalit to surfaces with punctures. In other words, it shows for a possibly punctured surface Σ that each member of a wide class of normal subgroups is geometric. This gives the first proof that every term in the Johnson filtration of a punctured surface is geometric. Proofs that the Torelli group $\mathcal{I}(\Sigma)$ and the Johnson kernel $\mathcal{J}(\Sigma)$ are geometric in this case are covered by the work of Kida [43].

In order to state this result we must first define the class of normal subgroups for which it holds. This definition is dependent on the supports of the elements contained in the subgroup.

Regions and small mapping classes

Let Σ be a surface of genus g and with n marked points. A *region* R of Σ is a connected, compact subsurface of Σ such that each component of its boundary is an essential simple closed curve. We write $g(R)$ and $n(R)$ for the genus of R and number of punctures in R respectively.

Suppose the support of a mapping class $[f] \in \mathrm{Mod}(\Sigma)$ is contained in a single boundary region R. If Σ is a sphere with punctures we say that $[f]$ is *small* if $n \geq 3n(R)-1$. If Σ is a torus with punctures we say that $[f]$ is small if $n \geq \max\{3n(R), 7\}$. Finally, in the general case we say $[f]$ is *small* if

$$g \geq g(R)+1 \quad \text{and} \quad n \geq 3n(R).$$

As well as proving that many normal subgroups are geometric, Theorem 1.2.1 also proves that the *abstract commensurator* group of such a normal subgroup is $\mathrm{Mod}^{\pm}(\Sigma)$. Recall, that for any group G we define $\mathrm{Comm}\, G$ to be the group of equivalence classes of isomorphisms between finite index subgroups of G, where two isomorphisms are equivalent if they agree on some finite index subgroup. A survey on abstract commensurator groups has been written by Studenmund [64]. Furthermore, recall that inherent in the definition of $\mathrm{Mod}^{\pm}(\Sigma)$ is that Σ has empty boundary.

Theorem 1.2.1. *Let Σ be a surface with punctures and let N be a normal subgroup of $\mathrm{Mod}^{\pm}(\Sigma)$ containing a small element. The natural homomorphisms*

$$\mathrm{Mod}^{\pm}(\Sigma) \to \mathrm{Aut}\, N \to \mathrm{Comm}\, N$$

are isomorphisms. Furthermore, if N is normal in $\mathrm{Mod}(\Sigma)$ but not in $\mathrm{Mod}^{\pm}(\Sigma)$ then the natural homomorphism

$$\mathrm{Mod}(\Sigma) \to \mathrm{Aut}\, N$$

is an isomorphism.

Collecting all the results discussed above, one may be fooled into thinking that *every* normal subgroup of $\mathrm{Mod}(\Sigma)$ is geometric. This is not true. The question surrounding non-geometric normal subgroups has been explored by Clay-Mangahas-Margalit [20] and Dahmani-Guirardel-Osin [21]. Encouragingly, the requirement for a normal subgroup to be non-geometric in these cases is similar to the the statement of Theorem 1.2.1, that is, they are determined by the relative topological size of the supports of their elements. It can be expected that the definition of small may be improved to the following:

Mutually small

Two mapping class $[f_1], [f_2] \in \mathrm{Mod}(\Sigma)$ are *mutually small* if their supports are subsurfaces R, Q that are disjoint up to homotopy and such that $\Sigma \setminus \{R, Q\}$ is *not* an annulus.

The conjecture below has an analogue for the closed surface case given by Brendle-Margalit [13].

Conjecture 1.2.2. *Let Σ be a possibly punctured surface without boundary. A normal subgroup of $\mathrm{Mod}(\Sigma)$ is geometric if and only if it contains a pair of mapping classes that are mutually small.*

A further conjecture by Clay-Mangahas-Margalit suggests the form all non-geometric normal subgroups can take [20]. Resolving both conjectures would give a complete picture of normal subgroups of mapping class groups for *all* surfaces with empty boundary. Techniques from this thesis may be of use in tackling this problem. It is likely however, that new ideas will also be required.

1.3 Genus zero case

Consider a surface $S_n = \Sigma_{0,n}$, that is, a n times punctured sphere. We can apply a version of Theorem 1.2.1 to this special case to arrive at a number of interesting results concerning *braid groups*. This application is viewed separately from the general case in a paper by the author [55].

1.3.1 Braid groups

The *braid group* on n-strands B_n is generated by $\sigma_1, \ldots, \sigma_{n-1}$ subject to the relations

$$\sigma_i\sigma_j = \sigma_j\sigma_i \text{ for } |i-j| > 1 \qquad \text{and} \qquad \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j \text{ for } |i-j| = 1.$$

The latter are known as the *braid relations*. The centre Z of B_n is generated by the element $(\sigma_1 \ldots \sigma_{n-1})^n$ [18]. We establish the automorphism group of any normal subgroup N of B_n such that $N \cap Z$ is trivial, providing N contains a product involving at most one third of the generators given above. This result is given explicitly in Corollary 1.3.3, the proof of which relies on an interpretation of B_n/Z as a subgroup of $\mathrm{Mod}(S_n)$.

Braid groups as mapping class groups

Let $\mathbf{D}_n$ be a disc with n punctures, that is, a surface homeomorphic to $\Sigma^1_{0,n}$. One can define an isomorphism from B_n to $\mathrm{Mod}(\mathbf{D}_n)$ such that the image of σ_i is a half twist. By collapsing the boundary $\partial\mathbf{D}_n$ to a point p, and considering p as a marked point of S_n, we see that $\mathrm{Mod}(S_n, p)$ is isomorphic to $\mathrm{Mod}(\mathbf{D}_n)$ modulo the subgroup generated by the Dehn twist about a curve isotopic $\partial\mathbf{D}_n$ [18]. This Dehn twist generates the centre Z of B_n and so it follows that

$$B_n/Z \cong \mathrm{Mod}(S_n, p).$$

Analogously to $\mathrm{Mod}(S_n, p)$, we define the subgroup $\mathrm{Mod}^{\pm}(S_n, p)$ of $\mathrm{Mod}^{\pm}(S_n)$ consisting of all elements that fix the marked point p. By removing the point p from the surface we get a natural inclusion map $\mathrm{Mod}(S_n, p) \hookrightarrow \mathrm{Mod}(S_{n+1})$. We can therefore define the following commutative diagram of inclusion maps:

$$\begin{tikzcd}
 & \mathrm{Mod}(S_n, p) \arrow[dl, hook] \arrow[dd, hook] \arrow[dr, hook] & \\
\mathrm{Mod}(S_{n+1}) \arrow[dr, hook] & & \mathrm{Mod}^{\pm}(S_n, p) \arrow[dl, hook] \\
 & \mathrm{Mod}^{\pm}(S_{n+1}). &
\end{tikzcd}$$

Note that if N is a subgroup of $\mathrm{Mod}(S_n, p)$ that is normal in both $\mathrm{Mod}(S_{n+1})$ and $\mathrm{Mod}^{\pm}(S_n, p)$ then it is also normal in $\mathrm{Mod}^{\pm}(S_{n+1})$. Furthermore, each subgroup in the diagram is of finite index. We define the injective homomorphism

$$\Psi : B_n/Z \hookrightarrow \mathrm{Mod}^{\pm}(S_{n+1})$$

by the isomorphism and inclusion maps discussed above. We use this interpretation to obtain the following result.

Theorem 1.3.1. *Let N be a normal subgroup of B_n/Z containing an element represented by a product of at most $(n-1)/3$ standard generators. Then* $\mathrm{Aut}\, N$ *is isomorphic to the normaliser of* $\Psi(N)$ *in* $\mathrm{Mod}^{\pm}(S_{n+1})$.

In other words, $\mathrm{Aut}\, N$ is isomorphic to one of $\mathrm{Mod}(S_n, p)$, $\mathrm{Mod}(S_{n+1})$, $\mathrm{Mod}^{\pm}(S_n, p)$, or $\mathrm{Mod}^{\pm}(S_{n+1})$. This result is a direct consequence of a version of Theorem 1.2.1 for punctured spheres and relies on the notions of small mapping classes discussed in Section 1.2. In order to arrive at the precise wording of Theorem 1.3.1 we note that a small element of $\mathrm{Mod}(S_{n+1})$ has support contained in a disc with $(n+2)/3$ punctures. We can therefore express this element as a product of at most $(n-1)/3$ half twists. If N is a small normal subgroup of $\mathrm{Mod}(S_{n+1}, p)$ this implies the corresponding subgroup of B_n/Z contains an element represented by a product of at most $(n-1)/3$ standard generators. We give a version of Theorem 1.2.1 for $\mathrm{Mod}(S_n, p)$ in Chapter 3 and in doing so complete the proof of Theorem 1.3.1.

As an example, we can apply Theorem 1.3.1 to the normal subgroup $\mathcal{BI}/Z$, where $\mathcal{BI}$ is the *braid Torelli group* (see [15]). It follows that $\mathrm{Aut}\,\mathcal{BI}/Z \cong \mathrm{Mod}^{\pm}(S_{n+1})$. Furthermore, this is also true for each of the congruence subgroups of $\mathcal{BI}$ modulo the centre Z.

We may also apply Theorem 1.3.1 to the group B_n/Z itself. We see that $\mathrm{Aut}\, B_n/Z \cong \mathrm{Mod}^{\pm}(S_{n+1}, p)$. This can also be shown using the fact that $\mathrm{Mod}(S_n, p)$ is a finite index subgroup of $\mathrm{Mod}^{\pm}(S_{n+1})$ [18]. Considering the braid relations, and the fact that $Z \cong \mathbb{Z}$, one is able to prove that the natural homomorphism

$$\mathrm{Aut}\, B_n \to \mathrm{Aut}\, B_n/Z$$

is an isomorphism. A short proof of this fact is given at the end of Chapter 3. The group $\mathrm{Mod}^{\pm}(S_n, p)$ can be generated by elements of $\mathrm{Mod}(S_n, p)$ and a single orientation-reversing element of $\mathrm{Mod}^{\pm}(S_n, p)$ and so we therefore recover the following isomorphism of Dyer-Grossman [23].

Corollary 1.3.2. *If $n \geq 4$ then*

$$\mathrm{Aut}\, B_n \cong \mathrm{Mod}^{\pm}(S_n, p) \cong B_n/Z \rtimes \mathbb{Z}/2\mathbb{Z}.$$

If N is any normal subgroup of B_n such that $N \cap Z = \{1\}$ then N is isomorphic to a normal subgroup of B_n/Z. It follows that Theorem 1.3.1 also gives the automorphism groups of subgroups of this type.

Corollary 1.3.3. *Let N be a normal subgroup of B_n containing a product of at most $(n-1)/3$ standard generators. If $N \cap Z = \{1\}$ then $\operatorname{Aut} N$ is isomorphic to the normaliser of $\Psi(N)$ in $\operatorname{Mod}^{\pm}(S_{n+1})$.*

In particular, the commutator subgroup $[B_n, B_n]$ has trivial intersection with the centre, leading to the following result.

Corollary 1.3.4. *If $n \geq 7$ then*

$$\operatorname{Aut}[B_n, B_n] \cong \operatorname{Mod}^{\pm}(S_n, p) \cong B_n/Z \rtimes \mathbb{Z}/2\mathbb{Z}.$$

This isomorphism was originally proved by Orevkov for $n \geq 4$ [58]. We note that each term further down the lower central series and the derived series of the braid group is equal to $[B_n, B_n]$, see for example [30]. The pure braid group PB_n is the kernel of the natural homomorphism from B_n to the symmetric group on the set of n elements. While Z is also the centre of PB_n, it has trivial intersection with its commutator subgroup $[PB_n, PB_n]$. Since PB_n is characteristic in B_n we are able to establish the automorphism group of $[PB_n, PB_n]$ and in fact we arrive at a more general result.

Corollary 1.3.5. *If $n \geq 7$ and Γ is any term in the lower central series or derived series of PB_n then $\operatorname{Aut}\Gamma$ is isomorphic to $\operatorname{Aut} PB_n/Z \cong \operatorname{Mod}^{\pm}(S_{n+1})$.*

Note that $[B_n, B_n]$, and each subgroup in the statement of Corollary 1.3.5, contains elements that can be written as a product of two standard generators. Therefore to apply Corollary 1.3.3 we require that $(n-1)/3 \geq 2$, that is, $n \geq 7$. The relationship between spheres with punctures and braid groups has also been studied by Bell-Margalit [4] [5], Charney-Crisp [18], and Leininger-Margalit [49].

1.3.2 The hyperelliptic Johnson filtration

A further application of Theorem 1.2.1 in the case of spheres actually gives us information about a family of subgroups of the group $\operatorname{Mod}(\Sigma_g)$, where Σ_g is a closed surface of genus g. Let $\iota : \Sigma_g \to \Sigma_g$ be a fixed hyperelliptic involution. We abuse notation by writing ι for the element of $\operatorname{Mod}(\Sigma_g)$ represented by this homeomorphism. The *hyperelliptic mapping class group* $\operatorname{SMod}(\Sigma_g)$ is the subgroup of $\operatorname{Mod}(\Sigma_g)$ consisting of all elements that commute with ι.

Mapping class groups of spheres and hyperelliptic mapping class groups are related by the following isomorphism;

$$\operatorname{Mod}(S_{2g+2}) \cong \operatorname{SMod}(\Sigma_g)/\langle \iota \rangle.$$

This is a consequence of the Birman-Hilden Theorem applied to the *hyperelliptic cover*. These notions are discussed in greater depth in Section 1.5 and Chapter 4.

Recall the definitions of the Torelli group and the Johnson filtration from Section 1.2. We define the *hyperelliptic Torelli group* and *hyperelliptic Johnson filtration* by the intersections

$$\mathcal{SI}(\Sigma_g) := \mathrm{SMod}(\Sigma_g) \cap \mathcal{I}(\Sigma_g) \quad \text{and} \quad \mathcal{SN}_k(\Sigma_g) := \mathrm{SMod}(\Sigma_g) \cap \mathcal{N}_k(\Sigma_g),$$

respectively. It is a result of Childers that $\mathrm{Aut}\,\mathcal{SI}(\Sigma_g) \cong \mathrm{SMod}^{\pm}(\Sigma_g)/\langle\iota\rangle$, when $g \geq 3$ [19]. Here, $\mathrm{SMod}^{\pm}(\Sigma_g)$ is the *extended hyperelliptic mapping class group*, the group generated by $\mathrm{SMod}(\Sigma_g)$ and a single orientation-reversing mapping class that commutes with ι. Following these results it is natural to ask what the automorphism groups are of terms appearing further down the hyperelliptic Johnson filtration. The following theorem answers this question and also gives the corresponding abstract commensurator groups.

Theorem 1.3.6. *For all surfaces Σ_g with $g \geq 6$ we have that*

$$\mathrm{Comm}\,\mathcal{SN}_k(\Sigma_g) \cong \mathrm{Aut}\,\mathcal{SN}_k(\Sigma_g) \cong \mathrm{SMod}^{\pm}(\Sigma_g)/\langle\iota\rangle.$$

To see how Theorem 1.2.1 implies this result we note that since the hyperelliptic involution ι does not act trivially on homology we have that $\mathcal{SI}(\Sigma_g)$ is isomorphic to a normal subgroup of $\mathrm{Mod}(S_{2g+2})$. Now, $\mathcal{SI}(\Sigma_g)$ is generated by Dehn twists about symmetric separating curves of Σ_g (separating curves that are fixed by ι) as shown by Brendle-Margalit-Putman [15]. Under the isomorphism of Birman-Hilden these generators map to squares of Dehn twists about curves separating an odd number of punctures in S_{2g+2}. If $g \geq 3$ then it follows that the image of $\mathcal{SI}(\Sigma_g)$ in $\mathrm{Mod}(S_{2g+2})$ contains a small element. The normaliser of this subgroup is $\mathrm{Mod}^{\pm}(S_{2g+2})$ and so from Theorem 1.2.1 we recover the result of Childers [19] that

$$\mathrm{Aut}\,\mathcal{SI}(\Sigma_g) \cong \mathrm{Mod}^{\pm}(S_{2g+2}) \cong \mathrm{SMod}^{\pm}(\Sigma_g)/\langle\iota\rangle.$$

Similarly, any term $\mathcal{SN}_k(\Sigma_g)$ is isomorphic to a normal subgroup of $\mathrm{Mod}(S_{2g+2})$. Using a construction similar to [25, Proof of Theorem 5.10] we can find an element of $\mathcal{SN}_k(\Sigma_g)$ whose support is contained in a genus two subsurface with one boundary component. This then corresponds to an element of $\mathrm{Mod}(S_{2g+2})$ whose support is contained in a disc with five punctures. From Theorem 1.2.1 and the definition of small for genus zero surfaces we have that if $2g+2 \geq 3(5)-1$ then

$$\mathrm{Comm}\,\mathcal{SN}_k(\Sigma_g) \cong \mathrm{Aut}\,\mathcal{SN}_k(\Sigma_g) \cong \mathrm{Mod}^{\pm}(S_{2g+2}) \cong \mathrm{SMod}^{\pm}(\Sigma_g)/\langle\iota\rangle,$$

which is precisely the statement of Theorem 1.3.6.

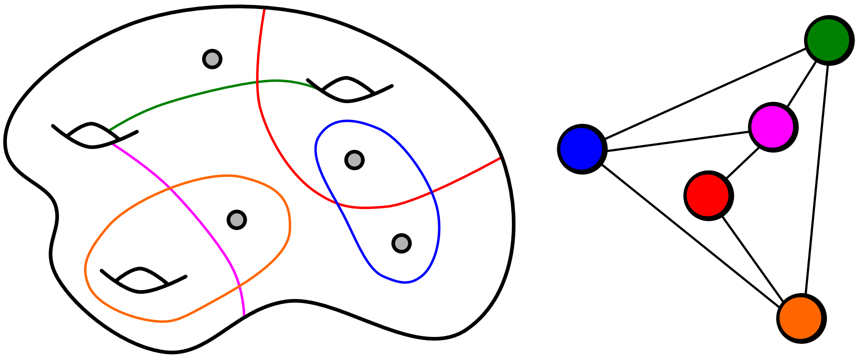

Figure 1.4: Five vertices in the curve complex $\mathcal{C}(\Sigma_{3,4})$ and corresponding curves in the surface.

1.4 The metaconjecture of Ivanov

Ivanov's proof that the mapping class group is geometric comes in two stages. The first stage is a study of the $\mathrm{Mod}^{\pm}(\Sigma)$-action on the *curve complex*. This complex and others that are related to it are discussed below. As in Sections 1.2 and 1.3.1 we assume that Σ is a surface with empty boundary.

1.4.1 The curve complex

A useful tool for the study of mapping class groups has been the curve complex $\mathcal{C}(\Sigma)$. This is a simplicial flag complex whose vertices correspond to all isotopy classes of essential simple closed curves in Σ. Two isotopy classes admit adjacent vertices when they contain disjoint representative curves.

Ivanov showed for a surface Σ of genus g and with n punctures that if either $g \geq 3$, or $g \geq 2$ and $n \geq 1$ then the natural homomorphism

$$\mathrm{Mod}^{\pm}(\Sigma) \to \mathrm{Aut}\,\mathcal{C}(\Sigma)$$

is in fact an isomorphism [37]. Combined results of Korkmaz [45] and Luo [50] later proved that equivalent results are true for tori with at least three punctures, spheres with at least five punctures, and no other surfaces. As discussed above, one of many applications of this result is that $\mathrm{Mod}(\Sigma)$ is geometric, that is, $\mathrm{Aut}\,\mathrm{Mod}(\Sigma) \cong \mathrm{Mod}^{\pm}(\Sigma)$.

Furthermore, in recent papers both Bavard-Dowdall-Rafi [3] and Hernandez-Morales-Valdez [32] prove that the natural homomorphism $\mathrm{Mod}^{\pm}(\Sigma) \to \mathrm{Aut}\,\mathcal{C}(\Sigma)$ is an isomorphism for any surface Σ of infinite type. This result answers a question of Patel-Vlamis in showing that for infinite type surfaces Σ and Σ' we have

$$\mathrm{Mod}(\Sigma) \cong \mathrm{Mod}(\Sigma') \quad \Rightarrow \quad \Sigma \cong \Sigma'.$$

Bavard-Dowdall-Rafi also use this reuslt to show that big mapping class groups are geometric [3].

1.4.2 Other complexes

An extreme generalisation of the curve complex is the *complex of domains* $\mathcal{D}(\Sigma)$, defined by McCarthy-Papadopoulos for surfaces of finite type [54]. Here, vertices correspond to the homotopy classes of all connected and compact subsurfaces in Σ with essential boundary. Again, adjacency between vertices is determined by disjoint representatives in the corresponding equivalence classes. The bijection between isotopy classes of curves and homotopy classes of annuli in Σ induces a natural inclusion of complexes $\mathcal{C}(\Sigma) \to \mathcal{D}(\Sigma)$. As well as this clear connection, McCarthy-Papadopoulos showed that when the genus of the surface is at least two and there is at most a single puncture, the natural homomorphism

$$\mathrm{Mod}^{\pm}(\Sigma) \to \mathrm{Aut}\,\mathcal{D}(\Sigma)$$

is an isomorphism [54, Theorem 1.1]. This result is then a generalisation of the results of Ivanov, Korkmaz and Luo mentioned above.

We can define similar natural inclusions into $\mathcal{D}(\Sigma)$ for many other complexes, such as: the complex of non-separating curves [35], the complex of separating curves [12], the truncated complex of domains [54], the arc complex [36], the arc and curve complex [46], and the complex of strongly separating curves [11]. Given certain restrictions on the genus and the number of punctures, for each of these complexes the natural homomorphism from $\mathrm{Mod}^{\pm}(\Sigma)$ to the automorphism group of the complex is an isomorphism. As with the application of Ivanov's theorem highlighted above, these results were used to show that certain normal subgroups of $\mathrm{Mod}(\Sigma)$ are geometric. Similar results were also shown to be true for the pants complex [52], the Torelli complex [43], and the ideal triangulation graph [47], although in these cases there is no natural inclusion of the vertex set into $\mathcal{D}(\Sigma)$. Following this work, Ivanov made a metaconjecture [25].

Metaconjecture 1.4.1 (Ivanov). *Every object naturally associated to a surface Σ and having a sufficiently rich structure has $\mathrm{Mod}^{\pm}(\Sigma)$ as its group of automorphisms. Moreover, this can be proved by a reduction to the theorem about automorphisms of $\mathcal{C}(\Sigma)$.*

While the language of the metaconjecture may seem vague, it reflects the breadth of results which it includes as evidence. Indeed, providing suitable definitions for "naturally associated" and "sufficiently rich" is one of the difficulties in resolving the metaconjecture. Brendle-Margalit defined a wide class of complexes associated to a closed surface Σ, where for each such complex we can define a natural inclusion into $\mathcal{D}(\Sigma)$. They then resolved the metaconjecture for such complexes [13]. We will extend this definition to surfaces with punctures. Theorem 1.4.2 resolves the metaconjecture for such complexes associated to surfaces, and is used in Chapter 3 to prove Theorem 1.2.1.

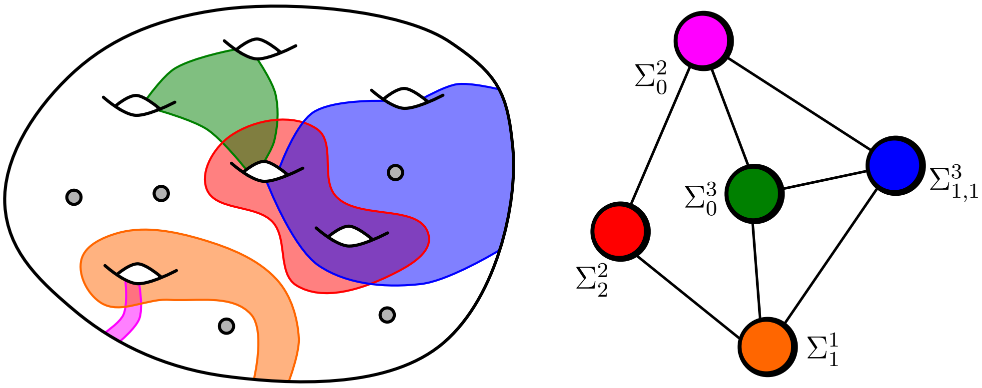

Figure 1.5: Five vertices in a complex of regions $\mathcal{C}_A(\Sigma_{6,5})$ and corresponding regions in the surface. Regions homeomorphic to Σ_0^2, Σ_1^1, Σ_2^2, Σ_0^3, and $\Sigma_{1,1}^3$ are represented in A.

1.4.3 Complexes of regions

Before we are able to state Theorem 1.4.2 we must first introduce some terminology.

Sets of regions

Let $\mathcal{R}(\Sigma)$ be the set of $\mathrm{Mod}^{\pm}(\Sigma)$-orbits of all regions of Σ. If R is a region and the $\mathrm{Mod}^{\pm}(\Sigma)$-orbit of R is an element of $A \subset \mathcal{R}(\Sigma)$ we say that R *represents* an element of a *set of regions* A, or that R is *represented* in A.

Complexes of regions

Given a set of regions $A \subset \mathcal{R}(\Sigma)$ we define the *complex of regions* $\mathcal{C}_A(\Sigma)$ to be a simplicial flag complex with vertices corresponding to the homotopy classes of all regions represented in A. We say that a vertex v *corresponds* to a region R if v corresponds to the equivalence class containing R. Two vertices are adjacent if they correspond to disjoint regions.

If $A \subset \mathcal{R}(\Sigma)$ is the set of essential annuli in Σ then $\mathcal{C}_A(\Sigma)$ is naturally isomorphic to the curve complex $\mathcal{C}(\Sigma)$. It is also clear that $\mathcal{C}_A(\Sigma) = \mathcal{D}(\Sigma)$ if $A = \mathcal{R}(\Sigma)$.

Exchange automorphisms

Let $\mathcal{X}$ be a simplicial complex. McCarthy-Papadopoulos define $\phi \in \operatorname{Aut}\mathcal{X}$ to be an *exchange automorphism* if there exist vertices $v_1, v_2 \in \mathcal{X}$ where $\phi(v_1) = v_2$, $\phi(v_2) = v_1$ and $\phi(v) = v$ for all other vertices v of $\mathcal{X}$ distinct from v_1 and v_2, see [54].

In Section 2.1.1 of this paper we discuss the fact that the presence of exchange automorphisms in a complex of regions is exactly the obstruction for the automorphism group of a complex of regions to be the extended mapping class group. In the closed surface case, Brendle-Margalit gave topological conditions on $A \subset \mathcal{R}(\Sigma)$ for the complex of regions $\mathcal{C}_A(\Sigma)$ to admit exchange automorphisms. We generalise these conditions below using the terminology of *corks* and *holes* that they introduced [13].

First, we define a *complementary region* Q of a region R to be a subsurface that is disjoint from R and homotopic to a component of $\Sigma \setminus R$. We say that a subsurface of R

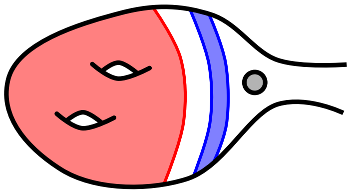

Figure 1.6: If no proper subsurface of a one boundary, genus two region with no punctures is represented in A then there exists a cork pair in $\mathcal{C}_A(\Sigma)$ corresponding to the two regions shown.

is *peripheral* if it is an annulus whose boundary components are isotopic to a boundary component of R.

Corks and holes

Let $A \subset \mathcal{R}(\Sigma)$. We say a vertex of $\mathcal{C}_A(\Sigma)$ is a *cork* if it corresponds to an annulus R with a complementary region Q represented in A with no proper, non-peripheral subsurface of Q represented in A. We call the vertices corresponding to R and Q a *cork pair*.

We say that a vertex v of $\mathcal{C}_A(\Sigma)$ is a *hole* if v corresponds to a region R that has a complementary region Q such that no subsurface of Q represents an element of A. Let v be a hole corresponding to a region R. Let Q_i be a region homotopic to such a complementary region of R such that the intersection of R and Q_i is an annulus. We define the *filling* of the hole v to be the union of R will all such Q_i. See Figure 1.7 for an example of two holes with equal fillings.

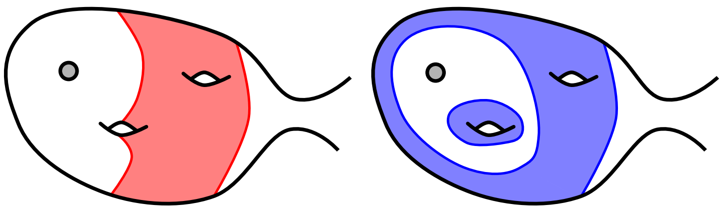

Figure 1.7: If there are no nonseparating annuli with one or no punctures represented in A then there are holes in $\mathcal{C}_A(\Sigma)$ corresponding to the two regions shown. Moreover, these holes have equal fillings; a region homeomorphic to $\Sigma^1_{2,1}$.

1.4.4 Statement of the theorem

We now give the statement of the main theorem regarding complexes of regions for a surface $\Sigma = \Sigma_{g,n}$.

Enveloping regions and small regions

Let $A \subset \mathcal{R}(\Sigma)$ and let R represent an element of A. We say that a region Q *covers* R if Q has a single boundary component and $R \subset Q$. Let g_R be the smallest genus of any region that covers R. Note that $g_R \geq g(R)$. We define a region $\widehat{R}$ to be an *enveloping*

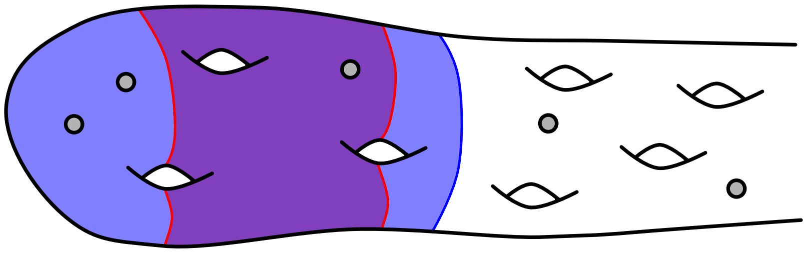

Figure 1.8: The picture shows a region R homeomorphic to $\Sigma^4_{1,1}$ in a surface Σ of genus at least 7 and at least 5 punctures. We see an enveloping region $\widehat{R}$ homeomorphic to $\Sigma^1_{3,3}$. In this case the enveloping region is unique up to homotopy.

region of R if $\widehat{R}$ has fewest punctures such that $\widehat{R}$ covers R and $g(\widehat{R}) = g_R$, see Figure 1.8. We say that R is a *core* region of A when for all regions Q representing an element of A we have that

$$\widehat{Q} \subset \widehat{R} \quad \Rightarrow \quad \widehat{Q} \simeq \widehat{R},$$

where '$\simeq$' is an equivalence relation on regions defined by the existence of a homotopy. Now, let Σ be a surface of genus $g > 0$ with $n > 0$ punctures. We call a core region R *g-small* if for all Q represented in A we have;

$$g(\widehat{Q}) \geq g(\widehat{R}), \qquad \text{and} \qquad n \geq \begin{cases} \max\{3n(\widehat{R}), 7\} \text{ if } g = 1, \text{ and} \\ 3n(\widehat{R}) \text{ if } g \geq 2. \end{cases}$$

Similarly, we call a core region R *n-small* if for all Q represented in A we have

$$n(\widehat{Q}) \geq n(\widehat{R}) \neq 1, \qquad \text{and} \qquad g \geq 3g(\widehat{R}) + 1.$$

In the case where either $n = 0$ or $g = 0$ the definitions of core, g-small, and n-small regions coincide. In particualr, every core region R is such that $\widehat{R}$ belongs to a single $\mathrm{Mod}^{\pm}(\Sigma)$-orbit. In these special cases we require that

$$g \geq 3g(\widehat{R}) + 1 \quad \text{or} \quad n \geq 3n(\widehat{R}) - 1,$$

for when $n = 0$ and $g = 0$ respectively. These special cases are addressed in papers by Brendle-Margalit [13] and the author [55].

For any surface Σ, if a vertex v of $\mathcal{C}_A(\Sigma)$ corresponds to a g-small region then we call it a g-small vertex. Similarly, a vertex corresponding to an n-small region is called an n-small vertex. We can now state the resolution of the metaconjecture for complexes of regions associated to surfaces with punctures.

Theorem 1.4.2. *Let Σ be a surface with punctures. Let $\mathcal{C}_A(\Sigma)$ be a complex of regions containing a vertex that is g-small and a vertex that is n-small. Then the natural homomorphism*

$$\eta_A : \mathrm{Mod}^{\pm}(\Sigma) \to \mathrm{Aut}\, \mathcal{C}_A(\Sigma)$$

is an isomorphism if and only if $\mathcal{C}_A(\Sigma)$ has no holes and no corks.

This result is used to prove Theorem 1.2.1 in Chapter 3. The reader may be wondering why the definition for *small regions* is considerably more obtuse than the definition for *small mapping classes* given in Section 1.2. This is because for a normal subgroup N containing a small mapping class we can build an associated complex of regions $\mathcal{C}_N(\Sigma)$ containing (not necessarily distinct) g-small and n-small vertices. See Chapter 3 for details on this construction. Conversely, if a normal subgroup contains two mapping classes whose supports are g-small and n-small, then there clearly exists a small mapping class in N. As however, one may want to consider a complex of regions not necessarily related to a normal subgroup, we give the necessary conditions on the complex in full detail.

1.5 The liftable and symmetric subgroups

As well as the mapping class group action on complexes, we may also use covering spaces to study the subgroup structure of mapping class groups. Unlike the previous sections, we now allow for surfaces with boundary.

Let $p : \widetilde{\Sigma} \to \Sigma$ be a regular, finite-sheeted covering space. We allow for the possibility of branched covering spaces. In such cases we write $\mathcal{B}$ for the set of branch points in Σ. Let $D < \mathrm{Homeo}^+(\widetilde{\Sigma})$ be the deck group. A homeomorphism $\widetilde{f} \in \mathrm{Homeo}^+(\widetilde{\Sigma})$ is *fibre-preserving* with respect to p if

$$p(x) = p(y) \quad \Rightarrow \quad p\widetilde{f}(x) = p\widetilde{f}(y) \quad \text{for all } x, y \in \widetilde{\Sigma}.$$

Define *the symmetric mapping class group* $\mathrm{SMod}(\widetilde{\Sigma})$ to be the subgroup of $\mathrm{Mod}(\widetilde{\Sigma})$ consisting of all mapping classes that are represented by fibre-preserving homeomorphisms. A homeomorphism $f \in \mathrm{Homeo}^+(\Sigma)$ *lifts* to a homeomorphism $\widetilde{f} \in \mathrm{Homeo}^+(\widetilde{\Sigma})$ if $p\widetilde{f} = fp$. Define *the liftable mapping class group* $\mathrm{LMod}(\Sigma, \mathcal{B})$ to be the subgroup of $\mathrm{Mod}(\Sigma, \mathcal{B})$ consisting of all mapping classes that are represented by homeomorphisms that lift to boundary preserving homeomorphisms.

In their seminal paper [7], Birman and Hilden generalised results from a series of papers in the 1970s to prove what is now known as the Birman-Hilden Theorem. It states that under mild conditions, for any regular, finite-sheeted covering space between closed surfaces $p : \widetilde{\Sigma} \to \Sigma$, with $\widetilde{\Sigma}$ of genus at least 2, that the quotient group $\mathrm{SMod}(\widetilde{\Sigma})/D$ is isomorphic to $\mathrm{LMod}(\Sigma, \mathcal{B})$. The conditions on the Birman-Hilden theorem were removed due to results of MacLachlan-Harvey [51] and Kerkhoff [42]. The fact that D is a subgroup of $\mathrm{SMod}(\widetilde{\Sigma})$ is not obvious and is a result of Birman-Hilden [7].

A survey article containing different approaches and applications of the Birman-Hilden Theorem by Margalit-Winarski gives a detailed account of this topic [53].

If the surface $\widetilde{\Sigma}$ has boundary then it follows that non-trivial elements of D are not representative homeomorphisms of mapping classes. This suggests that the statement of the theorem will be slightly different. The following result seems to be well known, though a proof is hard to come to by. As such, we fill this apparent gap in the literature by including a proof in Chapter 4.

Theorem 1.5.1. *Let $p : \widetilde{\Sigma} \to \Sigma$ be a regular, finite-sheeted, possibly branched covering space of surfaces with boundary. Then the groups* $\mathrm{LMod}(\Sigma, \mathcal{B})$ *and* $\mathrm{SMod}(\widetilde{\Sigma})$ *are isomorphic.*

The proof is similar in spirit to proofs given by Winarski [53] and Farb-Margalit [27]. Examples of some applications of the Birman-Hilden Theorem include Aramayona-Leininger-Souto [1], Bigelow-Budney [6], Brendle-Margalit [14], Brendle-Margalit-Putman [15], Endo [24], Morifuji [57], and Stukow [65] to name a few. Such endeavours have been fruitful due to considering covers where at least one of the liftable or symmetric mapping class groups coincide with the entire mapping class group of the surface. For example, in Section 1.3.6 we used the isomorphism in the hyperelliptic case

$$\mathrm{Mod}(S_{2g+2}) \cong \mathrm{Mod}(\Sigma_0, \mathcal{B}) = \mathrm{LMod}(\Sigma_0, \mathcal{B}) \cong \mathrm{SMod}(\Sigma_g)/\langle \iota \rangle,$$

owing to the fact that every mapping class is liftable in this case. We may then ask when does $\mathrm{LMod}(\Sigma, \mathcal{B}) = \mathrm{Mod}(\Sigma, \mathcal{B})$? When does $\mathrm{SMod}(\widetilde{\Sigma}) = \mathrm{Mod}(\Sigma)$? If equality is not achieved, when are the subgroups finite index?

If the surfaces are closed, then it is known that $\mathrm{LMod}(\Sigma, \mathcal{B})$ is finite index. It is also known in the case of the hyperelliptic involution that $\mathrm{SMod}(\widetilde{\Sigma})$ is infinite index when the genus of $\widetilde{\Sigma}$ is at least 2. Ghaswala-Winarski classified all cyclic branched covers of the sphere that satisfy $\mathrm{LMod}(\Sigma_0, \mathcal{B}) = \mathrm{Mod}(\Sigma_0, \mathcal{B})$ [29].

However, Birman-Hilden proved that if $\widetilde{\Sigma}$ is of genus at least 3 then there are no finite cyclic covers $p : \widetilde{\Sigma} \to \Sigma_0$ of a sphere with marked points such that $\mathrm{SMod}(\widetilde{\Sigma}) = \mathrm{Mod}(\widetilde{\Sigma})$ [7]. Included in their paper is the following remark:

The possibility remains that if we relax the requirements on $(p, \Sigma_0, \widetilde{\Sigma})$ to admit coverings of other Riemann surfaces, or to admit all regular coverings, or to admit non-regular coverings that we will have better luck. (However we conjecture that all such efforts will fail).

While this question is not addressed head on, our results agree with the opinion that mapping class groups rarely consist only of elements represented by fibre-preserving homeomorphisms with respect to a regular, finite-sheeted covering space.

1.5.1 A classification of regular, finite-sheeted covering spaces

For the remainder of this section, and in Chapter 4, we let $p : \widetilde{\Sigma} \to \Sigma$ be a regular, finite-sheeted, possibly branched cover of surfaces with boundary.

Theorem 1.5.2 classifies all such covers with the property that $\mathrm{LMod}(\Sigma, \mathcal{B}) = \mathrm{Mod}(\Sigma, \mathcal{B})$, and proves that $\mathrm{LMod}(\Sigma, \mathcal{B})$ is always finite index. Theorem 1.5.3 classifies all such covers where the equality $\mathrm{SMod}(\widetilde{\Sigma}) = \mathrm{Mod}(\widetilde{\Sigma})$ holds, and proves that if it does not, $\mathrm{SMod}(\widetilde{\Sigma})$ is an infinite index subgroup.

Burau covers

Pick a point $x \in \partial \mathbf{D}_n$ and let $\gamma_i \in \pi_1(\mathbf{D}_n, x)$ be the homotopy class of a loop surrounding only the ith puncture anti-clockwise. The set $\{\gamma_1, \dots, \gamma_n\}$ generates the fundamental group $\pi_1(\mathbf{D}_n, x)$. For each $k \geq 2$ we define a homomorphism;

$$q_k : \pi_1(\mathbf{D}_n, x) \to \mathbb{Z}/k\mathbb{Z}, \quad \text{by} \quad \gamma_i \mapsto 1,$$

for all i. The kernel of q_k determines a k-sheeted cyclic branched cover $p_k : \Sigma_g^m \to \Sigma_0^1$ branched at n points. Here, $m = \gcd(n, k)$ and it can be shown by an Euler characteristic argument that $g = \frac{1}{2}(nk - n - k - m + 2)$. We will call such a cover a *k-sheeted Burau cover*. This name owes to the intimate relationship between such covers and the *Burau representation* of the braid group, see [67].

Theorem 1.5.2. *Let $p : \widetilde{\Sigma} \to \Sigma$ be a regular, finite-sheeted, possibly branched covering space of surfaces with boundary. Then*

1. $\mathrm{LMod}(\Sigma, \mathcal{B}) = \mathrm{Mod}(\Sigma, \mathcal{B})$ *if and only if p is a Burau cover, and*

2. $\mathrm{LMod}(\Sigma, \mathcal{B})$ *is finite index in* $\mathrm{Mod}(\Sigma, \mathcal{B})$.

We now give the analogous result for the symmetric mapping class group.

Theorem 1.5.3. *Let $p : \widetilde{\Sigma} \to \Sigma$ be a regular, finite-sheeted, possibly branched covering space of surfaces with boundary. Then*

1. $\mathrm{SMod}(\widetilde{\Sigma}) = \mathrm{Mod}(\widetilde{\Sigma})$ *if and only if $\widetilde{\Sigma}$ is a disc, and annulus, or $p : \Sigma_1^1 \to \Sigma_0^1$ is the hyperelliptic cover, otherwise*

2. $\mathrm{SMod}(\widetilde{\Sigma})$ *is infinite index in* $\mathrm{Mod}(\widetilde{\Sigma})$.

Note that the hyperelliptic cover is in fact the 2-sheeted Burau cover. Combining these results we see that $p_2 : \Sigma_1^1 \to \Sigma_0^1$ is a regular, finite-sheeted covering space of surfaces with boundary where both symmetric and liftable mapping class groups coincide with the entire mapping class group. This fact gives rise to the following isomorphism of groups;

$$B_3 \cong \mathrm{Mod}(\mathbf{D}_3) \cong \mathrm{Mod}(\Sigma_0^1, \mathcal{B}) = \mathrm{LMod}(\Sigma_0^1, \mathcal{B}) \cong \mathrm{SMod}(\Sigma_1^1) = \mathrm{Mod}(\Sigma_1^1).$$

1.5.2 Braid group embeddings

As discussed in Section 1.3.1 the mapping class group of a disc $\mathrm{Mod}(\mathbf{D}_n)$ is isomorphic to the braid group B_n. The hyperelliptic cover and Theorem 1.5.1 (the Birman-Hilden Theorem) give a standard embedding of the braid group into a mapping class group, sending each generator σ_i to a Dehn twist. Embeddings of this type are said to be *geometric*.

A question of Wajnryb is whether or not there exist embeddings of the braid group that are non-geometric [68]. This question has been answered in the affirmative and examples of non-geometric embeddings have been given by Bödigheimer-Tillman [9], Kim-Song [44], Song [62], Song-Tillman [63], and Szepietowski [66].

Note that there is no relation between the notions of a geometric embedding of the braid group and a geometric normal subgroup of a mapping class group. The fact that two objects of study in this thesis have similar names is coincidental. These terms are left unchanged in order to stay aligned with the wider literature. Furthermore, there is little intersection between Chapter 3 and Chapter 4.

We may use Theorems 1.5.1 and 1.5.2 to construct a family of embeddings of the braid group induced by Burau covers as follows;

$$B_n \cong \mathrm{Mod}(\mathbf{D}_n) \cong \mathrm{Mod}(\Sigma_0^1, \mathcal{B}) = \mathrm{LMod}(\Sigma_0^1, \mathcal{B}) \cong \mathrm{SMod}(\Sigma_g^m) \hookrightarrow \mathrm{Mod}(\Sigma_g^m),$$

where g and m are given in the definition of Burau covers. The final section of Chapter 4 studies these embeddings.

Chains

We call a set of curves $\{\mathbf{c}_i\}_{i=1}^k$ a *k-chain* if $i(\mathbf{c}_i, \mathbf{c}_j) = 1$ if $j = i \pm 1$ and $i(\mathbf{c}_i, \mathbf{c}_j) = 0$ otherwise. Given a k-chain $\mathcal{C} = \{\mathbf{c}_i\}_{i=1}^k$ we call the product

$$T_{\mathcal{C}} := T_{\mathbf{c}_1} T_{\mathbf{c}_2} \dots T_{\mathbf{c}_k}$$

a *k-chain twist*, or simply a *chain twist*.

Theorem 1.5.4. *Given $n, k \geq 2$ there exists an injective group homomorphism*

$$\beta_k : B_n \to \mathrm{Mod}(\Sigma_g^m)$$

such that the image of each standard generator of the braid group is a $(k-1)$-chain twist.

When $k = 2$ the corresponding 1-chains are of course simple closed curves. It is clear that a k-chain twists is not equal to Dehn twist for $k \geq 2$. This implies that that β_2 gives the standard geometric embedding, while $k \geq 3$ gives non-geometric embeddings. Kim-Song independently arrived at the embedding β_3 in a recent paper [44].

