DIMENSION THEORY AND FRACTAL CONSTRUCTIONS BASED ON SELF-AFFINE CARPETS

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For my parents, Ailsa and Iain, and my sister, Cara,
whose love and encouragement sustain me.

Acknowledgments

It is a great pleasure to be able to take the time here to thank numerous friends for their help and encouragement over the last four years. First and foremost, my sincerest thanks go to my supervisors, Kenneth Falconer and Lars Olsen. I feel I have been given a near perfect balance of reassuring guidance and exciting freedom during my studies. In particular, I always felt supported by my supervisors and was fortunate enough to be able to meet with them as often as I liked, where I received countless vital pieces of advice and was guided towards various 'mathematical gems' I would have otherwise overlooked. That being said, I was also given a certain mathematical freedom to fully explore my own interests, knowing that I always had Kenneth and Lars in my corner should I wander off course. I must also thank the EPSRC for providing me with full financial support throughout my PhD studies.

Secondly, I would like to thank the many friends I have been lucky enough to share my life with during my time as a PhD student. There are too many names to list individually, but I would like to mention explicitly Andy Parkinson and Darren Kidney, who had to endure the experience of living with me at 7 Johnston Court. Not only did these heroic gentlemen carry out this task with grace and dignity, but they became two of the best friends I could wish for and made the whole process seem so simple and manageable within the grand scheme of things.

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This academic year I finally got around to joining the University Badminton Club. I would not quite call it a mid-life crisis, but to begin with it certainly seemed like an old man trying to pick up a former hobby with people far too young and fit for him but over the course of the year I became so fond of the club and its members that I deeply regret not joining it earlier. In any case, I would like to thank all my friends at the club in the 2012-2013 season for a great year and wish everyone the very best for the future.

I would like to extend my gratitude to everyone in the maths department in St Andrews for making my eight years as a student here such an outstanding experience. In particular, I would like to mention Mike Todd, Collin Bleak, Xiong Jin and Ábel Farkas, a.k.a. the original 'lunch bunch', for their unique blend of friendship, academic advice and lunchtime comedy.

The work in this thesis is based on five papers which I wrote on my own, but I have been very fortunate to collaborate with many wonderful mathematicians and friends during my PhD. Not least my supervisors, Kenneth Falconer and Lars Olsen, but also Richard Balka, Ábel Farkas, Andy Ferguson, James Hyde, Tuomas Orponen, Tuomas Sahlsten and Pablo Shmerkin. I have thoroughly enjoyed and learned from each collaboration and sincerely hope that we can work together again in the future.

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Last but not least, I would like to thank my family, Ailsa, Iain and Cara, to whom this thesis is dedicated. They know what for.

Declaration

I, Jonathan MacDonald Fraser, hereby certify that this thesis, which is approximately 31,500 words in length, has been written by me, that it is the record of work carried out by me and that it has not been submitted in any previous application for a higher degree.

I was admitted as a research student in September 2009 and as a candidate for the degree of Doctor of Philosophy in September 2009; the higher study for which this is a record was carried out in the University of St Andrews between 2009 and 2013.

Date: .... **Signature of Candidate:**..

I hereby certify that the candidate has fulfilled the conditions of the Resolution and Regulations appropriate for the degree of Doctor of Philosophy in the University of St Andrews and that the candidate is qualified to submit this thesis in application for that degree.

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Abstract

The aim of this thesis is to develop the dimension theory of self-affine carpets in several directions. Self-affine carpets are an important class of planar self-affine sets which have received a great deal of attention in the literature on fractal geometry over the last 30 years. These constructions are important for several reasons. In particular, they provide a bridge between the relatively well-understood world of self-similar sets and the far from understood world of general self-affine sets. These carpets are designed in such a way as to facilitate the computation of their dimensions, and they display many interesting and surprising features which the simpler self-similar constructions do not have. For example, they can have distinct Hausdorff and packing dimensions and the Hausdorff and packing measures are typically infinite in the critical dimensions. Furthermore, they often provide exceptions to the seminal result of Falconer from 1988 which gives the 'generic' dimensions of self-affine sets in a natural setting. The work in this thesis will be based on five research papers I wrote during my time as a PhD student, namely [Fr1, Fr2, Fr3, Fr4, Fr5].

The first contribution of this thesis will be to introduce a new class of self-affine carpets, which we call *box-like self-affine sets*, and compute their box and packing dimensions via a modified singular value function. This not only generalises current results on self-affine carpets, but also helps to reconcile the 'exceptional constructions' with Falconer's singular value function approach in the generic case. This will appear in Chapter 2 and is based on the paper [Fr1], which appeared in *Nonlinearity* in 2012.

In Chapter 3 we continue studying the dimension theory of self-affine sets by computing the Assouad and lower dimensions of certain classes. The Assouad and lower dimensions have not received much attention in the literature on fractals to date and their importance has been more related to quasi-conformal maps and embeddability problems. This appears to be changing, however, and so our results constitute a timely and important contribution to a growing body of literature on the subject. The material in this Chapter will be based on the paper [Fr4], which has been accepted for publication in *Transactions of the American Mathematical Society*.

In Chapters 4–6 we move away from the classical setting of iterated function systems to consider two more exotic constructions, namely, *inhomogeneous attractors* and *random 1-variable attractors*, with the aim of developing the dimension theory of self-affine carpets in these directions.

In order to put our work into context, in Chapter 4 we consider inhomogeneous self-similar sets and significantly generalise the results on box dimensions obtained by Olsen and Snigireva, answering several questions posed in the literature in the process. We then move to the self-affine setting and, in Chapter 5, investigate the dimensions of inhomogeneous self-affine carpets and prove that new phenomena can occur in this setting which do not occur in the setting of self-similar sets. The material in Chapter 4 will be based on the paper [Fr2], which appeared in *Studia Mathematica* in 2012, and the material in Chapter 5 is based on the paper [Fr5], which is in preparation.

Finally, in Chapter 6 we consider random self-affine sets. The traditional approach to random iterated function systems is probabilistic, but here we allow the randomness in the construction to be provided by the topological structure of the sample space, employing ideas from Baire category. We are able to obtain very general results in this setting, relaxing the conditions on the maps from 'affine' to 'bi-Lipschitz'. In order to get precise results on the Hausdorff and packing measures of typical attractors, we need to specialise to the setting of random self-similar sets and we show again that several interesting and new phenomena can occur when we relax to the setting of random self-affine carpets. The material in this Chapter will be based on the paper [Fr3], which has been accepted for publication by *Ergodic Theory and Dynamical Systems*.

Contents

1 Fractals and dimension theory

1.1 Fractal geometry

Since their popularisation in the 1970s via the works of Mandelbrot, for example [Ma1, Ma2], the mathematics of fractals has become important both in theory and in practice. The theoretical side has attracted a substantial amount of attention in the literature; connections being made with various areas of mathematics, including: fractal geometry, geometric measure theory, dynamical systems, number theory, differential equations and probability theory. However, the importance of fractals is not restricted to abstract mathematics, with many naturally occurring physical phenomena exhibiting a fractal structure such as graphs of random processes, percolation problems and fluid turbulence. Thus, understanding the geometric structure of fractals from a mathematical point of view can help in modelling real world phenomena which exhibit fractal properties. For example, modeling and understanding chaotic behaviour in numerous different areas of science often involves the use of fractal techniques, see [PJS, Mc]. Roughly speaking, a fractal is an object which is detailed on arbitrarily small scales, often with some degree of *self-similarity*. The most basic form of self-similarity is found in self-similar sets. These are sets which are made up of uniformly scaled down copies of themselves. Self-similar sets have been studied intensively over the past 30 years and are now relatively well-understood, except in the complicated 'overlapping' situation, see Section 1.3.2. A natural and important generalisation of *self-similar sets* are *self-affine sets*. These too are made up of scaled down copies of themselves, but the scaling may be by different amounts in different directions, and the copies can be skewed and sheared. This not only makes self-affine sets much more difficult to study than self-similar sets, but also makes them much more important as they occur more naturally in other areas of science. For example, a physical process governed by a non-conformal dynamical system will often display self-affinity on small scales. Self-affine sets, which are described in more detail in Section 1.3.3, are one of the key objects of study in this thesis.

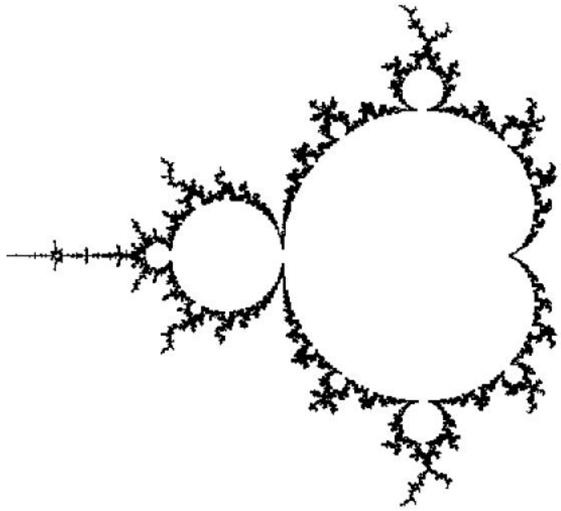 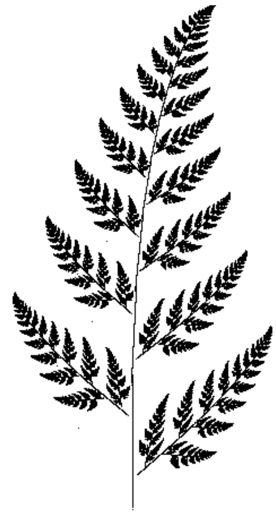

Figure 1: Two famous fractals. Left: The Mandelbrot set. Right: The (self-affine) Barnsley Fern.

1.2 Dimension theory

One of the most important notions connected with studying fractals from a rigorous mathematical point of view is that of *dimension*. Roughly speaking, a dimension is a (usually non-negative real) number which gives some geometric information concerning how the fractal set fills up space on small scales. There are many different notions of dimension and these notions often come in pairs. In this section we will give a rigorous account of the six notions of dimension that we will be concerned with in this thesis and compare some of their basic properties. These six notions are grouped naturally into three sets of two and so we thus divide this section accordingly. For a more detailed account of dimension theory and the interplay between the different notions, the reader is referred to the now standard texts [F8, Mat] and, for the less well-known Assouad and lower dimensions, the papers [Lu, Fr4].

The dimensions described here are very much metric quantities and, as such, throughout this section we will work in a metric space (X, d), which we assume to be compact for convenience, although this is not necessary. Other, non-metric, notions of dimension exist, such as the *topological dimension*, see [HW], although we will not concern ourselves with this here. Indeed, such non-metric notions are not really suitable for studying fractals; a fact highlighted by the elegant result of Luukkainen [Lu], which states that any non-empty separable metric space can be re-metrized with a compatible metric in such a way as to make all the six notions of dimension discussed here equal to the topological dimension and thus an integer. Luukkainen thus argues that there is no *topological* reason for a set to be called a fractal.

Central to the theory developed in this section are the notions of *covers* and *packings* of a set $F \subseteq X$ at some scale $\delta > 0$. A collection $\{U_i\}_{i \in I}$ of subsets of X will be called a δ-*cover* of F if each of the sets U_i is open and has diameter less than or equal to δ, and F is contained in the union $\bigcup_{i \in I} U_i$. Similarly, a collection $\{U_i\}_{i \in I}$ of subsets of X will be called a *centered* δ-*packing* of F if each of the sets U_i are closed balls with radius less than or equal to δ and centres in F. Analysing the behaviour of such covers and packings as δ converges to zero will be crucial in developing the theory of dimension.

1.2.1 Hausdorff and packing dimension

Hausdorff dimension, named after Felix Hausdorff, who introduced the notion in 1918 [Hau], is intrinsically linked with *packing dimension*, named due to its use of packings rather than the covers used to define Hausdorff dimension, which was introduced many years later in 1982 by Claude Tricot [T]. These two dimensions have probably received the most attention in the literature on fractals and have found their way into various different fields. They both have a convenient definition in terms of measures, which leads to a mathematically beautiful theory but can often make them very difficult to compute directly.

Let F be a subset of X. For $s \geqslant 0$ and $\delta > 0$ we define the δ-approximate s-dimensional Hausdorff measure of F by

$$\mathcal{H}^s_\delta(F) = \inf\left\{ \sum_{i \in I} |U_i|^s : \{U_i\}_{i\in I} \text{ is a countable } \delta\text{-cover of } F \right\}$$

and the s-dimensional Hausdorff (outer) measure of F by $\mathcal{H}^s(F) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(F)$. The Hausdorff dimension of F is

$$\dim_\mathrm{H} F = \inf\Big\{ s \geqslant 0 : \mathcal{H}^s(F) = 0 \Big\} = \sup\Big\{ s \geqslant 0 : \mathcal{H}^s(F) = \infty \Big\}.$$

If F is compact, then we may define the Hausdorff measure of F in terms of *finite* covers. Packing measure, defined in terms of *packings*, is a natural dual to Hausdorff measure, which was defined in terms of *covers*. For $s \geqslant 0$ and $\delta > 0$ we define the δ-approximate s-dimensional packing pre-measure of F by

$$\mathcal{P}^s_{0,\delta}(F) = \sup\left\{ \sum_{i \in I} |U_i|^s : \{U_i\}_{i\in I} \text{ is a countable centered } \delta\text{-packing of } F \right\}$$

and the s-dimensional packing pre-measure of F by $\mathcal{P}^s_0(F) = \lim_{\delta \to 0} \mathcal{P}^s_{0,\delta}(F)$. To ensure countable stability, the packing (outer) measure of F is defined by

$$\mathcal{P}^s(F) = \inf\left\{ \sum_i \mathcal{P}^s_0(F_i) : F \subseteq \bigcup_i F_i \right\}$$

and the packing dimension of F is

$$\dim_\mathrm{P} F = \inf\Big\{ s \geqslant 0 : \mathcal{P}^s(F) = 0 \Big\} = \sup\Big\{ s \geqslant 0 : \mathcal{P}^s(F) = \infty \Big\}.$$

The extra step in the definition of packing measure and dimension often makes it considerably more awkward to work with than Hausdorff measure. In a certain situation, this awkwardness can be overcome by an equivalent formulation of packing dimension given in the following section, see Proposition 1.1.

It is possible to consider a 'finer' definition of Hausdorff and packing dimension. We define a *gauge function* to be a function, $G : (0,\infty) \to (0,\infty)$, which is continuous, monotonically increasing, and satisfies $\lim_{t\to 0} G(t) = 0$. We then define the Hausdorff measure, packing pre-measure and packing measure with respect to the gauge G as

$$\mathcal{H}^G(F) = \lim_{\delta \to 0} \inf\left\{ \sum_{i \in I} G(|U_i|) : \{U_i\}_{i\in I} \text{ is a countable } \delta\text{-cover of } F \right\},$$

$$\mathcal{P}^G_0(F) = \lim_{\delta \to 0} \sup\left\{ \sum_{i \in I} G(|U_i|) : \{U_i\}_{i\in I} \text{ is a countable centered } \delta\text{-packing of } F \right\}$$

and

$$\mathcal{P}^G(F) = \inf\left\{ \sum_i \mathcal{P}^G_0(F_i) : F \subseteq \bigcup_i F_i \right\}$$

respectively. Note that if $G(t) = t^s$ then we obtain the standard Hausdorff and packing measures. The advantage of this approach is that, in the case where the measure of a set is zero or infinite in its dimension, one may be able to find an appropriate gauge for which the measure is positive and finite. For example, with probability 1, Brownian trails in $\mathbb{R}^2$ have Hausdorff dimension 2, but 2-dimensional Hausdorff measure equal to zero. However, with probability 1, they have positive and finite $\mathcal{H}^G$-measure with respect to the gauge $G(t) = t^2 \log(1/t) \log\log\log(1/t)$, see [F8, Chapter 16], and the references therein.

For a given gauge function G and a constant $c > 0$ we define

$$D^-(G, c) = \inf_{t>0} \frac{G(c\,t)}{G(t)} \qquad \text{and} \qquad D^+(G, c) = \sup_{t>0} \frac{G(c\,t)}{G(t)}.$$

Notice that, if $c \leqslant 1$, then $D^+(G, c) \leqslant 1$. It is easy to see that if $0 < D^-(G, c) \leqslant D^+(G, c) < \infty$ for some $c > 0$, then $0 < D^-(G, c) \leqslant D^+(G, c) < \infty$ for all $c > 0$ and in this case we say that the gauge is *doubling*. The standard gauge is clearly doubling, with $D^-(G, c) = D^+(G, c) = c^s$.

For a more detailed discussion of this finer approach to dimension, see [F8, Section 2.5] or [Rog, Chapter 2].

1.2.2 Box dimensions

A less sophisticated but nevertheless very useful notion of dimension is box dimension. The lower and upper box dimensions of a set $F \subseteq X$ are defined by

$$\underline{\dim}_{\mathrm{B}} F = \liminf_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta} \qquad \text{and} \qquad \overline{\dim}_{\mathrm{B}} F = \limsup_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta},$$

respectively, where $N_\delta(F)$ is the smallest number of sets required for a δ-cover of F. If $\underline{\dim}_{\mathrm{B}} F = \overline{\dim}_{\mathrm{B}} F$, then we call the common value the box dimension of F and denote it by $\dim_{\mathrm{B}} F$. It is useful to note that we can replace N_δ with a myriad of different definitions all based on covering or packing the set at scale δ, see [F8, Section 3.1]. For example, $N_r(F)$ can be taken as the maximal size of a centered δ-packing of F. We will usually denote this particular alternative definition by M_δ. If defining box dimension in a non-compact space, then usually one restricts to totally bounded sets in order to preclude the situation where $N_\delta(F) = \infty$.

One undesirable property of the box dimensions is that they are not countably stable, see Section 1.2.4. In order to remedy this, one could try to redefine box dimension by breaking the set up into countably many bits, taking the supremum of the box dimension of the bits and then taking the infimum over the different ways of splitting the set up. Amazingly, this new definition simply returns the packing dimension. We obtain

$$\dim_{\mathrm{P}} F = \inf \left\{ \sup_i \overline{\dim}_{\mathrm{B}} F_i : F \subseteq \bigcup_{i=1}^{\infty} F_i \right\}$$

where the infimum is taken over all countable partitions $\{F_i\}_i$ of F, see [F8, Chapter 3.4]. This alternative definition for packing dimension has the following very useful consequence.

Proposition 1.1. *Let $F \subseteq X$ be a compact set such that for every open set $U \subset X$ which intersects F, we have $\overline{\dim}_{\mathrm{B}}(F \cap U) = \overline{\dim}_{\mathrm{B}} F$. Then $\dim_{\mathrm{P}} F = \overline{\dim}_{\mathrm{B}} F$.*

For a proof of this see [F8, Chapter 3.4]. Finally we note that what we call the box dimension is sometimes referred to as the box-counting dimension, Minkowski dimension, or entropy dimension.

1.2.3 Assouad and lower dimension

If the dimensions described in the previous two sections give fine, but global, geometric information, then the Assouad and lower dimension give coarse, but localised, geometric information. As such we find their interplay with the other dimensions particularly fascinating.

The Assouad dimension was introduced by Patrice Assouad in the 1970s [A1, A2], see also [La1]. The *Assouad dimension* of a non-empty subset F of X is defined by

$$\dim_{\mathrm{A}} F \;=\; \inf \Bigg\{ \quad \alpha \quad : \text{ there exist constants } C,\, \rho > 0 \text{ such that,} \\ \text{for all } 0 < r < R \leqslant \rho, \text{ we have } \sup_{x \in F} N_r\big(B(x,R) \cap F\big) \;\leqslant\; C\bigg(\frac{R}{r}\bigg)^{\alpha} \quad \Bigg\}.$$

Although interesting in its own right, the importance of the Assouad dimension thus far has been its relationship with quasi-conformal mappings and embeddability problems rather than as a tool in the dimension theory of fractals, see [He, Lu, MT, Ro]. However, this seems to be changing, with several recent papers appearing which study Assouad dimension and its relationship with the other well-studied notions of dimension: Hausdorff, packing and box dimension; see, for example, [Fr4, KLV, M, O5, Ols]. The lower dimension is a natural dual to the Assouad dimension, and was introduced by Larman [La1], where it was called the *minimal dimensional number*, but it has been referred to by other names, for example: the *lower Assouad dimension* by Käenmäki, Lehrbäck and Vuorinen [KLV] and the *uniformity dimension* (Tuomas Sahlsten, personal communication). We decided on *lower dimension* to be consistent with the terminology used by Bylund and Gudayol in [ByG], but we wish to emphasise the relationship with the well-studied and popular Assouad dimension. The lower dimension of F is defined by

$$\dim_{\mathrm{L}} F \;=\; \sup \Bigg\{ \quad \alpha \quad : \text{ there exist constants } C,\, \rho > 0 \text{ such that,} \\ \text{for all } 0 < r < R \leqslant \rho, \text{ we have } \inf_{x \in F} N_r\big(B(x,R) \cap F\big) \;\geqslant\; C\bigg(\frac{R}{r}\bigg)^{\alpha} \quad \Bigg\}.$$

Indeed, the Assouad dimension and the lower dimension often behave as a pair, with many of their properties being intertwined. The lower dimension has received little attention in the literature on fractals, but despite this, we believe it is a very natural definition and should have a place in the study of dimension theory and fractal geometry. We summarise the key reasons for this below:

- The lower dimension is a natural dual to the well-studied Assouad dimension and dimensions often come in pairs. For example, the rich and complex interplay between Hausdorff dimension and packing dimension has become one of the key concepts in dimension theory. Also, the popular upper and lower box dimensions are a natural 'dimension pair'. Dimension pairs are important in several areas of geometric measure theory, such as the dimension theory of product spaces, see the discussion on products in Section 1.2.4.

- The lower dimension gives some important and easily interpreted information about the fine structure of the set. In particular, it identifies the parts of the set which are easiest to cover and gives a rigorous gauge of how efficiently the set can be covered in these areas.

- One might argue that the lower dimension is not a sensible tool for studying sets which are highly inhomogeneous in the sense of having some exceptional points around which the set is distributed very sparsely in comparison with the rest of the set. For instance, sets with isolated points have lower dimension equal to zero. However, it is perfect for studying attractors of iterated function systems (IFSs) as the IFS construction forces the set to have a certain degree of homogeneity. In fact the difference between the Assouad dimension and the lower dimension can give an insight into the amount of homogeneity present. For example, for self-similar sets satisfying the open set condition the two quantities are equal, indicating that the set is as homogeneous as possible. However, in Chapter 3 we will demonstrate that, for more complicated *self-affine sets* and *self-similar sets with overlaps*, the quantities can be, and often are, different.

The Assouad dimension and lower dimensions are much more sensitive to the local structure of the set around particular points, whereas the other dimensions give more global information. The Assouad dimension will be 'large' relative to the other dimensions if there are points around which the set is 'abnormally difficult' to cover and the lower dimension will be 'small' relative to the other dimensions if there are points around which the set is 'abnormally easy' to cover. This phenomenon is best illustrated by an example. Let $X = \{1/n : n \in \mathbb{N}\} \cup \{0\}$. Then

$$\dim_{\mathrm{L}} X \ = \ 0,$$

$$\underline{\dim}_{\mathrm{B}} X \ = \ \overline{\dim}_{\mathrm{B}} X \ = \ 1/2$$

and

$$\dim_{\mathrm{A}} X = 1.$$

The lower dimension is zero due to the influence of the isolated points in X. Indeed the set is locally very easy to cover around isolated points and it follows that if a set, X, has any isolated points, then $\dim_{\mathrm{L}} X = 0$. This could be viewed as an undesirable property for a 'dimension' to have because it causes it to be non-monotone and means that it can increase under Lipschitz mappings. We are not worried by this, however, as the geometric interpretation is clear and useful.

Finally, note that we can replace N_r in the definition of the Assouad and lower dimensions with any of the standard covering or packing functions, see [F8, Section 3.1]. For example, if F is a subset of Euclidean space, then $N_r(F)$ could denote the number of squares in an r-mesh orientated at the origin which intersect F or the maximum number of sets in an r-packing of F. We also obtain equivalent definitions if the ball $B(x, R)$ is taken to be open or closed, although we usually think of it as being closed.

1.2.4 Basic properties of the dimensions and some notation

In this section we will describe some basic properties which one might hope for a 'dimension' to satisfy and we will then summarise which of these properties are satisfied by the dimensions discussed in this chapter. In order to do this we need to introduce the notion of Lipschitz maps and the Hausdorff metric, which will be used throughout the thesis.

Let (X, d_X) (Y, d_Y) and be compact metric spaces. For a map $T : X \to Y$ define

$$\mathrm{Lip}^-(T) = \inf_{\substack{x,y\in X,\\ x\neq y}} \frac{d_Y\big(T(x),T(y)\big)}{d_X(x,y)} \qquad \text{and} \qquad \mathrm{Lip}^+(T) = \sup_{\substack{x,y\in X,\\ x\neq y}} \frac{d_Y\big(T(x),T(y)\big)}{d_X(x,y)}.$$

If $\mathrm{Lip}^+(T) < \infty$, then we say T is *Lipschitz* and if, in addition, $\mathrm{Lip}^-(T) > 0$, then we say T is *bi-Lipschitz*. If $\mathrm{Lip}^+(T) < 1$, then we say T is a *contraction* and if $\mathrm{Lip}^-(T) = \mathrm{Lip}^+(T)$, then we write $\mathrm{Lip}(T)$ to denote the common value and say that T is a *similarity*.

Write $\mathcal{K}(X)$ to denote the set of all non-empty compact subsets of X and endow $\mathcal{K}(X)$ with the *Hausdorff metric*, $d_{\mathcal{H}}$, defined by

$$d_{\mathcal{H}}(E, F) = \inf\{\varepsilon > 0 : E \subseteq F_\varepsilon \text{ and } F \subseteq E_\varepsilon\}$$

for $E, F \in \mathcal{K}(X)$ and where E_ε denotes the ε-neighbourhood of E. It turns out that $(\mathcal{K}(X), d_{\mathcal{H}})$ is a complete metric space. The following is a list of basic properties which dimensions may satisfy:

Monotonicity: dim is said to be monotone if $E \subseteq F \Rightarrow \dim E \leqslant \dim F$ for all $E, F \subseteq X$.

Finite stability: dim is said to be finitely stable if $\dim(E \cup F) = \max\{\dim E, \dim F\}$ for all $E, F \subseteq X$.

Countable stability: dim is said to be countably stable if $\dim \bigcup_i E_i = \sup_i \dim E_i$ for all countable collections of sets $\{E_i\}$ in X.

Stability under Lipschitz maps: dim is said to be stable under Lipschitz maps if $\dim f(E) \leqslant \dim E$ for all $E \subseteq X$ and all Lipschitz maps f on X.

Stability under bi-Lipschitz maps: dim is said to be stable under bi-Lipschitz maps if $\dim f(E) = \dim E$ for all $E \subseteq X$ and all bi-Lipschitz maps f on X.

Stability under taking closures: dim is said to be stable under taking closures if $\dim = \dim \overline{E}$ for all $E \subseteq X$.

Open set property: dim is said to satisfy the open set property if for any bounded open set $U \subset \mathbb{R}^n$, $\dim U = n$.

Measurability: dim is said to be measurable if it is a Borel measurable function from $(K(X), d_{\mathcal{H}})$ to $\mathbb{R}$.

The following table summarises which properties are satisfied by which dimensions.

Property	$\dim_{\mathrm{H}}$	$\dim_{\mathrm{P}}$	$\underline{\dim}_{\mathrm{B}}$	$\overline{\dim}_{\mathrm{B}}$	$\dim_{\mathrm{L}}$	$\dim_{\mathrm{A}}$
Monotone	✓	✓	✓	✓	×	✓
Finitely stable	✓	✓	×	✓	×	✓
Countably stable	✓	✓	×	×	×	×
Stable under Lipschitz maps	✓	✓	✓	✓	×	×
Stable under bi-Lipschitz maps	✓	✓	✓	✓	✓	✓
Stable under taking closures	×	×	✓	✓	✓	✓
Open set property	✓	✓	✓	✓	×	✓
Measurable	✓	×	✓	✓	✓	✓

For proofs of these facts, see [F8, Chapters 2-3] which gives details on the first seven properties for the Hausdorff, packing and box dimensions. For details on the Assouad dimension, see [Lu, Fr4], and for the lower dimension, see [Fr4]. The measurability property is somewhat more involved. For the Hausdorff, packing and box dimensions see [MM], and for the lower and Assouad dimensions see [Fr4].

One further geometric property that will be relevant in this thesis is how dimension behaves under taking the product of two metric spaces, (X, d_X) and (Y, d_Y). There are many natural 'product metrics' to impose on the product space $X \times Y$, but any reasonable choice is bi-Lipschitz equivalent to the metric $d_{X\times Y}$ on $X \times Y$ defined by

$$d_{X\times Y}\big((x_1, y_1), (x_2, y_2)\big) = \max\{d_X(x_1, x_2),\, d_Y(y_1, y_2)\},$$

for example, and so we need not specify which precise metric we use. 'Dimension pairs' are intimately related to the dimension theory of products and there is a pleasant symmetry in the formulae. We have

$$\dim_{\mathrm{H}} X + \dim_{\mathrm{H}} Y \;\leqslant\; \dim_{\mathrm{H}}(X \times Y) \;\leqslant\; \dim_{\mathrm{H}} X + \dim_{\mathrm{P}} Y \;\leqslant\; \dim_{\mathrm{P}}(X \times Y) \;\leqslant\; \dim_{\mathrm{P}} X + \dim_{\mathrm{P}} Y,$$

$$\underline{\dim}_{\mathrm{B}} X + \underline{\dim}_{\mathrm{B}} Y \;\leqslant\; \underline{\dim}_{\mathrm{B}}(X \times Y) \;\leqslant\; \underline{\dim}_{\mathrm{B}} X + \overline{\dim}_{\mathrm{B}} Y \;\leqslant\; \overline{\dim}_{\mathrm{B}}(X \times Y) \;\leqslant\; \overline{\dim}_{\mathrm{B}} X + \overline{\dim}_{\mathrm{B}} Y$$

and

$$\dim_{\mathrm{L}} X + \dim_{\mathrm{L}} Y \;\leqslant\; \dim_{\mathrm{L}}(X \times Y) \;\leqslant\; \dim_{\mathrm{L}} X + \dim_{\mathrm{A}} Y \;\leqslant\; \dim_{\mathrm{A}}(X \times Y) \;\leqslant\; \dim_{\mathrm{A}} X + \dim_{\mathrm{A}} Y.$$

The Hausdorff-packing result is due to Howroyd [How], the box dimension result is easily derived from the definition, and the lower-Assouad result was proved in [Fr4] - apart from the final inequality which is due to Assouad. Finally, we summarise the relationships between our six dimensions. For a set $F \subseteq X$, we have

$$\dim_{\mathrm{L}} F \leqslant \underline{\dim}_{\mathrm{B}} F \leqslant \overline{\dim}_{\mathrm{B}} F \leqslant \dim_{\mathrm{A}} F.$$

In general the lower dimension is not comparable to the Hausdorff dimension or packing dimension. However, if F is compact, then $\dim_{\mathrm{L}} F \leqslant \dim_{\mathrm{H}} F$. This was proved by Larman [La1, La2]. In particular, this means that the lower dimension provides a practical way of estimating the Hausdorff dimension of compact sets from below, which is often a difficult problem. For compact F, we have

$$\begin{array}{ccccccccc} & & & & \dim_{\mathrm{P}} F & & & & \\ & & & \rotatebox{30}{$\leqslant$} & & \rotatebox{-30}{$\leqslant$} & & & \\ \dim_{\mathrm{L}} F & \leqslant & \dim_{\mathrm{H}} F & & & & \overline{\dim}_{\mathrm{B}} F & \leqslant & \dim_{\mathrm{A}} F. \\ & & & \rotatebox{-30}{$\leqslant$} & & \rotatebox{30}{$\leqslant$} & & & \\ & & & & \underline{\dim}_{\mathrm{B}} F & & & & \end{array}$$

1.3 Iterated function systems

1.3.1 General iterated function systems and the symbolic space

Let (X, d) be a compact metric space. One of the most important ways of constructing fractals is via *iterated function systems*. An iterated function system (IFS) is a finite collection $\{S_i\}_{i \in \mathcal{I}}$ of contracting self-maps on X. It is a fundamental result in fractal geometry, dating back to Hutchinson's seminal 1981 paper [Hut], that for every IFS there exists a unique non-empty compact set F, called the *attractor*, which satisfies

$$F = \bigcup_{i \in \mathcal{I}} S_i(F).$$

This can be proved by an elegant application of Banach's contraction mapping theorem. Define a map $\Phi : \mathcal{K}(X) \to \mathcal{K}(X)$ by

$$\Phi(K) = \bigcup_{i \in \mathcal{I}} S_i(K).$$

It follows from the fact that each of the maps S_i is a contraction on (X, d) that Φ is a contraction on $(\mathcal{K}(X), d_{\mathcal{H}})$ and hence Φ has a unique fixed point $F \in \mathcal{K}(X)$.

If an IFS consists solely of *similarity* transformations, then the attractor is called a *self-similar set*. Likewise, if X is a Euclidean space and the mappings are all translate linear (*affine*) transformations, then the attractor is called *self-affine*. These classes of sets will be the fundamental objects of study in this thesis and will be discussed in more detail in Sections 1.3.2–1.3.3.

Often an attractor of an IFS has a more complicated structure and is more difficult to analyse if the pieces $\{S_i(F)\}_{i \in \mathcal{I}}$ overlap too much. As such, separation conditions are often imposed to make calculations easier. The following separation condition is fundamental in the theory of IFSs.

Definition 1.2. *An IFS, $\{S_i(F)\}_{i \in \mathcal{I}}$, with attractor F satisfies the* strong open set condition (SOSC)*, if there exists a non-empty open set, U, with $F \cap U \neq \emptyset$ and such that*

$$\bigcup_{i \in \mathcal{I}} S_i(U) \subseteq U$$

with the union disjoint.

A celebrated result of Schief [Sc1] is that the SOSC is equivalent to the weaker *open set condition* (OSC) if $X \subset \mathbb{R}^d$ and the maps in the IFS are similarities. The OSC is the same as the SOSC but without the requirement that $F \cap U \neq \emptyset$.

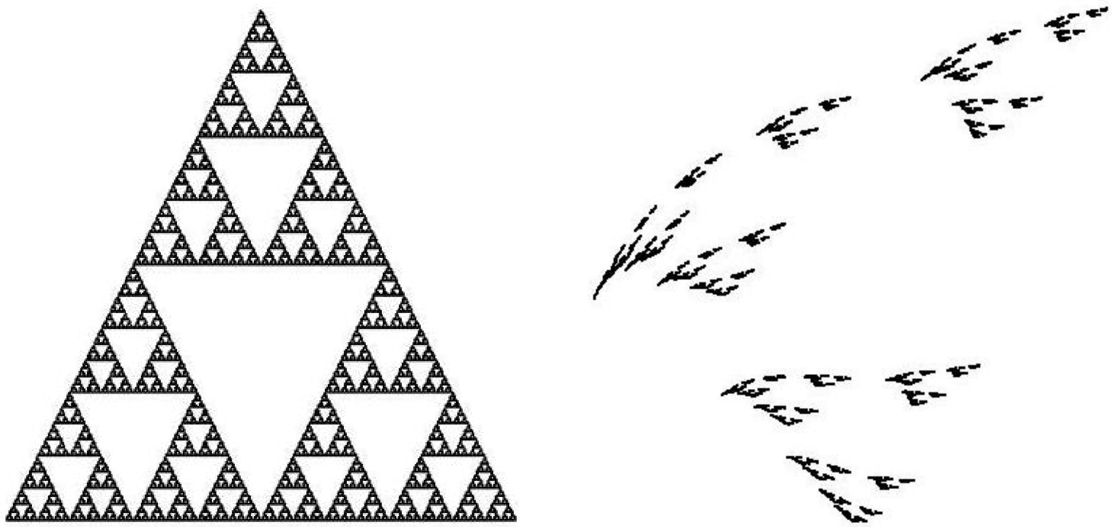

Figure 2: Two attractors of IFSs. Left: The Sierpiński Triangle. Right: An attractor of a nonlinear IFS. Observe that each of the attractors pictured above is made up of three scaled down copies of itself. The mappings used on the left are strict similarities and thus the Sierpiński Triangle is self-similar, whereas the mappings used on the right are more complicated nonlinear contractions.

Frequently in the study of attractors of IFSs, one uses a symbolic space built from the index set $\mathcal{I}$. The reason for this is that often it is more convenient to work with the geometry of this symbolic space than the actual geometry of the attractor and there is a straightforward way to transfer information from one space to the other. We will now briefly describe this technique and fix some notation which will be used throughout the thesis whenever a fixed IFS indexed by $\mathcal{I}$ is present. Let $\mathcal{I}^* = \bigcup_{k \geqslant 1} \mathcal{I}^k$ denote the set of all finite sequences with entries in $\mathcal{I}$ and for

$$\boldsymbol{i} = \big(i_1, i_2, \dots, i_k\big) \in \mathcal{I}^*$$

write

$$S_{\boldsymbol{i}} = S_{i_1} \circ S_{i_2} \circ \cdots \circ S_{i_k}.$$

Write $\mathcal{I}^{\mathbb{N}}$ to denote the set of all infinite $\mathcal{I}$-valued strings and for $\boldsymbol{i} \in \mathcal{I}^{\mathbb{N}}$ or $\mathcal{I}^l$ with $l \geqslant k$ write $\boldsymbol{i}|_k \in \mathcal{I}^k$ to denote the restriction of $\boldsymbol{i}$ to its first k entries. Let $\Pi : \mathcal{I}^{\mathbb{N}} \to F$ be the natural surjection from the 'symbolic' space to the 'geometric' space defined by

$$\Pi(\boldsymbol{i}) = \bigcap_{k \in \mathbb{N}} S_{\boldsymbol{i}|_k}(X).$$

For $\boldsymbol{i}, \boldsymbol{j} \in \mathcal{I}^*$, we will write $\boldsymbol{i} \prec \boldsymbol{j}$ if $\boldsymbol{j}|_k = \boldsymbol{i}$ for some $k \leqslant |\boldsymbol{j}|$, where $|\boldsymbol{j}|$ is the length of the string $\boldsymbol{j}$. For

$$\boldsymbol{i} = (i_1, i_2, \dots, i_{k-1}, i_k) \in \mathcal{I}^*$$

let

$$\overline{\boldsymbol{i}} = (i_1, i_2, \dots, i_{k-1}) \in \mathcal{I}^* \cup \{\omega\},$$

where ω is the empty word. For notational convenience the map S_ω is taken to be the identity map.

1.3.2 Self-similar sets

This thesis is primarily concerned with self-affine sets, however, *self-similar sets in Euclidean space* are a very special class of self-affine sets and so we will briefly discuss some of their key properties. Non-Euclidean self-similar sets will also crop up in a few contexts in later chapters.

The key reason why self-similar sets are so much easier to deal with than self-affine sets and more general attractors is that the images of the set under compositions of maps from the IFS form natural covers for the original set since everything scales down uniformly. This leads to a beautiful and simple formula for the dimensions of a self-similar set. Given an IFS, $\{S_i\}_{i \in \mathcal{I}}$, consisting of contracting similarities, the *similarity dimension* is defined to be the unique solution to the *Hutchinson-Moran formula*

$$\sum_{i \in \mathcal{I}} \mathrm{Lip}(S_i)^s = 1. \tag{1.1}$$

The similarity dimension is always an upper bound for the upper box dimension of the attractor (but not the Assouad dimension, see the example in Section 3.4.1). However, if the IFS satisfies the SOSC, then all of the dimensions discussed in this thesis are equal to the similarity dimension, see [F8, Section 9.3] and [Fr4] for the Assouad and lower dimension case. This formula first appeared in [Mo] (see also [E2, Chapter 13]) and later in [Hut] giving the Hausdorff dimension of Euclidean self-similar sets. For the non-Euclidean case, see [Sc2]. Furthermore, if the SOSC is satisfied then both the Hausdorff and packing measure of the attractor are positive and finite in the critical dimensions.

The case when the OSC is not satisfied is far from understood. In $\mathbb{R}^n$, a 'dimension drop' can occur if different iterates of maps in the IFS overlap exactly, but it is a major open problem to decide if this is the only way the dimension can drop, see for example [PS]. Recently an important step towards solving this conjecture has been made by Hochman [Ho]. Hochman verifies that the only cause for a dimension drop is exact overlaps, provided we are working in $\mathbb{R}$ and the defining parameters for the IFS are algebraic.

1.3.3 Self-affine sets

Self-affine sets are attractors of IFSs where all of the maps are contracting affine self-maps on some Euclidean space. An *affine* map is simply a map consisting of two parts: a linear part and a translation. Self-affine sets are notoriously difficult to handle in comparison with self-similar sets and there are still many fascinating open problems in the area. The study of the dimension theory of self-affine sets has really taken off in the literature since the early works of Mandelbrot in the mid-1980s. Since then, the study of self-affine sets has split into two parts: the *generic case* and the *specific case*. The generic case was pioneered by Falconer, beginning with the seminal papers [F2, F4] from 1988 and 1992 respectively. Here the linear parts of the mappings are fixed and the translates allowed to vary, and Falconer computed the dimensions for generic translations. The specific case began with the work of Bedford [Be1] and McMullen [McM] in 1984, where a much less general special case was considered, a class now known as the Bedford-McMullen carpets. The lack of generality in this specific case had the advantage that the simplicity of the model allowed exact calculation of the dimensions. This strategy led on to various different classes of self-affine carpet being introduced, with increasing levels of generality. We will describe both lines of research in detail in this section. The recent survey paper [F10] also describes both approaches as well as other areas connected with the dimension theory of self-affine sets.

The generic case: The *singular values* of a linear map, $A : \mathbb{R}^n \to \mathbb{R}^n$, are the positive square roots of the eigenvalues of $A^T A$. Viewed geometrically, these numbers are the lengths of the semi-axes of the image of the unit ball under A. Thus, roughly speaking, the singular values correspond to how much the map contracts (or expands) in different directions. For $s \in [0, n]$ define the *singular value function* $\phi^s(A)$ by

$$\phi^s(A) = \alpha_1 \alpha_2 \dots \alpha_{\lceil s \rceil - 1} \alpha_{\lceil s \rceil}^{s - \lceil s \rceil + 1} \tag{1.2}$$

where $\alpha_1 \geqslant \ldots \geqslant \alpha_n$ are the singular values of A. This function has played a vital role in the study of self-affine sets over the past 25 years. Let $\{A_i : i \in \mathcal{I}\}$ be a finite collection of contracting linear self-maps on $\mathbb{R}^n$, write $m = |\mathcal{I}|$ and let

$$d = d(A_i : i \in \mathcal{I}) = \inf \left\{ s : \sum_{k=1}^{\infty} \sum_{\mathcal{I}^k} \phi^s(A_{i_1} \circ \cdots \circ A_{i_k}) < \infty \right\}. \tag{1.3}$$

This number is called the *affinity dimension* of F and is always an upper bound for the upper box dimension of F, see [F2] and also [DO], but not the Assouad dimension of F, see Chapter 3. Moreover, Falconer proved the following celebrated result in the 1988 paper [F2]. We write $\mathcal{L}^{mn}$ to denote the m-fold product of n-dimensional Lebesgue measure, supported on the space $\times_{i \in \mathcal{I}} \mathbb{R}^n$.

Theorem 1.3. *Suppose each of the linear maps $\{A_i : i \in \mathcal{I}\}$ has Lipschitz constant strictly less than 1/2. Then, for $\mathcal{L}^{mn}$-almost all $(t_1, \ldots, t_m) \in \times_{i \in \mathcal{I}} \mathbb{R}^n$, the unique non-empty compact set F satisfying*

$$F = \bigcup_{i=1}^{m} (A_i + t_i)(F)$$

has

$$\dim_{\mathrm{B}} F = \dim_{\mathrm{P}} F = \dim_{\mathrm{H}} F = \min\{n, d\}.$$

In fact, the initial proof required that the Lipschitz constants be strictly less than 1/3 but this was relaxed to 1/2 by Solomyak [So], who also noted that 1/2 is the optimal constant, based on an example of Przytycki and Urbański [PU].

Figure 3: Three self-affine sets with the same linear part but different translations. Falconer's theorem implies that they all have the same Hausdorff dimension, unless of course we have been very unlucky and chosen some 'exceptional parameters'.

Despite the elegance of the above result, it seems difficult to calculate the exact dimension of a self-affine set in general. That being said, some work has been done on establishing sufficient conditions for the validity of Falconer's formula: see [Fr1, HL, KS] and Corollary 2.6 in Chapter 2, for example.

Given Theorem 1.3, a natural question is: can one remove the condition that the Lipschitz constants be strictly less than 1/2 by adding more randomness in the construction? This question was asked by Jordan, Pollicott and Simon [JPS], and moreover they gave a positive answer in the case where one randomly perturbs the translations at each stage of the construction. Indeed, the Lipschitz condition can be completely removed with the only requirement being that the maps are contractions. This idea was recently used by Falconer to study Bernoulli measures on such sets [F9], where the phrase '*almost self-affine set*' was coined to describe these systems.

The specific case: We first recall the construction introduced independently by Bedford and McMullen in 1984. Take the unit square, $[0,1]^2$, and divide it up into an $m \times n$ grid for some $m, n \in \mathbb{N}$ with $1 < m \leqslant n$. Then select a subset of the rectangles formed by the grid and consider the IFS consisting of the affine maps which map $[0,1]^2$ onto each chosen rectangle, preserving orientation. Bedford [Be1] and McMullen [McM] independently obtained explicit formulae for the box-counting, packing and Hausdorff dimensions of the attractor and, more recently, Mackay [M] computed the Assouad dimension. In general the Hausdorff dimension and box dimension can be different and can be strictly less than the affinity dimension. However, if the maps are chosen such that the projection onto the horizontal axis is an interval (having dimension 1), then the box dimension equals the affinity dimension. Our results, Corollaries 2.6 and 2.7 in Chapter 2, help to formalise this observation for a much larger class of self-affine sets. We will briefly recall the dimension formulae given by Bedford, McMullen and Mackay. For a Bedford-McMullen IFS $\{S_i\}_{i \in \mathcal{I}}$ with attractor F, let N be the number of maps in the IFS and for the ith column, let N_i be the number of maps chosen in that column and, finally, let $N_0 = \#\{i = 1, \dots, m : N_i \neq 0\}$.

Theorem 1.4 (Bedford-McMullen, Mackay). *Let F be a Bedford-McMullen carpet. Then*

$$\dim_{\mathrm{B}} F \;=\; \dim_{\mathrm{P}} F \;=\; \frac{\log N_0}{\log m} + \frac{\log(N/N_0)}{\log n},$$

$$\dim_{\mathrm{H}} F \;=\; \frac{\log \sum_{i=1}^{m} N_i^{\log m/\log n}}{\log m}$$

and

$$\dim_{\mathrm{A}} F \;=\; \frac{\log N_0}{\log m} + \max_{i=1,\dots,m} \frac{\log N_i}{\log n}.$$

Note that $s_1 = \frac{\log N_0}{\log m}$ is the dimension of the projection of F onto the horizontal axis and $\frac{\log N_i}{\log n}$ is the dimension of the self-similar attractor of the one-dimensional IFS induced by the ith column. These numbers can be computed via the Hutchinson-Moran formula (1.1). As such the box dimension of F satisfies

$$\sum_{i \in \mathcal{I}} (m^{-1})^{s_1} (n^{-1})^{\dim_{\mathrm{B}} F - s_1} = 1.$$

This version of the formula for box dimension is particularly useful to keep in mind whilst reading this thesis. Finally note that the Hausdorff, packing and box dimensions are equal if and only if N_i is constant whenever it is non-zero. In this case we say that the construction has *uniform vertical fibres*. In particular, in the typical situation, the Hausdorff, packing and Assouad dimensions are different, in stark contrast to the self-similar setting. Another important difference between self-affine carpets and self-similar sets satisfying the OSC is that the packing and Hausdorff measures need not be positive and finite. Indeed, Peres [P1, P2] proved that in the uniform vertical fibres case the measures are positive and finite, but in all other cases the measures are both infinite in the critical dimensions.