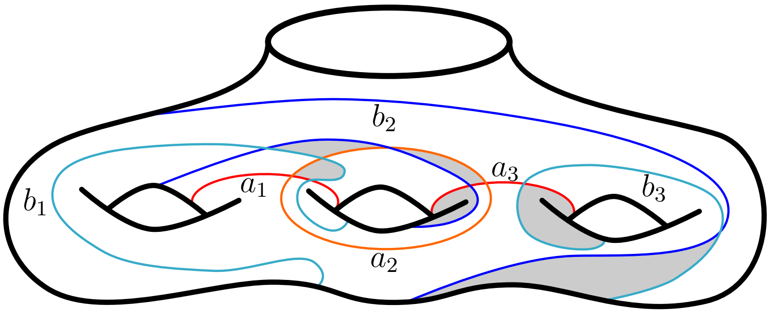

Figure 1.9: The two 3-chains $\mathcal{A} = \{a_1, a_2, a_3\}$ and $\mathcal{B} = \{b_1, b_2, b_3\}$ define chain twists $T_{\mathcal{A}} := T_{a_1}T_{a_2}T_{a_3}$ and $T_{\mathcal{B}} := T_{b_1}T_{b_2}T_{b_3}$ such that $T_{\mathcal{A}}T_{\mathcal{B}}T_{\mathcal{A}} = T_{\mathcal{B}}T_{\mathcal{A}}T_{\mathcal{B}}$.

Using these embeddings as inspiration, we define a combinatorial condition on two k-chains that imply their respective chain twists satisfy a braid relation. The condition is defined in Section 4.4.4 and proven to be sufficient in Proposition 4.4.6. Figure 1.9 shows two 3-chains satisfying the combinatorial condition, so their chain twists satisfy a braid relation.

Braid group embeddings are of interest in the wider literature, in particular the induced maps on the homology of groups of braid groups to mapping class groups, see Song-Tillman [63]. Another feature of interest is that if the target surface has a single boundary component then the embedding induces an action of the fundamental group which is a free group.

Chapter 2

Resolution of the metaconjecture

This chapter is dedicated to proving Theorem 1.4.2, that is, that the natural homomorphism

$$\eta_A : \mathrm{Mod}^{\pm}(\Sigma) \to \mathrm{Aut}\, \mathcal{C}_A(\Sigma)$$

is an isomorphism for surfaces with punctures. We first prove injectivity of the natural homomorphism in Section 2.1. Also in this section, we investigate exchange automorphisms, stating results of Brendle-Margalit which carry over to this more general case [13]. We then focus our attention on proving surjectivity. We begin by defining $\mathcal{S}$ to be the set of all $\mathrm{Mod}^{\pm}(\Sigma)$-orbits of separating curves in the surface. Recall that there is a bijection between the isotopy classes of curves and the homotopy classes of annuli. It follows then that for any given subset $X \subset \mathcal{S}$ we can define $\mathcal{C}_X(\Sigma)$ to be a subcomplex of the *separating curve complex* $\mathcal{C}_{\mathcal{S}}(\Sigma)$.

Each separating curve $\mathbf{c}$ has two *associated regions* homotopic to the components of $\Sigma \setminus \mathbf{c}$. We define the subset $\mathcal{S}(A) \subset \mathcal{S}$ to consist of all classes of separating curves whose associated regions both contain regions represented in $A \subset \mathcal{R}(\Sigma)$. The proof of Theorem 1.4.2 relies on the following result, the statement of which uses the definitions of g-small and n-small regions given in Section 1.4.4.

Theorem 2.0.1. *Let Σ be a surface with punctures and let $A \subset \mathcal{R}(\Sigma)$. If a g-small region and an n-small region are represented in A then the natural homomorphism*

$$\eta_{\mathcal{S}(A)} : \mathrm{Mod}^{\pm}(\Sigma) \to \mathrm{Aut}\, \mathcal{C}_{\mathcal{S}(A)}(\Sigma)$$

is an isomorphism.

In Section 2.2 we investigate the set $\mathcal{S}(A)$ and other related subsets of $\mathcal{S}$. Section 2.3 generalises the notion of *sharing pairs* (see [13] and [55]). Section 2.4 uses these definitions and a result of Kida [43] to prove Theorem 2.0.1. In Sections 2.5 and 2.6 we complete the proof of Theorem 1.4.2 using the notion of *dividing sets* introduced by Brendle-Margalit [13].

2.1 Injectivity and exchange automorphisms

In this section we prove that the homomorphism from Theorem 1.4.2 is injective. This result is in fact more general and will be used many times in this chapter. Following McCarthy-Papadopoulos [54, Section 4] and Brendle-Margalit [13, Section 2] we also look at the precise conditions for a complex of regions $\mathcal{C}_A(\Sigma)$ to admit exchange automorphisms.

Recall that the *link* $\mathrm{Lk}(v)$ of a vertex v is the set of all vertices that span an edge with v in the complex. The *star* of a vertex is the union of the vertex and its link.

Lemma 2.1.1. *Let Σ be a surface with punctures and let $A \subset \mathcal{R}(\Sigma)$. The natural homomorphism*

$$\eta_A : \mathrm{Mod}^{\pm}(\Sigma) \to \mathrm{Aut}\,\mathcal{C}_A(\Sigma)$$

is injective.

Proof. We begin by defining a homomorphism

$$\Theta : \mathrm{Aut}\,\mathcal{C}(\Sigma) \to \mathrm{Aut}\,\mathcal{C}_A(\Sigma)$$

as follows. Suppose v is vertex of $\mathcal{C}_A(\Sigma)$ corresponding to the region R. There exists a simplex σ of $\mathcal{C}(\Sigma)$ corresponding to the multicurve ∂R. For any $\phi \in \mathrm{Aut}\,\mathcal{C}(\Sigma)$ the image $\phi(\sigma)$ corresponds to a multicurve M of Σ. We define $\Theta(\phi)(v)$ to be the vertex corresponding to the region R' such that $\partial R' = M$. This makes sense since $\mathrm{Aut}\,\mathcal{C}(\Sigma)$ is isomorphic to $\mathrm{Mod}^{\pm}(\Sigma)$. That is, the multicurve M necessarily bounds a region in the $\mathrm{Mod}^{\pm}(\Sigma)$-orbit of R. Hence, the region R' is represented in A.

As $\eta : \mathrm{Mod}^{\pm}(\Sigma) \to \mathrm{Aut}\,\mathcal{C}(\Sigma)$ is the natural isomorphism it is clear that $\Theta \circ \eta = \eta_A$. It remains only to show that Θ is injective. To that end, suppose ϕ belongs to the kernel of Θ. Every simplex σ, as described above, must therefore be fixed by ϕ. We want to show that, in fact, ϕ fixes each vertex of σ. Suppose otherwise, that is, $\phi(v_1) = v_2$ where $v_1, v_2 \in \sigma$ and suppose v_i corresponds to the curve $\mathbf{c}_i$ where $\{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_m\}$ form boundary components for a region R represented in A. We can find a curve $\mathbf{c}_1'$ in the $\mathrm{Mod}^{\pm}(\Sigma)$-orbit of $\mathbf{c}_1$ such that $\mathbf{c}_1$ and $\mathbf{c}_1'$ have positive essential intersection and $\{\mathbf{c}_1', \mathbf{c}_2, \ldots, \mathbf{c}_m\}$ form boundary components for a region in the $\mathrm{Mod}^{\pm}(\Sigma)$-orbit of R. The mapping class $[f] = \eta^{-1}(\phi)$ maps $\mathbf{c}_1$ to $\mathbf{c}_2$ and so does not map $\mathbf{c}_1'$ to $\mathbf{c}_2$. It follows that there is a simplex in $\mathcal{C}(\Sigma)$ belonging to the $\mathrm{Mod}^{\pm}(\Sigma)$-orbit of σ that is not fixed by ϕ. This is contradiction and so we have that ϕ fixes every vertex in every simplex σ defined by a region represented in A.

We have now shown that for any vertex v corresponding to a curve $\mathbf{v}$ of $\mathcal{C}(\Sigma)$ there exist infinitely many vertices in $\mathrm{Lk}(v)$ that are fixed by ϕ. Namely, those vertices corresponding to boundary components of regions represented in A. Such vertices correspond to curves that fill the associated regions of $\mathbf{v}$. This implies that v is fixed by ϕ and hence ϕ is the identity, completing the proof. ∎

The proofs of the equivalent statements for the cases where $\Sigma = \Sigma_g$ and $\Sigma = \Sigma_{0,n}$ are given by Brendle-Margalit [13] and the author [55] respectively. All three proofs are of slightly different flavours. Going through the curve complex has two benefits. The first is that the same proof works for both surfaces of finite and infinite type. Methods from the other proofs are not so easily translated. The second benefit is that the method resonates with the second sentence of the metaconjecture. In some sense each "sufficiently rich" complex inherits all the structure of the curve complex. This idea is used throughout the remainder of this chapter when proving that η_A is surjective.

2.1.1 Exchange automorphisms

We now state two results concerning the exchange automorphisms of a complex of regions. Recall that $\phi \in \operatorname{Aut} \mathcal{C}_A(\Sigma)$ is an exchange automorphism if there exist vertices $v_1, v_2 \in \mathcal{C}_A(\Sigma)$ where $\phi(v_1) = v_2$, $\phi(v_2) = v_1$ and $\phi(v) = v$ for all other vertices v of $\mathcal{C}_A(\Sigma)$ distinct from v_1 and v_2. The results will be given without proof as the analogous results for closed surfaces carry over to the general case considered in this chapter. The following result is given in terms of holes and corks, whose definitions can be found in Section 1.4.

Theorem 2.1.2 (Brendle-Margalit)**.** *Let Σ be a surface of finite or infinite type and let $A \subset \mathcal{R}(\Sigma)$ such that $\mathcal{C}_A(\Sigma)$ is connected. Then $\mathcal{C}_A(\Sigma)$ admits exchange automorphisms if and only if it has a hole or a cork.*

As an example we consider the cork pair and the holes depicted in Figures 1.6 and 1.7. In Figure 1.6, any region that does not intersect the red region homeomorphic to $\Sigma^1_{2,0}$ is homotopic to a region that does not intersect the blue annulus. It follows then that the vertices corresponding to these regions have equal stars. In particular, they span an edge with each other. Similarly in Figure 1.7, every region that intersects the red region homeomorphic to $\Sigma^3_{1,0}$ intersects the blue region, also homemorphic $\Sigma^3_{1,0}$. This implies that the vertices corresponding to these regions have equal links. Here the vertices do not span an edge with each other. If two vertices have equal links or equal stars then we can define an exchange automorphism. The proof of [13, Theorem 2.1] tells us that if two vertices have equal links then the are holes, and if they have equal stars then they are cork pairs.

When the automorphism group of a complex of regions does have exchange automorphisms, Brendle-Margalit give us an explicit description of the automorphism group of the complex.

Theorem 2.1.3 (Brendle-Margalit)**.** *Let $\mathcal{C}_A(\Sigma)$ be a connected complex of regions with a g-small vertex and an n-small vertex. Then*

$$\operatorname{Aut} \mathcal{C}_A(\Sigma) \cong \operatorname{Ex} \mathcal{C}_A(\Sigma) \rtimes \operatorname{Mod}^{\pm}(\Sigma).$$

Here, $\mathrm{Ex}\,\mathcal{C}_A(\Sigma)$ is the subgroup of $\mathrm{Aut}\,\mathcal{C}_A(\Sigma)$ generated by all exchange automorphisms.

2.2 Characteristic vertex types

From now on we will assume the surface Σ is of genus $g \geq 0$ and has $n > 0$ punctures. Recall that a separating curve $\mathbf{c}$ is represented in $\mathcal{S}(A)$ when the associated regions of $\mathbf{c}$ contain regions that are represented in $A \subset \mathcal{R}(\Sigma)$. In Section 2.4 we will prove Theorem 2.0.1, that the natural homomorphism

$$\eta_{\mathcal{S}(A)} : \mathrm{Mod}^{\pm}(\Sigma) \to \mathrm{Aut}\,\mathcal{C}_{\mathcal{S}(A)}(\Sigma)$$

is an isomorphism. This proof relies on first showing that the natural homomorphism

$$\eta_X : \mathrm{Mod}^{\pm}(\Sigma) \to \mathrm{Aut}\,\mathcal{C}_X(\Sigma)$$

is an isomorphism, whenever X is an *extended set* of $\mathcal{S}(A)$. This is a particular type of subset of separating curves satisfying $\mathcal{S}(A) \subseteq X \subseteq \mathcal{S}$ which we define in Section 2.2.2. In this section we prove that the topology of the curves represented in X determine characteristic subsets of vertices in the complex $\mathcal{C}_X(\Sigma)$ when X is an *extended set*.

2.2.1 Minimum curves and lattices

In Section 1.4 we defined R to be a core region of $A \subset \mathcal{R}(\Sigma)$ if for all Q represented in A we have that if $\widehat{R}$ contains $\widehat{Q}$ as a subsurface then they are homotopic. Recall further, the definitions of g-small and n-small from Section 1.4.4.

We call a separating curve $\mathbf{c}$ in Σ a (k,l)-curve if it has a unique associated region of lowest genus $g(R) = k$ and l punctures. If both associated regions of $\mathbf{c}$ are of genus k then then we call $\mathbf{c}$ a (k,l)-curve the associated region of $\mathbf{c}$ with fewest punctures has exactly l punctures. If $\mathbf{c}$ is a (k,l)-curve then for any $[f] \in \mathrm{Mod}^{\pm}(\Sigma)$ we have that $[f](\mathbf{c})$ is a (k,l)-curve. We write $Z_{k,l}$ for the $\mathrm{Mod}^{\pm}(\Sigma)$-orbit of (k,l)-curves as an element of $\mathcal{S}$. We call any vertex of $\mathcal{C}_{\mathcal{S}}(\Sigma)$ that corresponds to a (k,l)-curve a (k,l)-vertex.

Given a subset $A \subset \mathcal{R}(\Sigma)$ we would like to know which values of k and l admit $Z_{k,l}$ as an element of $\mathcal{S}(A)$. Equivalently, for what values of k and l do (k,l)-vertices belong to the subcomplex $\mathcal{C}_{\mathcal{S}(A)}(\Sigma)$ of the separating curve complex? We begin with two fundamental topological types of curves represented in $\mathcal{S}(A)$. These are strongly linked to the definitions of g-small and n-small.

Minimum curves

Let $X \subset \mathcal{S}$ and let $\mathbf{c}$ be a separating curve represented in X with an associated region R. We call $\mathbf{c}$ a *minimum curve* of X if all curves that are represented in X and contained in R are isotopic to $\mathbf{c}$.

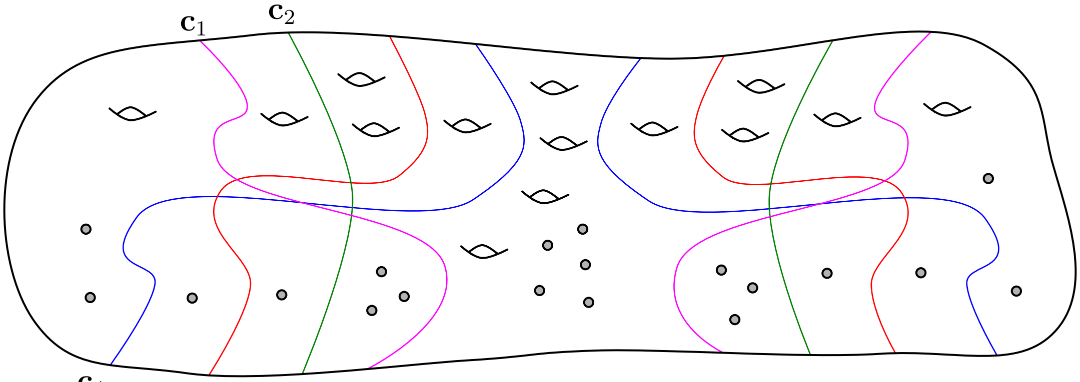

Figure 2.1: Two sets of four intersecting minimum curves in $\Sigma_{14,19}$. Each curve $\mathbf{c}_i$ has an associated region of genus $k_1 = 1$, $k_2 = 2$, $k_3 = 4$, $k_4 = 5$ with $n_1 = 7$, $n_2 = 4$, $n_3 = 3$, $n_4 = 2$ punctures.

Let Σ be a surface and let $A \subset \mathcal{R}(\Sigma)$ such that a g-small region and an n-small region are represented in A. Let

$$\mathcal{X} := \{R \subset \Sigma \mid R \text{ is an associated region of a curve in } \mathcal{S}(A)\}.$$

From the definition it follows that each curve represented in $\mathcal{S}(A)$ will correspond to two elements of $\mathcal{X}$. We now define the following values:

$$\begin{aligned}
k^g &:= \min\{g(R) \mid R \in \mathcal{X}\}, \\
l^g &:= \min\{n(R) \mid R \in \mathcal{X} \text{ and } g(R) = k^g\}, \\
l^n &:= \min\{n(R) \mid R \in \mathcal{X}\}, \text{ and} \\
k^n &:= \min\{g(R) \mid R \in \mathcal{X} \text{ and } n(R) = l^n\}.
\end{aligned}$$

Let $\mathbf{c}$ be a (k^g, l^g)-curve in $\mathcal{S}(A)$ and let R be its associated region of genus k^g with l^g punctures. By definition, all curves contained in R that are represented in $\mathcal{S}(A)$ are isotopic to $\mathbf{c}$. It follows then that $\mathbf{c}$ is the boundary of the unique enveloping region of some g-small region represented in A up to homotopy.

Similarly, let $\mathbf{c}$ now be a curve represented in $\mathcal{S}(A)$ and let R be its associated region of genus k^n with l^n punctures. All separating curves contained in R that are represented in A belong to the isotopy class of $\mathbf{c}$. As above, this implies that $\mathbf{c}$ is the boundary of the enveloping region of an n-small region of A. By the definition of n-small regions we have that $g > 2k^n$ and so (k^n, l^n)-curves are represented in $\mathcal{S}(A)$.

We now describe all minimum curves of $\mathcal{S}(A)$ explicitly. Let $\{\mathbf{c}_i\}_{i=1}^m$ be a set of curves represented in $\mathcal{S}(A)$ where each $\mathbf{c}_i$ is a (k_i, l_i)-curve. The curves $\{\mathbf{c}_i\}_{i=1}^m$ are a set of minimum curves of $\mathcal{S}(A)$ if

$$k^g = k_1 < k_2 < \cdots < k_m = k^n, \quad l^g = l_1 > l_2 > \cdots > l_m = l^n,$$

and for any curve $\mathbf{c}$ represented in $\mathcal{S}(A)$ we have that if $\mathbf{c}$ is a (k_i, l)-curve then $l \geq l_i$. See Figure 2.1 for a picture of generic minimum curves of of the set $\mathcal{S}(A)$.

The following definition gives a useful visual representation of the curves represented in a subset $X \subset \mathcal{S}$.

Lattices of separating curves

Let $X \subset \mathcal{S}$. We define the $\mathrm{Lat}(X)$ to be the integer lattice corresponding to elements of X. That is, if $Z_{k,l} \in X$ then $(k,l) \in \mathrm{Lat}(X) \subset \mathrm{Lat}(\mathcal{S})$, where

$$\mathrm{Lat}(\mathcal{S}) = \big([0, \lfloor g/2 \rfloor] \times [0,n]\big) \cap \big(\mathbb{Z} \times \mathbb{Z}\big).$$

We can equivalently define the numbers k^g, k^n, l^g and l^n using the visual language of lattices of separating curves. Define $H \subset \mathrm{Lat}(\mathcal{S}(A))$ to be the points closest to the horizontal axis and $V \subset \mathrm{Lat}(\mathcal{S}(A))$ to be the points closest to the vertical axis. The point (k^g, l^g) is the point in H closest to V and (k^n, l^n) is the point in V closest to H. We call a point (k,l) in $\mathrm{Lat}(X)$ a *minimum point* if (k,l)-curves are minimum curves of X.

For the special cases when $n = 0$ or $g = 0$ the minimum curves of a subset of separating curves $\mathcal{S}(A) \subset \mathcal{S}$ belong to the $\mathrm{Mod}^{\pm}(\Sigma)$-orbit of a single curve. In these cases the lattice $\mathrm{Lat}(\mathcal{S}(A))$ will consist of the integers in the interval $[k^g, \lfloor g/2 \rfloor]$ when $n = 0$, and the interval $[l^n, n - l^n]$ when $g = 0$.

The following lemma shows that a set of minimum curves of $\mathcal{S}(A)$ determines every curve represented in $\mathcal{S}(A)$.

Lemma 2.2.1. *Let $A \subset \mathcal{R}(\Sigma)$ such that a g-small region and an n-small region are represented in A. The curve $\mathbf{c}$ is represented in $\mathcal{S}(A)$ if and only if it separates two minimum curves.*

Proof. If $\mathbf{c}$ is a curve separating two minimum curves of $\mathcal{S}(A)$ then both associated regions of $\mathbf{c}$ contain an associated region of a minimum curve. Since both associated regions of a given minumum curve contain regions represented in A it follows that $\mathbf{c}$ is represented in $\mathcal{S}(A)$.

Now suppose that $\mathbf{c}$ is represented in $\mathcal{S}(A)$ and R, Q are the associated regions of $\mathbf{c}$ such that $g(R) \leq g(Q)$. We need to show that both R and Q contain a minimum curve. From the definition of k^g we have $g(R) \geq k^g = k_1$. If R does not contain a (k_1, l_1)-curve then $n(R) < l_1$ and so $g(R) \geq k_2$. If R does not contain a (k_2, l_2)-curve then $n(R) < l_2$ and so $g(R) \geq k_3$. If we continue this argument algorithmically it follows that if R does not contain any minimum curves then $n(R) < l_m = l^n$. This contradicts the minimality of l^n, so R must contain a minimum curve. Similarly we can show that Q contains a minimum curve. ■

It can now be checked that $\mathrm{Lat}(\mathcal{S}(A))$ is the set of integer coordinates contained in the shaded region of Figure 2.2.

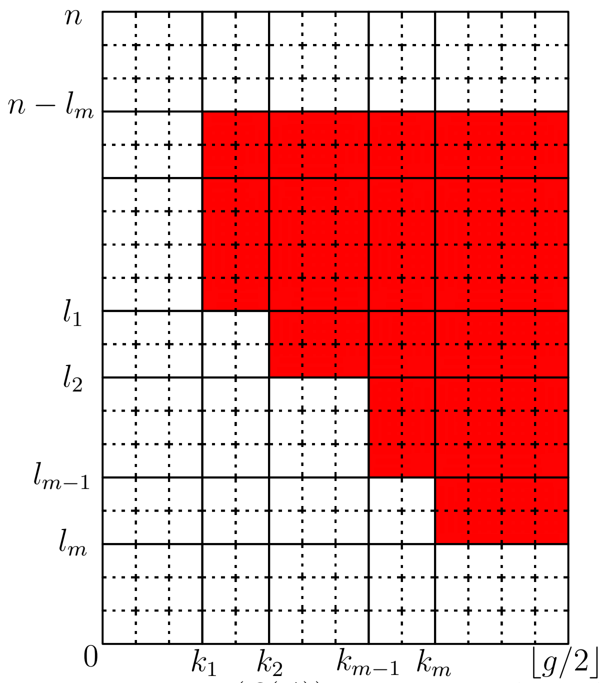

Figure 2.2: The lattice $\mathrm{Lat}(\mathcal{S}(A))$. Recall that $k_1 := k^g$ and $l_m := l^n$.

2.2.2 Extended sets and linear simplices

We now introduce a class of subsets of separating curves related to, but more general than $\mathcal{S}(A) \subset \mathcal{S}$. First, we define a *path* γ in $\mathrm{Lat}(X)$ to be a sequence of integer coordinates $\{(x_i, y_i)\}_{i=1}^{m}$ such that either $x_{i+1} = x_i$ and $|y_{i+1} - y_i| = 1$ or $|x_{i+1} - x_i| = 1$ and $y_{i+1} = y_i$. A path is said to be *strictly decreasing* if $x_{i+1} = x_i - 1$ or $y_{i+1} = y_i - 1$ for all $i \in \{1, \ldots, m-1\}$.

We define the *lower lattice* of a point (k, l) in $\mathrm{Lat}(X)$ to be $([0, k] \times [0, l]) \cap \mathrm{Lat}(X)$. If the lower lattice of (k, l) consists solely of the point itself then (k, l)-curves are minimum curves of X, that is, (k, l) is a minimum point.

Extended sets of separating curves

Let $A \subset \mathcal{R}(\Sigma)$ such that a g-small region and an n-small region are represented in A. Suppose we have a subset of separating curves X such that $\mathcal{S}(A) \subseteq X \subseteq \mathcal{S}$. We call X an *extended set* of $\mathcal{S}(A)$ if the following properties are satisfied;

1. for every minimum point (k_i, l_i) of $\mathrm{Lat}(X)$ there exists a minimum point (k_j, l_j) of $\mathrm{Lat}(\mathcal{S}(A))$ such that (k_i, l_i) is in the lower lattice of (k_j, l_j),

2. given (k_j, l_j) in $\mathrm{Lat}(X)$ and (k_i, l_i) in the lower lattice of (k_j, l_j), there exists a strictly decreasing path γ in $\mathrm{Lat}(X)$ connecting (k_j, l_j) and (k_i, l_i), and

3. if $l \geq n/2$ and $(\lfloor g/2 \rfloor, l) \in \mathrm{Lat}(X)$ then $(\lfloor g/2 \rfloor, n - l) \in \mathrm{Lat}(X)$.

Lemma 2.2.2. *Let $A \subset \mathcal{R}(\Sigma)$ such that a g-small region and an n-small region are represented in A. The set $\mathcal{S}(A)$ is an extended set of $\mathcal{S}(A)$.*

Proof. The first condition is clearly satisfied. To show that the second condition is met, let $(k_i, l_i), (k_j, l_j)$ be two points in $\mathrm{Lat}(\mathcal{S}(A))$ such that they are connected by a strictly decreasing path γ in $\mathrm{Lat}(\mathcal{S})$. Let $\mathbf{a}$ be a (k_i, l_i)-curve and let $\mathbf{b}$ be a (k_j, l_j)-curve such that $\mathbf{a}$ lies on in the associated region of $\mathbf{b}$ of genus k_j. From Lemma 2.2.1 we have that any curve separating $\mathbf{a}$ and $\mathbf{b}$ belongs to $\mathcal{S}(A)$. It follows that the path γ is contained in $\mathrm{Lat}(\mathcal{S}(A))$.

It remains to show that if $(\lfloor g/2 \rfloor, l) \in \mathrm{Lat}(\mathcal{S}(A))$ then $(\lfloor g/2 \rfloor, n-l) \in \mathrm{Lat}(\mathcal{S}(A))$ where $l \geq n/2$. We have that $l \geq l^n$ so $n-l \leq n-l^n$. If g is even then $l = n/2$, and the result is clear. Let g be odd and let $\mathbf{c}$ be a $((g-1)/2, l)$-curve. Let Q be the associated region of $\mathbf{c}$ with $n-l$ punctures. If Q does not contain a $((g-1)/2, n-l)$-curve that is represented in $\mathcal{S}(A)$ then there must be a minimum curve of $\mathcal{S}(A)$ with genus $(g+1)/2$. It follows that $k^n \geq (g+1)/2$, that is, $g \leq 2k^n - 1$. This contradicts the fact that each (k^n, l^n)-curve is the boundary of the enveloping region for some n-small region, completing the proof. ■

Given a subset of separating curves $X \subset \mathcal{S}$ it can be shown that simplices in $\mathcal{C}_X(\Sigma)$ correspond to multicurves of Σ by taking disjoint representatives of the isotopy classes of curves. We intend to make use of a multicurve $M \subset \Sigma$ that partitions the surface into subsurfaces, each with one or two boundary components. In Section 2.2.3 we use this to show that vertex types are preserved by automorphisms of the complex $\mathcal{C}_X(\Sigma)$, when X is an extended set. Recall that the link $\mathrm{Lk}(v)$ of a vertex v is the set of all vertices that span an edge with v.

Sides

Given a vertex v of a subcomplex of separating curves $\mathcal{C}_X(\Sigma)$ we say that vertices u, w lie on the *same side* of v if $u, w \in \mathrm{Lk}(v)$ and there exists another vertex in $\mathrm{Lk}(v)$ that does not span an edge with either u or w.

Linear simplices

We define a simplex σ of $\mathcal{C}_X(\Sigma)$ to be *linear* if there is a labeling of its vertices $v_0, \ldots, v_m$ such that v_{i-1} and v_{i+1} do not lie on the same side of v_i for all $i = 1, \ldots, m-1$. We call the vertices v_0 and v_m the *extreme vertices* of the linear simplex σ.

We say that a linear simplex is *maximal* if its vertices do not form a subset of another linear simplex. Any vertex that belongs to a maximal linear simplex is said to be *linear*. We say that a vertex v is an *increment* of u when v and u are adjacent in the ordering of some maximal linear simplex σ.

If $v_1, \ldots, v_m$ is a maximal linear simplex of a subcomplex of separating curves $\mathcal{C}_X(\Sigma)$ we can find a corresponding multicurve M in Σ. Let $\{R_i\}_{i=0}^{m}$ be the collection of regions defined by $\Sigma \setminus M$. When X is an extended set it is easy to see that two regions, R_0 and R_m, each have a single boundary component and all others have two boundary components. Moreover, every region with two boundary components is homeomorphic

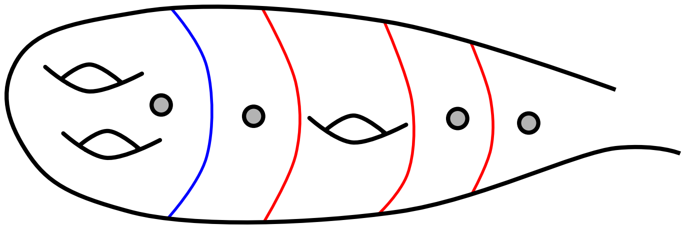

Figure 2.3: A typical maximal linear simplex of $\mathcal{C}_X(\Sigma)$ will correspond to curves of the form shown when X is an extended set of $\mathcal{S}(A)$.

to either $\Sigma^2_{1,0}$ or $\Sigma^2_{0,1}$, see Figure 2.3. This is a direct consequence of the existence of strictly decreasing paths in lattices of extended sets.

Lemma 2.2.3. *Let $A \subset \mathcal{R}(\Sigma)$ such that a g-small region and an n-small region are represented in A. Let $X \subset \mathcal{S}$ be an extended set of $\mathcal{S}(A)$. The vertex v of $\mathcal{C}_X(\Sigma)$ is an extreme vertex of a maximal linear simplex if and only if it corresponds to a minimum curve of X.*

Proof. The point $(\lfloor g/2 \rfloor, \lfloor n/2 \rfloor)$ belongs to $\mathrm{Lat}(\mathcal{S}(A))$. Since X is an extended set of $\mathcal{S}(A)$ we can construct a strictly decreasing path in $\mathrm{Lat}(\mathcal{S}(A))$ from $(\lfloor g/2 \rfloor, \lfloor n/2 \rfloor)$ to the point (k, l), where (k, l)-curves are minimum curves of X. The existence of this path defines a maximal linear simplex σ which proves the lemma. ■

Lemma 2.2.3 tells us that vertices corresponding to minimum curves are a characteristic subset. The next result will be used in Section 2.2.3 to prove that all vertex types are characteristic.

Lemma 2.2.4. *Let $A \subset \mathcal{R}(\Sigma)$ such that a g-small region and an n-small region are represented in A. If $X \subset \mathcal{S}$ is an extended set of $\mathcal{S}(A)$ then every vertex of $\mathcal{C}_X(\Sigma)$ is linear.*

Proof. Let v be a (k, l)-vertex in $\mathcal{C}_X(\Sigma)$. Let $K = \lfloor g/2 \rfloor - 1$ and consider the point $(K, L) \in \mathrm{Lat}(X)$ such that $L \geq \tilde{l}$ for all other points $(K, \tilde{l}) \in \mathrm{Lat}(X)$. Since X is an extended set of $\mathcal{S}(A)$ we can construct a strictly decreasing path from (K, L) to (k, l) and a strictly decreasing path from (k, l) to a minimum point of $\mathrm{Lat}(X)$. Since $L \geq n/2$ and X is an extended set of $\mathcal{S}(A)$ we have that $(K, n - L) \in \mathrm{Lat}(X)$ and so a strictly decreasing path exists between $(K, n-L)$ and a minimum point of $\mathrm{Lat}(X)$. The existence of these two paths, along with the path from $(\lfloor g/2 \rfloor, \lfloor n/2 \rfloor)$ to $(\lfloor g/2 \rfloor, n - L)$ is enough to show that an maximal linear simplex σ containing v exists in $\mathcal{C}_X(\Sigma)$. ■

2.2.3 Preservation by automorphisms

We will now study a property of subcomplexes of the separating curve complex that is necessary when proving Theorem 2.0.1. In the last section it was shown that there are sets of separating curves $X \subset \mathcal{S}$ such that every vertex of the complex $\mathcal{C}_X(\Sigma)$ is linear. We now discuss the two possible types of *increment* and show that each type is also characteristic in $\mathcal{C}_X(\Sigma)$.

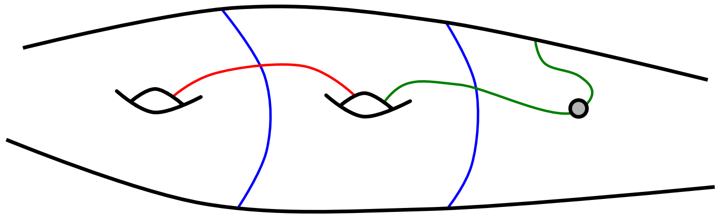

Figure 2.4: A typical genus increment.

Genus increments and puncture increments

Suppose u is (k_1, l_1)-vertex and v is a (k_2, l_2)-vertex such that u and v are adjacent in the ordering of some maximal linear simplex. It follows that either $k_1 = k_2 \pm 1$ or $l_1 = l_2 \pm 1$. In the first case we call u a *genus increment* of v and in the second case we call u a *puncture increment* of v.

Lemma 2.2.5. *Let $A \subset \mathcal{R}(\Sigma)$ such that a g-small region and an n-small region are represented in A. Let X be an extended region of $\mathcal{S}(A)$ and let ϕ be an automorphism of $\mathcal{C}_X(\Sigma)$. If a vertex v is a genus (resp. puncture) increment of a vertex u then $\phi(v)$ is a genus (resp. puncture) increment of $\phi(u)$.*

Proof. We claim that the vertex v is a genus increment of u if and only if there exist vertices x and y, such that the vertex set $\{u, v, x, y\}$ spans a square in $\mathcal{C}_X(\Sigma)$. The result follows from the claim. The forward implication of the claim is clear. We take appropriate Dehn twists or push maps of representative curves of u and v, see Figure 2.4.

Suppose now that the vertex v is an increment of the vertex u and such a x and y exist. Let R be the region with boundary components corresponding to u and v. Assume v is a puncture increment of u and take B to be the a regular neighbourhood union of the curves $\mathbf{u}$ and $\mathbf{x}$ which have minimal intersection where u, x correspond to $\mathbf{u}, \mathbf{x}$. One of the components of ∂B is the boundary of a disc $\mathbf{D}_1 \subset R$ with a single puncture. Any choice of y not adjacent to u will correspond to a curve that intersects $\mathbf{D}_1$; hence y is not adjacent to either x or v, a contradiction. ■

In Lemma 2.2.3 it was shown that automorphisms of $\mathcal{C}_X(\Sigma)$ preserve the set of vertices corresponding to minimum curves of the extended set X. We will now use Lemma 2.2.5 to go one step further to proving that each curve type determines a characteristic subset of vertices.

Lemma 2.2.6. *Let $A \subset \mathcal{R}(\Sigma)$ such that a g-small region and an n-small region are represented in A. Let $X \subset \mathcal{S}$ be an extended set of $\mathcal{S}(A)$. The (k, l)-vertices corresponding to minimum curves of X form a characteristic subset of $\mathcal{C}_X(\Sigma)$ for all possible values of k and l.*

Proof. In Lemma 2.2.3 it was shown that if the vertex v corresponds to a minimum curve then so does the vertex $\phi(v)$. We need to show that if v is a (k, l)-vertex then $\phi(v)$ is a (k, l)-vertex. There exist strictly decreasing paths in $\mathrm{Lat}(X)$ from (k^n, l^g) to any minimum point of $\mathrm{Lat}(X)$. Let the vertices v and $\phi(v)$ correspond to curves $\mathbf{v}$ and $\mathbf{v}'$. Since g-small and n-small regions are represented in A it follows that $g \geq 3k^g$ and $n \geq 3l^n$. Hence, there exists a minimum curve $\mathbf{c}$ in Σ disjoint from both $\mathbf{v}$ and $\mathbf{v}'$.

We claim that $\phi(v)$ is a (k,l)-vertex if and only if there exist maximal linear simplices

$$\sigma_u = \{u_0, u_1, \ldots, u_m\} \quad \text{and} \quad \sigma_w = \{w_0, w_1, \ldots, w_m\}$$

such that $u_0 = v$, $w_0 = \phi(v)$, u_m and w_m correspond to the curve $\mathbf{c}$, and both simplices consist of the same number of genus increments. The result then follows the claim.

To prove one direction of the claim we note that if $\mathbf{v}$ and $\mathbf{v}'$ are both (k,l)-curves lying in the same associated regions of $\mathbf{c}$ then we can define a maximal multicurve of separating curves between $\mathbf{c}$ and $\mathbf{v}$ (or $\mathbf{v}'$). Such that there exists a maximal linear simplex corresponding to these multicurves. The number of genus increments in each sequence will be equal.

Now, to prove the other direction let Q_u and Q_w be the highest genus non-separating regions associated to $\mathbf{v}$ and $\mathbf{v}'$. Let $\mathbf{c}$ be a $(\tilde{k}, \tilde{l})$-curve and suppose both simplices have m_g genus increments. We see that $g(Q_u) = g(Q_w) = \tilde{k} + m_g$. It follows that $\phi(v)$ is a (k,l)-vertex. ■

We can now finally prove that curve types determine characteristic subsets of vertices in the complex $\mathcal{C}_X(\Sigma)$.

Lemma 2.2.7. *Let $A \subset \mathcal{R}(\Sigma)$ such that a g-small region and an n-small region are represented in A. Let $X \subset \mathcal{S}$ be an extended set of $\mathcal{S}(A)$. The (k,l)-vertices form a characteristic subset of $\mathcal{C}_X(\Sigma)$ for all possible values of k and l.*

Proof. Let v be a (k,l)-vertex of $\mathcal{C}_X(\Sigma)$ corresponding to the curve $\mathbf{v}$ and let ϕ be an automorphism of $\mathcal{C}_X(\Sigma)$ such that $\phi(v)$ corresponds to the curve $\mathbf{v}'$. We need to show that $\mathbf{v}'$ is a (k,l)-curve. It follows from Lemma 2.2.4 that there exists some maximal linear simplex σ containing v. Suppose one of the extreme vertices is a $(\tilde{k}, \tilde{l})$-vertex distance t from v with respect to the ordering of σ. Now, Lemma 2.2.6 tells us that $\phi(\sigma)$ is a maximal linear simplex containing $\phi(v)$ and that one of the extreme vertices is a $(\tilde{k}, \tilde{l})$-vertex distance t from $\phi(v)$. Finally, it follows from Lemma 2.2.5 that $\mathbf{v}'$ is a (k,l)-curve and hence $\phi(v)$ is a (k,l)-vertex. ■

Note that in order to prove curve types determine characteristic subsets of vertices for surfaces of the form $\Sigma_{g,0}$ or $\Sigma_{0,n}$ we need only define maximal linear simplices. Indeed, all minimum curves are of the same topological type and there are only either genus increments or puncture increments. When these special cases are handled by Brendle-Margalit [13] and the author [55], the content of Section 2.2 can be reduced to a single lemma.

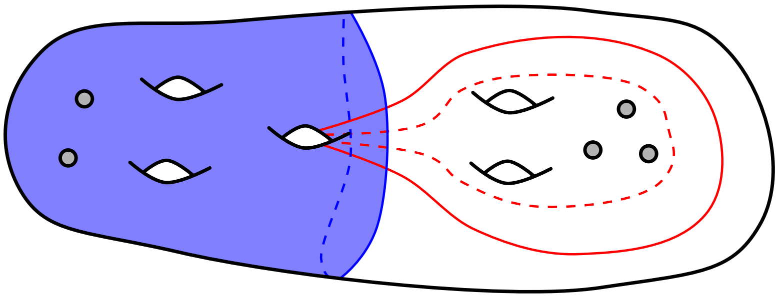

Figure 2.5: A vertex in $\mathcal{C}_X(\Sigma)$ corresponding to the red curve on the right projects to a non-separating arc.

2.3 Sharing pairs

In the last section we discussed subcomplexes of separating curves and in particular $\mathcal{C}_{\mathcal{S}(A)}(\Sigma)$. The purpose of this section is to show that certain intersection data is characteristic to the complexes we are interested in. We will generalise the notion of *sharing pairs* defined by Brendle-Margalit [13, Section 3] into three distinct flavours. In each case we say a pair of (k, l)-curves $\mathbf{a}, \mathbf{b}$ *share* a curve $\mathbf{c}$. If $\mathbf{c}$ is $(k-1, l)$-curve we call $\mathbf{a}, \mathbf{b}$ a *genus sharing pair*. If $\mathbf{c}$ is $(k, l-1)$-curve we call $\mathbf{a}, \mathbf{b}$ a *puncture sharing pair*. Finally, if $\mathbf{c}$ is $(k, l+1)$-curve we call $\mathbf{a}, \mathbf{b}$ a *reversed puncture sharing pair*.

Before we give the definition of *sharing pairs* we introduce *arcs* to facilitate the discussion. Let R be a surface with boundary. An *arc* in R is a continuous image of the interval whose endpoints map to the boundary of R. Let z be a vertex of $\mathcal{C}_X(\Sigma)$ corresponding to a curve with an associated region R of lowest genus with fewest punctures and let $SA(R)$ be the set whose elements are the, possibly empty, sets of arcs in R. We can define a projection map

$$\pi_z : \mathcal{C}_X(\Sigma) \to SA(R).$$

If v is a vertex of $\mathcal{C}_X(\Sigma)$ that shares an edge with z then $\pi_z(v) = \emptyset$. If v and z do not share an edge then v corresponds to a curve whose intersection with R is a nonempty collection of disjoint arcs.

We call an arc α *non-separating* if $R \setminus \alpha$ is a single connected surface, otherwise we call it *separating*. For a vertex $v \in \mathcal{C}_X(\Sigma)$, if the projection $\pi_z(v)$ is a set of non-separating arcs that belong to the same free isotopy class then it makes sense to think of $\pi_z(v)$ as a single non-separating arc up to isotopy. As we can see from Figure 2.5, it is possible for a vertex $v \in \mathcal{C}_X(\Sigma)$ to have a non-separating projection $\pi_z(v) \in SA(R)$.

2.3.1 Genus sharing pairs

The following definitions and Lemma 2.3.1 are necessary for the subsequent discussion of *genus sharing pairs*.

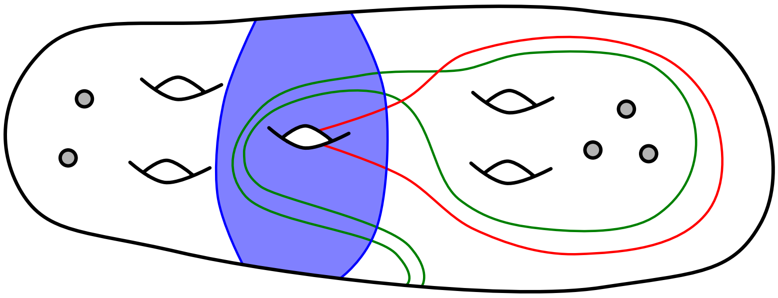

Figure 2.6: Two separating curves such that the union of their projections define the shaded torus with two boundary components, $\Sigma^2_{1,0}$.

Unlinked projections and handle pairs

For a surface R with one boundary component two vertices u, v of $\mathcal{C}_X(\Sigma)$ are said to have *unlinked projections* if there exists a connected segment of ∂R intersecting an arc of $\pi_z(u)$ twice but not intersecting $\pi_z(v)$.

The vertices u, v form a *handle pair* for R if $\pi_z(u)$ and $\pi_z(v)$ are distinct non-separating arcs of R with representatives that lie on some subsurface $Q \subset R$ such that $Q \cong \Sigma^2_{1,0}$, see Figure 2.6.

Recall that for a vertex v in $\mathcal{C}_X(\Sigma)$ we say that vertices u, w lie on the same side of v if $u, w \in \mathrm{Lk}(v)$ and there exists another vertex in $\mathrm{Lk}(v)$ which does not span an edge with either u or w. If v is a (k, l)-vertex then we say that a vertex lies on the *small side* of v if it does not lie on the same side as a $(k+1, l)$-vertex. The following result is analagous to a result of Brendle-Margalit in the closed case [13, Lemma 3.2].

Lemma 2.3.1. *Let $A \in \mathcal{R}(\Sigma)$ such that a g-small region and an n-small region are represented in A. Let $X \subset \mathcal{S}$ be an extended set of $\mathcal{S}(A)$ and let $\phi \in \mathrm{Aut}\,\mathcal{C}_X(\Sigma)$. Suppose z is a (k, l)-vertex and that $\mathcal{C}_X(\Sigma)$ contains $(k-1, l)$-vertices. Let u and v be two vertices of $\mathcal{C}_X(\Sigma)$ such that $\pi_z(u)$ and $\pi_z(v)$ are distinct, non-separating arcs.*

1. *The projection $\pi_{\phi(z)}(\phi(u))$ is a non-separating arc;*

2. *If $\pi_z(u)$ and $\pi_z(v)$ are unlinked non-separating arcs then $\pi_{\phi(z)}(\phi(u))$ and $\pi_{\phi(z)}(\phi(v))$ are unlinked non-separating arcs.*

3. *If $\pi_z(u)$ and $\pi_z(v)$ are a handle pair then $\pi_{\phi(z)}(\phi(u))$ and $\pi_{\phi(z)}(\phi(v))$ are a handle pair.*

Proof. Let R be a region of genus k with l punctures such that z corresponds to ∂R. For the first statement we claim that $\pi_z(u)$ is a non-separating arc if and only if there is more than one $(k-1, l)$-vertex in $\mathrm{Lk}(u)$ that lies on the small side of z.

To prove the forward direction we assume that $\pi_z(u)$ is a non-separating arc. It follows then that $R \setminus \pi_z(u) \cong \Sigma^2_{k-1,l}$. As there are infinitely many curves in $\Sigma^2_{k-1,l}$ separating the surface into a pair of pants and a surface homeomorphic to $\Sigma^1_{k-1,l}$, the implication is clear.

We deal with the other direction in two cases: either $\pi_z(u)$ contains the homotopy class of a separating arc or it contains more than one homotopy class of non-separating arcs. In the first case, if we cut R by a separating arc it results in two surfaces R_1 and R_2 homeomorphic to $\Sigma^1_{k_1,l_1}$ and $\Sigma^1_{k_2,l_2}$ with $k_2 \geq k_1$ and $k_1 + k_2 = k$ and $l_1 + l_2 = l$. If w is a vertex in $\mathrm{Lk}(u)$ on the small side of z then it must correspond to a curve contained in either R_1 or R_2. If w is a $(k-1, l)$-vertex then $k_1 = 1$, $l_1 = 0$ and so w is unique, a contradiction.

In the second case, suppose we cut along two distinct and disjoint non-separating arcs in R. Either we obtain a surface of genus $k-1$ and l punctures or we obtain one or two surfaces with less genus than $k-1$. That is, either there exists a single $(k-1, l)$-vertex adjacent to u on the small side of z or there are none. This completes the proof of the first statement.

To prove the second statement let u and v be adjacent vertices such that $\pi_z(u)$ and $\pi_z(v)$ are unlinked non-separating arcs. These arcs are distinct if and only if there exists a $(k-1, l)$-vertex of $\mathcal{C}_X(\Sigma)$ on the small side of z that is adjacent to u but not v. To prove the statement then we claim that the arcs $\pi_z(u)$ and $\pi_z(v)$ are linked if and only if there exists a $(k-1, l)$-vertex w in $\mathcal{C}_X(\Sigma)$ that lies on the small side of z and is adjacent to both u and v.

If we cut R along disjoint representatives of $\pi_z(u)$ and $\pi(v)$ then we either obtain a surface of genus $k-1$ and l punctures or we obtain one or two surfaces of genus less than $k-1$, depending on whether $\pi_z(u)$ and $\pi_z(v)$ are linked or unlinked. The claim follows similarly to the proof of the first statement.

Finally, we note that two non-separating arcs form a handle pair if and only if they are linked. This completes the proof. ∎

Genus sharing pairs

We say that two (k, l)-vertices form a (k, l)-*genus sharing pair* if they correspond to curves with geometric intersection number two and, of the four surfaces obtained by cutting Σ along the curves, one is homeomorphic to $\Sigma^1_{k-1,l}$ and two are homeomorphic to $\Sigma^1_{1,0}$.

If two vertices that form a genus sharing pair correspond to the curves $\mathbf{a}, \mathbf{b}$ we say that $\mathbf{a}, \mathbf{b}$ *share* the $(k-1, l)$-curve $\mathbf{c}$, where $\mathbf{c}$ is isotopic to the boundary curve of the region homeomorphic to $\Sigma^1_{k-1,l}$.

Lemma 2.3.2. *Let $A \subset \mathcal{R}(\Sigma)$ such that $X \subset \mathcal{S}$ is an extended set of $\mathcal{S}(A)$ and (k_0, l_0)-vertices are represented in $\mathcal{S}(A)$. Let u, v form a (k, l)-genus sharing pair. If $g \geq 2k_0 + k + 1$ and $n \geq 2l_0 + l$ then $\phi(u), \phi(v)$ form a (k, l)-genus sharing pair for all $\phi \in \operatorname{Aut} \mathcal{C}_X(\Sigma)$.*

Proof. We will show that two vertices u, v form a (k, l)-genus sharing pair if and only if there are (k_0, l_0)-vertices x_1, x_2, y_1, y_2 and a $(k+1, l)$-vertex z that satisfy the following properties.

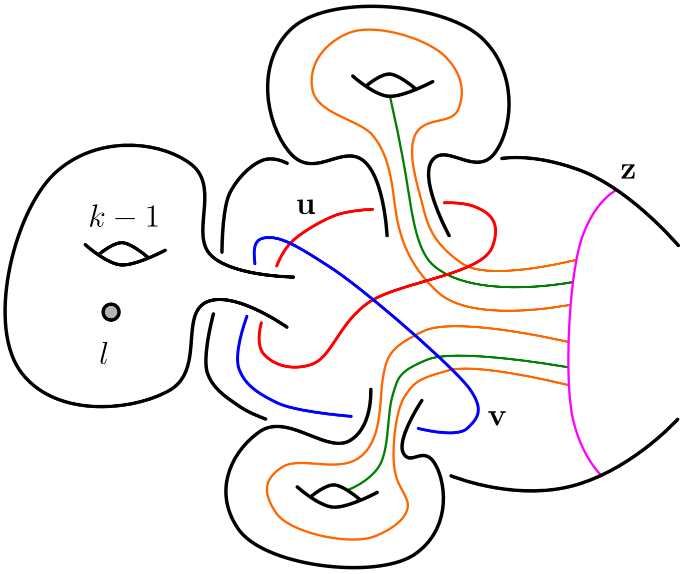

Figure 2.7: A (k,l)-genus sharing pair u,v corresponds to the red and blue curves. The vertex z corresponds to light blue curve. The arcs $\pi_z(x_1)$,$\pi_z(x_2)$,$\pi_z(y_1)$ and $\pi_z(y_2)$ are shown in green and orange.

1. Both u and v lie on the small side of z;
2. both x_1 and x_2 are adjacent to u, y_1 and y_2, but not v;
3. both y_1 and y_2 are adjacent to v, x_1 and x_2 but not u;
4. both pairs $\pi_z(x_1), \pi_z(x_2)$ and $\pi_z(y_1), \pi_z(y_2)$ are distinct handle pairs; and
5. the arcs $\pi_z(x_i)$ and $\pi_z(y_j)$ are unlinked for $i, j \in \{1, 2\}$.

The result then follows from Lemmas 2.2.7 and 2.3.1.

Suppose the vertices u, v form a (k,l)-genus sharing pair and correspond to the curves $\mathbf{u}, \mathbf{v}$. Up to homeomorphism there is a unique confuguration for the curves $\mathbf{u}, \mathbf{v}$ shown in Figure 2.7.

The curves $\mathbf{u}$ and $\mathbf{v}$ separate Σ into four regions which are homeomorphic to $\Sigma^1_{k-1,l}$, $\Sigma^1_{1,0}$, $\Sigma^1_{1,0}$ and $\Sigma^1_{g-k-1,n-l}$. Take R to be the complement of this final region in Σ and let z be the vertex corresponding to ∂R. We then define x_1, x_2, y_1, and y_2 to be (k_0, l_0)-vertices corresponding to the projected arcs shown in Figure 2.7. The chosen vertices satisfy the five conditions above.

Now suppose we have vertices $u, v, x_1, x_2, y_1, y_2, F$ and z satisfying the above conditions. By the fourth condition the arcs $\pi_z(x_1)$ and $\pi_z(x_2)$ are contained in some region $Q_x \cong \Sigma^2_{1,0}$. Denote the two boundary components of Q_x by $\partial_1 Q_x$ and $\partial_2 Q_x$. The vertex z must correspond to $\partial_1 Q_x$, which we label $\mathbf{z}$, and the arcs $\pi_z(x_1)$ and $\pi_z(x_2)$ have endpoints on $\mathbf{z}$. We want to show that the vertex u corresponds to $\partial_2 Q_x$.

The surface obtained by cutting along Q_x by $\pi_z(x_1)$ is homeomorphic to a pair of pants P. If we then cut P along $\pi_z(x_2)$ the resulting surface is an annulus. It follows that $\pi_z(x_1)$ and $\pi_z(x_2)$ fill Q_x. From the second condition we have that u corresponds to a curve $\mathbf{u}$ that is disjoint from Q_x. Since Q_x is of genus one the boundary component

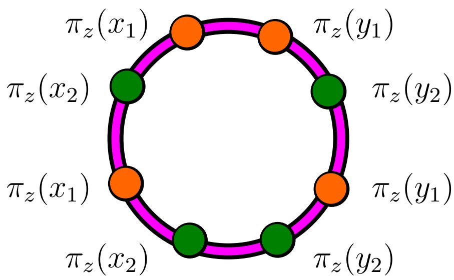

Figure 2.8: The curve $\mathbf{z}$ with indicated intersections with $\pi_z(x_1)$,$\pi_z(x_2)$,$\pi_z(y_1)$ and $\pi_z(y_2)$

$\partial_2 Q_x$ must be isotopic to $\mathbf{u}$. By symmetry, the vertex v must correspond to $\mathbf{v}$ the boundary component of the equivalent region Q_y not equal to ∂R.

From the fifth condition, we can view $\mathbf{z}$ as the circle in Figure 2.8. There exist segments γ_x and γ_y of $\mathbf{z}$, with $\gamma_x \cup \gamma_y = \mathbf{z}$, such that the arcs $\pi_z(x_1)$ and $\pi_z(x_2)$ have endpoints in γ_x and the arcs $\pi_z(y_1)$ and $\pi_z(y_2)$ have endpoints in γ_y. It follows that the intersection of $\pi_z(y_1)$ and $\pi_z(y_2)$ with Q_x is a set of four freely isotopic arcs. Since Q_y is a regular neighbourhood of the arcs $\pi_z(y_1)$ and $\pi_z(y_2)$ we have that the intersection of Q_x and Q_y is an annulus whose boundary components are isotopic to $\mathbf{z}$. The curves $\mathbf{u}$ and $\mathbf{v}$ must therefore have essential intersection two.

If two separating simple closed curves intersect in two points then they divide Σ into four regions, one of which must contain $\mathbf{z}$. It follows that one of these regions is of genus $k-1$ and has l punctures. Thus, u, v form a genus sharing pair. ■

2.3.2 Puncture sharing pairs

Before introducing the second type of sharing pair we note that if two arcs in a region R of Σ are separating and disjoint then they are necessarily unlinked.

Puncture sharing pairs

We say that two (k,l)-vertices form a (k,l)-*puncture sharing pair* if they correspond to curves with geometric intersection number two and, of the four surfaces obtained by cutting Σ along the curves, one is homeomorphic to $\Sigma^1_{k,l-1}$ and two are homeomorphic to $\Sigma^1_{0,1}$.

If two vertices that form a puncture sharing pair correspond to the curves $\mathbf{a}, \mathbf{b}$ we say that $\mathbf{a}, \mathbf{b}$ *share* the curve $\mathbf{c}$, where $\mathbf{c}$ is isotopic to the boundary curve of the region homeomorphic to $\Sigma^1_{k,l-1}$.

As well as pairs of (k,l)-curves sharing a $(k,l-1)$-curve, it will be necessary for Section 2.4 that we look at (k,l)-curves sharing a $(k,l+1)$-curve. The definition of these *reversed puncture sharing pairs* is essentially the same as that of puncture sharing pairs. Note that we define two types of puncture sharing pairs simply because our definition of the (k,l)-curves prioritises genus over number of punctures. This shift in perspective is needed in order to make certain lemmas easier to prove later on.

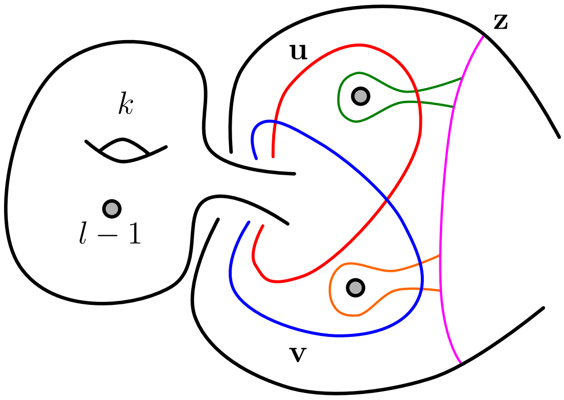

Figure 2.9: A (k, l)-puncture sharing pair u, v corresponds to the red and blue curves. The vertex z corresponds to light blue curve. The arcs $\pi_z(x)$ and $\pi_z(y)$ are shown in green and orange.

Reversed puncture sharing pairs

We say that two (k, l)-vertices form a (k, l)-*reversed puncture sharing pair* if they correspond to curves with geometric intersection number two and, of the four surfaces obtained by cutting Σ along the curves, one is homeomorphic to $\Sigma^1_{k,l+1}$ and two are homeomorphic to $\Sigma^1_{0,1}$.

As above, if two vertices that form a reversed puncture sharing pair correspond to the curves $\mathbf{a}, \mathbf{b}$ we say that $\mathbf{a}, \mathbf{b}$ *share* the curve $\mathbf{c}$, where $\mathbf{c}$ is isotopic to the boundary curve of the region homeomorphic to $\Sigma^1_{k,l+1}$.

Lemma 2.3.3. *Let $A \subset \mathcal{R}(\Sigma)$ such that $X \subset \mathcal{S}$ is an extended set of $\mathcal{S}(A)$ and (k_0, l_0)-vertices are represented in $\mathcal{S}(A)$. Let u, v form a (k, l)-puncture sharing pair. If $g \geq 2k_0 + k$ and $n \geq 2l_0 + l - 1$ then $\phi(u), \phi(v)$ form a (k, l)-puncture sharing pair for all $\phi \in \operatorname{Aut} \mathcal{C}_X(\Sigma)$.*

Proof. We will show that two vertices u, v form a (k, l)-puncture sharing pair if and only if there are (k_0, l_0)-vertices x and y and a $(k, l + 1)$-vertex z that satisfy the following properties.

1. Both u and v lie on the small side of z;

2. the vertex x is adjacent to u and y but not v;

3. the vertex y is adjacent to v and x but not u; and

4. both $\pi_z(x)$ and $\pi_z(y)$ are disjoint separating arcs.

The result then follows from Lemmas 2.2.7 and 2.3.1.

Suppose the vertices u, v form a (k, l)-puncture sharing pair and correspond to the curves $\mathbf{u}, \mathbf{v}$. Up to homeomorphism there is a unique configuration for the curves $\mathbf{u}, \mathbf{v}$ shown in Figure 2.9.

The curves $\mathbf{u}$ and $\mathbf{v}$ separate Σ into four regions which are homeomorphic to $\Sigma^1_{k,l-1}$, $\Sigma^1_{0,1}$, $\Sigma^1_{0,1}$ and $\Sigma^1_{g-k,n-l-1}$. Take R to be the complement of this final region in Σ and

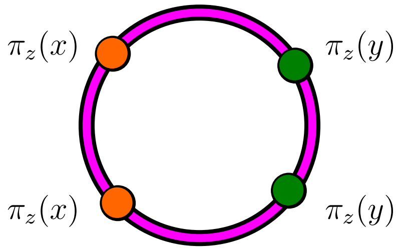

Figure 2.10: The curve $\mathbf{z}$ with indicated intersections with $\pi_z(x)$ and $\pi_z(y)$

let z be the vertex corresponding to ∂R. We then define x and y to be (k_0, l_0)-vertices corresponding to the projected arcs shown in Figure 2.9. The chosen vertices satisfy the five conditions above.

Now suppose we have vertices u, v, x, y and z satisfying the above conditions. By the first and second conditions the arc $\pi_z(x)$ is contained in some Q_x homeomorphic to an annulus with a single puncture. Denote the two boundary components of Q_x by $\partial_1 Q_x$ and $\partial_2 Q_x$. The vertex z must correspond to $\partial_1 Q_x$, which we label $\mathbf{z}$, and the arc $\pi_z(x)$ has endpoints on $\mathbf{z}$. We want to show that the vertex u corresponds to $\partial_2 Q_x$.

When we cut Q_x along the arc $\pi_z(x)$ we get two surfaces; an annulus and a disc with one puncture. The boundary of this annulus is isotopic to $\partial_2 Q_x$, a (k, l)-curve $\mathbf{u}$ that is contained in the associated region of $\mathbf{z}$ with genus k. This curve is unique and it follows that u corresponds to $\mathbf{u}$, hence $\partial_2 Q_x$. By symmetry, the vertex v must correspond to $\mathbf{v}$, the boundary component of the equivalent region Q_y not isotopic to $\mathbf{z}$.

From the fourth condition the curve $\mathbf{z}$ takes the form of the circle in Figure 2.10. There exist segments γ_x and γ_y of $\mathbf{z}$, with $\gamma_x \cup \gamma_y = \mathbf{z}$, such that the arcs $\pi_z(x)$ and $\pi_z(y)$ have endpoints in γ_x and γ_y respectively.

It follows that the intersection of the arc representing $\pi_z(y)$ with Q_x is a set of two freely isotopic arcs. If we cut along one of these arcs then since $\mathbf{u}$ and $\mathbf{v}$ must intersect they take the form shown in Figure 2.11 where they intersect exactly twice.

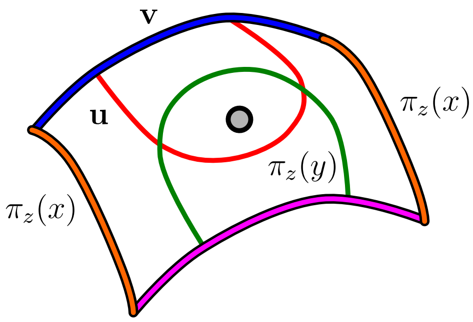

Figure 2.11: A subsurface homeomorphic to $\mathbf{D}_1$ with boundary is defined by the curves $\mathbf{v}$, $\mathbf{z}$ and the arc $\pi_z(x)$. The picture indicates that $\mathbf{u}$ and $\mathbf{v}$ intersect exactly twice.

If two separating simple closed curves intersect in two points then they divide Σ into four regions, one of which must contain $\mathbf{z}$. It follows that one of these regions is of genus k and has $l-1$ punctures. Thus, u, v form a genus sharing pair. ∎

Finally, we state the case of reversed puncture sharing pairs.

Lemma 2.3.4. *Let $A \subset \mathcal{R}(\Sigma)$ such that $X \subset \mathcal{S}$ is an extended set of $\mathcal{S}(A)$ and (k_0, l_0)-vertices are represented in $\mathcal{S}(A)$. Let u, v form a (k, l)-reversed puncture sharing pair. If $k \geq 2k_0$ and $l \geq 2l_0 - 1$ then $\phi(u), \phi(v)$ form a (k, l)-reversed puncture sharing pair for all $\phi \in \operatorname{Aut} \mathcal{C}_X(\Sigma)$.*

The bounds on k and l are obtained by replacing k and l in the bounds in the statement of Lemma 2.3.3 with $g - k$ and $n - l$. This lemma is just a rephrasing of Lemma 2.3.3 and so the proof is omitted.

2.3.3 Sharing triples

Let $A \subset \mathcal{R}(\Sigma)$ such that a g-small region and an n-small region are represented in A. Let $X \subset \mathcal{S}$ be an extended set of $\mathcal{S}(A)$. In this section we will prove that for a given automorphism of the complex $\mathcal{C}_X(\Sigma)$ we can extend it to an automorphism of $\mathcal{C}_Y(\Sigma)$, where Y is an extended set of $\mathcal{S}(A)$ containing X. We do this by first introducing the *graph of sharing pairs* and showing that it consists of infinitely many connected components, each representing a unique isotopy class of a shared curve.

Recall the set $Z_{k,l} \in \mathcal{S}(A)$ is the orbit of a (k, l)-curve. Suppose $Z_{k,l} \in X$. If $Y = X \cup \{Z_{k-1,l}\}$ then we call Y a *genus extension* of X. If $Y = X \cup \{Z_{k,l-1}\}$ then we call Y a *puncture extension* of X. If $Y = X \cup \{Z_{k,l+1}\}$ then we call Y a *reversed puncture extension* of X.

Graphs of sharing pairs

We call three vertices u, v, w a *sharing triple* if they form pairwise sharing pairs of the same type and correspond to curves that pairwise share the same curve. We construct a graph $\mathcal{SP}$ with vertices corresponding to all (k, l)-sharing pairs of the same type. Two vertices share an edge if they correspond to sharing pairs u, v and v, w, where u, v, w is a sharing triple. Note that this definition holds for the three types of sharing pair introduced so far in Section 2.3.

It is clear that if two vertices are connected then they correspond to sharing pairs that share the same curve. This implies that the graph $\mathcal{SP}$ is made up of various disconnected components, we will write $\mathcal{SP}(\mathbf{c})$ for the components relating to sharing pairs that correspond to pairs of curves that share the same curve $\mathbf{c}$. We will make use of a result of Putman concerning the connectivity of simplicial complexes [60].

Lemma 2.3.5 (Putman)**.** *Let G be a group acting on a simplicial complex X with v a fixed vertex in X^0. Let S be a set of generators of G and assume that;*

1. *for all $u \in X^0$, the orbit $G \cdot v$ intersects the connected component of X containing u, and*
2. *for all $s \in S^{\pm 1}$, there is a path P_s in X from v to $s \cdot v$.*

Then X is connected.

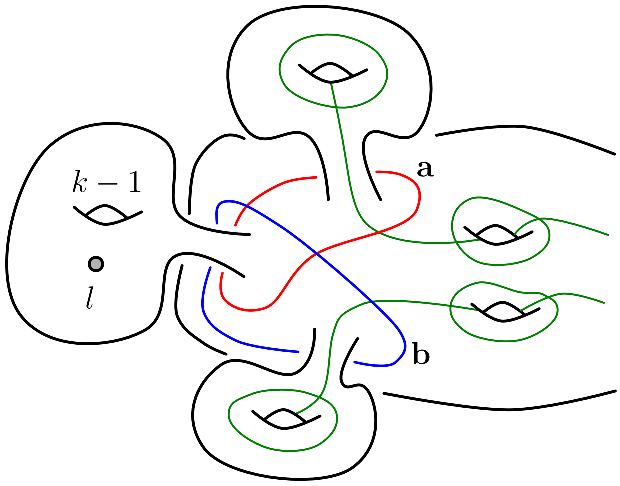

Figure 2.12: Generating twists and half twists with respect to a genus sharing pair.

Lemma 2.3.6. *Let Σ be a surface with $g \geq k+3$ and $n \geq l+2$. Let $\mathcal{SP}$ be any graph of (k,l)-sharing pairs such that a vertex of $\mathcal{SP}$ corresponds to curves that share a curve* $\mathbf{c}$. *Then the subgraph $\mathcal{SP}(\mathbf{c})$ is a single connected component of $\mathcal{SP}$.*

Proof. Let $\mathbf{a}, \mathbf{b}$ be (k,l)-curves that share the curve $\mathbf{c}$. Let R be the associated region of $\mathbf{c}$ that does not contain $\mathbf{a}$ or $\mathbf{b}$. Let $\mathrm{Mod}^{\pm}(\Sigma, R)$ be the subgroup of $\mathrm{Mod}^{\pm}(\Sigma)$ that fixes the subsurface R. For each vertex v in $\mathcal{SP}(\mathbf{c})$ there exists some $[f] \in \mathrm{Mod}^{\pm}(\Sigma, R)$ such that v corresponds to the curves $[f](\mathbf{a}), [f](\mathbf{b})$. This satisfies the first condition in Lemma 2.3.5 for the simplicial complex $\mathcal{SP}(\mathbf{c})$. It remains to show that the second condition is satisfied. This will be done in two cases; the first case concerns genus sharing pairs and the second concerns both puncture and reversed puncture sharing pairs. Suppose the vertices of $\mathcal{SP}$ correspond to genus sharing pairs. The groups $\mathrm{Mod}^{\pm}(\Sigma, R)$ and $\mathrm{Mod}^{\pm}(\Sigma^1_{g-(k-1),n-l})$ are isomorphic. It follows that there exists a finite generating set for $\mathrm{Mod}^{\pm}(\Sigma, R)$ consisting of Dehn twists about non-separating curves and half twists about $(0,2)$-curves, see Figure 2.12. We choose the set so that one non-separating curve intersects $\mathbf{a}$, one non-separating curve intersects $\mathbf{b}$, and all other curves are disjoint from both $\mathbf{a}$ and $\mathbf{b}$.

By symmetry it is enough to consider the single case where T is a Dehn twist about a non-separating curve intersecting $\mathbf{b}$ and disjoint from $\mathbf{a}$. It is clear that $T(\mathbf{a}) = \mathbf{a}$ and that $\mathbf{a}, T(\mathbf{b})$ share the curve $\mathbf{c}$. It remains to show that the vertices corresponding to $\mathbf{a}, \mathbf{b}$ and $\mathbf{a}, T(\mathbf{b})$ are connected in $\mathcal{SP}(\mathbf{c})$.

Given $g \geq k+3$ we can find a curve $\mathbf{d}$ such that that the vertex corresponding to $\mathbf{a}, \mathbf{d}$ is adjacent to the vertices corresponding to $\mathbf{a}, \mathbf{b}$ and $\mathbf{a}, T(\mathbf{b})$ in $\mathcal{SP}$, see Figure 2.13. By Lemma 2.3.5 the result holds for graphs of genus sharing pairs.

Now suppose the vertices of $\mathcal{SP}$ correspond to puncture sharing pairs. The groups $\mathrm{Mod}^{\pm}(\Sigma, R)$ and $\mathrm{Mod}^{\pm}(\Sigma^1_{g-k,n-(l-1)})$ are isomorphic. In the case where the vertices correspond to reversed puncture sharing pairs we have that $\mathrm{Mod}^{\pm}(\Sigma, R)$ is isomorphic to $\mathrm{Mod}^{\pm}(\Sigma^1_{k,l+1})$. In either case we can find a finite generating set for $\mathrm{Mod}^{\pm}(\Sigma, R)$ consisting of Dehn twists about non-separating curves and half twists about $(0,2)$-

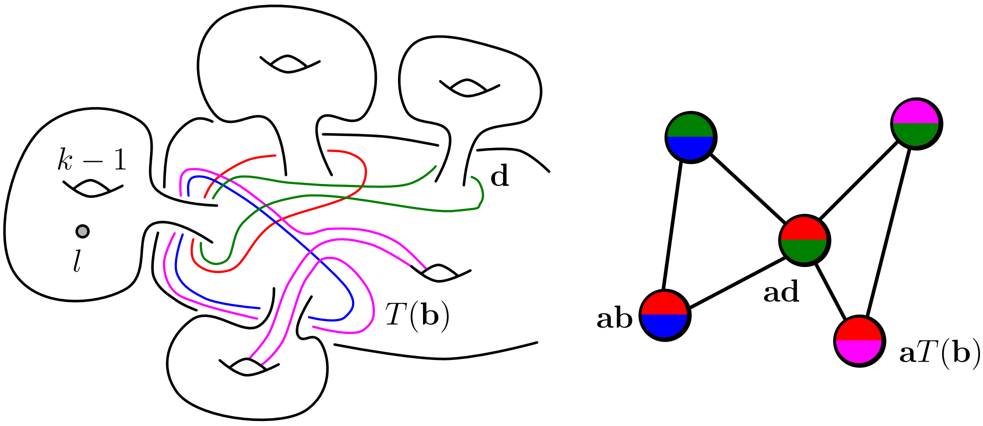

Figure 2.13: The vertex of $\mathcal{SP}(\mathbf{c})$ corresponding to the sharing pair $\mathbf{a}, \mathbf{d}$ spans an edge with both $\mathbf{a}, \mathbf{b}$ and $\mathbf{a}, T(\mathbf{b})$.

curves, see Figure 2.14. Again, in either case, we can choose the set so that one non-separating curve and one $(0, 2)$-curve intersect $\mathbf{b}$ and one $(0, 2)$-curve intersects both $\mathbf{a}$ and $\mathbf{b}$, all other curves are disjoint from both $\mathbf{a}$ and $\mathbf{b}$.

As before, if T is a Dehn twist about a non-separating curve intersecting $\mathbf{b}$ and disjoint from $\mathbf{a}$ then it is clear that $T(\mathbf{a}), T(\mathbf{b})$ share $\mathbf{c}$. Given $n \geq l+2$ we can find a curve $\mathbf{d}$ such that the vertex relating to $\mathbf{a}, \mathbf{d}$ is adjacent to the vertices relating to $\mathbf{a}, \mathbf{b}$ and $T(\mathbf{a}), T(\mathbf{b})$ in $\mathcal{SP}$. A similar argument follows for the half twist about the $(0, 2)$-curve intersecting $\mathbf{a}$ and not $\mathbf{b}$.

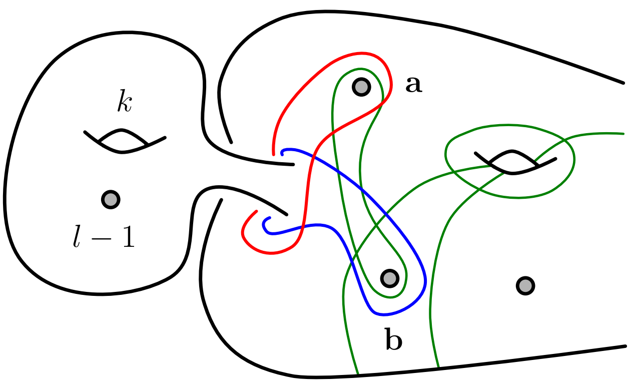

Figure 2.14: Generating twists with respect to a puncture sharing pair.

Finally, for the half twist H about a $(0, 2)$-curve intersecting both $\mathbf{a}$ and $\mathbf{b}$ it is clear that $H(\mathbf{a}), H(\mathbf{b})$ share $\mathbf{c}$. Furthermore, without loss of generality we can assume that $H(\mathbf{a}) = \mathbf{b}$. Given $n \geq l+2$ we can find a curve $\mathbf{d}$ such that the vertex corresponding $\mathbf{d}, \mathbf{b}$ is adjacent to both to $\mathbf{a}, \mathbf{b}$ and $H(\mathbf{b}), \mathbf{b}$, see Figure 2.15.

By Lemma 2.3.5 the result holds for graphs of puncture and reversed puncture sharing pairs. ■

We now prove a key step that we will use repeatedly when showing that the natural homomorphism

$$\eta_{\mathcal{S}(A)} : \mathrm{Mod}^{\pm}(\Sigma) \to \mathrm{Aut}\, \mathcal{C}_{\mathcal{S}(A)}(\Sigma)$$

is an isomorphism.

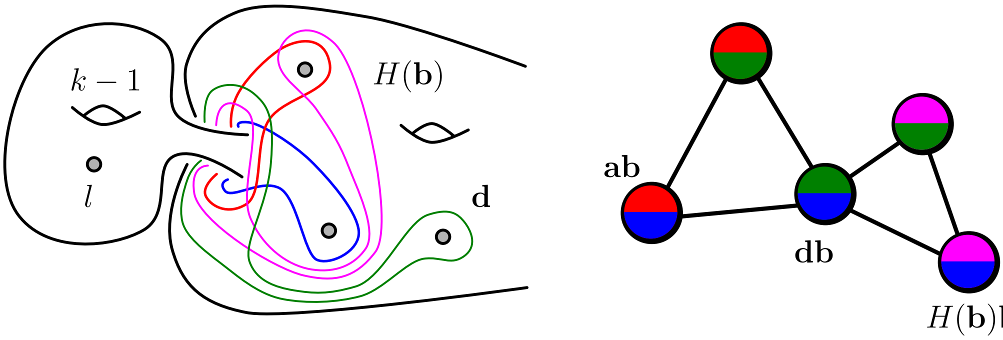

Figure 2.15: The vertex of $\mathcal{SP}(\mathbf{c})$ corresponding to the sharing pair $\mathbf{d}, \mathbf{b}$ spans an edge with both $\mathbf{a}, \mathbf{b}$ and $H(\mathbf{b}), \mathbf{b}$. Note also that $H(\mathbf{a}) = \mathbf{b}$.

Lemma 2.3.7. *Let $A \subset \mathcal{R}(\Sigma)$ and let X and Y be extended sets of $\mathcal{S}(A)$ such that $X \subset Y$ and $Y \setminus X = Z_{k,l}$. Suppose the natural homomorphism*

$$\eta_Y : \mathrm{Mod}^{\pm}(\Sigma) \to \mathrm{Aut}\,\mathcal{C}_Y(\Sigma)$$

is an isomorphism. If $(k+1, l)$-genus sharing pairs, $(k, l+1)$-puncture sharing pairs, or $(k, l-1)$-reversed puncture sharing pairs form characteristic subsets of $\mathcal{C}_X(\Sigma)$ then the natural homomorphism

$$\eta_X : \mathrm{Mod}^{\pm}(\Sigma) \to \mathrm{Aut}\,\mathcal{C}_X(\Sigma)$$

is an isomorphism.

Proof. By Lemma 2.1.1 the map η_X is injective. It remains to show that it is surjective. Let ϕ be an automorphism of $\mathrm{Aut}\,\mathcal{C}_X(\Sigma)$. By assumption $(k+1, l)$ genus sharing pairs, $(k, l+1)$ puncture sharing pairs, or $(k, l-1)$ reversed puncture sharing pairs are characteristic in $\mathcal{C}_X(\Sigma)$, therefore by Lemma 2.3.6 there exists a well defined automorphism $\widehat{\phi}$ of the vertices of $\mathcal{C}_Y(\Sigma)$ such that $\widehat{\phi}$ restricts to ϕ on the vertices of $\mathcal{C}_X(\Sigma)$. We will show that $\widehat{\phi}$ in fact extends to an automorphism of $\mathcal{C}_Y(\Sigma)$.

Suppose vertices u, v of $\mathcal{C}_Y(\Sigma)$ correspond to the curves $\mathbf{u}$ and $\mathbf{v}$. We need to show that the adjacency of u and v in the complex $\mathcal{C}_Y(\Sigma)$ is characteristic in its subcomplex $\mathcal{C}_X(\Sigma)$. If both u and v are vertices of $\mathcal{C}_X(\Sigma)$ then this is clear. Suppose neither u nor v are vertices of $\mathcal{C}_X(\Sigma)$, that is, they are both (k, l)-vertices. They are adjacent if and only if there are adjacent vertices w_1 and w_2 in $\mathcal{C}_X(\Sigma)$ such that w_1 corresponds to a curve that shares $\mathbf{u}$ and w_2 corresponds to a curve that shares $\mathbf{v}$.

Finally, suppose v is a vertex of $\mathcal{C}_X(\Sigma)$ and u is not, that is, u is a (k, l)-vertex. The vertices span an edge in $\mathcal{C}_Y(\Sigma)$ if there exists some vertex w spanning an edge with, or equal to, v in $\mathcal{C}_X(\Sigma)$ that corresponds to a curve that shares $\mathbf{u}$. As Y is an extended set of $\mathcal{S}(A)$ all such edges take this form. We have therefore shown that $\widehat{\phi} \in \mathrm{Aut}\,\mathcal{C}_Y(\Sigma)$.

By assumption there exists some $[f] \in \mathrm{Mod}^{\pm}(\Sigma)$ whose image in $\mathrm{Aut}\,\mathcal{C}_Y(\Sigma)$ is precisely $\widehat{\phi}$. Since the restriction of $\widehat{\phi}$ to $\mathcal{C}_X(\Sigma)$ is ϕ it follows that the image of $[f]$ in $\mathrm{Aut}\,\mathcal{C}_Y(\Sigma)$ is indeed ϕ. ∎

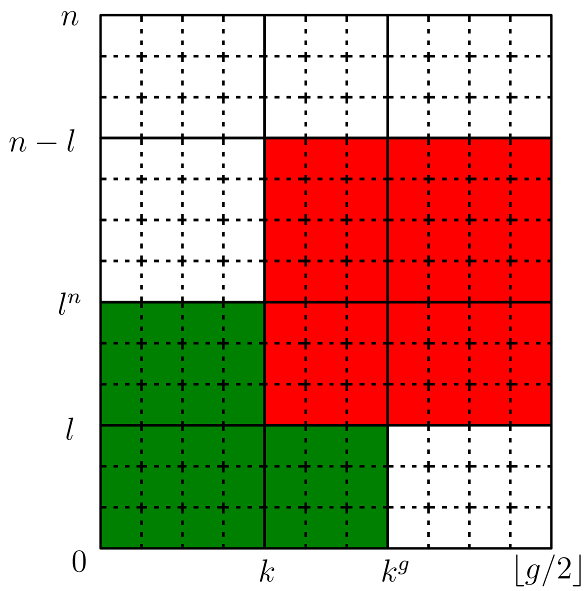

Figure 2.16: The lattice $\mathrm{Lat}(\mathcal{S}(A;k,l))$ corresponding to the (k,l)-extended set anchored by A.

2.4 Proof of Theorem 2.0.1

In this section we will prove Theorem 2.0.1 which says that the natural homomorphism

$$\eta_{\mathcal{S}(A)} : \mathrm{Mod}^{\pm}(\Sigma) \to \mathrm{Aut}\, \mathcal{C}_{\mathcal{S}(A)}(\Sigma)$$

is an isomorphism for $A \subset \mathcal{R}(\Sigma)$. This result plays in important role in the proof of Theorem 1.4.2, that

$$\eta_A : \mathrm{Mod}^{\pm}(\Sigma) \to \mathrm{Aut}\, \mathcal{C}_A(\Sigma)$$

is an isomorphism. We will first define a type of extended set $\mathcal{S}(A;k,l)$ of $\mathcal{S}(A)$. This allows us to use a double induction argument.

Recall that the minimum curves of a subset of $X \subset \mathcal{S}$ are the (k,l)-curves in the surface Σ such that

$$\big([0,k] \times [0,l]\big) \cap \mathrm{Lat}(X) = \{(k,l)\},$$

that is, the lower lattice of the point (k,l) in $\mathrm{Lat}(X)$ is the points (k,l) itself. The values k^g and l^n are the lowest genus and fewest number of punctures of the associated regions of the minimum curves of $\mathcal{S}(A)$.

The (k,l)-extended set anchored by A

Let $\mathcal{S}(A;k,l) \subset \mathcal{S}$ be an extended set of $\mathcal{S}(A)$ where $(\tilde{k},\tilde{l})$-curves are represented in $\mathcal{S}(A;k,l)$ if either $k \le \tilde{k} \le g/2$ and $l \le \tilde{l} \le n-l$ **or** $\tilde{k} \le k^g$ and $\tilde{l} \le l^n$, see Figure 2.16.

We will first prove two inductive steps in Lemmas 2.4.1 and 2.4.2. These results use Lemma 2.3.7 to relate the automorphism groups of complexes of the form $\mathcal{C}_{\mathcal{S}(A,k,l)}(\Sigma)$. Lemma 2.4.1 is an inductive step indexed by the value l and Lemma 2.4.2 is is an inductive step indexed by the value k. It has been shown by Kida that the natural homomorphism

$$\eta_{\mathcal{S}} : \mathrm{Mod}^{\pm}(\Sigma) \to \mathrm{Aut}\, \mathcal{C}_{\mathcal{S}(A;0,0)}(\Sigma)$$

is an isomorphism [43]. Here we are using the fact that $\mathcal{S}(A;0,0) = \mathcal{S}$ for any $A \subset \mathcal{R}(\Sigma)$. We then use Lemmas 2.4.1 and 2.4.2 to show that the natural homomorphism

$$\mathrm{Mod}^{\pm}(\Sigma) \to \mathrm{Aut}\, \mathcal{C}_{\mathcal{S}(A;k^g,l^n)}(\Sigma)$$

is an isomorphism. Finally we prove Theorem 2.0.1 using the isomorphism given above. Completing the proof is similar to that of the two inductive steps in spirit, although this part requires some more careful book-keeping. We begin by proving the two lemmas.

Lemma 2.4.1. *Let $A \subset \mathcal{R}(\Sigma)$ such that a g-small region and an n-small region are represented in A. For $1 \leq l \leq l^n$ if the natural homomorphism*

$$\eta_{\mathcal{S}(A;0,l-1)} \;:\; \mathrm{Mod}^{\pm}(\Sigma) \to \mathrm{Aut}\, \mathcal{C}_{\mathcal{S}(A;0,l-1)}(\Sigma)$$

is an isomorphism then the natural homomorphism

$$\eta_{\mathcal{S}(A;0,l)} \;:\; \mathrm{Mod}^{\pm}(\Sigma) \to \mathrm{Aut}\, \mathcal{C}_{\mathcal{S}(A;0,l)}(\Sigma)$$

is an isomorphism.

Proof. Let $X_0 = \mathcal{S}(A;0,l-1)$ and define X_i such that X_{i-1} is the reversed puncture extension of X_i, where

$$X_i = X_{i-1} \setminus Z_{i,n-(l-1)},$$

for all $i = 1, 2, \ldots, \lfloor g/2 \rfloor$. Now let $X_{\lfloor g/2 \rfloor} = Y_0$. Similarly we define Y_i such that Y_{i-1} is the puncture extension of Y_i where

$$Y_i = Y_{i-1} \setminus Z_{\lfloor g/2 \rfloor - i, l-1},$$

for all $i = 1, 2, \ldots, \lfloor g/2 \rfloor - k^g$, that is, $Y_{\lfloor g/2 \rfloor - k^g} = \mathcal{S}(A;0,l)$. By Lemma 2.2.7 the vertex types of each complex $\mathcal{C}_{X_i}(\Sigma)$ and $\mathcal{C}_{Y_i}(\Sigma)$ are preserved by automorphisms as each X_i and Y_i is an extended set of $\mathcal{S}(A)$. We will show that each natural homomorphism η_{X_i} and η_{Y_i} is an isomorphism. This is true for X_0 by assumption. Note that from the definition of n-small we have that $l^n \neq 1$, so $l^n \geq 2$.

We want to show that the $(\tilde{k}, n-l)$-reversed puncture sharing pairs form characteristic subsets in the complexes $\mathcal{C}_{X_i}(\Sigma)$ for all values $0 \leq \tilde{k} \leq \lfloor g/2 \rfloor$. Note that

$$\tilde{k} \geq 0 \qquad \text{and} \qquad n - l \geq 3l^n - 1 - l \geq 3.$$

Furthermore, since each complex contains $(0,2)$-vertices, we have that $\mathcal{C}_{X_i}(\Sigma)$ satisfies the conditions from Lemma 2.3.4. It then follows from induction and Lemma 2.3.7 that each natural homomorphism η_{X_i} is an isomorphism.

By construction, we now have that the homomorphism η_{Y_0} is an isomorphism. We now want to show that the $(\tilde{k}, l)$-puncture sharing pairs form characteristic subsets in the complexes $\mathcal{C}_{Y_i}(\Sigma)$ for all values $k^g \leq \tilde{k} \leq \lfloor g/2 \rfloor$. We have that

$$g \geq \tilde{k} \qquad \text{and} \qquad n \geq 3l^n - 1 \geq l + 3.$$

Once again, the complex $\mathcal{C}_{Y_i}(\Sigma)$ contains $(0,2)$-vertices. Lemma 2.3.3 implies that $(\tilde{k}, l)$-puncture sharing pairs form characteristic subsets. It now follows from induction and Lemma 2.3.7 that each natural homomorphism is an isomorphism. By definition of the sets Y_i the result is proved. ■

Lemma 2.4.2. *Let $A \subset \mathcal{R}(\Sigma)$ such that a g-small region and an n-small region are represented in A. For $1 \leq k \leq k^g$ if the natural homomorphism*

$$\eta_{\mathcal{S}(A;k-1,l)} \;:\; \mathrm{Mod}^{\pm}(\Sigma) \to \mathrm{Aut}\, \mathcal{C}_{\mathcal{S}(A;k-1,l)}(\Sigma)$$

is an isomorphism then the natural homomorphism

$$\eta_{\mathcal{S}(A;k,l)} \;:\; \mathrm{Mod}^{\pm}(\Sigma) \to \mathrm{Aut}\, \mathcal{C}_{\mathcal{S}(A;k,l)}(\Sigma)$$

is an isomorphism.

Proof. Let $X_0 = \mathcal{S}(A; k-1, l)$ and define X_i such that X_{i-1} is the genus extension of X_i, where

$$X_i = X_{i-1} \setminus Z_{k-1,n-l-i}$$

for $i = 1, \ldots, n-l-l^n$. That is, $X_{n-l-l^n} = \mathcal{S}(A; k, l)$. By Lemma 2.2.7 the vertex types of each complex $\mathcal{C}_{X_i}(\Sigma)$ are preserved by automorphisms as each X_i is an extended set of $\mathcal{S}(A)$. As with Lemma 2.4.1, we will show that each natural homomorphism η_{X_i} is an isomorphism. By assumption, this is true for X_0.

We want to show that the $(k, \tilde{l})$-genus sharing pairs form characteristic subsets in the complexes $\mathcal{C}_{X_i}(\Sigma)$ for all values $l^n \leq \tilde{l} \leq n - l$. We have that

$$g \geq 3k^g + 1 \geq k + 3 \qquad \text{and} \qquad n \geq \tilde{l}.$$

Since each complex contains $(1,0)$-vertices we have that $\mathcal{C}_{X_i}(\Sigma)$ satisfies the conditions in Lemma 2.3.2. It follows from induction and Lemma 2.3.7 that each homomorphism is an isomorphism. By definition of the X_i, we have proved the result. ■

Using the two inductive steps above we can now show that the natural homomorphsim

$$\eta_{\mathcal{S}(A;k^g,l^n)} : \mathrm{Mod}^{\pm}(\Sigma) \to \mathrm{Aut}\, \mathcal{C}_{\mathcal{S}(A;k^g,l^n)}(\Sigma)$$

is an isomorphism. It is useful at this stage to compare the lattices $\mathrm{Lat}(\mathcal{S}(A; k^g, l^n))$ and $\mathrm{Lat}(\mathcal{S}(A))$. This comparison is shown in Figure 2.17.

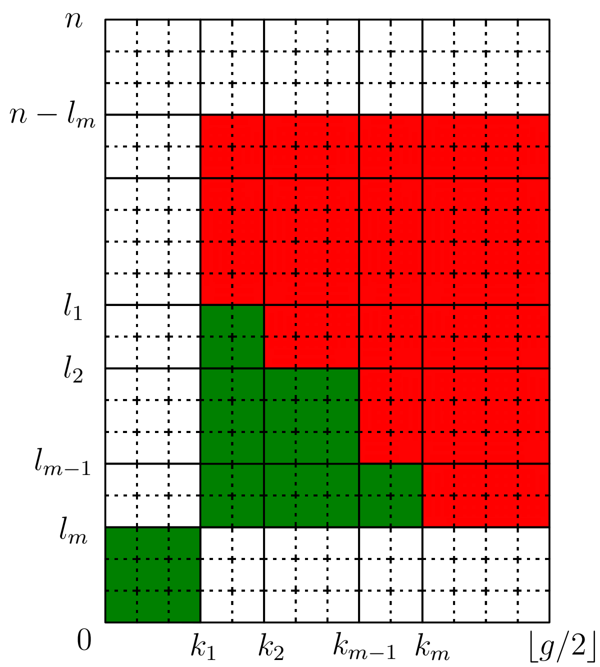

Figure 2.17: The integer lattice $\mathrm{Lat}(\mathcal{S}(A))$ is represented by the red region and the integer lattice $\mathrm{Lat}(\mathcal{S}(A; k^g, l^n))$ is represented by both shaded regions. Recall that $k_1 = k^g$ and $l_m = l^n$

We can now prove Theorem 2.0.1. This proof uses the same techniques as Lemmas 2.4.1 and 2.4.2.