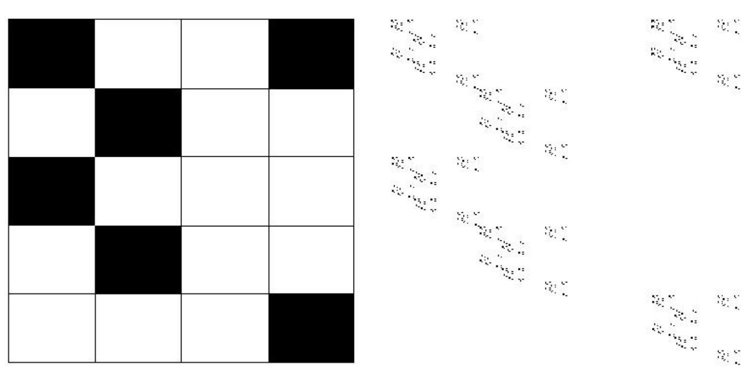

Figure 4: A self-affine Bedford-McMullen carpet with $m = 4$, $n = 5$. The shaded rectangles on the left indicate the 6 maps in the IFS. Note that this construction has uniform vertical fibres and thus the attractor has equal Hausdorff and packing dimension.

Gatzouras and Lalley [GL1] generalised the Bedford-McMullen construction by allowing the columns to have varying widths and be divided up, independently, with the only restriction being that the base of each chosen rectangle had to be strictly greater than the height. If we relax the 'strictly greater than' to 'greater than or equal' then we obtain a slightly more general class, not studied by Lalley and Gatzouras, which we will refer to as the extended Lalley-Gatzouras class. In the non-extended case they found an explicit formula for the box dimension and gave a variational principle for the Hausdorff dimension. Mackay [M] computed the Assouad dimension for this class. Barański [B2] studied another class where he divided the unit square up into an arbitrary mesh of rectangles by slicing horizontally and vertically a finite number of times (at least once in each direction). Again he gave an explicit formula for the box dimension and gave a variational principle for the Hausdorff dimension. Also, Feng and Wang [FeW] considered a construction where the rectangles did not have to be 'aligned' as in the Barański type IFSs. This added complication meant that the box dimension of the attractor was given in terms of the dimensions of its projection onto the horizontal and vertical axes, which may be difficult to compute as they are self-similar sets which no longer need to satisfy the OSC. The biggest difference between the Lalley-Gatzouras class and the Barański class is that for the Barański class the maps can sometimes contract more in the vertical direction and sometimes more in the horizontal direction. This property makes this class *much* more difficult to deal with and will be a common theme of this thesis. It is interesting to note that Bedford-McMullen obtained an explicit formula for the Hausdorff dimension, whereas in the other classes the Hausdorff dimension is only given via a variational principle and may be difficult to compute or even estimate. We do not present the results found by Lalley-Gatzouras, Barański and Feng-Wang here, but we will discuss some of them later in the thesis when they are especially relevant for our proofs.

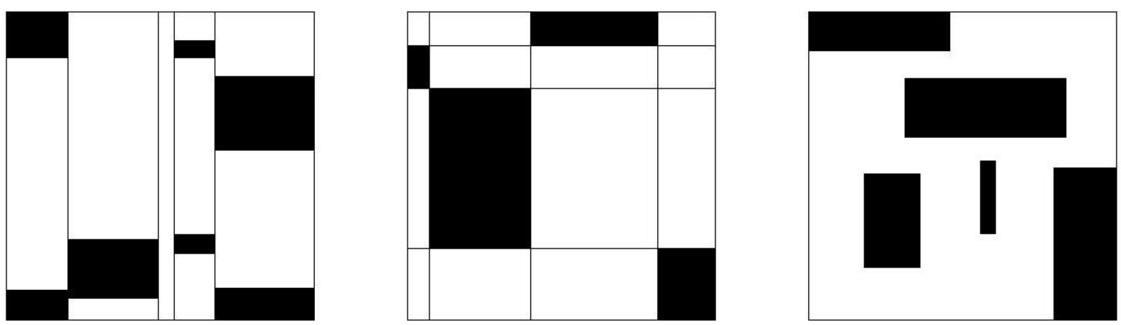

Figure 5: Three examples of IFSs of the types considered by Lalley-Gatzouras, Barański and Feng-Wang, respectively. The shaded rectangles represent the affine maps.

Barański [B2] also computes the Hausdorff, box and packing dimensions of a more general class of sets which he calls 'rectangle-like constructions'. These are not strictly self-affine, but their dimension theory can be modeled by a self-affine Barański type carpet; for example the flexed Sierpiński triangle, see [B2, Section 7] and [B1].

The Bedford-McMullen and Lalley-Gatzouras classes are discussed in some detail in [Pe], Section 16, as well as a generalisation known as 'geometric constructions with rectangles'. These are usually not self-affine, but display some of the same characteristics as the Lalley-Gatzouras class while allowing the rectangles in the construction to be slightly rotated. We note here that, although the rectangles can be slightly rotated, the construction forces the rotations to become less significant at later stages in the construction.

In all of the aforementioned examples the affine maps are orientation-preserving. In Chapter 2 of this thesis we relax this requirement by allowing the maps to have non-trivial rotational and reflectional components. We refer to the attractors of such systems as "box-like" sets and give their formal definition in Section 2.1.1. In Section 2.2 we compute the packing and box-counting dimensions by means of a pressure type formula based on the singular values of the maps. As in [FeW] the dimension of projections will be significant. In Chapter 3 we compute the Assouad and lower dimensions of the Lalley-Gatzouras and Barański carpets.

1.3.4 Inhomogeneous iterated function systems

Inhomogeneous iterated function systems are generalisations of the IFSs described above. Indeed, one might call the attractors of the standard systems *homogeneous attractors*. Let (X, d) be a compact metric space, let $\mathbb{I} = \{S_i\}_{i \in \mathcal{I}}$ be a standard IFS, and fix a compact set $C \subseteq X$, sometimes called the *condensation set*. Analogous to the homogeneous case, there is a unique non-empty compact set, F_C, satisfying

$$F_C = \bigcup_{i \in \mathcal{I}} S_i(F_C) \ \cup \ C, \tag{1.4}$$

which we refer to as the *inhomogeneous* attractor (with condensation C). Note that homogeneous attractors are inhomogeneous attractors with condensation equal to the empty set. From now on we will assume that the condensation set is non-empty. Inhomogeneous attractors were introduced and studied in [BD] and are also discussed in detail in [Ba2] where, among other things, Barnsley gives applications of these schemes to image compression. Define the *orbital set*, $\mathcal{O}$, by

$$\mathcal{O} \ = \ C \ \cup \ \bigcup_{i \in \mathcal{I}^*} S_i(C),$$

i.e., the union of the condensation set, C, with all images of C under compositions of maps in the IFS. The term *orbital set* was introduced in [Ba2] and it turns out that this set plays an important role in the structure of inhomogeneous attractors and, in particular,

$$F_C \ = \ F_\emptyset \cup \mathcal{O} \ = \ \overline{\mathcal{O}}, \tag{1.5}$$

where $F_\emptyset$ is the *homogeneous* attractor of the IFS, $\mathbb{I}$.

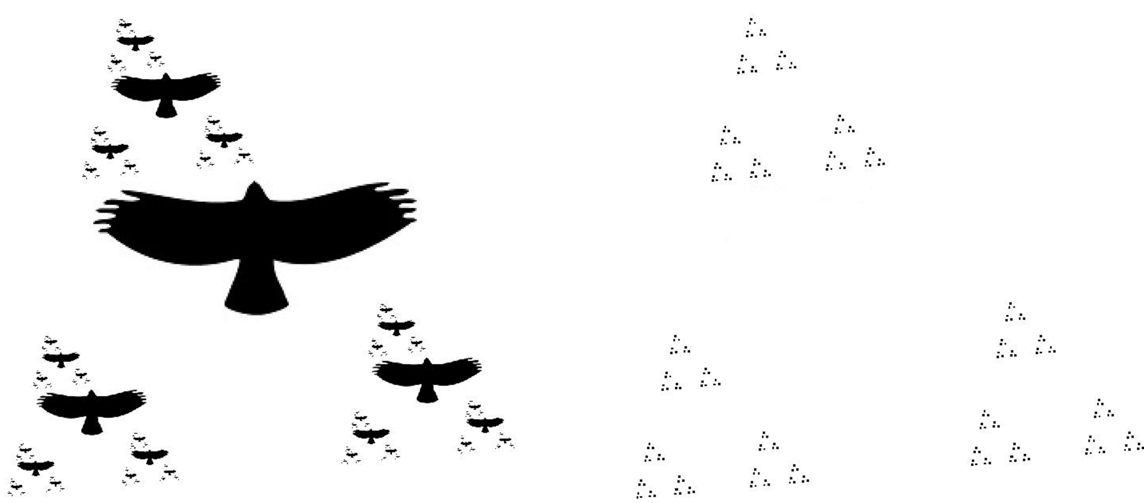

Figure 6: *A flock of birds from above* (left). The 'flock' is represented by an inhomogeneous self-similar set. The large bird in the middle is the condensation and there are three similarity mappings in the IFS all with contraction ratio 1/3. The corresponding homogeneous attractor is shown on the right. This is a totally disconnected self-similar set with all the dimensions equal to 1.

The relationship (1.5) was proved in [Sn, Lemma 3.9] in the case where X is a compact subset of $\mathbb{R}^d$ and the maps are similarities. We note here that their arguments easily generalise to obtain the general case stated above. When considering the dimension dim of F_C, one expects the relationship

$$\dim F_C = \max\{\dim F_\emptyset, \ \dim C\} \tag{1.6}$$

to hold. Indeed, if dim is countably stable, monotone and stable under Lipschitz maps, then

$$\begin{aligned}\max\{\dim F_\emptyset,\ \dim C\} \leqslant \dim F_C &= \dim(F_\emptyset \cup \mathcal{O}) \\ &= \max\Big\{\dim F_\emptyset,\ C \ \cup \bigcup_{i\in\mathcal{I}^*} S_i(C)\Big\} \\ &\leqslant \max\{\dim F_\emptyset,\ \dim C\}\end{aligned}$$

and so the formula holds trivially. Thus, studying the Hausdorff and packing dimensions of inhomogeneous attractors is equivalent to studying the Hausdorff and packing dimensions of the corresponding homogeneous attractor, and thus is not an interesting problem in its own right. However, the upper and lower box dimensions, Assouad dimension and lower dimension are not countably stable and so computing these dimensions in the inhomogeneous case is interesting and (perhaps) challenging, although one may still expect, somewhat naïvely, that the relationship (1.6) should hold for these dimensions.

In this thesis, we wish to investigate inhomogeneous attractors where the corresponding homogeneous attractor is a self-affine carpet. Since the Hausdorff and packing dimensions are easy to compute we will focus on the box dimensions, and in Chapter 5 we compute the box dimensions of inhomogeneous self-affine carpets in the Barański and Lalley-Gatzouras class. Despite our primary interest being the self-affine case, there are still some interesting open questions in the self-similar case and so to put our results on inhomogeneous self-affine carpets into context, we first investigate inhomogeneous self-similar sets in Chapter 4. Moreover, some of our results in the self-affine case rely on our results in Chapter 4.

The box dimensions were considered by Snigireva and Olsen. In [OSn, Corollary 2.6] and [Sn, Theorem 3.10 (2)] it was proved that if $X \subset \mathbb{R}^d$; each of the S_i are similarities; and the sets $S_1(F_C), \dots, S_N(F_C), C$ are pairwise disjoint, then

$$\overline{\dim}_{\mathrm{B}} F_C = \max\{\overline{\dim}_{\mathrm{B}} F_\emptyset,\ \overline{\dim}_{\mathrm{B}} C\}.$$

The authors then asked the following question, see [OSn, Question 2.7] and [Sn, Question 3.12].

Question 1.5. *Does the above formula for upper box dimension remain true if we relax the separation conditions to only the inhomogeneous open set condition (IOSC)?*

In Chapter 4 we give an affirmative answer to this question and, furthermore, prove that it holds assuming only that the IFS, $\mathbb{I}$, satisfies the strong open set condition (which is equivalent to the open set condition if $X \subset \mathbb{R}^d$), see Corollary 4.2, and even without assuming any separation conditions it holds *generically*, see Corollary 4.3. We remark here that the definitions of the IOSC given in [OSn, Sn] are slightly different. Rather than give both of the technical definitions we simply remark that we are able to answer Question 1.5 using significantly weaker separation conditions than either version of the IOSC. In particular, the condensation set can have arbitrary overlaps with the basic sets in the construction of the homogeneous attractor.

In [OSn, Sn] the authors also point out that they are not aware whether the corresponding formula holds for lower box dimension. The following question is asked in [Sn, Question 3.11].

Question 1.6. *If $X \subset \mathbb{R}^d$, each of the S_i are similarities and the sets $S_1(F_C), \dots, S_N(F_C), C$ are pairwise disjoint, then is it true that*

$$\underline{\dim}_{\mathrm{B}} F_C = \max\{\underline{\dim}_{\mathrm{B}} F_\emptyset,\ \underline{\dim}_{\mathrm{B}} C\}?$$

We prove that the answer to this question is *no*, see Theorem 4.9 and Proposition 4.12 (2). We also give some sufficient conditions for the answer to be *yes*, see Corollary 4.6, Corollary 4.11, Theorem 4.10 and Proposition 4.12 (1).

1.3.5 Random iterated function systems

Random iterated function systems (RIFSs) are another generalisation of the standard IFSs and are based on fixing a finite set of IFSs and then 'randomly choosing' which one to use at each stage in the construction. There are many different ways of defining this randomisation, in particular, see the work on V-variable fractals by Barnsley, Hutchinson and Stenflo [BHS1, BHS2]. To keep in line with this terminology we point out that in this thesis we are only concerned with 1-variable randomness, which we will now describe in detail.

Let (X, d) be a compact metric space. We define a random iterated function system (RIFS) to be a set $\underline{\mathbb{I}} = \{\mathbb{I}_1, \dots, \mathbb{I}_N\}$, where each $\mathbb{I}_i$ is a deterministic IFS, $\mathbb{I}_i = \{S_{i,j}\}_{j \in \mathcal{I}_i}$, for a finite index set, $\mathcal{I}_i$, and each map, $S_{i,j}$, is a contracting self-map on X. We define a continuum of attractors of $\underline{\mathbb{I}}$ in the following way. Let $D = \{1, \dots, N\}$, $\Omega = D^{\mathbb{N}}$ and let $\omega = (\omega_1, \omega_2, \dots) \in \Omega$. Define the *attractor of* $\underline{\mathbb{I}}$ *corresponding to* ω by

$$F_\omega = \bigcap_k \bigcup_{i_1 \in \mathcal{I}_{\omega_1}, \dots, i_k \in \mathcal{I}_{\omega_k}} S_{\omega_1, i_1} \circ \dots \circ S_{\omega_k, i_k}(X).$$

So, by 'randomly choosing' $\omega \in \Omega$, we 'randomly choose' an attractor F_ω. Attractors of RIFSs can enjoy a much richer and more complicated structure than attractors of IFSs. We provide some pictures to help illustrate this construction; see also Section 6.5.3.

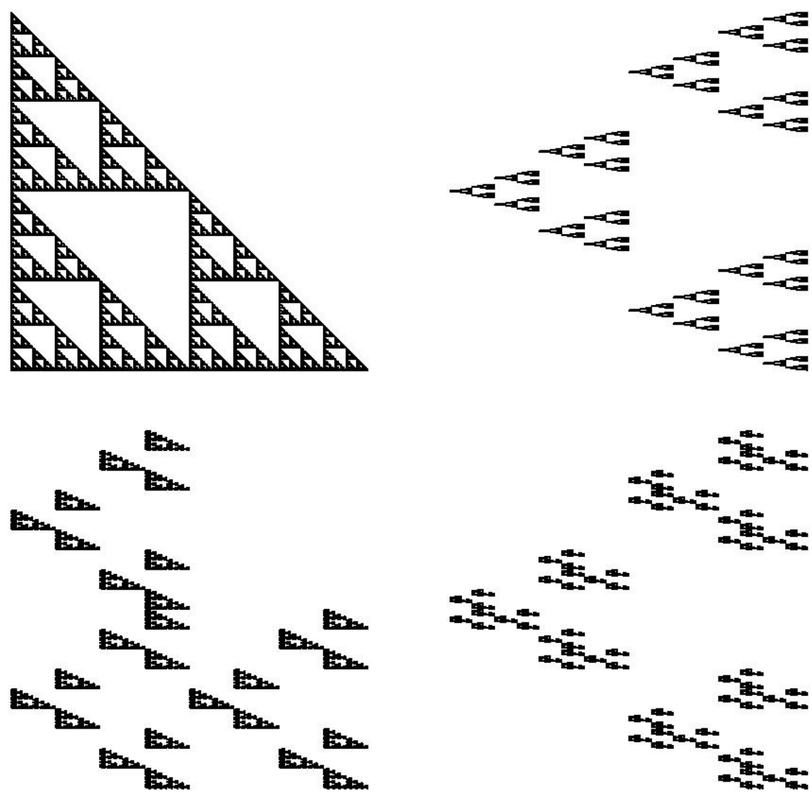

Figure 7: The attractors of deterministic IFSs $\mathbb{I}_1$ (top-left) and $\mathbb{I}_2$ (top-right) along with two random attractors of $\underline{\mathbb{I}} = \{\mathbb{I}_1, \mathbb{I}_2\}$ corresponding to $\omega = (1, 2, 2, 1, 1, 1, \dots)$ (bottom-left) and $\omega = (2, 2, 1, 2, 1, 2, \dots)$ (bottom-right). The attractor on the top-left is a variant of the Sierpiński Triangle and the attractor on the top-right is a Bedford-McMullen carpet.

We wish to make statements about the *generic* nature of F_ω. In particular, what is the generic dimension of F_ω? The most common approach to studying random fractals is to associate a probability measure with the space of possible attractors and then make *almost sure* statements. For some examples based on conformal systems, see [F1, LW, O1, BHS1, BHS2, Ba2]; and for non-conformal (self-affine) systems, see [GuLi1, GuLi2, GL2, O6, FO]. Associate a probability vector, $\underline{p} = (p_1, \ldots, p_N)$, with $\underline{\mathbb{I}}$. Then, to obtain our random attractor, we choose each entry in ω randomly and independently with respect to $\underline{p}$. This induces a probability measure, $\mathbb{P}$, on Ω given by

$$\mathbb{P} = \prod_{\mathbb{N}} \sum_{i=1}^{N} p_i \, \delta_i,$$

where δ_i is the Dirac measure concentrated at $i \in D = \{1, \ldots, N\}$. We say that a property of the random attractors is generic (in a probabilistic sense) if it occurs for $\mathbb{P}$-almost all $\omega \in \Omega$. This approach has attracted much attention in the literature with the ergodic theorem often playing a key role in the analysis, utilising the fact that $\mathbb{P}$ is ergodic with respect to the left shift on Ω. We give a couple of examples for which we will need to generalise the standard OSC to the RIFS situation in the following way.

Definition 1.7. *We say that $\underline{\mathbb{I}}$ satisfies the* uniform open set condition (UOSC)*, if each deterministic IFS satisfies the SOSC and the open set can be chosen uniformly, i.e. there exists a non-empty open set $U \subseteq X$ such that, for each $i \in D$, we have*

$$\bigcup_{j \in \mathcal{I}_i} S_{i,j}(U) \subseteq U$$

with the union disjoint.

The UOSC also appears in [BHS1], for example.

Theorem 1.8 ([Ha, BHS1])**.** *Let $\underline{\mathbb{I}} = \{\mathbb{I}_1, \ldots, \mathbb{I}_N\}$ be an RIFS consisting of similarity maps on $\mathbb{R}^n$ with associated probability vector $\boldsymbol{p} = (p_1, \ldots, p_N)$. Assume that $\mathbb{I}$ satisfies the UOSC and let s be the solution of*

$$\prod_{i=1}^{N} \left(\sum_{j \in \mathcal{I}_i} \mathrm{Lip}(S_{i,j})^s \right)^{p_i} = 1. \tag{1.7}$$

Then, for $\mathbb{P}$-almost all $\omega \in \Omega$, $\dim_{\mathrm{H}} F_\omega = \dim_{\mathrm{B}} F_\omega = \dim_{\mathrm{P}} F_\omega = s$.

Equation (1.7) should be viewed as a randomised version of the Hutchinson-Moran formula (1.1). Here the almost sure dimension is 'some sort of weighted average' of the dimensions of the attractors of $\mathbb{I}_i$. For a proof of Theorem 1.8, see [Ha, BHS1] or alternatively [Ba2, Chapter 5.7] and the references therein.

Self-affine sets often provide examples of strange behaviour not observed in the self-similar setting. We will now describe a random Bedford-McMullen carpet, a construction which will be used to provide several interesting examples in Chapter 6. Take N deterministic IFSs, $\mathbb{I}_i$, built by dividing the unit square into an $m_i \times n_i$ grid with $m_i \leqslant n_i$ and an associated probability vector $\boldsymbol{p} = (p_1, \ldots, p_N)$. The following dimension formula was given in [GuLi2] and can also be derived from results in [FO].

Theorem 1.9 ([FO], [GuLi2])**.** *For $j = 1 \ldots m_i$, let $N_{i,j} \in \{0, \ldots, m_i\}$ denote the number of rectangles chosen in the jth column in the ith IFS. Let*

$$\nu_1 = m_1^{p_1} \cdots m_N^{p_N} \qquad \text{and} \qquad \nu_2 = n_1^{p_1} \cdots n_N^{p_N}.$$

Then, for $\mathbb{P}$-almost all $\omega \in \Omega$,

$$\dim_{\mathrm{H}} F_\omega = \sum_{i=1}^{N} p_i \left(\frac{1}{\log \nu_1} \log \left(\sum_{j=1}^{m_i} N_{i,j}^{\log \nu_1 / \log \nu_2} \right) \right).$$

We note that in [FO] a higher dimensional analogue of Theorem 1.9 was obtained where one begins the construction with the unit cube in $\mathbb{R}^d$ rather than the unit square. Notice that if $m_i = m$ and $n_i = n$ for all i, then the above dimension formula simplifies to

$$\dim_{\mathrm{H}} F_\omega = \sum_{i=1}^{N} p_i \left(\frac{1}{\log m} \log \left(\sum_{j=1}^{m} N_{i,j}^{\log m/ \log n} \right) \right) = \sum_{i=1}^{N} p_i \, s_i,$$

where s_i is the Hausdorff dimension of the attractor of $\mathbb{I}_i$ given by Bedford and McMullen, see Theorem 1.4. In this case, the almost sure Hausdorff and box dimension were computed in [GuLi1]. If the m_i and n_i are not chosen uniformly, then we have a nonlinear dependence on the probability vector $\boldsymbol{p}$. An example using Theorem 1.9 will be given in Section 6.4.3.

In Chapter 6, we will study random self-affine carpets using an alternative notion of randomness, where ω is chosen according to the topological properties of Ω, rather than the probabilistic properties. This approach leads to a starkly different theory and in fact allows us to consider much more general systems than simply those based on self-affine carpets, although self-affine carpets will be used to demonstrate some key phenomena.

2 Box-like self-affine sets

2.1 Introduction

The aim of this chapter is to unify and generalise the classes of exceptional self-affine set described in Section 1.3.3 by considering a more general construction which, in particular, allows the generating maps to have non-trivial rotational and reflectional components. Our motivation is not simply generality for generality's sake but also to reconcile the 'exceptional constructions' with Falconer's almost sure formula. As such, we introduce a *modified singular value function* and use it to compute the packing and box-counting dimensions assuming a natural separation condition.

2.1.1 Box-like self-affine sets and notation

We call a self-affine set *box-like* if it is the attractor of an IFS consisting of contracting affine maps which take the unit square, $[0,1]^2$, to a rectangle with sides parallel to the axes. The affine maps which make up such an IFS are necessarily of the form $S = T \circ L + t$, where T is a contracting linear map of the form

$$T = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

for some $a, b \in (0,1)$; L is a linear isometry of the plane for which $L([-1,1]^2) = [-1,1]^2$; and $t \in \mathbb{R}^2$ is a translation vector. Note that there are 8 possible choices for L and if, for all maps in the IFS, we let L be the identity map, then we obtain the class of self-affine sets considered by Feng and Wang [FeW].

Let $\{S_i\}_{i \in \mathcal{I}}$ be an IFS consisting of maps of the form described above for some finite index set $\mathcal{I}$, with $|\mathcal{I}| \geqslant 2$, and let F be the corresponding attractor, i.e., the unique non-empty compact set satisfying

$$F = \bigcup_{i \in \mathcal{I}} S_i(F).$$

We refer to F as the *box-like self-affine set*. It is clear that we may choose a compact square $\mathcal{Q} \subset \mathbb{R}^2$ such that $\bigcup_{i \in \mathcal{I}} S_i(\mathcal{Q}) \subseteq \mathcal{Q}$. Without loss of generality we will assume throughout that we may choose $\mathcal{Q} = [0,1]^2$. Let

$$\mathcal{I}_A = \{i \in \mathcal{I} : S_i \text{ maps horizontal lines to horizontal lines}\}$$

and

$$\mathcal{I}_B = \{i \in \mathcal{I} : S_i \text{ maps horizontal lines to vertical lines}\}.$$

If $\mathcal{I}_B = \emptyset$, then we will say F is of *separated type* and otherwise we will say that F is of *non-separated type*. It will become clear why we make this distinction in the following section.

Let $\alpha_1(\boldsymbol{i}) \geqslant \alpha_2(\boldsymbol{i})$ be the singular values of the linear part of the map $S_{\boldsymbol{i}}$. Note that, for all $\boldsymbol{i} \in \mathcal{I}^*$, the singular values, $\alpha_1(\boldsymbol{i})$ and $\alpha_2(\boldsymbol{i})$, are just the lengths of the sides of the rectangle $S_{\boldsymbol{i}}\big([0,1]^2\big)$. Finally, let

$$\alpha_{\min} = \min\{\alpha_2(i) : i \in \mathcal{I}\}$$

and

$$\alpha_{\max} = \max\{\alpha_1(i) : i \in \mathcal{I}\}.$$

2.2 Results

In this section we will state our main results of this chapter. The dimension formula, which relies on the knowledge of the dimensions of the projection of F onto the horizontal and vertical axes, will be given in Section 2.2.1. In Section 2.2.2 we will discuss the problem of calculating the dimensions of the relevant projections.

2.2.1 The dimension formula

Let $\pi_1, \pi_2 : \mathbb{R}^2 \to \mathbb{R}$ be defined by $\pi_1(x,y) = x$ and $\pi_2(x,y) = y$ respectively. Also, let

$$s_1 = \dim_{\mathrm{B}} \pi_1(F)$$

and

$$s_2 = \dim_{\mathrm{B}} \pi_2(F).$$

It can be shown that both $\dim_{\mathrm{B}} \pi_1(F)$ and $\dim_{\mathrm{B}} \pi_2(F)$ exist using the 'implicit theorems' found in [F3, McL], or, alternatively, see Lemma 2.8 in Section 2.2.2. For $\boldsymbol{i} \in \mathcal{I}^*$, let $b(\boldsymbol{i}) = \big|\pi_1\big(S_{\boldsymbol{i}}\big([0,1]^2\big)\big)\big|$ and $h(\boldsymbol{i}) = \big|\pi_2\big(S_{\boldsymbol{i}}\big([0,1]^2\big)\big)\big|$ denote the length of the base and height of the rectangle $S_{\boldsymbol{i}}\big([0,1]^2\big)$ respectively and define $\pi_{\boldsymbol{i}} : \mathbb{R}^2 \to \mathbb{R}$ by

$$\pi_{\boldsymbol{i}} = \begin{cases} \pi_1 & \text{if } \boldsymbol{i} \in \mathcal{I}_A \text{ and } b(\boldsymbol{i}) \geqslant h(\boldsymbol{i}) \\ \pi_2 & \text{if } \boldsymbol{i} \in \mathcal{I}_A \text{ and } b(\boldsymbol{i}) < h(\boldsymbol{i}) \\ \pi_1 & \text{if } \boldsymbol{i} \in \mathcal{I}_B \text{ and } b(\boldsymbol{i}) < h(\boldsymbol{i}) \\ \pi_2 & \text{if } \boldsymbol{i} \in \mathcal{I}_B \text{ and } b(\boldsymbol{i}) \geqslant h(\boldsymbol{i}) \end{cases}$$

Finally, let $s(\boldsymbol{i}) = \dim_{\mathrm{B}} \pi_{\boldsymbol{i}} F$. In fact, $s(\boldsymbol{i})$ is simply the box dimension of the projection of $S_{\boldsymbol{i}}(F)$ onto the longest side of the rectangle $S_{\boldsymbol{i}}\big([0,1]^2\big)$ and is always equal to either s_1 or s_2.

For $s \geqslant 0$ and $\boldsymbol{i} \in \mathcal{I}^*$, we define the *modified singular value function*, ψ^s, of $S_{\boldsymbol{i}}$ by

$$\psi^s\big(S_{\boldsymbol{i}}\big) = \alpha_1(\boldsymbol{i})^{s(\boldsymbol{i})}\ \alpha_2(\boldsymbol{i})^{s-s(\boldsymbol{i})}, \tag{2.1}$$

and for $s \geqslant 0$ and $k \in \mathbb{N}$, we define a number Ψ_k^s by

$$\Psi_k^s = \sum_{\boldsymbol{i} \in \mathcal{I}^k} \psi^s(S_{\boldsymbol{i}}).$$

Lemma 2.1 (multiplicative properties)**.**

a) For $s \geqslant 0$ and $\boldsymbol{i}, \boldsymbol{j} \in \mathcal{I}^*$ we have

a1) If $s < s_1 + s_2$, then $\psi^s(S_{\boldsymbol{i}} \circ S_{\boldsymbol{j}}) \leqslant \psi^s(S_{\boldsymbol{i}})\,\psi^s(S_{\boldsymbol{j}})$;

a2) If $s = s_1 + s_2$, then $\psi^s(S_{\boldsymbol{i}} \circ S_{\boldsymbol{j}}) = \psi^s(S_{\boldsymbol{i}})\,\psi^s(S_{\boldsymbol{j}})$;

a3) If $s > s_1 + s_2$, then $\psi^s(S_{\boldsymbol{i}} \circ S_{\boldsymbol{j}}) \geqslant \psi^s(S_{\boldsymbol{i}})\,\psi^s(S_{\boldsymbol{j}})$.

b) For $s \geqslant 0$ and $k, l \in \mathbb{N}$ we have

b1) If $s < s_1 + s_2$, then $\Psi_{k+l}^s \leqslant \Psi_k^s\,\Psi_l^s$;

b2) If $s = s_1 + s_2$, then $\Psi_{k+l}^s = \Psi_k^s\,\Psi_l^s$;

b3) If $s > s_1 + s_2$, then $\Psi^s_{k+l} \geqslant \Psi^s_k \, \Psi^s_l$.

We will prove Lemma 2.1 in Section 2.4.2. It follows from Lemma 2.1 and standard properties of sub- and super-multiplicative sequences that we may define a function $P : [0, \infty) \to [0, \infty)$ by

$$P(s) = \lim_{k \to \infty} (\Psi^s_k)^{1/k}$$

where, in fact,

$$\lim_{k \to \infty} (\Psi^s_k)^{1/k} = \begin{cases} \inf_{k \in \mathbb{N}} (\Psi^s_k)^{1/k} & \text{if } s \in [0, s_1 + s_2) \\ \Psi^s_1 & \text{if } s = s_1 + s_2 \\ \sup_{k \in \mathbb{N}} (\Psi^s_k)^{1/k} & \text{if } s \in (s_1 + s_2, \infty) \end{cases}$$

Our function P is related to the notion of *topological pressure* in dynamical systems and in particular *non-additive* topological pressure. Although the spirit of this chapter is not dynamical, we can view our self-affine attractors as repellers of certain expanding dynamical systems (provided we have some separation conditions). For example, if a box-like self-affine set F satisfies the strong separation condition, then we can define an expanding map, f, on F such that the inverse branches of f coincide with the contractions in the defining IFS. Assuming $s \leqslant s_1 + s_2$ and writing $D_x f^n$ to denote the Jacobian of f^n at $x \in F$, the sequence $\big\{ \log \psi^s \big((D_x f^n)^{-1} \big) \big\}_n$ gives a subadditive valuation on F and a simple calculation yields

$$\log P(s) = P_{\text{top}}\big(f, \big\{ \log \psi^s \big((D_x f^n)^{-1} \big) \big\}\big)$$

where $P_{\text{top}}\big(f, \big\{ \log \psi^s \big((D_x f^n)^{-1} \big) \big\}\big)$ is the topological pressure of the sequence $\big\{ \log \psi^s \big((D_x f^n)^{-1} \big) \big\}_n$ with respect to f in the sense of Falconer [F5]. Non-additive versions of the thermodynamic formalism have attracted substantial attention over the last 15 years. For more details the reader is referred to, for example, [Pe, Ba, FH].

Lemma 2.2 (Properties of P)**.**

(1) For all $s, t \geqslant 0$ we have

$$\alpha^s_{\min} P(t) \, \leqslant \, P(s+t) \, \leqslant \, \alpha^s_{\max} P(t)$$

and furthermore, setting $t = 0$, for all $s \geqslant 0$ we have

$$0 \, < \, \alpha^s_{\min} P(0) \, \leqslant \, P(s) \, \leqslant \, \alpha^s_{\max} P(0) \, < \, \infty,$$

where $P(0) \in [|\mathcal{I}|, \infty)$ is a constant;

(2) P is continuous on $[0, \infty)$;

(3) P is strictly decreasing on $[0, \infty)$;

(4) There is a unique value $s \geqslant 0$ for which $P(s) = 1$.

We will prove Lemma 2.2 in Section 2.4.3. The following separation condition, which we will need to obtain the lower bound in our dimension result, was introduced in [FeW].

Definition 2.3. *An IFS $\{S_i\}_{i=1}^m$ satisfies the rectangular open set condition (ROSC) if there exists a non-empty open rectangle, $R = (a, b) \times (c, d) \subset \mathbb{R}^2$, such that $\{S_i(R)\}_{i=1}^m$ are pairwise disjoint subsets of R.*

We can now state our main result concerning the packing and box-counting dimensions for box-like self-affine sets.

Theorem 2.4. *Let F be a box-like self-affine set. Then $\dim_{\mathrm{P}} F = \overline{\dim}_{\mathrm{B}} F \leqslant s$ where $s \geqslant 0$ is the unique solution of $P(s) = 1$. Furthermore, if the ROSC is satisfied, then $\dim_{\mathrm{P}} F = \dim_{\mathrm{B}} F = s$.*

We will prove Theorem 2.4 in Section 2.5. We will now give two corollaries of Theorem 2.4 which show that the dimension formula can be simplified in certain situations. The first of which deals with the case where $s_1 = s_2$. This will occur, for example, if F is of non-separated type (see Lemma 2.8).

Corollary 2.5. *Let F be a box-like self-affine set which satisfies the ROSC and is such that $s_1 = s_2 =: t$. Then $\dim_P F = \dim_B F = s$, where s satisfies*

$$\lim_{k\to\infty} \left(\sum_{\boldsymbol{i}\in\mathcal{I}^k} \alpha_1(\boldsymbol{i})^t \ \alpha_2(\boldsymbol{i})^{s-t} \right)^{1/k} = 1.$$

The second corollary deals with the case where $s_1 = s_2 = 1$. Some easily verified sufficient conditions for this to occur are given in Lemma 2.9.

Corollary 2.6. *Let F be a box-like self-affine set which satisfies the ROSC and is such that $s_1 = s_2 = 1$. Then*

$$\dim_P F = \dim_B F = d$$

where d is the affinity dimension (1.3).

To prove Corollary 2.6 simply observe that, if $s_1 = s_2 = 1$, then our modified singular value function (2.1) coincides with the singular value function (1.2) in the range $s \in [1, 2]$. Furthermore, it is clear that the dimension lies in this range and therefore the unique value of s satisfying $P(s) = 1$ is the affinity dimension. The converse of Corollary 2.6 is not true. In particular, it is not true that, if both s_1 and s_2 are strictly less than 1, then the packing dimension is strictly less than the affinity dimension (for example, some self-similar sets). However, it is possible to give simple sufficient conditions for the packing dimension to drop below the affinity dimension. For example, if both s_1 and s_2 are strictly less than $\min\{1, d\}$, where d is the affinity dimension, and there exists a constant $\eta \in (0, 1)$ such that for all $k \in \mathbb{N}$ and all $\boldsymbol{i} \in \mathcal{I}^k$, $\alpha_2(\boldsymbol{i}) \leqslant \eta^k \alpha_1(\boldsymbol{i})$, then the packing dimension of the attractor is strictly less than the affinity dimension. To see this let $\epsilon = \min\{1, d\} - \max\{s_1, s_2\} > 0$ and d be the affinity dimension and note that

$$P(d) = \inf_k \left(\sum_{\boldsymbol{i}\in\mathcal{I}^k} \psi^d(S_{\boldsymbol{i}}) \right)^{1/k} \leqslant \ \inf_k \left(\sum_{\boldsymbol{i}\in\mathcal{I}^k} \phi^d(S_{\boldsymbol{i}}) \left(\frac{\alpha_2(\boldsymbol{i})}{\alpha_1(\boldsymbol{i})} \right)^{\epsilon} \right)^{1/k} \leqslant \eta^{\epsilon} \ < 1$$

from which it follows that $\dim_P F < d$.

Since $\overline{\dim}_B F \leqslant s_1 + s_2$, it is clear that the solution of $P(s) = 1$ always lies in the range $[0, s_1 + s_2]$. Even in the case where s_1 and s_2 can be computed it still may be very difficult to compute the solution of $P(s) = 1$ explicitly. However, since the solution lies in the submultiplicative region, it can be numerically estimated from above by considering the sequence $\{\hat{s}_k\}_{k\in\mathbb{N}}$ where each $\hat{s}_k$ is defined by $\Psi_k^{\hat{s}_k} = 1$ and is an upper bound for the dimension. Unfortunately, without establishing some sort of 'quasimultiplicativity' for the sequence $\{\Psi_k^s\}$, it is difficult to say anything about the rate of convergence of the sequence $\{\hat{s}_k\}$. In [FS] this problem was addressed in the case where standard singular value functions are used in place of modified singular value functions and a quasimultiplicativity condition was derived, provided that a certain technical condition was satisfied by the linear parts of the mappings.

We will now present one final corollary of Theorem 2.4 which shows that for a certain class of box-like self-affine sets of separated type the dimension may be calculated explicitly due to the modified singular value function being multiplicative for all s rather than sub- or supermultiplicative.

Corollary 2.7. *Let F be a box-like self-affine set of separated type which satisfies the ROSC. Furthermore, assume that each map, S_i, in the IFS has singular values $\alpha_1(i) \geqslant \alpha_2(i)$ where the larger singular value, $\alpha_1(i)$, corresponds to contracting in the horizontal direction. Then*

$$\dim_P F = \dim_B F = s$$

where s is the unique solution of

$$\sum_{i\in\mathcal{I}} \alpha_1(i)^{s_1} \ \alpha_2(i)^{s-s_1} = 1.$$

Furthermore, if $s_1 = 1$, then s is the affinity dimension.

Proof. It may be gleaned from the proof of Lemma 2.1 (a1), case (i), that, in the situation described above, the modified singular value function is multiplicative. It follows that the unique solution of $P(s) = 1$ satisfies $\Psi_1^s = 1$. □

Corollary 2.7 is similar to Corollary 1 in [FeW] but our result covers a much larger class of sets since we allow the maps in the IFS to have non-trivial reflectional and rotational components (whilst ensuring that F is of separated type). Although in a different context, a problem related to Corollary 2.7 was studied in [Hu]. There the author proved a version of Bowen's formula for a class of non-conformal C^2 expanding maps for which the expansion is stronger in one particular direction.

The idea to study box-like self-affine sets came from the paper [FO]. There the authors consider self-similar sets and, in particular, how varying the rotational or reflectional component of the mappings affects the attractor. Their approach relies on various group theoretic techniques. One thing to note is that, given the OSC, changing the rotational or reflectional component of the mappings in an IFS of similarities does not change the dimension. As we have shown (and unsurprisingly) the situation is more complicated in the self-affine case, see the examples below. It would be interesting to conduct an analysis similar to that found in [FO] in the self-affine case with the added complication that one could consider changes in dimension as well as changes in the symmetry of the (self-affine) attractor.

Finally, we remark that our dimension formula gives nothing but an upper bound for the *Hausdorff dimension* of box-like self-affine sets. It would be of great interest to investigate the Hausdorff dimension in more detail, especially in the non-separated case. Also, it would be interesting to allow our maps to include rotations other than multiples of $\pi/2$. In this situation the basic sets in the construction would cease to be rectangles and the non-separated/separated dichotomy would become more complicated as one might have to look to the dimension of projections in many different directions.

2.2.2 Dimensions of projections

The dimension formula given in Section 2.2.1 depends on knowledge of s_1 and s_2, i.e., the dimensions of the projections of F onto the horizontal and vertical axes, respectively. *A priori*, s_1 and s_2 are difficult to calculate explicitly, or even to obtain good estimates for. In this section we examine this problem and show that it is possible to compute s_1 and s_2 explicitly in a number of cases.

Lemma 2.8. *If F is of separated type, then $\pi_1(F)$ and $\pi_2(F)$ are self-similar sets. If F is of non-separated type, then $\pi_1(F)$ and $\pi_2(F)$ are a pair of graph-directed self-similar sets and, moreover, the associated adjacency matrix for the graph-directed system is irreducible. In this second case, it follows that $s_1 = s_2$.*

We will prove Lemma 2.8 in Section 2.4.1. It follows from Lemma 2.8 that the box dimensions of the projections exist and so s_1 and s_2 are well-defined. The problem with calculating the dimension of $\pi_1(F)$ and $\pi_2(F)$ is that the IFSs alluded to in Lemma 2.8 may not satisfy the open set condition (OSC), or graph-directed open set condition (GDOSC). However, in certain cases we will be able to invoke the *finite type* conditions introduced in [JY, LN, NW] and generalised to the graph-directed situation in [NWD]. In this situation, despite the possible failure of the OSC or GDOSC, we can view the projections as attractors of alternative IFSs or graph-directed IFSs where the necessary separation conditions *are* satisfied. We can then compute s_1 and s_2 using a standard formula, see, for example, [F7, Corollary 3.5]. An example of this will be given in Section 2.3.1. For more details on graph-directed sets, see [F7, Chapter 3]; [E1, Chapters 4 and 6], and [MW2].

There is one further situation where, even if the previously mentioned finite type conditions are not satisfied, we can still compute s_1 and s_2. In this case we will say that F is of *block type.*

Lemma 2.9 (block type)**.** *Let H be any closed, path connected set which contains F and is not contained in any vertical or horizontal line. If*

$$\pi_1\Big(\bigcup_{i\in\mathcal{I}} S_i(H)\Big) = \pi_1(H) \tag{2.2}$$

and

$$\pi_2\Big(\bigcup_{i\in\mathcal{I}} S_i(H)\Big) = \pi_2(H), \tag{2.3}$$

then $s_1 = s_2 = 1$.

Proof. This follows immediately since, by (2.2) and (2.3), $\pi_1(F)$ and $\pi_2(F)$ are intervals. □

2.3 Examples

In order to illustrate our results we will now present two examples and compute the packing and box dimensions. We will also examine what effect the rotational and reflectional components have on the dimension. In both cases it will be clear that the ROSC is satisfied, taking $R = (0,1)^2$. All rotations are taken to be clockwise about the origin and all numerical estimations were calculated in Maple using the method outlined at the end of Section 2.2.

2.3.1 Non-separated type

In this section we consider an example of a box-like self-affine set of non-separated type. Let F be the attractor of the IFS consisting of the maps which take $[0,1]^2$ to the 3 shaded rectangles on the left hand part of Figure 3, where the linear parts have been composed with: rotation by 270 degrees (top right); rotation by 90 degrees (bottom right); and reflection in the vertical axis (left).

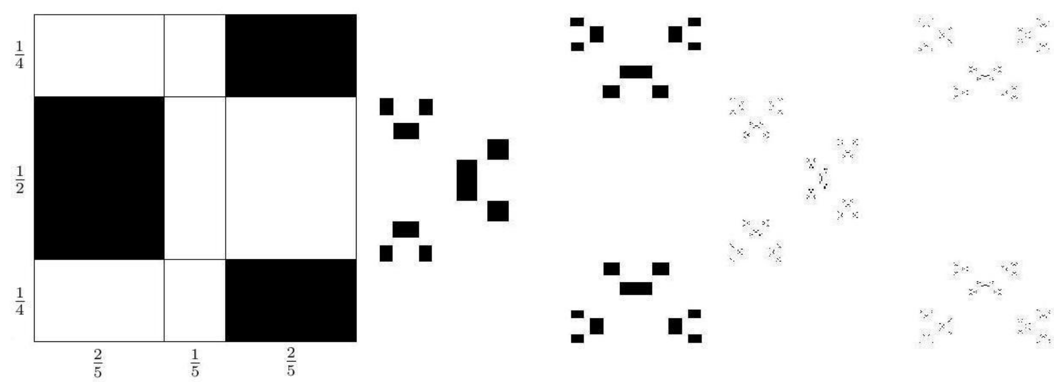

Figure 8: Levels 1, 3 and 7 in the construction of F.

Here, $\pi_1(F)$ and $\pi_2(F)$ are a pair of graph-directed self-similar sets of finite type. It is easy to see that in fact

$$\pi_1(F) = \big(\tfrac{2}{5} - \tfrac{2}{5}\pi_1(F)\big) \cup \big(\tfrac{2}{5}\pi_2(F) + \tfrac{3}{5}\big)$$

and

$$\pi_2(F) = \big(\tfrac{1}{4}\pi_1(F)\big) \cup \big(\tfrac{1}{2}\pi_2(F) + \tfrac{1}{4}\big) \cup \big(1 - \tfrac{1}{4}\pi_1(F)\big).$$

with the GDOSC satisfied for this system. The associated adjacency matrix is

$$A^{(t)} = \begin{pmatrix} (\frac{2}{5})^t & (\frac{2}{5})^t \\ & \\ 2\,(\frac{1}{4})^t & (\frac{1}{2})^t \end{pmatrix}$$

and solving $\rho\big(A^{(t)}\big) = 1$, where $\rho\big(A^{(t)}\big)$ is the spectral radius of $A^{(t)}$, for t yields $s_1 = s_2 =: t \approx 0.890959$, see [F7, Corollary 3.5]. Theorem 2.4 now gives that $\dim_{\mathrm{P}} F = \dim_{\mathrm{B}} F = s$ where $s \geqslant 0$ is the unique solution of

$$\lim_{k\to\infty} \bigg(\sum_{\boldsymbol{i} \in \mathcal{I}^k} \alpha_1(\boldsymbol{i})^t \, \alpha_2(\boldsymbol{i})^{s-t} \bigg)^{1/k} = 1.$$

If we considered the same construction but with no rotations or reflections, then we would have a self-affine set of the type considered by Barański. In this case, results in [B2] give us that the box dimension is approximately 1.11349, which is strictly larger than the dimension we obtained for our construction. To see this we computed $\hat{s}_{10} = 1.09557\dots$ which is an upper bound for s. In fact, our numerical estimates, $\{\hat{s}_k\}$, appear to converge to about 1.09.

2.3.2 Block type

In this section we consider an example of a box-like self-affine set of block type. Let F be the attractor of the IFS consisting of the maps S_1, S_2 and S_3 defined by

$$S_1 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{3}{10} \end{pmatrix} \circ R_1 + \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

$$S_2 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{5} \end{pmatrix} \circ R_2 + \begin{pmatrix} 1/4 \\ 7/10 \end{pmatrix},$$

and

$$S_3 = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{3}{5} \end{pmatrix} \circ R_3 + \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

where R_1 is reflection in the horizontal axis, R_2 is rotation by 90 degrees and R_3 is reflection in the vertical axis.

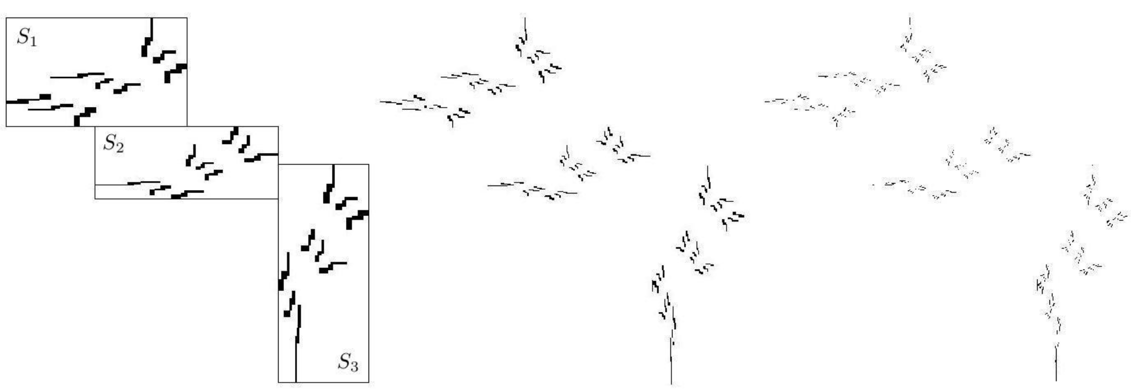

Figure 9: Levels 4, 5 and 6 in the construction of F. The boxes in the first image on the left indicate the mappings.