Proof of Theorem 2.0.1. As discussed above, from [43, Theorem 1.1], Lemmas 2.4.1, 2.4.2 and a double induction argument we have that the natural homomorphism

$$\eta_{\mathcal{S}(A;k^g,l^n)} : \mathrm{Mod}^{\pm}(\Sigma) \to \mathrm{Aut}\, \mathcal{C}_{\mathcal{S}(A;k^g,l^n)}(\Sigma)$$

is an isomorphism. Similar to the previous two lemmas, we can find a finite sequence of extended sets of $\mathcal{S}(A)$ which we label $X_0, \ldots, X_N$ where

$$X_0 = \mathcal{S}(A; k^g, l^n) \qquad \text{and} \qquad X_N = \mathcal{S}(A).$$

Furthermore, as indicated in Figure 2.17 we can choose each X_{i+1} such that it is a genus extension or puncture extension of X_i. In each case, the related (k, l)-sharing pairs will be such that $k \leq k^n$ and $l \leq l^g$. It follows that

$$g \geq 3k^n + 1 \geq k + 2k^n + 1 \qquad \text{and} \qquad n \geq 3l^g \geq l + 2l^g.$$

Therefore, from Lemmas 2.3.2 and 2.3.3, all (k, l)-sharing pairs form characteristic subsets of any complex $\mathcal{C}_{X_i}(\Sigma)$ containing (k, l)-vertices. By induction and Lemma 2.3.7 we have that the natural homomorphism

$$\eta_{X_N} : \mathrm{Mod}^{\pm}(\Sigma) \to \mathrm{Aut}\, \mathcal{C}_{X_N}(\Sigma)$$

is an isomorphism. By definition of X_N we have the required result. ∎

As mentioned throughout this section, the special cases of closed surfaces [13] and spheres with punctures [55] are much simpler. Indeed, the process of extended the isomorphism

$$\eta_{\mathcal{S}} : \mathrm{Mod}^{\pm}(\Sigma) \to \mathrm{Aut}\, \mathcal{C}_{\mathcal{S}}(\Sigma)$$

to an isomorphism

$$\eta_{\mathcal{S}(A)} : \mathrm{Mod}^{\pm}(\Sigma) \to \mathcal{C}_{\mathcal{S}(A)}(\Sigma)$$

involves defining a solitary type of sharing pair and then a single variable induction argument.

2.5 Complexes of dividing sets

The goal of this section is to connect Theorem 2.0.1 with complexes of regions. We do this by using a generalisation of separating curves for a surface Σ of strictly positive genus introduced by Brendle-Margalit [13, Section 4].

Dividing sets

A *dividing set* in Σ is a multicurve that divides the surface into exactly two regions. We allow for one of the regions to be an annulus, that is, the multicurve may consist of two isotopic non-separating curves. As with separating curves, the two regions obtained by cutting Σ along each curve in a dividing set $\mathbf{d}$ are the *associated regions* of $\mathbf{d}$. We say that two dividing sets are *nested* if one is contained entirely in one of the associated regions of the other, otherwise we say that they *intersect*. If two dividing sets intersect then their respective multicurves may intersect or they may not.

Let $\mathcal{DS}$ denote the set of all $\mathrm{Mod}^{\pm}(\Sigma)$-orbits of dividing sets in Σ. For a subset $D \subseteq \mathcal{DS}$ we define the simplicial flag complex $\mathcal{C}_D(\Sigma)$ analogously to complexes of regions. The vertices of $\mathcal{C}_D(\Sigma)$ correspond to all homotopy classes of dividing sets that represent elements of D. We say that a vertex corresponds to a dividing set if it corresponds to the equivalence class of that dividing set. Two vertices span an edge in $\mathcal{C}_D(\Sigma)$ if they correspond to nested dividing sets. As with complexes of regions there is a natural homomorphism

$$\eta_D : \mathrm{Mod}^{\pm}(\Sigma) \to \mathrm{Aut}\, \mathcal{C}_D(\Sigma)$$

for every subset $D \subseteq \mathcal{DS}$.

For $A \subset \mathcal{R}(\Sigma)$ we define $\partial A \subseteq \mathcal{DS}$ to be the subset consisting of dividing sets where each of the associated regions contain a region represented in A. We can also define $\partial D \subseteq \mathcal{DS}$ for any subset $D \subseteq \mathcal{DS}$ in the same way, that is, the subset ∂D consists of dividing sets with associated regions containing dividing sets represented in D. Notice that in the special case $\Sigma_{0,n}$ we have $\mathcal{S} = \mathcal{DS}$ and $\mathcal{S}(A) = \partial A$ for $A \subset \mathcal{R}(\Sigma)$. In general $\mathcal{S}(A) = \partial A \cap \mathcal{S}$.

Lemma 2.5.1. *Given $A \subset \mathcal{R}(\Sigma)$ the subset $\partial A \subseteq \mathcal{DS}$ satisfies the relation $\partial A = \partial(\partial A)$.*

Proof. Let $\mathbf{d}$ be represented in ∂A. There exist dividing sets homotopic to $\mathbf{d}$ that lie in each of its associated regions. It follows that $\mathbf{d}$ is represented in $\partial(\partial A)$. Now suppose $\mathbf{d}$ is represented in $\partial(\partial A)$ and suppose $\mathbf{d}$ separates dividing sets $\mathbf{d}_1, \mathbf{d}_2 \in \partial A$. There exists a region R_i represented in A that is contained in the associated region of $\mathbf{d}_i$ not containing $\mathbf{d}$ for $i = 1, 2$. It follows that $\mathbf{d}$ separates R_1 and R_2 and so is represented in ∂A. ■

2.5.1 The case with annular dividing sets

We call a dividing set $\mathbf{d}$ *annular* when $\mathbf{d}$ has an annular associated region. Clearly, there is a bijection between the isotopy classes of annular dividing sets and isotopy classes of non-separating curves. Suppose then that annular dividing sets are represented in $D \subseteq \mathcal{DS}$. It follows from [13, Lemma 4.1] that the vertices of $\mathcal{C}_D(\Sigma)$ that correspond to annular dividing sets form a characteristic subset. We thus obtain an injective homomorphism

$$\operatorname{Aut} \mathcal{C}_D(\Sigma) \hookrightarrow \operatorname{Aut} \mathcal{N}(\Sigma),$$

where $\mathcal{N}(\Sigma)$ is the complex of non-separating curves. From Lemma 2.1.1 and [35, Theorem 1.4] we have that the composition

$$\operatorname{Mod}^{\pm}(\Sigma) \hookrightarrow \operatorname{Aut} \mathcal{C}_D(\Sigma) \hookrightarrow \operatorname{Aut} \mathcal{N}(\Sigma) \xrightarrow{\cong} \operatorname{Mod}^{\pm}(\Sigma)$$

is injective and equal to the identity map, therefore $\operatorname{Mod}^{\pm}(\Sigma) \to \operatorname{Aut} \mathcal{C}_D(\Sigma)$ is an isomorphism. In the remainder of this section we will assume that annular dividing sets are not represented in $D \subseteq \mathcal{DS}$ and prove that the homomorphism is an isomorphism in this case as well.

2.5.2 Vertex types are characteristic

Assume throughout this section that no annular dividing sets are represented in $D \subset \mathcal{DS}$. We say that a vertex v of $\mathcal{C}_D(\Sigma)$ is *k-sided* if v corresponds to a dividing set $\mathbf{d}$ such that k of the associated regions contain a non-homotopic dividing set represented in $D \subset \mathcal{DS}$. Note that $k \in \{0, 1, 2\}$. We now generalise the notion of the enveloping regions given in Section 1.4.4.

Enveloping regions

The *enveloping region* of a 1-sided dividing set $\mathbf{d}$ is defined to be the enveloping region $\widehat{R}$ of a region R which is the associated region of $\mathbf{d}$ of lowest genus that has fewest punctures. An *enveloping region* of $D \subset \mathcal{DS}$, written as $\widehat{D}$, is a subregion of Σ with a single boundary component, lowest genus and fewest punctures such that $\widehat{D}$ contains an element of the $\mathrm{Mod}^{\pm}(\Sigma)$-orbit of $\mathbf{d}$ for every 1-sided element $\mathbf{d}$. Note that $\widehat{D}$ also contains the enveloping regions for all such 1-sided elements. Furthermore, if R is a core region represented in some $A \subset \mathcal{R}(\Sigma)$ then its enveloping region $\widehat{R}$ is also an enveloping region for a 1-sided dividing set.

As dividing sets are a generalisation of separating curves, we may employ similar techniques when studying complexes of dividing sets. In particular, just like vertices corresponding to separating curves, we can define *sides* of vertices corresponding to dividing sets by analysing their links as in Section 2.2.2. Explicitly, two vertices $u, w \in \mathrm{Lk}(v)$ lie on the same side of the vertex v if there exists another vertex in $\mathrm{Lk}(v)$ that does not span an edge with either u or w.

Now, for all $v \in \mathcal{C}_D(\Sigma)$ corresponding to a dividing set $\mathbf{v}$, we define $\delta(v)$ to be the number of components of $\mathbf{v}$. We say that a vertex $v \in \mathcal{C}_D(\Sigma)$ satisfying $\delta(v) = 1$ is of;

- type S_1 if v is 1-sided,
- type S_2 if every vertex on one side of v is of type S_1, and
- type S_3 otherwise.

Here, the letter 'S' indicates that v corresponds to a separating curve. If v is a vertex of $\mathcal{C}_D(\Sigma)$ such that $\delta(v) \geq 2$ then we say that v is of;

- type M_1 if v is 1-sided,
- type M_2 if every vertex on one side of v is of type S_1, and
- type M_3 otherwise.

The letter 'M' indicates that the vertex corresponds to a multicurve.

Our goal now is to show that vertices of type S_1, S_2 and S_3 form characteristic subsets of $\mathcal{C}_D(\Sigma)$ when $D = \partial D$. That is, separating curves determine a characteristic subset of vertices in $\mathcal{C}_D(\Sigma)$. We use the assumption that there are no annular dividing sets to define the *normal form* of a simplex [13, Section 4.2].

Normal form

Let σ be any simplex in the complex $\mathcal{C}_D(\Sigma)$ consisting of vertices $v_1, \ldots, v_m$. We call a collection of pairwise nested multicurves $\mathbf{v}_1, \ldots, \mathbf{v}_m$ a *normal form* representative for σ if each v_i corresponds to $\mathbf{v}_i$. We state the following result of Brendle-Margalit [13, Lemma 4.3].

Lemma 2.5.2 (Brendle-Margalit)**.** *Suppose $D \subset \mathcal{DS}$ and let σ be a simplex of $\mathcal{C}_D(\Sigma)$. There exists a normal form representative of σ, unique up to isotopy.*

Recall from Section 2.2 that a linear simplex is one with an ordering of the vertices determined by the sides of the corresponding curves. We use the same terminology in the case of dividing sets.

Linear simplices of $\mathcal{C}_D(\Sigma)$

A simplex σ of $\mathcal{C}_D(\Sigma)$ is *linear* if there is a labeling of its vertices $v_1, \ldots, v_m$ such that v_{i-1} and v_{i+1} do not lie on the same side of v_i for all $i = 1, \ldots, m-1$. We call the vertices v_0 and v_m the extreme vertices of the linear simplex σ.

As discussed in Section 2.2.2 and in [13, Lemma 4.5] we have the following result.

Lemma 2.5.3. *Let $\mathcal{C}_D(\Sigma)$ be a complex of dividing sets and let ϕ be an automorphism. If $\sigma = \{v_1, \ldots, v_m\}$ is a maximal linear simplex then $\phi(\sigma) = \{\phi(v_1), \ldots, \phi(v_m)\}$ is a maximal linear simplex.*

We now move on to showing that the various vertex types form characteristic subsets, beginning with vertices of type S_1.

Lemma 2.5.4. *Let Σ be a surface and let the subset $D \subset \mathcal{DS}$ satisfy $\partial D = D$. If $g \geq 3g(\widehat{D}) + 1$ and $n \geq 3n(\widehat{D})$ then the vertices of type S_1 form a characteristic subset of $\mathcal{C}_D(\Sigma)$.*

Proof. It follows from the definition of a maximal linear simplex that a vertex v is 1-sided if and only if it is an extreme vertex of some maximal linear simplex. We will show then that a vertex is of type M_1 if and only if it is 1-sided and there exist vertices u, w such that;

1. u and w span a triangle with v, and

2. any other 1-sided vertex spanning a triangle with u and w spans an edge with v.

To prove one direction suppose v is of type M_1 and corresponds to the multicurve $\mathbf{v}$. Let u be a vertex in the $\mathrm{Mod}^{\pm}(\Sigma)$-orbit of v corresponding to the multicurve $\mathbf{u}$ disjoint from $\mathbf{v}$ such that exactly one of the curves in $\mathbf{v}$ is isotopic to a curve in $\mathbf{u}$. Let R be the unique region defined by cutting along $\mathbf{u}$ and $\mathbf{v}$ that contains more than one dividing set represented in D. We now define w to be the vertex of $\mathcal{C}_D(\Sigma)$ corresponding to ∂R. Clearly the vertices u, v and w span a triangle. Now, any choice of 1-sided vertex, other than v, that spans a triangle with u and w must correspond to a dividing set contained in R. It follows then that any such vertex spans an edge with v.

Now assume that v is a vertex of type S_1 corresponding to $\mathbf{v}$ and let u and w be vertices corresponding to dividing sets $\mathbf{u}, \mathbf{w}$ satisfying the conditions above. Let R be the region of Σ with boundary defined by $\mathbf{v}$ and containing $\mathbf{u}, \mathbf{w}$. Since v does not correspond to $\mathbf{u}$ or $\mathbf{w}$ there exists an element in the $\mathrm{Mod}^{\pm}(\Sigma)$-orbit of $\mathbf{v}$ that is disjoint from $\mathbf{u}$ and $\mathbf{w}$ and intersects $\mathbf{v}$. This completes the proof. ∎

We treat the remaining cases seemingly out of order by first showing that vertices of S_3 form a characteristic subsets before dealing with vertices of type S_2.

Lemma 2.5.5. *Let Σ be a surface with punctures and let the subset $D \subset \mathcal{DS}$ satisfy $\partial D = D$. If $g \geq 3g(\widehat{D}) + 1$ and $n \geq 3n(\widehat{D})$ then the vertices of type S_3 form a characteristic subset of $\mathcal{C}_D(\Sigma)$.*

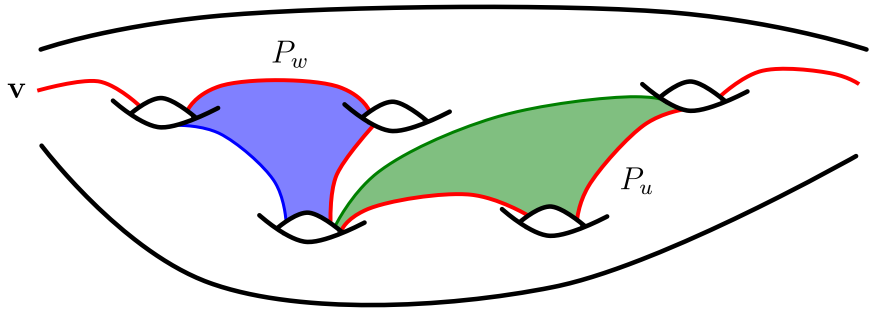

Figure 2.18: The red multicurve is $\mathbf{v}$. We see that we can construct the desired pairs of pants P_u and P_w.

Proof. It follows from Lemmas 2.5.3 and 2.5.4 that the sets S_1, M_1 and $S_2 \cup M_2$ form characteristic subsets of $\mathcal{C}_D(\Sigma)$. It remains only to show that we can distinguish between vertices of type M_3 and vertices of type S_3. We claim that a vertex v is of type M_3 if and only if ;

1. there exist two vertices u and w that span a triangle with v, and
2. there exists exactly one vertex not adjacent to v that spans an triangle with both u and w.

To prove the forward direction of the claim we assume v is of type M_3 and consider three cases separately; $\delta(v) \geq 4$, $\delta(v) = 3$, and $\delta(v) = 2$. In each case we will construct dividing sets $\mathbf{u}$ and $\mathbf{w}$ that are not on the same side of a dividing set $\mathbf{v}$, where v corresponds to $\mathbf{v}$. In order to define the unique dividing set implicit in the claim we require that the (possibly connected) subsurface bounded by $\mathbf{u}$ and $\mathbf{v}$ is a collection of annuli and a single pair of pants P_u. We define the pair of pants P_w related to the dividing set $\mathbf{w}$ in the same way. Here, we go against convention slightly by defining a pair pants to be homeomorphic to either $\Sigma^3_{0,0}$ or $\Sigma^2_{0,1}$. Furthermore, we require that if a component curve of $\mathbf{v}$ bounds P_u (or P_w) it must bound an annulus with $\mathbf{w}$ (or $\mathbf{u}$). Given such curves we then let the vertices u and w correspond to the dividing sets $\mathbf{u}$ and $\mathbf{w}$. The unique vertex spanning edges with u and w but not v must correspond to the dividing set

$$\{\mathbf{u} \cap \mathbf{w}\} \cup \{\partial P_u \setminus \mathbf{v}\} \cup \{\partial P_w \setminus \mathbf{v}\}.$$

An example is shown in Figure 2.18.

First we consider the case where $\delta(v) \geq 4$. The pair of pants P_u will consist either of three boundary components, or two boundary components and a single puncture. Suppose that such a P_u does not exist, then every dividing set $\mathbf{d}$ nested with $\mathbf{v}$ will be isotopic to $\mathbf{v}$. This contradicts our assumption that v is 2-sided. Similarly, we can find a pair of pants P_w satsifying the conditions above, see Figure 2.18.

Now let $\delta(v) = 3$ and let R_u and R_w be the two associated regions of $\mathbf{v}$ such that $g(R_w) \geq g(R_u)$. Suppose we can choose a dividing set $\mathbf{u}$ in R_u with four components, two of which are isotopic to distinct components of $\mathbf{v}$. Since v is 2-sided we can find an appropriate choice of $\mathbf{w}$ contained in R_w where either $\mathbf{w}$ has two components and

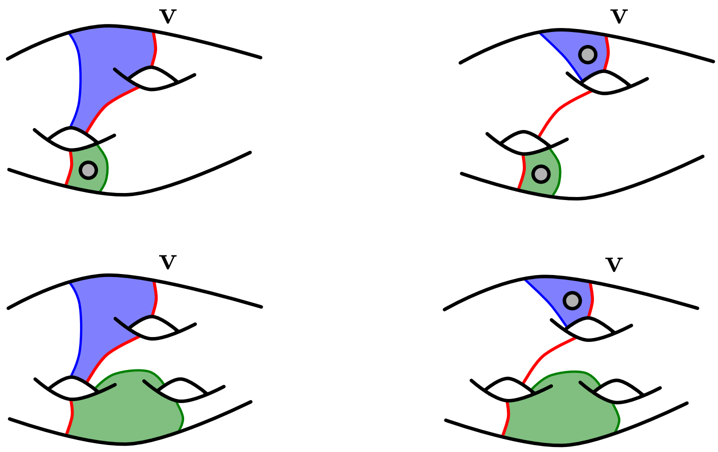

Figure 2.19: The red multicurve with three components is $\mathbf{v}$. We see that we can always construct desired pairs of pants P_u and P_w.

P_w is homeomorphic to $\Sigma^3_{0,0}$ or $\mathbf{w}$ has three components and P_w is homeomorphic to $\Sigma^2_{0,1}$. This is shown in the bottom two configurations in Figure 2.19 where $\mathbf{u}$ is the dividing set on the right and $\mathbf{w}$ is on the left. Similarly, suppose we can choose $\mathbf{u}$ with three boundary components, two of which belong to $\mathbf{v}$ and where the region P_u is homeomorphic to $\Sigma^2_{0,1}$. Once again, as v is 2-sided, there is an appropriate choice of $\mathbf{w}$ in R_w. A picture can be seen in the top two configurations in Figure 2.19, again $\mathbf{u}$ is on the right and $\mathbf{w}$ is the dividing set on the on the left.

If neither choice of $\mathbf{u}$ exists it follows that there are no 1-sided vertices of $\mathcal{C}_D(\Sigma)$ corresponding to dividing sets in R_u with associated region of lower genus or fewer punctures than R_u, and so $g(\widehat{D}) \geq g(R_u)$ and $n(\widehat{D}) \geq n(R_u)$. We now have the inequalities;

$$\begin{aligned} n(R_w) = n - n(R_u) &\geq 3n(\widehat{D}) - n(R_u), \\ &\geq 3n(R_u) - n(R_u), \\ &= 2n(R_u), \\ \text{so } n(R_w) &> n(R_u). \end{aligned}$$

We can choose a dividing set with two components $\mathbf{u}$ contained in R_u with associated side $Q_u \subset R_u$. Furthermore, we choose $\mathbf{u}$ such that $g(Q_u) = g(R_u)$ and $n(Q_u) = n(R_u)$. This is always possible as v is 2-sided. It follows that there exists a dividing set $\mathbf{d}$ in the $\mathrm{Mod}^{\pm}(\Sigma)$-orbit of $\mathbf{u}$ represented in R_w. Since $n(R_w) > n(R_u)$ it follows that $\mathbf{v}$ and $\mathbf{d}$ bound a region with at least one puncture. We now set $\mathbf{w}$ to be the dividing set separating $\mathbf{v}$ and $\mathbf{d}$ with side $Q_w \subset R_w$ such that $\mathbf{w}$ has three components, the same genus as R_w, and $n(Q_w) = n(R_w) - 1$. It is clear now that P_w is homeomorphic to $\Sigma^2_{0,1}$. This is shown in the top left picture of Figure 2.19, where $\mathbf{u}$ is the dividing set on the left and $\mathbf{w}$ is on the right.

Now we deal with the case where $\delta(v) = 2$. If both associated regions of $\mathbf{v}$ contain dividing sets $\mathbf{u}$ and $\mathbf{w}$ such that P_u and P_w are homeomorphic to $\Sigma^2_{0,1}$ then we are

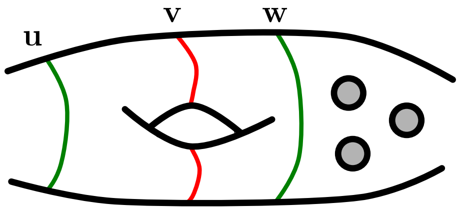

Figure 2.20: When the surface is of genus one there is a vertex of type M_2 adjacent to a vertex of type S_1 and S_3 in a maximal linear simplex. Moreover, there is no other vertex that corresponds to a multi-curve in this simplex.

done. If this is not the case then since v is of type M_3 there exists a vertex in $\mathcal{C}_D(\Sigma)$ spanning an edge with v that is either of type S_2 or S_3. Any such vertex is not 1-sided and so we can find a dividing set $\mathbf{u}$ with three components, two of which are shared by $\mathbf{v}$. As before we can therefore find the desired pairs of pants P_u and P_w.

We now assume that v is a vertex of type S_3. If u and w lie on the same side of v then up to relabeling there are infinitely many vertices in the $\mathrm{Mod}^{\pm}(\Sigma)$-orbit of v spanning edges with u and w but not with v. Suppose then that u and w lie on different sides of v. If the vertices u, v correspond to the dividing sets $\mathbf{u}, \mathbf{v}$ then the subsurface bounded by these curves cannot be an annulus, as $\mathbf{v}$ is a separating curve. It follows that there are infinitely many vertices in the $\mathrm{Mod}^{\pm}(\Sigma)$-orbit of v spanning edges with u and w but not with v. This completes the proof. ■

Finally we complete the proof that the vertices of $\mathcal{C}_D(\Sigma)$ corresponding to separating curves form characteristic subsets by distinguishing vertices of type S_2 and M_2.

Lemma 2.5.6. *Let Σ be a surface that is not a torus with six punctures. Let $D \subset \mathcal{DS}$ with $\partial D = D$. If $g \geq 3g(\widehat{D}) + 1$ and $n \geq 3n(\widehat{D})$ then the vertices of type S_2 form a characteristic subset of $\mathcal{C}_D(\Sigma)$.*

Proof. Following Lemmas 2.5.3 and 2.5.4 we need only distinguish vertices of type S_2 from vertices of type M_2. We deal with the cases $g = 1$ and $g \geq 2$ separately. To that end, let $g = 1$. We claim that a vertex v is of type M_2 if and only if there there exists a vertex u of type S_1 and a vertex w of type S_3 such that;

1. the vertices u, v, w are adjacent in a maximal linear simplex, and
2. there is no vertex x of type M_3 such that u, v, w, x are adjacent in a maximal linear simplex.

To prove one direction of the claim we let v be a vertex of type M_2 corresponding to the dividing set $\mathbf{v}$. By definition we can choose a vertex u of type S_1 as above. Let $\mathbf{w}$ be a separating curve in Σ that bounds a pair of pants with $\mathbf{v}$ such that the vertex w corresponding to $\mathbf{w}$ is distinct from u. Suppose w is of type S_1 or S_2. Since $g = 1$ both u and w must represent curves bounding discs with $n(\widehat{D})$ or $n(\widehat{D}) + 1$ punctures and so $n = 2n(\widehat{D})$ or $2n(\widehat{D}) + 1$. However, $n \geq 3n(\widehat{D})$, a contradiction as $n(\widehat{D}) \geq 2$. So w must be of type S_3 and u, v, w are the first three vertices in a maximal simplex. As $g = 1$ it is clear that any vertex x cannot be of type M_1, M_2 or M_3.

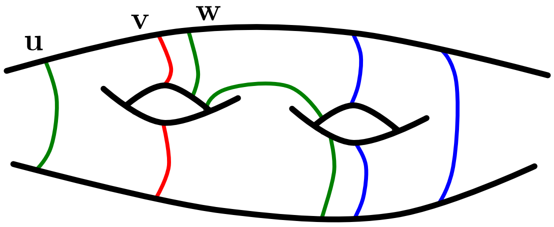

Figure 2.21: When the genus of the surface is at least two there is vertex w of type M_3 such that $\delta(w) = 3$ adjacent to a vertex v of type M_2 in a maximal linear simplex Moreover, any choice of a subsequent vertex in this simplex must correspond to a multi-curve. In particular, it is not of type S_3.

Now assume v is type S_2. Up to the action of $\mathrm{Mod}^{\pm}(\Sigma)$ there is a unique choice of a vertex u of type S_1. Again, up to the action of $\mathrm{Mod}^{\pm}(\Sigma)$ there is a unique choice of w, whereby v and w correspond to curves $\mathbf{v}$ and $\mathbf{w}$ that bound a region homeomorphic to $\Sigma^2_{0,1}$. Since $g = 1$ and $g \geq 3g(\widehat{D}) + 1$, it follows that the associated region R of $\mathbf{w}$ not containing $\mathbf{v}$ is such that $g(R) = 1$. We can therefore define a vertex x of type M_1, M_2, or M_3 such that u, v, w, x are adjacent in a maximal linear simplex. Since $g = 1$, hence $g(\widehat{D}) = 0$, the vertex x cannot be of type M_1. Let x correspond to the dividing set $\mathbf{x}$. If x is of type M_2 then using similar arguments as above we see that the two associated regions of $\mathbf{x}$ are homeomorphic to $\Sigma^2_{0,n(\widehat{D})+2}$ and $\Sigma^2_{0,n(\widehat{D})}$, so $n = 2n(\widehat{D}) + 2$. Since $n \geq 3n(\widehat{D})$, we have that $n(\widehat{D}) = 2$, hence $n = 6$. Since Σ is not a torus with six punctures we have that x is of type M_3, a contradiction.

We now move onto the case when $g \geq 2$. We claim that a vertex v is of type M_2 if and only if there exists a vertex u of type S_1 and a vertex w of type M_3 such that;

1. the vertices u, v, w are adjacent in a maximal linear simplex, and

2. there is no vertex x of type S_3 such that u, v, w, x are adjacent in a maximal linear simplex.

First we let v be a vertex of type M_2. As above we choose u to be one of infinitely many vertices of type S_1 such that u, v form the first two vertices in some maximal linear simplex. Let u and v correspond to the dividing sets $\mathbf{u}$ and $\mathbf{v}$. Since $g \geq 2$ and $g \geq 3g(\widehat{D}) + 1$, the associated region R of $\mathbf{v}$ not containing $\mathbf{u}$ is such that $g(R) \geq 2$. We can therefore find a dividing set $\mathbf{w}$ with three components that bounds an annulus and a pair of pants with $\mathbf{v}$, see Figure 2.21. By the definition of $\widehat{D}$ we have that there is a vertex w of $\mathcal{C}_D(\Sigma)$ corresponding to $\mathbf{w}$. Moreover, w is of type M_3. It is clear that u, v, w form the first three vertices of some maximal linear simplex. Any choice of vertex x such that u, v, w, x are adjacent in a maximal linear complex σ cannot correspond to a separating curve, otherwise σ would not be maximal, again see Figure 2.21.

For the other direction we assume that v is of type S_2. Up to the action of $\mathrm{Mod}^{\pm}(\Sigma)$ there are unique choices of both u and w. Note that $\delta(w) = 2$ and v and w must correspond to dividing sets $\mathbf{v}$ and $\mathbf{w}$ that bound a pair of pants. It easy to see that we can find a separating curve $\mathbf{x}$ that bounds a pair of pants with $\mathbf{w}$. By definition of $\widehat{D}$

there is a vertex x of $\mathcal{C}_D(\Sigma)$ corresponding to $\mathbf{x}$. The associated region of $\mathbf{x}$ containing $\mathbf{v}$ is of genus at most $g(\widehat{D})+1$ and has at most $n(\widehat{D})+1$ punctures. The other associated region is then of genus at least $2g(\widehat{D})$ and has at least $2n(\widehat{D})-1$ punctures. It follows that x must be of type S_3, which is a contradiction. ∎

The condition that Σ is not a torus with six punctures is equivalent to the second requirement in the definition of n-small from Section 1.4.4. In practice this rules out very few cases. An example of such a case is the arc complex on the surface of genus 1 with less than seven punctures.

2.5.3 The case without annular dividing sets

We may now connect the preceding results on complexes of dividing sets with Theorem 2.0.1, this then provides our starting point when discussing complexes of regions in the next section.

Lemma 2.5.7. *Let $A \subset \mathcal{R}(\Sigma)$ such that a g-small region and an n-small region are represented in A. The natural homomorphism*

$$\eta_{\partial A} : \mathrm{Mod}^{\pm}(\Sigma) \to \mathrm{Aut}\, \mathcal{C}_{\partial A}(\Sigma)$$

is an isomorphism.

Proof. By Lemma 2.1.1 we have that $\eta_{\partial A}$ is injective. We want then to show that $\eta_{\partial A}$ is surjective. Let $\phi \in \mathrm{Aut}\, \mathcal{C}_{\partial A}(\Sigma)$. By Lemmas 2.5.4, 2.5.5 and 2.5.6, ϕ restricts to an automorphism $\widehat{\phi}$ of $\mathcal{C}_{\mathcal{S}(A)}(\Sigma)$. Here we think of $\mathcal{C}_{\mathcal{S}(A)}(\Sigma)$ as a full subcomplex of $\mathcal{C}_{\partial A}(\Sigma)$. By Theorem 2.0.1 there exists a mapping class $[f] \in \mathrm{Mod}^{\pm}(\Sigma)$ such that $\eta_{\mathcal{S}(A)}([f]) = \widehat{\phi}$. We need to show that $\eta_{\partial A}([f]) = \phi$.

It suffices to show that an automorphism of $\mathcal{C}_{\partial A}(\Sigma)$ restricting to the identity on $\mathcal{C}_{\mathcal{S}(A)}(\Sigma)$ must be the identity. To do this, we show by induction on the distance from a vertex of $\mathcal{C}_{\mathcal{S}(A)}(\Sigma)$ and then since $\mathcal{C}_{\partial A}(\Sigma)$ is connected the result follows.

By assumption, the automorphism restricts to the identity for all vertices distance zero from $\mathcal{C}_{\mathcal{S}(A)}(\Sigma)$. Assume then that the automorphism restricts to the identity for all vertices of $\mathcal{C}_{\partial A}(\Sigma)$ distance i from $\mathcal{C}_{\mathcal{S}(A)}(\Sigma)$. We deal with the inductive step separately for 1-sided vertices and 2-sided vertices.

Let v be a 1-sided vertex of $\mathcal{C}_{\partial A}(\Sigma)$ that is distance $i+1$ from a vertex of $\mathcal{C}_{\mathcal{S}(A)}(\Sigma)$. Let w be a vertex of $\mathcal{C}_{\partial A}(\Sigma)$ spanning an edge with v and distance i from a vertex of $\mathcal{C}_{\mathcal{S}(A)}(\Sigma)$. Let v, w correspond to $\mathbf{v}, \mathbf{w}$. There exist elements of the $\mathrm{Mod}^{\pm}(\Sigma)$-orbit of $\mathbf{w}$ that fill the associated region of $\mathbf{v}$ containing $\mathbf{w}$. The vertex v is 1-sided, hence is the unique vertex whose link contains vertices corresponding to such dividing sets. It follows that the automorphism must also fix v.

Assume now that v is a 2-sided vertex that is distance $i+1$ from $\mathcal{C}_{\mathcal{S}(A)}(\Sigma)$. Let w be a vertex of $\mathcal{C}_{\partial A}(\Sigma)$ adjacent to v. Let u be any vertex of $\mathcal{C}_{\partial A}(\Sigma)$ that is not on the same side of v as u. We can show using a similar method that v must be fixed by the automorphism, completing the proof. ■

2.6 Complexes of regions

In this section we will complete the resolution of the metaconjecture in the case of surfaces with punctures, that is we prove Theorem 1.4.2. A key step to this result is invoking Lemma 2.5.7 given at the end of the previous section. We relate the sets $\partial A \subseteq \mathcal{DS}$ and $A \subset \mathcal{R}(\Sigma)$ by observing a bijection between sets of joins of vertices in the complex $\mathcal{C}_A(\Sigma)$ and the vertices of the complex $\mathcal{C}_{\partial A}(\Sigma)$. This allows us to construct an injective homomorphism

$$\partial : \operatorname{Aut} \mathcal{C}_A(\Sigma) \to \operatorname{Aut} \mathcal{C}_{\partial A}(\Sigma).$$

We then consider the injective homomorphism $\eta_{\partial A}^{-1} \circ \partial \circ \eta_A$ and show that it is the identity of $\operatorname{Mod}^{\pm}(\Sigma)$. First we define the map

$$\Phi : \Big\{\text{vertices of } \mathcal{C}_{\partial A}(\Sigma)\Big\} \to \Big\{\text{subcomplexes of } \mathcal{C}_A(\Sigma)\Big\}.$$

Given a vertex v of $\mathcal{C}_{\partial A}(\Sigma)$ corresponding to a dividing set $\mathbf{v}$, define $\Phi(v)$ to be the full subcomplex of $\mathcal{C}_A(\Sigma)$ spanned by the vertices that correspond to regions contained in the associated regions of $\mathbf{v}$.

2.6.1 Joins

Recall that a subcomplex $X \subset \mathcal{C}_A(\Sigma)$ is a *join* if X is spanned by disjoint subsets of vertices $V_1, \ldots, V_m$, such that every vertex in V_i spans an edge with every vertex in V_j for all $i \neq j$. Assuming the subcomplex spanned by the vertices in V_i is not itself a join for each i, we say that the X has m *component subcomplexes*. If a component subcomplex consists of a solitary vertex we call it a *singular* component, and a *non-singular* component otherwise. We say that a join X is k*-sided* if X has exactly k non-singular components.

In the following three lemmas we show that the image of each vertex of $\mathcal{C}_{\partial A}(\Sigma)$, with respect to the map Φ, forms a characteristic join of vertices in $\mathcal{C}_A(\Sigma)$. We deal with three cases; *strong* 2-sided vertices, *weak* 2-sided vertices, and finally all 1-sided vertices.

Strong 2-sided vertices

Recall from Section 2.5 that a vertex of $\mathcal{C}_{\partial A}(\Sigma)$ is 2-sided if it corresponds to a dividing set such that each of the associated regions contains a non-homotopic dividing set of ∂A. We call a 2-sided vertex of $\mathcal{C}_{\partial A}(\Sigma)$ *strong* if there are infinitely many vertices on each of its sides, otherwise we call it *weak*. Note that all 2-sided vertices are strong, unless one of the associated regions is homeomorphic to either Σ_0^3 or $\Sigma_{0,1}^2$, and ∂A contains annular dividing sets. Furthermore, ∂A contains annular dividing sets if and only if A contains non-separating annuli.

We begin by characterising the image of all strong 2-sided vertices of $\mathcal{C}_{\partial A}(\Sigma)$ under the map Φ. To that end, we say that a 2-sided join X in $\mathcal{C}_A(\Sigma)$ is *maximal* if there exist no vertices z in $\mathcal{C}_A(\Sigma) \setminus X$ such that $X \cup \{z\}$ spans a 2-sided join.

Lemma 2.6.1. *If the complex $\mathcal{C}_A(\Sigma)$ has no holes and no corks then the restriction of the map*

$$\Phi : \Big\{\textit{strong 2-sided vertices of } \mathcal{C}_{\partial A}(\Sigma)\Big\} \to \Big\{\textit{maximal 2-sided joins of } \mathcal{C}_A(\Sigma)\Big\}$$

is a bijection.

Proof. We must first show that this map makes sense, that is, that for any strong 2-sided vertex $v \in \mathcal{C}_{\partial A}(\Sigma)$ the the subcomplex $\Phi(v)$ is a maximal 2-sided join. Let v correspond to the dividing set $\mathbf{v}$ and suppose L and R are the two associated regions of $\mathbf{v}$. We write V_L for the subcomplex spanned by vertices corresponding to non-peripheral regions of L. We define V_M to be the subcomplex spanned by peripheral regions of L (and R) and define V_R analogously to V_L. Note that V_M may be empty. Now, there are no vertices in $\Phi(v)$ that are not contained in either V_L, V_M, or V_R. Furthermore every vertex of V_L spans an edge with every vertex of V_M and V_R. The same is true for V_M and V_R and so $\Phi(v) = V_L * V_M * V_R$, a join. By definition of a strong 2-sided vertex, L and R are filled by regions represented in A. It follows that V_L and V_R are non-singular components of $\Phi(v)$. Furthermore, it is clear that each vertex of V_M spans an edge with every other vertex of V_M, and so $\Phi(v)$ is a 2-sided join.

Suppose $\Phi(v)$ is not a maximal 2-sided join. Then there exists a vertex z not in $\Phi(v)$ such that $\Phi(v) \cup \{z\}$ spans a 2-sided join. Every vertex that is not in $\Phi(v)$ corresponds to a region that intersects both L and R. It follows that the subcomplex spanned by V_L, V_R, and the vertex z is not a join, which is a contradiction. It follows that $\Phi(v)$ is indeed maximal.

It remains to show that all maximal 2-sided joins of $\mathcal{C}_A(\Sigma)$ are of this form. Let $X = V_1 * V_2 * \cdots * V_m$ be a such a join, where V_1 and V_2 are the two non-singular components. Each V_i corresponds to a subsurface R_i of Σ, that is, the vertices in V_i correspond to regions that fill R_i. Now, both R_1 and R_2 are non-separating and the complement $\{R_i\}_{i=1}^m$ must be a collection of annuli, as otherwise X cannot be maximal. Now, for $i > 2$ each component V_i is a single vertex. If this vertex does not correspond to an annulus then we can find a region represented in A that intersects R_i and either R_1 or R_2. The subcomplex spanned by X and a vertex corresponding to this region is 2-sided join and so X is not maximal, a contradiction. Similarly, it must be that each annulus R_i, for $i > 2$, has boundary components that are isotopic to boundary components of R_1 and R_2. It follows then R_1 has boundary components that are isotopic to a 2-sided dividing set in ∂A. ∎

Weak 2-sided vertices

We now move on to the weak 2-sided vertices of $\mathcal{C}_{\partial A}(\Sigma)$. Recall that these only occur when one of the associated regions is a pair of pants or a punctured annulus, and non-separating annuli are represented in A.

Suppose X is a 1-sided join with more than two components, that is, one non-singular component and at least two singular components. Let $u, w \in X$ be two such singular components. We say that X is a *filling* join if there are no vertices $x, y \in \mathcal{C}_A(\Sigma)$ such that $\{u, w, x, y\}$ spans a square. We call a filling join X *maximal* if there exist no vertices z in $\mathcal{C}_A(\Sigma) \setminus X$ such that $X \cup \{z\}$ spans a filling join or a 2-sided join.

Lemma 2.6.2. *If the complex $\mathcal{C}_A(\Sigma)$ has no holes and no corks then the restriction of the map*

$$\Phi : \Big\{\textit{weak 2-sided vertices of } \mathcal{C}_{\partial A}(\Sigma)\Big\} \to \Big\{\textit{maximal filling joins of } \mathcal{C}_A(\Sigma)\Big\}$$

is a bijection.

Proof. Let v be a weak 2-sided vertex of $\mathcal{C}_{\partial A}$ corresponding to the dividing set $\mathbf{v}$. Let L be the associated region of $\mathbf{v}$ region homeomorphic to either Σ_0^3 or $\Sigma_{0,1}^2$ and let R be the other associated region of $\mathbf{v}$. Let V_R be the subcomplex spanned by vertices corresponding to regions contained in R. Similarly, let V_M be the subcomplex spanned by the vertices corresponding to annuli with boundary components isotopic to boundary components of R. Finally, define V_L to be the possibly empty subcomplex consisting of the single vertex corresponding to L. Note that L contains no other regions represented in A. It follows then that $\Phi(v)$ is equal to the join $V_L * V_M * V_R$ and that each vertex in V_M spans an edge with all other vertices in $\Phi(v)$. Furthermore, V_R contains infinitely many vertices and is not a join, so $\Phi(v)$ is a 1-sided join in $\mathcal{C}_A(\Sigma)$. Now, let u and w be any two distinct vertices of $V_L \cup V_M$ corresponding to regions $Q_u, Q_w \subseteq L$. If $Q_u \simeq L$ then there is no vertex z that spans an edge with both u and w. If Q_u and Q_w are annuli then any region intersecting Q_u and not Q_w must also intersect every region that intersects Q_w. It follows that there are no vertices x, y that span a square with u, w, hence $\Phi(v)$ is a filling join of $\mathcal{C}_A(\Sigma)$.

Suppose now that $\Phi(v)$ is not maximal. Then there exists a vertex z not in $\Phi(v)$ that spans a filling join or a 2-sided join with $\Phi(v)$. However, every vertex that is not in $\Phi(v)$ also fails to span an edge with one of the vertices in V_M. If $L \cong \Sigma_{0,1}^2$ then it follows that if $X := \Phi(v) \cup \{z\}$ spans a join, it must span a 1-sided join with a sole singular component subcomplex. In particular X is not a filling join or a 2-sided, hence $\Phi(v)$ is maximal. Suppose then that $L \cong \Sigma_0^3$ and X spans a join that is 1-sided. The join X must have two singular components u, w corresponding to non-separating annuli. Furthermore, the complement of two such annuli in Σ is a single connected region. We can therefore find vertices $x, y \in \mathcal{C}_A(\Sigma)$ corresponding to annuli that span a square with u and w. It follows that X is not a filling join, hence $\Phi(v)$ is maximal.

It remains to show that all maximal filling joins correspond to weak 2-sided dividing sets. Let $X = V * v_1 * \cdots * v_m$ be such a join where V is the non-singular component and each v_i is a vertex. Suppose the subcomplex V is spanned by region in the subsurface R. Since X is 1-sided, R is connected. Furthermore, since X is a filling join, R is non-separating. Suppose both v_1 and v_2 do not correspond to annuli. There exists a region represented in A that is disjoint from R and intersects both of these non-annular regions. It follows that there exists a vetex $z \in \mathcal{C}_A(\Sigma) \setminus X$ that spans a 2-sided join with X, which is a contradiction. Thus, we have shown that at most one singular component of X corresponds to a region other than an annulus. If v_1 corresponds to annulus that is not peripheral in R then the $\mathrm{Mod}^{\pm}(\Sigma)$-orbit of this annulus fills the complementary region of R. As above, this contradicts the maximality of X.

Suppose now that there are at least four singular components of X that correspond to annuli. Any region with four boundary components contains a non-peripheral annulus. Since R is non-separating, it has a unique complementary region Q with at least four boundary components. Any such region is filled by non-separating annuli hence X is not maximal. We have therefore proven that there are at most three singular vertices of X that correspond to annuli. Similar to the above argument, if Q has three boundary components and contains a puncture then it is filled by non-separating annuli, hence X is not maximal. If Q has two boundary components it must contain a puncture, otherwise X would only have two components. It follows that Q is either a pair of pants or a punctured annulus, completing the proof. ■

All 1-sided vertices

Finally, we deal with the 1-sided vertices of $\mathcal{C}_{\partial A}(\Sigma)$. If a 1-sided join X has two component subcomplexes, that is, one singular component and one non-singular component, we call it *perfect*. A perfect join X is *maximal* if there exist no vertices z in $\mathcal{C}_A(\Sigma) \setminus X$ such that $X \cup \{z\}$ spans a join.

Lemma 2.6.3. *If the complex $\mathcal{C}_A(\Sigma)$ has no holes and no corks then the restriction of the map*

$$\Phi : \Big\{ 1\text{-}sided\ vertices\ of\ \mathcal{C}_{\partial A}(\Sigma) \Big\} \to \Big\{ maximal\ perfect\ joins\ of\ \mathcal{C}_A(\Sigma) \Big\}$$

is a bijection.

Proof. We begin by showing that $\Phi(v)$ is a maximal perfect join if v is a 1-sided vertex corresponding to the dividing set $\mathbf{v}$. Let L and R be the two associated regions of $\mathbf{v}$ such that L does not contain any non-homotopic dividing sets. By the definition of ∂A, this implies that if Q is a region in L represented in A then $\mathbf{v} \subseteq \partial Q$. Since the complex $\mathcal{C}_A(\Sigma)$ does not contain any holes, it must be that $\mathbf{v} = \partial Q$, hence $Q \simeq L$ or Q is an annulus. If Q is an annulus then either $Q \simeq L$ (a non-separating annulus) or L is not represented in A, as $\mathcal{C}_A(\Sigma)$ has no corks. If Q is not an annulus we arrive at the same conclusion and hence $\Phi(v)$ is a perfect join. To see that it is maximal we note that any vertex z not in $\Phi(v)$ cannot span an edge with the singular component of $\Phi(v)$, hence $\Phi(v)$ and z do not span a join.

It remains to show that every maximal perfect join $X = V * u$ is of this form. Let V correspond to the region R. Since X is a maximal perfect join R must be a connected non-separating subsurface. Let L be the complementary region of R. If L contains a dividing set $\mathbf{d} \in \partial A$ that is not homotopic the boundary of L then it must contain more than one region represented in A. It follows that we can find a vertex of $\mathcal{C}_A(\Sigma)$ that is not in X yet spans a join with X. This contradicts the maximality of X, so L only contains dividing sets homotopic to its boundary, that is, X is the image of a 1-sided dividing set. ■

2.6.2 Completing the proof

As a consequence of Lemmas 2.6.1, 2.6.2, and 2.6.3 we have that an automorphism of $\mathcal{C}_A(\Sigma)$ induces an automorphism on the vertices of $\mathcal{C}_{\partial A}(\Sigma)$. We will now show that this automorphism extends to an automorphism of the entire complex. To that end, we say that a subcomplex V of $\mathcal{C}_{\partial A}(\Sigma)$ is *compatible* with a subcomplex W if $V = V_1 * V_2$ where V_1 is not empty and $V_1 \subseteq W$ [13, Section 5]. We can now state the following results of Brendle-Margalit. These facts are vital in proving Theorem 1.4.2.

Lemma 2.6.4 (Brendle-Margalit)**.** *Let u and v be vertices of the connected complex of dividing sets $\mathcal{C}_{\partial A}(\Sigma)$. Then u and v span an edge if and only if $\Phi(u)$ is compatible with $\Phi(v)$.*

In other words, vertices u and v correspond to nested dividing sets if and only if their images in Φ are compatible.

Lemma 2.6.5 (Brendle-Margalit)**.** *Let $A \subset \mathcal{R}(\Sigma)$ so that $\mathcal{C}_A(\Sigma)$ has no holes, no corks, and is connected.*

1. *Let R be represented in A; then there is a simplex in $\mathcal{C}_{\partial A}(\Sigma)$ that corresponds to ∂R.*

2. *The complex $\mathcal{C}_{\partial A}(\Sigma)$ is connected.*

Before completing the proof of the main theorem of this chapter we note that there is a partial order on vertices of $\mathcal{C}_A(\Sigma)$. We say that $u \preceq v$ if the link of v is contained in the link of u. A vertex is *link-minimal* when it is minimal with respect to this ordering. Similar to the definition for dividing sets we say that a vertex of $\mathcal{C}_A(\Sigma)$ is 1-*sided* if it corresponds to a region R such that exactly one of its complementary regions contains a region represented in A. Finally, if a vertex v corresponds to an annulus then we call v an *annular vertex.*

Proof of Theorem 1.4.2. Let $\mathcal{C}_A(\Sigma)$ be a connected complex of regions with a g-small vertex, an n-small vertex, and no holes or corks. We would like to show that

$$\eta_A : \mathrm{Mod}^{\pm}(\Sigma) \to \mathrm{Aut}\,\mathcal{C}_A(\Sigma)$$

is an isomorphism. It follows from Lemma 2.1.1 that η_A is injective. It remains to show that it is surjective. We have from Lemmas 2.6.1, 2.6.2, 2.6.3, and 2.6.4 that there is a well-defined map

$$\partial : \mathrm{Aut}\,\mathcal{C}_A(\Sigma) \to \mathrm{Aut}\,\mathcal{C}_{\partial A}(\Sigma)$$

where the the image under $\partial(\phi)$ of a vertex corresponding to a dividing set is determined by the image under ϕ of the corresponding maximal join. We will show that ∂ is injective.

Suppose $\partial(\phi)$ is the identity. Let v be a 1-sided, annular vertex of $\mathcal{C}_A(\Sigma)$ corresponding to an annulus with boundary components isotopic to the curve $\mathbf{v}$, we need to show that $\phi(v) = v$. Let the regions R and Q be the associated regions of $\mathbf{v}$. We want to find a vertex of $\mathcal{C}_{\partial A}(\Sigma)$ corresponding to a curve which is not isotopic to the curve $\mathbf{v}$. Since $\mathcal{C}_A(\Sigma)$ is connected then, up to renaming regions, it contains a vertex w corresponding to a subsurface of Q that is not homotopic to R. If w corresponds to an annulus then the desired vertex of $\mathcal{C}_{\partial A}(\Sigma)$ corresponds to the isotopy class of the boundary components of the annulus. If w does not correspond to an annulus then from Lemma 2.6.5 we can find the desired vertex. We do not consider the case where w corresponds to the region Q itself as $\mathcal{C}_A(\Sigma)$ does not contain corks.

Having found vertex in $\mathcal{C}_{\partial A}(\Sigma)$ that corresponds to a non-peripheral curve in Q we deduce that there exist vertices of $\mathcal{C}_{\partial A}(\Sigma)$ which correspond to curves that fill Q. Each of these vertices is fixed by $\partial(\phi)$ by assumption and so it follows that $\phi(v)$ corresponds to a region disjoint from Q. Since v is a 1-sided vertex and $\mathcal{C}_A(\Sigma)$ has no holes we have that $\phi(v) = v$. It can be shown using a similar argument that if v is a 1-sided, non-annular, link-minimal vertex of $\mathcal{C}_A(\Sigma)$ then we can deduce that $\phi(v) = v$.

Now assume that v is any other vertex of $\mathcal{C}_A(\Sigma)$. Let Q be a complementary region of a region R_v, such that v corresponds to R_v. Since v is not a 1-sided, annular vertex and $\mathcal{C}_A(\Sigma)$ does not contain holes, we have that there exist vertices that span edges with v and that correspond to regions contained in Q. We will label the set of all such vertices $\mathcal{Q}$.

Suppose vertices $u, w \in \mathcal{Q}$ correspond to regions $R_u, R_w \subset Q$. Define $\widetilde{R}_u, \widetilde{R}_w$ to be the lowest genus nonseparating regions in Q with fewest punctures that contain R_u and R_w respectively. Writing $u \leq w$ if $\widehat{R}_u \subseteq \widehat{R}_w$ up to homotopy, we see $\leq$ is a partial order on the vertices of $\mathcal{Q}$. We claim that a $\leq$-minimal vertex of $\mathcal{Q}$ is either a 1-sided, annular vertex or it is a 1-sided, non-annular, link-minimal vertex.

We prove the claim in three steps. First we assume that u is a 2-sided, non-annular vertex. Let P be a complementary region of R_u that does not contain the boundary of Q. There are no holes in $\mathcal{C}_A(\Sigma)$, so there must exist a vertex w of $\mathcal{C}_A(\Sigma)$ corresponding to a subsurface of P. This implies that $w < u$.

In the second case we assume that u is a 1-sided, non-annular vertex that is not link-minimal. Since u is non-annular and there are no holes in $\mathcal{C}_A(\Sigma)$ the region R_u must be nonseparating. However u is not link-minimal, so there must be a vertex w such that $R_w \subset R_u$, hence $w < u$.

Finally we suppose that u is a 2-sided, annular vertex. Denote by P the complementary region of R_u that does not contain the boundary of Q. Since $\mathcal{C}_A(\Sigma)$ has no corks, there must be a vertex w of $\mathcal{C}_A(\Sigma)$ represented by a proper subsurface of P. Once again, since $\mathcal{C}_A(\Sigma)$ has neither holes nor corks, it must be that there exists a vertex $w < u$.

We have therefore characterised all $\leq$-minimal vertices. We now have that one of the following conditions hold;

1. there exist 1-sided, annular vertices and 1-sided, non-annular, link-minimal vertices in $\mathcal{Q}$ corresponding to regions that fill the region Q,

2. there exists a 1-sided annular vertex of $\mathcal{C}_A(\Sigma)$ corresponding to the boundary of Q and no vertices of $\mathcal{C}_A(\Sigma)$ correspond to non-peripheral subsurfaces of Q, or

3. the region Q is homeomorphic to Σ_0^2, $\Sigma_{0,1}^2$, or Σ_0^3 and $\mathcal{Q}$ contains 1-sided annular vertices corresponding to non-separating annuli.

Indeed, if there exists a $\leq$-minimal vertex of $\mathcal{Q}$ corresponding to a non-peripheral region in Q then by the above claim we must be in the first case. If however, all $\leq$-minimal vertices of $\mathcal{Q}$ are peripheral then the boundary of Q may be connected or it may not. If it is connected then since there are no corks, the region Q is not represented in A and we are in the second case. If the boundary of Q is not connected then each of its boundary components must be nonseparating curves and hence nonseparating annuli are represented in A. If Q is homeomorphic to anything other than an annulus, a punctured annulus, or a pair of paints then we can find nonseparating annuli in that fill Q. This contradicts our assumption that all $\leq$-minimal vertices are peripheral, so we must be in the third case.

Given a vertex v, let $\mathcal{V}$ be the set of all the 1-sided, annular vertices and 1-sided, non-annular, link-minimal vertices in the link of v. We can now conclude that v is the unique vertex of $\mathcal{C}_A(\Sigma)$ whose link $\mathrm{Lk}(v)$ contains $\mathcal{V}$. Since we have shown that such vertices are fixed by ϕ it follows that $\phi(v) = v$. Hence ∂ is injective.

From Lemma 2.6.5 the complex $\mathcal{C}_{\partial A}(\Sigma)$ is connected and by Lemma 2.5.7 the natural homomorphism $\eta_{\partial A}$ is an isomorphism. The diagram

$$\begin{tikzcd}
\mathrm{Mod}^{\pm}(\Sigma) \arrow[rr, hook, "\eta_A"] \arrow[dr, "\cong", "\eta_{\partial A}"'] & & \mathrm{Aut}\,\mathcal{C}_A(\Sigma) \arrow[dl, hook', "\partial"] \\
& \mathrm{Aut}\,\mathcal{C}_{\partial A}(\Sigma) &
\end{tikzcd}$$

is commutative from the definition of ∂. Finally we may conclude that both η_A and ∂ are isomorphisms, completing the proof. $\blacksquare$

Chapter 3

Geometric normal subgroups

Throughout this chapter we will assume that all surfaces have empty boundary. We write g for the genus of a surface Σ and allow for the possibility of n punctures. We prove Theorem 1.2.1 which determines the automorphism groups of many normal subgroups of $\mathrm{Mod}(\Sigma)$. A crucial ingredient to the proof is an application of Theorem 1.4.2, which we proved in the last chapter. At the end we include the proof of a result discussed in Section 1.3.1. Namely, that the automorphism groups $\mathrm{Aut}\, B_n$ and $\mathrm{Aut}\, B_n/Z$ are isomorphic, where Z is the centre of the braid group B_n.

A great deal of the mathematical machinery used to prove Theorem 1.2.1 was developed by Brendle-Margalit to prove the analogous result for closed surfaces. In fact, many of the lemmas and proofs from their paper carry over to this more general case. For this reason we state a number of results and give appropriate references to the paper of Brendle-Margalit [13, Section 6].

Before moving on to the proof we note that the results on braid groups discussed in Section 1.3.1 rely on a slightly different version of Theorem 1.2.1.

Proposition 3.0.1. *Let N be a normal subgroup of $\mathrm{Mod}(S_n, p)$ such that N contains an element with support contained in a disc disjoint from p with at most $n/3$ punctures. Then $\mathrm{Aut}\, N$ is isomorphic to the normaliser of N in $\mathrm{Mod}^{\pm}(S_{n+1})$. Furthermore, if N is normal in $\mathrm{Mod}^{\pm}(S_{n+1})$ then the group of abstract commensurators $\mathrm{Comm}\, N$ is $\mathrm{Mod}^{\pm}(S_{n+1})$.*

The proof of Proposition 3.0.1 can be found in a paper by the author [55]. There is very little difference between this special case and the results of this chapter.

3.1 Method

We will define a complex of regions $\mathcal{C}_N(\Sigma)$ related to a normal subgroup N of $\mathrm{Mod}(\Sigma)$. The vertices of $\mathcal{C}_N(\Sigma)$ will correspond to the supports of so-called *basic subgroups* of N. We first state some results which arise from the study of the Nielsen-Thurston classification of elements of $\mathrm{Mod}(\Sigma)$.

Let R be region of Σ. A *partial pseudo-Anosov* element of $\mathrm{Mod}(\Sigma)$ is the image of a pseudo-Anosov element of $\mathrm{Mod}(R)$ under the map $\mathrm{Mod}(R) \to \mathrm{Mod}(\Sigma)$ induced by the inclusion of R in the surface Σ. The region R is called the *support* of the partial pseudo-Anosov element and is unique up to isotopy; see [8] for more details.

3.1.1 Pure mapping classes

Using the terminology of Ivanov [38] we call a mapping class $[f]$ *pure* if it can be written as a product $[f_1] \dots [f_k]$ where;

1. each $[f_i]$ is a partial pseudo-Anosov element or a power of a Dehn twist; and
2. if $i \neq j$ then the supports of $[f_i]$ and $[f_j]$ have disjoint non-homotopic representatives.

The $[f_i]$ are called the *components* of $[f]$. We note that the pure elements of $\mathrm{Mod}^{\pm}(\Sigma)$ are also elements of $\mathrm{Mod}(\Sigma)$. Furthermore, while the components of a pure mapping class are not canonical in general, the support of a pure element (the union of the supports of the components) is canonical [8].

Pure subgroups

We call a subgroup of $\mathrm{Mod}(\Sigma)$ *pure* if each of its elements is pure. The support of a pure subgroup is well-defined and is invariant under passing to finite index subgroups.

Let $\mathcal{B}$ be a finite set of marked points in a surface Σ. We denote by $\mathrm{PMod}(\Sigma, \mathcal{B})$ the subgroup of $\mathrm{Mod}(\Sigma, \mathcal{B})$ consisting of elements that induce the trivial permutation of the marked points in $\mathcal{B}$. If R is a component of the support of a pure subgroup H, then there is a well defined *reduction map*

$$H \to \mathrm{PMod}(\overline{R}, \mathcal{B})$$

where $\overline{R}$ is the surface obtained by collapsing each boundary component of R to a marked point and $\mathcal{B}$ is the set of such points.

Recall that two pseudo-Anosov elements with equal support are independent if their corresponding foliations are distinct. Similarly two partial pseudo-Anosov elements are *independent* if their images under the reduction map are independent pseudo-Anosov elements of $\mathrm{PMod}(\overline{R}, \mathcal{B})$, otherwise they are *dependent*.

We are now able to state a key fact concerning the commutativity of pure elements. Crucially, this only applies to pure elements that belong to a pure subgroup. The result follows from the ideas given above and in [8] and [38, Lemma 5.10].

Fact 3.1.1. *Two elements of a pure subgroup commute if and only if;*

1. *the supports of their components are pairwise disjoint or equal; and*

2. *in the case where two partial pseudo-Anosov components have equal support, the components are dependent.*

In particular if two pure elements commute then all of their nontrivial powers commute and if two pure elements do not commute then all of their nontrivial powers fail to commute.

Note that this is not the case for all non-commuting pure mapping classes. Paris-Bonatti construct examples of this type, however such elements do not generate a pure subgroup [10].

The next lemma uses Fact 3.1.1 to relate the centraliser of a non-abelian pure subgroup H and the complement of one of the components of its support.

Lemma 3.1.2 (Brendle-Margalit)**.** *Let H be a pure non-abelian subgroup of* $\mathrm{Mod}(\Sigma)$*. Then there is a component R of the support of H so that the reduction map $H \to$* $\mathrm{PMod}(\overline{R}, \mathcal{B})$ *has non-abelian image. For any such R, the centraliser of H is supported in the complement of R.*

The proof of Lemma 3.1.2 is given in [13, Lemma 6.2]. In general a subgroup of a normal subgroup N does not have connected support. As, however, we are aiming to define a complex of regions where vertices correspond to subgroups of N this issue must now be addressed.

Lemma 3.1.3 (Brendle-Margalit)**.** *Let N be a pure normal subgroup of* $\mathrm{Mod}(\Sigma)$ *and let G be a finite index subgroup of N. Let $[f]$ be an element of G and let R be a region of the surface so that some component of $[f]$ has support contained as a non-peripheral subsurface of R and all other components have support that is either contained in or disjoint from R. Let J be the subgroup of G consisting of all elements supported in R. Then*

1. *the subgroup J is not abelian;*
2. *the subgroup J contains an element with support R; and*
3. *the centraliser $C_G(J)$ is supported in the complement of R.*

The proof of this lemma depends upon the fact that N is pure. If $[f]$ is an element of N with a component whose support is a non-annular region R then the *commutator trick* allows us to find an element of N whose support is R. The proof is detailed in the paper of Brendle-Margalit along with a more in depth discussion of the *commutator trick* [13, Section 6.2].

3.2 Vertices of the complex

We shall turn our attention to the vertices of the complex related to N. First, recall from Section 1.2 that a mapping class is small if its support is contained in a single boundary region R such that; $n \geq 3n(R) - 1$ when $g = 0$, $n \geq \max\{3n(R), 7\}$ when $g = 1$, and in the general case we say a mapping class is is small if

$$g \geq g(R) + 1 \quad \text{and} \quad n \geq 3n(R).$$

Furthermore, recall from Section 1.4.4 that R a core region of $A \subset \mathcal{R}(\Sigma)$ is one where

$$\widehat{R} \subseteq \widehat{Q} \quad \Rightarrow \quad \widehat{R} \simeq \widehat{Q},$$

for any region Q represented in A.

The complex of regions $\mathcal{C}_N(\Sigma)$ we construct in the next section will have vertices that correspond to the supports of so-called *basic subgroups*. Our goal now is to show that these supports are regions of the surface, that the extended mapping class group acts on the set of supports, and that there is a *basic subgroup* of N whose support is a core region. By definition, if one such support is a core region and N contains a small element, our complex $\mathcal{C}_N(\Sigma)$ will contain both a g-small and an n-small vertex.

Basic subgroups

There exists a strict partial order on subgroups of G as follows:

$$H \prec H' \quad \text{if} \quad C_G(H') \subsetneq C_G(H).$$

This means that '$\prec$' is a transitive binary relation, but no subgroup is related to itself. We say a subgroup of G is a *basic subgroup* if among all non-abelian subgroups of G it is minimal with respect to the strict partial order.

The next lemma tells us that the supports of basic subgroups are indeed suitable candidates from which we can build a complex of regions. We define $A_N \subset \mathcal{R}(\Sigma)$ to be the set containing the $\mathrm{Mod}^{\pm}(\Sigma)$-orbits of the supports of all basic subgroups.

Lemma 3.2.1 (Brendle-Margalit)**.** *Let N be a pure normal subgroup of* $\mathrm{Mod}(\Sigma)$ *that contains a small element and let G be a finite index subgroup of N.*

1. *The support of a basic subgroup of G is a non-annular region of Σ.*

2. *If B is a basic subgroup of N then $B \cap G$ is a basic subgroup of G; similarly, any basic subgroup of G is a basic subgroup of N.*

3. *N contains a basic subgroup whose support is core region of A_N.*

4. $\mathrm{Mod}^{\pm}(\Sigma)$ *acts on the set of supports of basic subgroups of G.*

Proof. To prove the first statement we let H be a basic subgroup of G. The support of H is not empty and is not an annulus as H is not abelian by the definition of basic subgroup. Suppose the support of H is the entire surface Σ. From Lemma 3.1.2 it follows that $C_G(H)$ is trivial. Now, since G is finite index it contains a small element

$[f]$ supported in a single boundary region R such that $g \geq 3(R)$ and $n \geq 3(R) - 1$. From Lemma 3.1.3 we can find a non-abelian subgroup J containing an element $[j]$ with support R. The centraliser $C_G(J)$ contains an element of the form $[hjh^{-1}]$ for some $[h] \in \mathrm{Mod}^{\pm}(\Sigma)$. Since J is not trivial this contradicts the fact that H is basic, thus the support of H is not the entire surface.

In order to finish the proof of the first statement we need to show that the support of H is connected, hence a region. Assume then that the support of H is not connected. By Lemma 3.1.2 there exists a component R of the support of H so that the image of the reduction map $H \to \mathrm{Mod}(\overline{R}, \mathcal{B})$ is not abelian. There must then be an element of H with a component whose support is a non-peripheral subsurface of R. This satisfies the conditions of Lemma 3.1.3 so the subgroup $J \subset G$ consisting of all elements supported in R is not abelian. By Lemma 3.1.2 the centraliser $C_G(H)$ is supported in the complement of R, hence $C_G(H) \subseteq C_G(J)$. In order to contradict our assumption that the support of H is not connected we will show that there is an element of $C_G(J) \setminus C_G(H)$. Thus, implying $J \prec H$.

Let $[h]$ be an element of H whose support is not contained in R. There exists a non-annular region Q disjoint from R that contains a component of the support of $[h]$ as a non-peripheral subsurface. Once again, this satisfies Lemma 3.1.3 so there exists an element of $C_G(J)$ not in $C_G(H)$. This contradicts the minimality of the basic subgroup H, so the support of H must be connected. This completes the proof of the first statement.

The proofs of the second statement and fourth statement are the same as the analogous result of Brendle-Margalit [13, Lemma 6.4].

Now we will prove the third statement. We have that G contains a non-trivial pure element $[f]$ that is small. Let the support of this $[f]$ be the region Q. We can find a region R_1 contained in Q satisfying the conditions of Lemma 3.1.3. Let

$$J_{R_1} = \{[h] \in G \mid \text{the support of } [h] \text{ is contained in } R_1\}.$$

Now, from Lemma 3.1.3, J_{R_1} is not abelian and $C_G(J_{R_1})$ is precisely the elements of G with support disjoint from R_1. We will show that J_{R_1} contains a basic subgroup. We assume that J_{R_1} is not itself minimal, so it contains a non-abelian subgroup $J'_{R_1} \prec J_{R_1}$. By Lemma 3.1.2 the support of J'_{R_1} has a component that is a subsurface R_2 which is non-peripheral in R_1. We define J_{R_2} in the same way as J_{R_1} and we see that $J_{R_2} \prec J_{R_1}$. Repeating this process algorithmically we will arrive at a basic subgroup H of J_{R_1} whose support is a core region of A_N in Q. By the second statement of this lemma H is also basic in G, completing the proof. ■

3.3 A complex of regions for a normal subgroup

Let N be a fixed pure normal subgroup of $\mathrm{Mod}(\Sigma)$ with a small element. We can now define a complex of regions associated to N. We first define $\mathcal{C}_N^\sharp(\Sigma)$ to be the complex of regions whose vertices correspond to the supports of the basic subgroups of N. From Lemma 3.2.1(1) we have that this complex of regions has no corks and from Lemma 3.2.1(3) we have that it contains a both a g-small and an n-small vertex. This complex may however be disconnected and may contain holes.

Recall from Section 1.4.3 that if v is a hole corresponding to the region R then the filling of v is the union of R and the complementary regions containing no subregions represented in A. If a complex of regions has holes then we define its *filling* to be the complex defined by replacing holes with vertices corresponding to their fillings.

We define $\mathcal{C}_N^\flat(\Sigma)$ to be the filling of $\mathcal{C}_N^\sharp(\Sigma)$. Brendle-Margalit show that the complex $\mathcal{C}_N^\flat(\Sigma)$ has no holes, no corks, and contains a small vertex [13, Lemma 2.4]. Furthermore, they show that the small vertices of $\mathcal{C}_N^\flat(\Sigma)$ lie in the same connected component of the complex [13, Lemma 6.5]. We define this connected component to be the complex of regions $\mathcal{C}_N(\Sigma)$. It is easy to check that since $\mathcal{C}_N^\flat(\Sigma)$ has no holes and no corks the complex $\mathcal{C}_N(\Sigma)$ has no holes and no corks. We have therefore proven the following result.

Proposition 3.3.1. *Let N be a pure normal subgroup of* $\mathrm{Mod}(\Sigma)$ *that contains a small element. Then the natural map*

$$\mathrm{Mod}^{\pm}(\Sigma) \to \mathrm{Aut}\,\mathcal{C}_N(\Sigma)$$

is an isomorphism.

3.3.1 Action of the group of abstract commensurators

Recall from Section 1.2 that the group of abstract commensurators is the group of equivalence classes of isomorphisms between finite index subgroups of G, where two isomorphisms are equivalent if they agree on some finite index subgroup. Now, for any basic subgroup B we define v_B to be the vertex corresponding to the support of B. We can define the map

$$\mathrm{Comm}\,N \to \mathrm{Aut}\,\mathcal{C}_N(\Sigma)$$

as follows: if $\alpha : H_1 \to H_2$ is an isomorphism between finite index subgroups of N and $\alpha_\star$ is the image in $\mathrm{Aut}\,\mathcal{C}_N(\Sigma)$ of the equivalence class of α, then for any basic subgroup B of N we have

$$\alpha_\star\big(v_B\big) = v_{\alpha(B\cap H_1)}.$$

The fact that this makes sense is a consequence of Lemma 3.2.1(2). Explicitly, $B \cap H_1$ is a basic subgroup of H_1 and since α is an isomorphism, $\alpha(B\cap H_1)$ is a basic subgroup of H_2, therefore it is a basic subgroup of N.

As well as being a set map, Brendle-Margalit show that this construction yields a well defined group homomorphism [13, Proposition 6.8].

Proposition 3.3.2 (Brendle-Margalit)**.** *Let N be a pure normal subgroup of* $\mathrm{Mod}(\Sigma)$ *that contains a small element. The map*

$$\mathrm{Comm}\, N \to \mathrm{Aut}\, \mathcal{C}_N(\Sigma).$$

defined above is a well defined homomorphism.

3.4 Geometric normal subgroups with small elements

In this section we will prove Theorem 1.2.1 which states that any member of a wide class of normal subgroups of $\mathrm{Mod}(\Sigma)$ are geometric. To simplify notation, we will denote mapping classes by lower case letters, for example $f \in \mathrm{Mod}(\Sigma)$. Note that in other sections of this thesis we tend to write homeomorphisms in this way. We denote by α_f the automorphism of $\mathrm{Mod}(\Sigma)$ given by conjugation by $f \in \mathrm{Mod}^{\pm}(\Sigma)$. If f belongs to the normaliser of N then we may consider α_f as an element of $\mathrm{Aut}\, N$. If there is a restriction of α_f that is an isomorphism between finite index subgroups of N then we can think of the equivalence class $[\alpha_f]$ as an element of $\mathrm{Comm}\, N$. For $f \in \mathrm{Mod}(\Sigma)$ let $f_\star$ be its image in $\mathrm{Aut}\, \mathcal{C}_N(S_n)$ by the isomorphism in Proposition 3.3.1. Throughout this section we will use the fact that $f_\star(v_B) = v_{fBf^{-1}}$ [13, Lemma 6.7(4)].

Proof of Theorem 1.2.1. Let N be a normal subgroup of $\mathrm{Mod}(\Sigma)$. Let $\mathcal{M}$ be the normaliser of N in $\mathrm{Mod}^{\pm}(\Sigma)$, either $\mathrm{Mod}(\Sigma)$ or $\mathrm{Mod}^{\pm}(\Sigma)$. Let P be a pure normal subgroup of finite index in $\mathcal{M}$ (this always exists, as shown by Ivanov [38]). We define the following sequence of homomorphisms;

$$\mathcal{M} \xrightarrow{\Phi_1} \mathrm{Aut}\, N \xrightarrow{\Phi_2} \mathrm{Comm}\, N \xrightarrow{\Phi_3} \mathrm{Comm}\, N \cap P \xrightarrow{\Phi_4} \mathrm{Aut}\, \mathcal{C}_{N \cap P}(\Sigma) \xrightarrow{\Phi_5} \mathrm{Mod}^{\pm}(\Sigma).$$

- The map Φ_1 is defined so that $\Phi_1(f) = \alpha_f$,
- the map Φ_2 sends an element of $\mathrm{Aut}\, N$ to its equivalence class in $\mathrm{Comm}\, N$,
- the map Φ_3 sends an element of $\mathrm{Comm}\, N$ to the equivalence class of any restriction that is an isomorphism between finite index subgroups of $N \cap P$,
- the map Φ_4 is defined in Proposition 3.3.2, and
- the map Φ_5 is defined in Proposition 3.3.1.

We claim that each Φ_i is injective and that

$$\Phi_5 \circ \Phi_4 \circ \Phi_3 \circ \Phi_2 \circ \Phi_1 = \mathrm{Id}_{\mathcal{M}}\, .$$

We show that Φ_1 is injective by using similar methods to that of Lemma 2.1.1. If $f \in \mathcal{M}$ commutes with an element $h \in N$ then the reduction system of h is fixed by f. As there exists an element of N with nonempty reduction systems, elements of the $\mathcal{M}$-orbit of this reduction system fills an associated region of an arbitrary curve $\mathbf{c}$ and hence fixes $\mathbf{c}$. Similar to Lemma 2.1.1, the injectivity of Φ_1 follows.

Now let $\alpha \in \operatorname{Aut} N$ be an element of the kernel of Φ_2 and let $f, h \in N$ where h is pseudo-Anosov. Since $\Phi_2(\alpha)$ is the identity, the restriction of α to a finite index subgroup G of N is also the identity. There is some $m > 0$ so that both h^m and $(fhf^{-1})^m$ lie in G and so

$$fh^m f^{-1} = (fhf^{-1})^m = \alpha((fhf^{-1})^m) = \alpha(f)\alpha(h^m)\alpha(f)^{-1} = \alpha(f)h^m\alpha(f)^{-1}.$$

It follows that $f^{-1}\alpha(f)$ commutes with h^m and so fixes the unstable foliation of h. Since h was arbitrary and N is normal in $\mathcal{M}$ it follows that $f^{-1}\alpha(f)$ fixes all curves. Again, this implies that $f^{-1}\alpha(f)$ is the identity and so $\alpha(f) = f$ for all $f \in N$.

Now, the map Φ_3 is an isomorphism since a finite index subgroup of $N \cap P$ is also finite index subgroup of N.

To show that Φ_4 is injective we first fix an isomorphism $\alpha : H_1 \to H_2$ between finite index subgroups that represents an element $[\alpha]$ of $\operatorname{Comm} N \cap P$. We denote the image of $[\alpha]$ in $\operatorname{Aut} \mathcal{C}_{N\cap P}(\Sigma)$ by $\alpha_\star$ as in Proposition 3.3.2. Suppose $\alpha_\star$ is the identity. We will show that $[\alpha]$ is the identity. Let $h \in H_1$. Since the natural map $\operatorname{Mod}^{\pm}(\Sigma) \to \operatorname{Aut} \mathcal{C}_{N\cap P}(\Sigma)$ is an isomorphism it suffices to show that the images of $\alpha(h)$ and h are equal. We denote these images $\alpha(h)_\star$ and $h_\star$ respectively.

Let B be a basic subgroup of N. By Lemma 3.2.1 (2) we can assume that $B \subset H_1$. It follows that

$$\begin{aligned} h_\star(v_B) &= v_{hBh^{-1}} = \alpha_\star(v_{hBh^{-1}}) = v_{\alpha(hBh^{-1})} = v_{\alpha(h)\alpha(B)\alpha(h)^{-1}} \\ &= \alpha(h)_\star(v_{\alpha(B)}) = \alpha(h)_\star\alpha_\star(v_B) = \alpha(h)_\star(v_B). \end{aligned}$$

It follows that $h_\star = \alpha(h)_\star$, as desired.

As mentioned previously, the map Φ_5 is an isomorphism due to Proposition 3.3.1. By construction, the composition is the identity on $\mathcal{M}$ and the claim follows. This completes the proof of the first statement, that is, if $\mathcal{M} = \operatorname{Mod}^{\pm}(\Sigma)$ then the natural homomorphisms

$$\operatorname{Mod}^{\pm}(\Sigma) \to \operatorname{Aut} N \to \operatorname{Comm} N$$

are isomorphisms. Furthermore, if $\operatorname{Comm} N \cong \mathcal{M} = \operatorname{Mod}(\Sigma)$ then the composition is again the identity which implies that $\operatorname{Mod}(\Sigma) \to \operatorname{Aut} N$ is an isomorphism.

The final case we must consider is when $\mathcal{M} = \operatorname{Mod}(\Sigma)$, but the image of $\Phi_5 \circ \Phi_4 \circ \Phi_3$

is $\mathrm{Mod}^{\pm}(\Sigma)$. We can now construct the following commutative diagram;

$$\begin{tikzcd}
 & \mathrm{Mod}^{\pm}(\Sigma) \arrow[ddd, "\Phi_6"] & \\
\mathrm{Aut}\,\mathcal{C}_{N\cap P}(\Sigma) \arrow[ur, "\Phi_5", "\cong"'] & & \mathrm{Mod}(\Sigma) \arrow[ul, hook'] \arrow[d, hook, "\Phi_1"] \\
\mathrm{Comm}\,N\cap P \arrow[u, hook, "\Phi_4"] & & \mathrm{Aut}\,N \arrow[dl, hook', "\Phi_2"] \\
 & \mathrm{Comm}\,N \arrow[ul, "\Phi_3", "\cong"'] &
\end{tikzcd}$$

where Φ_6 is the natural homomorphism. We will show that Φ_6 is the left inverse of $\Phi_5 \circ \Phi_4 \circ \Phi_3$, that is,

$$\Phi_6 \circ \Phi_5 \circ \Phi_4 \circ \Phi_3([\alpha]) = [\alpha].$$

We fix an isomorphsim $\alpha : H_1 \to H_2$ between finite index subgroups that represents an element $[\alpha] \in \mathrm{Comm}\,N \cap P \cong \mathrm{Comm}\,N$. As usual we denote by $\alpha_\star$ the image of $[\alpha]$ in $\mathrm{Aut}\,\mathcal{C}_{N\cap P}(\Sigma)$. Assume that $\Phi_5(\alpha_\star) = f \in \mathrm{Mod}^{\pm}(\Sigma)$. We need to show that $[\alpha]$ is equal to the restriction of α_f, the conjugation map defined by f.

To that end, let $h \in H_1$. We want to show that $\alpha(h) = fhf^{-1}$. For any $j \in \mathrm{Mod}^{\pm}(\Sigma)$ let $j_\star$ be its image in $\mathrm{Aut}\,\mathcal{C}_{N\cap P}(\Sigma)$. In particular, we have that $\alpha_\star = f_\star$. Since Φ_5 is an isomorphism it suffices to show that $(fhf^{-1})_\star = \alpha(h)_\star$.

Let B be a basic subgroup of N. Without loss of generality we assume that B is contained in H_1. We now have

$$\begin{aligned}
(fh)_\star\big(v_B\big) &= f_\star h_\star\big(v_B\big) = \alpha_\star h_\star\big(v_B\big) = \alpha_\star\big(v_{hBh^{-1}}\big) = v_{\alpha(hBh^{-1})} \\
&= v_{\alpha(h)\alpha(B)\alpha(h^{-1})} = \alpha(h)_\star\big(v_{\alpha(B)}\big) = \alpha(h)_\star\alpha_\star\big(v_B\big) \\
&= \alpha(h)_\star f_\star\big(v_B\big) = (\alpha(h)f)_\star\big(v_B\big).
\end{aligned}$$

It follows that $(fh)_\star = (\alpha(h)f)_\star$, that is, $\alpha(h) = fhf^{-1}$, as required. Thus, we have shown that Φ_6 is a left inverse of $\Phi_5 \circ \Phi_4 \circ \Phi_3$. Since the diagram commutes however, we have that it is an isomorphism. We can use this to prove that Φ_1 is surjective, hence an isomorphism. Let $\alpha \in \mathrm{Aut}\,N$, denote $\Phi_2(\alpha)$ by $[\alpha]$, and let $f = \Phi_5 \circ \Phi_4 \circ \Phi_3([\alpha])$. From the argument above we have that $\Phi_6(f) = [\alpha]$, that is, $[\alpha_f] = [\alpha]$. We will now show that $\alpha_f = \alpha$. Since $\mathrm{Mod}(\Sigma)$ is the normaliser of N and the diagram commutes, this implies that α belongs to the image of Φ_1 and that it is surjective.

We want to show that $\alpha(j) = fjf^{-1}$ for all $j \in N$. Since $[\alpha] = [\alpha_f]$, there is a finite index subgroup G of N such that the restriction of α to G agrees with the restriction of α_f to G. Let h be a pseudo-Anosov element of N. There is some $m > 0$ such that h^m and jh^mj^{-1} lie in G. We have that $\alpha(h^m) = fh^mf^{-1}$ and $\alpha(jh^mj^{-1}) = fjh^mj^{-1}f^{-1}$ and so

$$fjh^mj^{-1}f^{-1} = \alpha(jh^mj^{-1}) = \alpha(j)\alpha(h^m)\alpha(j)^{-1} = \alpha(j)fh^mf^{-1}\alpha(j)^{-1}.$$

Hence

$$(f^{-1}\alpha(j)^{-1}fj)h^m = h^m(f^{-1}\alpha(j)^{-1}fj),$$

and therefore $f^{-1}\alpha(j)^{-1}fj$ fixes the unstable foliation of h. As above, this implies that $f^{-1}\alpha(j)^{-1}fj$ is the identity. So $\alpha(j) = fjf^{-1}$, as desired. ■

3.5 Automorphisms of the braid group

Recall that Theorem 1.3.1 establishes the automorphism group $\operatorname{Aut} N$ for a normal subgroup N of the braid group B_n that contains no central elements. In particular, this result does not apply to B_n itself. We now address this technical issue in order to recover a result of Dyer-Grossman that computes $\operatorname{Aut} B_n$ [23].

Characteristic subgroups

Let G be any group. Given a characteristic subgroup, that is a subgroup $C < G$ such that $\phi(C) = C$ for all $\phi \in \operatorname{Aut} G$, there is a well defined natural group homomorphism $\operatorname{Aut} G \to \operatorname{Aut} G/C$. This is of use to us because the centre of a group is always characteristic.

It is an interesting property of the braid group that the automorphism groups of B_n and B_n/Z are isomorphic, see the work of Charney-Crisp [18] for example. We will show this isomorphism directly using the natural homomorphism discussed above.

Theorem 3.5.1. *For $n \geq 3$ the natural homomorphism*

$$\operatorname{Aut} B_n \to \operatorname{Aut} B_n/Z$$

is an isomorphism.

Proof. First we will show that the homomorphism is surjective. Any $\overline{\phi} \in \operatorname{Aut} B_n/Z$ is determined by where it sends the generators $\{[\sigma_i]\}$. Every element of $[\sigma_i]$ can be written as $\delta_i z^k$ for some fixed $\delta_i \notin Z$, and where z is the generator of Z. It can be seen that the corresponding well defined set map $\phi : B_n \to B_n$, given by $\phi(\sigma_i) = \delta_i$, is an isomorphism. This defines a right inverse for the natural homomorphism $\operatorname{Aut} B_n \to \operatorname{Aut} B_n/Z$ and hence proves it is surjective.

To show that it is injective suppose $\phi \in \operatorname{Aut} B_n$ is sent to the trivial automorphism of B_n/Z. It follows that for all i there exists some $z_i \in Z$ such that $\phi(\sigma_i) = z_i\sigma_i$. Since ϕ is a homomorphism and each z_i is central we have that

$$\begin{aligned}
\phi(\sigma_i\sigma_{i+1}\sigma_i) &= \phi(\sigma_i)\phi(\sigma_{i+1})\phi(\sigma_i) \\
&= z_i\sigma_i z_{i+1}\sigma_{i+1}z_i\sigma_i \\
&= z_i^2 z_{i+1}\sigma_i\sigma_{i+1}\sigma_i.
\end{aligned}$$

Similarly $\phi(\sigma_{i+1}\sigma_i\sigma_{i+1}) = z_i z_{i+1}^2 \sigma_{i+1}\sigma_i\sigma_{i+1}$. Since $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ it must be that $z_i = z_{i+1}$ for all i.

Since Z is characteristic and isomorphic to $\mathbb{Z}$, either $\phi(z) = z$ or $\phi(z) = z^{-1}$ for all $z \in Z$. In the second case, since $(\sigma_1 \dots \sigma_{n-1})^n \in Z$ we have that

$$(\sigma_1 \dots \sigma_{n-1})^{-n} = \phi((\sigma_1 \dots \sigma_{n-1})^n) = z_i^{n(n-1)}(\sigma_1 \dots \sigma_{n-1})^n.$$

Hence $z_i^{(1-n)} = (\sigma_1 \dots \sigma_{n-1})^2$. This implies that $(\sigma_1 \dots \sigma_{n-1})^2 \in Z$ and so 2 must be a multiple of n, which is contradiction as $n \geq 3$, so $\phi(z) = z$. Similar to the argument above it follows that

$$(\sigma_1 \dots \sigma_{n-1})^n = z_i^{n(n-1)}(\sigma_1 \dots \sigma_{n-1})^n.$$

We see then that $z_i^{n(n-1)}$ is the identity. Since $n \geq 3$ it must be that z_i is the identity and so $\phi(\sigma_i) = \sigma_i$ for all i, completing the proof. ■

Integral to this proof is the structure of $\mathbb{Z}$ and the braid relation $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$. If N is normal subgroup of B_n with $N \cap Z \cong \mathbb{Z}$ then we cannot generalise the above proof unless similar tricks are found to mirror this manipulation of the braid relations.

Chapter 4

Subgroups from covering spaces

The work presented in this chapter is joint with Tyrone Ghaswala of the University of Manitoba. In it, we analyse two subgroups of mapping class groups that arise naturally from covering spaces; namely the liftable mapping class group and the symmetric mapping class group. In particular we restrict our focus to case where the surfaces have non-empty boundary.

We begin in Section 4.1 by giving a proof of the Birman-Hilden Theorem for surfaces with boundary. In Section 4.2 we review and prove some results concerning the fundamental groupoid, a useful tool in studying mapping class groups of covering spaces. In Section 4.3 we prove Theorems 1.5.2 and 1.5.3, which classify all cases where either of the liftable or symmetric mapping class groups coincide with the mapping class groups of their respective surfaces. Finally, we apply some of these results and techniques in Section 4.4 to investigate an infinite family of braid group embeddings into mapping class groups of surfaces.

4.1 The Birman-Hilden Theorem with boundary

In this section we give a proof of the Birman-Hilden Theorem in the case of surfaces with boundary. The proof relies upon a version of the *Alexander trick* for arcs and *multi-arcs* on surfaces. These are analogous to the more well known notions of curves and multi-curves. This approach is similar in nature to the proofs given by Winarski [53] and by Farb-Margalit for the hyperelliptic case [27]. We begin with some background results on the level of homeomorphisms.

4.1.1 Covering spaces of surfaces with boundary

Let $p : \widetilde{\Sigma} \to \Sigma$ be a regular, finite-sheeted, possibly branched covering space with deck group $D < \mathrm{Homeo}^+(\widetilde{\Sigma})$. Recall that a homeomorphism $\tilde{f} \in \mathrm{Homeo}^+(\widetilde{\Sigma})$ is fibre-preserving with respect to p if

$$p(x) = p(y) \quad \Rightarrow \quad p\tilde{f}(x) = p\tilde{f}(y) \quad \text{for all } x, y \in \widetilde{\Sigma}.$$

The set of all fibre-preserving homeomorphisms of $\mathrm{Homeo}^+(\widetilde{\Sigma})$ forms a subgroup denoted $\mathrm{SHomeo}^+(\widetilde{\Sigma})$. Here, the 'S' indicates the intrinsic symmetry of the elements of $\mathrm{SHomeo}^+(\widetilde{\Sigma})$ with respect to p. If $\mathcal{B}$ is the set of branch points in Σ then there is a well defined homomorphism

$$\Pi : \mathrm{SHomeo}^+(\widetilde{\Sigma}) \to \mathrm{Homeo}^+(\Sigma, \mathcal{B})$$

given by $\Pi(\tilde{f})(x) := p\tilde{f}(\tilde{x})$ where $\tilde{x} \in \widetilde{\Sigma}$ is any element such that $p(\tilde{x}) = x$. It is easy to see that for any $d \in D$ we have that $\Pi(d)$ is the trivial homeomorphism of Σ and in fact we have that $D = \ker \Pi$. We will write f for the image $\Pi(\tilde{f})$ and it follows that the square

$$\begin{array}{ccc} \widetilde{\Sigma} & \xrightarrow{\tilde{f}} & \widetilde{\Sigma} \\ \downarrow p & & \downarrow p \\ \Sigma & \xrightarrow{f} & \Sigma \end{array}$$

commutes. Furthermore, $\mathrm{SHomeo}^+(\widetilde{\Sigma})$ is the normaliser of D in $\mathrm{Homeo}^+(\widetilde{\Sigma})$. One direction of this proof follows from the commuting square given above and the other uses the fact that the homomorphism Π is well defined.

As stated previously, our focus is on mapping class groups of surfaces with boundary. Recall from Chapter 1 that the mapping class group of a surface with boundary includes only the isotopy classes of homeomorphisms that fix the boundary pointwise. We shall therefore consider the intersection

$$\mathrm{SHomeo}^+(\widetilde{\Sigma}, \partial\widetilde{\Sigma}) := \mathrm{SHomeo}^+(\widetilde{\Sigma}) \cap \mathrm{Homeo}^+(\widetilde{\Sigma}, \partial\widetilde{\Sigma}),$$

that is, the subgroup of fibre-preserving homeomorphisms that fix the boundary pointwise. Note that all such homeomorphisms are orientation-preserving, so the superscript '+' is superfluous. Furthermore, since D is finite, no nontrivial element of D is an element of $\mathrm{SHomeo}^+(\widetilde{\Sigma}, \partial\widetilde{\Sigma})$. We do however have the following relationship with the deck group.