It is clear that F is of block type, taking $H = [0,1]^2$ in Lemma 2.9, and so $s_1 = s_2 = 1$. Theorem 2.4 now gives that $\dim_{\mathrm{P}} F = \dim_{\mathrm{B}} F = s$ where $s \geqslant 0$ is the unique solution of

$$\lim_{k\to\infty} \bigg(\sum_{\boldsymbol{i} \in \mathcal{I}^k} \alpha_1(\boldsymbol{i})^1 \, \alpha_2(\boldsymbol{i})^{s-1} \bigg)^{1/k} = 1,$$

which, by Corollary 2.6, coincides with the affinity dimension. Again, let us consider the same construction but with no rotational or reflectional components in the mappings. In this case we have a self-affine set of the type considered by Feng and Wang and results in [FeW] give that the box dimension is approximately 1.18405, which is again larger than for our construction. To see this we computed $\hat{s}_{10} = 1.17348\dots$ which is an upper bound for s. In fact, our numerical estimates, $\{\hat{s}_k\}$, appear to converge to about 1.16.

2.4 Proofs of preliminary lemmas

Note that we prove Lemma 2.8 before Lemma 2.1 because we will use Lemma 2.8 in the proof of Lemma 2.1.

2.4.1 Proof of Lemma 2.8

Let $\mathbb{I}_1, \mathbb{I}_2 : [0,1] \to [0,1]^2$ be defined by

$$\mathbb{I}_1(x) = (x, 0)$$

and

$$\mathbb{I}_2(y) = (0, y).$$

Also, for $i \in \mathcal{I}$ and $a, b \in \{1, 2\}$, we define a contracting similarity mapping $\tilde{S}_i^{a,b} : [0,1] \to [0,1]$ by

$$\tilde{S}_i^{a,b} = \pi_a \circ S_i \circ \mathbb{I}_b.$$

For certain choices of a, b and i the image $\tilde{S}_i^{a,b}([0,1])$ is a singleton, but we will not be interested in these maps. Also, let $X = \pi_1(F)$ and $Y = \pi_2(F)$. It is clear that

$$X = \Big(\bigcup_{i \in \mathcal{I}_A} \tilde{S}_i^{1,1}(X) \Big) \cup \Big(\bigcup_{i \in \mathcal{I}_B} \tilde{S}_i^{1,2}(Y) \Big) \tag{2.4}$$

and

$$Y = \Big(\bigcup_{i \in \mathcal{I}_A} \tilde{S}_i^{2,2}(Y) \Big) \cup \Big(\bigcup_{i \in \mathcal{I}_B} \tilde{S}_i^{2,1}(X) \Big). \tag{2.5}$$

It follows that if $\mathcal{I}_B = \emptyset$, then X and Y are the self-similar attractors of the IFSs $\{\tilde{S}_i^{1,1}\}_{i \in \mathcal{I}}$ and $\{\tilde{S}_i^{2,2}\}_{i \in \mathcal{I}}$ respectively and if $\mathcal{I}_B \neq \emptyset$, then X and Y are a pair of graph-directed self-similar sets with an associated adjacency matrix defined by (2.4–2.5). This matrix is clearly irreducible since the existence of an element in $\mathcal{I}_B$ ensures that we can find a directed cycle in the associated directed graph which contains both vertices. This proves Lemma 2.8. $\square$

2.4.2 Proof of Lemma 2.1

We will first prove part (a) by a case by case analysis. Part (b) will then follow easily.

Proof of (a).

a1) Let $s \in [0, s_1 + s_2)$ and let $\boldsymbol{i}, \boldsymbol{j} \in \mathcal{I}^*$. Firstly, assume that F is of non-separated type. It follows from Lemma 2.8 that $s_1 = s_2 =: t$. We have

$$\begin{aligned}
\psi^s(S_{\boldsymbol{i}} \circ S_{\boldsymbol{j}}) &= \alpha_1(\boldsymbol{ij})^t \alpha_2(\boldsymbol{ij})^{s-t} \\
&= \Big(\alpha_1(\boldsymbol{ij})\, \alpha_2(\boldsymbol{ij})\Big)^{s-t} \alpha_1(\boldsymbol{ij})^{2t-s} \\
&= \Big(\alpha_1(\boldsymbol{i})\, \alpha_2(\boldsymbol{i}) \alpha_1(\boldsymbol{j})\, \alpha_2(\boldsymbol{j})\Big)^{s-t} \alpha_1(\boldsymbol{ij})^{2t-s} \\
&\leqslant \Big(\alpha_1(\boldsymbol{i})\, \alpha_2(\boldsymbol{i})\Big)^{s-t} \Big(\alpha_1(\boldsymbol{j}) \alpha_2(\boldsymbol{j})\Big)^{s-t} \Big(\alpha_1(\boldsymbol{i})\, \alpha_1(\boldsymbol{j})\Big)^{2t-s} \qquad \text{since } 2t - s > 0 \\
&= \psi^s(S_{\boldsymbol{i}})\, \psi^s(S_{\boldsymbol{j}})
\end{aligned}$$

proving (a1) in the non-separated case. Secondly, assume that F is of separated type and assume, in addition, that $b(\boldsymbol{i}) \geqslant h(\boldsymbol{i})$, recalling that $b(\boldsymbol{i})$ and $h(\boldsymbol{i})$ are the lengths of the base and height of the rectangle $S_{\boldsymbol{i}}\big([0,1]^2\big)$ respectively. The case where $b(\boldsymbol{i}) < h(\boldsymbol{i})$ is analogous. We now have the following three cases:

(i) $b(\boldsymbol{j}) \geqslant h(\boldsymbol{j})$ and $b(\boldsymbol{ij}) \geqslant h(\boldsymbol{ij})$;

(ii) $b(\boldsymbol{j}) < h(\boldsymbol{j})$ and $b(\boldsymbol{ij}) \geqslant h(\boldsymbol{ij})$;

(iii) $b(\boldsymbol{j}) < h(\boldsymbol{j})$ and $b(\boldsymbol{ij}) < h(\boldsymbol{ij})$.

The key property that we will utilise here is that, since F is of separated type, $b(\boldsymbol{ij}) = b(\boldsymbol{i})\, b(\boldsymbol{j})$ and $h(\boldsymbol{ij}) = h(\boldsymbol{i})\, h(\boldsymbol{j})$. Note that this precludes the case: $b(\boldsymbol{j}) \geqslant h(\boldsymbol{j})$ and $b(\boldsymbol{ij}) < h(\boldsymbol{ij})$. To complete the proof of (a1) we will show that, in each of the above cases (i-iii), we have

$$\frac{\psi^s(S_{\boldsymbol{i}} \circ S_{\boldsymbol{j}})}{\psi^s(S_{\boldsymbol{i}})\, \psi^s(S_{\boldsymbol{j}})} \leqslant 1.$$

(i) We have

$$\frac{\psi^s(S_{\boldsymbol{i}} \circ S_{\boldsymbol{j}})}{\psi^s(S_{\boldsymbol{i}})\, \psi^s(S_{\boldsymbol{j}})} = \frac{b(\boldsymbol{ij})^{s_1} h(\boldsymbol{ij})^{s-s_1}}{b(\boldsymbol{i})^{s_1} h(\boldsymbol{i})^{s-s_1} b(\boldsymbol{j})^{s_1} h(\boldsymbol{j})^{s-s_1}} = 1.$$

(ii) Similarly

$$\frac{\psi^s(S_{\boldsymbol{i}} \circ S_{\boldsymbol{j}})}{\psi^s(S_{\boldsymbol{i}})\, \psi^s(S_{\boldsymbol{j}})} = \frac{b(\boldsymbol{ij})^{s_1} h(\boldsymbol{ij})^{s-s_1}}{b(\boldsymbol{i})^{s_1} h(\boldsymbol{i})^{s-s_1} h(\boldsymbol{j})^{s_2} b(\boldsymbol{j})^{s-s_2}} = \left(\frac{b(\boldsymbol{j})}{h(\boldsymbol{j})}\right)^{s_1+s_2-s} \leqslant 1.$$

(iii) Finally

$$\frac{\psi^s(S_{\boldsymbol{i}} \circ S_{\boldsymbol{j}})}{\psi^s(S_{\boldsymbol{i}})\, \psi^s(S_{\boldsymbol{j}})} = \frac{h(\boldsymbol{ij})^{s_2} b(\boldsymbol{ij})^{s-s_2}}{b(\boldsymbol{i})^{s_1} h(\boldsymbol{i})^{s-s_1} h(\boldsymbol{j})^{s_2} b(\boldsymbol{j})^{s-s_2}} = \left(\frac{h(\boldsymbol{i})}{b(\boldsymbol{i})}\right)^{s_1+s_2-s} \leqslant 1.$$

The proofs of (a2) and (a3) are similar and, therefore, omitted.

Proof of (b).

This follows easily by noting that, for all $k, l \in \mathbb{N}$, we have

$$\Psi^s_{k+l} = \sum_{\boldsymbol{i} \in \mathcal{I}^{k+l}} \psi^s(S_{\boldsymbol{i}}) = \sum_{\boldsymbol{i} \in \mathcal{I}^k} \sum_{\boldsymbol{j} \in \mathcal{I}^l} \psi^s(S_{\boldsymbol{i}} \circ S_{\boldsymbol{j}})$$

and

$$\Psi^s_k\, \Psi^s_l = \left(\sum_{\boldsymbol{i} \in \mathcal{I}^k} \psi^s(S_{\boldsymbol{i}})\right)\left(\sum_{\boldsymbol{i} \in \mathcal{I}^l} \psi^s(S_{\boldsymbol{j}})\right) = \sum_{\boldsymbol{i} \in \mathcal{I}^k} \sum_{\boldsymbol{j} \in \mathcal{I}^l} \psi^s(S_{\boldsymbol{i}})\, \psi^s(S_{\boldsymbol{j}})$$

and applying part (a). □

2.4.3 Proof of Lemma 2.2

(1) Let $s, t \in [0, \infty)$. We have

$$\begin{aligned} P(s+t) &= \lim_{k\to\infty} \left(\sum_{\boldsymbol{i} \in \mathcal{I}^k} \alpha_1(\boldsymbol{i})^{s(\boldsymbol{i})}\ \alpha_2(\boldsymbol{i})^{s+t-s(\boldsymbol{i})}\right)^{1/k} \\ &\leqslant \lim_{k\to\infty} \left(\alpha_{\max}^{ks} \sum_{\boldsymbol{i} \in \mathcal{I}^k} \alpha_1(\boldsymbol{i})^{s(\boldsymbol{i})}\ \alpha_2(\boldsymbol{i})^{t-s(\boldsymbol{i})}\right)^{1/k} \\ &= \alpha_{\max}^s P(t). \end{aligned}$$

The proof of the left hand inequality is similar. Furthermore, note that

$$\infty \ > \ \inf_{k\geqslant 0} (\Psi^0_k)^{\frac{1}{k}} \ = \ P(0) \ = \ \lim_{k\to\infty} \left(\sum_{\boldsymbol{i} \in \mathcal{I}^k} \alpha_1(\boldsymbol{i})^{s(\boldsymbol{i})}\ \alpha_2(\boldsymbol{i})^{-s(\boldsymbol{i})}\right)^{1/k} \ \geqslant \ \lim_{k\to\infty} \left(\sum_{\boldsymbol{i} \in \mathcal{I}^k} 1\right)^{1/k} \ = \ |\mathcal{I}|$$

and together with setting $t = 0$ above gives the second chain of inequalities.

(2) The continuity of P follows immediately from (1).

(3) Let $t, \varepsilon \geqslant 0$. Since $P(t+\varepsilon), P(t) \in (0,\infty)$, by (1) we have

$$\frac{P(t+\varepsilon)}{P(t)} \leqslant \alpha_{\max}^{\varepsilon} < 1$$

and so P is strictly decreasing on $[0,\infty)$.

(4) It follows from (1) that $P(0) \geqslant |\mathcal{I}| > 1$ and that $P(s) < 1$ for sufficiently large s. These facts, combined with parts (2) and (3), imply that there is a unique value of s for which $P(s) = 1$. □

2.5 Proof of Theorem 2.4

We will now prove our main result, that the packing and box-counting dimensions of F are equal to the unique s which satisfies $P(s) = 1$. We will prove this in the box dimension case and it is well-known that, since F is compact and every open ball centered in F contains a bi-Lipschitz image of F, $\dim_{\mathrm{P}} F = \overline{\dim}_{\mathrm{B}} F$, see [F8, Corollary 3.9] or Proposition 1.1 .

Let $s \geqslant 0$ be the unique solution of $P(s) = 1$. For $\delta \in (0,1]$ we define the δ-*stopping*, $\mathcal{I}_\delta$, as follows:

$$\mathcal{I}_\delta = \big\{ \boldsymbol{i} \in \mathcal{I}^* : \alpha_2(\boldsymbol{i}) < \delta \leqslant \alpha_2(\overline{\boldsymbol{i}}) \big\}.$$

Note that for $\boldsymbol{i} \in \mathcal{I}_\delta$ we have

$$\alpha_{\min}\,\delta \leqslant \alpha_2(\boldsymbol{i}) < \delta. \tag{2.6}$$

Lemma 2.10. *Let $t \geqslant 0$.*

(1) If $t > s$, then there exists a constant $K(t) < \infty$ such that

$$\sum_{\boldsymbol{i}\in\mathcal{I}_\delta} \psi^t(S_{\boldsymbol{i}}) \leqslant K(t)$$

for all $\delta \in (0,1]$.

(2) If $t < s$, then there exists a constant $L(t) > 0$ such that

$$\sum_{\boldsymbol{i}\in\mathcal{I}_\delta} \psi^t(S_{\boldsymbol{i}}) \geqslant L(t)$$

for all $\delta \in (0,1]$.

Proof. (1) Let $t > s$ and $\delta \in (0,1]$. We have

$$\sum_{\boldsymbol{i}\in\mathcal{I}_\delta} \psi^t(S_{\boldsymbol{i}}) \leqslant \sum_{\boldsymbol{i}\in\mathcal{I}^*} \psi^t(S_{\boldsymbol{i}}) = \sum_{k=1}^{\infty} \sum_{\boldsymbol{i}\in\mathcal{I}^k} \psi^t(S_{\boldsymbol{i}}) = \sum_{k=1}^{\infty} \Psi_k^t < \infty$$

since $\lim_{k\to\infty} (\Psi_k^t)^{1/k} = P(t) < 1$. The result follows, setting $K(t) = \sum_{k=1}^{\infty} \Psi_k^t$.

(2) Let $t < s$. Consider two cases according to whether t is in the submultiplicative region $[0, s_1 + s_2]$, or supermultiplicative region $(s_1 + s_2, \infty)$. We will be able to deduce retrospectively that $s \leqslant s_1 + s_2$ and so the second case is, in fact, vacuous. It would be possible to prove part (2) only in the submultiplicative case and then deduce that the dimension is $\min\{s, s_1 + s_2\}$, but in order to conclude that the dimension is simply s, we include the proof in the supermultiplicative case.

(i) $0 \leqslant t \leqslant s_1 + s_2$. We remark that an argument similar to the following was used in [F2].

Let $\delta \in (0,1]$ and assume that

$$\sum_{\boldsymbol{i} \in \mathcal{I}_\delta} \psi^t(S_{\boldsymbol{i}}) \leqslant 1. \tag{2.7}$$

To obtain a contradiction we will show that this implies that $t \geqslant s$. Let $k(\delta) = \max\{|\boldsymbol{i}| : \boldsymbol{i} \in \mathcal{I}_\delta\}$, where $|\boldsymbol{i}|$ denotes the length of the string $\boldsymbol{i}$, and define $\mathcal{I}_{\delta,k}$ by

$$\mathcal{I}_{\delta,k} = \big\{ \boldsymbol{i}_1 \dots \boldsymbol{i}_m \quad : \quad \boldsymbol{i}_j \in \mathcal{I}_\delta \text{ for all } j = 1, \dots, m,$$

$$|\boldsymbol{i}_1 \dots \boldsymbol{i}_m| \leqslant k \text{ but } |\boldsymbol{i}_1 \dots \boldsymbol{i}_m \boldsymbol{i}_{m+1}| > k \text{ for some } \boldsymbol{i}_{m+1} \in \mathcal{I}_\delta \big\}.$$

For all $\boldsymbol{i} \in \mathcal{I}^*$ we have, by the submultiplicativity of ψ^t,

$$\begin{aligned} \sum_{\boldsymbol{j} \in \mathcal{I}_\delta} \psi^t(S_{\boldsymbol{i}\boldsymbol{j}}) &\leqslant \sum_{\boldsymbol{j} \in \mathcal{I}_\delta} \psi^t(S_{\boldsymbol{i}})\, \psi^t(S_{\boldsymbol{j}}) \\ &= \psi^t(S_{\boldsymbol{i}}) \sum_{\boldsymbol{j} \in \mathcal{I}_\delta} \psi^t(S_{\boldsymbol{j}}) \\ &\leqslant \psi^t(S_{\boldsymbol{i}}) \end{aligned}$$

by (2.7). It follows by repeated application of the above that, for all $k \in \mathbb{N}$,

$$\sum_{\boldsymbol{i} \in \mathcal{I}_{\delta,k}} \psi^t(S_{\boldsymbol{i}}) \leqslant 1. \tag{2.8}$$

Let $\boldsymbol{i} \in \mathcal{I}^k$ for some $k \in \mathbb{N}$. It follows that $\boldsymbol{i} = \boldsymbol{j}_1 \boldsymbol{j}_2$ for some $\boldsymbol{j}_1 \in \mathcal{I}_{\delta,k}$ and some $\boldsymbol{j}_2 \in \mathcal{I}^* \cup \{\omega\}$ with $|\boldsymbol{j}_2| \leqslant k(\delta)$ and by the submultiplicativity of ψ^t,

$$\psi^t(S_{\boldsymbol{i}}) \;=\; \psi^t(S_{\boldsymbol{j}_1 \boldsymbol{j}_2}) \;\leqslant\; \psi^t(S_{\boldsymbol{j}_1})\, \psi^t(S_{\boldsymbol{j}_2}) \;\leqslant\; c_{k(\delta)}\, \psi^t(S_{\boldsymbol{j}_1}),$$

where $c_{k(\delta)} = \max\{\psi^t(S_{\boldsymbol{i}}) : |\boldsymbol{i}| \leqslant k(\delta)\} < \infty$ is a constant which depends only on δ. Since there are at most $|\mathcal{I}|^{k(\delta)+1}$ elements $\boldsymbol{j}_2 \in \mathcal{I}^* \cup \{\omega\}$ with $|\boldsymbol{j}_2| \leqslant k(\delta)$ we have

$$\Psi_k^t \;=\; \sum_{\boldsymbol{i} \in \mathcal{I}^k} \psi^t(S_{\boldsymbol{i}}) \;\leqslant\; |\mathcal{I}|^{k(\delta)+1}\, c_{k(\delta)} \sum_{\boldsymbol{i} \in \mathcal{I}_{\delta,k}} \psi^t(S_{\boldsymbol{i}}) \;\leqslant\; |\mathcal{I}|^{k(\delta)+1}\, c_{k(\delta)}$$

by (2.8). Since this is true for all $k \in \mathbb{N}$ we have

$$P(t) = \lim_{k \to \infty} \left(\Psi_k^t \right)^{1/k} \leqslant 1$$

from which it follows that $t \geqslant s$. So, if $t \leqslant s_1 + s_2$, then we may set $L(t) = 1$.

(ii) $t > s_1 + s_2$.

Since $t < s$ it follows that $\sum_{\boldsymbol{i} \in \mathcal{I}^k} \psi^t(S_{\boldsymbol{i}}) \to \infty$ as $k \to \infty$. Therefore, we may fix a $k \in \mathbb{N}$ such that

$$\sum_{\boldsymbol{i} \in \mathcal{I}^k} \psi^t(S_{\boldsymbol{i}}) \geqslant 1. \tag{2.9}$$

Fix $\delta \in (0,1]$ and define

$$\mathcal{I}_{k,\delta} = \big\{ \boldsymbol{i}_1 \dots \boldsymbol{i}_m \quad : \quad \boldsymbol{i}_j \in \mathcal{I}^k \text{ for all } j = 1, \dots, m,$$

$$\alpha_2(\boldsymbol{i}_1 \dots \boldsymbol{i}_m) \geqslant \delta \text{ but } \alpha_2(\boldsymbol{i}_1 \dots \boldsymbol{i}_m \boldsymbol{i}_{m+1}) < \delta \text{ for some } \boldsymbol{i}_{m+1} \in \mathcal{I}^k \big\}.$$

For all $\boldsymbol{i} \in \mathcal{I}^*$ we have, by the supermultiplicativity of ψ^t,

$$\begin{aligned}\sum_{\boldsymbol{j}\in\mathcal{I}^k} \psi^t(S_{\boldsymbol{i}\boldsymbol{j}}) &\geqslant \sum_{\boldsymbol{j}\in\mathcal{I}^k} \psi^t(S_{\boldsymbol{i}})\,\psi^t(S_{\boldsymbol{j}})\\ &= \psi^t(S_{\boldsymbol{i}}) \sum_{\boldsymbol{j}\in\mathcal{I}^k} \psi^t(S_{\boldsymbol{j}})\\ &\geqslant \psi^t(S_{\boldsymbol{i}})\end{aligned}$$

by (2.9). It follows by repeated application of the above that

$$\sum_{\boldsymbol{i}\in\mathcal{I}_{k,\delta}} \psi^t(S_{\boldsymbol{i}}) \geqslant 1. \tag{2.10}$$

Let $\boldsymbol{i} \in \mathcal{I}_\delta$. It follows that $\boldsymbol{i} = \boldsymbol{j}_1\boldsymbol{j}_2$ for some $\boldsymbol{j}_1 \in \mathcal{I}_{k,\delta}$ and some $\boldsymbol{j}_2 \in \mathcal{I}^*$. Since $\alpha_2(\boldsymbol{i}) \geqslant \delta\,\alpha_{\min}$ by (2.6) and $\alpha_2(\boldsymbol{j}_1) \leqslant \delta\alpha_{\min}^{-k}$ we have

$$\alpha_2(\boldsymbol{j}_1) \leqslant \alpha_2(\boldsymbol{i})\alpha_{\min}^{-(k+1)} \leqslant \alpha_2(\boldsymbol{j}_1)\alpha_{\max}^{|\boldsymbol{j}_2|}\alpha_{\min}^{-(k+1)} \tag{2.11}$$

which yields $|\boldsymbol{j}_2| \leqslant (k+1)\frac{\log\alpha_{\min}}{\log\alpha_{\max}}$. Setting $c_k = \min\left\{\psi^t(S_{\boldsymbol{i}}) : |\boldsymbol{i}| \leqslant (k+1)\frac{\log\alpha_{\min}}{\log\alpha_{\max}}\right\} > 0$, it follows from (2.11), (2.10) and the supermultiplicativity of ψ^t that

$$\sum_{\boldsymbol{i}\in\mathcal{I}_\delta} \psi^t(S_{\boldsymbol{i}}) \geqslant c_k \sum_{\boldsymbol{i}\in\mathcal{I}_{k,\delta}} \psi^t(S_{\boldsymbol{i}}) \geqslant c_k.$$

We have now proved part (2) setting $L(t) = \min\{1,\, c_k\} = c_k$. Note that although $L(t)$ appears to depend on k, recall that we fixed k at the beginning of the proof of (2)(ii) and the choice of k depended only on t. □

We are now ready to prove Theorem 2.4. It follows immediately from the definition of box dimension that for all $\varepsilon > 0$ there exists a constant $C_\varepsilon \geqslant 1$ such that for all $\delta \in (0, \alpha_{\min}^{-1}]$ we have

$$\tfrac{1}{C_\varepsilon}\,\delta^{-s_1+\varepsilon/2} \leqslant N_\delta(\pi_1 F) \leqslant C_\varepsilon\,\delta^{-s_1-\varepsilon/2} \tag{2.12}$$

and

$$\tfrac{1}{C_\varepsilon}\,\delta^{-s_2+\varepsilon/2} \leqslant N_\delta(\pi_2 F) \leqslant C_\varepsilon\,\delta^{-s_2-\varepsilon/2}. \tag{2.13}$$

For $\boldsymbol{i} \in \mathcal{I}^*$, we will write $F_{\boldsymbol{i}} = S_{\boldsymbol{i}}(F)$.

Upper bound (assuming no separation conditions)

Let $\varepsilon > 0$, $\delta \in (0,1]$ and suppose that, for each $\boldsymbol{i} \in \mathcal{I}_\delta$, $\{U_{\boldsymbol{i},j}\}_{j=1}^{N_\delta(F_{\boldsymbol{i}})}$ is a δ-cover of $F_{\boldsymbol{i}}$. Since $F = \bigcup_{\boldsymbol{i}\in\mathcal{I}_\delta} F_{\boldsymbol{i}}$ it follows that

$$\bigcup_{\boldsymbol{i}\in\mathcal{I}_\delta} \bigcup_{j=1}^{N_\delta(F_{\boldsymbol{i}})} \{U_{\boldsymbol{i},j}\}$$

is a δ-cover for F. Hence,

$$\begin{aligned} 0 \leqslant \delta^{s+\varepsilon} N_\delta(F) &\leqslant \delta^{s+\varepsilon} \sum_{\boldsymbol{i}\in\mathcal{I}_\delta} N_\delta\big(F_{\boldsymbol{i}}\big)\\ &= \delta^{s+\varepsilon} \sum_{\boldsymbol{i}\in\mathcal{I}_\delta} N_{\delta/\alpha_1(\boldsymbol{i})}\big(\pi_{\boldsymbol{i}} F\big) && \text{since } \alpha_2(\boldsymbol{i}) < \delta\\ &\leqslant \delta^{s+\varepsilon} \sum_{\boldsymbol{i}\in\mathcal{I}_\delta} C_\varepsilon \left(\frac{\delta}{\alpha_1(\boldsymbol{i})}\right)^{-s(\boldsymbol{i})-\varepsilon/2} && \text{by (2.12–2.13)}\end{aligned}$$

$$\begin{aligned}
&\leqslant\; C_\varepsilon\, \alpha_{\min}^{-s-\epsilon} \sum_{\boldsymbol{i}\in\mathcal{I}_\delta} \alpha_1(\boldsymbol{i})^{s(\boldsymbol{i})+\varepsilon/2}\alpha_2(\boldsymbol{i})^{s+\varepsilon-s(\boldsymbol{i})-\varepsilon/2} && \text{by (2.6)}\\
&\leqslant\; C_\varepsilon\, \alpha_{\min}^{-s-\epsilon} \sum_{\boldsymbol{i}\in\mathcal{I}_\delta} \psi^{s+\epsilon/2}(\boldsymbol{i})\\
&\leqslant\; C_\varepsilon\, \alpha_{\min}^{-s-\varepsilon}\, K\big(s+\tfrac{\varepsilon}{2}\big)
\end{aligned}$$

by Lemma 2.10 (1). It follows that $\overline{\dim}_{\mathrm{B}} F \leqslant s+\varepsilon$ and, since $\varepsilon > 0$ was arbitrary, we have the desired upper bound.

Lower bound (assuming the ROSC)

Let $\varepsilon \in (0,s)$, $\delta \in (0,1]$ and U be any closed square of sidelength δ. Also, let R be the open rectangle used in the ROSC and let r_- denote the length of the shortest side of R. Finally, let

$$M = \min\big\{n \in \mathbb{N} : n \geqslant (\alpha_{\min} r_-)^{-1} + 2\big\}.$$

Since $\{S_{\boldsymbol{i}}(R)\}_{\boldsymbol{i}\in\mathcal{I}_\delta}$ is a collection of pairwise disjoint open rectangles each with shortest side having length at least $\alpha_{\min}\delta r_-$, it is clear that U can intersect no more than M^2 of the sets $\{F_{\boldsymbol{i}}\}_{\boldsymbol{i}\in\mathcal{I}_\delta}$. It follows that, using the δ-mesh definition of N_δ, we have

$$\sum_{\boldsymbol{i}\in\mathcal{I}_\delta} N_\delta\big(F_{\boldsymbol{i}}\big) \leqslant M^2\, N_\delta(F).$$

This yields

$$\begin{aligned}
\delta^{s-\varepsilon} N_\delta(F) \;&\geqslant\; \delta^{s-\varepsilon}\, \tfrac{1}{M^2} \sum_{\boldsymbol{i}\in\mathcal{I}_\delta} N_\delta\big(F_{\boldsymbol{i}}\big)\\
&=\; \delta^{s-\varepsilon}\, \tfrac{1}{M^2} \sum_{\boldsymbol{i}\in\mathcal{I}_\delta} N_{\delta/\alpha_1(\boldsymbol{i})}\big(\pi_{\boldsymbol{i}} F\big) && \text{since } \alpha_2(\boldsymbol{i}) < \delta\\
&\geqslant\; \delta^{s-\varepsilon}\, \tfrac{1}{M^2} \sum_{\boldsymbol{i}\in\mathcal{I}_\delta} \tfrac{1}{C_\varepsilon} \left(\frac{\delta}{\alpha_1(\boldsymbol{i})}\right)^{-s(\boldsymbol{i})+\varepsilon/2} && \text{by (2.12–2.13)}\\
&\geqslant\; \tfrac{1}{M^2 C_\varepsilon} \sum_{\boldsymbol{i}\in\mathcal{I}_\delta} \alpha_2(\boldsymbol{i})^{s-\varepsilon}\, \alpha_{\min}\, \alpha_2(\boldsymbol{i})^{-s(\boldsymbol{i})+\varepsilon/2}\, \alpha_1(\boldsymbol{i})^{s(\boldsymbol{i})-\varepsilon/2} && \text{by (2.6)}\\
&=\; \tfrac{1}{M^2 C_\varepsilon}\, \alpha_{\min} \sum_{\boldsymbol{i}\in\mathcal{I}_\delta} \alpha_1(\boldsymbol{i})^{s(\boldsymbol{i})-\varepsilon/2}\, \alpha_2(\boldsymbol{i})^{s-\varepsilon/2-s(\boldsymbol{i})}\\
&\geqslant\; \tfrac{1}{M^2 C_\varepsilon}\, \alpha_{\min} \sum_{\boldsymbol{i}\in\mathcal{I}_\delta} \psi^{s-\varepsilon/2}(\boldsymbol{i})\\
&\geqslant\; \tfrac{1}{M^2}\, \tfrac{1}{C_\varepsilon}\, \alpha_{\min}\, L\big(s-\tfrac{\varepsilon}{2}\big)
\end{aligned}$$

by Lemma 2.10 (2). It follows that $\underline{\dim}_{\mathrm{B}} F \geqslant s-\varepsilon$ and, since $\varepsilon \in (0,s)$ was arbitrary, we have the desired lower bound. $\square$

3 Assouad and lower dimensions of self-affine carpets

3.1 Introduction

In this chapter we continue to investigate the dimension theory of the standard self-affine carpets. Recently, Mackay [M] computed the Assouad dimension for the Lalley-Gatzouras class which contains the Bedford-McMullen class. We will compute the Assouad dimension and lower dimension for the Barański class, which also contains the Bedford-McMullen class, and we will complement Mackay's result by computing the lower dimension for the Lalley-Gatzouras class. We also devote some time to comparing these new dimension formulae with the formulae for the Hausdorff and packing dimensions.

3.2 Dimension results for self-affine carpets

In this section we state our main results on the Assouad and lower dimensions of self-affine sets. Self-affine sets often exhibit a high degree of inhomogeneity because the mappings can stretch by different amounts in different directions. We will only consider self-affine carpets which are attractors of IFSs in the Lalley-Gatzouras or Barański class, which have at least one map which is not a similarity. The reason we assume that one of the mappings is not a similarity is so that the sets are genuinely self-affine. The dimension theory for genuinely self-affine sets is very different from self-similar sets and we intentionally keep the two classes separate, see [Fr4] for more details on the self-similar situation. We will divide the class of self-affine carpets into three subclasses, *horizontal*, *vertical* and *mixed*, which will be described below.

In order to state our results, we need to introduce some notation. Throughout this section F will be a self-affine carpet which is the attractor of an IFS $\{S_i\}_{i \in \mathcal{I}}$ for some finite index set $\mathcal{I}$, with $|\mathcal{I}| \geqslant 2$. The maps S_i in the IFS will be translate linear orientation-preserving contractions on $[0,1]^2$ of the form

$$S_i\big((x,y)\big) = (c_i x, d_i y) + \boldsymbol{t}_i$$

for some contraction constants $c_i \in (0,1)$ in the horizontal direction and $d_i \in (0,1)$ in the vertical direction and a translation $\boldsymbol{t}_i \in \mathbb{R}^2$. We will say that F is of *horizontal type* if $c_i \geqslant d_i$ for all $i \in \mathcal{I}$; of *vertical type* if $c_i \leqslant d_i$ for all $i \in \mathcal{I}$; and of *mixed type* if F falls into neither the horizontal or vertical classes. We remark here that the horizontal and vertical classes are equivalent as one can just rotate the unit square by 90^{o} to move from one class to the other. The horizontal (and hence also vertical) class is precisely the Lalley-Gatzouras class and the Barański class is split between vertical, horizontal and mixed, with carpets of mixed type being considerably more difficult to deal with and represent the major advancement of the work of Barański [B2] over the much earlier work by Lalley and Gatzouras [GL1].

Let π_1 denote the projection mapping from the plane to the horizontal axis and π_2 denote the projection mapping from the plane to the vertical axis. Also, for $i \in \mathcal{I}$ let

$$\mathrm{Slice}_{1,i}(F) = \text{the vertical slice of } F \text{ through the fixed point of } S_i$$

and let

$$\mathrm{Slice}_{2,i}(F) = \text{the horizontal slice of } F \text{ through the fixed point of } S_i.$$

Note that the sets $\pi_1(F)$, $\pi_2(F)$, $\mathrm{Slice}_{1,i}(F)$ and $\mathrm{Slice}_{2,i}(F)$ are self-similar sets satisfying the open set condition and so their box dimension can be computed via the Hutchinson-Moran formula. We can now state our dimension results.

Theorem 3.1. *Let F be a self-affine carpet. If F is of horizontal type, then*

$$\dim_{\mathrm{A}} F \;=\; \dim_{\mathrm{B}} \pi_1(F) \;+\; \max_{i \in \mathcal{I}} \dim_{\mathrm{B}} \mathrm{Slice}_{1,i}(F);$$

if F is of vertical type, then

$$\dim_{\mathrm{A}} F \;=\; \dim_{\mathrm{B}} \pi_2(F) \;+\; \max_{i \in \mathcal{I}} \dim_{\mathrm{B}} \mathrm{Slice}_{2,i}(F);$$

and if F is of mixed type, then

$$\dim_{\mathrm{A}} F \;=\; \max_{i \in \mathcal{I}} \max_{j=1,2} \Big(\dim_{\mathrm{B}} \pi_j(F) \;+\; \dim_{\mathrm{B}} \mathrm{Slice}_{j,i}(F) \Big).$$

We will prove Theorem 3.1 for the mixed class in Section 3.6.2 and for the horizontal and vertical classes in Section 3.6.3. If F is in the (non-extended) Lalley-Gatzouras class, then the above result was obtained in [M].

Theorem 3.2. *Let F be a self-affine carpet. If F is of horizontal type, then*

$$\dim_{\mathrm{L}} F \;=\; \dim_{\mathrm{B}} \pi_1(F) \;+\; \min_{i \in \mathcal{I}} \dim_{\mathrm{B}} \mathrm{Slice}_{1,i}(F);$$

if F is of vertical type, then

$$\dim_{\mathrm{L}} F \;=\; \dim_{\mathrm{B}} \pi_2(F) \;+\; \min_{i \in \mathcal{I}} \dim_{\mathrm{B}} \mathrm{Slice}_{2,i}(F);$$

and if F is of mixed type, then

$$\dim_{\mathrm{L}} F \;=\; \min_{i \in \mathcal{I}} \min_{j=1,2} \Big(\dim_{\mathrm{B}} \pi_j(F) \;+\; \dim_{\mathrm{B}} \mathrm{Slice}_{j,i}(F) \Big).$$

We will prove Theorem 3.2 for the mixed class in Section 3.6.4 and for the horizontal and vertical classes in Section 3.6.5. We remark here that the formulae presented in Theorems 3.1 and 3.2 are completely explicit and can be computed easily to any required degree of accuracy. It is interesting to investigate conditions for which the dimensions discussed here are equal or distinct. Mackay [M] noted a fascinating dichotomy for the Lalley-Gatzouras class in that either the Hausdorff dimension, box dimension and Assouad dimension are all distinct or are all equal. We obtain the following extension of this result.

Corollary 3.3. *Let F be a self-affine carpet in the horizontal or vertical class. Then either*

$$\dim_{\mathrm{L}} F < \dim_{\mathrm{H}} F < \dim_{\mathrm{B}} F < \dim_{\mathrm{A}} F$$

or

$$\dim_{\mathrm{L}} F = \dim_{\mathrm{H}} F = \dim_{\mathrm{B}} F = \dim_{\mathrm{A}} F.$$

We will prove Corollary 3.3 in Section 3.6.6. It is natural to wonder if this dichotomy also holds for the mixed class. In fact it does not and in Section 3.4.2 we provide an example of a self-affine set in the mixed class for which $\dim_{\mathrm{L}} F < \dim_{\mathrm{H}} F = \dim_{\mathrm{B}} F = \dim_{\mathrm{A}} F$. We do obtain the following slightly weaker result.

Corollary 3.4. *Let F be a self-affine carpet. Then either*

$$\dim_{\mathrm{L}} F < \dim_{\mathrm{B}} F$$

or

$$\dim_{\mathrm{L}} F = \dim_{\mathrm{H}} F = \dim_{\mathrm{B}} F = \dim_{\mathrm{A}} F.$$

We will prove Corollary 3.4 in Section 3.6.7. Theorems 3.1-3.2 provide explicit means to estimate, or at least obtain non-trivial bounds for, the Hausdorff dimension and box dimension. The formulae for the box dimensions given in [GL1, B2] are completely explicit, but the formulae for the Hausdorff dimensions are not explicit and are often difficult to evaluate. As such, our results concerning lower dimension provide completely explicit and easily computable lower bounds for the Hausdorff dimension. Finally we note that, despite how apparently easy it is to have lower dimension equal to zero, it is easy to see from Theorem 3.2 that the lower dimension of a self-affine carpet is always strictly positive, indeed, bigger than or equal to $\min\{\dim_{\mathrm{B}} \pi_1(F), \dim_{\mathrm{B}} \pi_2(F)\}$. A set with strictly positive lower dimension is called *uniformly perfect* and we note that in fact all self-affine sets which are not just a singleton are uniformly perfect, see [XYS].

3.3 Dimension results for quasi-self-similar sets

In this section we take a short detour to investigate the Assouad and lower dimensions of sets displaying some degree of quasi-self-similarity, in particular, self-similar sets. We do this for two reasons: to put our results on self-affine sets into context; and because we will use the results in this section to portray interesting phenomena later in the chapter. We will be particularly interested in conditions which guarantee the equality of certain dimensions. Throughout this section (X, d) will be a compact metric space. Recall that (X, d) is called *Ahlfors regular* if $\dim_{\mathrm{H}} X < \infty$ and there exists a constant $\lambda > 0$ such that, writing $\mathcal{H}^{\dim_{\mathrm{H}} X}$ to denote the Hausdorff measure in the critical dimension,

$$\tfrac{1}{\lambda}\, r^{\dim_{\mathrm{H}} X} \;\leqslant\; \mathcal{H}^{\dim_{\mathrm{H}} X}\big(B(x,r)\big) \;\leqslant\; \lambda\, r^{\dim_{\mathrm{H}} X}$$

for all $x \in X$ and all $0 < r < \mathrm{diam}(X)$, see [He, Chapter 8]. A metric space is called *locally Ahlfors regular* if the above estimates on the measure of balls holds for sufficiently small $r > 0$. It is easy to see that a compact locally Ahlfors regular space is Ahlfors regular. In a certain sense Ahlfors regular spaces are the most homogeneous spaces. This is reflected in the following proposition.

Proposition 3.5. *If (X, d) is Ahlfors regular, then*

$$\dim_{\mathrm{L}} X \;=\; \dim_{\mathrm{A}} X.$$

For a proof of this see, for example, [ByG]. We will now consider the Assouad and lower dimensions of quasi-self-similar sets, which are a natural class of sets exhibiting a high degree of homogeneity. We will define quasi-self-similar sets via the implicit theorems of Falconer [F3] and McLaughlin [McL]. These results allow one to deduce facts about the dimensions and measures of a set without having to calculate them explicitly. This is done by showing that, roughly speaking, parts of the set can be 'mapped around' onto other parts without too much distortion.

Definition 3.6. *A non-empty compact set $F \subseteq (X, d)$ is called quasi-self-similar if there exists $a > 0$ and $r_0 > 0$ such that the following two conditions are satisfied:*

(1) for every set U that intersects F with $|U| \leqslant r_0$, there is a mapping $g : F \cap U \to F$ satisfying

$$a\,|U|^{-1}\,|x-y| \;\leqslant\; |g(x)-g(y)| \qquad (x, y \in F \cap U)$$

(2) for every closed ball B with centre in F and radius $r \leqslant r_0$, there is a mapping $g : F \to F \cap B$ satisfying

$$a\,r\,|x-y| \;\leqslant\; |g(x)-g(y)| \qquad (x, y \in F)$$

Writing $s = \dim_{\mathrm{H}} F$, it was shown in [McL, F3] that condition (1) is enough to guarantee that $\mathcal{H}^s(F) \geqslant a^s > 0$ and $\underline{\dim}_{\mathrm{B}} F = \overline{\dim}_{\mathrm{B}} F = s$ and it was shown in [F3] that condition (2) is enough to guarantee $\mathcal{H}^s(F) \leqslant 4^s\, a^{-s} < \infty$ and $\underline{\dim}_{\mathrm{B}} F = \overline{\dim}_{\mathrm{B}} F = s$. Also see [F7, Chapter 3]. Here we extend these implicit results to include the Assouad and lower dimensions.

Theorem 3.7. *Let F be a non-empty compact subset of X.*

(1) If F satisfies condition (1) in the definition of quasi-self-similar, then

$$\dim_{\mathrm{L}} F \leqslant \dim_{\mathrm{H}} F = \dim_{\mathrm{P}} F = \dim_{\mathrm{B}} F = \dim_{\mathrm{A}} F.$$

(2) If F satisfies condition (2) in the definition of quasi-self-similar, then

$$\dim_{\mathrm{L}} F = \dim_{\mathrm{H}} F = \dim_{\mathrm{P}} F = \dim_{\mathrm{B}} F \leqslant \dim_{\mathrm{A}} F.$$

(3) If F satisfies conditions (1) and (2) in the definition of quasi-self-similar, then we have

$$\dim_{\mathrm{L}} F = \dim_{\mathrm{H}} F = \dim_{\mathrm{P}} F = \dim_{\mathrm{B}} F = \dim_{\mathrm{A}} F$$

and moreover, F is Ahlfors regular.

The proof of Theorem 3.7 is fairly straightforward, but we defer it to Section 3.7. We obtain the following corollary which gives useful relationships between the Assouad, lower and Hausdorff dimensions in a variety of contexts.

Corollary 3.8. *The following classes of sets are Ahlfors regular and, in particular, have equal Assouad and lower dimension:*

(1) self-similar sets satisfying the open set condition;

(2) graph-directed self-similar sets satisfying the graph-directed open set condition;

(3) mixing repellers of $C^{1+\alpha}$ conformal mappings on Riemann manifolds;

(4) Bedford's recurrent sets satisfying the open set condition, see [Be2].

The following classes of sets have equal Assouad dimension and Hausdorff dimension:

(5) sub-self-similar sets satisfying the open set condition, see [F6];

(6) boundaries of self-similar sets satisfying the open set condition.

The following classes of sets have equal lower dimension and Hausdorff dimension regardless of separation conditions:

(7) self-similar sets;

(8) graph-directed self-similar sets;

(9) Bedford's recurrent sets, see [Be2];

Proof. This follows immediately from Theorem 3.7 and the fact that the sets in each of the classes (1)-(4) are quasi-self-similar, see [F3]; the sets in each of the classes (5)-(6) satisfy condition (1) in the definition of quasi-self-similar, see [F3, F6] and the sets in each of the classes (7)-(9) satisfy condition (2) in the definition of quasi-self-similar, see [F3]. □

We do not claim that all the information presented in the above corollary is new. For example, the fact that self-similar sets satisfying the open set condition are Ahlfors regular dates back to Hutchinson, see [Hut]. Also, Olsen [O5] recently gave a direct proof that graph-directed self-similar sets (more generally, graph-directed Moran constructions) have equal Hausdorff dimension and Assouad dimension. Corollary 3.8 unifies previous results and demonstrates further that sets with equal Assouad dimension and lower dimension should display a high degree of homogeneity.

Finally, we remark that Theorem 3.7 is sharp, in that the inequalities in parts (1) and (2) cannot be replaced with equalities in general. To see this note that the inequality in (1) is sharp as the unit interval union a single isolated point satisfies condition (1) in the definition of quasi-self-similar, but has lower dimension strictly less than Hausdorff dimension; and the inequality in (2) is sharp because self-similar sets which do not satisfy the open set condition can have Assouad dimension strictly larger than Hausdorff dimension and such sets satisfy condition (2) in the definition of quasi-self-similar. We will prove this latter fact in Section 3.4.1 by providing an example.

3.4 Examples

In this section we give two examples and compute their Assouad and lower dimensions. Each example is designed to illustrate an important phenomenon.

3.4.1 A self-similar set with overlaps

Self-similar sets with overlaps are currently at the forefront of research on fractals and are notoriously difficult to deal with. For example, a recent paper of Hochman [Ho] has made a major contribution to the famous problem of when a 'dimension drop' can occur, in particular, when the Hausdorff dimension of a self-similar subset of the line can be strictly less than the minimum of the similarity dimension and one. In this section we provide an example of a self-similar set $F \subset [0, 1]$ with overlaps for which

$$\dim_{\mathrm{L}} F = \dim_{\mathrm{H}} F = \dim_{\mathrm{B}} F < \dim_{\mathrm{A}} F.$$

This answers a question of Olsen [O5, Question 1.3] which asked if it was possible to find a graph-directed Moran fractal F with $\dim_{\mathrm{B}} F < \dim_{\mathrm{A}} F$. Self-similar sets are the most commonly studied class of graph-directed Moran fractals, see [O5] for more details. We also use this example to show that Assouad dimension can increase under Lipschitz maps and, in particular, projections. Although this example is slightly incongruous with the material in this chapter, we note that constructing 'weak tangents' to sets will be an important technique for us in the coming sections and so we include this section to give a straightforward example of the power of weak tangents in a simpler setting.

Let $\alpha, \beta, \gamma \in (0, 1)$ be such that $(\log \beta)/(\log \alpha) \notin \mathbb{Q}$ and define similarity maps S_1, S_2, S_3 on $[0, 1]$ as follows

$$S_1(x) = \alpha x, \qquad S_2(x) = \beta x \qquad \text{and} \qquad S_3(x) = \gamma x + (1 - \gamma).$$

Let F be the self-similar attractor of $\{S_1, S_2, S_3\}$. We will now prove that $\dim_{\mathrm{A}} F = 1$ and, in particular, the Assouad dimension is independent of α, β, γ provided they are chosen with the above property. We will use the following proposition due to Mackay and Tyson, see [MT, Proposition 6.1.5].