Proposition 4.1.1. *Let C be the centraliser of the deck group D in* $\mathrm{Homeo}(\widetilde{\Sigma})$. *Then*

$$\mathrm{SHomeo}^+(\widetilde{\Sigma}, \partial\widetilde{\Sigma}) = C \cap \mathrm{Homeo}^+(\widetilde{\Sigma}, \partial\widetilde{\Sigma}).$$

Proof. It suffices to show that any homeomorphism $\tilde{f}$ that fixes the boundary and is in the normaliser of D is in the centraliser of D. Let $\tilde{x} \in \partial\widetilde{\Sigma}$ and let $d \in D$. It is clear then that $d(\tilde{x}) \in \partial\widetilde{\Sigma}$. Since $\tilde{f}$ fixes the boundary pointwise we have $\tilde{f}^{-1}d\tilde{f}(\tilde{x}) = \tilde{f}^{-1}d(\tilde{x}) = d(\tilde{x})$. Since $\tilde{f}$ is in the normaliser of D, $\tilde{f}^{-1}d\tilde{f} \in D$. Since the deck group acts freely on $\partial\widetilde{\Sigma}$ we have $\widetilde{f}^{-1}d\widetilde{f} = d$, completing the proof. ■

Recall that we define $\mathcal{B}$ to be the set of branch points in Σ. We say that a homeomorphism $f \in \mathrm{Homeo}^+(\Sigma, \mathcal{B})$ *lifts* if there exists a homeomorphism $\tilde{f} \in \mathrm{Homeo}^+(\widetilde{\Sigma})$

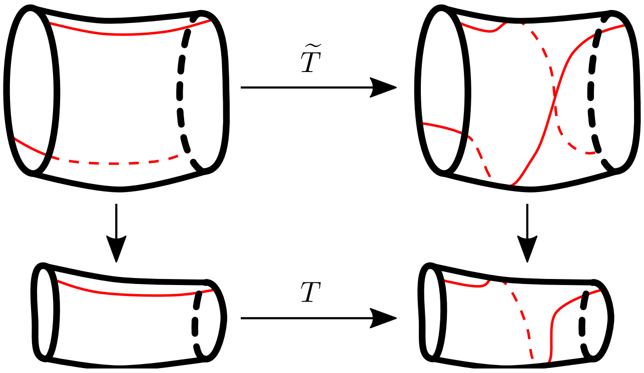

Figure 4.1: A 2-sheeted cover $p : \Sigma_0^2 \to \Sigma_0^2$. The lift of Dehn twist $T \in \mathrm{LHomeo}^+(\Sigma_0^2)$ does not preserve the boundary pointwise.

such that $p\tilde{f} = fp$. In this case we say that $\tilde{f}$ is *a lift* of f. As with fibre-preserving homeomorphisms, the set $\mathrm{LHomeo}^+(\Sigma, \mathcal{B}) \subset \mathrm{Homeo}^+(\Sigma, \mathcal{B})$, consisting of homeomorphisms that lift, forms a subgroup. It follows then that

$$\Pi : \mathrm{SHomeo}^+(\widetilde{\Sigma}) \to \mathrm{LHomeo}^+(\Sigma, \mathcal{B})$$

is surjective. Any fibre-preserving homeomorphism of $\widetilde{\Sigma}$ that fixes $\partial\widetilde{\Sigma}$ pointwise must project to a homeomorphism of Σ that fixes $\partial\Sigma$ pointwise. Since the only element of $D < \mathrm{SHomeo}^+(\widetilde{\Sigma})$ that fixes $\partial\widetilde{\Sigma}$ pointwise is the identity, restricting the domain of Π gives us an injective homomorphism

$$\Pi : \mathrm{SHomeo}^+(\widetilde{\Sigma}, \partial\widetilde{\Sigma}) \to \mathrm{LHomeo}^+(\Sigma, \mathcal{B}) \cap \mathrm{Homeo}^+(\Sigma, \partial\Sigma).$$

We shall define $\mathrm{LHomeo}^+(\Sigma, \partial\Sigma, \mathcal{B}) := \Pi(\mathrm{SHomeo}^+(\widetilde{\Sigma}, \partial\widetilde{\Sigma}))$. That is, the set of homeomorphisms of Σ fixing $\partial\Sigma$ pointwise that lift to homeomorphisms of $\widetilde{\Sigma}$ that fix $\partial\widetilde{\Sigma}$ pointwise. While it is tempting to define $\mathrm{LHomeo}^+(\Sigma, \partial\Sigma, \mathcal{B})$ simply as the intersection of the sets $\mathrm{LHomeo}^+(\Sigma, \mathcal{B})$ and $\mathrm{Homeo}^+(\Sigma, \partial\Sigma)$, in general there exist boundary preserving homeomorphisms of Σ that lift to homeomorphisms of $\widetilde{\Sigma}$ that do not fix the boundary $\partial\widetilde{\Sigma}$.

For example, Figure 4.1 shows a 2-sheeted covering space $p : \Sigma_0^2 \to \Sigma_0^2$. The deck group is generated by the hyperelliptic involution defined by rotation about the central vertical axis of the annulus. While this is a well defined homeomorphism of Σ_0^2, it does not preserve the boundary pointwise. We see then that $T \in \mathrm{LHomeo}^+(\Sigma_0^2, \mathcal{B})$ but $T \notin \mathrm{LHomeo}^+(\Sigma_0^2, \partial\Sigma_0^2, \mathcal{B})$. Of course, as this cover is unbranched the set of branch points $\mathcal{B}$ is empty.

From our definitions it is clear that

$$\Pi : \mathrm{SHomeo}^+(\widetilde{\Sigma}, \partial\widetilde{\Sigma}) \to \mathrm{LHomeo}^+(\Sigma, \partial\Sigma, \mathcal{B})$$

is an isomorphism. Now, for any surface we may define the natural homomorphism

$$\mathfrak{P} : \mathrm{Homeo}^+(\Sigma, \partial\Sigma) \to \mathrm{Mod}(\Sigma).$$

What the Birman-Hilden Theorem (Theorem 1.5.1) says is that the corresponding groups

$$\mathrm{SMod}(\widetilde{\Sigma}) := \mathfrak{P}\big(\mathrm{SHomeo}^+(\widetilde{\Sigma}, \partial\widetilde{\Sigma})\big) \quad \text{and} \quad \mathrm{LMod}(\Sigma, \mathcal{B}) := \mathfrak{P}\big(\mathrm{LHomeo}^+(\Sigma, \partial\Sigma, \mathcal{B})\big)$$

are also isomorphic. The rest of Section 4.1 will be dedicated to proving this result.

4.1.2 Multi-arcs

Recall that we define an arc in Σ to be a continuous map $\hat{\alpha} : [0,1] \hookrightarrow \Sigma$ if $\{\hat{\alpha}(0), \hat{\alpha}(1)\} \subset \partial\Sigma$ and $\hat{\alpha}(t) \notin \partial\Sigma$ for all $t \in (0,1)$. We will abuse notation by writing $\hat{\alpha}$ for the image of the arc in Σ. We call the points $\hat{\alpha}(0)$ and $\hat{\alpha}(1)$ the *endpoints* of the arc $\hat{\alpha}$.

We call two arcs $\hat{\alpha}$ and $\hat{\beta}$ isotopic if they have the same endpoints and there exists an isotopy between them that only passes through arcs with the same endpoints. If $\hat{\alpha}$ and $\hat{\beta}$ are isotopic we write $\hat{\alpha} \sim \hat{\beta}$.

Lemma 4.1.2. *Let $\hat{\alpha}$ and $\hat{\beta}$ be isotopic arcs in a surface Σ. There exists an arc $\hat{\gamma}$ disjoint from both $\hat{\alpha}$ and $\hat{\beta}$ everywhere except the endpoints such that $\hat{\alpha} \sim \hat{\gamma} \sim \hat{\beta}$.*

Proof. Let $\hat{\alpha}$ and $\hat{\beta}$ have endpoints at $x, y \in \partial\Sigma$. If they are disjoint everywhere except at x and y then there exists an isotopy passing thorough some arc $\hat{\gamma}$ satisfying the above conditions.

Suppose then that $\hat{\alpha}$ and $\hat{\beta}$ intersect away from the endpoints. There exists a disc $R \subset \Sigma$ containing both $\hat{\alpha}$ and $\hat{\beta}$ such that

$$R \cap \partial\Sigma = R \cap \hat{\alpha} = R \cap \hat{\beta} = \{x, y\}.$$

We define $\hat{\gamma}$ to be one of the two segments of ∂R with endpoints at x and y, completing the proof. ∎

We define a *multi-arc* α in Σ to be a set of arcs $\{\hat{\alpha}_i\}_{i=1}^N$ where $\hat{\alpha}_i \cap \hat{\alpha}_j = \emptyset$ for all distinct values of i and j. We call $\hat{\alpha}_i$ a *component arc* of α for each $i \in \{1, \ldots, N\}$. Note that component arcs of a multi-arc can never intersect at the boundary $\partial\Sigma$.

Two multi-arcs α and β are isotopic if they have the same number of component arcs and for each component arc of α there is a unique isotopic component arc of β. Let α and β be isotopic such that $\hat{\alpha}_i \sim \hat{\beta}_i$ are the corresponding component arcs where $i = 1, 2, ..., N$. If $\hat{\alpha}_i \cap \hat{\beta}_j = \emptyset$ for all $i \neq j$ then we say that α and β are *distinctly isotopic*.

We will now state and prove a version of the Alexander trick for multi-arcs.

Lemma 4.1.3 (The Alexander trick). *Let Σ be a surface with boundary and let $f \in \mathrm{Homeo}^+(\Sigma, \partial\Sigma)$. Let $\alpha = \{\hat{\alpha}_1, \ldots, \hat{\alpha}_N\}$ be a multi-arc in Σ.*

1. *If there is a permutation σ of $\{1, ..., N\}$ so that $f(\hat{\alpha}_i)$ is isotopic to $\hat{\alpha}_{\sigma(i)}$ for each i, then the multi-arcs $f(\alpha)$ and α are isotopic.*

If we regard α as a disconnected graph Γ in Σ, with vertices at the endpoints of arcs, then the composition of f with this isotopy gives an automorphism $\bar{f}$ of Γ.

2. *Suppose now that α fills Σ. If $\bar{f}$ fixes each vertex and edge of Γ with orientations, then f is isotopic to the identity.*

Proof. The arcs $f(\hat{\alpha}_i)$ and $\hat{\alpha}_{\sigma(i)}$ are isotopic for each i and so the multi-arcs $f(\cup\hat{\alpha}_i)$ and $\cup\hat{\alpha}_{\sigma(i)}$ are isotopic. Since $\cup\hat{\alpha}_{\sigma(i)} = \cup\hat{\alpha}_i$ this proves the first statement.

As in the statement of the lemma, the homeomorphism f induces an automorphism $\bar{f} \in \mathrm{Aut}(\Gamma)$. Since $\bar{f}$ fixes the vertices and edges of Γ with orientations and f is orientation-preserving, it follows that f preserves the sides of Γ. So, up to isotopy, f fixes Γ pointwise and sends each component of $\Sigma \setminus \Gamma$ to itself.

To prove the second statement, we have that $\Sigma \setminus \Gamma$ is homeomorphic to the disjoint union of finitely many discs and discs with a single marked point. Every homeomorphism of such a disc is isotopic to the identity homeomorphism and so it follows that f is isotopic to the identity homeomorphism of Σ. ■

Symmetry

We call a multi-arc *symmetric* with respect to a regular, finite-sheeted cover $p : \tilde{\Sigma} \to \Sigma$ with deck group D if for all $d \in D$ we have that $d(\alpha) = \alpha$.

A symmetric multi-arc α is called *self symmetric* if there exists some component arc of α, say $\hat{\alpha}_i$, such that $\hat{\alpha}_i(0) = d(\hat{\alpha}_i(1))$ for some $d \in D$. Note that self symmetric multi-arcs are precisely those with a component arc that intersects a fixed point of an order two element of D. If a symmetric multi-arc is not self symmetric we call it *properly symmetric*.

Lemma 4.1.4. *Let α and β be two properly symmetric multi-arcs in $\tilde{\Sigma}$. If α and β are distinctly isotopic then there is an isotopy between $p(\alpha)$ and $p(\beta)$ that does not pass through a branch point.*

Proof. We claim that no isotopy between α and β can pass through a self symmetric multi-arc. Assume such an isotopy exists. Then both α and β are isotopic to some self symmetric mult-arc δ. Let $\hat{\delta}_i$ be a component arc such that $\hat{\delta}_i(0) = d(\hat{\delta}_i(1))$ for some $d \in D$ and $\hat{\alpha}_i$ be the unique component arc of α isotopic to $\hat{\delta}_i$. It follows that $\hat{\alpha}_i(0) = \hat{\delta}_i(0) = d(\hat{\delta}_i(1)) = d(\hat{\alpha}_i(1))$. Since α is properly symmetric we arrive at a contradiction, proving the claim.

Since α and β are distinctly isotopic, we can define a symmetric multi-arc γ, not intersecting but isotopic to both α and β where each component arc is defined as in

Lemma 4.1.2. Let $\{B_i\}_{i=1}^N$ be the set of bigons defined by the isotopic components $\{\hat{\alpha}_i\}_{i=1}^N$ and $\{\hat{\gamma}_i\}_{i=1}^N$. Both multi-arcs are properly symmetric, so for any choice of i the bigon $d(B_i)$ is distinct from B_i for all non trivial $d \in D$. It follows then that for each bigon there is a fundamental domain containing it, and so $p(B_i)$ is a bigon in Σ for all values of i. If $p(B_j)$ contains a branch point for some $j \in \{1, 2, ..., N\}$ then $\{B_i\}_{i=1}^N$ defines an isotopy of α and γ passing through a self symmetric mult-arc, which we have shown to be a contradiction. The set $\{p(B_i)\}_{i=1}^N$ therefore contains $N/|D|$ distinct bigons in Σ that define an isotopy $p(\alpha) \sim p(\gamma)$. Similarly $p(\beta) \sim p(\gamma)$ and the result follows. ■

4.1.3 Proof of the Birman-Hilden Theorem with boundary

Recall that the homomorphism $\Pi : \mathrm{SHomeo}^+(\widetilde{\Sigma}) \to \mathrm{Homeo}^+(\Sigma, \mathcal{B})$ restricts to an isomorphism

$$\mathrm{SHomeo}^+(\widetilde{\Sigma}, \partial\widetilde{\Sigma}) \to \mathrm{LHomeo}^+(\Sigma, \partial\Sigma, \mathcal{B}).$$

We now prove Theorem 1.5.1 by showing that the isomorphism above induces an isomorphism of $\mathrm{SMod}(\Sigma)$ and $\mathrm{LMod}(\Sigma, \mathcal{B})$.

Proof of Theorem 1.5.1. We take the surjective homomorphism

$$\mathrm{SHomeo}^+(\widetilde{\Sigma}, \partial\Sigma) \to \mathrm{LHomeo}^+(\Sigma, \partial\Sigma, \mathcal{B}) \to \mathrm{LMod}(\Sigma, \mathcal{B})$$

defined by $\mathfrak{P} \circ \Pi$. We would like to show that the kernel of this map is precisely the set of elements isotopic to the identity. First, note that any isotopy between a homeomorphism $f \in \mathrm{LHomeo}^+(\Sigma, \partial\Sigma, \mathcal{B})$ and the identity lifts to an isotopy between $\tilde{f}$ and the identity. It follows that the kernel of $\mathfrak{P} \circ \Pi$ is a set consisting of such elements.

Assume then that $\tilde{f} \in \mathrm{SHomeo}(\widetilde{\Sigma}, \partial\widetilde{\Sigma})$ is isotopic to the identity. Choose a multi-arc $\alpha = \{\hat{\alpha}_i\}$ in Σ such that $\Sigma \setminus \{\hat{\alpha}_i\}$ is a collection of discs, each of which contains at most one branch point. Define multi-arcs in $\widetilde{\Sigma}$ by $\alpha_i = p^{-1}(\hat{\alpha}_i)$ and $\beta_i = \tilde{f}p^{-1}(\hat{\alpha}_i)$ for each i. By definition, α_i and β_i are properly symmetric and distinctly isotopic for each i. Now, we can apply Lemma 4.1.4 to see that

$$\hat{\alpha}_i = p(\alpha_i) \sim p(\beta_i) = p\tilde{f}(\alpha_i) = fp(\alpha_i) = f(\hat{\alpha}_i).$$

Furthermore, this isotopy does not pass through a branch point. Applying the Alexander trick (Lemma 4.1.3) to the multi-arc α we conclude that f is isotopic to the identity of $\mathrm{LHomeo}^+(\Sigma, \partial\Sigma, \mathcal{B})$. Therefore the homeomorphism $\tilde{f}$ belongs to the kernel of $\mathfrak{P} \circ \Pi$ if and only if it is isotopic to the identity of $\mathrm{SHomeo}^+(\widetilde{\Sigma}, \partial\widetilde{\Sigma})$. By the first isomorphism theorem we conclude that

$$\Pi : \mathrm{SMod}(\widetilde{\Sigma}) \to \mathrm{LMod}(\Sigma, \mathcal{B})$$

is a well defined isomorphism. ■

4.2 Groupoids

In this section we introduce the fundamental groupoid of a surface $\widetilde{\Sigma}$. Roughly, this can be thought of as a fundamental group of $\widetilde{\Sigma}$ with multiple basepoints. It is useful tool for studying a regular finite-sheeted covering $p : \widetilde{\Sigma} \to \Sigma$ as its automorphism group contains both $\mathrm{Mod}(\widetilde{\Sigma})$ and the deck group D associated to the cover. This makes it a natural choice for studying the relationship between mapping classes and covering spaces.

4.2.1 Connected groupoids

A *groupoid* is a small category (a category whose classes of objects and homomorphisms are sets) where every morphism is an isomorphism. Equivalently, a groupoid $\mathcal{G}$ is a disjoint collection of sets $\{G_{ij}\}_{i,j\in I}$ together with an associative partial operation $\cdot : G_{ij} \times G_{jk} \to G_{ik}$ such that;

1. for each $i \in I$ there is an identity $e_i \in G_{ii}$ such that $e_i g = g$ and $he_i = h$ for all g, h such that the products $e_i g$ and he_i are defined, and
2. for each $g \in G_{ij}$ there is an inverse $g^{-1} \in G_{ji}$ such that $gg^{-1} = e_i$ and $g^{-1}g = e_j$.

We will call I the *object set* of $\mathcal{G}$. If $|I| = 1$ then we recover the definition of a group. A groupoid is *connected* if $G_{ij} \neq \emptyset$ for all $i, j \in I$. Notice that G_{ii} is a group for all $i \in I$. Furthermore, if $\mathcal{G}$ is connected then the groups G_{ii} and G_{jj} are isomorphic for all $i, j \in I$. We call these groups the *vertex groups* of $\mathcal{G}$.

Let $\mathcal{G}$ be a connected groupoid and fix $0 \in I$. For each $i \in I$ choose an element $\iota_i \in G_{0i}$ such that $\iota_0 = e_0 \in G_{00}$. Given such a choice, the groupoid $\mathcal{G}$ is generated by the vertex group G_{00} and the set $\{\iota_i\}_{i\in I}$. In fact, every element in G_{ij} can be uniquely written as the product $\iota_i^{-1} g \iota_j$ for some $g \in G_{00}$. We call $\{\iota_i\}_{i\in I}$ a *star* based at $0 \in I$. From now on we will assume $\mathcal{G}$ is a connected groupoid.

Subgroupoids and quotients

A *subgroupoid* $\mathcal{H} < \mathcal{G}$ is a collection of sets $\{H_{ij}\}_{i,j\in J}$ for some non-empty $J \subset I$ where $H_{ij} \subset G_{ij}$ such that $\mathcal{H}$ is a groupoid with the inherited operation from $\mathcal{G}$. A subgroupoid $\mathcal{H} < \mathcal{G}$ is *normal* if $g^{-1}H_{ii}g \subset H_{jj}$ for all $g \in G_{ij}$. It follows that $h \mapsto g^{-1}hg$ is an isomorphism of the vertex groups H_{ii} and H_{jj} of $\mathcal{H}$.

Of interest to this chapter will be connected normal subgroupoids of connected groupoids. To that end, let $\mathcal{H}$ be a connected normal subgroupoid of $\mathcal{G}$. Construct the *quotient groupoid* $\mathcal{G}/\mathcal{H}$ to be a group, that is, a groupoid with one object, as follows. We say for elements $g_1, g_2 \in \mathcal{G}$ that $g_1 \sim g_2$ if there exists $h_1, h_2 \in \mathcal{H}$ such that $g_1 = h_1 g_2 h_2$. We write $[g_1]$ for the equivalence class of g_1 and we call the equivalence classes the *cosets* of $\mathcal{H}$ in $\mathcal{G}$. We define the elements of the quotient $\mathcal{G}/\mathcal{H}$ to be the cosets of $\mathcal{H}$ in $\mathcal{G}$. Define an operation on the cosets by $[g_1][g_2] = [g_1 h g_2]$ for some $h \in \mathcal{H}$. This is a well defined group operation on $\mathcal{G}/\mathcal{H}$.

Automorphisms of groupoids

We now restrict our attention to connected groupoids $\mathcal{G}$ whose object sets are finite. Let $\mathcal{G}$ and $\mathcal{H}$ be groupoids with finite object sets I and J respectively. A *morphism* $\phi : \mathcal{G} \to \mathcal{H}$ is a function $\hat{\phi} : I \to J$ together with a family of functions $\hat{\phi}_{ij} : G_{ij} \to H_{\hat{\phi}(i)\hat{\phi}(j)}$ for all $i, j \in I$ such that $\hat{\phi}_{ij}(g_1)\hat{\phi}_{jk}(g_2) = \hat{\phi}_{ik}(g_1 g_2)$ for all $g_1 \in G_{ij}$, $g_2 \in G_{jk}$. It follows that $\hat{\phi}_{ii}(e_i) = e_{\hat{\phi}(i)}$ and that $\hat{\phi}_{ji}(g^{-1}) = \hat{\phi}_{ij}(g)^{-1}$ for all $i, j \in I$ an $g \in G_{ij}$. We will suppress the subscripts and simply write $\hat{\phi}_{ij}(g)$ as $\phi(g)$.

An *automorphism* of $\mathcal{G}$ is a morphism $\phi : \mathcal{G} \to \mathcal{G}$ with a two-sided inverse. The set of automorphisms of $\mathcal{G}$ forms a group under composition, denoted $\mathrm{Aut}(\mathcal{G})$.

We define the *pure automorphism group* of $\mathcal{G}$ by

$$\mathrm{PAut}(\mathcal{G}) := \{\phi \in \mathrm{Aut}(\mathcal{G}) : \hat{\phi}(i) = i \text{ for all } i \in I\}.$$

If $\mathcal{H} < \mathcal{G}$ is a normal subgroupoid and $\phi \in \mathrm{Aut}(\mathcal{G})$ such that $\phi(\mathcal{H}) \subset \mathcal{H}$, then ϕ induces an automorphism $\overline{\phi} \in \mathrm{Aut}(\mathcal{G}/\mathcal{H})$ given by $\overline{\phi}([a]) = [\phi(a)]$. We can now define the *liftable automorphism group* $\mathrm{LAut}_{\mathcal{H}}(\mathcal{G}) < \mathrm{PAut}(\mathcal{G})$ by

$$\mathrm{LAut}_{\mathcal{H}}(\mathcal{G}) = \{\phi \in \mathrm{PAut}(\mathcal{G}) : \phi(\mathcal{H}) = \mathcal{H} \text{ and } \overline{\phi} = \mathrm{id} \in \mathrm{Aut}(\mathcal{G}/\mathcal{H})\}.$$

As the notation suggests, this subgroup is linked to the liftable mapping class group. The next lemma, due to Ghaswala, gives the index of $\mathrm{LAut}_{\mathcal{H}}(\mathcal{G})$ in $\mathrm{PAut}(\mathcal{G})$ [28, Lemma 6.1.3].

Lemma 4.2.1 (Ghaswala)**.** *Let $\mathcal{G}$ be a connected groupoid with object set I and let $\mathcal{H}$ be a connected normal subgroupoid. Let $G = G_{00}$ and $H = H_{00}$. Suppose G is finitely generated and H is finite index in G. Then $\mathrm{LAut}_{\mathcal{H}}(\mathcal{G})$ is finite index in $\mathrm{PAut}(\mathcal{G})$.*

In the next section we define the groupoid of most importance to this chapter.

4.2.2 The fundamental groupoid

Here, we give an introduction to the *fundamental groupoid* of a topological space X. In practice, we are only concerned with the case where X is a surface with boundary. For a deeper study of this topic see Brown [17, Chapter 6] and Higgens [33, Chapter 6].

Let X be a topological space and $A \subset X$ a subset. The fundamental groupoid $\pi_1(X, A)$ is the set of homotopy classes of arcs whose endpoints are in A. The partial operation is defined by concatenation of arcs and in particular the object set of $\pi_1(X, A)$ is A. When A is a single point $x \in X$ we recover the standard fundamental group $\pi_1(X, x)$.

Now, if $f : X \to X$ is a homeomorphism preserving the subset A then it induces an automorphism $f_* : \pi_1(X, A) \to \pi_1(X, A)$. Furthermore, if two homeomorphisms are homotopic relative to A then the induced groupoid automorphisms are equal.

The fundamental groupoid arises from the study of covering spaces. Indeed, let $p : \widetilde{X} \to X$ be a covering space and let $x \in X$ be the basepoint of the fundamental group $\pi_1(X, x)$. An interesting and natural object to study is the fundamental groupoid $\pi_1(\widetilde{X}, p^{-1}(x))$.

We can define source and target maps by $s([\hat{\alpha}]) = \hat{\alpha}(0)$ and $t([\hat{\alpha}]) = \hat{\alpha}(1)$ for any arc $\hat{\alpha}$. For any $x \in A$, we define the sets

$$S(x) := \{g \in \pi_1(X, A) : s(g) = x\} \quad \text{and} \quad T(x) := \{g \in \pi_1(X, A) : t(g) = x\}.$$

The next lemma gives us the first tool in studying groupoids arising from covering spaces.

Lemma 4.2.2. *Let $p : \widetilde{X} \to X$ be a covering space, $A \subset X$ a subset, and $\pi_1(\widetilde{X}, p^{-1}(A))$, $\pi_1(X, A)$ the corresponding fundamental groupoids. The maps $p_* : S(\tilde{x}) \to S(p(\tilde{x}))$ and $p_* : T(\tilde{x}) \to T(p(\tilde{x}))$ are bijections for all $\tilde{x} \in p^{-1}(A)$.*

The result follows from the path and homotopy lifting properties for covering spaces and the details are left to the reader. Let $p : \widetilde{X} \to X$ be a regular, finite-sheeted, covering space with deck group D. Let $A = \{x_0, \dots, x_{m-1}\} \subset X$, and $B = p^{-1}(A) \subset \widetilde{X}$. For each $i \in \{0, \dots, m-1\}$, choose $\tilde{x}_i \in p^{-1}(x_i)$. Let $\widetilde{A} = \{\tilde{x}_0, \dots, \tilde{x}_{m-1}\}$. We now define the following groupoids.

$$\mathcal{G} := \pi_1(X, A), \quad \mathcal{H} := p_*(\pi_1(\widetilde{X}, \widetilde{A})), \quad \mathcal{K} := \pi_1(\widetilde{X}, B).$$

The deck group D acts freely on B and so D injects into $\mathrm{Aut}(\mathcal{K})$. In order to make the notation of the following definition simpler, we consider D as a subgroup of $\mathrm{Aut}(\mathcal{K})$.

$$\mathrm{SAut}(\mathcal{K}) := \mathrm{PAut}(\mathcal{K}) \cap C_{\mathrm{Aut}(\mathcal{K})}(D) = \{\tilde{\phi} \in \mathrm{PAut}(\mathcal{K}) : [\tilde{\phi}, d] = 1 \text{ for all } d \in D\}.$$

The covering space p is regular so $\mathcal{H}$ is a normal subgroupoid of $\mathcal{G}$. Recall the definition of the subgroup $\mathrm{LAut}_{\mathcal{H}}(\mathcal{G}) < \mathrm{Aut}(\mathcal{G})$ from Section 4.2.1. The next lemma is a kind of Birman-Hilden theorem for groupoid automorphisms. This leads us to reuse the notation for the homomorphism between symmetric and liftable mapping class groups. To that end, we define a map $\Pi : \mathrm{SAut}(\mathcal{K}) \to \mathrm{LAut}_{\mathcal{H}}(\mathcal{G})$ by $\Pi(\tilde{\phi})(g) = p_*\tilde{\phi}(\tilde{g})$ where $\tilde{g}$ is any element of $p_*^{-1}(g)$.

Lemma 4.2.3. *The group homomorphism*

$$\Pi : \mathrm{SAut}(\mathcal{K}) \to \mathrm{LAut}_{\mathcal{H}}(\mathcal{G})$$

is an isomorphism.

Throughout this proof, we will use without mention the fact that $p_* : \mathcal{K} \to \mathcal{G}$ satisfies the conditions from Lemma 4.2.2. Furthermore, given an element $g \in \mathcal{G}$ we call the elements of $p_*^{-1}(g)$ the lifts of g.

Proof of Lemma 4.2.3. First we prove that this makes sense as a well defined set map. Suppose $\tilde{g}_1$ and $\tilde{g}_2$ are two lifts of $g \in \mathcal{G}$. There exists some $d \in D$ such that $\tilde{g}_1 = d(\tilde{g}_2)$. It follows that for $\tilde{\phi} \in \mathrm{SAut}(\mathcal{K})$

$$p_*\tilde{\phi}(\tilde{g}_1) = p_*\tilde{\phi}d(\tilde{g}_2) = p_*d\tilde{\phi}(\tilde{g}_2) = p_*\tilde{\phi}(\tilde{g}_2).$$

We now let $\tilde{g}, \tilde{h}$ be lifts of g, h respectively such that $\tilde{g}\tilde{h} \in \mathcal{K}$ and is a lift of gh. We have

$$\Pi(\tilde{\phi})(gh) = p_*(\tilde{\phi})(\tilde{g}\tilde{h}) = p_*\tilde{\phi}(\tilde{g})p_*\tilde{\phi}(\tilde{h}) = \Pi(\tilde{\phi})(g)\Pi(\tilde{\phi})(h),$$

hence $\Pi(\tilde{\phi})$ is a well defined groupoid morphism. To see that it is an isomorphism we note that

$$\Pi(\tilde{\phi})\Pi(\tilde{\phi}^{-1})(g) = \Pi(\tilde{\phi})(p_*\tilde{\phi}^{-1}(\tilde{g})) = p_*\tilde{\phi}(\widetilde{p_*\tilde{\phi}^{-1}(\tilde{g})}) = p_*\tilde{\phi}\tilde{\phi}^{-1}(\tilde{g}) = p_*(\tilde{g}) = g,$$

so $\Pi(\tilde{\phi}^{-1})$ is a right inverse for $\Pi(\tilde{\phi})$. It can also been seen that $\Pi(\tilde{\phi}^{-1})$ is a left inverse, and so $\Pi(\tilde{\phi})$ is an isomorphism. Hence, Π is a well defined map from $\mathrm{SAut}(\mathcal{K})$ to $\mathrm{Aut}(\mathcal{G})$.

Let $g \in \mathcal{G}$ and let $\tilde{\phi} \in \mathrm{SAut}(\mathcal{K})$ as above. In particular $\tilde{\phi}$ is pure and so $s(\tilde{\phi}(\tilde{g})) = s(\tilde{g})$. Now,

$$s(g) = s(p_*(\tilde{g})) = s(p_*\tilde{\phi}(\tilde{g})) = s(\Pi(\tilde{\phi})(g)),$$

therefore $\Pi(\tilde{\phi}) \in \mathrm{PAut}(\mathcal{G})$. We still need to show that $\Pi(\tilde{\phi}) \in \mathrm{LAut}_{\mathcal{H}}(\mathcal{G})$. To that end, let $g \in \mathcal{G}$. We will show that there exists some $h \in \mathcal{H}$ such that $\Pi(\tilde{\phi})(g) = hg$, hence $\tilde{\phi}$ induces the identity element of $\mathrm{Aut}(\mathcal{G}/\mathcal{H})$. Suppose then that $\tilde{g}$ is a lift of g such that $s(\tilde{g}) \in \widetilde{A}$ and that $h = p_*(\tilde{\phi}(\tilde{g})\tilde{g}^{-1})$. By definition of $\mathcal{H}$ we have that $h \in \mathcal{H}$ and so

$$hg = p_*(\tilde{\phi}(\tilde{g})\tilde{g}^{-1})g = p_*\tilde{\phi}(\tilde{g})p_*\tilde{g}^{-1}p_*\tilde{g} = p_*\tilde{\phi}(g) = \Pi(\tilde{\phi})(g).$$

This completes the proof that $\Pi : \mathrm{SAut}(\mathcal{K}) \to \mathrm{LAut}_{\mathcal{H}}(\mathcal{G})$ is a well defined set map. Furthermore, we have that

$$\Pi(\tilde{\phi}_1\tilde{\phi}_2)(g) = p_*(\tilde{\phi}_1\tilde{\phi}_2)(\tilde{g}) = p_*\tilde{\phi}_1 p_*\tilde{\phi}_2(\tilde{g}) = \Pi(\tilde{\phi}_1)\Pi(\tilde{\phi}_2)(g),$$

and so Π is a group homomorphism.

Given an element $g \in \mathcal{G}$ we define $\tilde{g}_x$ to be the unique lift of g such that $s(\tilde{g}_x) = x$. We will now construct a set map which we optimistically label $\Pi^{-1} : \mathrm{LAut}_{\mathcal{H}}(\mathcal{G}) \to \mathrm{SAut}(\mathcal{K})$ and define it by $\Pi^{-1}(\phi)(k) = (\widetilde{\phi p_*(k)})_{s(k)}$. In particular this means that k and $\Pi^{-1}(\phi)(k)$ start at the same basepoint. We will now show that $t(k) = t(\Pi^{-1}(\phi)(k))$, that is, k and $\Pi^{-1}(\phi)(k)$ terminate at the same basepoint.

Let $k \in \mathcal{K}$ such that $s(k) = d_1(\tilde{x}_i)$ and $t(k) = d_2(\tilde{x}_j)$ for $d_1, d_2 \in D$ and $\tilde{x}_i, \tilde{x}_j \in A$. By definition we have that $s(p_*(k)) = x_i$ and $t(p_*(k)) = x_j$. Now, since $\phi \in \mathrm{PAut}(\mathcal{G})$ it follows that $s(\phi p_*(k)) = x_i$ and $t(\phi p_*(k)) = x_j$ also. Furthermore, since $\phi \in \mathrm{LAut}_{\mathcal{H}}(\mathcal{G})$ there exists $h_1, h_2 \in \mathcal{H}$ such that $\phi p_*(k) = h_1 p_*(k) h_2$. Now, define $\tilde{h}_1$ and $\tilde{h}_2$ to be the unique lifts of h_1 and h_2 such that $s(\tilde{h}_1) = t(\tilde{h}_1) = \tilde{x}_i$ and $s(\tilde{h}_2) = t(\tilde{h}_2) = \tilde{x}_j$ as in the definition of $\mathcal{H}$. We now have

$$p_*(\Pi^{-1}(\phi)(k)) = \phi p_*(k) = h_1 p_*(k) h_2 = p_*(\tilde{h}_1)p_*(k)p_*(\tilde{h}_2) = p_*(d_1(\tilde{h}_1)kd_2(\tilde{h}_2))$$

and $s(\Pi^{-1}(\phi)(k)) = s(d_1(\tilde{h}_1)) = d_1(\tilde{x}_i)$, so $\Pi^{-1}(\phi)(k) = d_1(\tilde{h}_1)kd_2(\tilde{h}_2)$. We also have that $t(\Pi^{-1}(\phi)(k)) = t(d_1(\tilde{h}_1)kd_2(\tilde{h}_2)) = t(d_2(\tilde{h}_2)) = d_2(\tilde{x}_j)$. Thus, we have shown that k and $\Pi^{-1}(\phi)(k)$ start and terminate at the same basepoints.

In order to show that $\Pi^{-1}(\phi) : \mathcal{K} \to \mathcal{K}$ is a groupoid homomorphism, we choose $k, l \in \mathcal{K}$ such that the product kl is well defined. Since $t(k) = t(\Pi^{-1}(\phi)(k))$ and $s(l) = s(\Pi^{-1}(\phi)(l))$, the product $\Pi^{-1}(\phi)(k)\Pi^{-1}(\phi)(l)$ is well defined in $\mathcal{K}$. Therefore,

$$p_*(\Pi^{-1}(\phi)(k)\Pi^{-1}(\phi)(l)) = \phi p_*(k)\phi p_*(l) = \phi p_*(kl) = p_*(\Pi^{-1}(\phi)(kl)).$$

Since $s(\Pi^{-1}(\phi)(k)\Pi^{-1}(\phi)(l)) = s(\Pi^{-1}(\phi)(kl))$ we can conclude that $\Pi^{-1}(\phi)(k)\Pi^{-1}(\phi)(l)$ is equal to $\Pi^{-1}(\phi)(kl)$ and hence $\Pi^{-1}(\phi)$ is indeed a groupoid homomorphism. Similar to the argument above, it can be shown that $\Pi^{-1}(\phi^{-1})$ is a right and left inverse of $\Pi^{-1}(\phi)$, therefore both are automorphisms of $\mathcal{K}$. It has already been shown that $\Pi^{-1}(\phi)$ preserves source and target basepoints and so it follows that $\Pi^{-1}(\phi) \in \mathrm{PAut}(\mathcal{K})$.

For any $d \in D$ we have that

$$p_*(\Pi^{-1}(\phi)d(k)) = \phi p_* d(k) = \phi p_*(k) = p_*(\Pi^{-1}(\phi)(k)) = p_*(d\Pi^{-1}(\phi)(k)).$$

Furthermore, $s(\Pi^{-1}(\phi)d(k)) = s(d(k)) = s(d\Pi^{-1}(\phi)(k))$ so $\Pi^{-1}(\phi)d = d\Pi^{-1}(\phi)$, that is, $\Pi^{-1}(\phi) \in \mathrm{SAut}(\mathcal{K})$. This completes the proof that $\Pi^{-1} : \mathrm{LAut}_{\mathcal{H}}(\mathcal{G} \to \mathrm{SAut}(\mathcal{K})$ is a well defined set map.

It only remains to show that Π^{-1} is in fact an inverse to Π, as the notation suggests. We have that

$$\Pi(\Pi^{-1}(\phi))(g) = p_*(\Pi^{-1}(\phi))(\tilde{g}) = p_*(\widetilde{\phi p_*(\tilde{g})})_{s(\tilde{g})} = \phi p_*(\tilde{g}) = \phi(g)$$

and so $\Pi\Pi^{-1} = \mathrm{id}$. Furthermore,

$$p_*(\Pi^{-1}(\Pi(\tilde{\phi}))(k)) = \Pi(\tilde{\phi})p_*(k) = p_*\tilde{\phi}(\widetilde{p_*(k)}) = p_*\tilde{\phi}(k),$$

and since $s(\tilde{\phi}(k)) = s(k) = s(\Pi^{-1}(\Pi(\tilde{\phi}))(k))$ it follows that $\Pi^{-1}\Pi(\tilde{\phi})(k) = \tilde{\phi}(k)$, that is, $\Pi^{-1}\Pi = \mathrm{id}$. This completes the proof that $\Pi : \mathrm{SAut}(\mathcal{K}) \to \mathrm{LAut}_{\mathcal{H}}(\mathcal{G})$ is an isomorphism. ■

4.2.3 The liftable mapping class group

In the last section we looked at two subgroups of groupoid automorphisms that arise naturally from the study of covering spaces of topological spaces. We can now apply this knowledge to the regular, finite-sheeted covers of surfaces with boundary that are of interest to this chapter. To that end, let Σ be a surface with $m > 0$ boundary components and let each boundary component of Σ contain a basepoint x_i where $i \in \{0, \ldots, m-1\}$. Consider the fundamental groupoid $\pi_1(\Sigma, \{x_0, \ldots, x_{m-1}\})$. It easy to see that the natural group homomorphism

$$\mathrm{Mod}(\Sigma) \to \mathrm{PAut}(\pi_1(\Sigma, \{x_0, \ldots, x_{m-1}\})).$$

is injective. For a mapping class in $\mathrm{Mod}(\Sigma)$ represented by the homeomorphism f we write f_* for the induced automorphism on the groupoid.

Suppose the cover $p : \widetilde{\Sigma} \to \Sigma$ is branched at finitely many points $\mathcal{B} \subset \Sigma$. We will write $\widetilde{\Sigma}^\circ = \widetilde{\Sigma} \setminus p^{-1}(\mathcal{B})$ and $\Sigma^\circ = \Sigma \setminus \mathcal{B}$ and abusing notation, denote the resulting unbranched cover $p : \widetilde{\Sigma}^\circ \to \Sigma^\circ$. As discussed in Chapter 1 we have an isomorphism $\mathrm{LMod}(\Sigma, \mathcal{B}) \cong \mathrm{LMod}(\Sigma^\circ)$, and therefore by Theorem 1.5.1 we have $\mathrm{SMod}(\widetilde{\Sigma}) \cong \mathrm{SMod}(\widetilde{\Sigma}^\circ)$. Some results are easier to prove in the unbranched case and these isomorphisms allow us to do so without loss of generality.

As above suppose Σ° has m boundary components and let $A = \{x_0, x_1, \ldots, x_{m-1}\} \subset \partial\Sigma^\circ$ be such that each boundary component contains exactly one of the x_i. For each x_i, choose a point $\tilde{x}_i \in p^{-1}(x_i)$ and let $\widetilde{A} = \{\tilde{x}_0, \tilde{x}_1, \ldots, \tilde{x}_{m-1}\} \subset \partial\widetilde{\Sigma}^\circ$. Let $B = p^{-1}(A)$ and denote the fundamental groupoids $\mathcal{G} = \pi_1(\Sigma^\circ, A)$, $\mathcal{H} = p_*\pi_1(\widetilde{\Sigma}^\circ, \widetilde{A})$, and $\mathcal{K} = \pi_1(\widetilde{\Sigma}^\circ, B)$ as in Section 4.2.2.

We have a useful commutative diagram

$$\begin{tikzcd}
\mathrm{SMod}(\widetilde{\Sigma}) \arrow[r, "\Psi"] \arrow[d, "\Pi"] & \mathrm{SAut}(\mathcal{K}) \arrow[d, "\Pi"] \\
\mathrm{LMod}(\Sigma, \mathcal{B}) \arrow[r, "\Psi"] & \mathrm{LAut}_{\mathcal{H}}(\mathcal{G})
\end{tikzcd}$$

where the horizontal injections are given by the action of the mapping class group on the fundamental groupoid discussd above, and the vertical maps are the Birman-Hilden isomorphisms from Theorem 1.5.1 and Lemma 4.2.3. Note that the horizontal maps are actually the composition of the action on the fundamental groupoid with the isomorphisms $\mathrm{SMod}(\widetilde{\Sigma}) \cong \mathrm{SMod}(\widetilde{\Sigma}^\circ)$ and $\mathrm{LMod}(\Sigma, \mathcal{B}) \cong \mathrm{LMod}(\Sigma^\circ)$.

Theorem 4.2.4 (Ghaswala)**.** *Using the notation introduced above, the liftable mapping class group* $\mathrm{LMod}(\Sigma, \mathcal{B})$ *is given by*

$$\{[f] \in \mathrm{Mod}(\Sigma, \mathcal{B}) : f_* \in \mathrm{LAut}_{\mathcal{H}}(\mathcal{G})\}.$$

The proof of Theorem 4.2.4 can be found in Section 6.2 of Ghaswala's thesis [28]. Heuristically, this is true as being in $\mathrm{LAut}_{\mathcal{H}}(\mathcal{G})$ ensures a trivial permutation of $\widetilde{A}$, hence B. This, in turn, ensures that the mapping class $[f]$ preserves the boundary pointwise. The result gives the following useful corollary which we will use in the proof of Theorem 1.5.2.

Corollary 4.2.5. *Suppose Σ has one boundary component. Choose a basepoint $x_0 \in \partial\Sigma^\circ$ and some $\tilde{x}_0 \in p^{-1}(x)$. Then*

$$\mathrm{LMod}(\Sigma, \mathcal{B}) = \{[f] \in \mathrm{Mod}(\Sigma, \mathcal{B}) : qf_* = q\}$$

where $q : \pi_1(\Sigma^\circ, x_0) \to \pi_1(\Sigma^\circ, x_0)/p_*\pi_1(\widetilde{\Sigma}^\circ, \tilde{x}_0)$ is the quotient map.

We also arrive at a direct way to check whether or not a mapping class lifts in the case where Σ has $m > 1$ boundary components. Choose a point $x_0 \in \partial\Sigma$ and a lift $\tilde{x}_0 \in p^{-1}(x_0)$. Furthermore, choose a generating set $\{\gamma_1, \ldots, \gamma_N\}$ of $\pi_1(\Sigma^\circ, x_0)$. The cover is regular and so we let

$$q : \pi_1(\Sigma^\circ, x_0) \to \pi_1(\Sigma^\circ, x_0)/p_*(\pi_1(\widetilde{\Sigma}^\circ, \tilde{x}_0)) \cong D$$

be the quotient map as in the statement of Corollary 4.2.5, where D is the deck group. Let x_i be a point on a unique boundary component for all $i \in \{1, \ldots, m-1\}$. Let the set $\{\iota_i\}_{i=1}^{m-1}$ be a star in the fundamental groupoid $\pi_1(\Sigma^\circ, \{x_0, \ldots, x_{m-1}\})$ such that ι_i is represented by an arc starting at x_0 and ending at x_i. Recall from Section 4.2.1 that $\pi_1(\Sigma^\circ, x_0)$ and $\{\iota_i\}_{i=1}^{m-1}$ generate the fundamental groupoid $\mathcal{G} = \pi_1(\Sigma^\circ, \{x_0, \ldots, x_{m-1}\})$. Given an element $[f] \in \mathrm{Mod}(\Sigma, \mathcal{B})$ we have that $f_*(\iota_j) = a_j\iota_j$ for some a_j in the fundamental group $\pi_1(\Sigma^\circ, x_0)$ [28].