Proposition 3.9 (Mackay-Tyson). *Let $X \subset \mathbb{R}$ be compact and let F be a compact subset of X. Let T_k be a sequence of similarity maps defined on $\mathbb{R}$ and suppose that $T_k(F) \cap X \to_{d_{\mathcal{H}}} \hat{F}$ for some non-empty compact set $\hat{F} \in \mathcal{K}(X)$. Then $\dim_{\mathrm{A}} \hat{F} \leqslant \dim_{\mathrm{A}} F$. The set $\hat{F}$ is called a* weak tangent *to F.*

We will now show that $[0, 1]$ is a weak tangent to F in the above sense. Let $X = [0, 1]$ and assume without loss of generality that $\alpha < \beta$. For each $k \in \mathbb{N}$ let T_k be defined by

$$T_k(x) = \beta^{-k} x.$$

We will now show that $T_k(F) \cap [0, 1] \to_{d_{\mathcal{H}}} [0, 1]$. Since

$$E_k \ := \ \big\{\alpha^m \beta^n : m \in \mathbb{N}, n \in \{-k, -k+1, \dots, \infty\}\big\} \cap [0, 1] \ \subset \ T_k(F) \cap [0, 1]$$

for each k it suffices to show that $E_k \to_{d_{\mathcal{H}}} [0, 1]$. Indeed, we have

$$\begin{aligned} E_k \quad &\to_{d_{\mathcal{H}}} \quad \overline{\bigcup_{k \in \mathbb{N}} E_k} \cap [0, 1] \\ &= \quad \overline{\{\alpha^m \beta^n : m \in \mathbb{N}, n \in \mathbb{Z}\}} \cap [0, 1] \\ &= \quad [0, 1]. \end{aligned}$$

It now follows from Proposition 3.9 that $\dim_{\mathrm{A}} F = 1$. To see why $\overline{\{\alpha^m \beta^n : m \in \mathbb{N}, n \in \mathbb{Z}\}} \cap [0, 1] = [0, 1]$ we apply Dirichlet's Theorem in the following way. It suffices to show that

$$\{m \log \alpha + n \log \beta : m \in \mathbb{N}, n \in \mathbb{Z}\}$$

is dense in $(-\infty, 0)$. We have

$$m \log \alpha + n \log \beta = n \log \alpha \left(\frac{m}{n} + \frac{\log \beta}{\log \alpha} \right)$$

and Dirichlet's Theorem gives that there exists infinitely many n such that

$$\left| \frac{m}{n} + \frac{\log \beta}{\log \alpha} \right| < 1/n^2$$

for some m, see [Sch, Theorem 1A, Corollary 1B]. Since $\log \beta / \log \alpha$ is irrational, we may choose m, n to make

$$0 < |m \log \alpha + n \log \beta| < \frac{|\log \alpha|}{n}$$

with n arbitrarily large, and thus make $m \log \alpha + n \log \beta$ arbitrarily small. We can therefore find arithmetic progressions $\{\varepsilon k : k = -1, -2, \dots\} \subset \{m \log \alpha + n \log \beta : m \in \mathbb{N}, n \in \mathbb{Z}\}$ for arbitrarily small $\varepsilon > 0$, which gives density and completes the proof.

Clearly we may choose α, β, γ with the desired properties making the similarity dimension arbitrarily small. In particular, the similarity dimension is the unique solution, s, of

$$\alpha^s + \beta^s + \gamma^s = 1$$

and if we choose α, β, γ such that $s < 1$, then it follows from Corollary 3.8 (7), the above argument, and the fact that the similarity dimension is an upperbound for the upper box dimension of any self-similar set, that

$$\dim_{\mathrm{L}} F = \dim_{\mathrm{H}} F = \dim_{\mathrm{B}} F \leqslant s < 1 = \dim_{\mathrm{A}} F.$$

We give an example with $s \approx 0.901$ below.

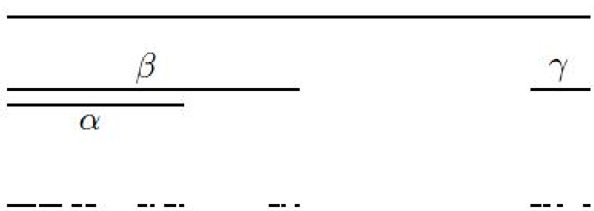

Figure 10: The first level iteration and the final attractor for the self-similar set with $\alpha = 2^{-\sqrt{3}}$, $\beta = 1/2$ and $\gamma = 1/10$. The tangent structure can be seen emerging around the origin.

The construction in this section has another interesting consequence. Let $\alpha, \beta, \gamma \in (0, 1)$ be chosen as before and consider the similarity maps T_1, T_2, T_3 on $[0, 1]^2$ defined as follows

$$T_1(x, y) = (\alpha x, \alpha y), \qquad T_2(x, y) = (\beta x, \beta y) + (0, 1 - \beta) \qquad \text{and} \qquad T_3(x) = (\gamma x, \gamma y) + (1 - \gamma, 0)$$

and let E be the attractor of $\{T_1, T_2, T_3\}$. Now if α, β, γ are chosen such that $\alpha+\beta, \beta+\gamma, \alpha+\gamma \leqslant 1$ and with the similarity dimension $s < 1$, then $\{T_1, T_2, T_3\}$ satisfies the open set condition and therefore by Corollary 3.8 (7) the Assouad dimension of E is equal to s defined above. However, note that F is the projection of E onto the horizontal axis but $\dim_{\mathrm{A}} F > \dim_{\mathrm{A}} E$. This shows that Assouad dimension can increase under Lipschitz maps. This is already known, see [Lu, Example A.6 2], however, our example extends this idea in two directions as we show that the Assouad dimension can increase under Lipschitz maps on *Euclidean space* and under *projections*, which are a very restricted class of Lipschitz maps.

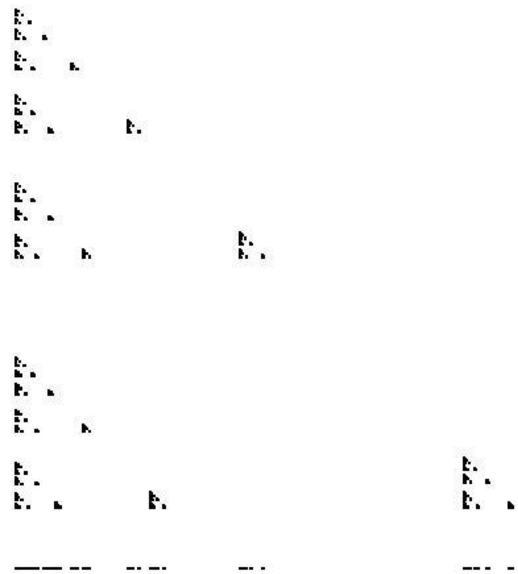

Figure 11: The set E and its projection F for $\alpha = 2^{-\sqrt{3}}$, $\beta = 1/2$ and $\gamma = 1/10$.

3.4.2 A self-affine carpet in the mixed class

In this section we will give an example of a self-affine carpet in the mixed class for which $\dim_{\mathrm{L}} F < \dim_{\mathrm{H}} F = \dim_{\mathrm{B}} F = \dim_{\mathrm{A}} F$. This is not possible in the horizontal or vertical classes by Corollary 3.3 and thus demonstrates that new phenomena can occur in the mixed class. In particular, the dichotomy seen in Corollary 3.3 does not extend to this case.

For this example we will let $\{S_i\}_{i\in\mathcal{I}}$ be an IFS of affine maps corresponding to the shaded rectangles in Figure 1 below. Here we have divided the unit square horizontally in the ratio $1/5 : 4/5$ and vertically into four strips each of height $1/4$.

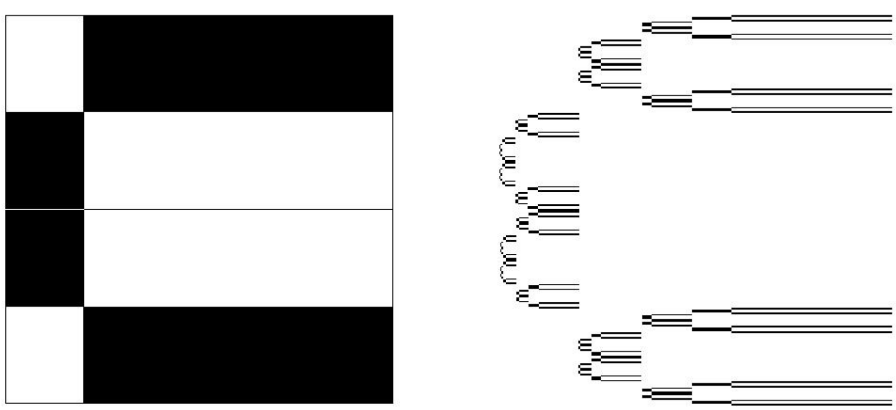

Figure 12: The defining pattern for the IFS (left) and the corresponding attractor (right).

It is easy to see that

$$\dim_{\mathrm{B}} \pi_1(F) \;=\; \dim_{\mathrm{B}} \pi_2(F) \;=\; 1,$$

$$\dim_{\mathrm{B}} \mathrm{Slice}_{1,i}(F) \;=\; 0.5 \qquad \text{and} \qquad \dim_{\mathrm{B}} \mathrm{Slice}_{2,i}(F) \;=\; 0$$

for all $i \in \mathcal{I}$ and therefore by Theorems 3.1-3.2, we have $\dim_{\mathrm{L}} F = 1$ and $\dim_{\mathrm{A}} F = 1.5$. Furthermore, the formulae in [B2] plus a simple calculation gives $\dim_{\mathrm{B}} F = \dim_{\mathrm{H}} F = 1.5$.

3.5 Open questions and discussion

In this section we will briefly outline what we believe are the key questions for the future and discuss some of the interesting points raised by the results in this chapter.

There are many natural ways to attempt to generalise our results on the Assouad and lower dimensions of self-affine sets. Firstly, one could try to compute the dimensions of more general carpets.

Question 3.10. *What is the Assouad dimension and lower dimension of the more general self-affine carpets considered by Feng and Wang [FeW] and Fraser [Fr1]?*

Whilst the classes of self-affine sets considered in [FeW, Fr1] are natural generalisations of the Lalley-Gatzouras and Barański classes, one notable difference is that there is no obvious analogue of *approximate squares*, on which the methods used in this chapter heavily rely. In order to generalise our results one may need to 'mimic' approximate squares in a delicate manner or adopt a different approach. Perhaps the most interesting direction for generalisation would be to look at arbitrary self-affine sets in a generic setting.

Question 3.11. *Can we say something about the Assouad dimension and lower dimension of self-affine sets in the generic case in the sense of Falconer, see [F2] and Theorem 1.3, or Jordan-Pollicott-Simon, see [JPS]?*

An interesting consequence of Mackay's results [M] and Theorem 3.1 is that the Assouad dimension, unlike the upper box dimension, is not bounded above by the *affinity dimension* (1.3). Are the Assouad and lower dimensions almost surely equal? If they are, then this almost sure value must indeed be the affinity dimension. If they are not almost surely equal, then are they at least almost surely equal to two different constants?

In the study of fractals one is often concerned with measures supported on sets rather than sets themselves. Although their definitions depend only on the structure of the set, the Assouad and lower dimensions have a fascinating link with certain classes of measures. Luukkainen and Saksman [LuS] (see also [KV]) proved that the Assouad dimension of a compact metric space X is the infimum of $s \geqslant 0$ such that there exists a locally finite measure μ on X and a constant $c_s > 0$ such that for any $0 < \rho < 1$, $x \in X$ and $r > 0$

$$\mu(B(x,r)) \;\leqslant\; c_s \rho^{-s} \mu(B(x,\rho r)). \tag{3.1}$$

Dually, Bylund and Gudayol [ByG] proved that the lower dimension of a compact metric space X is the supremum of $s \geqslant 0$ such that there exists a locally finite measure μ on X and a constant $d_s > 0$ such that for any $0 < \rho < 1$, $x \in X$ and $r > 0$

$$\mu(B(x,r)) \;\geqslant\; d_s \rho^{-s} \mu(B(x,\rho r)). \tag{3.2}$$

As such, our results give the existence of measures supported on self-affine carpets with useful scaling properties. In particular, if F is a self-affine carpet, then for each $s > \dim_{\mathrm{A}} F$ there exists a measure supported on F satisfying (3.1) and for each $s < \dim_{\mathrm{L}} F$ there exists a measure supported on F satisfying (3.2). It is natural to ask if 'sharp' measures exist.

Question 3.12. *Let F be a self-affine carpet. Does there exist a measure supported on F satisfying (3.1) for $s = \dim_{\mathrm{A}} F$ and a measure supported on F satisfying (3.2) for $s = \dim_{\mathrm{L}} F$?*

As mentioned above, it is interesting to examine the relationship between the Assouad and lower dimensions and the other dimensions discussed here. In particular, for a given class of sets one can ask *what relationships are possible between the dimensions?* For example, for Ahlfors regular sets all the dimensions are necessarily equal. The following table summarises the possible relationships between the Assouad and lower dimensions and the box dimension for the classes of sets we have been most interested in.

Configuration	horizontal/vertical class	mixed class	self-similar class
$\dim_{\mathrm{L}} F = \dim_{\mathrm{B}} F = \dim_{\mathrm{A}} F$	possible	possible	possible
$\dim_{\mathrm{L}} F = \dim_{\mathrm{B}} F < \dim_{\mathrm{A}} F$	not possible	not possible	possible
$\dim_{\mathrm{L}} F < \dim_{\mathrm{B}} F = \dim_{\mathrm{A}} F$	not possible	possible	not possible
$\dim_{\mathrm{L}} F < \dim_{\mathrm{B}} F < \dim_{\mathrm{A}} F$	possible	possible	not possible

The information presented in this table can be gleaned from Corollary 3.8, Corollary 3.3, Corollary 3.4 and the examples in Sections 3.4.1 and 3.4.2. Interestingly, the configuration $\dim_{\mathrm{L}} F < \dim_{\mathrm{B}} F = \dim_{\mathrm{A}} F$ is possible for self-affine carpets, but not for self-similar sets (even with overlaps) and the configuration $\dim_{\mathrm{L}} F = \dim_{\mathrm{B}} F < \dim_{\mathrm{A}} F$ is not possible for self-affine carpets, but is possible for self-similar sets with overlaps. Roughly speaking, the reason for this is that the non-uniform scaling present in self-affine carpets allows one to 'spread' the set out making certain places easier to cover and thus making the lower dimension drop and one can use overlaps to 'pile' the set up making certain places harder to cover and thus raising the Assouad dimension. It would be interesting to add Hausdorff dimension to the above analysis, but there are some configurations for which we have been unable to determine if they are possible or not.

Question 3.13. *Are any of the entries marked with a question mark in the following table possible in the relevant class of sets? The rest of the entries may be gleaned from Corollary 3.8, Corollary 3.3, Corollary 3.4 and the examples in Sections 3.4.1 and 3.4.2.*

Configuration	horizontal/vertical class	mixed class	self-similar
$\dim_{\mathrm{L}} F = \dim_{\mathrm{H}} F = \dim_{\mathrm{B}} F = \dim_{\mathrm{A}} F$	possible	possible	possible
$\dim_{\mathrm{L}} F = \dim_{\mathrm{H}} F = \dim_{\mathrm{B}} F < \dim_{\mathrm{A}} F$	not possible	not possible	possible
$\dim_{\mathrm{L}} F = \dim_{\mathrm{H}} F < \dim_{\mathrm{B}} F = \dim_{\mathrm{A}} F$	not possible	?	not possible
$\dim_{\mathrm{L}} F = \dim_{\mathrm{H}} F < \dim_{\mathrm{B}} F < \dim_{\mathrm{A}} F$	not possible	?	not possible
$\dim_{\mathrm{L}} F < \dim_{\mathrm{H}} F = \dim_{\mathrm{B}} F = \dim_{\mathrm{A}} F$	not possible	possible	not possible
$\dim_{\mathrm{L}} F < \dim_{\mathrm{H}} F = \dim_{\mathrm{B}} F < \dim_{\mathrm{A}} F$	not possible	?	not possible
$\dim_{\mathrm{L}} F < \dim_{\mathrm{H}} F < \dim_{\mathrm{B}} F = \dim_{\mathrm{A}} F$	not possible	?	not possible
$\dim_{\mathrm{L}} F < \dim_{\mathrm{H}} F < \dim_{\mathrm{B}} F < \dim_{\mathrm{A}} F$	possible	possible	not possible

3.6 Proofs

3.6.1 Preliminary results and approximate squares

In this section we will introduce some notation and give some basic technical lemmas. Let F be a self-affine carpet, which is the attractor of an IFS $\{S_i\}_{i\in\mathcal{I}}$. We will assume that F is not contained in a horizontal or vertical line, as otherwise it is self-similar and the results are obvious. Write $\alpha_1(\boldsymbol{i}) \geqslant \alpha_2(\boldsymbol{i})$ for the singular values of the linear part of the map $S_{\boldsymbol{i}}$. Note that, for all $\boldsymbol{i} \in \mathcal{I}^*$, the singular values, $\alpha_1(\boldsymbol{i})$ and $\alpha_2(\boldsymbol{i})$, are just the lengths of the sides of the rectangle $S_{\boldsymbol{i}}\big([0,1]^2\big)$. Also, let

$$\alpha_{\min} = \min\{\alpha_2(i) : i \in \mathcal{I}\}$$

and

$$\alpha_{\max} = \max\{\alpha_1(i) : i \in \mathcal{I}\}.$$

A subset $\mathcal{I}_0 \subset \mathcal{I}^*$ is called a *stopping* if for every $\boldsymbol{i} \in \mathcal{I}^*$ either there exists $\boldsymbol{j} \in \mathcal{I}_0$ such that $\boldsymbol{i} \prec \boldsymbol{j}$ or there exists a unique $\boldsymbol{j} \in \mathcal{I}_0$ such that $\boldsymbol{j} \prec \boldsymbol{i}$. An important class of stoppings will be ones where the members are chosen to have some sort of approximate property in common. In particular, *r-stoppings* are stoppings where the smallest sides of the corresponding rectangles are approximately equal to r. For $r \in (0,1]$ we define the *r-stopping*, $\mathcal{I}_r$, by

$$\mathcal{I}_r = \big\{\boldsymbol{i} \in \mathcal{I}^* : \alpha_2(\boldsymbol{i}) < r \leqslant \alpha_2(\overline{\boldsymbol{i}})\big\}.$$

Note that for $\boldsymbol{i} \in \mathcal{I}_r$ we have

$$\alpha_{\min}\, r \leqslant \alpha_2(\boldsymbol{i}) < r. \tag{3.3}$$

We will now fix some notation for the dimensions of the various projections and slices we will be interested in. Let

$$s_1 = \dim_{\mathrm{B}} \pi_1(F),$$

$$s_2 = \dim_{\mathrm{B}} \pi_2(F),$$

$$t_1 = \max_{i \in \mathcal{I}} \ \dim_{\mathrm{B}} \mathrm{Slice}_{1,i}(F),$$

$$t_2 = \max_{i \in \mathcal{I}} \ \dim_{\mathrm{B}} \mathrm{Slice}_{2,i}(F),$$

$$u_1 = \min_{i \in \mathcal{I}} \ \dim_{\mathrm{B}} \mathrm{Slice}_{1,i}(F),$$

and

$$u_2 = \min_{i \in \mathcal{I}} \ \dim_{\mathrm{B}} \mathrm{Slice}_{2,i}(F).$$

Note that all of these values can be easily computed as they are the dimensions of self-similar sets satisfying the open set condition. We will be particularly interested in estimating the precise value of the covering function N_r applied to the projections. It follows immediately from the definition of box dimension that for all $\varepsilon > 0$ there exists a constant $C_\varepsilon \geqslant 1$ such that for all $r \in (0,1]$ we have

$$\tfrac{1}{C_\varepsilon} \, r^{-s_1+\varepsilon} \leqslant N_r(\pi_1 F) \leqslant C_\varepsilon \, r^{-s_1-\varepsilon} \tag{3.4}$$

and

$$\tfrac{1}{C_\varepsilon} \, r^{-s_2+\varepsilon} \leqslant N_r(\pi_2 F) \leqslant C_\varepsilon \, r^{-s_2-\varepsilon}. \tag{3.5}$$

Since the basic rectangles in the construction of F often become very long and thin, they do not provide 'natural' covers for F, unlike in the self-similar setting. For this reason, we need to introduce *approximate squares*, which are now a standard concept in the study of self-affine carpets. The basic idea is to group together the construction rectangles into collections that look roughly like a square. Let $\boldsymbol{i} \in \mathcal{I}^{\mathbb{N}}$ and $r > 0$. Let $k_1(\boldsymbol{i}, r)$ equal the unique number $k \in \mathbb{N}$ such that

$$c_{\boldsymbol{i}|_{k+1}} < r \leqslant c_{\boldsymbol{i}|_k}$$

and $k_2(\boldsymbol{i}, r)$ equal the unique number $k \in \mathbb{N}$ such that

$$d_{\boldsymbol{i}|_{k+1}} < r \leqslant d_{\boldsymbol{i}|_k}.$$

Finally, we define the approximate square $Q(\boldsymbol{i}, r)$ 'centred' at $\Pi(\boldsymbol{i})$, with 'radius' r in the following way. If $k_1(\boldsymbol{i}, r) < k_2(\boldsymbol{i}, r)$, then

$$Q(\boldsymbol{i}, r) = S_{\boldsymbol{i}|_{k_1(i,r)}}\big([0,1]^2\big) \ \cap \ \Big\{x \in [0,1]^2 : \pi_1(x) \in \pi_1\big(S_{\boldsymbol{i}|_{k_2(i,r)}}\big([0,1]^2\big)\big)\Big\},$$

if $k_1(\boldsymbol{i}, r) > k_2(\boldsymbol{i}, r)$, then

$$Q(\boldsymbol{i}, r) = S_{\boldsymbol{i}|_{k_2(i,r)}}\big([0,1]^2\big) \ \cap \ \Big\{x \in [0,1]^2 : \pi_2(x) \in \pi_2\big(S_{\boldsymbol{i}|_{k_1(i,r)}}\big([0,1]^2\big)\big)\Big\},$$

and if $k_1(\boldsymbol{i}, r) = k_2(\boldsymbol{i}, r) = k$, then

$$Q(\boldsymbol{i}, r) = S_{\boldsymbol{i}|_k}\big([0,1]^2\big).$$

We will write

$$\mathcal{I}_{Q(\boldsymbol{i},r)} = \big\{\boldsymbol{j} \in \mathcal{I}^{\max\{k_1(\boldsymbol{i},r),\, k_2(\boldsymbol{i},r)\}} : S_{\boldsymbol{j}}(F) \subseteq Q(\boldsymbol{i}, r)\big\}.$$

The following lemma gives some of the basic properties of approximate squares.

Lemma 3.14. *Let $\boldsymbol{i} \in \mathcal{I}^{\mathbb{N}}$ and $r > 0$.*

(1) If $k_1(\boldsymbol{i}, r) \geqslant k_2(\boldsymbol{i}, r)$, then for $\boldsymbol{j} \in \mathcal{I}_{Q(\boldsymbol{i},r)}$ we have

$$r \ \leqslant \ c_{\boldsymbol{j}} \ \leqslant \ c_{\max}^{-1} \, r.$$

(2) If $k_1(\boldsymbol{i}, r) \leqslant k_2(\boldsymbol{i}, r)$, then for $\boldsymbol{j} \in \mathcal{I}_{Q(\boldsymbol{i},r)}$ we have

$$r \ \leqslant \ d_{\boldsymbol{j}} \ \leqslant \ d_{\max}^{-1} \, r.$$

(3) *The approximate square* $Q(\boldsymbol{i}, r)$ *is a rectangle with sides parallel to the coordinate axes and with base length in the interval* $[r, c_{\max}^{-1} r]$ *and height in the interval* $[r, d_{\max}^{-1} r]$*, so is indeed* approximately *a square.*

(4) *We have*

$$Q(\boldsymbol{i}, r) \subset B\big(\Pi(\boldsymbol{i}), \sqrt{2}\, \alpha_{\min}^{-1} r\big).$$

(5) *For any* $x \in F$*, the ball* $B(x, r)$ *can be covered by at most 9 approximate squares of radius* r *and the constant 9 is sharp.*

Proof. These facts follow immediately from the definition of approximate squares and are omitted. $\square$

Note that (4) and (5) together imply that we may replace $N_r(B(x, R))$ with $N_r(Q(\boldsymbol{i}, R))$ in the definitions of Assouad and lower dimension. Barański [B2] defined the numbers D_A and D_B to be the unique real numbers satisfying

$$\sum_{i \in \mathcal{I}} c_i^{s_1} d_i^{D_A - s_1} = 1 \qquad \text{and} \qquad \sum_{i \in \mathcal{I}} d_i^{s_2} c_i^{D_B - s_2} = 1$$

respectively. He then proved that $\dim_{\mathrm{B}} F = \max\{D_A,\, D_B\}$. The following lemma relates the numbers D_A and D_B to the numbers s_1, s_2, u_1, u_2, t_1 and t_2.

Lemma 3.15. *We have*

$$s_1 + u_1 \ \leqslant\ D_A \ \leqslant\ s_1 + t_1$$

and

$$s_2 + u_2 \ \leqslant\ D_B \ \leqslant\ s_2 + t_2.$$

Proof. We will prove that $s_1 + u_1 \leqslant D_A$. The other inequalities are proved similarly. Suppose that $D_A < s_1 + u_1$. Write m and n for the number of non-empty columns and rows respectively, counting columns from the left and rows from the bottom, and for $i \in \{1, \dots, m\}$ write

$$\mathcal{C}_i = \{j \in \mathcal{I} : S_j([0,1]^2) \text{ is found in the } i\text{th non-empty column of the defining pattern}\}$$

and for $i \in \{1, \dots, n\}$ write

$$\mathcal{R}_i = \{j \in \mathcal{I} : S_j([0,1]^2) \text{ is found in the } i\text{th non-empty row of the defining pattern}\}.$$

A useful consequence of splitting $\mathcal{I}$ up into columns and rows is that if i, j are in the same column, then $c_i = c_j$ and if i, j are in the same row, then $d_i = d_j$. As such, for $i \in \{1, \dots, m\}$ we will write $\hat{c}_i$ for the common base length in the ith column and for $i \in \{1, \dots, n\}$ we will write $\hat{d}_i$ for the common height in the ith row. We have

$$1 \ =\ \sum_{i \in \mathcal{I}} c_i^{s_1} d_i^{D_A - s_1} \ >\ \sum_{i \in \mathcal{I}} c_i^{s_1} d_i^{s_1 + u_1 - s_1} \ =\ \sum_{i=1}^{m} \hat{c}_i^{s_1} \sum_{j \in \mathcal{C}_i} d_j^{u_1} \ \geqslant\ \sum_{i=1}^{m} \hat{c}_i^{s_1} \ =\ 1$$

which is a contradiction. $\square$

Lemma 3.16. *Let* $\mathcal{I}_0$ *be a stopping. Then*

$$\sum_{\boldsymbol{i} \in \mathcal{I}_0} c_{\boldsymbol{i}}^{s_1} d_{\boldsymbol{i}}^{D_A - s_1} = \sum_{\boldsymbol{i} \in \mathcal{I}_0} d_{\boldsymbol{i}}^{s_2} c_{\boldsymbol{i}}^{D_B - s_2} = 1$$

Proof. This follows immediately from the definitions of D_A and D_B. $\square$

Let $r > 0$ and $\boldsymbol{i} = (i_1, i_2, \dots) \in \mathcal{I}^{\mathbb{N}}$. We call $\mathcal{I}_0 \subset \mathcal{I}^*$ a $Q(\boldsymbol{i}, r)$-*pseudo stopping* if the following conditions are satisfied.

(1) For each $\boldsymbol{j} \in \mathcal{I}_0$, we have $\boldsymbol{i}_{\min\{k_1(\boldsymbol{i}, r), k_2(\boldsymbol{i}, r)\}} \prec \boldsymbol{j}$,

(2) For each $\boldsymbol{j} \in \mathcal{I}_0$, we have $|\boldsymbol{j}| \leqslant \max\{k_1(\boldsymbol{i}, r), k_2(\boldsymbol{i}, r)\}$,

(3) For every $\boldsymbol{i}' \in \mathcal{I}^{\max\{k_1(i,r),k_1(i,r)\}}$ there exists a unique $\boldsymbol{j} \in \mathcal{I}_0$ such that $\boldsymbol{j} \prec \boldsymbol{i}'$.

The important feature of a $Q(\boldsymbol{i}, r)$-pseudo stopping, $\mathcal{I}_0$, is that the sets $\{S_{\boldsymbol{j}}([0,1]^2)\}_{\boldsymbol{j}\in\mathcal{I}_0}$ intersect the approximate square $Q(\boldsymbol{i}, r)$ in such a way as to induce natural IFSs of similarities on $[0,1]$. For instance, if $\max\{k_1(\boldsymbol{i}, r), k_2(\boldsymbol{i}, r)\} = k_1(\boldsymbol{i}, r)$, then each of the base lengths of the sets $\{S_{\boldsymbol{j}}([0,1]^2)\}_{\boldsymbol{j}\in\mathcal{I}_0}$ are greater than or equal to the base length of the approximate square. We then focus on the vertical lengths and, after scaling these up by the height of $Q(\boldsymbol{i}, r)$, use these as similarity ratios for a set of 1-dimensional contractions on $[0,1]$. It is easy to see that in this 'vertical case', the similarity dimension of the induced IFS lies in the interval $[u_1, t_1]$. This trick is illustrated in the following lemma and will be used frequently in the subsequent proofs.

Lemma 3.17. *Let $r > 0$, $\boldsymbol{i} \in \mathcal{I}^{\mathbb{N}}$ and let $\mathcal{I}_0$ be a $Q(\boldsymbol{i}, r)$-pseudo stopping and assume that $k_1(\boldsymbol{i}, r) \geqslant k_2(\boldsymbol{i}, r)$. Then, for any $t \geqslant t_1$, we have*

$$\sum_{\boldsymbol{j}\in\mathcal{I}_0} (d_{\boldsymbol{j}}/r)^t \;\leqslant\; d_{\min}^{-t}$$

and for any $u \leqslant u_1$, we have

$$\sum_{\boldsymbol{j}\in\mathcal{I}_0} (d_{\boldsymbol{j}}/r)^u \;\geqslant\; 1.$$

Proof. This proof is straightforward and we will only sketch it. Let $t \geqslant t_1$ and let $k_1 = k_1(\boldsymbol{i}, r) \geqslant k_2(\boldsymbol{i}, r) = k_2$. We have

$$\begin{aligned}
\sum_{\boldsymbol{j}\in\mathcal{I}_0} (d_{\boldsymbol{j}}/r)^t &= \sum_{\boldsymbol{j}\in\mathcal{I}_0} (d_{i_1}\dots d_{j_{k_2}}/r)^t\,(d_{j_{k_2+1}}\dots d_{j_{k_1}})^t \\
&\leqslant d_{\min}^{-t} \sum_{\boldsymbol{j}\in\mathcal{I}_0} (d_{j_{k_2+1}}\dots d_{j_{k_1}})^t \qquad \text{by Lemma 3.14 (2)} \\
&\leqslant d_{\min}^{-t}
\end{aligned}$$

since viewing the $(d_{j_{k_2+1}}\dots d_{j_{k_1}})$ as contraction ratios of a 1-dimensional IFS of similarities and noting that this IFS has similarity dimension less than or equal to t_1, yields

$$\sum_{\boldsymbol{i}\in\mathcal{I}_0} (d_{j_{k_2+1}}\dots d_{j_{k_1}})^t \;\leqslant\; 1.$$

The second estimate is similar. For $u \leqslant u_1$, we have

$$\begin{aligned}
\sum_{\boldsymbol{j}\in\mathcal{I}_0} (d_{\boldsymbol{j}}/r)^u &= \sum_{\boldsymbol{j}\in\mathcal{I}_0} (d_{j_1}\dots d_{j_{k_2}}/r)^u\,(d_{j_{k_2+1}}\dots d_{j_{k_1}})^u \\
&\geqslant \sum_{\boldsymbol{i}\in\mathcal{I}_0} (d_{j_{k_2+1}}\dots d_{j_{k_1}})^u \qquad \text{by Lemma 3.14 (2)} \\
&\geqslant 1
\end{aligned}$$

since viewing the $(d_{j_{k_2+1}}\dots d_{j_{k_1}})$ as contraction ratios of a 1-dimensional IFS of similarities and noting that this IFS has similarity dimension greater than or equal to u_1, yields

$$\sum_{\boldsymbol{j}\in\mathcal{I}_0} (d_{j_{k_2+1}}\dots d_{j_{k_1}})^u \;\geqslant\; 1.$$

This completes the proof. $\square$

Note that there are obvious analogues of the above Lemma in the case $k_1(\boldsymbol{i}, r) < k_2(\boldsymbol{i}, r)$, but we omit them here. Two natural examples of $Q(\boldsymbol{i}, r)$-pseudo stoppings are the 'extreme cases' $\{\boldsymbol{i}|_{\min\{k_1(i,r),\,k_2(i,r)\}}\}$ and $\mathcal{I}_{Q(i,r)}$. We give an example of an intermediate $Q(\boldsymbol{i}, r)$-pseudo stopping in the following figure.

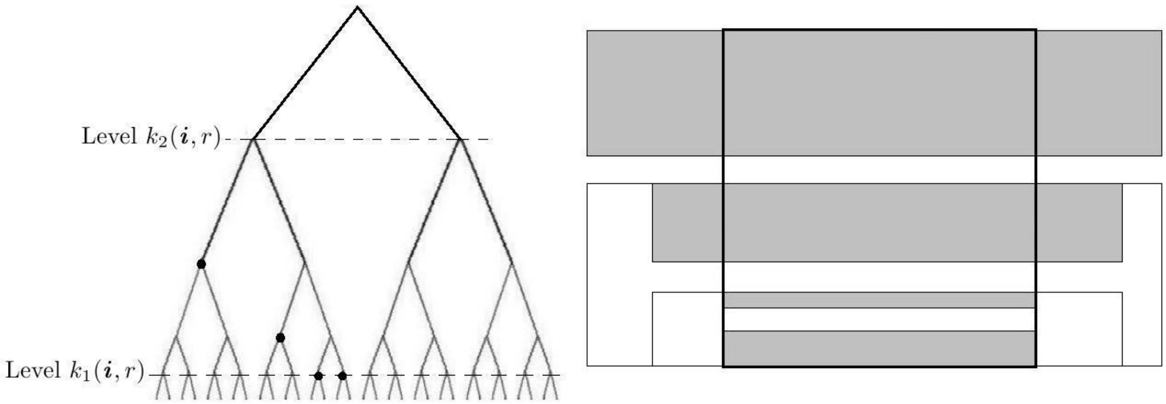

Figure 13: A binary tree giving a graphical representation of a pseudo stopping, $\mathcal{I}_0$, with black dots representing the elements of the pseudo stopping (left) and an indication of how the grey rectangles $\{S_{\boldsymbol{j}}([0,1]^2)\}_{\boldsymbol{j}\in\mathcal{I}_0}$ intersect the approximate square $Q(\boldsymbol{i}, r)$ (right).

3.6.2 Proof of Theorem 3.1 for the mixed class

Upper bound. The key to proving the upper bound for $\dim_{\mathrm{A}} F$ is to find the appropriate way to cover approximate squares. Fix $\boldsymbol{i}' \in \mathcal{I}^{\mathbb{N}}$, $R > 0$ and $r \in (0, R)$ and consider the approximate square $Q(\boldsymbol{i}', R)$. Without loss of generality assume that $k_1(\boldsymbol{i}', R) \geqslant k_2(\boldsymbol{i}', R)$ and to simplify notation let $k = k_1(\boldsymbol{i}', R)$. Furthermore we may assume that there exists $j_1, j_2 \in \mathcal{I}$ such that $c_{j_1} > d_{j_1}$ and $c_{j_2} < d_{j_2}$ as otherwise we are in the horizontal or vertical class, which will be dealt with in Section 3.6.3. Let $s = \max_{i\in\mathcal{I}} \max_{j=1,2} \big(\dim_{\mathrm{B}} \pi_j(F) + \dim_{\mathrm{B}} \mathrm{Slice}_{j,i}(F)\big)$. It suffices to prove that for all $\varepsilon \in (0,1)$, there exists a constant $C(\varepsilon)$ such that

$$N_r\Big(Q(\boldsymbol{i}', R) \cap F\Big) \leqslant C(\varepsilon) \left(\frac{R}{r}\right)^{s+\varepsilon}.$$

Let $\varepsilon \in (0,1)$. Writing

$$\mathcal{I}_Q = \mathcal{I}_{Q(\boldsymbol{i}',R)} = \big\{\boldsymbol{j} \in \mathcal{I}^k : S_{\boldsymbol{j}}(F) \subseteq Q(\boldsymbol{i}', R)\big\},$$

we first split the approximate square $Q(\boldsymbol{i}', R)$ up as

$$Q(\boldsymbol{i}', R) \cap F = \bigcup_{\boldsymbol{i}\in\mathcal{I}_Q} S_{\boldsymbol{i}}(F).$$

Secondly, we group together the sets $S_{\boldsymbol{i}}(F)$ for which $d_{\boldsymbol{i}} < r$ and cover their union separately. Within the other sets, $S_{\boldsymbol{i}}(F)$, we iterate the IFS until one side of the rectangle $S_{\boldsymbol{ij}}\big([0,1]^2\big) \supseteq S_{\boldsymbol{ij}}(F)$ is smaller than r. This is reminiscent of the techniques used in Chapter 2. We then cover each of the resulting copies of F individually. This is especially convenient because covering the part of F which lies in such a rectangle by sets of radius r is equivalent to covering a scaled down copy of the projection of F onto either the horizontal or vertical axis. Finally, we split the sets $S_{\boldsymbol{ij}}(F)$ which we are covering individually into two groups according to whether the short side of $S_{\boldsymbol{ij}}\big([0,1]^2\big)$ is vertical or horizontal. We have

$$\begin{aligned} N_r\Big(Q(\boldsymbol{i}', R) \cap F\Big) &\leqslant N_r\Bigg(\bigcup_{\substack{\boldsymbol{i}\in\mathcal{I}_Q:\\ d_{\boldsymbol{i}}<r}} S_{\boldsymbol{i}}(F)\Bigg) + \sum_{\substack{\boldsymbol{i}\in\mathcal{I}_Q:\\ d_{\boldsymbol{i}}\geqslant r}} \sum_{\substack{\boldsymbol{j}\in\mathcal{I}^*:\\ \boldsymbol{ij}\in\mathcal{I}_r}} N_r\Big(S_{\boldsymbol{ij}}(F)\Big) \\ &\leqslant N_r\Bigg(\bigcup_{\substack{\boldsymbol{i}\in\mathcal{I}_Q:\\ d_{\boldsymbol{i}}<r}} S_{\boldsymbol{i}}(F)\Bigg) \end{aligned}$$

$$+ \sum_{\substack{\boldsymbol{i}\in\mathcal{I}_Q:\\ d_{\boldsymbol{i}}\geqslant r}} \sum_{\substack{\boldsymbol{j}\in\mathcal{I}^*:\\ \boldsymbol{ij}\in\mathcal{I}_r,\\ \alpha_2(\boldsymbol{ij})=d_{\boldsymbol{ij}}}} N_r\Big(S_{\boldsymbol{ij}}(F)\Big) + \sum_{\substack{\boldsymbol{i}\in\mathcal{I}_Q\\ d_{\boldsymbol{i}}\geqslant r}} \sum_{\substack{\boldsymbol{j}\in\mathcal{I}^*:\\ \boldsymbol{ij}\in\mathcal{I}_r,\\ \alpha_2(\boldsymbol{ij})=c_{\boldsymbol{ij}}}} N_r\Big(S_{\boldsymbol{ij}}(F)\Big)$$

$$= N_r\left(\bigcup_{\substack{\boldsymbol{i}\in\mathcal{I}_Q:\\ d_{\boldsymbol{i}}<r}} S_{\boldsymbol{i}}(F) \right)$$

$$+ \sum_{\substack{\boldsymbol{i}\in\mathcal{I}_Q:\\ d_{\boldsymbol{i}}\geqslant r}} \sum_{\substack{\boldsymbol{j}\in\mathcal{I}^*:\\ \boldsymbol{ij}\in\mathcal{I}_r,\\ \alpha_2(\boldsymbol{ij})=d_{\boldsymbol{ij}}}} N_{r/c_{\boldsymbol{ij}}}\big(\pi_1(F)\big) + \sum_{\substack{\boldsymbol{i}\in\mathcal{I}_Q:\\ d_{\boldsymbol{i}}\geqslant r}} \sum_{\substack{\boldsymbol{j}\in\mathcal{I}^*:\\ \boldsymbol{ij}\in\mathcal{I}_r,\\ \alpha_2(\boldsymbol{ij})=c_{\boldsymbol{ij}}}} N_{r/d_{\boldsymbol{ij}}}\big(\pi_2(F)\big)$$

Now that we have established a natural way to cover $Q(\boldsymbol{i}', R)$, we need to show that this yields the correct estimates. We will analyse each of the three above terms separately. Write

$$\mathcal{I}_Q^{<r} = \big\{\boldsymbol{i} \in \mathcal{I}_Q : d_{\boldsymbol{i}} < r\big\}.$$

For the first term, we have

$$\begin{aligned}
N_r\left(\bigcup_{\substack{\boldsymbol{i}\in\mathcal{I}_Q:\\ d_{\boldsymbol{i}}<r}} S_{\boldsymbol{i}}(F) \right) &= N_r\left(\bigcup_{\boldsymbol{i}\in\mathcal{I}_Q^{<r}} S_{\boldsymbol{i}}(F) \right) \\
&= N_r\left(\bigcup_{\substack{\boldsymbol{j}\in\mathcal{I}_r:\\ \exists \boldsymbol{i}\in\mathcal{I}_Q^{<r},\ \boldsymbol{j}\prec\boldsymbol{i}}} S_{\boldsymbol{j}}(F)\cap Q(\boldsymbol{i}', R) \right) \\
&\leqslant \sum_{\substack{\boldsymbol{j}\in\mathcal{I}_r:\\ \exists \boldsymbol{i}\in\mathcal{I}_Q^{<r},\ \boldsymbol{j}\prec\boldsymbol{i}}} N_r\Big(S_{\boldsymbol{j}}(F)\cap Q(\boldsymbol{i}', R)\Big) \\
&= \sum_{\substack{\boldsymbol{j}\in\mathcal{I}_r:\\ \exists \boldsymbol{i}\in\mathcal{I}_Q^{<r},\ \boldsymbol{j}\prec\boldsymbol{i}}} N_{r/c_{\boldsymbol{j}}}\big(\pi_1(F)\big) \\
&\leqslant \sum_{\substack{\boldsymbol{j}\in\mathcal{I}_r:\\ \exists \boldsymbol{i}\in\mathcal{I}_Q^{<r},\ \boldsymbol{j}\prec\boldsymbol{i}}} C_\varepsilon \left(\frac{c_{\boldsymbol{j}}}{r}\right)^{s_1+\varepsilon} && \text{by (3.4)} \\
&\leqslant C_\varepsilon\, c_{\max}^{-2} \left(\frac{R}{r}\right)^{s_1+t_1+\varepsilon} \sum_{\substack{\boldsymbol{j}\in\mathcal{I}_r:\\ \exists \boldsymbol{i}\in\mathcal{I}_Q^{<r},\ \boldsymbol{j}\prec\boldsymbol{i}}} (r/R)^{t_1} && \text{by Lemma 3.14(1)} \\
&\leqslant C_\varepsilon\, c_{\max}^{-2}\, \alpha_{\min}^{-1} \left(\frac{R}{r}\right)^{s_1+t_1+\varepsilon} \sum_{\substack{\boldsymbol{j}\in\mathcal{I}_r:\\ \exists \boldsymbol{i}\in\mathcal{I}_Q^{<r},\ \boldsymbol{j}\prec\boldsymbol{i}}} (d_{\boldsymbol{j}}/R)^{t_1} && \text{by (3.3)} \\
&\leqslant C_\varepsilon\, c_{\max}^{-2}\, \alpha_{\min}^{-1}\, d_{\min}^{-t_1} \left(\frac{R}{r}\right)^{s_1+t_1+\varepsilon}
\end{aligned}$$

by Lemma 3.17 since $\{\boldsymbol{j} \in \mathcal{I}_r : \text{ there exists } \boldsymbol{i} \in \mathcal{I}_Q^{<r} \text{ such that } \boldsymbol{j} \prec \boldsymbol{i}\}$ is clearly contained in some $Q(\boldsymbol{i}', R)$-pseudo stopping. For the second term, by (3.4), we have

$$\begin{aligned}
\sum_{\substack{\boldsymbol{i}\in\mathcal{I}_Q:\\ d_{\boldsymbol{i}}\geqslant r}} \sum_{\substack{\boldsymbol{j}\in\mathcal{I}^*:\\ \boldsymbol{ij}\in\mathcal{I}_r,\\ \alpha_2(\boldsymbol{ij})=d_{\boldsymbol{ij}}}} N_{r/c_{\boldsymbol{ij}}}\big(\pi_1(F)\big) &\leqslant \sum_{\substack{\boldsymbol{i}\in\mathcal{I}_Q:\\ d_{\boldsymbol{i}}\geqslant r}} \sum_{\substack{\boldsymbol{j}\in\mathcal{I}^*:\\ \boldsymbol{ij}\in\mathcal{I}_r,\\ \alpha_2(\boldsymbol{ij})=d_{\boldsymbol{ij}}}} C_\varepsilon \left(\frac{c_{\boldsymbol{i}}c_{\boldsymbol{j}}}{r}\right)^{s_1+\varepsilon} \\
&\leqslant C_\varepsilon \left(\frac{1}{r}\right)^{s_1+\varepsilon} \sum_{\substack{\boldsymbol{i}\in\mathcal{I}_Q:\\ d_{\boldsymbol{i}}\geqslant r}} c_{\boldsymbol{i}}^{s_1+\varepsilon} \sum_{\substack{\boldsymbol{j}\in\mathcal{I}^*:\\ \boldsymbol{ij}\in\mathcal{I}_r,\\ \alpha_2(\boldsymbol{ij})=d_{\boldsymbol{ij}}}} c_{\boldsymbol{j}}^{s_1} \\
&\leqslant C_\varepsilon \left(\frac{1}{r}\right)^{s_1+\varepsilon} \sum_{\substack{\boldsymbol{i}\in\mathcal{I}_Q:\\ d_{\boldsymbol{i}}\geqslant r}} (c_{\max}^{-1} R)^{s_1+\varepsilon} \sum_{\substack{\boldsymbol{j}\in\mathcal{I}^*:\\ \boldsymbol{ij}\in\mathcal{I}_r,\\ \alpha_2(\boldsymbol{ij})=d_{\boldsymbol{ij}}}} c_{\boldsymbol{j}}^{s_1} \left(d_{\boldsymbol{i}}\, d_{\boldsymbol{j}}\, r^{-1}\, \alpha_{\min}^{-1}\right)^{t_1} \\
&\qquad\qquad \text{by Lemma 3.14(1) and (3.3)} \\
&\leqslant C_\varepsilon\, c_{\max}^{-2}\, \alpha_{\min}^{-1} \left(\frac{R}{r}\right)^{s_1+\varepsilon} \left(\frac{1}{r}\right)^{t_1} \sum_{\substack{\boldsymbol{i}\in\mathcal{I}_Q:\\ d_{\boldsymbol{i}}\geqslant r}} d_{\boldsymbol{i}}^{t_1} \sum_{\substack{\boldsymbol{j}\in\mathcal{I}^*:\\ \boldsymbol{ij}\in\mathcal{I}_r,\\ \alpha_2(\boldsymbol{ij})=d_{\boldsymbol{ij}}}} c_{\boldsymbol{j}}^{s_1}\, d_{\boldsymbol{j}}^{t_1} \\
&\leqslant C_\varepsilon\, c_{\max}^{-2}\, \alpha_{\min}^{-1} \left(\frac{R}{r}\right)^{s_1+t_1+\varepsilon} \sum_{\boldsymbol{i}\in\mathcal{I}_Q} (d_{\boldsymbol{i}}/R)^{t_1} \sum_{\substack{\boldsymbol{j}\in\mathcal{I}^*:\\ \boldsymbol{ij}\in\mathcal{I}_r}} c_{\boldsymbol{j}}^{s_1}\, d_{\boldsymbol{j}}^{D_A-s_1} \\
&\qquad\qquad \text{by Lemma 3.15} \\
&\leqslant C_\varepsilon\, c_{\max}^{-2}\, \alpha_{\min}^{-1} \left(\frac{R}{r}\right)^{s_1+t_1+\varepsilon} \sum_{\boldsymbol{i}\in\mathcal{I}_Q} (d_{\boldsymbol{i}}/R)^{t_1} \qquad \text{by Lemma 3.16} \\
&\leqslant C_\varepsilon\, c_{\max}^{-2}\, \alpha_{\min}^{-1}\, d_{\min}^{-t_1} \left(\frac{R}{r}\right)^{s_1+t_1+\varepsilon}
\end{aligned}$$

by Lemma 3.17 since $\mathcal{I}_Q$ is a $Q(\boldsymbol{i}', R)$-pseudo stopping. Finally, for the third term, by (3.5), we have

$$\begin{aligned}
\sum_{\substack{\boldsymbol{i}\in\mathcal{I}_Q:\\ d_{\boldsymbol{i}}\geqslant r}} \sum_{\substack{\boldsymbol{j}\in\mathcal{I}^*:\\ \boldsymbol{ij}\in\mathcal{I}_r,\\ \alpha_2(\boldsymbol{ij})=c_{\boldsymbol{ij}}}} N_{r/d_{\boldsymbol{ij}}}\big(\pi_2(F)\big) &\leqslant \sum_{\boldsymbol{i}\in\mathcal{I}_Q} \sum_{\substack{\boldsymbol{j}\in\mathcal{I}^*:\\ \boldsymbol{ij}\in\mathcal{I}_r,\\ \alpha_2(\boldsymbol{ij})=c_{\boldsymbol{ij}}}} C_\varepsilon \left(\frac{d_{\boldsymbol{i}}d_{\boldsymbol{j}}}{r}\right)^{s_2+\varepsilon} \\
&\leqslant C_\varepsilon \left(\frac{1}{r}\right)^{s_2+\varepsilon} \sum_{\boldsymbol{i}\in\mathcal{I}_Q} d_{\boldsymbol{i}}^{s_2+\varepsilon} \sum_{\substack{\boldsymbol{j}\in\mathcal{I}^*:\\ \boldsymbol{ij}\in\mathcal{I}_r,\\ \alpha_2(\boldsymbol{ij})=c_{\boldsymbol{ij}}}} d_{\boldsymbol{j}}^{s_2} \\
&\leqslant C_\varepsilon \left(\frac{1}{r}\right)^{s_2+\varepsilon} \sum_{\boldsymbol{i}\in\mathcal{I}_Q} d_{\boldsymbol{i}}^{s_2+\varepsilon} \sum_{\substack{\boldsymbol{j}\in\mathcal{I}^*:\\ \boldsymbol{ij}\in\mathcal{I}_r,\\ \alpha_2(\boldsymbol{ij})=c_{\boldsymbol{ij}}}} d_{\boldsymbol{j}}^{s_2} \left(c_{\boldsymbol{j}}\, c_{\boldsymbol{i}}\, r^{-1}\, \alpha_{\min}^{-1}\right)^{t_2} \qquad \text{by (3.3)}
\end{aligned}$$

$$
\begin{aligned}
&= C_\varepsilon\, \alpha_{\min}^{-1} \left(\frac{1}{r}\right)^{s_2+\varepsilon} \left(\frac{1}{r}\right)^{t_2} \sum_{\boldsymbol{i}\in\mathcal{I}_Q} d_{\boldsymbol{i}}^{s_2+\varepsilon}\, c_{\boldsymbol{i}}^{t_2} \sum_{\substack{\boldsymbol{j}\in\mathcal{I}^*:\\ \boldsymbol{ij}\in\mathcal{I}_r,\\ \alpha_2(\boldsymbol{ij})=c_{\boldsymbol{ij}}}} d_{\boldsymbol{j}}^{s_2}\, c_{\boldsymbol{j}}^{t_2} \\
&\leqslant C_\varepsilon\, \alpha_{\min}^{-1} \left(\frac{1}{r}\right)^{s_2+t_2+\varepsilon} R^{s_2+\varepsilon} \sum_{\boldsymbol{i}\in\mathcal{I}_Q} (d_{\boldsymbol{i}}/R)^{s_2}\, (c_{\max}^{-1}R)^{t_2} \sum_{\substack{\boldsymbol{j}\in\mathcal{I}^*:\\ \boldsymbol{ij}\in\mathcal{I}_r}} d_{\boldsymbol{j}}^{s_2}\, c_{\boldsymbol{j}}^{D_B-s_2} \\
&\qquad \text{by Lemma 3.14 (1) and Lemma 3.15} \\
&\leqslant C_\varepsilon\, c_{\max}^{-2}\, \alpha_{\min}^{-1} \left(\frac{R}{r}\right)^{s_2+t_2+\varepsilon} \sum_{\boldsymbol{i}\in\mathcal{I}_Q} (d_{\boldsymbol{i}}/R)^{s_2} \qquad \text{by Lemma 3.16} \\
&\leqslant C_\varepsilon\, c_{\max}^{-2}\, \alpha_{\min}^{-1}\, d_{\min}^{-s_2} \left(\frac{R}{r}\right)^{s_2+t_2+\varepsilon}
\end{aligned}
$$

by Lemma 3.17 since $\mathcal{I}_Q$ is a $Q(\boldsymbol{i}', R)$-pseudo stopping and $s_2 \geqslant t_1$. Since both s_1+t_1 and s_2+t_2 are less than or equal to s, combining the above estimates for the three terms appearing in the natural cover for $Q(\boldsymbol{i}', R)$ which were introduced at the beginning of the proof yields

$$
N_r\Big(Q(\boldsymbol{i}', R) \cap F\Big) \;\leqslant\; 3\, C_\varepsilon\, c_{\max}^{-2}\, \alpha_{\min}^{-1}\, d_{\min}^{-1} \left(\frac{R}{r}\right)^{s+\varepsilon}
$$

which upon letting $\varepsilon \to 0$ gives the desired upper bound. □

Lower bound. The proof of the lower bound will employ some of the techniques used in Mackay [M]. In particular, we will construct *weak tangents* with the desired dimension. Weak tangents were used in Section 3.4.1 to find a lower bound for the dimension of a self-similar set with overlaps. Here we require the 2 dimensional version, which also follows from [MT, Proposition 6.1.5], which we state here for the benefit of the reader.

Proposition 3.18 (Mackay-Tyson)**.** *Let $X \subset \mathbb{R}^2$ be compact and let F be a compact subset of X. Let T_k be a sequence of similarity maps defined on $\mathbb{R}^2$ and suppose that $T_k(F) \cap X \to_{d_{\mathcal{H}}} \hat{F}$. Then $\dim_{\mathrm{A}} \hat{F} \leqslant \dim_{\mathrm{A}} F$.*

The set $\hat{F}$ in the above lemma is called a *weak tangent* to F. We are now ready to prove the lower bound.

Proof. Let F be a self-affine set in the mixed class. Without loss of generality we may assume that

$$
\max_{i\in\mathcal{I}} \max_{j=1,2} \Big(\dim_{\mathrm{B}} \pi_j(F) \,+\, \dim_{\mathrm{B}} \mathrm{Slice}_{j,i}(F)\Big) = \dim_{\mathrm{B}} \pi_1(F) + \mathrm{Slice}_{1,i}(F)
$$

for some $i \in \mathcal{I}$ which we now fix. Also fix $j \in \mathcal{I}$ with $c_j > d_j$ which we may assume exists as otherwise we are in the horizontal or vertical class, which will be dealt with in the following section. Let $k \in \mathbb{N}$, let

$$
\boldsymbol{i}(k) = (\underbrace{j, j, \dots, j}_{k \text{ times}}, i, i, \dots) \in \mathcal{I}^{\mathbb{N}}
$$

and let $X = [0, c_i^{-1}] \times [0,1]$. We will consider the sequence of approximate squares $\{Q(\boldsymbol{i}(k), d_j^k)\}_k$. Note that for $k \in \mathbb{N}$, we have $k_2(\boldsymbol{i}(k), d_j^k) = k$ and let $k_1(\boldsymbol{i}(k), d_j^k) = k + l(k)$ for $l(k) \in \mathbb{N}$ satisfying

$$
c_j^k c_i^{l(k)+1} < d_j^k \leqslant c_j^k c_i^{l(k)}.
$$

For each $k \in \mathbb{N}$, let T_k be the unique homothetic similarity on $\mathbb{R}^2$ with similarity ratio d_j^{-k} which maps the approximate square $Q(\boldsymbol{i}(k), d_j^k)$ to $[0, d_j^{-k} c_i^k c_j^{l(k)}] \times [0,1] \subseteq X$, mapping the left vertical side of $Q(\boldsymbol{i}(k), d_j^k)$ to $\{0\} \times [0,1]$.

Note that we may take a subsequence of the T_k such that $d_j^{-k} c_i^k c_j^{l(k)} \to 1$. To see this observe that if $\log(d_j/c_j)/(\log c_i) \in \mathbb{Q}$, then there exists a subsequence where $d_j^{-k} c_i^k c_j^{l(k)} = 1$ for all k and if $\log(d_j/c_j)/(\log c_i) \notin \mathbb{Q}$, then the $d_j^{-k} c_i^k c_j^{l(k)}$ are uniformly distributed on $(1, c_i^{-1})$. Using this and the fact that $\big(\mathcal{K}(X), d_\mathcal{H}\big)$ is compact, we may extract a subsequence of the T_k for which $T_k(F) \cap X$ converges to a weak tangent $\hat{F} \subseteq X$ and $d_j^{-k} c_i^k c_j^{l(k)} \to 1$.

Lemma 3.19. *The weak tangent $\hat{F}$ constructed above contains the set $\pi_1(F) \times \pi_2\big(\mathrm{Slice}_{1,i}(F)\big)$.*

Proof. It suffices to show that $T_k\big(Q(\boldsymbol{i}(k), d_j^k) \cap F\big)$ converges to $\pi_1(F) \times \pi_2\big(\mathrm{Slice}_{1,i}(F)\big)$ in the Hausdorff metric. The IFS $\mathcal{I}$ induces an IFS of similarities on the vertical slice through $\Pi(\boldsymbol{i})$ (which is the fixed point of S_i). It is easy to see that the attractor of this IFS is isometric to $\mathrm{Slice}_{1,i}(F)$. Let E_k denote the $l(k)$th level in the construction of $\mathrm{Slice}_{1,i}(F)$ via the induced IFS. We claim that the set $T_k\big(Q(\boldsymbol{i}(k), d_j^k) \cap F\big)$ will never be further away than $d_j^{-k} c_i^k c_j^{l(k)} - 1 \ + \ d_{\max}^{l(k)}$ from the set $\pi_1(F) \times \pi_2(E_k)$ in the Hausdorff metric. To see this observe that if we scale $T_k\big(Q(\boldsymbol{i}(k), c_j^k) \cap F\big)$ horizontally by $d_j^k c_i^{-k} c_j^{-l(k)}$ it becomes a set, $\pi_1(F) \times H_k$ for some set $H_k \subseteq \pi_2(E_k)$ with the property that H_k intersects every basic interval in $\pi_2(E_k)$. Since each basic interval in $\pi_2(E_k)$ has length no greater than $d_{\max}^{l(k)}$ we have that $\pi_1(F) \times \pi_2(E_k)$ is contained in the $d_{\max}^{l(k)}$ neighbourhood of $\pi_1(F) \times H_k$. Hence

$$\begin{aligned} d_\mathcal{H}\Big(T_k\big(Q(\boldsymbol{i}(k), d_j^k) \cap F\big), \ \pi_1(F) \times \pi_2(E_k)\Big) \ &\leqslant \ d_\mathcal{H}\Big(T_k\big(Q(\boldsymbol{i}(k), d_j^k) \cap F\big)\big), \ \pi_1(F) \times H_k\Big) \\ &\qquad + d_\mathcal{H}\Big(\pi_1(F) \times H_k, \ \pi_1(F) \times \pi_2(E_k)\Big) \\ &\leqslant \ d_j^{-k} c_i^k c_j^{l(k)} - 1 \ + \ d_{\max}^{l(k)}. \end{aligned}$$

It follows from the claim that

$$\begin{aligned} d_\mathcal{H}\Big(T_k\big(Q(\boldsymbol{i}(k), d_j^k) \cap F\big), \ \ \pi_1(F) \times \pi_2\big(\mathrm{Slice}_j(F)\big)\Big) \ &\leqslant \ d_\mathcal{H}\Big(T_k\big(Q(\boldsymbol{i}(k), d_j^k) \cap F\big), \ \ \pi_1(F) \times \pi_2(E_k)\Big) \\ &\qquad + d_\mathcal{H}\Big(\pi_1(F) \times \pi_2(E_k), \ \ \pi_1(F) \times \pi_2\big(\mathrm{Slice}_j(F)\big)\Big) \\ &\leqslant \ \Big(d_j^{-k} c_i^k c_j^{l(k)} - 1 \ + \ d_{\max}^{l(k)}\Big) \ + \ d_{\max}^{l(k)} \\ &\to \ 0 \end{aligned}$$

as $k \to \infty$, since

$$l(k) \ > \ k\frac{\log(d_j/c_j)}{\log c_i} - 1 \to \infty \qquad \text{and} \qquad d_j^{-k} c_i^k c_j^{l(k)} \to 1$$

which completes the proof of Lemma 3.19. □

We can now complete the proof of the lower bound by estimating the Assouad dimension of F from below, using the fact that $\hat{F}$ is a product of two self-similar sets.

$$\begin{aligned} \dim_\mathrm{B} \pi_1(F) + \dim_\mathrm{B} \mathrm{Slice}_{1,i}(F) \ &= \ \dim_\mathrm{B} \Big(\pi_1(F) \times \pi_2\big(\mathrm{Slice}_{1,i}(F)\big)\Big) \\ &\leqslant \ \dim_\mathrm{B} \hat{F} \qquad \text{by Lemma 3.19} \\ &\leqslant \ \dim_\mathrm{A} \hat{F} \ \leqslant \ \dim_\mathrm{A} F \end{aligned}$$

by Proposition 3.18. □

3.6.3 Proof of Theorem 3.1 for the horizontal and vertical classes

This is similar to the proof in the mixed case and so we only briefly discuss it.

Upper bound. We break up $N_r\big(Q(\boldsymbol{i}, R) \cap F\big)$ in the same way except in this case we may omit either the second or the third term as the smallest singular value always corresponds to either the vertical contraction (in the horizontal class) or the horizontal contraction (in the vertical class). The rest of the proof proceeds in the same way.

Lower bound. One can construct a weak tangent with the required dimension. The key difference to Mackay's argument [M] is that, since we may be in the extended Lalley-Gatzouras case, we may not be able to fix a map at the beginning to 'follow into the construction'. Either one can iterate the IFS to find a genuinely affine map which one can 'follow in' to find a weak tangent with dimension arbitrarily close to the required dimension, or one can follow our proof in the previous section and choose a genuinely affine map for the first k stages and then switch to a map in the correct column.

3.6.4 Proof of Theorem 3.2 for the mixed class

Upper bound. Since lower dimension is a natural dual to Assouad dimension and tends to 'mirror' the Assouad dimension in many ways, one might expect, given that weak tangents provide a very natural way to find *lower bounds* for Assouad dimension, that weak tangents might provide a way of giving *upper bounds* for lower dimension. In particular, in light of Proposition 3.18, one might naïvely expect the following statement to be true:

"Let $X \subset \mathbb{R}^2$ be compact and let F be a compact subset of X. Let T_k be a sequence of similarity maps defined on $\mathbb{R}^2$ and suppose that $T_k(F) \cap X \to_{d_\mathcal{H}} \hat{F}$. Then $\dim_\mathrm{L} \hat{F} \geqslant \dim_\mathrm{L} F$."

However, it is easy to see that this is false as one can often find weak tangents with isolated points, and hence lower dimension equal to zero, even if the original set has positive lower dimension. However, we will now state and prove what we believe is the natural analogue of Proposition 3.18 for the lower dimension. Note that we also give a slight strengthening of Proposition 3.18 in that we relax the conditions on the maps $\{T_k\}$ from *similarity maps* to certain classes of *bi-Lipschitz maps*. This change is specifically designed to deal with the lower dimension because we can now make the weak tangent precisely equal to the limit of scaled versions of approximate squares. This is necessary because lower dimension is not monotone and so an analogue of Lemma 3.19 would not suffice. We include the statement for Assouad dimension for completeness. The key feature for the lower dimension result is the existence of a constant $\theta \in (0, 1]$ with the properties described below. This is required to prevent the unwanted introduction of isolated points or indeed any points around which the set is inappropriately easy to cover. We call the 'tangents' described in the following proposition *very weak tangents*.

Proposition 3.20 (very weak tangents)**.** *Let $X \subset \mathbb{R}^n$ be compact and let F be a compact subset of X. Let T_k be a sequence of bi-Lipschitz maps defined on $\mathbb{R}^n$ with Lipschitz constants $a_k, b_k \geqslant 1$ such that*

$$a_k|x-y| \;\leqslant\; |T_k(x) - T_k(y)| \;\leqslant\; b_k|x-y| \qquad (x, y \in \mathbb{R}^n)$$

and

$$\sup_k b_k/a_k \;=\; C_0 \;<\; \infty$$

and suppose that $T_k(F) \cap X \to_{d_\mathcal{H}} \hat{F}$. Then

$$\dim_\mathrm{A} \hat{F} \;\leqslant\; \dim_\mathrm{A} F.$$

If, in addition, there exists a uniform constant $\theta \in (0, 1]$ such that for all $r \in (0, 1]$ and $\hat{x} \in \hat{F}$, there exists $\hat{y} \in \hat{F}$ such that $B(\hat{y}, r\theta) \subseteq B(\hat{x}, r) \cap X$; then

$$\dim_\mathrm{L} F \;\leqslant\; \dim_\mathrm{L} \hat{F} \;\leqslant\; \dim_\mathrm{A} \hat{F} \;\leqslant\; \dim_\mathrm{A} F.$$

Proof. Let $F \subseteq X$ be a compact set and assume that $\dim_{\mathrm{L}} F > 0$. If $\dim_{\mathrm{L}} F = 0$, then the lower estimate is trivial. Let $\hat{F}$ be a very weak tangent to F, as described above, and let $\alpha, \beta \in (0,\infty)$ with $\alpha < \dim_{\mathrm{L}} F \leqslant \dim_{\mathrm{A}} F < \beta$. It follows from the fact that the T_k are bi-Lipschitz maps with Lipschitz constants $b_k \geqslant a_k \geqslant 1$ satisfying $\sup_k b_k/a_k = C_0 < \infty$ that there exists uniform constants $C_1, C_2, \rho > 0$ such that for all $k \in \mathbb{N}$, all $0 < r < R \leqslant \rho$ and all $x \in T_k(F)$ we have

$$C_1 \, C_0^{-\alpha} \left(\frac{R}{r}\right)^{\alpha} \;\leqslant\; N_r\big(B(x,R) \cap T_k(F)\big) \;\leqslant\; C_2 \, C_0^{\beta} \left(\frac{R}{r}\right)^{\beta}.$$

Fix $0 < r < R \leqslant \rho$ and fix $\hat{x} \in \hat{F}$. Choose $k \in \mathbb{N}$ such that $d_{\mathcal{H}}(T_k(F) \cap X, \hat{F}) < r/2$. It follows that there exists $x \in T_k(F) \cap X$ such that $B(\hat{x},R) \cap \hat{F} \subseteq B(x,2R)$ and hence, given any $r/2$-cover of $B(x,2R) \cap T_k(F)$, we may find an r-cover of $B(\hat{x},R) \cap \hat{F}$ by the same number of sets. Thus

$$N_r\big(B(\hat{x},R) \cap \hat{F}\big) \;\leqslant\; N_{r/2}\big(B(x,2R) \cap T_k(F)\big) \;\leqslant\; C_2 \, C_0^{\beta} \left(\frac{2R}{r/2}\right)^{\beta} \;=\; C_2 \, C_0^{\beta} \, 4^{\beta} \left(\frac{R}{r}\right)^{\beta}$$

which proves that $\dim_{\mathrm{A}} \hat{F} \;\leqslant\; \dim_{\mathrm{A}} F$.

For the lower estimate assume that there exists $\theta \in (0,1]$ satisfying the above property and fix $\hat{x} \in \hat{F}$. We may thus find $\hat{y} \in \hat{F}$ such that $B(\hat{y},R\theta) \subseteq B(\hat{x},R) \cap X$. Choose $k \in \mathbb{N}$ such that $d_{\mathcal{H}}(T_k(F) \cap X, \hat{F}) < \min\{r/2, R\theta/2\}$. It follows that there exists $y \in T_k(F) \cap X$ such that $B(y,R\theta/2) \subseteq B(\hat{y},R\theta) \subseteq B(\hat{x},R) \cap X$ and hence, given any r-cover of $B(\hat{x},R) \cap \hat{F}$, we may find an $2r$-cover of $B(y,R\theta/2) \cap T_k(F) \cap X = B(y,R\theta/2) \cap T_k(F)$ by the same number of sets. Thus

$$N_r\big(B(\hat{x},R) \cap \hat{F}\big) \;\geqslant\; N_{2r}\big(B(y,R\theta/2) \cap T_k(F)\big) \;\geqslant\; C_1 \, C_0^{-\alpha} \left(\frac{R\theta/2}{2r}\right)^{\alpha} \;=\; C_1 \, C_0^{-\alpha} \, (\theta/4)^{\alpha} \left(\frac{R}{r}\right)^{\alpha}$$

which proves that $\dim_{\mathrm{L}} \hat{F} \;\geqslant\; \dim_{\mathrm{L}} F$. □

We will now turn to the proof of Theorem 3.2. Let F be a self-affine set in the mixed class. Without loss of generality we may assume that

$$\min_{i \in \mathcal{I}} \min_{j=1,2} \Big(\dim_{\mathrm{B}} \pi_j(F) \,+\, \dim_{\mathrm{B}} \mathrm{Slice}_{j,i}(F)\Big) = \dim_{\mathrm{B}} \pi_1(F) + \mathrm{Slice}_{1,i}(F)$$

for some $i \in \mathcal{I}$ which we now fix. Also assume that the column in the construction pattern which contains the rectangle corresponding to i contains at least one other rectangle. Why we assume this will become clear during the proof and we will deal with the other case afterwards. Now fix $j \in \mathcal{I}$ with $c_j > d_j$ which we may assume exists as otherwise we are in the horizontal or vertical class, which will be dealt with in the following section. Let $k \in \mathbb{N}$ and let

$$\boldsymbol{i}(k) = (\underbrace{j, j, \dots, j}_{k \text{ times}}, i, i, \dots) \in \mathcal{I}^{\mathbb{N}}$$

and let $X = [0,1]^2$. We will consider the sequence of approximate squares $\{Q(\boldsymbol{i}(k), d_j^k)\}_k$. For each $k \in \mathbb{N}$, let T_k be the unique linear bi-Lipschitz map on $\mathbb{R}^2$ which maps the approximate square $Q(\boldsymbol{i}(k), d_j^k)$ to X, mapping the left vertical side of $Q(\boldsymbol{i}(k), d_j^k)$ to $\{0\} \times [0,1]$ and the bottom side of $Q(\boldsymbol{i}(k), d_j^k)$ to $[0,1] \times \{0\}$. Note that this sequence of maps $\{T_k\}$ satisfies the requirements of Proposition 3.20 with $C_0 = \alpha_{\min}^{-1}$, say. Since $\big(\mathcal{K}(X), d_{\mathcal{H}}\big)$ is compact, we may extract a subsequence of the T_k for which $T_k(F) \cap X$ converges to a very weak tangent $\hat{F} \subseteq X$.

Lemma 3.21. *The very weak tangent, $\hat{F}$, constructed above is equal to $\pi_1(F) \times \pi_2\big(\mathrm{Slice}_{1,i}(F)\big)$ and, furthermore, there exists $\theta \in (0,1]$ with the desired property from Proposition 3.20.*

Proof. To show that $\hat{F} = \pi_1(F) \times \pi_2\big(\mathrm{Slice}_{1,i}(F)\big)$, it suffices to show that $T_k\big(Q(\boldsymbol{i}(k), d_j^k) \cap F\big)$ converges to $\pi_1(F) \times \pi_2\big(\mathrm{Slice}_{1,i}(F)\big)$ in the Hausdorff metric. This follows by a virtually identical argument to that used in the proof of Lemma 3.19 and is therefore omitted. It remains to show that there exists $\theta \in (0,1]$ such that for all $r \in (0,1]$ and $\hat{x} \in \hat{F}$, there exists $\hat{y} \in \hat{F}$ such that $B(\hat{y}, r\theta) \subseteq B(\hat{x}, r) \cap X$.

We will first prove that the one dimensional analogue of this property holds for self-similar subsets of $[0,1]$. In particular, let $E \subseteq [0,1]$ be the self-similar attractor of an IFS consisting of $N \geqslant 2$ homothetic similarities with similarity ratios $\{c_1, \dots, c_N\}$ ordered from left to right by translation vector and write $c_{\min}$ for the smallest contraction ratio. We will prove that there exists $\theta \in (0,1]$ such that for all $r \in (0,1]$ and $x \in E$, there exists $y \in E$ such that $B(y, r\theta) \subseteq B(x,r) \cap [0,1]$. If $E \subseteq (0,1)$, then we may choose $\theta = \inf_{x\in E, y=0,1} |x-y| > 0$ and then for any $x \in E$, we may choose $y = x$. Thus we assume without loss of generality that $0 \in E$ and so c_1 is the contraction ratio of a map which fixes 0. Also, write $z = \sup_{x \in E}|x|$. It suffices to prove the result in the case $x = 0$ and $r \in (0, z]$. Observe that $c_1^k z \in E$ for all $k \in \mathbb{N}_0$ and let

$$k = \min\left\{ l \in \mathbb{N}_0 : c_1^l z < r(1-c_{\min})\right\},$$

$y = c_1^k z \in E$ and $\theta = c_{\min}(1-c_{\min})$. To see that this choice of y and θ works, observe that

$$y + \theta r = c_1^k z + c_{\min}(1-c_{\min})r < r(1-c_{\min}) + c_{\min} r = r$$

and

$$y - \theta r = c_1^k z - c_{\min}(1-c_{\min})r \geqslant c_1 r(1-c_{\min}) - c_{\min}(1-c_{\min})r \geqslant r(1-c_{\min})(c_1 - c_{\min}) \geqslant 0$$

and so $B(y, r\theta) \subseteq B(x,r) \cap [0,1] = [0,r)$. Finally, observe that our set $\hat{F}$ is the product of two self-similar sets, E_1 and E_2, of the above form with constants θ_1 and θ_2 giving the desired one dimensional property. Now let $r \in (0,1]$ and $\hat{x} = (x_1, x_2) \in \hat{F} = E_1 \times E_2$. By the above argument, there exists $y_1 \in E_1$ and $y_2 \in E_2$ such that $B(y_1, r\theta_1) \subseteq B(x_1, r) \cap [0,1]$ and $B(y_2, r\theta_2) \subseteq B(x_2, r) \cap [0,1]$. Setting $\hat{y} = (y_1, y_2) \in \hat{F}$, it follows that $B\big(\hat{y}, r\min(\theta_1, \theta_2)\big) \subseteq B(\hat{x}, r) \cap [0,1]^2$, which completes the proof. $\square$

We can now complete the proof of the upper bound by estimating the lower dimension of F from above using the fact that $\hat{F}$ is a very weak tangent to F and is the product of two self-similar sets. We have

$$\begin{aligned} \dim_{\mathrm{B}} \pi_1(F) + \dim_{\mathrm{B}} \mathrm{Slice}_{1,i}(F) &= \dim_{\mathrm{B}} \Big(\pi_1(F) \times \pi_2\big(\mathrm{Slice}_{1,i}(F)\big)\Big) \\ &= \dim_{\mathrm{B}} \hat{F} \qquad \text{by Lemma 3.21} \\ &\geqslant \dim_{\mathrm{L}} \hat{F} \;\geqslant\; \dim_{\mathrm{L}} F \end{aligned}$$

by Proposition 3.20. Finally, we have to deal with the case where i corresponds to a rectangle which is alone in some column in the construction. In this case we can construct a very weak tangent to F as above, but we may not be able to find a constant θ with the desired properties. In particular, if the rectangle corresponding to i is at the top or bottom of the column, then the very weak tangent will lie on the boundary of X. However, this problem is easily overcome. Let $\varepsilon > 0$ and note that by iterating the IFS we may produce a new IFS, corresponding to $\mathcal{I}'$, with the same attractor which has some $i' \in \mathcal{I}'$ for which $\dim_{\mathrm{B}} \pi_1(F) + \mathrm{Slice}_{1,i'}(F) < \dim_{\mathrm{B}} \pi_1(F) + \mathrm{Slice}_{1,i}(F) + \varepsilon$ and does not correspond to a rectangle which is in a column by itself. We can then construct a very weak tangent to F in the above manner with dimension ε-close to the desired dimension which is sufficient to complete the proof of the upper bound.

Lower bound. The following proof is in the same spirit as the proof of the *upper* bound in Theorem 3.1. Fix $\boldsymbol{i}' \in \mathcal{I}^{\mathbb{N}}$, $R > 0$ and $r \in (0,R)$ and as before we will consider the approximate square $Q(\boldsymbol{i}', R)$. Without loss of generality assume that $k_1(\boldsymbol{i}', R) \geqslant k_2(\boldsymbol{i}', R)$ and let $k = k_1(\boldsymbol{i}', R)$. Furthermore we may assume that there exists $j_1, j_2 \in \mathcal{I}$ such that $c_{j_1} > d_{j_1}$ and $c_{j_2} < d_{j_2}$ as otherwise we are not in the mixed class. Let

$$s = \min_{i \in \mathcal{I}} \min_{j=1,2} \Big(\dim_{\mathrm{B}} \pi_j(F) \,+\, \dim_{\mathrm{B}} \mathrm{Slice}_{j,i}(F)\Big).$$

It suffices to prove that, for all $\varepsilon \in (0,1)$, there exists a constant $C(\varepsilon)$ such that

$$N_r\Big(Q(\boldsymbol{i}', R) \cap F\Big) \;\geqslant\; C(\varepsilon) \left(\frac{R}{r}\right)^{s-\varepsilon}.$$

Let $\varepsilon \in (0,1)$. As before, writing

$$\mathcal{I}_Q = \mathcal{I}_{Q(\boldsymbol{i}',R)} = \big\{ \boldsymbol{j} \in \mathcal{I}^k : S_{\boldsymbol{j}}(F) \subseteq Q(\boldsymbol{i}', R) \big\},$$

and

$$\mathcal{I}_Q^{<r} = \big\{ \boldsymbol{i} \in \mathcal{I}_Q : d_{\boldsymbol{i}} < r \big\},$$

we have

$$N_r\Big(Q(\boldsymbol{i}',R) \cap F\Big) \quad = \quad N_r\Bigg(\bigcup_{\boldsymbol{i} \in \mathcal{I}_Q^{<r}} S_{\boldsymbol{i}}(F) \ \cup \bigcup_{\substack{\boldsymbol{i} \in \mathcal{I}_Q: \\ d_{\boldsymbol{i}} \geqslant r}} S_{\boldsymbol{i}}(F) \Bigg).$$

At this point in the proof of the upper bound in Theorem 3.1, we iterated the IFS within each of the sets $\{S_{\boldsymbol{i}}(F) : \boldsymbol{i} \in \mathcal{I}_Q \text{ s.t. } d_{\boldsymbol{i}} \geqslant r\}$ to decompose F into small 'rectangular parts' with smallest side comparable to r. If we proceed in this way here, then the 'third term' causes problems. In particular, we end up with a term containing

$$\sum_{\boldsymbol{i} \in \mathcal{I}_Q} (d_{\boldsymbol{i}}/R)^{s_2 - \varepsilon}$$

which we wish to bound from below, but cannot as $s_2 - \varepsilon$ may be too large. This problem does not occur in the proof of the upper bound in Theorem 3.1 as the term

$$\sum_{\boldsymbol{i} \in \mathcal{I}_Q} (d_{\boldsymbol{i}}/R)^{s_2 + \varepsilon}$$

can be bounded from *above*. This was a surprising and interesting complication. To overcome this we need to engineer it so that the third term disappears. As such we will iterate only using maps $S_{\boldsymbol{i}}$ which have $c_{\boldsymbol{i}} \geqslant d_{\boldsymbol{i}}$. Fortunately we are able to do this by introducing a new IFS $\{S_{\boldsymbol{i}}\}_{\boldsymbol{i} \in \mathcal{I}_\varepsilon}$ with the properties outlined in the following lemma.

Lemma 3.22. *There exists an IFS $\{S_{\boldsymbol{i}}\}_{\boldsymbol{i} \in \mathcal{I}_\varepsilon}$ of affine maps on $[0,1]^2$ with attractor F_ε which has the following properties:*

(1) F_ε is of 'horizontal type', i.e., $c_{\boldsymbol{i}} \geqslant d_{\boldsymbol{i}}$ for all $\boldsymbol{i} \in \mathcal{I}_\varepsilon$,

(2) $\mathcal{I}_\varepsilon$ is a subset of some stopping $\mathcal{I}'$ created from the original IFS,

(3) F_ε is a subset of F and is such that $\dim_{\mathrm{B}} F_\varepsilon \geqslant s_1 + u_1 - \varepsilon$.

Proof. Let

$$\mathcal{I}_0 = \{\boldsymbol{i} \in \mathcal{I}^* : c_{\boldsymbol{i}} \geqslant d_{\boldsymbol{i}} \text{ and } \nexists \boldsymbol{j} \prec \boldsymbol{i} \text{ s.t. } \boldsymbol{j} \neq \boldsymbol{i} \text{ and } c_{\boldsymbol{j}} \geqslant d_{\boldsymbol{j}}\}$$

and let

$$\mathcal{I}_k = \{\boldsymbol{i} \in \mathcal{I}_0 : |\boldsymbol{i}| \leqslant k\}.$$

It is clear that $\mathcal{I}_k$ satisfies properties (1) and (2) and that k can be chosen large enough to ensure that property (3) is satisfied. □

We treat $\mathcal{I}_\varepsilon$ like $\mathcal{I}$ and write $(\mathcal{I}_\varepsilon)^* = \bigcup_{k \geqslant 1} (\mathcal{I}_\varepsilon)^k$ to denote the set of all finite sequences with entries in $\mathcal{I}_\varepsilon$ and

$$\alpha_{\varepsilon,\min} = \min\{\alpha_2(\boldsymbol{i}) : \boldsymbol{i} \in \mathcal{I}_\varepsilon\} > 0,$$

which clearly depends on ε. We have

$$N_r\Big(Q(\boldsymbol{i}',R) \cap F\Big) \quad = \quad N_r\Bigg(\bigcup_{\substack{\boldsymbol{j} \in \mathcal{I}_r: \\ \exists \boldsymbol{i} \in \mathcal{I}_Q^{<r}, \\ \boldsymbol{j} \prec \boldsymbol{i}}} \Big(S_{\boldsymbol{j}}(F) \cap Q(\boldsymbol{i}',R)\Big) \ \cup \bigcup_{\substack{\boldsymbol{i} \in \mathcal{I}_Q: \\ d_{\boldsymbol{i}} \geqslant r}} \ \bigcup_{\substack{\boldsymbol{j} \in (\mathcal{I}_\varepsilon)^*: \\ \alpha_2(\boldsymbol{i}\boldsymbol{j}) < r \leqslant \alpha_2(\boldsymbol{i}\bar{\boldsymbol{j}})}} S_{\boldsymbol{i}\boldsymbol{j}}(F) \Bigg).$$

Let U be any $r \times r$ closed square with sides parallel to the coordinate axes and let

$$M_\varepsilon = \min\{n \in \mathbb{N} : n \geqslant \alpha_{\varepsilon,\min}^{-1} + 2\}.$$

Observe that each of the sets $S_{\boldsymbol{j}}(F) \cap Q(\boldsymbol{i}', R)$ and $S_{\boldsymbol{ij}}(F)$ inside the above unions lies in a rectangle whose smallest side is of length at least $\alpha_{\varepsilon,\min} r$ and the interiors of these rectangles are pairwise disjoint. It follows from this that U can intersect no more than M_ε^2 of the sets $S_{\boldsymbol{j}}(F) \cap Q(\boldsymbol{i}', R)$ and $S_{\boldsymbol{ij}}(F)$. Hence, using the r-grid definition of N_r,

$$\begin{aligned} M_\varepsilon^2\, N_r\Big(Q(\boldsymbol{i}', R) \cap F\Big) \quad &\geqslant \sum_{\substack{\boldsymbol{j} \in \mathcal{I}_r: \\ \exists \boldsymbol{i} \in \mathcal{I}_Q^{<r}, \\ \boldsymbol{j} \prec \boldsymbol{i}}} N_r\Big(S_{\boldsymbol{j}}(F) \cap Q(\boldsymbol{i}', R)\Big) \;+\; \sum_{\substack{\boldsymbol{i} \in \mathcal{I}_Q: \\ d_{\boldsymbol{i}} \geqslant r}} \; \sum_{\substack{\boldsymbol{j} \in (\mathcal{I}_\varepsilon)^*: \\ \alpha_2(\boldsymbol{ij}) < r \leqslant \alpha_2(\boldsymbol{i}\bar{\boldsymbol{j}})}} N_r\Big(S_{\boldsymbol{ij}}(F)\Big) \\ &\geqslant \sum_{\substack{\boldsymbol{j} \in \mathcal{I}_r: \\ \exists \boldsymbol{i} \in \mathcal{I}_Q^{<r}, \\ \boldsymbol{j} \prec \boldsymbol{i}}} N_{r/c_{\boldsymbol{j}}}(\pi_1(F)) \;+\; \sum_{\substack{\boldsymbol{i} \in \mathcal{I}_Q: \\ d_{\boldsymbol{i}} \geqslant r}} \; \sum_{\substack{\boldsymbol{j} \in (\mathcal{I}_\varepsilon)^*: \\ \alpha_2(\boldsymbol{ij}) < r \leqslant \alpha_2(\boldsymbol{i}\bar{\boldsymbol{j}})}} N_{r/c_{\boldsymbol{ij}}}\big(\pi_1(F)\big) \end{aligned}$$

As before, we will analyse each of the above terms separately. For the first term, we have

$$\begin{aligned} \sum_{\substack{\boldsymbol{j} \in \mathcal{I}_r: \\ \exists \boldsymbol{i} \in \mathcal{I}_Q^{<r}, \\ \boldsymbol{j} \prec \boldsymbol{i}}} N_{r/c_{\boldsymbol{j}}}(\pi_1(F)) \quad &\geqslant \sum_{\substack{\boldsymbol{j} \in \mathcal{I}_r: \\ \exists \boldsymbol{i} \in \mathcal{I}_Q^{<r}, \\ \boldsymbol{j} \prec \boldsymbol{i}}} \tfrac{1}{C_\varepsilon} \left(\frac{R}{r}\right)^{s_1 - \varepsilon} \qquad \text{by (3.4) and Lemma 3.14 (1)} \\ &\geqslant \tfrac{1}{C_\varepsilon} \left(\frac{R}{r}\right)^{s_1 + u_1 - \varepsilon} \sum_{\substack{\boldsymbol{j} \in \mathcal{I}_r: \\ \exists \boldsymbol{i} \in \mathcal{I}_Q^{<r}, \\ \boldsymbol{j} \prec \boldsymbol{i}}} (r/R)^{u_1} \\ &\geqslant \tfrac{1}{C_\varepsilon} \left(\frac{R}{r}\right)^{s - \varepsilon} \sum_{\substack{\boldsymbol{j} \in \mathcal{I}_r: \\ \exists \boldsymbol{i} \in \mathcal{I}_Q^{<r}, \\ \boldsymbol{j} \prec \boldsymbol{i}}} (d_{\boldsymbol{j}}/R)^{u_1} \qquad \text{by (3.3)} \end{aligned}$$

For the second term, we have

$$\begin{aligned} \sum_{\substack{\boldsymbol{i} \in \mathcal{I}_Q: \\ d_{\boldsymbol{i}} \geqslant r}} \; \sum_{\substack{\boldsymbol{j} \in (\mathcal{I}_\varepsilon)^*: \\ \alpha_2(\boldsymbol{ij}) < r \leqslant \alpha_2(\boldsymbol{i}\bar{\boldsymbol{j}})}} N_{r/c_{\boldsymbol{ij}}}\big(\pi_1(F)\big) \quad &\geqslant \sum_{\substack{\boldsymbol{i} \in \mathcal{I}_Q: \\ d_{\boldsymbol{i}} \geqslant r}} \; \sum_{\substack{\boldsymbol{j} \in (\mathcal{I}_\varepsilon)^*: \\ \alpha_2(\boldsymbol{ij}) < r \leqslant \alpha_2(\boldsymbol{i}\bar{\boldsymbol{j}})}} \tfrac{1}{C_\varepsilon} \left(\frac{c_{\boldsymbol{i}} c_{\boldsymbol{j}}}{r}\right)^{s_1 - \varepsilon} \qquad \text{by (3.4)} \\ &\geqslant \tfrac{1}{C_\varepsilon} \left(\frac{1}{r}\right)^{s_1 - \varepsilon} \sum_{\substack{\boldsymbol{i} \in \mathcal{I}_Q: \\ d_{\boldsymbol{i}} \geqslant r}} c_{\boldsymbol{i}}^{s_1 - \varepsilon} \sum_{\substack{\boldsymbol{j} \in (\mathcal{I}_\varepsilon)^*: \\ \alpha_2(\boldsymbol{ij}) < r \leqslant \alpha_2(\boldsymbol{i}\bar{\boldsymbol{j}})}} c_{\boldsymbol{j}}^{s_1} \\ &\geqslant \tfrac{1}{C_\varepsilon} \left(\frac{R}{r}\right)^{s_1 - \varepsilon} \sum_{\substack{\boldsymbol{i} \in \mathcal{I}_Q: \\ d_{\boldsymbol{i}} \geqslant r}} \; \sum_{\substack{\boldsymbol{j} \in (\mathcal{I}_\varepsilon)^*: \\ \alpha_2(\boldsymbol{ij}) < r \leqslant \alpha_2(\boldsymbol{i}\bar{\boldsymbol{j}})}} c_{\boldsymbol{j}}^{s_1} \big(d_{\boldsymbol{i}} d_{\boldsymbol{j}} r^{-1}\big)^{u_1} \\ &\qquad \text{by Lemma 3.14 (1) and since } r > \alpha_2(\boldsymbol{ij}) = d_{\boldsymbol{i}} d_{\boldsymbol{j}} \\ &\geqslant \tfrac{1}{C_\varepsilon} \left(\frac{R}{r}\right)^{s_1 - \varepsilon} \left(\frac{1}{r}\right)^{u_1} \sum_{\substack{\boldsymbol{i} \in \mathcal{I}_Q: \\ d_{\boldsymbol{i}} \geqslant r}} d_{\boldsymbol{i}}^{u_1} \sum_{\substack{\boldsymbol{j} \in (\mathcal{I}_\varepsilon)^*: \\ \alpha_2(\boldsymbol{ij}) < r \leqslant \alpha_2(\boldsymbol{i}\bar{\boldsymbol{j}})}} c_{\boldsymbol{j}}^{s_1} d_{\boldsymbol{j}}^{u_1} \end{aligned}$$

$$\geqslant \tfrac{1}{C_\varepsilon}\left(\frac{R}{r}\right)^{s_1+u_1-\varepsilon} \sum_{\substack{\boldsymbol{i}\in\mathcal{I}_Q: \\ d_{\boldsymbol{i}}\geqslant r}} (d_{\boldsymbol{i}}/R)^{u_1} \sum_{\substack{\boldsymbol{j}\in(\mathcal{I}_\varepsilon)^*: \\ \alpha_2(\boldsymbol{ij})<r\leqslant\alpha_2(\boldsymbol{i}\bar{\boldsymbol{j}})}} c_{\boldsymbol{j}}^{s_1} d_{\boldsymbol{j}}^{\dim_{\mathrm{B}} F_\varepsilon+\varepsilon-s_1}$$

by Lemma 3.22 (3)

$$\geqslant \tfrac{1}{C_\varepsilon}\left(\frac{R}{r}\right)^{s_1+u_1-\varepsilon} \sum_{\substack{\boldsymbol{i}\in\mathcal{I}_Q: \\ d_{\boldsymbol{i}}\geqslant r}} (d_{\boldsymbol{i}}/R)^{u_1}\ d_{\boldsymbol{j}}^{\varepsilon} \sum_{\substack{\boldsymbol{j}\in(\mathcal{I}_\varepsilon)^*: \\ \alpha_2(\boldsymbol{ij})<r\leqslant\alpha_2(\boldsymbol{i}\bar{\boldsymbol{j}})}} c_{\boldsymbol{j}}^{s_1} d_{\boldsymbol{j}}^{\dim_{\mathrm{B}} F_\varepsilon-s_1}$$

$$\geqslant \tfrac{1}{C_\varepsilon}\,\alpha_{\varepsilon,\min}\left(\frac{R}{r}\right)^{s-2\varepsilon} \sum_{\substack{\boldsymbol{i}\in\mathcal{I}_Q: \\ d_{\boldsymbol{i}}\geqslant r}} (d_{\boldsymbol{i}}/R)^{u_1}$$

by Lemma 3.16 and the fact that $d_{\boldsymbol{j}} \geqslant (r/d_{\boldsymbol{i}})\,\alpha_{\varepsilon,\min} \geqslant (r/R)\,\alpha_{\varepsilon,\min}$. Combining the estimates for the two terms introduced above yields

$$\begin{aligned} M_\varepsilon^2\, N_r\Big(Q(\boldsymbol{i}',R)\cap F\Big) &\geqslant \tfrac{1}{C_\varepsilon}\left(\frac{R}{r}\right)^{s-\varepsilon} \sum_{\substack{\boldsymbol{j}\in\mathcal{I}_r: \\ \exists \boldsymbol{i}\in\mathcal{I}_Q^{<r}, \\ \boldsymbol{j}\prec\boldsymbol{i}}} (d_{\boldsymbol{j}}/R)^{u_1} + \tfrac{1}{C_\varepsilon}\,\alpha_{\varepsilon,\min}\left(\frac{R}{r}\right)^{s-2\varepsilon} \sum_{\substack{\boldsymbol{i}\in\mathcal{I}_Q: \\ d_{\boldsymbol{i}}\geqslant r}} (d_{\boldsymbol{i}}/R)^{u_1} \\ &\geqslant \tfrac{1}{C_\varepsilon}\,\alpha_{\varepsilon,\min}\left(\frac{R}{r}\right)^{s-2\varepsilon} \sum_{\boldsymbol{j}\in\mathcal{I}_0} (d_{\boldsymbol{j}}/R)^{u_1} \end{aligned}$$

where

$$\mathcal{I}_0 \ := \ \{\boldsymbol{j}\in\mathcal{I}_r : \exists \boldsymbol{i}\in\mathcal{I}_Q^{<r} \text{ s.t. } \boldsymbol{j}\prec\boldsymbol{i}\} \ \cup\ \{\boldsymbol{i}\in\mathcal{I}_Q : d_{\boldsymbol{i}}\geqslant r\}.$$

Observe that $\mathcal{I}_0$ is a $Q(\boldsymbol{i}',R)$-pseudo stopping, and so by Lemma 3.17 we have

$$\sum_{\boldsymbol{j}\in\mathcal{I}_0} (d_{\boldsymbol{j}}/R)^{u_1} \geqslant 1$$

which yields

$$N_r\Big(Q(\boldsymbol{i}',R)\cap F\Big) \ \geqslant\ \tfrac{1}{M_\varepsilon^2}\,\tfrac{1}{C_\varepsilon}\,\alpha_{\varepsilon,\min}\left(\frac{R}{r}\right)^{s-\varepsilon}.$$

It follows that $\dim_{\mathrm{L}} F \geqslant s-2\varepsilon$ and letting $\varepsilon\to 0$ completes the proof. □

3.6.5 Proof of Theorem 3.2 for the horizontal and vertical classes

This is similar to the proof in the mixed case, so we only briefly discuss it.

Upper bound. As in the mixed class, one can construct a *lower weak tangent* with the required dimension. The proof is slightly simpler in that for the horizontal class, for example, we necessarily have that $s = \dim_{\mathrm{B}}\pi_1(F) + \mathrm{Slice}_{1,i}(F)$ for some $i\in\mathcal{I}$ and that there exists $j\in\mathcal{I}$ with $c_j > d_j$.

Lower bound. The proof is greatly simplified in this case because we do not have the added complication alluded to above. In particular, we do not have to introduce the 'horizontal subsystem' $\mathcal{I}_\varepsilon$, and we can just iterate using $\mathcal{I}$ as before with no third term appearing.