Proposition 4.2.6 (Ghaswala)**.** *A mapping class $[f]$ is in $\mathrm{LMod}(\Sigma)$ if and only if;*

1. *for all i we have $qf_*(\gamma_i) = q(\gamma_i)$, and*
2. *if $f_*(\iota_j) = a_j\iota_j$ then $a_j \in \ker q$ for all j.*

Proof. Choose a lift $\tilde{x}_0 \in p^{-1}(x_0)$. For all i choose lifts $\tilde{\iota}_i$ of ι_i such that $s(\tilde{\iota}_i) = \tilde{x}_0$ and let $\tilde{x}_i = t(\tilde{\iota}_i)$. Define the groupoids

$$\mathcal{G} = \pi_1(\Sigma^\circ, \{x_0, \ldots, x_{m-1}\}) \quad \text{and} \quad \mathcal{H} = p_*\pi_1(\widetilde{\Sigma}^\circ, \{\tilde{x}_0, \ldots, \tilde{x}_{m-1}\}),$$

as before. Theorem 4.2.4 states that

$$[f] \in \mathrm{LMod}(\Sigma) \quad \Leftrightarrow \quad f_* \in \mathrm{LAut}_{\mathcal{H}}(\mathcal{G}).$$

We notice that $qf_*(\gamma_i) = q(\gamma_i)$ for all i if and only if f_* acts trivially on the cosets of $p_*\pi_1(\widetilde{\Sigma}^\circ, \tilde{x}_0)$ in $\pi_1(\Sigma^\circ, x_0)$. Furthermore, if $a_j \in \ker q$ then $a_j \in p_*\pi_1(\widetilde{\Sigma}^\circ, \tilde{x}_0)$ for each j. The result follows from the fact that $\mathcal{G}$ is generated by the sets $\{\gamma_1, \ldots, \gamma_N\}$ and $\{\iota_1, \ldots, \iota_{m-1}\}$. ■

4.3 Classification results

In this section we give proofs of Theorems 1.5.2 and 1.5.3. Recall that these results give the necessary and sufficient conditions for the liftable and symmetric mapping class

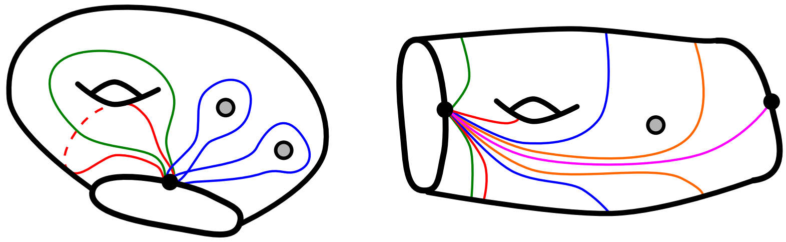

Figure 4.2: A basic generating set of $\pi_1(\Sigma^1_{1,2}, x_0)$ and an essential generating set of $\pi_1(\Sigma^2_{1,1}, x_0)$.

group to coincide with their respective mapping class groups. In the first instance we make extensive use of the results concerning the fundamental groupoid discussed in the last section. We prove the second classification by showing that if $\mathrm{SMod}(\widetilde{\Sigma}) = \mathrm{Mod}(\widetilde{\Sigma})$ then the action on the first homology of $\widetilde{\Sigma}$ provides restrictions on the possible covering spaces.

4.3.1 The case where everything lifts

For any element c of $\pi_1(\Sigma^\circ, x_0)$ we will abuse notation by writing c for a loop in Σ° representing c. Furthermore, if c is a simple loop, we will write $\mathbf{c}$ for the corresponding unique free isotopy class of simple closed curves in Σ°.

If $m > 1$ let $\{\iota_1, \ldots, \iota_{m-1}\}$ be a star of the fundamental groupoid $\pi_1(\Sigma^\circ, A)$ based at x_0 where each representative of ι_i terminates at a point x_i on a unique boundary component ∂_i of Σ°. We now give two types of generating set for the fundamental group of a surface with boundary.

Basic and essential generating sets

If $m = 1$ we call a finite generating set $\mathcal{S} = \{a_1, b_1, \ldots, a_g, b_g, \gamma_1, \ldots, \gamma_n\}$ of $\pi_1(\Sigma^\circ, x_0)$ a *basic* generating set if each element is represented by a simple loop, each γ_i bounds a subsurface homeomorphic to a punctured disk, and

$$i(\mathbf{a}_i, \mathbf{a}_j) = i(\mathbf{b}_i, \mathbf{b}_j) = 0 \text{ for all } i, j,$$

$$i(\mathbf{a}_i, \mathbf{b}_j) = \begin{cases} 1 \text{ if } i = j, & \text{and} \\ 0 \text{ otherwise.} \end{cases}$$

If $m > 1$ we call a finite generating set $\mathcal{S}$ of $\pi_1(\Sigma^\circ, x_0)$ *essential* with respect to the star $\{\iota_i\}_{i=1}^{m-1}$ if for all $c \in \mathcal{S}$ there exists an $i \in \{1, \ldots, m-1\}$ such that $i(\iota_i, c) = 1$. Here, the name comes from the fact that each generator has essential intersection with at least one element of the star.

The fact that basic generating sets exist is well known; see [31, Section 1.2]. The next lemma shows the existence of essential generating sets of fundamental groups.

Lemma 4.3.1. *Let Σ° be a surface with $m > 1$ boundary components and let $x_0 \in \partial\Sigma^\circ$. There exists an essential generating set $\mathcal{S}$ of $\pi_1(\Sigma^\circ, x_0)$ with respect to a star $\{\iota_i\}_{i=1}^{m-1}$.*

Proof. It follows from [31, Section 1.2] that we have a generating set

$$\{a_1, b_1, \ldots, a_g, b_g, c_1, \ldots, c_{m-1}, \gamma_1 \ldots \gamma_n\},$$

where each c_i is represented by a simple loop around a boundary component other than the unique component ∂_0 containing x_0 and each γ_i is represented by a simple loop around a puncture. Furthermore, we can choose elements c_i such that $i(\iota_i, \mathbf{c}_i) = 1$ for all $i \in \{1, \ldots, m-1\}$. To see this we will abuse notation by writing ι_i for a representative of the element of the star. The only isotopy class of simple closed curve represented in a regular neighbourhood of $\iota_i \cup \partial_i$ contains the simple loop around a boundary component which we can define to be a representative of $\mathbf{c}_i$.

For each i there exist representatives of the groupoid elements γ_i and ι_j that do not intersect for any choice of j. If $\mathbf{d}$ is the free isotopy class of curves corresponding to $\gamma_i c_1$ it follows that $i(\iota_1, \mathbf{d}) = 1$ for all i. Similar arguments can be used to show that

$$\{a_1c_1, b_1c_1, \ldots, a_gc_1, b_gc_1, c_1, \ldots, c_{m-1}, \gamma_1c_1 \ldots \gamma_nc_1\}$$

is an essential generating with respect to the star $\{\iota_i\}_{i=1}^{m-1}$. ■

We now move on to proving Theorem 1.5.2. As such, we return to the setting of the original branched cover $p : \widetilde{\Sigma} \to \Sigma$ branched at $\mathcal{B} \subset \Sigma \setminus \partial\Sigma$. If $[f] \in \mathrm{Mod}(\Sigma, \mathcal{B})$ we will abuse notation and denote by $f_* \in \mathrm{Aut}(\pi_1(\Sigma^\circ, A))$ the automorphism induced by any representative homeomorphism for $[f]$. The abuse of notation is legal since any representative homeomorphism for $[f]$ fixes A pointwise, and isotopic homeomorphisms induce the same groupoid automorphism.

Proof of Theorem 1.5.2. We first prove that $\mathrm{LMod}(\Sigma, \mathcal{B}) = \mathrm{Mod}(\Sigma, \mathcal{B})$ if and only if $p : \widetilde{\Sigma} \to \Sigma$ is a Burau cover. Let $p : \widetilde{\Sigma}^\circ \to \mathbf{D}_n$ be the associated unbranched cover of a Burau cover. Let $x_0 \in \partial\mathbf{D}_n$ and let $\{\gamma_1, \ldots, \gamma_n\}$ be a basic generating set for $\pi_1(\mathbf{D}_n, x_0)$. Let $H_{\mathbf{c}_i} \in \mathrm{Mod}(\mathbf{D}_n)$ be a half twist whose support (a disk with two punctures) intersects all representative loops of γ_i and γ_{i+1} and is disjoint from representatives of γ_j for all $j \neq i, i+1$. If we consider each $H_{\mathbf{c}_i}$ as an automorphism of $\pi_1(\mathbf{D}_n, x_0)$ we can assume that $H_{\mathbf{c}_i}(\gamma_i) = \gamma_i\gamma_{i+1}\gamma_i^{-1}$ and $H_{\mathbf{c}_i}(\gamma_{i+1}) = \gamma_i$. From the definition of Burau covers we have that

$$qH_{\mathbf{c}_i}(\gamma_i) = q(\gamma_i\gamma_{i+1}\gamma_i^{-1}) = q(\gamma_i)q(\gamma_{i+1})q(\gamma_i^{-1}) = 1,$$

$$qH_{\mathbf{c}_i}(\gamma_{i+1}) = q(\gamma_i) = 1, \text{ and}$$

$$qH_{\mathbf{c}_i}(\gamma_j) = q(\gamma_j) = 1 \text{ for all } j \neq i, i+1.$$

By Corollary 4.2.5 we have that $H_{\mathbf{c}_i} \in \mathrm{LMod}(\mathbf{D}_n)$. It follows from the fact that the set $\{H_{\mathbf{c}_1}, \ldots, H_{\mathbf{c}_{n-1}}\}$ generates $\mathrm{Mod}(\mathbf{D}_n)$ that every mapping class lifts, that is, $\mathrm{LMod}(\mathbf{D}_n) = \mathrm{Mod}(\mathbf{D}_n)$. As discussed at the beginning of Section 4.2.3 this is equivalent to showing that $\mathrm{LMod}(\Sigma_0^1, \mathcal{B}) = \mathrm{Mod}(\Sigma_0^1, \mathcal{B})$.

To prove the other direction we first assume that Σ has $m > 1$ boundary components. Let $\{\iota_i\}_{i=1}^{m-1}$ be a star in the fundamental groupoid and let $\mathcal{S}$ be an essential generating set of $\pi_1(\Sigma^\circ, x_0)$ as in Lemma 4.3.1. Let $c \in \mathcal{S}$ and let $T_{\mathbf{c}}$ be the corresponding Dehn twist about a simple clsoed curve freely isotopic to c. Without loss of generality we have that $T_{\mathbf{c}}(\iota_i) = c\iota_i$, as an element of the fundamental groupoid for some i. If $T_{\mathbf{c}} \in \mathrm{LMod}(\Sigma^\circ)$ then from Proposition 4.2.6 it follows that $c \in p_*\pi_1(\widetilde{\Sigma}^\circ, \widetilde{x}_0)$ and since $c \in \mathcal{S}$ was arbitrary we have that every element of $\mathcal{S}$ belongs to $p_*\pi_1(\widetilde{\Sigma}^\circ, \widetilde{x}_0)$, hence the image of the quotient map is trivial. This is a contradiction and so we have shown that Σ must have a single boundary component.

We now assume that Σ has positive genus g. Let $\mathcal{S}$ be a basic generating set of $\pi_1(\Sigma^\circ, x_0)$, we denote by $\gamma_1, \dots, \gamma_n \in \mathcal{S}$ the simple loops around the removed branch points. For all $i \in \{1, \dots, n\}$ we can find an element $[f] \in \mathrm{Mod}(\Sigma^\circ)$ such that $f_*(\gamma_i) = \gamma_1$. It follows from Corollary 4.2.5 that $q(\gamma_i) = qf_*(\gamma_i) = q(\gamma_1)$ for all i.

We label the $2g$ elements of $\mathcal{S}$ that do not bound a disk by $a_1, b_1, \dots, a_g, b_g$. Each representative loop belongs to a different free isotopy class of non-separating simple closed curves and so a_i, b_i belong to the $\mathrm{Mod}(\Sigma^\circ)$-orbit of a_1 for all $i = 1, \dots, g$. Similarly, as representative loops of the elements $\gamma_1 b_1 a_1$ and $b_1 a_1$ belong to free isotopy classes of non-separating simple closed curves we have that there exist $[f], [h] \in \mathrm{Mod}(\Sigma^\circ)$ such that $f_*(a_1) = \gamma_1 b_1 a_1$ and $h_*(b_1) = b_1 a_1$. We now have that

$$q(b_1) = qh_*(b_1) = q(b_1 a_1) = q(b_1)q(a_1).$$

Hence $q(a_1)$, and therefore each $q(a_i)$ and $q(b_i)$, is equal to the identity element of the deck group D for any value of i. Furthermore, it follows that

$$q(\gamma_1) = q(\gamma_1)q(b_1)q(a_1) = q(\gamma_1 b_1 a_1) = qf_*(a_1) = q(a_1).$$

Again, this implies that the image of the quotient map is trivial, which is a contradiction. The genus of Σ must therefore be zero and, as shown above, Σ has a single boundary component, that is, Σ is a disk. We have already shown that $q(\gamma_i) = q(\gamma_1)$ for all $i = 1, \dots, n$ and so it follows that $p : \widetilde{\Sigma} \to \Sigma$ is a Burau cover.

We would also like to show that $\mathrm{LMod}(\Sigma, \mathcal{B})$ is always finite-index in $\mathrm{Mod}(\Sigma, \mathcal{B})$. Recall the definitions of the fundamental groupoids $\mathcal{H}$ and $\mathcal{G}$ from Section 4.2.3. Let $\Psi : \mathrm{Mod}(\Sigma, \mathcal{B}) \to \mathrm{PAut}(\mathcal{G})$ be the injective homomorphism given by the action of $\mathrm{Mod}(\Sigma, \mathcal{B})$ on the fundamental groupoid $\mathcal{G}$. By Theorem 4.2.4 it follows that $\Psi(\mathrm{LMod}(\Sigma, \mathcal{B}))$ is contained in $\mathrm{LAut}_{\mathcal{H}}(\mathcal{G})$. We have

$$\begin{aligned}
[\mathrm{Mod}(\Sigma, \mathcal{B}), \mathrm{LMod}(\Sigma, \mathcal{B})] &= [\Psi(\mathrm{Mod}(\Sigma, \mathcal{B})), \Psi(\mathrm{LMod}(\Sigma, \mathcal{B}))] \\
&\leq [\mathrm{PAut}(\mathcal{G}) : \mathrm{LAut}_{\mathcal{H}}(\mathcal{G})] \\
&< \infty,
\end{aligned}$$

where the last inequality is by Lemma 4.2.1. ■

While we have shown that there are infinitely many covering spaces with the property that $\mathrm{LMod}(\Sigma, \mathcal{B}) = \mathrm{Mod}(\Sigma, \mathcal{B})$, it is clear that this occurs only in a distinct minority of cases. We will see in the next section that the conditions for a covering space to satisfy $\mathrm{SMod}(\widetilde{\Sigma}) = \mathrm{SMod}(\widetilde{\Sigma})$ are even more severe.

4.3.2 The case where everything is symmetric

In this section we prove Theorem 1.5.3. In particular, we show that the symmetric mapping class group $\mathrm{SMod}(\widetilde{\Sigma})$ coincides with the mapping class group $\mathrm{Mod}(\widetilde{\Sigma})$ in a very small number of cases. To prove the result we make use of the mapping class group action on homology. Recall that the Lefschetz fixed point theorem for smooth manifolds states

$$\sum_{p \in \mathrm{fix}(f)} i(f,p) = \sum_{i=0}^{\infty} (-1)^i \operatorname{tr}(f_* : H_i(\Sigma; \mathbb{Q}) \to H_i(\Sigma; \mathbb{Q})).$$

where $i(f,p)$ is the index of the fixed point p of the homeomorphism f [48]. We will apply this result to our context of surfaces with boundary.

Lemma 4.3.2. *Let Σ be an oriented surface with boundary. Let f be a finite-order, orientation-preserving homeomorphism of Σ. Then the fixed points of f are isolated and the number of fixed points is equal to*

$$1 - \operatorname{tr}(f_* : H_1(\Sigma; \mathbb{Z}) \to H_1(\Sigma; \mathbb{Z})).$$

Proof. We will first prove that the fixed points are isolated. Let f have order k and let μ be a Riemannian metric on Σ. Define the Riemannian metric

$$\overline{\mu} := \sum_{i=1}^{k} (f^k)^* \mu.$$

Then $f^*\overline{\mu} = \overline{\mu}$ and so f is an isometry. Since f is orientation-preserving its fixed points must be isolated. Let $p \in \Sigma$ be such a fixed point and let $T_p\Sigma \cong \mathbb{R}^2$ be the tangent space. Now, all orientation-preserving isometries of $\mathbb{R}^2$ that fix the origin are rotations about the origin. We therefore have that f induces a rotation $T_p\Sigma \to T_p\Sigma$ and so $i(f,p) = 1$.

For a surface with boundary, $H_i(\Sigma; \mathbb{Q}) \cong \{0\}$ for all $i \geq 2$. Furthermore, $H_0(\Sigma; \mathbb{Q}) \cong \mathbb{Q}$ and $f_* : H_0(\Sigma; \mathbb{Q}) \to H_0(\Sigma; \mathbb{Q})$ is the identity map. It follows that $\operatorname{tr}(f_* : H_0(\Sigma; \mathbb{Q}) \to H_0(\Sigma; \mathbb{Q})) = 1$. Note that since the first homology group is free abelian we may replace the coefficients with $\mathbb{Z}$. Finally, since the index of each fixed point is 1, we have that the number of fixed points is equal to

$$1 - \operatorname{tr}(f_* : H_1(\Sigma; \mathbb{Z}) \to H_1(\Sigma; \mathbb{Z}))$$

completing the proof. ■

Corollary 4.3.3. *Suppose Σ is an orientable surface of genus g with $m \geq 1$ boundary components other than a disk or an annulus. Let $f \in \mathrm{Homeo}^+(\Sigma)$ be a finite-order, orientation-preserving homeomorphism of Σ. Then f acts non-trivially on $H_1(\Sigma; \mathbb{Z})$.*

Proof. If f acts trivially on $H_1(\Sigma; \mathbb{Z})$ then $\mathrm{tr}(f_* : H_1(\Sigma; \mathbb{Z}) \to H_1(\Sigma; \mathbb{Z})) = 2g+m-1$ by choice of a natural basis of $H_1(\Sigma; \mathbb{Z})$. By Lemma 4.3.2 we must have $1-(2g+m-1) \geq 0$ and so $2g + m \leq 2$. This only occurs when $g = 0$ and $m = 1, 2$, or equivalently, when Σ is a disk an annulus. ■

The next result shows that a hyperelliptic involution has a unique action on homology up to conjugation. The proof follows from Lemma 4.3.2 and an argument similar to that of [27, Proposition 7.15].

Lemma 4.3.4. *Let Σ be a surface of genus $g \geq 1$ with a single boundary component. Suppose $f_1, f_2 \in \mathrm{Homeo}^+(\Sigma)$ are order 2 homeomorphisms such that $(f_1)_* = (f_2)_* = -I : H_1(\Sigma; \mathbb{Z}) \to H_1(\Sigma; \mathbb{Z})$. Then f_1 and f_2 are conjugate in $\mathrm{Homeo}^+(\Sigma)$.*

Throughout the proof of Theorem 1.5.3 we will repeatedly use the fact that if $\mathbf{c}$ is an isotopy class of simple closed curves then every power of the Dehn twist $T_{\mathbf{c}}$ is an element of $\mathrm{SMod}(\widetilde{\Sigma})$ if and only if $d(\mathbf{c}) = \mathbf{c}$ for all $d \in D$, where D is the deck group.

Proof of Theorem 1.5.3. We start by proving that $\mathrm{SMod}(\widetilde{\Sigma}) = \mathrm{Mod}(\widetilde{\Sigma})$ in the three cases stated in the theorem. First, if $\widetilde{\Sigma}$ is a disk then $\mathrm{SMod}(\widetilde{\Sigma}) = \mathrm{Mod}(\widetilde{\Sigma})$ trivially. If $\widetilde{\Sigma}$ is an annulus, let $\mathbf{c}$ be the unique unoriented isotopy class of an essential simple closed curve. For every homeomorphism f of $\widetilde{\Sigma}$ we have that $f(\mathbf{c}) = \mathbf{c}$. Since $\mathrm{Mod}(\widetilde{\Sigma}) = \langle T_{\mathbf{c}} \rangle$ it follows that $\mathrm{SMod}(\widetilde{\Sigma}) = \mathrm{Mod}(\widetilde{\Sigma})$. Finally, suppose $\widetilde{\Sigma}$ is a torus with a single boundary component, and let $\iota \in \mathrm{Homeo}^+(\widetilde{\Sigma})$ be a hyperelliptic involution. There exist two simple closed curves $\mathbf{a}$ and $\mathbf{b}$ whose isotopy classes are fixed by ι such that the Dehn twists $T_{\mathbf{a}}$, $T_{\mathbf{b}}$ generate $\mathrm{Mod}(\widetilde{\Sigma})$. Therefore we have that $\mathrm{SMod}(\widetilde{\Sigma}) = \mathrm{Mod}(\widetilde{\Sigma})$.

Conversely, suppose $\mathrm{SMod}(\widetilde{\Sigma}) = \mathrm{Mod}(\widetilde{\Sigma})$ and $\widetilde{\Sigma}$ is neither a disk nor an annulus. Suppose $\widetilde{\Sigma}$ is a surface of genus g with $m \geq 1$ boundary components. There is a generating set $\mathcal{S} = \{a_1, \ldots, a_{2g}, x_0, \ldots, x_{m-1}\}$ of $H_1(\widetilde{\Sigma}; \mathbb{Z})$ where each generator a_i is represented by an essential simple closed curve and each x_i is the homology class of a curve isotopic to a boundary component.

Let d be a non-trivial element of the deck group D. It must be that d preserves the unoriented isotopy class of every essential simple closed curve and so we see that $d_* : H_1(\widetilde{\Sigma}; \mathbb{Z}) \to H_1(\widetilde{\Sigma}; \mathbb{Z})$ is given by the diagonal matrix

$$\begin{bmatrix} \epsilon_1 & & \\ & \ddots & \\ & & \epsilon_{2g+m-1} \end{bmatrix}$$

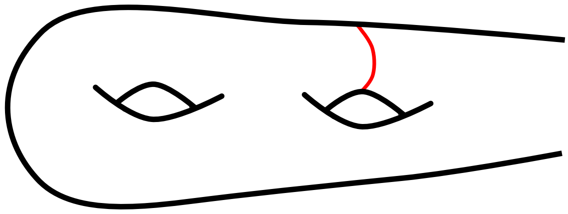

Figure 4.3: The picture shows a curve whose isotopy class is not preserved by the hyperelliptic involution for a surface with boundary Σ of genus at least two.

with respect to the generating set $\mathcal{S}$, where $\epsilon_i = \pm 1$ for all i. However, since d is orientation-preserving, it must preserve the orientation of every boundary component, therefore $\epsilon_i = 1$ for all $i > 2g$.

We now argue that $\widetilde{\Sigma}$ must have exactly one boundary component. Since d is not the identity, Lemma 4.3.3 implies that there must be at least one $i \in \{1, \ldots, 2g\}$ such that $\epsilon_i = -1$. If $m \geq 2$ then there is at least one element in $\mathcal{S}$ that is the homology class of a boundary component. Consider the homology classes $x_0 + a_i$ and $x_0 - a_i$. One of these is the homology class of an essential simple closed curve $\mathbf{c}$. Since $d_*(x_0 \pm a_i) = x_0 \mp a_i$ we have that $d_*(\mathbf{c}) \neq \pm\mathbf{c}$ and so the unoriented isotopy class of $\mathbf{c}$ is not preserved by d. Therefore $T_{\mathbf{c}} \notin \mathrm{SMod}(\widetilde{\Sigma})$ and so $\mathrm{SMod}(\widetilde{\Sigma}) \neq \mathrm{Mod}(\widetilde{\Sigma})$, a contradiction. It follows then that $m = 1$.

The next step is to show d is a hyperelliptic involution. Suppose not, then by Lemma 4.3.3 there are $i, j \in \{1, \ldots, 2g\}$ such that $\epsilon_i = 1$ and $\epsilon_j = -1$. Similar to the above argument, we can find a curve $\mathbf{c}$ such that $T_{\mathbf{c}} \notin \mathrm{SMod}(\widetilde{\Sigma})$. This implies that

$$d_* = -I : H_1(\widetilde{\Sigma}; \mathbb{Z}) \to H_1(\widetilde{\Sigma}; \mathbb{Z})$$

for all non-trivial $d \in D$. By Lemma 4.3.4 we may conclude that D is generated by a hyperelliptic involution ι. Finally, if $g \geq 2$ then we can find a curve that is not fixed by ι (see Figure 4.3), completing the proof of the first statement.

If $\mathrm{SMod}(\widetilde{\Sigma}) \neq \mathrm{Mod}(\widetilde{\Sigma})$ then we may choose a Dehn twist $T_{\mathbf{c}} \notin \mathrm{SMod}(\widetilde{\Sigma})$ and note that each power belongs to a different coset of $\mathrm{SMod}(\widetilde{\Sigma})$ in $\mathrm{Mod}(\widetilde{\Sigma})$. Since Dehn twists have infinite order, it follows that $\mathrm{SMod}(\widetilde{\Sigma})$ is infinite-index in $\mathrm{Mod}(\widetilde{\Sigma})$. ■

Remark 4.3.5. Suppose $p : \widetilde{\Sigma} \to \Sigma$ is a finite-sheeted, regular, possibly branched cover of surfaces without boundary with deck group D. Combining the proof of Theorem 4 in [7] with the Neilsen realisation theorem for finite groups [42] allows one to conclude that $\mathrm{SMod}(\widetilde{\Sigma})$ is the normaliser of D in $\mathrm{Mod}(\widetilde{\Sigma})$.

When the surfaces in question have boundary, then D is not a subgroup of $\mathrm{Mod}(\widetilde{\Sigma})$. However, D and $\mathrm{Mod}(\Sigma)$ are both subgroups of $\mathrm{Aut}(\mathcal{K})$, where $\mathcal{K}$ is the groupoid defined in Section 4.2.2.

In light of both the normaliser result just stated for closed surfaces, and Theorem 4.2.4 for $\mathrm{LMod}(\Sigma, \mathcal{B})$, we conjecture that $\mathrm{SMod}(\widetilde{\Sigma})) = \{[f] \in \mathrm{Mod}(\widetilde{\Sigma}) : f_* \in \mathrm{SAut}(\mathcal{K})\}$. Unfortunately, a proof seems out of reach at the moment.

4.4 Non-geometric embeddings of braid groups

In this section we will investigate a family of injective homomorphisms from the braid group to mapping class groups. We will refer to such a homomorphism as a *braid group embedding.* We first recall the definition of the Burau covers from Section 1.

Burau covers

Pick a point $x \in \partial \mathbf{D}_n$ and let $\gamma_i \in \pi_1(\mathbf{D}_n, x)$ be the homotopy class of a loop surrounding solely the ith puncture anti-clockwise. Then $\{\gamma_1, \ldots, \gamma_n\}$ generates $\pi_1(\mathbf{D}_n, x)$. For each $k \geq 2$, define a homomorphism

$$\begin{aligned} q_k : \pi_1(\mathbf{D}_n, x) &\to \mathbb{Z}/k\mathbb{Z} \\ \gamma_i &\mapsto 1 \end{aligned}$$

for all i. The kernel of q_k determines a k-sheeted cyclic branched cover $p_k : \Sigma_g^m \to \Sigma_0^1$ branched at n points. Here $m = \gcd(n, k)$ and $g = 1 - \frac{1}{2}(k + n - nk + m)$.

In Theorem 1.5.2 it was shown that $\mathrm{LMod}(\Sigma, \mathcal{B}) = \mathrm{Mod}(\Sigma, \mathcal{B})$ if and only if Σ is a disk and $p_k : \Sigma_g^m \to \Sigma_0^1$ is a k-sheeted Burau cover. We can therefore define the following braid group embedding;

$$\beta_k : B_n \cong \mathrm{Mod}(\Sigma_0^1, \mathcal{B}) = \mathrm{LMod}(\Sigma_0^1, \mathcal{B}) \cong \mathrm{SMod}(\Sigma_g^m) \hookrightarrow \mathrm{Mod}(\Sigma_g^m).$$

The first isomorphism is well known, the equality comes from Theorem 1.5.2, and the second isomorphism is a consequence of the Birman-Hilden theorem.

Let $\{\sigma_1, \ldots, \sigma_{n-1}\}$ be the standard generators of B_n. It is known that $\beta_2(\sigma_i) = T_{\mathbf{c}_i}$ where $\mathbf{c}_i$ is some non-separating curve for all $i \in \{1, \ldots, n-1\}$. Furthermore, the deck group $D \cong \mathbb{Z}/2\mathbb{Z}$ is generated by a hyperelliptic involution [27, Section 9.4].

In this section we will describe the image of the standard braid generators under β_k where $k \geq 3$. In particular we show that β_k is a non-geometric embedding of the braid group, that is, $\beta_k(\sigma_i)$ is *not* a Dehn twist. In order to describe the image of a single braid generator it suffices to consider the embeddings

$$\beta_{2g+1} : B_2 \hookrightarrow \mathrm{Mod}(\Sigma_g^1) \quad \text{and} \quad \beta_{2g+2} : B_2 \hookrightarrow \mathrm{Mod}(\Sigma_g^2),$$

for integers $g > 0$. In other words, we will study the Burau covers

$$p_{2g+1} : \Sigma_g^1 \to \Sigma_0^1 \quad \text{and} \quad p_{2g+2} : \Sigma_g^2 \to \Sigma_0^1,$$

in each case branched at two points. We will deal with the two cases separately although the techniques used in each case are similar.

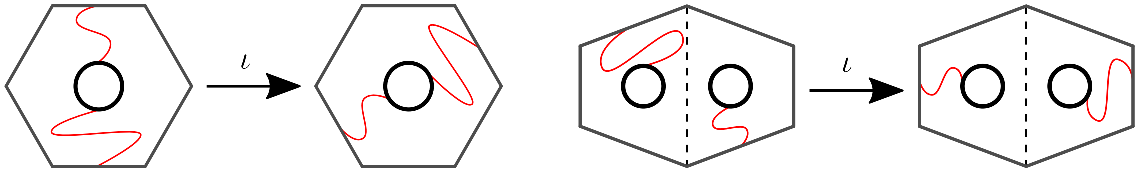

Figure 4.4: A generator of the deck group $D \cong \mathbb{Z}/3\mathbb{Z}$ of the 3-sheeted Burau cover, and a generator of the deck group $D \cong \mathbb{Z}/4\mathbb{Z}$ of the 4-sheeted Burau cover.

4.4.1 Odd Burau

First we consider the braid group embedding β_{2g+1} given above. We will define an element N of $\mathrm{Mod}(\Sigma_g^1)$ and then prove that the isomorphism

$$\Pi : \mathrm{SMod}(\Sigma_g^1) \to \mathrm{LMod}(\Sigma_0^1, \mathcal{B})$$

sends N to the standard generator of $\mathrm{Mod}(\Sigma_0^1, \mathcal{B})$. Recall that we can represent a closed surface of genus g by a regular $(4g+2)$-gon, centred at the origin, with opposite sides identified. If we remove an open disk about the centre we arrive at a representation of Σ_g^1. Label this representation P and let ι be the anti-clockwise rotation of P about its centre by $2\pi/(2g+1)$. The two unique vertices of P are fixed by ι. We see that the quotient space $\Sigma_g^1/\langle\iota\rangle$ is homeomorphic to Σ_0^1, and the quotient map is a covering map branched at two points. Furthermore, around both fixed points ι is locally a rotation by $2\pi/(2g+1)$ anti-clockwise, therefore the associated covering space is the $(2g+1)$-sheeted Burau cover of Σ_0^1 with deck group $D \cong \mathbb{Z}/(2g+1)\mathbb{Z}$.

We will write $p_{2g+1} : \Sigma_g^{1\circ} \to \mathbf{D}_2$ for the assocatied unbranched cover. Let $x \in \partial\mathbf{D}_2$ and let a and b be elements of $\pi_1(\mathbf{D}_2, x)$ such that a is represented by a loop that surrounds a single marked point and b is represented by a loop isotopic to $\partial\mathbf{D}_2$ as in Figure 4.5(i).

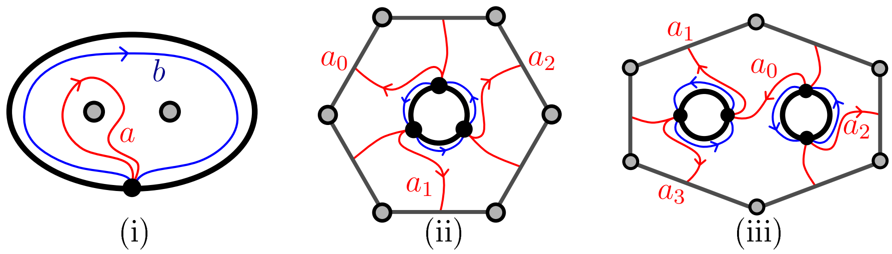

Figure 4.5: (i) Generators of the fundamental group of $\mathbf{D}_2$. (ii) Generators for the fundamental groupoid $\pi_1(\Sigma_1^{1\circ}, \{\tilde{x}_0, \tilde{x}_1, \tilde{x}_2\})$. (iii) Generators for $\pi_1(\Sigma_1^{2\circ}, \{\tilde{x}_0, \tilde{x}_1, \tilde{x}_2, \tilde{x}_3\})$.

The elements a and b generate $\pi_1(\mathbf{D}_2, x)$. Denote the full preimage $p_{2g+1}^{-1}(x)$ by $\{\widetilde{x}_i\}$ indexed by elements of $\mathbb{Z}/(2g+1)\mathbb{Z}$ such that $\iota(\widetilde{x}_i) = \widetilde{x}_{i+1}$. Similarly we define $(p_{2g+1})_*^{-1}(a) = \{a_i\}$ and $(p_{2g+1})_*^{-1}(b) = \{b_i\}$ such that $\iota_*(a_i) = a_{i+1}$ and $\iota_*(b_i) = b_{i+1}$, see Figure 4.5(ii). The set $\{a_i, b_i\}$, indexed by elements of $\mathbb{Z}/(2g+1)\mathbb{Z}$, generates the fundamental groupoid $\pi_1(\Sigma_g^{1\circ}, \{\widetilde{x}_i\})$, a fact which follows from Lemma 4.2.2.

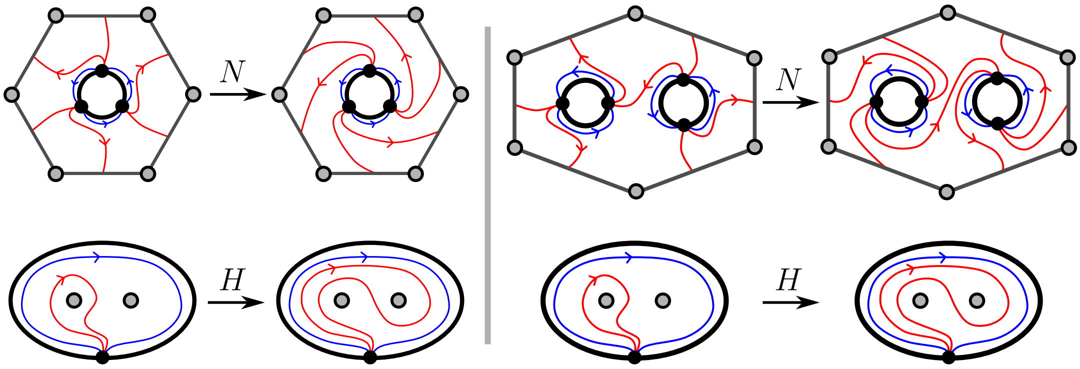

Figure 4.6: The mapping class $H \in \mathrm{Mod}(\mathbf{D}_2)$ and its lifts in the 3 and 4-sheeted Burau covers. Lemma 4.4.1 shows that N is the lift of H.

The odd notch

We let N denote the mapping class in $\mathrm{Mod}(\Sigma_g^1)$ represented by the homeomorphism that rotates the edges of P by $2\pi/(4g+2)$ and fixes the single boundary component at the centre. See Figure 4.6 for an image of N when $g = 1$.

In the following lemma we will write H for the half twist in $\mathrm{Mod}(\Sigma_0^1, \mathcal{B})$ and for the induced automorphism of $\pi_1(\mathbf{D}_2, x)$ such that $H(a) = ba^{-1}$.

Lemma 4.4.1. *Given the Burau cover $p_{2g+1} : \Sigma_g^1 \to \Sigma_0^1$ the half twist $H \in \mathrm{Mod}(\Sigma_0^1, \mathcal{B})$ lifts to the mapping class $N \in \mathrm{Mod}(\Sigma_g^1)$.*

Proof. Let $\mathcal{G} = \pi_1(\mathbf{D}_2, x)$ and let $\mathcal{K}$ be the fundamental groupoid $\pi_1(\Sigma_g^{1\circ}, \{\widetilde{x}_i\})$. We will abuse notation by writing N for its image in $\mathrm{Aut}(\mathcal{K})$ and H for its image in $\mathrm{Aut}(\mathcal{G})$ under the injective natural homomorphisms. We need to show that $N \in \mathrm{SAut}(\mathcal{K})$ as defined in Section 4.2.2. Since the deck group D is generated by ι, this is equivalent to showing that $N\iota = \iota N$ as automorphisms of $\mathcal{K}$. It can be seen from Figure 4.6 that $N(a_i) = b_i a_{i+1+g}^{-1}$ and that $N(b_i) = b_i$.

It follows then that

$$N\iota(a_i) = N(a_{i+1}) = b_{i+1}a_{i+2+g}^{-1} = \iota(b_i a_{i+1+g}^{-1}) = \iota N(a_i), \qquad \text{and}$$

$$N\iota(b_i) = N(b_{i+1}) = b_{i+1} = \iota(b_i) = \iota N(b_i).$$

Since the set $\{a_i, b_i\}$ generates the fundamental groupoid, we are done.

We will now show that the image of N in $\mathrm{LAut}_{\mathcal{H}}(\mathcal{G})$ under the isomorphism Π of Lemma 4.2.3 is equal to H. This makes sense since $\mathrm{LMod}(\Sigma_0^1, \mathcal{B}) = \mathrm{Mod}(\Sigma_0^1, \mathcal{B})$ and so from Theorem 4.2.4, we conclude that $H \in \mathrm{LAut}_{\mathcal{H}}(\mathcal{G})$. We now have

$$\Pi(N)(a) = p_* N(a_i) = p_*(b_i a_{i+1+g}^{-1}) = ba^{-1} = H(a), \qquad \text{and}$$

$$\Pi(N)(b) = p_* N(b_i) = p_*(b_i) = b = H(b).$$

So $\Pi(N) = H$ and since the diagram

$$\begin{tikzcd} \mathrm{SMod}(\Sigma_g^1) \arrow[r, hook, "\Psi"] \arrow[d, "\Pi", "\cong"'] & \mathrm{SAut}(\mathcal{K}) \arrow[d, "\Pi", "\cong"'] \\ \mathrm{Mod}(\Sigma_0^1, \mathcal{B}) \arrow[r, hook, "\Psi"] & \mathrm{LAut}_{\mathcal{H}}(\mathcal{G}) \end{tikzcd}$$

commutes, the mapping class N is indeed the lift of the half twist H. ■

4.4.2 Even Burau

We will now move on to the braid group embedding $\beta_{2g+2} : B_2 \hookrightarrow \mathrm{Mod}(\Sigma_g^2)$ given above. As in the odd case we will define an element of $\mathrm{Mod}(\Sigma_g^2)$ and then prove that it is the lift of a half twist H. We take H to be the half twist such that $H_*(a) = ba^{-1}$ for $a, b \in \pi_1(\mathbf{D}_2, x)$ as before. We want to find a polygonal representation of Σ_g^2. We take a regular $(4g+2)$-gon with opposite sides identified. This time, we remove two open disks as shown in Figure 4.4 and label the representation P.

We define an order $2g+2$ homeomorphism ι as follows:

1. Cut P along a straight line connecting the top and bottom vertices and label the resulting $(2g+2)$-gons P_L and P_R.

2. Rotate both P_L and P_R anti-clockwise by $2\pi/(2g+2)$ and re-attach them along the straight line connecting top and bottom vertices.

3. Rotate P by π.

While this homeomorphism of Σ_g^2 is substantially more complicated than the one described in the Section 4.4.1 it shares many properties. Both vertices of P are fixed by ι however, this time, locally ι is a clockwise rotation by $2\pi/(2g+2)$. It follows that the quotient space $\Sigma_g^2/\langle\iota\rangle$ is homeomorphic to Σ_0^1 and the associated covering space is the $(2g+2)$-sheeted Burau cover of Σ_0^1 with deck group $D \cong \mathbb{Z}/(2g+2)\mathbb{Z}$.

The even notch

We define $N \in \mathrm{Mod}(\Sigma_g^2)$ to be the mapping class represented by the homeomorphism that rotates the edges of both P_L and P_R by $2\pi/(2g+2)$ and fixes the boundary components. See Figure 4.6 for an image of N when $g = 1$.

Using the same method as the proof of Lemma 4.4.1 we arrive at the following result.

Lemma 4.4.2. *Given the Burau cover $p_{2g+2} : \Sigma_g^2 \to \Sigma_0^1$ the half twist $H \in \mathrm{Mod}(\Sigma_0^1, \mathcal{B})$ lifts to the mapping class $N \in \mathrm{Mod}(\Sigma_g^1)$.*

The proof of Lemma 4.4.2 is identical to the odd case, except that while $N(b_i) = b_i$ as before, we now have $N(a_i) = b_i a_{i+1}^{-1}$.

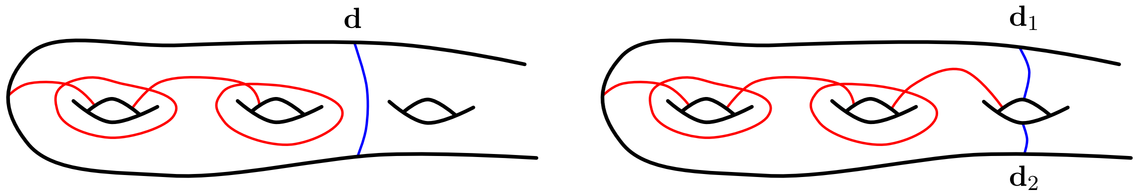

Figure 4.7: A 4-chain and a 5-chain. The 4-chain has a single boundary curve and the 5-chain has two boundary curves.

4.4.3 Chain twists

We will now describe the two maps defined in the previous section as products of Dehn twists. We will often abuse notation by referring to an isotopy class of curves by the name of a single representative curve.

Chains

Recall that a sequence of curves $\{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_k\}$ is called a k-chain if $i(\mathbf{c}_i, \mathbf{c}_j) = 1$ if $j = i \pm 1$ and $i(\mathbf{c}_i, \mathbf{c}_j) = 0$ otherwise. If $k = 2g$ for some g then the closed neighbourhood of $\cup \mathbf{c}_i$ is a subsurface homeomorphic to Σ_g^1 with boundary component isotopic to the curve $\mathbf{d}$. Furthermore, if $k = 2g+1$ then the closed neighbourhood of $\cup \mathbf{c}_i$ is a subsurface homeomorphic to Σ_g^2 with boundary components $\mathbf{d}_1$ and $\mathbf{d}_2$ (see Figure 4.7).

By considering the braid group embedding $\beta_2 : B_{k+1} \hookrightarrow \widetilde{\Sigma}$ it can be shown that

$$(T_{\mathbf{c}_1} T_{\mathbf{c}_2} \ldots T_{\mathbf{c}_{2g}})^{4g+2} = T_{\mathbf{d}} \quad \text{and} \quad (T_{\mathbf{c}_1} T_{\mathbf{c}_2} \ldots T_{\mathbf{c}_{2g+1}})^{2g+2} = T_{\mathbf{d}_1} T_{\mathbf{d}_2},$$

see Farb-Margalit [27, Section 4.4] for more details. Given a k-chain $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_k\}$, we call the product $T_{\mathcal{C}} := T_{\mathbf{c}_1} T_{\mathbf{c}_2} \ldots T_{\mathbf{c}_k}$ a *k-chain twist* (or a *chain twist*). Now, let $p_{2g+1} : \Sigma_g^1 \to \Sigma_0^1$ be a Burau cover and let N be the lift of the half twist as discussed in Lemma 4.4.1. If the curve $\mathbf{d}$ is isotopic to the boundary of Σ_g^1 then it is clear that

$$N^{4g+2} = T_{\mathbf{d}} = (T_{\mathcal{C}})^{4g+2}$$

for any $2g$-chain $\mathcal{C}$ in Σ_g^1. Similarly, suppose $p_{2g+2} : \Sigma_g^2 \to \Sigma_0^1$ is a Burau cover and N is the lift of the half twist discussed in Lemma 4.4.2. If the curves $\mathbf{d}_1, \mathbf{d}_2$ are isotopic to the boundary components of Σ_g^2 then we have

$$N^{2g+2} = T_{\mathbf{d}_1} T_{\mathbf{d}_2} = (T_{\mathcal{C}})^{2g+2}$$

for any $(2g+1)$-chain $\mathcal{C}$ in Σ_g^2. In Proposition 4.4.3 we will prove that as well as having the same power as a chain twist, the notch N is in fact equal to a chain twist in both the odd and even cases, proving Theorem 1.5.4. This implies that there exist chains $\mathcal{A}, \mathcal{B}$ (of any length) whose corresponding chain twists $T_{\mathcal{A}}, T_{\mathcal{B}}$ satisfy the braid relation.