3.6.6 Proof of Corollary 3.3

In this section we will rely on some results from [GL1] which, technically speaking, were not proved in the extended Lalley-Gatzouras case. However, it is easy to see that their arguments can be extended to cover this situation and give the results we require. Without loss of generality, let F be a self-affine attractor of an IFS in the horizontal class, and assume that $\dim_{\mathrm{L}} F < \dim_{\mathrm{A}} F$. It follows from Theorems 3.1 and 3.2 that

$$\min_{i \in \mathcal{I}} \ \dim_{\mathrm{B}} \mathrm{Slice}_{1,i}(F) < \max_{i \in \mathcal{I}} \ \dim_{\mathrm{B}} \mathrm{Slice}_{1,i}(F)$$

which means that we do not have *uniform vertical fibres*, and it follows from [GL1] that $\dim_{\mathrm{H}} F < \dim_{\mathrm{B}} F$. We will now show that $\dim_{\mathrm{L}} F < \dim_{\mathrm{H}} F$. We will use the formula for the Hausdorff dimension given in [GL1], so we must briefly introduce some notation. Suppose we have m non-empty columns in the construction and we have chosen n_i rectangles from the ith column. For the jth rectangle in the ith column, write c_i for the length of the base and d_{ij} for the height. Notice that the length of the base depends only on which column we are in. Then the Hausdorff dimension of F is given by

$$\dim_{\mathrm{H}} F = \max \left\{ \frac{\sum_i \sum_j p_{ij} \log p_{ij}}{\sum_i \sum_j p_{ij} \log d_{ij}} + \sum_i q_i \log q_i \left(\frac{1}{\sum_i q_i \log c_i} - \frac{1}{\sum_i \sum_j p_{ij} \log d_{ij}} \right) \right\}$$

where the maximum is taken over all associated probability distributions $\{p_{ij}\}$ on the set $\big\{(i,j) : i \in \{1, \dots, m\}, j \in \{1, \dots, n_i\}\big\}$ and $q_i = \sum_j p_{ij}$. Notice that this formula may be rewritten as

$$\max \left\{ \frac{\sum_i \sum_j p_{ij} \log \big(q_i/p_{ij}\big)}{\sum_i \sum_j p_{ij} \log (d_{ij})^{-1}} + \frac{\sum_i q_i \log q_i}{\sum_i q_i \log c_i} \right\}$$

which clearly demonstrates that if we continuously decrease a particular d_{ij}, then the Hausdorff dimension continuously decreases. Note that we may continuously decrease any particular d_{ij} without affecting any other rectangle in the construction.

If $\min_{i \in \mathcal{I}} \ \dim_{\mathrm{B}} \mathrm{Slice}_{1,i}(F) = 0$, then the result is clear. However, if $\min_{i \in \mathcal{I}} \ \dim_{\mathrm{B}} \mathrm{Slice}_{1,i}(F) > 0$, then, although we have already noted that F does not have uniform horizontal fibres, we may continuously decrease the d_{ij} to obtain a new IFS with index set $\mathcal{I}_1$, with the same number of rectangles and the same base lengths, which has an attractor F_1 where $\dim_{\mathrm{B}} \mathrm{Slice}_{1,j}(F_1) = \min_{i \in \mathcal{I}} \ \dim_{\mathrm{B}} \mathrm{Slice}_{1,i}(F)$ for each $j \in \mathcal{I}_1$. It follows from the above argument and Theorems 3.1 and 3.2 that

$$\dim_{\mathrm{L}} F = \dim_{\mathrm{H}} F_1 < \dim_{\mathrm{H}} F.$$

It remains to show that $\dim_{\mathrm{B}} F < \dim_{\mathrm{A}} F$. However, this follows from a dual argument observing that the box dimension of F is given by the unique solution s of

$$\sum_{i=1}^{m} \sum_{j=1}^{n_i} c_i^{s_1} d_{ij}^{s-s_1} = 1$$

(see [GL1] for the basic case or [FeW, Fr1] for the extended case), and so we may continuously *increase* the d_{ij} independently (while keeping $d_{ij} \leqslant c_i$) and, if necessary, add new maps to certain columns, to form a new construction F_2 with uniform vertical fibres and such that

$$\dim_{\mathrm{B}} F < \dim_{\mathrm{B}} F_2 = \dim_{\mathrm{A}} F.$$

3.6.7 Proof of Corollary 3.4

Let F be in the mixed class. The result for the horizontal and vertical classes follows from Corollary 3.3. Suppose F is such that $\dim_{\mathrm{L}} F = \dim_{\mathrm{B}} F$. It follows from Lemma 3.15 that $D_A = D_B = \dim_{\mathrm{L}} F \leqslant s_j + \dim_{\mathrm{B}} \mathrm{Slice}_{j,i}(F)$ for all $j \in \{1,2\}$ and $i \in \mathcal{I}$. Hence, using the notation from the proof of Lemma 3.15,

$$1 \;=\; \sum_{i\in\mathcal{I}} c_i^{s_1} d_i^{D_A - s_1} \;\geqslant\; \sum_{i=1}^{m} \hat{c}_i^{s_1} \sum_{j\in\mathcal{C}_i} d_j^{\dim_{\mathrm{B}} \mathrm{Slice}_{1,j}(F)} \;=\; 1$$

and

$$1 \;=\; \sum_{i\in\mathcal{I}} d_i^{s_2} c_i^{D_B - s_2} \;\geqslant\; \sum_{i=1}^{n} \hat{d}_i^{s_2} \sum_{j\in\mathcal{R}_i} c_j^{\dim_{\mathrm{B}} \mathrm{Slice}_{2,j}(F)} \;=\; 1$$

and so we have equality throughout in the above two lines which implies that

$$D_A = D_B = \dim_{\mathrm{L}} F = \max_{i\in\mathcal{I}} \max_{j=1,2} \Big(\dim_{\mathrm{B}} \pi_j(F) \;+\; \dim_{\mathrm{B}} \mathrm{Slice}_{j,i}(F) \Big) = \dim_{\mathrm{A}} F,$$

which completes the proof. We remark here that the key reason that a symmetric argument cannot be used to show that if $\dim_{\mathrm{A}} F = \dim_{\mathrm{B}} F$, then $\dim_{\mathrm{A}} F = \dim_{\mathrm{L}} F$, is that $\dim_{\mathrm{A}} F = \dim_{\mathrm{B}} F$ only implies that $\max\{D_A, D_B\} \geqslant s_j + \dim_{\mathrm{B}} \mathrm{Slice}_{j,i}(F)$ for all $j \in \{1,2\}$ and $i \in \mathcal{I}$. Indeed, such an implication is not true, as shown by the example in Section 3.4.2.

3.7 Proof of Theorem 3.7

In this section we will prove Theorem 3.7. Let (X,d) be a metric space, and let F be a compact subset of (X,d). It follows immediately from the definition of box dimension that for all $\varepsilon, \rho > 0$ there exists a constant $C_{\varepsilon,\rho} \geqslant 1$ such that for all $r \in (0,\rho]$ we have

$$\tfrac{1}{C_{\varepsilon,\rho}}\, r^{-\underline{\dim}_{\mathrm{B}} F + \varepsilon} \;\leqslant\; N_r(F) \;\leqslant\; C_{\varepsilon,\rho}\, r^{-\overline{\dim}_{\mathrm{B}} F - \varepsilon}. \tag{3.6}$$

Recall that for a map $f : A \to B$, for metric spaces (A, d_A), (B, d_B) we will write

$$\mathrm{Lip}^-(f) = \inf_{\substack{x,y\in A:\\ x\neq y}} \frac{d_B\big(f(x), f(y)\big)}{d_A(x,y)}.$$

Proof of (1). Suppose F satisfies (1) from Definition 3.6 with given parameters a, r_0 and write $s = \dim_{\mathrm{H}} F = \overline{\dim}_{\mathrm{B}} F$. Let $0 < r < R \leqslant r_0/2$ and $x \in F$. By condition (1) in the definition of quasi-self-similar, there exists an injection $g_1 : B(x,r) \cap F \to F$ with $\mathrm{Lip}^-(g_1) \geqslant a\,(2R)^{-1}$. If $\{U_i\}$ is an $ar/2R$ cover of $g_1(B(x,r)\cap F)$, then $\{g_1^{-1}(U_i)\}$ is an r cover of $B(x,r)\cap F$. It follows from this and (3.6) that

$$N_r\big(B(x,r)\cap F\big) \;\leqslant\; N_{ar/2R}\big(g_1(B(x,r)\cap F)\big) \;\leqslant\; N_{ar/2R}(F) \;\leqslant\; C_{\varepsilon,a/2}(2/a)^{s+\varepsilon} \left(\frac{R}{r}\right)^{s+\varepsilon}$$

which gives that $\dim_{\mathrm{A}} F \leqslant s + \varepsilon$ and letting $\varepsilon \to 0$ completes the proof. □

Proof of (2). Suppose F satisfies (2) from Definition 3.6 with given parameters a, r_0 and write $s = \dim_{\mathrm{H}} F = \underline{\dim}_{\mathrm{B}} F$. Let $0 < r < R \leqslant r_0/2$ and $x \in F$. By condition (2) in the definition of quasi-self-similar, there exists an injection $g_2 : F \to B(x,r)\cap F$ with $\mathrm{Lip}^-(g_2) \geqslant aR$. If $\{U_i\}$ is an r cover of $g_2(F)$, then $\{g_2^{-1}(U_i)\}$ is an r/aR cover of F. It follows from this and (3.6) that

$$N_r\big(B(x,r)\cap F\big) \;\geqslant\; N_r\big(g_2(F)\big) \;\geqslant\; N_{r/aR}(F) \;\geqslant\; \tfrac{1}{C_{\varepsilon,1/a}}\, a^{s-\varepsilon} \left(\frac{R}{r}\right)^{s-\varepsilon}$$

which gives that $\dim_{\mathrm{L}} F \geqslant s - \varepsilon$ and letting $\varepsilon \to 0$ completes the proof. □

Proof of (3). Let $F \subseteq (X,d)$ be a quasi-self-similar set with given parameters a, r_0 from Definition 3.6 and write $s = \dim_{\mathrm{H}} F$. Note that it follows from (1)-(2) above that $\dim_{\mathrm{L}} F = \dim_{\mathrm{A}} F$; however, it does not follow immediately that F is Ahlfors regular, so we will prove that now. It follows from the results in [F3] that

$$a^s \;\leqslant\; \mathcal{H}^s(F) \;\leqslant\; 4^s\, a^{-s}. \tag{3.7}$$

Let $r \in (0,\, r_0/2)$ and $x \in F$ and consider the set $B(x,r) \cap F := B(x,r) \cap F$. By condition (1) in the definition of quasi-self-similar, there exists a map $g_1 : B(x,r) \cap F \to F$ with $\mathrm{Lip}^-(g_1) \geqslant a\,(2r)^{-1}$. It follows from this, (3.7) and the scaling property for Hausdorff measure, that

$$\mathcal{H}^s(B(x,r) \cap F) \;\leqslant\; \mathrm{Lip}^-(g_1)^{-s}\, \mathcal{H}^s\big(g_1(B(x,r) \cap F)\big) \;\leqslant\; a^{-s}\,(2r)^s\, \mathcal{H}^s(F) \;\leqslant\; 8^s\, a^{-2s}\, r^s. \tag{3.8}$$

Furthermore, by condition (2) in the definition of quasi-self-similar, there exists a map $g_2 : F \to B(x,r) \cap F$ with $\mathrm{Lip}^-(g_2) \geqslant a\,r$. It follows from this, (3.7) and the scaling property for Hausdorff measure, that

$$\mathcal{H}^s(B(x,r) \cap F) \;\geqslant\; \mathcal{H}^s\big(g_2(F)\big) \;\geqslant\; \mathrm{Lip}^-(g_2)^s\, \mathcal{H}^s(F) \;\geqslant\; a^s\, r^s\, \mathcal{H}^s(F) \;\geqslant\; a^{2s}\, r^s. \tag{3.9}$$

It follows from (3.8) and (3.9) that F is locally Ahlfors regular setting $\lambda = 8^s\, a^{-2s}$ and, since F is compact, we have that it is, in fact, Ahlfors regular. $\square$

4 Inhomogeneous self-similar sets

4.1 Introduction

In this chapter we investigate the upper and lower box dimensions of inhomogeneous self-similar sets. We extend some results of Olsen and Snigireva [OSn, Sn] by computing the upper box dimensions assuming some mild separation conditions. We show that in our setting the upper box dimension behaves in the same way as the countably stable dimensions, in particular the relationship (1.6) holds. Secondly, we investigate the more difficult problem of computing the lower box dimension. We give some non-trivial bounds on the lower box dimension and prove that it does not behave as the other dimensions. In particular, the lower box dimension is not in general the maximum of the lower box dimensions of the homogeneous self-similar set and the condensation set. We introduce a quantity which we call the *covering regularity exponent* which is designed to give information about the oscillatory behaviour of the covering function N_δ and use it to study the lower box dimensions. We believe the covering regularity exponent will be a useful quantity in other circumstances where one needs finer information about the asymptotic properties of N_δ, or indeed other function where the asymptotic oscillations are important.

4.2 Results

In this section we will state our main results for this chapter. Let (X, d) be a compact metric space. Fix an IFS $\mathbb{I} = \{S_1, \dots, S_N\}$ where each S_i is a similarity on (X, d), fix a non-empty compact condensation set $C \subseteq X$ and let s denote the similarity dimension of $F_\emptyset$. Our results concerning upper box dimension will be given in Section 4.2.1 and those concerning lower box dimension will be given in Section 4.2.2. We will write $B(x, r)$ to denote the open ball of radius r centered at x.

4.2.1 Upper box dimension

In this section we significantly generalise the results in [OSn, Sn] concerning upper box dimension, which were obtained as Corollaries to results on the L^q-dimensions of inhomogeneous self-similar measures. Our proofs are direct and deal only with sets. Our first result bounds the upper box dimension of an inhomogeneous self-similar set, without assuming any separation conditions.

Theorem 4.1. *We have*

$$\max\{\overline{\dim}_{\mathrm{B}} F_\emptyset,\ \overline{\dim}_{\mathrm{B}} C\} \ \leqslant\ \overline{\dim}_{\mathrm{B}} F_C \ \leqslant\ \max\{s,\ \overline{\dim}_{\mathrm{B}} C\}.$$

Although the bounds given in Theorem 4.1 are not tight in general, we can apply them in two useful situations to obtain an exact result. The following Corollary answers Question 1.5 in the affirmative and, in fact, proves something stronger in that the separation conditions can be severely weakened and we can work in an arbitrary compact metric space.

Corollary 4.2. *Suppose that the IFS, $\mathbb{I}$, satisfies the SOSC. Then*

$$\overline{\dim}_{\mathrm{B}} F_C = \max\{\overline{\dim}_{\mathrm{B}} F_\emptyset,\ \overline{\dim}_{\mathrm{B}} C\}.$$

Proof. This follows immediately from Theorem 4.1 since if $\mathbb{I}$ satisfies the SOSC, then $s = \overline{\dim}_{\mathrm{B}} F_\emptyset$, see [Sc2]. □

Of course, if $X \subseteq \mathbb{R}^d$ then the SOSC is equivalent to the OSC. We can also obtain an exact result in a generic sense.

Corollary 4.3. *Let $d \in \mathbb{N}$ and fix linear contracting similarities, $\{T_1, \dots, T_N\}$, each mapping $\mathbb{R}^d$ to itself, and assume that $\mathrm{Lip}(T_i) < 1/2$ for all i and fix a compact condensation set $C \subset \mathbb{R}^d$. For $\mathbf{t} = (t_1, \dots, t_N) \in \times_{i=1}^N \mathbb{R}^d$, let $F_{\mathbf{t},\emptyset}$ denote the homogeneous attractor satisfying*

$$F_{\mathbf{t},\emptyset} = \bigcup_{i=1}^{N} \Big(T_i(F_{\mathbf{t},\emptyset}) + t_i \Big)$$

and let $F_{\mathbf{t},C}$ denote the inhomogeneous attractor satisfying

$$F_{\mathbf{t},C} \ = \ \bigcup_{i=1}^{N} \Big(T_i(F_{\mathbf{t},C}) + t_i \Big) \ \cup \ C.$$

Then, writing $\mathcal{L}^{dN}$ for the N-fold product of d-dimensional Lebesgue measure, we have

$$\overline{\dim}_{\mathrm{B}} F_{\mathbf{t},C} = \max\{\overline{\dim}_{\mathrm{B}} F_{\mathbf{t},\emptyset},\ \overline{\dim}_{\mathrm{B}} C\}$$

for $\mathcal{L}^{dN}$-almost all $\mathbf{t} = (t_1, \dots, t_N) \in \times_{i=1}^N \mathbb{R}^d$.

Proof. This follows immediately from Theorem 4.1 since, for $\mathcal{L}^{dN}$-almost all $\mathbf{t} = (t_1, \dots, t_N) \in \times_{i=1}^N \mathbb{R}^d$, we have that $\overline{\dim}_{\mathrm{B}} F_{\mathbf{t},\emptyset}$ is equal to the solution of

$$\sum_{i=1}^{N} \mathrm{Lip}(T_i)^s = 1$$

which is also the similarity dimension of $F_{\mathbf{t},\emptyset}$. This is a special case of Falconer's theorem, Theorem 1.3. □

We conclude this section with two open questions:

Question 4.4. *Is it true that*

$$\overline{\dim}_{\mathrm{B}} F_C = \max\{\overline{\dim}_{\mathrm{B}} F_\emptyset,\ \overline{\dim}_{\mathrm{B}} C\}$$

even if $\overline{\dim}_{\mathrm{B}} F_\emptyset < s$? In particular, such systems cannot satisfy the SOSC.

Question 4.5. *Is it true that*

$$\overline{\dim}_{\mathrm{B}} F_C = \max\{\overline{\dim}_{\mathrm{B}} F_\emptyset,\ \overline{\dim}_{\mathrm{B}} C\}$$

even if F_C is a more general inhomogeneous attractor, i.e., if the contractions are not similarities?

We will address Question 4.5 in Chapter 5 for certain classes of inhomogeneous self-affine sets.

4.2.2 Lower box dimension

In this section we examine the lower box dimension. Theorem 4.1 gives us the following immediate Corollary which gives (basically trivial) bounds on the lower box dimension.

Corollary 4.6. *We have*

$$\max\{\underline{\dim}_{\mathrm{B}} F_\emptyset,\ \underline{\dim}_{\mathrm{B}} C\} \ \leqslant\ \underline{\dim}_{\mathrm{B}} F_C \ \leqslant\ \overline{\dim}_{\mathrm{B}} F_C \ \leqslant\ \max\{s,\ \overline{\dim}_{\mathrm{B}} C\}$$

and if $\mathbb{I}$ *satisfies the SOSC, then*

$$\max\{\underline{\dim}_{\mathrm{B}} F_\emptyset,\ \underline{\dim}_{\mathrm{B}} C\} \ \leqslant\ \underline{\dim}_{\mathrm{B}} F_C \ \leqslant\ \overline{\dim}_{\mathrm{B}} F_C \ \leqslant\ \max\{\underline{\dim}_{\mathrm{B}} F_\emptyset,\ \overline{\dim}_{\mathrm{B}} C\}.$$

So we can compute the lower box dimension in three easy cases:

(1) If the box dimension of C *exists and* $\dim_{\mathrm{B}} C \geqslant s$*, then*

$$\underline{\dim}_{\mathrm{B}} F_C = \overline{\dim}_{\mathrm{B}} F_C = \dim_{\mathrm{B}} C;$$

(2) If the box dimension of C *exists and* $\mathbb{I}$ *satisfies the SOSC, then*

$$\underline{\dim}_{\mathrm{B}} F_C = \overline{\dim}_{\mathrm{B}} F_C = \max\{\dim_{\mathrm{B}} F_\emptyset,\ \dim_{\mathrm{B}} C\};$$

(3) If $\mathbb{I}$ *satisfies the SOSC and* $\overline{\dim}_{\mathrm{B}} C \leqslant s$*, then*

$$\underline{\dim}_{\mathrm{B}} F_C = \overline{\dim}_{\mathrm{B}} F_C = s = \dim_{\mathrm{B}} F_\emptyset.$$

Note that in each of the above cases the answer to Question 1.6 is yes, i.e.,

$$\underline{\dim}_{\mathrm{B}} F_C = \max\{\underline{\dim}_{\mathrm{B}} F_\emptyset,\ \underline{\dim}_{\mathrm{B}} C\}.$$

Even when $\mathbb{I}$ satisfies the SOSC, computing $\underline{\dim}_{\mathrm{B}} F_C$ appears to be a subtle and difficult problem if $\max\{s, \underline{\dim}_{\mathrm{B}} C\} < \overline{\dim}_{\mathrm{B}} C$. We will now briefly outline the reason for this. Firstly, note that since lower box dimension is stable under taking closures, it follows from (1.5) that

$$\underline{\dim}_{\mathrm{B}} F_C = \underline{\dim}_{\mathrm{B}} \overline{\mathcal{O}} = \underline{\dim}_{\mathrm{B}} \mathcal{O}.$$

We can thus restrict our attention to the orbital set. However, computing the dimension of $\mathcal{O}$ is difficult as it consists of copies of C scaled by different amounts. If the box dimension of C does not exist, then the growth of the function $N_\delta(C)$ can vary wildly as $\delta \to 0$. It turns out that the lower box dimension of $\mathcal{O}$ depends not only on $\underline{\dim}_{\mathrm{B}}$, $\overline{\dim}_{\mathrm{B}}$ and s, but also on the behaviour of the function $\delta \mapsto N_\delta(C)$. In order to analyse the behaviour of $N_\delta(C)$, we introduce a quantity which we call the *covering regularity exponent* (CRE). For $t \geqslant 0$ and $\delta \in (0, 1]$, the (t, δ)-CRE of C is defined as

$$p_{t,\delta}(C) = \sup\left\{p \in [0, 1] : N_{\delta^p}(C) \geqslant \delta^{-pt}\right\} \tag{4.1}$$

and the t-CRE is

$$p_t(C) = \liminf_{\delta \to 0}\ p_{t,\delta}(C).$$

Roughly speaking, $p_{t,\delta}(C)$ tells you at scale δ how much you have to 'scale up' to find a scale $\delta_0 \geqslant \delta$ where you need at least δ_0^{-t} sets to cover C, i.e., how far back you have to go to find a scale where the set is 'hard' to cover. In fact, the smaller $p_{t,\delta}(C)$ is, the further you have to go back. The constant $p_t(C)$ tells you the 'furthest away' you ever are from a scale where your set is 'hard to cover', as you let δ tend to zero. The following Lemma gives some simple but useful properties of the CREs.

Lemma 4.7.

(1) For all $t, \delta > 0$*, we have* $p_{t,\delta}(C), p_t(C) \in [0,1]$*;*

(2) $p_t(C)$ *is decreasing in* t *and if* $t < \underline{\dim}_{\mathrm{B}} C$*, then* $p_t(C) = 1$ *and if* $t > \overline{\dim}_{\mathrm{B}} C$*, then* $p_t(C) = 0$*;*

(3) For all $\delta > 0$ *we have*

$$N_{\delta^{p_{t,\delta}(C)}}(C) \geqslant \delta^{-p_{t,\delta}(C)t},$$

i.e., the supremum in (4.1) is obtained;

(4) For all $t > \underline{\dim}_{\mathrm{B}} C$*, we have*

$$p_t(C) \ \leqslant \ \frac{\underline{\dim}_{\mathrm{B}} C}{t} \ < \ 1;$$

(5) For $\underline{\dim}_{\mathrm{B}} C < s < t < \overline{\dim}_{\mathrm{B}} C$ *we have*

$$p_t(C) \ \leqslant \ \frac{s}{t}\, p_s(C);$$

(6) Suppose X *is doubling, i.e. has finite Assouad dimension. For all* $t \in (\underline{\dim}_{\mathrm{B}} C, \overline{\dim}_{\mathrm{B}} C)$*, we have*

$$p_t(C) \ \leqslant \ \frac{\underline{\dim}_{\mathrm{B}} C}{t} \frac{\dim_{\mathrm{A}} X - t}{\dim_{\mathrm{A}} X - \underline{\dim}_{\mathrm{B}} C}.$$

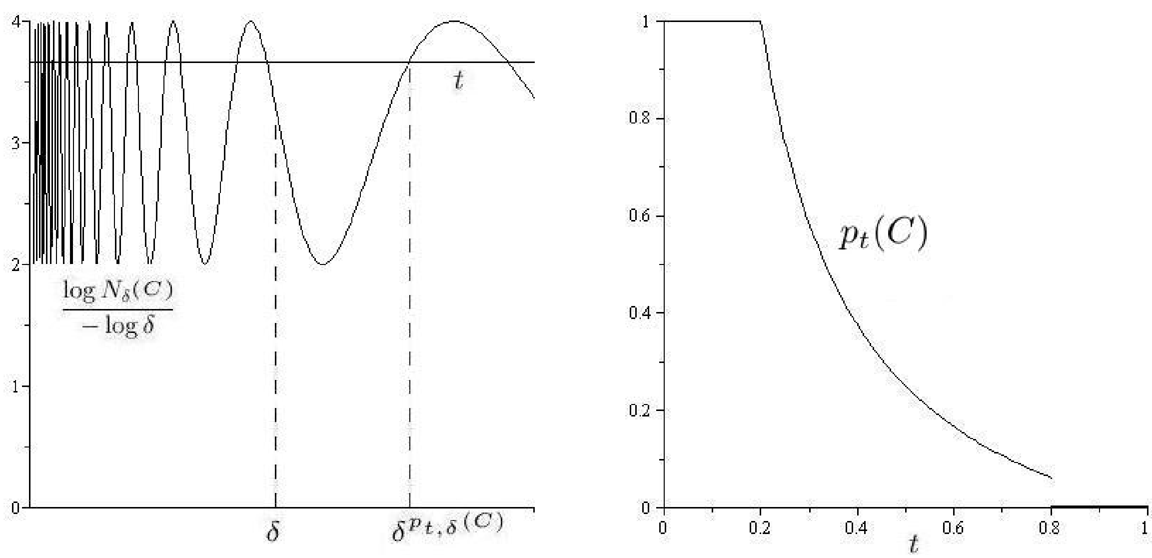

Figure 14: Left: A plot of $\log N_\delta(C)/(-\log \delta)$ for a set C with distinct upper and lower box dimension. A horizontal line is included at a value t between the upper and lower box dimensions. At the indicated point, δ, we have that $N_\delta(C) < \delta^{-t}$ and so we have to 'scale up' to $\delta_0 = \delta^{p_{t,\delta}(C)}$ to find a scale where $N_{\delta_0}(C) \geqslant \delta_0^{-t}$. Right: A typical graph of $p_t(C)$ for a set C with lower box dimension 0.2 and upper box dimension 0.8.

We will prove Lemma 4.7 in Section 4.3.2. We will now use the CREs to obtain non-trivial bounds on the lower box dimension of F_C. From now on we will assume that we are in the difficult case: $\max\{s, \underline{\dim}_{\mathrm{B}} C\} < \overline{\dim}_{\mathrm{B}} C$.

We adapt the SOSC to the case of inhomogeneous attractors in the following way.

Definition 4.8. *An IFS,* $\{S_1, \dots, S_N\}$*, together with a compact set* $C \subseteq X$*, satisfies the* condensation open set condition (COSC)*, if the IFS,* $\{S_1, \dots, S_N\}$*, satisfies the SOSC and the open set,* U*, can be chosen such that* $C \subseteq \overline{U}$*.*

The following theorem gives a *lower bound* on the lower box dimension of F_C and gives some sufficient conditions for the answer to Question 1.6 to be *no*.

Theorem 4.9. *Suppose (X, d) is Ahlfors regular and that $\mathbb{I}$ together with C satisfies the COSC. For all $t \geqslant 0$ we have*

$$\underline{\dim}_{\mathrm{B}} F_C \;\geqslant\; p_t(C)\, t \;+\; (1 - p_t(C))\, s.$$

In particular, if for some $t > \max\{s, \underline{\dim}_{\mathrm{B}} C\}$ we have

$$p_t(C) > \max\left\{0,\ \frac{\underline{\dim}_{\mathrm{B}} C - s}{t - s}\right\},$$

then

$$\underline{\dim}_{\mathrm{B}} F_C > \max\{\underline{\dim}_{\mathrm{B}} F_\emptyset,\, \underline{\dim}_{\mathrm{B}} C\}.$$

We will prove Theorem 4.9 in Section 4.3.4. The next theorem gives an *upper bound* on the lower box dimension of F_C and gives some sufficient conditions for the answer to Question 1.6 to be *yes*.

Theorem 4.10. *For all $t > \max\{s, \underline{\dim}_{\mathrm{B}} C\}$ we have*

$$\underline{\dim}_{\mathrm{B}} F_C \;\leqslant\; \max\{t,\, s + p_t(C)\, t\}$$

and, in particular, if $p_t(C) = 0$ for $t > \max\{s, \underline{\dim}_{\mathrm{B}} C\}$, then

$$\underline{\dim}_{\mathrm{B}} F_C \;\leqslant\; \max\{s,\, \underline{\dim}_{\mathrm{B}} C\}$$

and if, furthermore, the SOSC is satisfied, then

$$\underline{\dim}_{\mathrm{B}} F_C \;=\; \max\{\underline{\dim}_{\mathrm{B}} F_\emptyset,\, \underline{\dim}_{\mathrm{B}} C\}.$$

We will prove Theorem 4.10 in Section 4.3.5. We obtain the following (perhaps surprising) corollary in a very special case.

Corollary 4.11. *If $\underline{\dim}_{\mathrm{B}} C = 0$ and $\mathbb{I}$ satisfies the SOSC, then*

$$\underline{\dim}_{\mathrm{B}} F_C \;=\; \max\{\underline{\dim}_{\mathrm{B}} F_\emptyset,\, \underline{\dim}_{\mathrm{B}} C\} = \underline{\dim}_{\mathrm{B}} F_\emptyset = s.$$

Proof. This follows immediately from Theorem 4.10 since Lemma 4.7 (4) gives that $p_t(C) = 0$ for all $t > 0$. □

The following proposition proves the existence of compact sets with the extremal behaviour described in Theorems 4.9–4.10. In particular, Proposition 4.12 (2) combined with Theorem 4.9 gives a negative answer to Question 1.6.

Proposition 4.12. *Let $X = [0, 1]^d$ for some $d \in \mathbb{N}$.*

(1) For all $0 < b < t < B \leqslant d$, there exists a compact set $C \subseteq X$ such that $\underline{\dim}_{\mathrm{B}} C = b < B = \overline{\dim}_{\mathrm{B}} C$ and $p_t(C) = 0$ for all $t \geqslant b$;

(2) For all $0 < b < B \leqslant d$, there exists a compact set $C \subseteq X$ such that $\underline{\dim}_{\mathrm{B}} C = b < B = \overline{\dim}_{\mathrm{B}} C$ and

$$p_t(C) = \frac{b}{t}\,\frac{d - t}{d - b}.$$

for all $t \in (b, B)$. In particular, such a C shows that the upper bound in Lemma 4.7 (6) is sharp.

We will prove Proposition 4.12 in Section 4.3.6. Although we specialise to the case where X is the unit cube, the result applies in much more general situations. However, as we only require them to provide examples, we omit any further technical details.

The case where the condensation set is constructed as in Proposition 4.12 (2) is an interesting case. Not only does it provide a negative answer to Question 1.6, but we also obtain an explicit (non-trivial) formula for $p_t(C)$. We obtain the following corollary in this situation.

Corollary 4.13. *Let $X = [0,1]^d$, let $\mathbb{I} = \{S_1, \dots, S_N\}$ be an IFS of similarities on X and fix a non-empty compact set $C \subset [0,1]^d$ such that*

$$p_t(C) = \frac{\underline{\dim}_{\mathrm{B}} C}{t} \frac{d-t}{d - \underline{\dim}_{\mathrm{B}} C}$$

for all $t \in (\underline{\dim}_{\mathrm{B}} C, \overline{\dim}_{\mathrm{B}} C)$. Furthermore assume that $\mathbb{I}$ together with C satisfies the COSC. Then

$$\frac{\underline{\dim}_{\mathrm{B}} C}{t} \frac{d-t}{d - \underline{\dim}_{\mathrm{B}} C} \ (t - s) + s \ \leqslant \ \underline{\dim}_{\mathrm{B}} F_C \ \leqslant \ \max\Big\{t, \ s + \underline{\dim}_{\mathrm{B}} C \frac{d-t}{d - \underline{\dim}_{\mathrm{B}} C}\Big\}$$

for all $t \in (\underline{\dim}_{\mathrm{B}} C, \overline{\dim}_{\mathrm{B}} C)$.

Write $L(t)$ and $U(t)$ for the lower and upper bounds for $\underline{\dim}_{\mathrm{B}} F_C$ given in the above Corollary. We will now provide a plot of these as functions of t in two typical situations. Of course, the best lower and upper bounds for $\underline{\dim}_{\mathrm{B}} F_C$ are really the supremum and infimum of $L(t)$ and $U(t)$ respectively. In both cases we let $X = [0,1]^5$. For the plot on the left, we let $\underline{\dim}_{\mathrm{B}} C = 1$, $s = 1.5$ and $\overline{\dim}_{\mathrm{B}} C = 4.5$. For the plot on the right, we let $\underline{\dim}_{\mathrm{B}} C = s = 1$ and $\overline{\dim}_{\mathrm{B}} C = 2$. In the first case, the trivial bounds from Corollary 4.6 have been improved from $[1.5, 4.5]$ to $[1.756, 2.2]$ and, in the second case, the trivial bounds have been improved from $[1, 2]$ to $[1.375, 1.8]$.

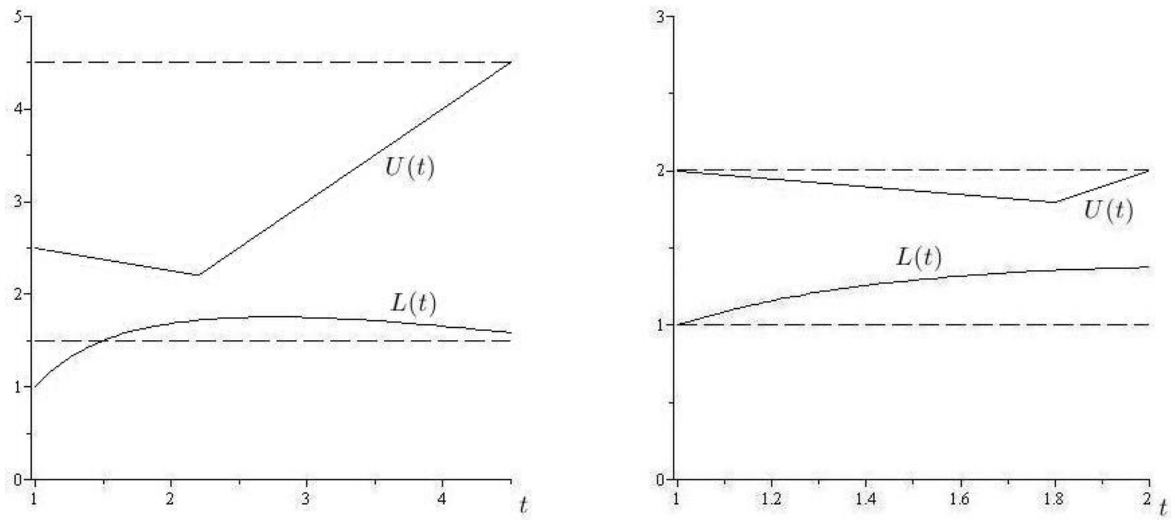

Figure 15: Two graphs showing the upper and lower bounds on the lower box dimension of F_C. $U(t)$ and $L(t)$ are plotted as solid lines, and the trivial bounds from Corollary 4.6 are plotted as dashed lines. We can clearly see a significant improvement on the trivial bounds, and in both cases $\underline{\dim}_{\mathrm{B}} F_C > \max\{\underline{\dim}_{\mathrm{B}} F_\emptyset, \underline{\dim}_{\mathrm{B}} C\}$.

We will present one final corollary which summarises the 'bad behaviour' of the lower box dimension of inhomogeneous self-similar sets.

Corollary 4.14. *Regardless of separation conditions, the lower box dimension of F_C is not in general given by a function of the numbers:*

$$\underline{\dim}_{\mathrm{B}} C, \ \overline{\dim}_{\mathrm{B}} C, \ \dim_{\mathrm{H}} C, \ \dim_{\mathrm{P}} C, \ \dim_{\mathrm{B}} F_\emptyset \ \textit{and} \ \ s.$$

This is in stark contrast to the situation for the countably stable dimensions and the upper box dimension.

Proof. This follows from the results in this section. $\square$

4.3 Proofs

4.3.1 Preliminary results and notation

Fix an IFS $\mathbb{I} = \{S_1, \ldots, S_N\}$ where each S_i is a similarity and fix a compact condensation set, $C \subseteq X$. Write $\mathcal{I} = \{1, \ldots, N\}$, $L_{\min} = \min_{i \in \mathcal{I}} \operatorname{Lip}(S_i)$ and $L_{\max} = \max_{i \in \mathcal{I}} \operatorname{Lip}(S_i)$. For $\delta \in (0, 1]$, define a δ-stopping, $\mathcal{I}_\delta$, by

$$\mathcal{I}_\delta = \left\{ \boldsymbol{i} \in \mathcal{I}^* : \operatorname{Lip}(S_{\boldsymbol{i}}) < \delta \leqslant \operatorname{Lip}(S_{\overline{\boldsymbol{i}}}) \right\},$$

where we assume for convenience that $\operatorname{Lip}(S_\omega) = 1$, where ω is the empty word.

Lemma 4.15. *For all $\delta \in (0, 1]$, we have*

$$\delta^{-s} \ \leqslant \ |\mathcal{I}_\delta| \ \leqslant \ L_{\min}^{-s} \, \delta^{-s}.$$

Proof. Repeated application of the Hutchinson-Moran formula (1.1) gives

$$\sum_{\boldsymbol{i} \in \mathcal{I}_\delta} \operatorname{Lip}(S_{\boldsymbol{i}})^s = 1$$

from which we deduce

$$1 = \sum_{\boldsymbol{i} \in \mathcal{I}_\delta} \operatorname{Lip}(S_{\boldsymbol{i}})^s \geqslant \sum_{\boldsymbol{i} \in \mathcal{I}_\delta} (\delta \, L_{\min})^s = |\mathcal{I}_\delta| \, (\delta \, L_{\min})^s \tag{4.2}$$

and

$$1 = \sum_{\boldsymbol{i} \in \mathcal{I}_\delta} \operatorname{Lip}(S_{\boldsymbol{i}})^s \leqslant \sum_{\boldsymbol{i} \in \mathcal{I}_\delta} \delta^s = |\mathcal{I}_\delta| \, \delta^s. \tag{4.3}$$

The desired upper and lower bounds now follow from (4.2) and (4.3) respectively. □

Lemma 4.16. *For all $t > s$ we have*

$$\sum_{\boldsymbol{i} \in \mathcal{I}^*} \operatorname{Lip}(S_{\boldsymbol{i}})^t \ = \ b_t \ < \ \infty$$

for some constant b_t depending only on t.

Proof. This is a standard fact but we include the simple proof for completeness and to define the constant b_t. Since $t > s$ we have $\sum_{i \in \mathcal{I}} \operatorname{Lip}(S_i)^t < 1$. It follows that

$$\sum_{\boldsymbol{i} \in \mathcal{I}^*} \operatorname{Lip}(S_{\boldsymbol{i}})^t = \sum_{k=1}^{\infty} \sum_{\boldsymbol{i} \in \mathcal{I}^k} \operatorname{Lip}(S_{\boldsymbol{i}})^t = \sum_{k=1}^{\infty} \left(\sum_{i \in \mathcal{I}} \operatorname{Lip}(S_i)^t \right)^k < \infty,$$

which proves the Lemma, setting $b_t = \sum_{k=1}^{\infty} \left(\sum_{i \in \mathcal{I}} \operatorname{Lip}(S_i)^t \right)^k$. □

Lemma 4.17. *For all $\delta \in (0, 1)$, we have*

$$|\{\boldsymbol{i} \in \mathcal{I}^* : \delta \leqslant \operatorname{Lip}(S_{\boldsymbol{i}})\}| \ \leqslant \ \frac{\log \delta}{\log L_{\max}} \ \delta^{-s}.$$

Proof. Let $\delta \in (0, 1)$ and suppose $\boldsymbol{i} \in \mathcal{I}^*$ is such that $\delta \leqslant \operatorname{Lip}(S_{\boldsymbol{i}})$. It follows that $\delta \leqslant L_{\max}^{|\boldsymbol{i}|}$ and hence

$$|\boldsymbol{i}| \leqslant \frac{\log \delta}{\log L_{\max}}. \tag{4.4}$$

Repeatedly applying the Hutchison-Moran formula (1.1) gives

$$\frac{\log \delta}{\log L_{\max}} \ \geqslant \ \sum_{\substack{l \in \mathbb{N}: \\ l \leqslant \frac{\log \delta}{\log L_{\max}}}} 1 \ \geqslant \ \sum_{\substack{l \in \mathbb{N}: \\ l \leqslant \frac{\log \delta}{\log L_{\max}}}} \sum_{\boldsymbol{i} \in \mathcal{I}^l} \operatorname{Lip}(S_{\boldsymbol{i}})^s$$

$$\geqslant \sum_{\substack{l\in\mathbb{N}:\\ l\leqslant \frac{\log\delta}{\log L_{\max}}}} \sum_{\substack{\boldsymbol{i}\in\mathcal{I}^l:\\ \delta\leqslant \mathrm{Lip}(S_{\boldsymbol{i}})}} \mathrm{Lip}(S_{\boldsymbol{i}})^s$$

$$\geqslant \sum_{\substack{l\in\mathbb{N}:\\ l\leqslant \frac{\log\delta}{\log L_{\max}}}} \sum_{\substack{\boldsymbol{i}\in\mathcal{I}^l:\\ \delta\leqslant \mathrm{Lip}(S_{\boldsymbol{i}})}} \delta^s$$

$$= \left|\{\boldsymbol{i}\in\mathcal{I}^* : \delta \leqslant \mathrm{Lip}(S_{\boldsymbol{i}})\}\right| \delta^s$$

by (4.4), which proves the result. □

4.3.2 Proof of Lemma 4.7

Proof of (1): This follows immediately from the definition of $p_{t,\delta}(C)$ and the fact that the set

$$\left\{p\in[0,1] : N_{\delta^p}(C)\geqslant \delta^{-pt}\right\}$$

is never empty as it always contains the point 0.

Proof of (2): It is clear that $p_t(C)$ is decreasing in t. If $t < \underline{\dim}_{\mathrm{B}} C$, then there exists $\delta_0\in(0,1]$ such that for all $\delta<\delta_0$ we have

$$N_\delta(C)\geqslant \delta^{-t}$$

which implies that if $\delta<\delta_0$, then $p_{t,\delta}(C)=1$, which completes the proof. The proof that if $t>\overline{\dim}_{\mathrm{B}} C$, then $p_t(C)=0$ is similar and omitted.

Proof of (3): Let $t>0$ and $\delta\in(0,1]$ and without loss of generality assume that $p_{t,\delta}(C)>0$. Assume that the required supremum is not obtained and thus, by the definition of $p_{t,\delta}(C)$, we may choose arbitrarily small $\varepsilon\in\big(0,p_{t,\delta}(C)\big)$, such that

$$N_{\delta^{p_{t,\delta}(C)-\varepsilon}}(C)\geqslant \delta^{-(p_{t,\delta}(C)-\varepsilon)t}. \tag{4.5}$$

It follows from this that

$$N_{\delta^{p_{t,\delta}(C)}}(C)\geqslant N_{\delta^{p_{t,\delta}(C)-\varepsilon}}(C)\geqslant \delta^{-(p_{t,\delta}(C)-\varepsilon)t}=\delta^{-p_{t,\delta}(C)t}\,\delta^{\varepsilon t}$$

and letting $\varepsilon\to0$ through values satisfying (4.5) proves the result by contradiction.

Proof of (4): Let $t>\underline{\dim}_{\mathrm{B}} C$ and $\varepsilon\in(0,t-\underline{\dim}_{\mathrm{B}} C)$. By the definition of lower box dimension, there exists arbitrarily small $\delta>0$ such that

$$N_\delta(C)\leqslant \delta^{-(\underline{\dim}_{\mathrm{B}} C+\varepsilon)}.$$

Fix such a $\delta\in(0,1)$ and since $N_\delta(C)$ increases as δ decreases,

$$\delta^{-p_{t,\delta}(C)t}\leqslant N_{\delta^{p_{t,\delta}(C)}}(C)\leqslant N_\delta(C)\leqslant \delta^{-(\underline{\dim}_{\mathrm{B}} C+\varepsilon)}.$$

Taking logs and dividing by $-t\log\delta$ yields

$$p_{t,\delta}(C)\leqslant \frac{\underline{\dim}_{\mathrm{B}} C+\varepsilon}{t}$$

and since we can find arbitrarily small δ satisfying the above inequality, the desired upper bound follows.

Proof of (5): Let $\underline{\dim}_{\mathrm{B}} C < s < t < \overline{\dim}_{\mathrm{B}} C$. It follows from Lemma 4.7 (4) above that $p_s(C) < 1$ and so we may choose $\varepsilon \in (0, 1 - p_s(C)]$. It follows that there exists $\delta \in (0, \varepsilon)$ such that $p_{s,\delta}(C) < p_s(C) + \varepsilon \leqslant 1$. This implies that

$$N_{\delta^{p_s(C)+\varepsilon}}(C) < \delta^{-(p_s(C)+\varepsilon)s}.$$

Using this, Lemma 4.7 (3), and the fact that $N_\delta(C)$ increases as δ decreases, we have

$$\delta^{-p_{t,\delta}(C)t} \leqslant N_{\delta^{p_{t,\delta}(C)}}(C) \leqslant N_{\delta^{p_s(C)+\varepsilon}}(C) < \delta^{-(p_s(C)+\varepsilon)s}.$$

Taking logs and dividing by $-t \log \delta$ yields

$$p_{t,\delta}(C) \;\leqslant\; \frac{s}{t}\,(p_s(C) + \varepsilon)$$

and since we can find arbitrarily small δ satisfying the above inequality, the desired upper bound follows.

Proof of (6): Let $t \in (\underline{\dim}_{\mathrm{B}} C, \overline{\dim}_{\mathrm{B}} C)$ and $\varepsilon \in (0, t - \underline{\dim}_{\mathrm{B}} C)$. Following the argument used in the proof of Lemma 4.7 (4), we can find arbitrarily small $\delta \in (0,1)$ such that

$$N_\delta(C) \leqslant \delta^{-(\underline{\dim}_{\mathrm{B}} C + \varepsilon)} \qquad \text{and} \qquad p_{t,\delta}(C) \leqslant \frac{\underline{\dim}_{\mathrm{B}} C + \varepsilon}{t} \leqslant 1. \tag{4.6}$$

Fix such a δ. By the definition of Assouad dimension, it follows that there exists constants $K \geqslant 1$ and $\rho \in (0, 1]$ such that any ball of radius $\delta < \rho$ can be covered by fewer than

$$K \left(\frac{\delta}{\delta_0}\right)^{\dim_{\mathrm{A}} X} \tag{4.7}$$

balls of radius $\delta_0 \leqslant \delta < \rho$. Let

$$m = \max\left\{1,\ \frac{\log K}{(\dim_{\mathrm{A}} X - t)\log \delta} + \frac{\dim_{\mathrm{A}} X - \underline{\dim}_{\mathrm{B}} C - \varepsilon}{\dim_{\mathrm{A}} X - t}\right\}. \tag{4.8}$$

Let $\delta' = \delta^q \in (\delta^m, \delta)$ for some $q \in (1, m)$. A simple calculation combining (4.6, 4.7, 4.8) yields that

$$N_{\delta'}(C) \;=\; N_{\delta^q}(C) \;\leqslant\; K\Big(\frac{\delta}{\delta^q}\Big)^{\dim_{\mathrm{A}} X} N_\delta(C) \;\leqslant\; K\Big(\frac{\delta}{\delta^q}\Big)^{\dim_{\mathrm{A}} X} \delta^{-(\underline{\dim}_{\mathrm{B}} C + \varepsilon)} \;<\; \delta^{-qt} = (\delta')^{-t}.$$

Note that if $m = 1$, then this is vacuously true, but indeed $m > 1$ for sufficiently small ε and δ. It follows that

$$N_{\delta'}(C) < (\delta')^{-t}$$

for all $\delta' \in (\delta^m, \delta) \cup [\delta, \delta^{p_{t,\delta}(C)}) = (\delta^m, \delta^{p_{t,\delta}(C)})$. This, combined with the fact that

$$N_{(\delta^m)^{p_{t,\delta}(C)/m}}(C) \;=\; N_{\delta^{p_{t,\delta}(C)}}(C) \;\geqslant\; \delta^{-p_{t,\delta}(C)t} \;=\; (\delta^m)^{-(p_{t,\delta}(C)/m)t}$$

by the definition of $p_{t,\delta}(C)$, yields that $p_{t,\delta^m}(C) = p_{t,\delta}(C)/m$. Hence

$$p_{t,\delta^m}(C) \;=\; \frac{p_{t,\delta}(C)}{m} \;\leqslant\; \frac{\underline{\dim}_{\mathrm{B}} C + \varepsilon}{t} \left(\frac{\log K}{(\dim_{\mathrm{A}} X - t)\log\delta} + \frac{\dim_{\mathrm{A}} X - \underline{\dim}_{\mathrm{B}} C - \varepsilon}{\dim_{\mathrm{A}} X - t}\right)^{-1}$$

by (4.6, 4.8). Letting $\delta \to 0$ through values satisfying (4.6) yields

$$p_t(C) \;\leqslant\; \frac{\underline{\dim}_{\mathrm{B}} C + \varepsilon}{t}\,\frac{\dim_{\mathrm{A}} X - t}{\dim_{\mathrm{A}} X - \underline{\dim}_{\mathrm{B}} C - \varepsilon}$$

and finally letting $\varepsilon \to 0$ we have

$$p_t(C) \;\leqslant\; \frac{\underline{\dim}_{\mathrm{B}} C}{t}\,\frac{\dim_{\mathrm{A}} X - t}{\dim_{\mathrm{A}} X - \underline{\dim}_{\mathrm{B}} C}$$

as required. $\square$

4.3.3 Proof of Theorem 4.1

By monotonicity of upper box dimension, we have $\max\{\overline{\dim}_{\mathrm{B}} F_\emptyset,\ \overline{\dim}_{\mathrm{B}} C\} \leqslant \overline{\dim}_{\mathrm{B}} F_C$. We will now prove the other inequality. Since upper box dimension is finitely stable, it suffices to show that

$$\overline{\dim}_{\mathrm{B}} \mathcal{O} \leqslant \max\{s,\ \overline{\dim}_{\mathrm{B}} C\}.$$

Let $t > \max\{s,\ \overline{\dim}_{\mathrm{B}} C\}$. It follows from the definition of upper box dimension that there exists a constant $c_t > 0$ such that

$$N_\delta(C) \leqslant c_t\, \delta^{-t} \tag{4.9}$$

for all $\delta \in (0, 1]$. Also note that since X is compact, the number of balls of radius 1 required to cover X is a finite constant $N_1(X)$. Let $\delta \in (0, 1]$. We have

$$\begin{aligned}
N_\delta(\mathcal{O}) &= N_\delta\left(C \cup \bigcup_{i \in \mathcal{I}^*} S_i(C) \right) \\
&\leqslant \sum_{\substack{i \in \mathcal{I}^*: \\ \delta \leqslant \mathrm{Lip}(S_i)}} N_\delta\big(S_i(C)\big) \ + \ N_\delta\left(\bigcup_{\substack{i \in \mathcal{I}^*: \\ \delta > \mathrm{Lip}(S_i)}} S_i(C) \right) \ + \ N_\delta(C) \\
&\leqslant \sum_{\substack{i \in \mathcal{I}^*: \\ \delta \leqslant \mathrm{Lip}(S_i)}} N_{\delta/\mathrm{Lip}(S_i)}(C) \ + \ N_\delta\left(\bigcup_{i \in \mathcal{I}_\delta} S_i(X) \right) \ + \ N_\delta(C) \\
&\leqslant \sum_{\substack{i \in \mathcal{I}^*: \\ \delta \leqslant \mathrm{Lip}(S_i)}} c_t\, \big(\delta/\mathrm{Lip}(S_i)\big)^{-t} \ + \ \sum_{i \in \mathcal{I}_\delta} N_{\delta/\mathrm{Lip}(S_i)}(X) \ + \ c_t\, \delta^{-t} \qquad \text{by (4.9)} \\
&\leqslant c_t\, \delta^{-t} \sum_{\substack{i \in \mathcal{I}^*: \\ \delta \leqslant \mathrm{Lip}(S_i)}} \mathrm{Lip}(S_i)^t \ + \ N_1(X)\, |\mathcal{I}_\delta| \ + \ c_t\, \delta^{-t} \\
&\leqslant c_t\, \delta^{-t} \sum_{i \in \mathcal{I}^*} \mathrm{Lip}(S_i)^t \ + \ N_1(X)\, L_{\min}^{-s}\, \delta^{-s} \ + \ c_t\, \delta^{-t} \qquad \text{by Lemma 4.15} \\
&\leqslant \big(c_t\, b_t \ + \ N_1(X)\, L_{\min}^{-s} \ + \ c_t\big)\, \delta^{-t}
\end{aligned}$$

by Lemma 4.16, from which it follows that $\overline{\dim}_{\mathrm{B}} F_C = \overline{\dim}_{\mathrm{B}} \mathcal{O} \leqslant t$ and since t can be chosen arbitrarily close to $\max\{s,\ \overline{\dim}_{\mathrm{B}} C\}$, we have proved the theorem. $\square$

4.3.4 Proof of Theorem 4.9

Suppose (X, d) is Ahlfors regular and that $\mathbb{I}$, together with C, satisfies the COSC. We begin with two simple technical lemmas.

Lemma 4.18. *Let $a, b > 0$, let $\{U_i\}$ be a collection of disjoint open subsets of X and suppose that each U_i contains a ball of radius ar and is contained in a ball of radius br. Then any ball of radius r intersects no more than*

$$\lambda^2 \left(\frac{1 + 2b}{a} \right)^{\dim_{\mathrm{H}} X}$$

of the closures $\{\overline{U}_i\}$.

This is a trivial modification of a standard result in Euclidean space, see [F8, Lemma 9.2], but for completeness we include the simple proof.

Proof. For each i let B_i denote the ball of radius ar contained in U_i and note that these balls are pairwise disjoint. Fix $x \in X$ and suppose $B(x,r) \cap \overline{U}_i \neq \emptyset$ for some i. It follows that $\overline{U}_i \subseteq B\big(x, (1+2b)r\big)$. Suppose the number of i such that $B(x,r) \cap \overline{U}_i \neq \emptyset$ is equal to N. Then

$$N \, \tfrac{1}{\lambda} \, (ar)^{\dim_{\mathrm{H}} X} \;\leqslant\; \sum_{i : B(x,r) \cap \overline{U}_i \neq \emptyset} \mathcal{H}^{\dim_{\mathrm{H}} X}\big(B_i\big) \;\leqslant\; \mathcal{H}^{\dim_{\mathrm{H}} X}\Big(B\big(x,(1+2b)r\big)\Big) \;\leqslant\; \lambda \left((1+2b)r\right)^{\dim_{\mathrm{H}} X}$$

and solving for N proves the lemma. $\square$

Lemma 4.19. *Let $\delta \in (0,1]$ and $\boldsymbol{i}, \boldsymbol{j} \in \mathcal{I}_\delta$ with $\boldsymbol{i} \neq \boldsymbol{j}$. Writing U for the open set used in the COSC, we have*

$$S_{\boldsymbol{i}}(U) \cap S_{\boldsymbol{j}}(U) = \emptyset.$$

Proof. This is a simple consequence of the COSC (in fact the OSC is enough) and the fact that neither $\boldsymbol{i}$ nor $\mathbf{j}$ is a subword of the other. $\square$

We now turn to the proof of Theorem 4.9.

Proof. If $0 \leqslant t \leqslant \max\{s, \underline{\dim}_{\mathrm{B}} C\}$, then the result is clearly true (and not an improvement on Corollary 4.6) so assume that $t > \max\{s, \underline{\dim}_{\mathrm{B}} C\}$ and let $\varepsilon \in (0,1]$. Choose $\delta_0 \in (0,1]$ such that for all $\delta \in (0, \delta_0]$ we have $p_{t,\delta}(C) \geqslant p_t(C) - \varepsilon$. Fix $\delta \in (0, \delta_0]$ and finally, to simplify notation, write $p_{t,\delta} = p_{t,\delta}(C)$ and $p_t = p_t(C)$. We will now consider two cases.

Case 1: Assume that $\delta^{1-p_{t,\delta}} \, L_{\min}^{-1} \leqslant 1$.

Let U be the open set used for the COSC and choose $a, b > 0$ such that U contains a ball of radius a and is contained in a ball of radius b. It follows that for each $\boldsymbol{i} \in \mathcal{I}(\delta^{1-p_{t,\delta}} \, L_{\min}^{-1})$ the image $S_{\boldsymbol{i}}(U)$ is an open set which contains a ball of radius $a \, \delta^{1-p_{t,\delta}}$ and is contained in a ball of radius $b \, L_{\min}^{-1} \, \delta^{1-p_{t,\delta}}$. Furthermore, it follows from Lemma 4.19 that the sets

$$\big\{ S_{\boldsymbol{i}}(U) : \boldsymbol{i} \in \mathcal{I}(\delta^{1-p_{t,\delta}} \, L_{\min}^{-1}) \big\}$$

are pairwise disjoint. Since, for each $\boldsymbol{i} \in \mathcal{I}(\delta^{1-p_{t,\delta}} \, L_{\min}^{-1})$, we have $S_{\boldsymbol{i}}(C) \subseteq \overline{S_{\boldsymbol{i}}(U)}$, it follows from Lemma 4.18 that any ball of radius $\delta^{1-p_{t,\delta}}$, and hence any set of diameter δ, can intersect no more than

$$\kappa := \lambda^2 \left(\frac{1 + 2bL_{\min}^{-1}}{a} \right)^{\dim_{\mathrm{H}} X}$$

of the sets

$$\big\{ S_{\boldsymbol{i}}(C) : \boldsymbol{i} \in \mathcal{I}(\delta^{1-p_{t,\delta}} \, L_{\min}^{-1}) \big\}.$$

Hence

$$\begin{aligned}
N_\delta(\mathcal{O}) = N_\delta\left(C \cup \bigcup_{\boldsymbol{i} \in \mathcal{I}^*} S_{\boldsymbol{i}}(C) \right) \;&\geqslant\; \kappa^{-1} \sum_{\boldsymbol{i} \in \mathcal{I}(\delta^{1-p_{t,\delta}} \, L_{\min}^{-1})} N_\delta\big(S_{\boldsymbol{i}}(C)\big) \\
&=\; \kappa^{-1} \sum_{\boldsymbol{i} \in \mathcal{I}(\delta^{1-p_{t,\delta}} \, L_{\min}^{-1})} N_{\delta/\mathrm{Lip}(S_{\boldsymbol{i}})}(C) \\
&\geqslant\; \kappa^{-1} \sum_{\boldsymbol{i} \in \mathcal{I}(\delta^{1-p_{t,\delta}} \, L_{\min}^{-1})} N_{\delta^{p_{t,\delta}}}(C) \\
&\geqslant\; \kappa^{-1} \, \delta^{-t p_{t,\delta}} \; |\mathcal{I}(\delta^{1-p_{t,\delta}} \, L_{\min}^{-1})| \qquad \text{by Lemma 4.7 (3)} \\
&\geqslant\; \kappa^{-1} \, \delta^{-t p_{t,\delta}} \; (\delta^{1-p_{t,\delta}} \, L_{\min}^{-1})^{-s} \qquad \text{by Lemma 4.15} \\
&=\; \kappa^{-1} \, L_{\min}^{s} \, \delta^{-\left(p_{t,\delta} t + (1-p_{t,\delta}) s\right)}
\end{aligned}$$

$$\geqslant \quad \kappa^{-1} L_{\min}^{s} \, \delta^{-\big((p_t-\varepsilon)t+(1-(p_t-\varepsilon))s\big)}$$

from which it follows that $\underline{\dim}_{\mathrm{B}} F_C = \underline{\dim}_{\mathrm{B}} \mathcal{O} \geqslant (p_t - \varepsilon)t + (1 - (p_t - \varepsilon))s$.

Case 2: Assume that $\delta^{1-p_{t,\delta}} L_{\min}^{-1} > 1$.

Note that our assumption implies that $1 \geqslant \delta^{-(1-p_{t,\delta})s} L_{\min}^{s}$. It follows that

$$N_\delta(\mathcal{O}) \;\geqslant\; N_{\delta^{p_{t,\delta}}}(C) \;\geqslant\; \delta^{-p_{t,\delta} t} \;\geqslant\; \delta^{-(1-p_{t,\delta})s} L_{\min}^{s} \, \delta^{-p_{t,\delta} t} \;\geqslant\; L_{\min}^{s} \, \delta^{-\big((p_t-\varepsilon)t+(1-(p_t-\varepsilon))s\big)}$$

from which it follows that $\underline{\dim}_{\mathrm{B}} \mathcal{O} \geqslant (p_t - \varepsilon)t + (1 - (p_t - \varepsilon))s$.

Combining Cases 1–2 and letting ε tend to zero proves the theorem. □

4.3.5 Proof of Theorem 4.10

We begin with a simple technical Lemma.

Lemma 4.20. *Let $t \geqslant 0$. If $p_t(C) < 1$, then for all $\varepsilon \in \big(0, 1 - p_t(C)\big)$, there exists $\delta \in (0, \varepsilon)$ such that*

$$p_t(C) - \varepsilon < p_{t,\delta}(C) < p_t(C) + \varepsilon$$

and, for all $\delta_0 \in [\delta, \delta^{p_t(C)}]$, we have

$$N_{\delta_0}(C) \leqslant \delta_0^{-t}.$$

Proof. Since $p_t(C) < 1$, it follows that for all $\varepsilon \in \big(0, 1 - p_t(C)\big)$, there exists $\delta \in (0, \varepsilon)$ such that $p_t(C) - \varepsilon < p_{t,\delta}(C) < p_t(C) + \varepsilon < 1$. By the definition of $p_{t,\delta}(C)$ this implies that for all $\delta_0 \in [\delta, \delta^{p_t(C)+\varepsilon}]$ we have

$$N_{\delta_0}(C) \leqslant \delta_0^{-t}$$

which completes the proof. □

We will now turn to the proof of Theorem 4.10.

Proof. Let $t > \max\{s, \underline{\dim}_{\mathrm{B}} C\}$. By Lemma 4.7 (4), we have $p_t(C) \leqslant \underline{\dim}_{\mathrm{B}} C/t < 1$ and so by Lemma 4.20, for all $\varepsilon \in \big(0, 1 - p_t(C)\big)$, there exists $\delta \in (0, \varepsilon)$ such that

$$p_t(C) - \varepsilon < p_{t,\delta}(C) < p_t(C) + \varepsilon \tag{4.10}$$

and for all $\delta_0 \in [\delta, \delta^{p_t(C)}]$ we have

$$N_{\delta_0}(C) \leqslant \delta_0^{-t}. \tag{4.11}$$

Fix $\varepsilon \in (0, 1 - p_t(C))$ and choose $\delta \in (0, \varepsilon)$ satisfying (4.10, 4.11). Write $p_{t,\delta} = p_{t,\delta}(C)$ and $p_t = p_t(C)$. We have

$$\begin{aligned}
N_\delta(\mathcal{O}) \;&=\; N_\delta\Big(C \cup \bigcup_{\boldsymbol{i} \in \mathcal{I}^*} S_{\boldsymbol{i}}(C)\Big) \\
&\leqslant \sum_{\substack{\boldsymbol{i} \in \mathcal{I}^*: \\ \delta^{1-p_{t,\delta}-\varepsilon} \leqslant \mathrm{Lip}(S_{\boldsymbol{i}}) < 1}} N_\delta\big(S_{\boldsymbol{i}}(C)\big) \;+\; \sum_{\substack{\boldsymbol{i} \in \mathcal{I}^*: \\ \delta \leqslant \mathrm{Lip}(S_{\boldsymbol{i}}) < \delta^{1-p_{t,\delta}-\varepsilon}}} N_\delta\big(S_{\boldsymbol{i}}(C)\big) \\
&\qquad + N_\delta\left(\bigcup_{\substack{\boldsymbol{i} \in \mathcal{I}^*: \\ \mathrm{Lip}(S_{\boldsymbol{i}}) < \delta}} S_{\boldsymbol{i}}(C)\right) \;+\; N_\delta(C) \\
&\leqslant \sum_{\substack{\boldsymbol{i} \in \mathcal{I}^*: \\ \delta^{1-p_{t,\delta}-\varepsilon} \leqslant \mathrm{Lip}(S_{\boldsymbol{i}}) < 1}} N_{\delta/\mathrm{Lip}(S_{\boldsymbol{i}})}(C) \;+\; \sum_{\substack{\boldsymbol{i} \in \mathcal{I}^*: \\ \delta \leqslant \mathrm{Lip}(S_{\boldsymbol{i}}) < \delta^{1-p_{t,\delta}-\varepsilon}}} N_{\delta/\mathrm{Lip}(S_{\boldsymbol{i}})}(C)
\end{aligned}$$

$$
\begin{aligned}
&\qquad + N_\delta\left(\bigcup_{\boldsymbol{i}\in\mathcal{I}_\delta} S_{\boldsymbol{i}}(X)\right) + N_\delta(C)\\
&\leqslant \sum_{\substack{\boldsymbol{i}\in\mathcal{I}^*:\\ \delta^{1-p_{t,\delta}-\varepsilon}\leqslant \mathrm{Lip}(S_i)<1}} \big(\delta/\mathrm{Lip}(S_{\boldsymbol{i}})\big)^{-t} + \sum_{\substack{\boldsymbol{i}\in\mathcal{I}^*:\\ \delta\leqslant \mathrm{Lip}(S_i)<\delta^{1-p_{t,\delta}-\varepsilon}}} N_{\delta^{p_{t,\delta}+\varepsilon}}(C)\\
&\qquad + \sum_{\boldsymbol{i}\in\mathcal{I}_\delta} N_{\delta/\mathrm{Lip}(S_{\boldsymbol{i}})}(X) + \delta^{-t} \qquad \text{by (4.10, 4.11)}\\
&\leqslant \delta^{-t} \sum_{\substack{\boldsymbol{i}\in\mathcal{I}^*:\\ \delta^{1-p_{t,\delta}-\varepsilon}\leqslant \mathrm{Lip}(S_i)<1}} \mathrm{Lip}(S_{\boldsymbol{i}})^t + \sum_{\substack{\boldsymbol{i}\in\mathcal{I}^*:\\ \delta\leqslant \mathrm{Lip}(S_i)<\delta^{1-p_{t,\delta}-\varepsilon}}} \delta^{-(p_{t,\delta}+\varepsilon)t} + N_1(X)\,|\mathcal{I}_\delta|\\
&\qquad + \delta^{-t} \qquad \text{by (4.10, 4.11)}\\
&\leqslant \delta^{-t}\sum_{\boldsymbol{i}\in\mathcal{I}^*}\mathrm{Lip}(S_{\boldsymbol{i}})^t + |\{\boldsymbol{i}\in\mathcal{I}^*:\delta\leqslant\mathrm{Lip}(S_{\boldsymbol{i}})\}|\,\delta^{-(p_{t,\delta}+\varepsilon)t}\\
&\qquad + N_1(X)\,\delta^{-s} + \delta^{-t} \qquad \text{by Lemma 4.15}\\
&\leqslant \big(b_t+N_1(X)+1\big)\,\delta^{-t} + \frac{\log\delta}{\log L_{\max}}\,\delta^{-s}\,\delta^{-(p_{t,\delta}+\varepsilon)t} \qquad \text{by Lemmas 4.16 and 4.17}\\
&\leqslant \big(b_t+N_1(X)+1\big)\,\delta^{-t} + \frac{\log\delta}{\log L_{\max}}\,\delta^{-(s+(p_t+2\varepsilon)t)}
\end{aligned}
$$

from which it follows that $\underline{\dim}_{\mathrm{B}}F_C = \underline{\dim}_{\mathrm{B}}\mathcal{O} \leqslant \max\{t,\ s+(p_t+2\varepsilon)t\}$ and letting ε tend to zero yields the desired upper bound. Note that we do not obtain an upper bound for the upper box dimension here as we only find a sequence of δs tending to zero for which the above estimate holds. $\square$

4.3.6 Proof of Proposition 4.12

Let $X=[0,1]^d$ for some $d\in\mathbb{N}$ and let $0<b<B\leqslant d$. We will first describe a general way of constructing sets $C\subseteq[0,1]^d$ which gives us the required control over the oscillations of the function $N_\delta(C)$.

For $k\in\mathbb{N}$, let $\mathcal{Q}_k$ be the set of closed $2^{-k}\times\cdots\times 2^{-k}$ cubes formed by imposing a 2^{-k} grid on $[0,1]^d$ orientated at the origin. For each k select a subset of these cubes and call their union Q_k. We assume that $[0,1]^d\supseteq Q_1\supseteq Q_2\supseteq\dots$ and that if a cube is chosen at the kth step, then at least one sub-cube is chosen at the $(k+1)$th stage. Finally, we set $C=\cap_{k\in\mathbb{N}}Q_k$. Let $M_{2^{-k}}(C)$ denote the number of cubes in $\mathcal{Q}_k$ which intersect C. We will only choose cubes at the kth level in two different ways:

Method 1: at the $(k+1)$th stage we choose *precisely one* cube from each kth level cube;

and

Method 2: at the $(k+1)$th stage we choose *all* sub-cubes from within each kth level cube.

For $\delta\in(0,1)$, let $k(\delta)=\max\{k\in\mathbb{N}\cup 0:\delta\leqslant 2^{-k}\}$. It is easy to see that

$$3^{-d}\,M_{2^{-k(\delta)}}(C)\leqslant N_\delta(C)\leqslant M_{2^{-(k(\delta)+1+d)}}(C).$$

Also, for all $k\in\mathbb{N}$,

$$M_{2^{-k}}(C)\leqslant M_{2^{-(k+1)}}(C)\leqslant 2^d\,M_{2^{-k}}(C)$$

and these bounds are tight as if at the $(k+1)$th stage we use Method 1, then we attain the left hand bound, and if at the $(k+1)$th stage we use Method 2, then we attain the right hand bound.

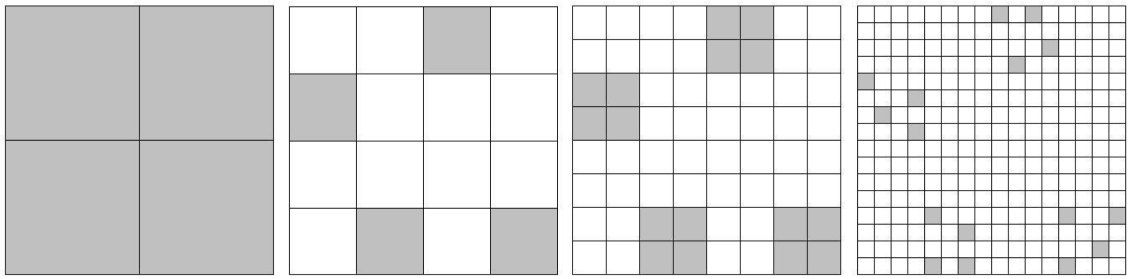

Figure 16: The first four steps in the construction of a compact set $C \subset [0,1]^2$ using Methods 2, 1, 2, 1 respectively.

Proof of (1): The key to constructing a compact set $C \subseteq X$ with $p_t(C) = 0$ for all $t \geqslant b$ is to force $N_\delta(C)$ to be strictly smaller than δ^{-b} for increasingly long periods of time as $\delta \to 0$. Let $\mathcal{N}(2,k)$ denote the number of times we use Method 2 in the first k steps in the construction of C and let

$$\overline{\mathcal{N}}(2) = \limsup_{k\to\infty} \frac{\mathcal{N}(2,k)}{k}$$

and

$$\underline{\mathcal{N}}(2) = \liminf_{k\to\infty} \frac{\mathcal{N}(2,k)}{k}.$$

Observe that

$$M_{2^{-k}}(C) = 2^{d\mathcal{N}(2,k)}$$

and hence

$$\underline{\dim}_{\mathrm{B}} C = d\underline{\mathcal{N}}(2) \qquad \text{and} \qquad \overline{\dim}_{\mathrm{B}} C = d\overline{\mathcal{N}}(2). \tag{4.12}$$

Also observe that if $\delta > 0$ is such that $\mathcal{N}(2, k(\delta)+d+1) < bk(\delta)/d$, then

$$N_\delta(C) \leqslant M_{2^{-(k(\delta)+1+d)}}(C) = 2^{d\mathcal{N}(2,k(\delta)+d+1)} < 2^{bk(\delta)} \leqslant \delta^{-b}. \tag{4.13}$$

It is clear that we may alternate between Methods 1 and 2 when constructing C in such a way as to ensure that

$$\overline{\mathcal{N}}(2) = B/d, \qquad \underline{\mathcal{N}}(2) = b/d$$

and for infinitely many $k_0 \in \mathbb{N}$, we have, for all $k = k_0, \dots, k_0^2$, that

$$\mathcal{N}(2, k+d+1) < bk/d.$$

It follows from (4.12) and (4.13) that such a compact set C has the desired properties. To show that $p_t(C) = 0$ for all $t \geqslant b$ it suffices to prove that $p_b(C) = 0$ since $p_t(C)$ is decreasing in t (Lemma 4.7 (2)). To see that $p_b(C) = 0$ observe that if $\delta > 0$ is such that $k(\delta) = k_0^2$ for such a k_0 described above, then

$$N_{\delta'}(C) < (\delta')^{-b}$$

for all $\delta' \in [\delta, 2^{-k_0}]$ by (4.13). Hence,

$$(2^{-k_0^2})^{p_{b,\delta}(C)} \geqslant \delta^{p_{b,\delta}(C)} \geqslant 2^{-k_0}$$

which yields $p_{b,\delta}(C) \leqslant 1/k_0$ and letting k_0 tend to infinity (and thus δ tend to zero) proves that $p_b(C) = 0$. $\square$

Proof of (2): The key to constructing a compact set $C \subseteq X$ with

$$p_t(C) = \frac{b}{t}\,\frac{d-t}{d-b}$$

for all $t \in (b, B)$ is to force $N_\delta(C)$ to oscillate as fast as possible as $\delta \to 0$. We alternate between choosing cubes according to Method 1 and 2 as fast as we can making sure that the lower box dimension is b and the upper box dimension is B. Unfortunately, there is a bound on how quickly we can do this (seen in Lemma 4.7 (6)). We construct C in the following way. Use Method 1 from step 1 until k_1 where $k_1 \in \mathbb{N}$ is the first time that

$$M_{2^{-k_1}}(C) \leqslant 2^{k_1 b}$$

then change to Method 2 from step $k_1 + 1$ until $k_2 > k_1$ where $k_2 \in \mathbb{N}$ is the next occasion where

$$M_{2^{-k_2}}(C) \geqslant 3^d\, 2^B\, 2^{k_2 B}$$

then change back to Method 1. Repeat this process as $k \to \infty$ to obtain an infinite increasing sequence $\{k_n\}_{n\in\mathbb{N}}$ where

$$M_{2^{-k_{2n-1}}}(C) \leqslant 2^{k_{2n-1} b} \tag{4.14}$$

and

$$M_{2^{-k_{2n}}}(C) \geqslant 3^d\, 2^{k_{2n} B} \tag{4.15}$$

for each $n \in \mathbb{N}$. Furthermore, it is clear that

$$2^{-b}\, 2^{kb} \leqslant M_{2^{-k}}(C) \leqslant 3^d\, 2^d\, 2^{kB}$$

for all $k \in \mathbb{N}$ and it follows from this and (4.14, 4.15) that $b = \underline{\dim}_{\mathrm{B}} C < \overline{\dim}_{\mathrm{B}} C = B$. Let $t \in (b, B)$ and observe that

$$p_t(C) \leqslant \frac{b}{t}\,\frac{d-t}{d-b}$$

by Lemma 4.7 (6). We will now show the opposite inequality. For each k_{2n} above, let $\overline{k_{2n}}$ be the biggest integer less than or equal to $Bt^{-1}k_{2n}$ and let $\underline{k_{2n}}$ be the smallest integer greater than or equal to $(d-B)(d-t)^{-1}k_{2n}$. It follows that for each $n \in \mathbb{N}$ we have

$$N_{2^{-\overline{k_{2n}}}}(C) \geqslant 3^{-d}\, M_{2^{-\overline{k_{2n}}}}(C) \geqslant 3^{-d}\, M_{2^{-k_{2n}}}(C) \geqslant 3^{-d}\, 3^d\, 2^{k_{2n} B} \geqslant 2^{\overline{k_{2n}}\, t} = \left(2^{-\overline{k_{2n}}}\right)^{-t}$$

and

$$N_{2^{-\underline{k_{2n}}}}(C) \geqslant 3^{-d}\, M_{2^{-\underline{k_{2n}}}}(C) \geqslant 3^{-d}\, 2^{(\underline{k_{2n}} - k_{2n})d} M_{2^{-k_{2n}}}(C) \geqslant 3^{-d}\, 2^{(\underline{k_{2n}} - k_{2n})d} 3^d\, 2^{k_{2n} B} \geqslant \left(2^{-\underline{k_{2n}}}\right)^{-t}.$$

Clearly for $\delta \in (2^{-\overline{k_{2n}}}, 2^{-\underline{k_{2n}}})$ we have $N_\delta(C) \geqslant \delta^{-t}$. This implies that, asymptotically, $p_{t,\delta}(C)$ cannot be smaller than the case where $\delta = 2^{-\underline{k_{2(n+1)}}}$ and, writing $p = p_{t, 2^{-\underline{k_{2(n+1)}}}}(C)$,

$$2^{-\underline{k_{2(n+1)}}\,p} = 2^{-\overline{k_{2n}}},$$

i.e. if $p = \overline{k_{2n}} / \underline{k_{2(n+1)}}$. This yields

$$\begin{aligned} p_t(C) \;&\geqslant\; \liminf_{n\to\infty} \frac{\overline{k_{2n}}}{\underline{k_{2(n+1)}}} \;\geqslant\; \liminf_{n\to\infty} \frac{k_{2n}}{k_{2(n+1)}} \frac{\Big(B/t - 1/k_{2n}\Big)}{\Big((d-B)/(d-t) + 1/k_{2(n+1)}\Big)} \\ &\geqslant\; \frac{B}{t}\,\frac{d-t}{d-B}\, \liminf_{n\to\infty} \frac{k_{2n}}{k_{2(n+1)}}. \end{aligned} \tag{4.16}$$

Fix $n \in \mathbb{N}$ and observe that

$$2^{(k_{2(n+1)}-k_{2n+1})d}\,2^{k_{2n+1}b-b} \leqslant 2^{(k_{2(n+1)}-k_{2n+1})d}\,M_{2^{-k_{2n+1}}}(C) \leqslant M_{2^{-k_{2(n+1)}}}(C) \;\leqslant\; 3^d\,2^d\,2^{k_{2(n+1)}B}$$

$$\leqslant\; 2^{3d+k_{2(n+1)}B}$$

from which it follows that

$$(k_{2(n+1)}-k_{2n+1})d + k_{2n+1}b - b \;\leqslant\; 3d + k_{2(n+1)}B$$

and hence

$$\frac{k_{2n+1}}{k_{2(n+1)}} \;\geqslant\; \frac{d-B}{d-b} - \frac{b+3d}{k_{2(n+1)}(d-b)}. \tag{4.17}$$

Also, we have

$$2^{-b}\,2^{(k_{2n+1}-1)b} \leqslant M_{2^{-(k_{2n+1}-1)}}(C) = M_{2^{-k_{2n}}}(C) \leqslant 3^d\,2^d\,2^{k_{2n}B} \leqslant 2^{3d+k_{2n}B}$$

from which it follows that

$$(k_{2n+1}-1)b - b \;\leqslant\; 3d + k_{2n}B$$

and hence

$$\frac{k_{2n}}{k_{2n+1}} \;\geqslant\; \frac{b}{B} - \frac{2b+3d}{k_{2n+1}B}. \tag{4.18}$$

It follows from (4.16, 4.17, 4.18) that

$$p_t(C) \;\geqslant\; \frac{B}{t}\,\frac{d-t}{d-B}\,\liminf_{n\to\infty}\frac{k_{2n}}{k_{2(n+1)}} \;\geqslant\; \frac{B}{t}\,\frac{d-t}{d-B}\,\frac{b}{B}\,\frac{d-B}{d-b} \;=\; \frac{b}{t}\,\frac{d-t}{d-b}$$

which is the desired lower bound and completes the proof. $\square$

5 Inhomogeneous self-affine carpets

5.1 Introduction

In this chapter we continue to investigate inhomogeneous attractors, focusing on verifying or disproving the relationship (1.6) for dimensions which are not countably stable. In particular, we analyse inhomogeneous self-affine carpets of the Barański and Lalley-Gatzouras class and consider the box dimensions.

We find that the relationship (1.6) does not hold in general (even though the IFSs involved satisfy the OSC) and give some specific conditions for (1.6) to hold, or not hold, depending on the dimensions of the projections of the condensation set C. Thus the self-affine case displays new phenomena not observed in the self-similar setting.

5.2 Results

In this section we will state our results. Let $\mathbb{I} = \{S_i\}_{i \in \mathcal{I}}$ be an IFS in the Barański or Lalley-Gatzouras class and fix a compact condensation set $C \subseteq [0,1]^2$. We adopt the terminology used in Chapter 3 to split up such self-affine carpets into horizontal, vertical or mixed classes and assume that at least one of the mappings S_i is not a similarity. If all the maps are similarities, then we are in the setting of inhomogeneous self-similar sets, which was dealt with in the previous chapter. Also, as in Chapter 3, we write c_i and d_i to denote the horizontal and vertical contractions of the map S_i. Let $F_\emptyset$ denote the homogeneous attractor of $\mathbb{I}$ and F_C denote the inhomogeneous attractor of $\mathbb{I}$ together with C. We will say that F_C is in the horizontal/vertical/mixed class if $F_\emptyset$ is in the horizontal/vertical/mixed class.

As in the dimension theory of homogeneous self-affine carpets, the dimensions of orthogonal projections play an important role. Let π_1, π_2 denote the orthogonal projections from the plane onto the first and second coordinates respectively. Write

$$s_1(F_\emptyset) = \dim_{\mathrm{B}} \pi_1(F_\emptyset),$$

$$s_2(F_\emptyset) = \dim_{\mathrm{B}} \pi_2(F_\emptyset),$$

$$\underline{s}_1(C) = \underline{\dim}_{\mathrm{B}} \pi_1(C),$$

$$\overline{s}_1(C) = \overline{\dim}_{\mathrm{B}} \pi_1(C),$$

$$\underline{s}_2(C) = \underline{\dim}_{\mathrm{B}} \pi_2(C)$$

and

$$\overline{s}_2(C) = \overline{\dim}_{\mathrm{B}} \pi_2(C).$$

Note that $s_1(F_\emptyset)$ and $s_2(F_\emptyset)$ exist and can be easily computed because they are the box dimensions of self-similar sets satisfying the OSC, whereas the equalities $\underline{s}_1(C) = \overline{s}_1(C)$ and $\underline{s}_2(C) = \overline{s}_2(C)$ may not hold, even if the box dimension of C exists. Finally, let $\underline{s}_A$, $\overline{s}_A$, $\underline{s}_B$ and $\overline{s}_B$, be the unique solutions of

$$\sum_{i\in\mathcal{I}} c_i^{\max\{s_1(F_\emptyset),\underline{s}_1(C)\}} d_i^{\underline{s}_A - \max\{s_1(F_\emptyset),\underline{s}_1(C)\}} = 1,$$

$$\sum_{i\in\mathcal{I}} c_i^{\max\{s_1(F_\emptyset),\overline{s}_1(C)\}} d_i^{\overline{s}_A - \max\{s_1(F_\emptyset),\overline{s}_1(C)\}} = 1,$$

$$\sum_{i\in\mathcal{I}} d_i^{\max\{s_2(F_\emptyset),\underline{s}_2(C)\}} c_i^{\underline{s}_B - \max\{s_2(F_\emptyset),\underline{s}_2(C)\}} = 1$$

and

$$\sum_{i\in\mathcal{I}} d_i^{\max\{s_2(F_\emptyset),\overline{s}_2(C)\}} c_i^{\overline{s}_B - \max\{s_2(F_\emptyset),\overline{s}_2(C)\}} = 1,$$

respectively. If $\underline{s}_A = \overline{s}_A$ or $\underline{s}_B = \overline{s}_B$, then write s_A and s_B respectively for the common values. Unfortunately, we need to make the following assumption to obtain a sharp formula for the upper box dimension of F_C.

Assumption (A): $\dim_{\mathrm{A}} \pi_i(C) \leqslant \max\{\overline{s}_i(C), s_i(F_\emptyset)\}$, for $i = 1, 2$.

We can now state our results.

Theorem 5.1. *Assume (A). If F_C is in the horizontal class, then*

$$\max\{\underline{s}_A, \underline{\dim}_{\mathrm{B}} C\} \leqslant \underline{\dim}_{\mathrm{B}} F_C \leqslant \overline{\dim}_{\mathrm{B}} F_C = \max\{\overline{s}_A, \overline{\dim}_{\mathrm{B}} C\}.$$

If F_C is in the vertical class, then

$$\max\{\underline{s}_B, \underline{\dim}_{\mathrm{B}} C\} \leqslant \underline{\dim}_{\mathrm{B}} F_C \leqslant \overline{\dim}_{\mathrm{B}} F_C = \max\{\overline{s}_B, \overline{\dim}_{\mathrm{B}} C\}.$$

If F_C is in the mixed class, then

$$\max\{\underline{s}_A, \underline{s}_B, \underline{\dim}_{\mathrm{B}} C\} \leqslant \underline{\dim}_{\mathrm{B}} F_C \leqslant \overline{\dim}_{\mathrm{B}} F_C = \max\{\overline{s}_A, \overline{s}_B, \overline{\dim}_{\mathrm{B}} C\}.$$

Furthermore, if we do not assume (A), then the same results are true but with the final equalities replaced by greater than or equal to.

We will prove Theorem 5.1 in Section 5.4. It is regrettable that we need to assume (A) and we certainly conjecture that it is not required. In a certain sense it is not important that we need this assumption, because the main purpose of this chapter is to show that the expected relationship (1.6) can fail for inhomogeneous carpets and, since we only need assumption (A) for the *upper bound*, this does not change the situations where we can demonstrate this failure. Also, we note that without assumption (A) our methods would yield an upper bound for $\overline{\dim}_{\mathrm{B}} F_C$ where we use the Assouad dimensions of projects instead of upper box dimensions in the definition of $\overline{s}_A$ and $\overline{s}_B$, but we omit further discussion of this.

Notice that we obtain a precise formula for the upper box dimension, but only estimates for the lower box dimension. We saw in Chapter 4 that calculating the lower box dimension of inhomogeneous attractors is a subtle and difficult problem, even in the simpler setting of self-similar sets. To obtain better estimates here, one could analyse the behaviour of the oscillations of the function $\delta \mapsto N_\delta(C)$ using CREs, for example, but we do not pursue this and instead focus more on the upper box dimension and, in particular, the fact that the relationship (1.6) can fail. The following corollaries are immediate and include some simple sufficient conditions for the relationship (1.6) to hold, or not hold.

Corollary 5.2. *Suppose the box dimensions of C and the orthogonal projections of C exist and assume (A). If F_C is in the horizontal class, then*

$$\dim_{\mathrm{B}} F_C \;=\; \max\{s_A, \dim_{\mathrm{B}} C\}.$$

If F_C is in the vertical class, then

$$\dim_{\mathrm{B}} F_C \;=\; \max\{s_B, \dim_{\mathrm{B}} C\}.$$

If F_C is in the mixed class, then

$$\dim_{\mathrm{B}} F_C \;=\; \max\{s_A, s_B, \dim_{\mathrm{B}} C\}.$$

Corollary 5.3. *Assuming (A), the relationship (1.6) holds for upper box dimension, i.e.*

$$\overline{\dim}_{\mathrm{B}} F_C \;=\; \max\{\overline{\dim}_{\mathrm{B}} F_\emptyset, \overline{\dim}_{\mathrm{B}} C\},$$

in each of the following cases:

(1) If F_C is in the horizontal class and $\overline{s}_1(C) \leqslant s_1(F_\emptyset)$

(2) If F_C is in the vertical class and $\overline{s}_2(C) \leqslant s_2(F_\emptyset)$

(3) If F_C is in the mixed class, $\overline{s}_1(C) \leqslant s_1(F_\emptyset)$ and $\overline{s}_2(C) \leqslant s_2(F_\emptyset)$

Corollary 5.4. *The relationship (1.6) fails for lower box dimension, i.e.*

$$\underline{\dim}_{\mathrm{B}} F_C \;>\; \max\{\underline{\dim}_{\mathrm{B}} F_\emptyset, \underline{\dim}_{\mathrm{B}} C\},$$

in each of the following cases:

(1) If F_C is in the horizontal class, $\underline{s}_A \geqslant \underline{\dim}_{\mathrm{B}} C$ and $\underline{s}_1(C) > s_1(F_\emptyset)$

(2) If F_C is in the vertical class, $\underline{s}_B \geqslant \underline{\dim}_{\mathrm{B}} C$ and $\underline{s}_2(C) > s_2(F_\emptyset)$

(3) If F_C is in the mixed class, $\max\{\underline{s}_A, \underline{s}_A\} \geqslant \underline{\dim}_{\mathrm{B}} C$, $\underline{s}_1(C) > s_1(F_\emptyset)$ and $\underline{s}_2(C) > s_2(F_\emptyset)$.

Similarly, the relationship (1.6) fails for upper box dimension, i.e.

$$\overline{\dim}_{\mathrm{B}} F_C \;>\; \max\{\overline{\dim}_{\mathrm{B}} F_\emptyset, \overline{\dim}_{\mathrm{B}} C\},$$

in each of the following cases:

(1) If F_C is in the horizontal class, $\overline{s}_A \geqslant \overline{\dim}_{\mathrm{B}} C$ and $\overline{s}_1(C) > s_1(F_\emptyset)$

(2) If F_C is in the vertical class, $\overline{s}_B \geqslant \overline{\dim}_{\mathrm{B}} C$ and $\overline{s}_2(C) > s_2(F_\emptyset)$

(3) If F_C is in the mixed class, $\max\{\overline{s}_A, \overline{s}_A\} \geqslant \overline{\dim}_{\mathrm{B}} C$, $\overline{s}_1(C) > s_1(F_\emptyset)$ and $\overline{s}_2(C) > s_2(F_\emptyset)$.

In a certain sense, Corollary 5.4 is the most interesting as it gives simple, and easily constructible, conditions for the relationship (1.6) to fail. We will construct such an example in section 5.3.1.

Although the underlying homogeneous IFSs automatically satisfy the OSC, it is worth remarking that our results impose no further separation conditions concerning the condensation set C. In particular, C may have arbitrary overlaps with $F_\emptyset$.

It would be interesting to extend the results in this case to the more general carpets introduced by Feng and Wang or, indeed, the box-like sets we introduced in Chapter 2. However, there are some additional difficulties in these cases. Indeed, the Feng-Wang case is intimately related to the question of whether the relationship (1.6) holds for self-similar sets not satisfying the OSC, see Question 4.4 in Chapter 4. In particular, the sets $\pi_1(F_C)$ and $\pi_2(F_C)$ are inhomogeneous self-similar sets and knowledge of their dimension is crucial in the subsequent proofs. Furthermore, in the box-like case, one would need to extend the results on inhomogeneous self-similar sets to the graph-directed case. There is certainly scope for future research here, and it is easily seen that our methods give solutions to the more general problem in certain cases and can always provide non-trivial estimates; however, we omit further discussion.

5.3 Examples

5.3.1 Inhomogeneous fractal combs

We give a construction of an inhomogeneous Bedford-McMullen carpet, which we refer to as an *inhomogeneous fractal comb* which exhibits some interesting properties. The underlying homogeneous IFS will be a Bedford-McMullen construction where the unit square has been divided into 2 columns of width $1/2$, and $n > 2$ rows of height $1/n$. The IFS is then made up of all the maps which correspond to the left hand column. The condensation set for this construction is taken as $C = [0,1] \times \{0\}$, i.e. the base of the unit square. The inhomogeneous attractor is termed the inhomogeneous fractal comb and is denoted by F_C^n.

It follows from Theorem 5.1 that $\underline{\dim}_B F_C^n = \overline{\dim}_B F_C^n$ is the unique solution of

$$n\, 2^{-1}\, n^{1-s} = 1$$

which gives

$$\underline{\dim}_B F_C^n \;=\; \overline{\dim}_B F_C^n \;=\; 2 - \log 2/\log n \;>\; 1.$$

However,

$$\max\{\overline{\dim}_B F_\emptyset,\ \overline{\dim}_B C\} \;=\; 1$$

and thus our fractal comb provides a simple example showing that the 'expected relationship' for upper box dimension (1.6) can fail for self-affine sets, even if the homogeneous IFS satisfies the OSC. This is in stark contrast to the self-similar setting, see Corollary 4.2.

This example has another interesting property: it shows that $\overline{\dim}_B F_C^n$ does not just depend on the sets $F_\emptyset$ and C, but also depends on the IFS itself. To see this observe that $F_\emptyset$ and C do not depend on n, but $\overline{\dim}_B F_C^n$ does. In fact $F_\emptyset = \{0\} \times [0,1]$, i.e. the left hand side of the unit square, for any n. Again, this behaviour is not observed in the self-similar setting.

Finally, observe that, although the inhomogeneous fractal combs are subsets of $\mathbb{R}^2$ and the expected box dimension is 1, we can find examples where the achieved box dimension is arbitrarily close to 2, demonstrating that, in this case, there is no limit to how 'badly' the relationship (1.6) can fail.

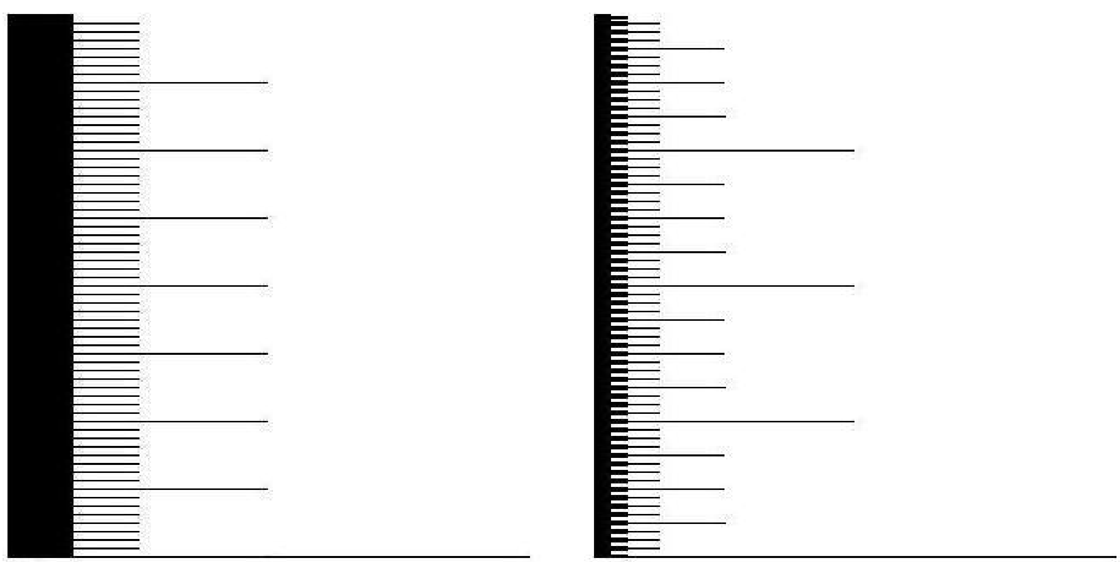

Figure 17: Two fractal combs: the inhomogeneous fractal combs F_C^8, with box dimension 5/3 (left); and F_C^4, with box dimension 3/2 (right).

5.3.2 A more exotic example

In this section we provide an example with a slightly more exotic looking structure. Despite this, however, it is perhaps less interesting than the previous one as the relationship (1.6) holds. The underlying homogeneous IFS will be in the mixed class and will consist of the mappings

$$S_1 = \begin{pmatrix} \frac{3}{10} & 0 \\ 0 & \frac{3}{10} \end{pmatrix},$$

$$S_2 = \begin{pmatrix} \frac{3}{10} & 0 \\ 0 & \frac{7}{10} \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{3}{10} \end{pmatrix}$$

and

$$S_3 = \begin{pmatrix} \frac{7}{10} & 0 \\ 0 & \frac{3}{10} \end{pmatrix} + \begin{pmatrix} \frac{3}{10} \\ 0 \end{pmatrix}$$

and the condensation set will be the Sierpiński triangle.