In Section 4.4.4 we will give the explicit combinatorial data required for two k-chains to admit chain twists satisfying the braid relation. Furthermore, we show that

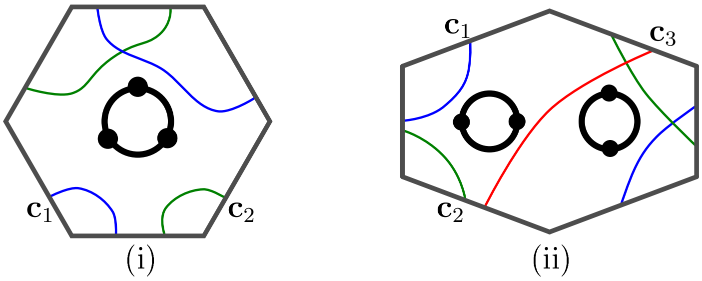

Figure 4.8: The 2-chain and 3-chain shown have corresponding chain twists equal to the notch N coming from the 3-fold and 4-fold Burau covers respectively.

this data encodes the braid relation on the level of Dehn twists.

Proposition 4.4.3. *Given a Burau cover* $p_k : \widetilde{\Sigma} \to \Sigma_0^1$ *the half twist* $H \in \mathrm{Mod}(\Sigma_0^1, \mathcal{B})$ *lifts to a* $(k-1)$*-chain twist.*

Proof. Given Lemmas 4.4.1 and 4.4.2 we need only show that the mapping class $N \in \mathrm{Mod}(\Sigma_g^m)$ is equal to a $(k-1)$-chain twist $T_{\mathcal{C}}$, for some $(k-1)$-chain $\mathcal{C}$. To do this we will show that the image of $T_{\mathcal{C}}$ and N are equal in the group $\mathrm{Aut}(\mathcal{K})$ where $\mathcal{K}$ is a fundamental groupoid of Σ_g^m with basepoints on all boundary components.

To that end, let $k = 2g+1$ and let $\mathcal{K}$ be the fundamental groupoid of Σ_g^1 generated by the set depicted in Figure 4.5(ii). Note that this set was also used to define a fundamental groupoid of $\Sigma_g^{1^\circ}$. In this setting however, the vertices of the polygon are not punctures. Furthermore, in order to facilitate the proof we change the indexing so that $\alpha_{2i} := a_i$. We define the curve $\mathbf{c}_0$ uniquely by the groupoid element $\alpha_0\alpha_1 b_0^{-1}$. We then define $\mathbf{c}_i$ to be $N^i(\mathbf{c}_0)$ for all $i \in \mathbb{Z}/(2g+1)\mathbb{Z}$, see Figure 4.8(i). Now, we define the $2g$-chain $\mathcal{C} := \{\mathbf{c}_1, \ldots, \mathbf{c}_{2g}\}$. In fact, we may choose $\mathcal{C}$ to be any $2g$-chain consisting of the curves $\mathbf{c}_i$. Now, by construction it can be seen that

$$\begin{aligned} T_{\mathbf{c}_i}(\alpha_i) &= N(\alpha_i), \\ T_{\mathbf{c}_j}(\alpha_i) &= \alpha_i \text{ for } j > i, \\ T_{\mathbf{c}_j} N(\alpha_i) &= N(\alpha_i) \text{ for } j < i. \end{aligned}$$

It follows then that for any $i \in \{1, \ldots, 2g\}$ we have

$$\begin{aligned} T_{\mathcal{C}}(\alpha_i) &= T_{\mathbf{c}_1} T_{\mathbf{c}_2} \ldots T_{\mathbf{c}_{2g}}(\alpha_i) \\ &= T_{\mathbf{c}_1} T_{\mathbf{c}_2} \ldots T_{\mathbf{c}_i}(\alpha_i) \\ &= T_{\mathbf{c}_1} T_{\mathbf{c}_2} \ldots T_{\mathbf{c}_{i-1}} N(\alpha_i) = N(\alpha_i). \end{aligned}$$

It remains to show that $T_{\mathcal{C}}(\alpha_0) = N(\alpha_0)$. The curve $\mathbf{c}_{2g}$ intersects the representative of α_0 once and by definition $i(\mathbf{c}_i, \mathbf{c}_{i+1}) = 1$. It therefore follows that the product $T_{\mathcal{C}}$ adds a copy of each of the $\mathbf{c}_i$ to α_0. It is shown in Figure 4.9 that this is in fact equal to $N(\alpha_0)$ in the case where $g = 1$, and indeed, this is true for any $g > 0$. It follows that $T_{\mathcal{C}} = N$ as groupoid automorphisms, and hence, they are equal as elements of $\mathrm{Mod}(\Sigma_g^1)$. The case where $k = 2g+2$ is similar to that of $2g+1$. We can use the

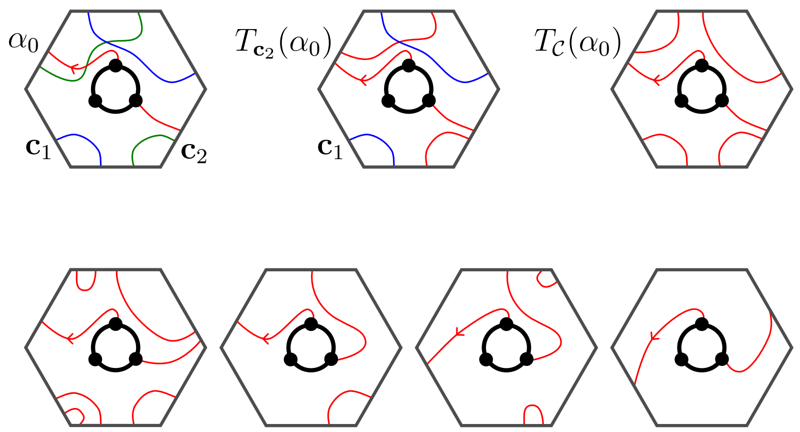

Figure 4.9: All arcs on the bottom row of hexagons are isotopic to the arc shown in the hexagon on the top right. The arcs $T_{\mathcal{C}}(\alpha_0)$ and $N(\alpha_0)$ are isotopic and so are equal as elements of the fundamental groupoid.

groupoid generators shown in Figure 4.5(iii) and the details are left to the reader. ■

Note that Proposition 4.4.3 proves Theorem 1.5.4 in that it completely determines the image of each braid group generator σ_i by the homomorphism β_k.

4.4.4 Intersection data

In Section 4.4.3 we saw that half twists lift to $(k-1)$-chain twists with respect to any k-sheeted Burau cover where $k \geq 3$. We will now explicitly describe a sufficient combinatorial condition for two chains that implies their chain twists satisfy the braid relation. We will assume that $k \geq 3$ for the remainder of this section.

Bracelets

Let $\mathcal{C} = \{\mathbf{c}_1, \ldots, \mathbf{c}_{k-1}\}$ be a $(k-1)$-chain and let $\mathbf{c}_0 = T_{\mathcal{C}}(\mathbf{c}_{k-1})$. We call the set $\{\mathbf{c}_i : i \in \mathbb{Z}/k\mathbb{Z}\}$ a *k-bracelet* (or a *bracelet*). Such a bracelet is called the *bracelet completion* of the chain $\mathcal{C}$.

Lemma 4.4.4. *Let $\mathcal{C} = \{\mathbf{c}_1, \ldots, \mathbf{c}_{k-1}\}$ be a chain and $\{\mathbf{c}_i : i \in \mathbb{Z}/k\mathbb{Z}\}$ the bracelet completion of $\mathcal{C}$. Then*

(i) $i(\mathbf{c}_i, \mathbf{c}_j) = \begin{cases} 1 & \textit{if } i = j-1, j+1 \\ 0 & \textit{otherwise,} \end{cases}$

(ii) $T_{\mathbf{c}_i} T_{\mathbf{c}_{i+1}} \cdots T_{\mathbf{c}_{i-2}} = T_{\mathbf{c}_{i+1}} T_{\mathbf{c}_{i+2}} \cdots T_{\mathbf{c}_{i-1}}$ *for all $i \in \mathbb{Z}/k\mathbb{Z}$.*

Proof. Note that $T_{\mathbf{c}_0} = T_{\mathbf{c}_1} \cdots T_{\mathbf{c}_{k-1}} T_{\mathbf{c}_{k-2}}^{-1} \cdots T_{\mathbf{c}_1}^{-1}$. Any solution to the word problem for the braid group (Dehornoy's handle reduction [22] for example) can be used to show $[T_{\mathbf{c}_0}, T_{\mathbf{c}_i}] = 1$ if $i \neq 1, k-1$ and $T_{\mathbf{c}_0} T_{\mathbf{c}_i} T_{\mathbf{c}_0} = T_{\mathbf{c}_i} T_{\mathbf{c}_0} T_{\mathbf{c}_i}$ if $i = 1, k-1$. This proves property (i).

For (ii), note $T_{\mathbf{c}_0}\cdots T_{\mathbf{c}_{k-2}} = T_{\mathbf{c}_1}\cdots T_{\mathbf{c}_{k-1}}$ from the definition of $\mathbf{c}_0$. Suppose now $T_{\mathbf{c}_j}\cdots T_{\mathbf{c}_{j-2}} = T_{\mathbf{c}_{j+1}}\cdots T_{\mathbf{c}_{j-1}}$ for some $j \in \mathbb{Z}/k\mathbb{Z}$. Then

$$\begin{aligned} T_{\mathbf{c}_{j+2}}\cdots T_{\mathbf{c}_j} &= T_{\mathbf{c}_{j+1}}^{-1} T_{\mathbf{c}_{j+1}} T_{\mathbf{c}_{j+2}}\cdots T_{\mathbf{c}_{j-1}} T_{\mathbf{c}_j} \\ &= T_{\mathbf{c}_{j+1}}^{-1} T_{\mathbf{c}_j} T_{\mathbf{c}_{j+1}}\cdots T_{\mathbf{c}_{j-2}} T_{\mathbf{c}_j} \\ &= T_{\mathbf{c}_j} T_{\mathbf{c}_{j+1}} T_{\mathbf{c}_j}^{-1} T_{\mathbf{c}_{j+2}}\cdots T_{\mathbf{c}_{j-2}} T_{\mathbf{c}_j} \\ &= T_{\mathbf{c}_j}\cdots T_{\mathbf{c}_{j-2}} \end{aligned}$$

completing the proof. ■

Note that property (ii) in Lemma 4.4.4 implies that a k-bracelet is the completion of any of the $(k-1)$-chains obtained by deleting a curve. Abusing notation, suppose $\mathcal{C}$ is a k-bracelet. In light of this fact, define the *bracelet twist* $T_{\mathcal{C}}$ as the chain twist about any of the $(k-1)$-chains obtained by deleting a curve from $\mathcal{C}$.

Mesh intersection

Let $\mathcal{A} = \{\mathbf{a}_i : i \in \mathbb{Z}/k\mathbb{Z}\}$ and $\mathcal{B} = \{\mathbf{b}_j : j \in \mathbb{Z}/k\mathbb{Z}\}$ be two k-bracelets. We say $\mathcal{A}$ and $\mathcal{B}$ have *mesh intersection* if there exists $t \in \mathbb{Z}/k\mathbb{Z}$ such that

$$i(\mathbf{a}_i, \mathbf{b}_{j+t}) = \begin{cases} 1 & \text{if } i = j, j+1 \\ 0 & \text{otherwise.} \end{cases}$$

In practice, we may simply relabel the curves in $\mathcal{B}$ and assume $t = 0$.

Fix $k \geq 3$ and let $\beta_k : B_3 \to \mathrm{Mod}(\Sigma_g^m)$ be the embedding of the braid group arising from the k-sheeted Burau cover. Proposition 4.4.3 shows that each standard generator is sent to a $(k-1)$-chain twist. The proof proceeds by first constructing a set of k curves, and then arbitrarily discarding one. The set of k curves constructed is in fact the completion of the $(k-1)$-chain. Furthermore, it can be checked that β_k sends the two standard generators to bracelet twists about bracelets with mesh intersection. In fact, in the discussion following the statement of Proposition 4.4.6 we will see that if $\mathcal{A}$ and $\mathcal{B}$ are two bracelets with mesh intersection such that all $2k$ curves are distinct, then there exists a Burau cover that lifts two half twists satisfying a braid relation to $T_{\mathcal{A}}$ and $T_{\mathcal{B}}$. It follows from Theorem 1.5.2 that $T_{\mathcal{A}}$ and $T_{\mathcal{B}}$ satisfy the braid relation. By leveraging the algebraic properties of Dehn twists, we may arrive at the same conclusion without mention of such a covering space.

Theorem 4.4.5. *If two k-bracelets $\mathcal{A}$ and $\mathcal{B}$ have mesh intersection then $T_{\mathcal{A}}T_{\mathcal{B}}T_{\mathcal{A}} = T_{\mathcal{B}}T_{\mathcal{A}}T_{\mathcal{B}}$.*

Proof. Be relabelling the curves in $\mathcal{B}$, we may assume $t = 0$ in the definition of mesh

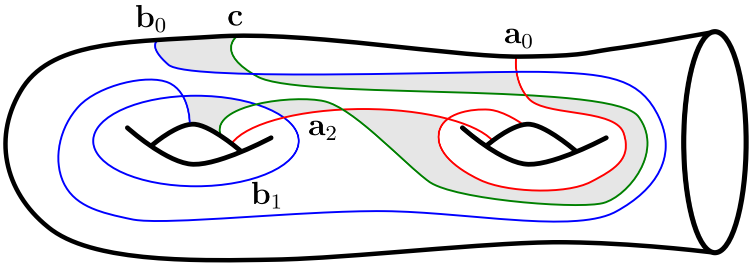

Figure 4.10: Two 3-bracelets $\mathcal{A} = \{\mathbf{a}_0, \mathbf{c}, \mathbf{a}_2\}$ and $\mathcal{B} = \{\mathbf{b}_0, \mathbf{b}_1, \mathbf{c}\}$ with mesh intersection.

intersection. We first show that $T_{\mathcal{A}}T_{\mathcal{B}}T_{\mathbf{a}_i} = T_{\mathbf{b}_i}T_{\mathcal{A}}T_{\mathcal{B}}$ as follows;

$$\begin{aligned}
T_{\mathcal{A}}T_{\mathcal{B}}T_{\mathbf{a}_i} &= T_{\mathbf{a}_{i+1}} \cdots T_{\mathbf{a}_{i-1}} T_{\mathbf{b}_i} \cdots T_{\mathbf{b}_{i-2}} T_{\mathbf{a}_i} \\
&= T_{\mathbf{a}_{i+1}} T_{\mathbf{b}_i} T_{\mathbf{a}_{i+2}} \cdots T_{\mathbf{a}_{i-1}} T_{\mathbf{b}_{i+1}} \cdots T_{\mathbf{b}_{i-2}} T_{\mathbf{a}_i} \\
&= T_{\mathbf{a}_{i+1}} T_{\mathbf{b}_i} T_{\mathbf{a}_{i+2}} \cdots T_{\mathbf{a}_{i-1}} T_{\mathbf{a}_i} T_{\mathbf{b}_{i+1}} \cdots T_{\mathbf{b}_{i-2}} \\
&= T_{\mathbf{a}_{i+1}} T_{\mathbf{b}_i} T_{\mathbf{a}_{i+1}} \cdots T_{\mathbf{a}_{i-1}} T_{\mathbf{b}_{i+1}} \cdots T_{\mathbf{b}_{i-2}} \\
&= T_{\mathbf{b}_i} T_{\mathbf{a}_{i+1}} T_{\mathbf{b}_i} T_{\mathbf{a}_{i+2}} \cdots T_{\mathbf{a}_{i-1}} T_{\mathbf{b}_{i+1}} \cdots T_{\mathbf{b}_{i-2}} \\
&= T_{\mathbf{b}_i} T_{\mathbf{a}_{i+1}} T_{\mathbf{a}_{i+2}} \cdots T_{\mathbf{a}_{i-1}} T_{\mathbf{b}_i} T_{\mathbf{b}_{i+1}} \cdots T_{\mathbf{b}_{i-2}} \\
&= T_{\mathbf{b}_i} T_{\mathcal{A}} T_{\mathcal{B}}.
\end{aligned}$$

The second and third equalities come from the intersection data of curves $\mathbf{b}_i$ and $\mathbf{a}_{i-1}$ respectively. The fourth equality comes from property (ii) of Lemma 4.4.4. The fifth and sixth equalities come from property (i) of Lemma 4.4.4 applied to $\mathcal{B}$.

This allows us to achieve the braid relation as follows:

$$\begin{aligned}
T_{\mathcal{A}}T_{\mathcal{B}}T_{\mathcal{A}} &= T_{\mathcal{A}}T_{\mathcal{B}}T_{\mathbf{a}_0}T_{\mathbf{a}_1} \cdots T_{\mathbf{a}_{k-2}} \\
&= T_{\mathbf{b}_0}T_{\mathcal{A}}T_{\mathcal{B}}T_{\mathbf{a}_1} \cdots T_{\mathbf{a}_{k-2}} \\
&= \cdots \\
&= T_{\mathbf{b}_0}T_{\mathbf{b}_1} \cdots T_{\mathbf{b}_{k-2}} T_{\mathcal{A}}T_{\mathcal{B}} \\
&= T_{\mathcal{B}}T_{\mathcal{A}}T_{\mathcal{B}}. \qquad \blacksquare
\end{aligned}$$

When $k \geq 4$, it can be shown that if two k-bracelets have mesh intersection, then all $2k$ curves in question are distinct. However, this is not the case when $k = 3$. Figure 4.10 shows two 3-bracelets $\{\mathbf{a}_i : i \in \mathbb{Z}/3\mathbb{Z}\}$ and $\{\mathbf{b}_i : i \in \mathbb{Z}/3\mathbb{Z}\}$ with mesh intersection such that $\mathbf{a}_1 = \mathbf{b}_2$.

Intersection data for chains

We now shift our attention to finding a sufficient combinatorial condition for two $(k-1)$-chains $\mathcal{A}$ and $\mathcal{B}$ to have the property that their bracelet completions have mesh intersection. We will then be able to conclude, by Theorem 4.4.5, that the two chain twists $T_{\mathcal{A}}$ and $T_{\mathcal{B}}$ satisfy a braid relation.

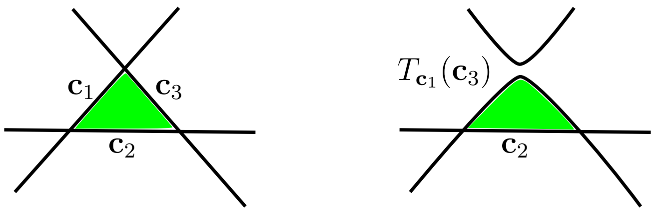

Figure 4.11: The triple $(\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3)$ bounds a positively oriented triangle. The right image shows the bigon between $T_{\mathbf{c}_1}(\mathbf{c}_3)$ and $\mathbf{c}_2$, implying $i(T_{\mathbf{c}_1}(\mathbf{c}_3), \mathbf{c}_2) = 0$.

Suppose $\alpha_1, \alpha_2, \alpha_3$ are curves on a surface in minimal position such that $i(\alpha_i, \alpha_j) = 1$ if $i \neq j$. The graph given by the three curves defines two triangles and a hexagon on the surface. Suppose one of the triangles bounds a disk $\mathbf{D}$. We say the triple $(\alpha_1, \alpha_2, \alpha_3)$ *bounds a positively oriented triangle* if you can traverse $\partial\mathbf{D}$ in a anti-clockwise direction from the intersection point $x \in \alpha_1 \cap \alpha_3$ and travel along a segment of α_1, then a segment of α_2, then a segment of α_3 in that order and return to x. See Figure 4.11 for a local picture of three curves bounding a positively oriented triangle.

We say a triple $(\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3)$ of isotopy classes of curves bounds a positively oriented triangle if there exist representatives γ_i of $\mathbf{c}_i$ such that $(\gamma_1, \gamma_2, \gamma_3)$ bounds a positively oriented triangle.

Note that if $(\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3)$ bounds a positively oriented triangle and $\sigma \in S_3$ is a permutation, then $(\mathbf{c}_{\sigma(1)}, \mathbf{c}_{\sigma(2)}, \mathbf{c}_{\sigma(3)})$ bounds a positively oriented triangle if and only if σ is an even permutation.

The importance of the definition of a triple bounding a positively oriented triangle is that if $(\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3)$ bounds a positively oriented triangle, then $i(T_{\mathbf{c}_1}(\mathbf{c}_3), \mathbf{c}_2) = 0$. This can be seen in Figure 4.11.

Proposition 4.4.6. *Suppose* $\mathcal{A} = \{\mathbf{a}_1, \ldots, \mathbf{a}_{k-1}\}$ *and* $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_{k-1}\}$ *are two* $(k-1)$*-chains with the property that*

(i) $i(\mathbf{a}_i, \mathbf{b}_j) = \begin{cases} 1 & \textit{if } i = j, j+1 \\ 0 & \textit{otherwise} \end{cases}$, *and*

(ii) *The triples* $(\mathbf{a}_i, \mathbf{b}_i, \mathbf{a}_{i+1})$ *and* $(\mathbf{b}_i, \mathbf{a}_{i+1}, \mathbf{b}_{i+1})$ *bound positively oriented triangles for all* $i \in \{1, \ldots, k-2\}$.

Then the chain twists $T_{\mathcal{A}}$ *and* $T_{\mathcal{B}}$ *satisfy a braid relation.*

If each of the $2k-2$ curves in $\mathcal{A}$ and $\mathcal{B}$ are distinct then we may view them as depicted Figure 4.12.

Defining Σ_g^m to be the regular neighbourhood of the curves and triangles it can be seen that $m = \gcd(3, k)$. Furthermore, an Euler characteristic argument shows that $g = k-2$ if $m = 3$ and $g = k-1$ if $m = 1$. These are precisely the values of m and g that give rise to a k-sheeted Burau cover $p_k : \Sigma_g^m \to \Sigma_0^1$ with three branch points. By using a variation of the change of coordinates principle (see [27, Section 1.3.2]) we may conclude that $T_{\mathcal{A}}$ and $T_{\mathcal{B}}$ are lifts of half twists that satisfy a braid relation. Hence from Theorem 1.5.2 we conclude that $T_{\mathcal{A}}$ and $T_{\mathcal{B}}$ satisfy a braid relation.

Unlike the discussion above, the following proof of Proposition 4.4.6 does not make use of the Birman-Hilden Theorem. As such, it provides a more intrinsic perspective of chain twists satisfying a braid relation, and deals with the case when the curves in $\mathcal{A}$ and $\mathcal{B}$ are not distinct.

Proof of Proposition 4.4.6. Let $\mathbf{a}_0 = T_{\mathcal{A}}(\mathbf{a}_{k-1})$ and $\mathbf{b}_0 = T_{\mathcal{B}}(\mathbf{b}_{k-1})$. To ease notation let $\Delta = T_{\mathbf{b}_1} \cdots T_{\mathbf{b}_{k-2}}$ and $\nabla = T_{\mathbf{b}_2} \cdots T_{\mathbf{b}_{k-1}}$. Note that $T_{\mathbf{b}_0} = \Delta T_{\mathbf{b}_{k-1}} \Delta^{-1} = \nabla^{-1} T_{\mathbf{b}_1} \nabla$.

By Theorem 4.4.5 it suffices to show

$$i(\mathbf{a}_0, \mathbf{b}_j) = \begin{cases} 1 & \text{if } j = 0, k-1 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad i(\mathbf{a}_i, \mathbf{b}_0) = \begin{cases} 1 & \text{if } i = 0, 1 \\ 0 & \text{otherwise.} \end{cases}$$

Let $i \neq 0, 1$. Since $(\mathbf{b}_j, \mathbf{a}_{j+1}, \mathbf{b}_{j+1})$ bounds a positively triangle for all $j \in \{1, \ldots, k-2\}$, we have $i(T_{\mathbf{b}_j}(\mathbf{b}_{j+1}), \mathbf{a}_{j+1}) = 0$ so $[T_{\mathbf{b}_j} T_{\mathbf{b}_{j+1}} T_{\mathbf{b}_j}^{-1}, T_{\mathbf{a}_{j+1}}] = 1$. Rearranging and relabelling we get

$$T_{\mathbf{b}_i}^{-1} T_{\mathbf{b}_{i-1}}^{-1} T_{\mathbf{a}_i} T_{\mathbf{b}_{i-1}} T_{\mathbf{b}_i} = T_{\mathbf{b}_{i-1}}^{-1} T_{\mathbf{a}_i} T_{\mathbf{b}_{i-1}}$$

for all $i \neq 0, 1$. We have

$$\begin{aligned}
[T_{\mathbf{b}_0}, T_{\mathbf{a}_i}] &= \Delta T_{\mathbf{b}_{k-1}} \Delta^{-1} T_{\mathbf{a}_i} \Delta T_{\mathbf{b}_{k-1}}^{-1} \Delta^{-1} T_{\mathbf{a}_i}^{-1} \\
&= \Delta T_{\mathbf{b}_{k-1}} T_{\mathbf{b}_{k-2}}^{-1} \cdots T_{\mathbf{b}_{i+1}}^{-1} T_{\mathbf{b}_i}^{-1} T_{\mathbf{b}_{i-1}}^{-1} T_{\mathbf{a}_i}^{-1} \\
&\qquad T_{\mathbf{b}_{i-1}} T_{\mathbf{b}_i} T_{\mathbf{b}_{i+1}} \cdots T_{\mathbf{b}_{k-2}} T_{\mathbf{b}_{k-1}}^{-1} \Delta^{-1} T_{\mathbf{a}_i}^{-1} \\
&= \Delta T_{\mathbf{b}_{k-1}} T_{\mathbf{b}_{k-2}}^{-1} \cdots T_{\mathbf{b}_{i+1}}^{-1} T_{\mathbf{b}_{i-1}}^{-1} T_{\mathbf{a}_i}^{-1} T_{\mathbf{b}_{i-1}} T_{\mathbf{b}_{i+1}} \cdots T_{\mathbf{b}_{k-2}} T_{\mathbf{b}_{k-1}}^{-1} \Delta^{-1} T_{\mathbf{a}_i}^{-1} \\
&= T_{\mathbf{b}_1} \cdots T_{\mathbf{b}_{i-1}} T_{\mathbf{b}_i} T_{\mathbf{b}_{i-1}}^{-1} T_{\mathbf{a}_i}^{-1} T_{\mathbf{b}_{i-1}} T_{\mathbf{b}_i}^{-1} T_{\mathbf{b}_{i-1}}^{-1} \cdots T_{\mathbf{b}_1}^{-1} T_{\mathbf{a}_i}^{-1} \\
&= T_{\mathbf{b}_1} \cdots T_{\mathbf{b}_{i-1}} T_{\mathbf{b}_{i-1}}^{-1} T_{\mathbf{a}_i} T_{\mathbf{b}_{i-1}} T_{\mathbf{b}_{i-1}}^{-1} \cdots T_{\mathbf{b}_1}^{-1} T_{\mathbf{a}_i}^{-1} \\
&= T_{\mathbf{b}_1} \cdots T_{\mathbf{b}_{i-2}} T_{\mathbf{a}_i} T_{\mathbf{b}_{i-2}}^{-1} \cdots T_{\mathbf{b}_1}^{-1} T_{\mathbf{a}_i}^{-1} \\
&= 1.
\end{aligned}$$

Therefore $i(\mathbf{a}_i, \mathbf{b}_0) = 0$. When $i = 1$ we have

$$\begin{aligned}
T_{\mathbf{a}_1} T_{\mathbf{b}_0} T_{\mathbf{a}_1} &= T_{\mathbf{a}_1} \nabla^{-1} T_{\mathbf{b}_1} \nabla T_{\mathbf{a}_1} \\
&= \nabla^{-1} T_{\mathbf{a}_1} T_{\mathbf{b}_1} T_{\mathbf{a}_1} \nabla \\
&= \nabla^{-1} T_{\mathbf{b}_1} T_{\mathbf{a}_1} T_{\mathbf{b}_1} \nabla \\
&= \nabla^{-1} T_{\mathbf{b}_1} \nabla T_{\mathbf{a}_1} \nabla^{-1} T_{\mathbf{b}_1} \nabla \\
&= T_{\mathbf{b}_0} T_{\mathbf{a}_1} T_{\mathbf{b}_0}
\end{aligned}$$

so $i(\mathbf{a}_1, \mathbf{b}_0) = 1$. Similar arguments show $i(\mathbf{a}_0, \mathbf{b}_j) = 0$ for $j \neq 0, k-1$ and $i(\mathbf{a}_0, \mathbf{b}_{k-1}) = 1$.

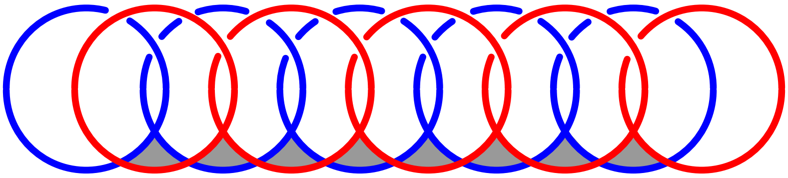

Figure 4.12: Intersecting circles describing two 5-chains satisfying the conditions of Proposition 4.4.6. The regular neighbourhood of this collection of curves and triangles is homeomorphic to Σ_4^3.

It remains to show $i(\mathbf{a}_0, \mathbf{b}_0) = 1$. We have

$$\begin{aligned} T_{\mathbf{a}_0} T_{\mathbf{b}_0} T_{\mathbf{a}_0} &= T_{\mathbf{a}_0} \Delta T_{\mathbf{b}_{k-1}} \Delta^{-1} T_{\mathbf{a}_0} \\ &= \Delta T_{\mathbf{a}_0} T_{\mathbf{b}_{k-1}} T_{\mathbf{a}_0} \Delta^{-1} \\ &= \Delta T_{\mathbf{b}_{k-1}} T_{\mathbf{a}_0} T_{\mathbf{b}_{k-1}} \Delta^{-1} \\ &= \Delta T_{\mathbf{b}_{k-1}} \Delta^{-1} T_{\mathbf{a}_0} \Delta T_{\mathbf{b}_{k-1}} \Delta^{-1} \\ &= T_{\mathbf{b}_0} T_{\mathbf{a}_0} T_{\mathbf{b}_0} \end{aligned}$$

completing the proof. ■

See Figure 1.9 for two 3-chains on Σ_3^1 satisfying the conditions of Lemma 4.4.6. The positively oriented triangles are shaded in grey.

4.4.5 Open Questions

Here are a few natural questions relating to the braid group embeddings constructed above. Recall that for each $k \geq 3$ and $n \geq 2$ we have constructed an embedding $\beta_k : B_n \hookrightarrow \mathrm{Mod}(\Sigma_g^m)$ arising from the k-sheeted Burau cover. Here, $m = \gcd(n, k)$ and $g = 1 - \frac{1}{2}(k + n + m - nk)$.

Necessity of mesh intersection

When two simple closed curves a and b on a surface intersect once, then T_a and T_b satisfy a braid relation. In fact, this condition is necessary. That is, $T_a T_b T_a = T_b T_a T_b$ if and only if $i(a, b) = 1$ (see [27, §3.5]). The next question asks the analogous question for chain twists.

Question 4.4.7. *Suppose $\mathcal{A}$ and $\mathcal{B}$ are k-chains for $k \geq 2$, and let $T_{\mathcal{A}}$ and $T_{\mathcal{B}}$ be the corresponding chain twists. Is it true that if $T_{\mathcal{A}} T_{\mathcal{B}} T_{\mathcal{A}} = T_{\mathcal{B}} T_{\mathcal{A}} T_{\mathcal{B}}$, then the bracelet completions of $\mathcal{A}$ and $\mathcal{B}$ have mesh intersection?*

Automorphisms of free groups

For a surface Σ with non-empty boundary, there is a homomorphism $\mathrm{Mod}(\Sigma) \to \mathrm{Aut}(\pi_1(\Sigma))$ given by the action of $\mathrm{Mod}(\Sigma)$ on the fundamental group of Σ with a basepoint on the boundary. For a surface of genus g and m boundary components,

$\pi_1(\Sigma_g^m) \cong F_{2g+m-1}$. Precomposing with the braid group embeddings above, we get an induced homomorphism from the braid group into the automorphism group of a free group.

Question 4.4.8. *Let $k \geq 3$. For each $n \geq 2$ there is a homomorphism $\phi_{n,k} : B_n \to F_{(n-1)(k-1)}$. What can be said about this family of homomorphisms? Do they give rise to new embeddings of the braid group in $\mathrm{Aut}(F_n)$?*

Triviality of the induced map on stable homology

There is a geometric embedding $B_{2g} \hookrightarrow \mathrm{Mod}(\Sigma_g^1)$ for each g. This family of embeddings gives a map from $B_\infty = \lim_{g\to\infty} B_{2g}$ to $\Gamma_\infty = \lim_{g\to\infty} \mathrm{Mod}(\Sigma_g^1)$. In the 1980s J. Harer conjectured that the induced map on stable homology $H_*(B_\infty; \mathbb{Z}/2\mathbb{Z}) \to H_*(\Gamma_\infty; \mathbb{Z}/2\mathbb{Z})$ is trivial. The conjecture was proved by Song and Tillman in [63, Theorem 1.1]. A stronger version of Harer's conjecture was proved for a large family of non-geometric embeddings of the braid group in [9].

Question 4.4.9. *Fix $k > 3$. Is the map on stable homology $H_*(B_\infty; \mathbb{Z}/2\mathbb{Z}) \to H_*(\Gamma_\infty; \mathbb{Z}/2\mathbb{Z})$ induced by the embeddings $\beta_k : B_n \hookrightarrow \mathrm{Mod}(\Sigma_g^m)$ trivial?*

Note that this question is answered affirmatively for stable homology with any coefficients when $k = 3$ by Kim-Song [44, Theorem 3.4].

Classifying braid embeddings

There are now infinite families of non-geometric embeddings of braid groups in mapping class groups.

Question 4.4.10. *Is there a classification of all possible conjugacy classes of embeddings of the braid group in the mapping class group?*

The proof of Theorem 4.4.5 suggests a way to construct more examples as follows.

Suppose we have two subsets of mapping classes $\{\phi_i\}$ and $\{\theta_i\}$ indexed by $\mathbb{Z}/k\mathbb{Z}$ such that

$$\phi_i\phi_j = \begin{cases} \phi_j\phi_i\phi_j\phi_i^{-1} & \text{if } j = i-1, i+1 \\ \phi_j\phi_i & \text{otherwise,} \end{cases}$$

$$\theta_i\theta_j = \begin{cases} \theta_j\theta_i\theta_j\theta_i^{-1} & \text{if } j = i-1, i+1 \\ \theta_j\theta_i & \text{otherwise.} \end{cases}$$

Suppose further that the following relations are satisfied;

$$\phi_i\theta_j = \begin{cases} \theta_j\phi_i\theta_j\phi_i^{-1} & \text{if } i = j, j+1 \\ \theta_j\phi_i & \text{otherwise,} \end{cases}$$

and for any $i \in \mathbb{Z}/k\mathbb{Z}$ we have

$$\Phi := \phi_i \dots \phi_{i-2} = \phi_{i+1} \dots \phi_{i-1} \quad \text{and} \quad \Theta := \theta_i \dots \theta_{i-2} = \theta_{i+1} \dots \theta_{i-1}.$$

Then the products Φ and Θ satisfy the braid relation, that is, $\Phi\Theta\Phi = \Theta\Phi\Theta$.

We conjecture however, that this is only possible when each ϕ_i and θ_i is a Dehn twist and the corresponding sets of curves are bracelets with mesh intersection. There is no particular reason to assume otherwise, except to satisfy our own insatiable desire for pattern.

Bibliography

[1] Javier Aramayona, Christopher J. Leininger, and Juan Souto. Injections of mapping class groups. *Geom. Topol.*, 13(5):2523–2541, 2009.

[2] Hyman Bass and Alexander Lubotzky. Linear-central filtrations on groups. *Contemp. Math.*, 169, 1994.

[3] Juliette Bavard, Spencer Dowdall, and Kasra Rafi. Isomorphisms between big mapping class groups. *pre-print*, 2017.

[4] Robert Bell and Dan Margalit. Braid groups and the co-Hopfian property. *Journal of Algebra*, 303:275-294, 2006.

[5] Robert Bell and Dan Margalit. Injections of Artin groups. *Commenarii Mathemetici Helvetici*, 82:725-751, 2007.

[6] Stephen J. Bigelow and Ryan D. Budney. The mapping class group of a genus two surface is linear. *Algebr. Geom. Topol.*, 1:699–708, 2001.

[7] Joan S. Birman and Hugh M. Hilden. On isotopies of homeomorphisms of Riemann surfaces. *Annals of Mathematical Studies*, 97:424-439, 1973.

[8] Joan S. Birman, Alex Lubotzky, and John McCarthy. Abelian and solvable subgroups of the mapping class groups. *Duke Mathematics Journal*, (4) 50: 1107–1120, 1983.

[9] Carl-Friedrich Bödigheimer and Ulrike Tillmann. Embeddings of braid groups into mapping class groups and their homology. In *Configuration spaces*, volume 14 of *CRM Series*, pages 173–191. Ed. Norm., Pisa, 2012.

[10] Christian Bonatti and Luis Paris. Roots in the mapping class groups. *Proc. Lond. Math. Soc.*, 98:471-903, 2009.

[11] Brian Bowditch. Rigidity of the strongly separating curve graph. *Michigan Mathematical Journal*, (65):813-832, 2016.

[12] Tara E. Brendle and Dan Margalit. Commensurations of the Johnson kernel. *Geometry and Topology*, (8):1361-1384, 2004.

[13] Tara E. Brendle and Dan Margalit. Normal subgroups of mapping class groups and the metaconjecture of Ivanov. *arxiv:1710.08929*, 2017.

[14] Tara E. Brendle and Dan Margalit. The level four braid group. *J. Reine Angew. Math.*, 735:249–264, 2018.

[15] Tara E. Brendle, Dan Margalit, and Andrew Putman. Generators for the hyperelliptic Torelli group and the kernel of the Burau representation at t = -1. *Inventiones Mathematicae*, 200(1):263-310, 2015.

[16] Martin Bridson, Alexandra Pettet, and Juan Souto. The abstract commensurator of the Johnson kernels. *in preperation.*

[17] Ronald Brown. *Topology and groupoids.* BookSurge, LLC, Charleston, SC, 2006. Third edition of ıt Elements of modern topology [McGraw-Hill, New York, 1968; MR0227979], With 1 CD-ROM (Windows, Macintosh and UNIX).

[18] Ruth Charney and John Crisp. Automorphism groups of some affine and finite type Artin groups. *Mathematical Research Letters*, 12:321–333, 2005.

[19] Leah R. Childers. The automorphism group of the hyperelliptic Torelli group. *New York Journal of Mathematics*, 23:671-697, 2017.

[20] Matt Clay, Johanna Mangahas, and Dan Margalit. Normal right-angled Artin subgroups of mapping class groups. *In preperation.*

[21] F. Dahmani, V. Guirardel, and D. Osin. Hyperbolically embedded subgroups and rotating families in groups acting on hyperbolic spaces. *Memoirs of the American Mathematical Society*, 245(1156):v+152, 2017.

[22] Patrick Dehornoy. A fast method for comparing braids. *Adv. Math.*, 125(2):200–235, 1997.

[23] J. L. Dyer and E. K. Grossman. The automorphism group of the braid groups. *American Journal of Mathematics*, 103:1151-1169, 1981.

[24] Hisaaki Endo. Meyer's signature cocycle and hyperelliptic fibrations. *Math. Ann.*, 316(2):237–257, 2000.

[25] Benson Farb. Fifteen problems about the mapping class groups. *Proceedings of Symposia in Pure Mathematics*, 74 - Problems on Mapping Class Groups and Related Topics, 2006.

[26] Benson Farb and Nikolai V. Ivanov. The Torelli geometry and its applications. research announcement. *Mathematical Research Letters*, 293-301, 2005.

[27] Benson Farb and Dan Margalit. *A primer on mapping class groups.* Princeton University Press, 2012.

[28] Tyrone Ghaswala. *The Liftable Mapping Class Group*. PhD thesis, 2017.

[29] Tyrone Ghaswala and Rebecca R. Winarski. Lifting homeomorphisms and cyclic branched covers of spheres. *Michigan Math. J.*, 66(4):885–890, 2017.

[30] E. A. Gorin and V Ja. Lin. Algebraic equations with continuous coefficients and some problems of the algebraic theory of braids. *Math. USSR Sbornik*, 7:569-596, 1969.

[31] Allen Hatcher. *Algebraic topology*. Cambridge University Press, 2002.

[32] Jesus Hernández Hernández, Isreal Morales, and Ferrán Valdez. Isomorphisms between curve graphs of infinite-type surfaces are geometric. *arXiv:1706.03697*, 2017.

[33] Philip J. Higgins. *Notes on categories and groupoids*. Van Nostrand Reinhold Co., London-New York-Melbourne, 1971. Van Nostrand Rienhold Mathematical Studies, No. 32.

[34] Stephen P. Humphries. Generators for the mapping class group. *Lecture Notes in Mathematics*, 772, 1979.

[35] Elmas Irmak. Complexes of nonseparating curves and mapping class groups. *Michigan Mathematical Journal*, (54):81-110, 2006.

[36] Elmas Irmak and John D. McCarthy. Injective simplicial maps of the arc complex. *Turkish Journal of Mathematics*, (34):2055-2077, 2010.

[37] Nikolai V. Ivanov. Automorphisms of complexes of curves and of Teichmüller spaces. *International Mathematics Research Notices*, (14):651-666, 1997.

[38] Nocolai V. Ivanov. *Subgroups of Teichmüller Modular Groups*. American Mathematical Society, 1992.

[39] Denis Johnson. The structure of the Torelli group. I. a finite set of generators for $\mathcal{I}$. *Annals of Mathematical Studies*, 118(3):423-442, 1983.

[40] Denis Johnson. The structure of the Torelli group. II. a characterization of the group generated by twists on bounding curves. *Topology*, 24(2):113-126, 1985.

[41] Yasushi Kasahara. An expansion of the Jones representation of genus 2 and the Torelli group. *Algebr. Geom. Topol.*, 1:39-55, 2001.

[42] Steven P. Kerckhoff. The Nielsen realization problem. *Annals of Mathematical Studies*, 117(2):235-265, 1983.

[43] Yoshikata Kida. Automorphisms of the Torelli complex and the complex of separating curves. *Journal of the Mathematical Society of Japan*, 63, 2011.

[44] B. C. Kim and Y. Song. Configuration space, moduli space and 3-fold covering space. *arXiv:1802.00594v1*, 2018.

[45] Mustafa Korkmaz. Automorphisms of complexes of curves on punctured spheres and on punctured tori. *Topology and its applications*, 95, 1999.

[46] Mustafa Korkmaz and Athanase Papadopoulos. On the arc and curve complex. *Mathematical Proceedings of the Cambridge Philosophical Society*, (148):473-483, 2010.

[47] Mustafa Korkmaz and Athanase Papadopoulos. On the ideal triangulation graph of a punctured surface. *Annalesde l'Institut Fourier*, (62):1367-1384, 2012.

[48] S. Lefschetz. On the fixed point formula. *Annals of Mathematics*, 38(4):819–822, 1937.

[49] Chris Leininger and Dan Margalit. Abstract commensurators of braid groups. *Journal of Algebra*, 299(2):447-455, 2006.

[50] Feng Luo. Automorphisms of the complex of curves. *Topology*, 39, 2000.

[51] C. MacLachlan and W. J. Harvey. On mapping-class groups and Teichmü ller spaces. *Proceedings of the London Mathematical Society*, 30(part 4):496-512, 1975.

[52] Dan Margalit. Automorphisms of the pants complex. *Geometry and Topology*, (8):1361-1384, 2004.

[53] Dan Margalit and Rebecca Winarski. The Birman-Hilden theory. *arXiv:1703.03448v1*, 2017.

[54] John D. McCarthy and Athanase Papadopolous. Simplicial actions of mapping class groups. *Handbook of Teichmüller theory*, 3, 2013.

[55] Alan McLeay. Normal subgroups of the braid group and the metaconjecture of Ivanov. *arxiv:1801.05209*, 2017.

[56] Geoffrey Mess. The Torelli groups for genus 2 and 3 surfaces. *Topology*, 31(4):775-790, 1992.

[57] Takayuki Morifuji. On Meyer's function of hyperelliptic mapping class groups. *J. Math. Soc. Japan*, 55(1):117–129, 2003.

[58] Stepen Orevkov. Automorphism group of the commutator subgroup of the braid group. *Annales de la Faculté des Sciences de Toulouse*, to appear.

[59] Andrew Putman. Cutting and pasting the Torelli group. *Geom. Topol.*, 11:829-865, 2007.

[60] Andrew Putman. A note on the connectivity of certain complexes associated to surfaces. *L'Enseignement Mathématique*, (2) 54: 287-301, 2008.

[61] Andrew Putman. Small generating sets for the Torelli group. *Geom. Topol.*, 16(1):111-125, 2012.

[62] Yongjin Song. The braidings in the mapping class groups of surfaces. *J. Korean Math. Soc.*, 50(4):865–877, 2013.

[63] Yongjin Song and Ulrike Tillman. Braids, mapping class groups, and categorical delooping. *Math. Ann.*, 339(2):377–393, 2007.

[64] Daniel Studenmund. Abstract commensurators: a brief survey. 2014.

[65] Michal Stukow. Conjugacy classes of finite subgroups of certain mapping class groups. *Turkish J. Math.*, 28(2):101–110, 2004.

[66] Błażej Szepietowski. Embedding the braid group in mapping class groups. *Publ. Mat.*, 54(2):359–368, 2010.

[67] Vladimir Turaev. Faithful linear representations of the braid groups. *arXiv:math/0006202*, 2000.

[68] Bronislaw Wajnryb. Relations in the mapping class group. In *Problems on mapping class groups and related topics*, volume 74 of *Proc. Sympos. Pure Math.*, pages 115–120. Amer. Math. Soc., Providence, RI, 2006.