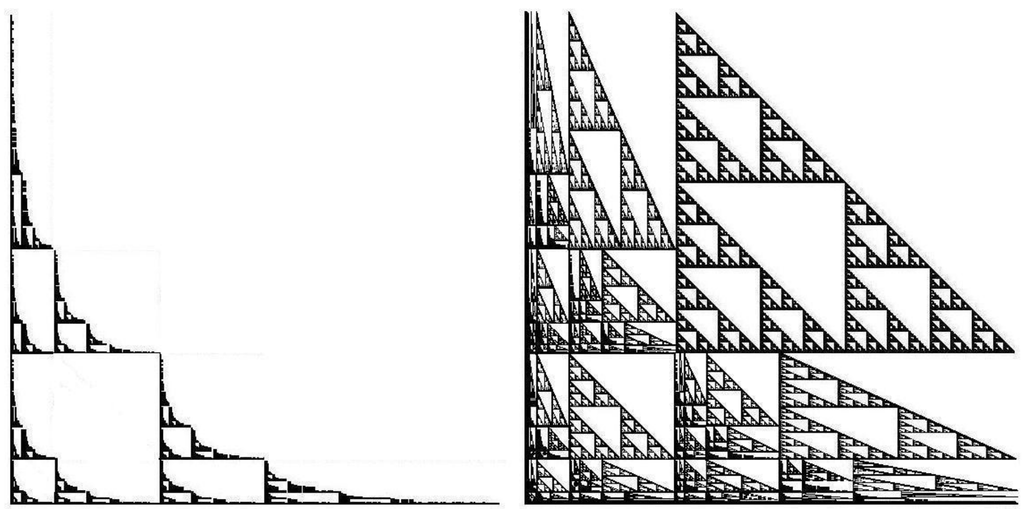

Figure 18: The homogeneous Barański type carpet described above (left) and the corresponding inhomogeneous carpet with condensation set based on the Sierpiński triangle (right).

One can easily see that $s_A = s_B = s$ is the solution of

$$\left(\tfrac{3}{10}\right)\left(\tfrac{3}{10}\right)^{s-1} + \left(\tfrac{3}{10}\right)\left(\tfrac{7}{10}\right)^{s-1} + \left(\tfrac{7}{10}\right)\left(\tfrac{3}{10}\right)^{s-1} = 1$$

which is $s = 1.2647\ldots$ and it follows from Theorem 5.1 that

$$\underline{\dim}_{\mathrm{B}} F_C = \overline{\dim}_{\mathrm{B}} F_C = \max\{s, \dim_{\mathrm{B}} C\} = \max\{s, \log 3/\log 2\} = \frac{\log 3}{\log 2} = 1.5849\ldots$$

5.4 Proofs

5.4.1 Preliminary results

In this section we will introduce some notation and establish some simple estimates before beginning the main proofs.

For $\boldsymbol{i} \in \mathcal{I}^*$, let

$$\pi_{\boldsymbol{i}} = \begin{cases} \pi_1 & \text{if } c_{\boldsymbol{i}} \geqslant d_{\boldsymbol{i}} \\ \pi_2 & \text{if } c_{\boldsymbol{i}} < d_{\boldsymbol{i}} \end{cases}$$

$$s_{\boldsymbol{i}}(F_\emptyset) = \begin{cases} s_1(F_\emptyset) & \text{if } c_{\boldsymbol{i}} \geqslant d_{\boldsymbol{i}} \\ s_2(F_\emptyset) & \text{if } c_{\boldsymbol{i}} < d_{\boldsymbol{i}} \end{cases}$$

$$\overline{s}_{\boldsymbol{i}}(C) = \begin{cases} \overline{s}_1(C) & \text{if } c_{\boldsymbol{i}} \geqslant d_{\boldsymbol{i}} \\ \overline{s}_2(C) & \text{if } c_{\boldsymbol{i}} < d_{\boldsymbol{i}} \end{cases}$$

$$\underline{s}_{\boldsymbol{i}}(C) = \begin{cases} \underline{s}_1(C) & \text{if } c_{\boldsymbol{i}} \geqslant d_{\boldsymbol{i}} \\ \underline{s}_2(C) & \text{if } c_{\boldsymbol{i}} < d_{\boldsymbol{i}} \end{cases}$$

The sets $\pi_1(\overline{\mathcal{O}})$ and $\pi_2(\overline{\mathcal{O}})$ are inhomogeneous self-similar sets with condensation sets $\pi_1(C)$ and $\pi_2(C)$ respectively. The underlying IFSs (derived in the obvious way from the original IFS) satisfy the OSC and so it follows from Corollary 4.6 that

$$\max\{s_1(F_\emptyset), \underline{s}_1(C)\} \ \leqslant\ \underline{\dim}_{\mathrm{B}}\pi_1(\mathcal{O}) \ \leqslant\ \overline{\dim}_{\mathrm{B}}\pi_1(\mathcal{O}) \ =\ \max\{s_1(F_\emptyset), \overline{s}_1(C)\}$$

and

$$\max\{s_2(F_\emptyset), \underline{s}_2(C)\} \ \leqslant\ \underline{\dim}_{\mathrm{B}}\pi_2(\mathcal{O}) \ \leqslant\ \overline{\dim}_{\mathrm{B}}\pi_2(\mathcal{O}) \ =\ \max\{s_2(F_\emptyset), \overline{s}_2(C)\}.$$

It follows that, for all $\varepsilon \in (0,1]$, there exists a constant $C_\varepsilon \geqslant 1$ such that for all $\boldsymbol{i} \in \mathcal{I}^*$ and all $\delta \in (0, \alpha_{\min}^{-1}]$ we have

$$C_\varepsilon^{-1}\delta^{-\max\{\underline{s}_{\boldsymbol{i}}(C), s_{\boldsymbol{i}}(F_\emptyset)\}+\varepsilon} \ \leqslant\ N_\delta\big(\pi_{\boldsymbol{i}}(\mathcal{O})\big) \ \leqslant\ C_\varepsilon \delta^{-\max\{\overline{s}_{\boldsymbol{i}}(C), s_{\boldsymbol{i}}(F_\emptyset)\}-\varepsilon} \tag{5.1}$$

For $\delta \in (0,1]$ we define the δ-*stopping*, $\mathcal{I}_\delta$, as follows:

$$\mathcal{I}_\delta = \big\{\boldsymbol{i} \in \mathcal{I}^* : \alpha_2(\boldsymbol{i}) < \delta \leqslant \alpha_2(\overline{\boldsymbol{i}})\big\}.$$

Note that for $\boldsymbol{i} \in \mathcal{I}_\delta$ we have

$$\alpha_{\min}\,\delta \leqslant \alpha_2(\boldsymbol{i}) < \delta. \tag{5.2}$$

5.4.2 Proof of the lower bound for the lower box dimension in Theorem 5.1

In this section we will prove that if F is in the mixed class, then $\max\{\underline{s}_A, \underline{s}_B, \underline{\dim}_{\mathrm{B}}C\} \leqslant \underline{\dim}_{\mathrm{B}}F_C$. The proof of the analogous inequality for the horizontal and vertical classes is similar and omitted. Since lower box dimension is monotone, it suffices to show that $\max\{\underline{s}_A, \underline{s}_B\} \leqslant \underline{\dim}_{\mathrm{B}}F_C$, and we will assume without loss of generality that $\max\{\underline{s}_A, \underline{s}_B\} = \underline{s}_A$.

Let $\varepsilon \in (0, \underline{s}_A)$, $\delta \in (0,1]$ and U be any closed square of sidelength δ. Also, let

$$M = \min\big\{n \in \mathbb{N} : n \geqslant \alpha_{\min}^{-1} + 2\big\}.$$

Since $\{S_{\boldsymbol{i}}\big([0,1]^2\big)\}_{\boldsymbol{i}\in\mathcal{I}_\delta}$ is a collection of pairwise disjoint open rectangles each with shortest side having length at least $\alpha_{\min}\delta$, it is clear that U can intersect no more than M^2 of the sets $\{S_i(\mathcal{O})\}_{\boldsymbol{i}\in\mathcal{I}_\delta}$ since $S_i(\mathcal{O}) \subseteq S_{\boldsymbol{i}}\big([0,1]^2\big)$ for all $\boldsymbol{i} \in \mathcal{I}_\delta$. It follows that, using the δ-mesh definition of N_δ, we have

$$\sum_{\boldsymbol{i}\in\mathcal{I}_\delta} N_\delta\big(S_{\boldsymbol{i}}(\mathcal{O})\big) \ \leqslant\ M^2\, N_\delta\left(\bigcup_{\boldsymbol{i}\in\mathcal{I}_\delta} S_i(\mathcal{O})\right) \ \leqslant\ M^2\, N_\delta(\mathcal{O}).$$

This yields

$$\begin{aligned}
N_\delta(\mathcal{O}) \ &\geqslant\ \tfrac{1}{M^2} \sum_{\boldsymbol{i}\in\mathcal{I}_\delta} N_\delta\big(S_{\boldsymbol{i}}(\mathcal{O})\big) \\
&=\ \tfrac{1}{M^2} \sum_{\boldsymbol{i}\in\mathcal{I}_\delta} N_{\delta/\alpha_1(i)}\big(\pi_{\boldsymbol{i}}(\mathcal{O})\big) \qquad\qquad \text{since } \alpha_2(\boldsymbol{i}) < \delta \\
&\geqslant\ \tfrac{1}{M^2} \sum_{\boldsymbol{i}\in\mathcal{I}_\delta} C_\varepsilon^{-1} \left(\frac{\alpha_1(\boldsymbol{i})}{\delta}\right)^{\max\{\underline{s}_i(C), s_i(F_\emptyset)\}-\varepsilon} \qquad\qquad \text{by (5.1)} \\
&=\ \tfrac{1}{M^2 C_\varepsilon}\ \delta^{-\underline{s}_A+\varepsilon} \sum_{\boldsymbol{i}\in\mathcal{I}_\delta} \alpha_1(\boldsymbol{i})^{\max\{\underline{s}_i(C), s_i(F_\emptyset)\}}\, \delta^{\underline{s}_A - \max\{\underline{s}_i(C), s_i(F_\emptyset)\}}
\end{aligned}$$

$$\geqslant \quad \tfrac{1}{M^2 C_\varepsilon}\ \delta^{-\underline{s}_A+\varepsilon} \sum_{\boldsymbol{i}\in\mathcal{I}_\delta} \alpha_1(\boldsymbol{i})^{\max\{\underline{s}_i(C),s_i(F_\emptyset)\}}\alpha_2(\boldsymbol{i})^{\underline{s}_A-\max\{\underline{s}_i(C),s_i(F_\emptyset)\}}$$

by (5.2). We now claim that for all $\boldsymbol{i}\in\mathcal{I}_\delta$ we have

$$\alpha_1(\boldsymbol{i})^{\max\{\underline{s}_i(C),s_i(F_\emptyset)\}}\alpha_2(\boldsymbol{i})^{\underline{s}_A-\max\{\underline{s}_i(C),s_i(F_\emptyset)\}} \ \geqslant\ c_{\boldsymbol{i}}^{\max\{\underline{s}_1(C),s_1(F_\emptyset)\}} d_{\boldsymbol{i}}^{\underline{s}_A-\max\{\underline{s}_1(C),s_1(F_\emptyset)\}}.$$

If $c_{\boldsymbol{i}}\geqslant d_{\boldsymbol{i}}$, then we trivially have equality, so assume that $c_{\boldsymbol{i}}<d_{\boldsymbol{i}}$, in which case

$$\begin{aligned}
\alpha_1(\boldsymbol{i})^{\max\{\underline{s}_i(C),s_i(F_\emptyset)\}}\alpha_2(\boldsymbol{i})^{\underline{s}_A-\max\{\underline{s}_i(C),s_i(F_\emptyset)\}} &= d_{\boldsymbol{i}}^{\max\{\underline{s}_2(C),s_2(F_\emptyset)\}} c_{\boldsymbol{i}}^{\underline{s}_A-\max\{\underline{s}_2(C),s_2(F_\emptyset)\}}\\
&= c_{\boldsymbol{i}}^{\max\{\underline{s}_1(C),s_1(F_\emptyset)\}} d_{\boldsymbol{i}}^{\underline{s}_A-\max\{\underline{s}_1(C),s_1(F_\emptyset)\}}\\
&\qquad \cdot\left(\frac{d_{\boldsymbol{i}}}{c_{\boldsymbol{i}}}\right)^{\max\{\underline{s}_1(C),s_1(F_\emptyset)\}+\max\{\underline{s}_2(C),s_2(F_\emptyset)\}-\underline{s}_A}\\
&\geqslant c_{\boldsymbol{i}}^{\max\{\underline{s}_1(C),s_1(F_\emptyset)\}} d_{\boldsymbol{i}}^{\underline{s}_A-\max\{\underline{s}_1(C),s_1(F_\emptyset)\}}
\end{aligned}$$

since it is easily seen that $\underline{s}_A\leqslant \max\{\underline{s}_1(C),s_1(F_\emptyset)\}+\max\{\underline{s}_2(C),s_2(F_\emptyset)\}$. Combining this with the above estimate yields

$$\begin{aligned}
N_\delta(\mathcal{O}) &\geqslant \tfrac{1}{M^2 C_\varepsilon}\ \delta^{-\underline{s}_A+\varepsilon}\sum_{\boldsymbol{i}\in\mathcal{I}_\delta} c_{\boldsymbol{i}}^{\max\{\underline{s}_1(C),s_1(F_\emptyset)\}} d_{\boldsymbol{i}}^{\underline{s}_A-\max\{\underline{s}_1(C),s_1(F_\emptyset)\}}\\
&= \tfrac{1}{M^2 C_\varepsilon}\ \delta^{-(\underline{s}_A-\varepsilon)}
\end{aligned}$$

by repeated application of the definition of $\underline{s}_A$. This proves that $\underline{\dim}_{\mathrm{B}} F_C=\underline{\dim}_{\mathrm{B}}\mathcal{O}\geqslant \underline{s}_A-\varepsilon$ and letting ε tend to zero gives the desired lower bound. $\square$

5.4.3 Proof of the upper bound for the upper box dimension in Theorem 5.1

In this section we will prove that if F_C is in the mixed class and satisfies assumption (A), then $\overline{\dim}_{\mathrm{B}} F_C\leqslant \max\{\overline{s}_A,\overline{s}_B,\overline{\dim}_{\mathrm{B}} C\}$. The proof of the analogous inequality for the horizontal and vertical classes is similar and omitted. Let $s=\max\{\overline{s}_A,\overline{s}_B,\overline{\dim}_{\mathrm{B}} C\}$ and $\varepsilon>0$. Since $F_C=\overline{\mathcal{O}}$ and upper box dimension is stable under taking closures, it suffices to estimate $\overline{\dim}_{\mathrm{B}}\mathcal{O}$. We have

$$\begin{aligned}
N_\delta(\mathcal{O}) &= N_\delta\Bigg(\bigcup_{\boldsymbol{i}\in\mathcal{I}_\delta} S_{\boldsymbol{i}}(\mathcal{O}) \ \cup \bigcup_{\substack{\boldsymbol{i}\in\mathcal{I}^*\cup\{\omega\}:\\ \alpha_2(\boldsymbol{i})>\delta}} S_{\boldsymbol{i}}(C)\Bigg)\\
&\leqslant \sum_{\boldsymbol{i}\in\mathcal{I}_\delta} N_\delta\big(S_{\boldsymbol{i}}(\mathcal{O})\big) \ +\sum_{\substack{\boldsymbol{i}\in\mathcal{I}^*\cup\{\omega\}:\\ \alpha_2(\boldsymbol{i})>\delta}} N_\delta\big(S_{\boldsymbol{i}}(C)\big).
\end{aligned}$$

We will analyse these two terms separately. For the first term

$$\begin{aligned}
\sum_{\boldsymbol{i}\in\mathcal{I}_\delta} N_\delta\big(S_{\boldsymbol{i}}(\mathcal{O})\big) &= \sum_{\boldsymbol{i}\in\mathcal{I}_\delta} N_{\delta/\alpha_1(\boldsymbol{i})}\big(\pi_{\boldsymbol{i}}(\mathcal{O})\big) && \text{since } \alpha_2(\boldsymbol{i})<\delta\\
&\leqslant \sum_{\boldsymbol{i}\in\mathcal{I}_\delta} C_\varepsilon\left(\frac{\alpha_1(\boldsymbol{i})}{\delta}\right)^{\max\{\overline{s}_i(C),s_i(F_\emptyset)\}+\varepsilon} && \text{by (5.1)}\\
&= C_\varepsilon\,\delta^{-s-\varepsilon}\sum_{\boldsymbol{i}\in\mathcal{I}_\delta}\alpha_1(\boldsymbol{i})^{\max\{\overline{s}_i(C),s_i(F_\emptyset)\}}\delta^{s-\max\{\overline{s}_i(C),s_i(F_\emptyset)\}}\\
&\leqslant C_\varepsilon\,\alpha_{\min}^{-2}\,\delta^{-s-\varepsilon}\sum_{\boldsymbol{i}\in\mathcal{I}_\delta}\alpha_1(\boldsymbol{i})^{\max\{\overline{s}_i(C),s_i(F_\emptyset)\}}\alpha_2(\boldsymbol{i})^{s-\max\{\overline{s}_i(C),s_i(F_\emptyset)\}} && \text{by (5.2)}
\end{aligned}$$

$$\leqslant \quad C_\varepsilon\, \alpha_{\min}^{-2}\, \delta^{-s-\varepsilon} \left(\sum_{\boldsymbol{i}\in\mathcal{I}_\delta} c_{\boldsymbol{i}}^{\max\{\overline{s}_1(C),s_1(F_\emptyset)\}} d_{\boldsymbol{i}}^{\overline{s}_A-\max\{\overline{s}_1(C),s_1(F_\emptyset)\}} + \sum_{\boldsymbol{i}\in\mathcal{I}_\delta} d_{\boldsymbol{i}}^{\max\{\overline{s}_2(C),s_2(F_\emptyset)\}} c_{\boldsymbol{i}}^{\overline{s}_B-\max\{\overline{s}_2(C),s_2(F_\emptyset)\}} \right)$$

$$\leqslant \quad 2\, C_\varepsilon\, \alpha_{\min}^{-2}\, \delta^{-(s+\varepsilon)}$$

by repeated application of the definitions of $\overline{s}_A$ and $\overline{s}_B$. The second term is awkward as we have to estimate $N_\delta\big(S_{\boldsymbol{i}}(C)\big)$ for $\boldsymbol{i}$ with various different values of $\alpha_2(\boldsymbol{i}) > \delta$. This is the only occasion in the proof where we require assumption (A).

Lemma 5.5. *Assume (A), let $\varepsilon > 0$ and let $\delta \in (0,1]$. There exists a constant $D_\varepsilon > 0$ such that for all $\boldsymbol{i} \in \mathcal{I}^* \cup \{\omega\}$ such that $\alpha_2(\boldsymbol{i}) > \delta$, we have*

$$N_\delta\big(S_{\boldsymbol{i}}(C)\big) \;\leqslant\; D_\varepsilon\, \delta^{-(s+\varepsilon)}\, \alpha_1(\boldsymbol{i})^{\max\{\overline{s}_i(C),s_i(F_\emptyset)\}} \alpha_2(\boldsymbol{i})^{s+\varepsilon/2-\max\{\overline{s}_i(C),s_i(F_\emptyset)\}}$$

Proof. First take a cover of C by fewer than

$$D_{1,\varepsilon}\left(\frac{\alpha_2(\boldsymbol{i})}{\delta}\right)^{s+\varepsilon}$$

balls of diameter $\delta/\alpha_2(\boldsymbol{i})$, where $D_{1,\varepsilon}$ is a universal constant depending only on ε. Taking images of these sets under $S_{\boldsymbol{i}}$ gives a cover of $S_{\boldsymbol{i}}(C)$ by ellipses with minor axis δ and major axis $\delta\alpha_1(\boldsymbol{i})/\alpha_2(\boldsymbol{i})$. Projecting each of these ellipses under $\pi_{\boldsymbol{i}}$ gives an interval of length $\delta\alpha_1(\boldsymbol{i})/\alpha_2(\boldsymbol{i})$, the intersection of which with $\pi_{\boldsymbol{i}}\big(S_{\boldsymbol{i}}(C)\big)$ may be covered by fewer than

$$D_{2,\varepsilon}\left(\frac{\delta\alpha_1(\boldsymbol{i})/\alpha_2(\boldsymbol{i})}{\delta}\right)^{\dim_{\mathrm{A}} \pi_i(C)+\varepsilon/2} \;=\; D_{2,\varepsilon}\left(\frac{\alpha_1(\boldsymbol{i})}{\alpha_2(\boldsymbol{i})}\right)^{\dim_{\mathrm{A}} \pi_i(C)+\varepsilon/2}$$

intervals of radius δ, where $D_{2,\varepsilon}$ is a universal constant depending only on ε. Pulling each of these intervals back up to $S_i(C)$ and applying assumption (A) gives a δ cover of $S_{\boldsymbol{i}}(C)$ by fewer than

$$D_{1,\varepsilon}\left(\frac{\alpha_2(\boldsymbol{i})}{\delta}\right)^{s+\varepsilon} D_{2,\varepsilon}\left(\frac{\alpha_1(\boldsymbol{i})}{\alpha_2(\boldsymbol{i})}\right)^{\dim_{\mathrm{A}} \pi_i(C)+\varepsilon/2}$$

$$\leqslant D_{1,\varepsilon}\, D_{2,\varepsilon}\, \delta^{-(s+\varepsilon)}\, \alpha_1(\boldsymbol{i})^{\max\{\overline{s}_i(C),s_i(F_\emptyset)\}} \alpha_2(\boldsymbol{i})^{s+\varepsilon/2-\max\{\overline{s}_i(C),s_{\mathrm{i}}(F_\emptyset)\}}$$

which proves the lemma. □

We can now estimate the awkward second term. We have

$$\sum_{\substack{\boldsymbol{i}\in\mathcal{I}^*\cup\{\omega\}:\\ \alpha_2(\boldsymbol{i})>\delta}} N_\delta\big(S_{\boldsymbol{i}}(C)\big) \quad\leqslant\quad D_\varepsilon\, \delta^{-(s+\varepsilon)} \sum_{\substack{\boldsymbol{i}\in\mathcal{I}^*\cup\{\omega\}:\\ \alpha_2(\boldsymbol{i})>\delta}} \alpha_1(\boldsymbol{i})^{\max\{\overline{s}_i(C),s_i(F_\emptyset)\}} \alpha_2(\boldsymbol{i})^{s+\varepsilon/2-\max\{\overline{s}_i(C),s_i(F_\emptyset)\}}$$

by Lemma 5.5

$$\leqslant \quad D_\varepsilon\, \delta^{-(s+\varepsilon)} \sum_{k=0}^{\infty} \alpha_{\max}^{k\varepsilon/2} \left(\sum_{\boldsymbol{i}\in\mathcal{I}^k} c_{\boldsymbol{i}}^{\max\{\overline{s}_1(C),s_1(F_\emptyset)\}} d_{\boldsymbol{i}}^{\overline{s}_A-\max\{\overline{s}_1(C),s_1(F_\emptyset)\}} + \sum_{\boldsymbol{i}\in\mathcal{I}^k} d_{\boldsymbol{i}}^{\max\{\overline{s}_2(C),s_2(F_\emptyset)\}} c_{\boldsymbol{i}}^{\overline{s}_B-\max\{\overline{s}_2(C),s_2(F_\emptyset)\}} \right)$$

$$\leqslant \quad 2\, D_\varepsilon\, \delta^{-(s+\varepsilon)} \sum_{k=0}^{\infty} \left(\alpha_{\max}^{\varepsilon/2}\right)^k \qquad \text{by the definitions of } \overline{s}_A \text{ and } \overline{s}_B$$

$$\leqslant \quad \frac{2\,D_\varepsilon}{1-\alpha_{\max}^{\varepsilon/2}}\,\delta^{-(s+\varepsilon)}.$$

Combining the two estimates given above yields

$$N_\delta(\mathcal{O}) \;\leqslant\; \left(2\,C_\varepsilon\,\alpha_{\min}^{-2} \;+\; \frac{2\,D_\varepsilon}{1-\alpha_{\max}^{\varepsilon/2}}\right)\delta^{-(s+\varepsilon)}$$

which proves that $\overline{\dim}_{\mathrm{B}} F_C = \overline{\dim}_{\mathrm{B}} \mathcal{O} \leqslant s+\varepsilon$ and letting ε tend to zero gives the desired upper bound.

5.4.4 Proof of the lower bound for the upper box dimension in Theorem 5.1

In this section we will prove the lower bounds for the the upper box dimension of inhomogeneous self-affine carpets, which, combined with the upper bound in the previous section, yields a precise formula. We will begin by proving the result in a special case.

Proposition 5.6. *Let F_C be in the horizontal class and assume that $c_i = c \geqslant d = d_i$ for all $i \in \mathcal{I}$. Then*

$$\overline{\dim}_{\mathrm{B}} F_C \geqslant \max\{\overline{s}_A, \overline{\dim}_{\mathrm{B}} C\}.$$

Proof. Since upper box dimension is monotone, it suffices to show that $\overline{\dim}_{\mathrm{B}} F_C \geqslant \overline{s}_A$. Since $\pi_1(\mathcal{O})$ is an inhomogeneous self-similar set which satisfies the OSC, we know from Corollary 4.2 that $\overline{\dim}_{\mathrm{B}} \pi_1(\mathcal{O}) = \max\{\overline{s}_1(C), s_1(F_\emptyset)\}$. It follows that for all $\varepsilon > 0$ we can find infinitely many $k \in \mathbb{N}$ such that

$$N_{(d/c)^k}\big(\pi_1(\mathcal{O})\big) \geqslant \big((d/c)^k\big)^{-(\max\{\overline{s}_1(C), s_1(F_\emptyset)\}-\varepsilon)}. \tag{5.3}$$

Fix such a k, let $\varepsilon \in (0, \underline{s}_A)$, and U be any closed square of sidelength d^k. Since $\{S_{\boldsymbol{i}}\big([0,1]^2\big)\}_{\boldsymbol{i} \in \mathcal{I}^k}$ is a collection of pairwise disjoint open rectangles each with shortest side having length d^k which is strictly less than the longer side, it is clear that U can intersect no more than 6 of the sets $\{S_i(\mathcal{O})\}_{\boldsymbol{i} \in \mathcal{I}^k}$ since $S_i(\mathcal{O}) \subseteq S_{\boldsymbol{i}}\big([0,1]^2\big)$ for all $\boldsymbol{i} \in \mathcal{I}^k$. It follows that, using the δ-mesh definition of N_δ, we have

$$\sum_{\boldsymbol{i} \in \mathcal{I}^k} N_{d^k}\big(S_{\boldsymbol{i}}(\mathcal{O})\big) \;\leqslant\; 6\,N_{d^k}\left(\bigcup_{\boldsymbol{i} \in \mathcal{I}^k} S_i(\mathcal{O})\right) \;\leqslant\; 6\,N_{d^k}(\mathcal{O}).$$

This yields

$$\begin{aligned}
N_{d^k}(\mathcal{O}) \;&\geqslant\; \tfrac{1}{6} \sum_{\boldsymbol{i} \in \mathcal{I}^k} N_{d^k}\big(S_{\boldsymbol{i}}(\mathcal{O})\big) \\
&=\; \tfrac{1}{6} \sum_{\boldsymbol{i} \in \mathcal{I}^k} N_{(d/c)^k}\big(\pi_1(\mathcal{O})\big) \qquad\qquad \text{since } \alpha_2(\boldsymbol{i}) = d^k \\
&\geqslant\; \tfrac{1}{6} \sum_{\boldsymbol{i} \in \mathcal{I}^k} \big((d/c)^k\big)^{-(\max\{\overline{s}_1(C), s_1(F_\emptyset)\}-\varepsilon)} \qquad\qquad \text{by (5.3)} \\
&\geqslant\; \tfrac{1}{6}\,(d^k)^{-\overline{s}_A+\varepsilon} \left(\sum_{i \in \mathcal{I}} c^{\max\{\overline{s}_1(C), s_1(F_\emptyset)\}}\, d^{\overline{s}_A - \max\{\overline{s}_1(C), s_1(F_\emptyset)\}}\right)^k \\
&\geqslant\; \tfrac{1}{6}\,(d^k)^{-(\overline{s}_A-\varepsilon)}
\end{aligned}$$

by the definition of $\overline{s}_A$, which proves that $\overline{\dim}_{\mathrm{B}} F_C \geqslant \overline{\dim}_{\mathrm{B}} \mathcal{O} \geqslant \overline{s}_A - \varepsilon$ and letting ε tend to zero gives the desired lower bound. $\square$

We will now use Proposition 5.6 to prove the result in the general case. The key idea is to approximate the IFS ‘from within’ by subsytems which fall into the subclass used in Proposition 5.6. This approach is reminiscent of that used by Ferguson, Jordan and Shmerkin when studying projections of carpets [FJS, Lemma 4.3]. There the authors prove that for all $\varepsilon > 0$ any Lalley-Gatzouras or Barański system, $\mathbb{I}$, has a finite subsystem $\mathbb{J}_\varepsilon \subseteq \mathbb{I}^m$ (for some $m \in \mathbb{N}$), with the following properties: $\mathbb{J}_\varepsilon$ consists only of maps with linear part of the form

$$\begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}$$

for some constants $c, d \in (0, 1)$ depending on ε; the Hausdorff dimension of the attractor of $\mathbb{J}_\varepsilon$ is no more than ε smaller than the Hausdorff dimension of the attractor of $\mathbb{I}$; and $\mathbb{J}_\varepsilon$ has uniform fibres (either vertical or horizontal, depending on the relative size of c and d). It is interesting to note that one cannot approximate the box and packing dimensions 'from within' in the same way. To see this observe that in the uniform fibres case the Hausdorff, box and packing dimensions coincide. As such if these dimensions did not coincide in the original construction, then one cannot find subsystems for which they coincide but get arbitrarily close to the box dimension. It is natural to ask if one can do this if the uniform fibres condition is dropped. We have been unable to show this and it seems that the problem is somehow linked to the fact that the packing dimension does not behave well with respect to fixing prescribed frequencies of maps in the IFS. For examples of such bad behaviour, we note that for Bedford-McMullen carpets there does not usually exist a Bernoulli measure with full packing dimension and the packing spectrum of Bernoulli measures supported on self-affine carpets need not peak at the ambient packing dimension (Thomas Jordan, personal communication). In contrast to this, there is always a Bernoulli measure with full Hausdorff dimension and the Hausdorff spectrum always peaks at the ambient Hausdorff dimension, [Ki, JR]. Also, see the related work of Nielsen [N] on subsets of carpets consisting of points where the digits in the expansions occur with prescribed frequencies. Fortunately, for the purposes of this chapter, we do not need to approximate the box dimension from within, but rather approximate the quantities $\overline{s}_A$ and $\overline{s}_B$, which we *can* do.

Proposition 5.7. *Let F_C be an inhomogeneous self-affine carpet in the horizontal or mixed class and assume that $\overline{s}_1(C) \geqslant s_1(F_\emptyset)$. Then for all $\varepsilon > 0$, there exists a finite subsystem $\mathbb{J}_\varepsilon = \{S_{\boldsymbol{i}}\}_{\boldsymbol{i} \in \mathcal{J}_\varepsilon}$ for some $\mathcal{J}_\varepsilon \subseteq \mathcal{I}^m$ and $m \in \mathbb{N}$, with the property that for all $\boldsymbol{i} \in \mathcal{J}_\varepsilon$ we have $c_{\boldsymbol{i}} = c$, $d_{\boldsymbol{i}} = d$ for some constants $c, d \in (0, 1)$ depending on ε; and the number $\overline{s}_A$ defined by $\mathcal{J}_\varepsilon$ is no more than ε smaller than the number $\overline{s}_A$ defined by $\mathcal{I}$.*

Proof. We will use a version of Stirling's approximation for the logarithm of large factorials. This states that for all $n \in \mathbb{N} \setminus \{1\}$ we have

$$n \log n - n \ \leqslant \ \log n! \ \leqslant \ n \log n - n + \log n. \tag{5.4}$$

For $i \in \mathcal{I}$, let

$$p_i = c_i^{\overline{s}_1(C)} d_i^{\overline{s}_A - \overline{s}_1(C)}$$

and for $k \in \mathbb{N}$, let

$$m(k) = \sum_{i \in \mathcal{I}} \lfloor p_i k \rfloor \in \mathbb{N}$$

and note that $k - |\mathcal{I}| \leqslant m(k) \leqslant k$. Consider the $m(k)$th iteration of $\mathcal{I}$ and let

$$\mathcal{J}_k = \Big\{ \boldsymbol{j} = (j_1, \dots, j_{m(k)}) \in \mathcal{I}^{m(k)} : \#\{n : j_n = i\} = \lfloor p_i k \rfloor \Big\}.$$

It is straightforward to see that

$$|\mathcal{J}_k| = \frac{m(k)!}{\prod_{i \in \mathcal{I}} \lfloor p_i k \rfloor !} \tag{5.5}$$

and for each $\boldsymbol{j} \in \mathcal{J}_k$ we have

$$c_{\boldsymbol{j}} = \prod_{i \in \mathcal{I}} c_i^{\lfloor p_i k \rfloor} =: c$$

and

$$d_{\boldsymbol{j}} = \prod_{i \in \mathcal{I}} d_i^{\lfloor p_i k \rfloor} =: d.$$

Indeed, these facts were observed in [FJS]. We can now use this information to estimate the number $\overline{s}_A$ corresponding to $\mathcal{J}_k$, which we will denote by $\overline{s}_A(\mathcal{J}_k)$ to differentiate it from the number $\overline{s}_A$

corresponding to $\mathcal{I}$, which we will denote by $\overline{s}_A(\mathcal{I})$. Since $\mathcal{J}_k$ is a subsystem of $\mathcal{I}$ and since $\overline{s}_1(C) \geqslant s_1(F_\emptyset)$, it follows by definition that

$$\begin{aligned}
\overline{s}_A(\mathcal{I}) \ \geqslant \ \overline{s}_A(\mathcal{J}_k) \quad &= \quad \frac{\log|\mathcal{J}_k|}{-\log d} \ + \ \overline{s}_1(C)\left(1 - \frac{\log c}{\log d}\right) \\
&= \quad \frac{\log m(k)! - \sum_{i \in \mathcal{I}} \log\lfloor p_i k \rfloor !}{-\log d} \ + \ \overline{s}_1(C)\left(1 - \frac{\log c}{\log d}\right) \qquad \text{by (5.5)} \\
&\geqslant \quad \frac{m(k)\log m(k) - m(k) - \sum_{i \in \mathcal{I}} \Big(\lfloor p_i k \rfloor \log\lfloor p_i k \rfloor - \lfloor p_i k \rfloor + \log\lfloor p_i k \rfloor\Big)}{-\log d} \\
&\qquad + \overline{s}_1(C)\left(1 - \frac{\log c}{\log d}\right) \qquad \text{by Stirling's approximation (5.4)} \\
&= \quad \frac{m(k)\log m(k) - \sum_{i \in \mathcal{I}} \lfloor p_i k \rfloor \log\lfloor p_i k \rfloor}{-\log d} \ + \ \overline{s}_1(C)\left(1 - \frac{\log c}{\log d}\right) \\
&\qquad + \frac{\sum_{i \in \mathcal{I}} \log\lfloor p_i k \rfloor}{\log d} \\
&\geqslant \quad \frac{m(k)\log m(k) - \sum_{i \in \mathcal{I}} \lfloor p_i k \rfloor \log k c_i^{\overline{s}_1(C)} d_i^{\overline{s}_A(\mathcal{I}) - \overline{s}_1(C)}}{-\log d} \ + \ \overline{s}_1(C)\left(1 - \frac{\log c}{\log d}\right) \\
&\qquad + \frac{\sum_{i \in \mathcal{I}} \log\lfloor p_i k \rfloor}{\log d} \\
&\geqslant \quad \frac{-\sum_{i \in \mathcal{I}} \lfloor p_i k \rfloor \log c_i^{\overline{s}_1(C)} d_i^{\overline{s}_A(\mathcal{I}) - \overline{s}_1(C)}}{-\log d} \ + \ \overline{s}_1(C)\left(1 - \frac{\log c}{\log d}\right) \\
&\qquad + \frac{\sum_{i \in \mathcal{I}} \log\lfloor p_i k \rfloor - m(k)\log(m(k)/k)}{\log d} \\
&= \quad \overline{s}_1(C)\frac{-\sum_{i \in \mathcal{I}} \lfloor p_i k \rfloor \log c_i}{-\log d} \ + \ \Big(\overline{s}_A(\mathcal{I}) - \overline{s}_1(C)\Big)\frac{-\sum_{i \in \mathcal{I}} \lfloor p_i k \rfloor \log d_i}{-\log d} \\
&\qquad + \overline{s}_1(C)\left(1 - \frac{\log c}{\log d}\right) \ + \ \frac{\sum_{i \in \mathcal{I}} \log\lfloor p_i k \rfloor - m(k)\log(m(k)/k)}{\log d} \\
&= \quad \overline{s}_1(C)\frac{\log c}{\log d} \ + \ \Big(\overline{s}_A(\mathcal{I}) - \overline{s}_1(C)\Big) \ + \ \overline{s}_1(C)\left(1 - \frac{\log c}{\log d}\right) \\
&\qquad + \frac{\sum_{i \in \mathcal{I}} \log\lfloor p_i k \rfloor - m(k)\log(m(k)/k)}{\log d} \\
&= \quad \overline{s}_A(\mathcal{I}) \ + \ \frac{\sum_{i \in \mathcal{I}} \log\lfloor p_i k \rfloor - m(k)\log(m(k)/k)}{\log d} \\
&\to \quad \overline{s}_A(\mathcal{I})
\end{aligned}$$

as $k \to \infty$. It follows that for any $\varepsilon > 0$, we can choose k large enough to ensure that the IFS $\mathbb{J}_k = \{S_i\}_{i \in \mathcal{J}_k}$ satisfies the properties required by $\mathbb{J}_\varepsilon$, which completes the proof. □

We can now complete the proof of the lower bound for the upper box dimension in Theorem 5.1. We will prove this in the case when F_C is an inhomogeneous self-affine carpet in the horizontal class or in the mixed class with $\overline{s}_A \geqslant \overline{s}_B$. The other cases can clearly be shown by a symmetric argument.

Proof. We wish to show that $\overline{\dim}_{\mathrm{B}} F_C \geqslant \max\{\overline{s}_A, \overline{\dim}_{\mathrm{B}} C\}$. If $\overline{s}_1(C) \leqslant s_1(F_\emptyset)$, then the result follows by the monotonicity of upper box dimension since in this case $\overline{s}_A \leqslant \overline{\dim}_{\mathrm{B}} F$. If $\overline{s}_1(C) > s_1(F_\emptyset)$, then we may apply Propositions 5.6-5.7 in the following way. Let $\varepsilon > 0$. Then by Proposition 5.7 there exists a subsystem $\mathcal{J}_\varepsilon$ of the type considered in Proposition 5.6 for which the number $\overline{s}_A = \overline{s}_A(\mathcal{J}_\varepsilon)$ defined by the system $\mathcal{J}_\varepsilon$ is no more than ε smaller than the number $\overline{s}_A = \overline{s}_A(\mathcal{I})$ defined for the original system $\mathcal{I}$. Writing $F_C(\mathcal{J}_\varepsilon)$ for the attractor of the IFS corresponding to $\mathcal{J}_\varepsilon$, it follows from Proposition 5.6 that

$$\overline{\dim}_{\mathrm{B}} F_C \ \geqslant\ \overline{\dim}_{\mathrm{B}} F_C(\mathcal{J}_\varepsilon) \ \geqslant\ \overline{s}_A(\mathcal{J}_\varepsilon) \ \geqslant\ \overline{s}_A(\mathcal{I}) - \varepsilon$$

and letting ε tend to zero completes the proof. □

6 Dimension and measure for typical random fractals

6.1 Introduction

In this chapter we consider the dimension and measure of typical attractors of very general random iterated function systems (RIFSs). Much work has been done on computing the 'almost sure' dimensions of these random attractors, where 'almost sure' refers to a probability measure on the sample space Ω induced from a probability vector associated with the finite list of IFSs, see Section 1.3.5. One expects the dimension to be 'some sort of weighted average' of the dimensions corresponding to the attractors of the deterministic IFSs. Here we consider a *topological* approach, based on Baire category, to computing the generic dimensions and obtain results in stark contrast to those obtained using the probabilistic approach. We are able to obtain very general results, only requiring that our maps are bi-Lipschitz and assuming no separation conditions. We compute the typical Hausdorff, packing and box dimensions of the random attractors (in the sense of Baire) and also study the typical Hausdorff and packing measures with respect to different gauge functions. Finally, we give a number of illustrative examples based on self-affine carpets.

We find that the *dimensions* of typical attractors behave rather well. In particular, the typical Hausdorff and lower box dimension are always as small as possible and the typical packing and upper box dimensions are always as large as possible. In comparison, the typical Hausdorff and packing *measures* behave rather badly. We provide examples where the typical Hausdorff measure in the critical dimension is as small as possible and examples where it is as large as possible (with similar examples concerning packing measure). We find that in the simpler setting of random self-similar sets, the behaviour of the typical Hausdorff and packing measures is more predictable.

6.1.1 The topological approach to randomness

In this chapter we will investigate the generic dimension and measure of F_ω from a topological point of view using Baire category. In this section we will recall the basic definitions and theorems.

Let (X, d) be a complete metric space. A set $N \subseteq X$ is *nowhere dense* if for all $x \in N$ and for all $r > 0$ there exists a point $y \in X \setminus N$ and $t > 0$ such that

$$B(y, t) \subseteq B(x, r) \setminus N.$$

A set M is said to be *of the first category*, or, *meagre*, if it can be written as a countable union of nowhere dense sets. We think of a meagre set as being *small* and the complement of a meagre set as being *big*. A set $T \subseteq X$ is *residual* or *co-meagre*, if $X \setminus T$ is meagre. A property is called *typical* if the set of points which have the property is residual. In Section 6.3 we will use the following theorem to test for typicality without mentioning it explicitly.

Theorem 6.1. *In a complete metric space, a set T is residual if and only if T contains a countable intersection of open dense sets or, equivalently, T contains a dense G_δ subset of X.*

Proof. See [Ox]. □

In order to consider typical properties of members of Ω, we need to topologize Ω in a suitable way. We do this by equipping it with the metric d_Ω where, for $u = (u_1, u_2, \dots) \neq v = (v_1, v_2, \dots) \in \Omega$,

$$d_\Omega(u, v) = 2^{-k}$$

where $k = \min\{n \in \mathbb{N} : u_n \neq v_n\}$. The space (Ω, d_Ω) is complete. For a more detailed account of Baire category the reader is referred to [Ox].

It is worth noting that one could also formulate the topological approach using the set $\{F_\omega : \omega \in \Omega\}$, equipped with the Hausdorff metric, instead of Ω. In fact, this leads to an equivalent analysis but, since we do not use this approach directly, we defer discussion of it until Section 6.6 (9).

6.2 Results

In this section we state our results. In Section 6.2.1 we state results which apply in very general circumstances, namely, the random iterated function systems introduced in Section 1.1. Theorem 6.2 is the main result of the chapter and gives the typical Hausdorff, packing and upper and lower box dimensions of F_ω and, furthermore, gives sufficient conditions for the typical Hausdorff and packing measures with respect to any (doubling) gauge function to be zero or infinite. In Section 6.2.2 we specialise to the self-similar setting.

6.2.1 Results in the general setting

Let $\mathbb{I} = \{\mathbb{I}_1, \dots, \mathbb{I}_N\}$ be a RIFS where all the maps involved are bi-Lipschitz. This is a very general setting and includes all random self-similar sets and random self-affine sets as well as many other nonlinear examples. Our main result is the following.

Theorem 6.2. *Let $G : (0, \infty) \to (0, \infty)$ be a gauge function.*

(1) If $\inf_{u \in \Omega} \mathcal{H}^G(F_u) = 0$, then for a typical $\omega \in \Omega$, we have $\mathcal{H}^G(F_\omega) = 0$;

(2) If G is doubling and $\sup_{u \in \Omega} \mathcal{P}^G(F_u) = \infty$, then for a typical $\omega \in \Omega$, we have $\mathcal{P}^G(F_\omega) = \infty$;

(3) The typical Hausdorff dimension is infimal, i.e., for a typical $\omega \in \Omega$, we have

$$\dim_{\mathrm{H}} F_\omega = \inf_{u \in \Omega} \dim_{\mathrm{H}} F_u;$$

(4) The packing dimension and upper box dimension are supremal and, in fact, for a typical $\omega \in \Omega$, we have

$$\overline{\dim}_{\mathrm{B}} F_\omega = \dim_{\mathrm{P}} F_\omega = \sup_{u \in \Omega} \overline{\dim}_{\mathrm{B}} F_u = \sup_{u \in \Omega} \dim_{\mathrm{P}} F_u;$$

(5) The lower box dimension is infimal, i.e, for a typical $\omega \in \Omega$, we have

$$\underline{\dim}_{\mathrm{B}} F_\omega = \inf_{u \in \Omega} \underline{\dim}_{\mathrm{B}} F_u.$$

We will prove Theorem 6.2 part (1) in Section 6.3.2; part (2) in Section 6.3.3; and part (5) in Section 6.3.4. Choosing G such that $G(t) = t^s$, part (3) follows from part (1) and part (4) follows from part (2) combined with the observation that the packing and upper box dimension coincide for all random attractors, see Lemma 6.10.

At first sight, it is slightly unsatisfactory that in Theorem 6.2 part (1) we do not get a precise value for the typical Hausdorff measure if the infimal Hausdorff measure is positive and finite; and similarly, in part (2) we do not get a precise value for the typical packing measure if the supremal packing measure is positive and finite. In keeping with the rest of the results and what is 'usually' expected when dealing with Baire category, one might expect that either: the typical Hausdorff measure will be the infimal value and the typical packing measure will be the supremal value; or, even though F_ω will typically be 'small' in terms of Hausdorff dimension and 'large' in terms of packing dimension, due to the influence of deterministic IFSs with non-extremal attractors, they will be 'large' in terms of Hausdorff measure and 'small' in terms of packing measure. Surprisingly, both of these phenomena are possible. In the following two theorems we identify a large class of RIFS where the second type of behaviour occurs. Theorem 6.4 refers to Hausdorff measure and Theorem 6.5 refers to packing measure. Before stating the result, we need to introduce a separation condition.

Definition 6.3. *Let μ be a Borel measure supported on X. We say that $\mathbb{I}$ satisfies the* μ-measure separated condition (μ-MSC)*, if, for all $\omega \in \Omega$, $l \in D$ and $i, j \in \mathcal{I}_l$ with $i \neq j$, we have*

$$\mu\big(S_{l,i}(F_\omega) \cap S_{l,j}(F_\omega)\big) = 0.$$

The μ-MSC means that μ will be additive on the subsets of F_ω corresponding to images of finite (distinct) sequences of maps, $S_{\omega_1,i_1}, \dots, S_{\omega_k,i_k}$. We will use the μ-MSC with μ equal to either the Hausdorff or packing measure.

Theorem 6.4. *Write $h = \inf_{u\in\Omega} \dim_{\mathrm{H}} F_u$ and assume that $\mathbb{I}$ satisfies the $\mathcal{H}^h$-MSC and that there exists $v = (v_1, v_2, \dots) \in \Omega$ such that*

$$\lim_{l\to\infty} \sum_{j_1\in\mathcal{I}_{v_1},\dots,j_l\in\mathcal{I}_{v_l}} \mathrm{Lip}^-(S_{v_1,j_1} \circ \dots \circ S_{v_l,j_l})^h = \infty. \tag{6.1}$$

Then,

(1) If $\inf_{u\in\Omega} \mathcal{H}^h(F_u) = 0$, then for a typical $\omega \in \Omega$, we have $\mathcal{H}^h(F_\omega) = 0$;

(2) If $\inf_{u\in\Omega} \mathcal{H}^h(F_u) > 0$, then for a typical $\omega \in \Omega$, we have $\mathcal{H}^h(F_\omega) = \infty$.

Note that part (1) follows from Theorem 6.2 without the additional assumptions given above. We will prove Theorem 6.4 (2) in Section 6.3.5. Although condition (6.1) seems a little contrived, what it really means is that, for some $v \in \Omega$, we can give a simple lower bound for the Hausdorff dimension of F_v which is strictly bigger than the infimal Hausdorff dimension, h.

Theorem 6.5. *Write $p = \sup_{u\in\Omega} \dim_{\mathrm{P}} F_u$ and assume that there exists $v = (v_1, v_2, \dots) \in \Omega$ such that*

$$\lim_{k\to\infty} \sum_{j_1\in\mathcal{I}_{v_1},\dots,j_k\in\mathcal{I}_{v_k}} \mathrm{Lip}^+(S_{v_1,j_1} \circ \dots \circ S_{v_k,j_k})^p = 0. \tag{6.2}$$

Then,

(1) If $\sup_{u\in\Omega} \mathcal{P}^p(F_u) = \infty$, then for a typical $\omega \in \Omega$, we have $\mathcal{P}^p(F_\omega) = \infty$;

(2) If $\sup_{u\in\Omega} \mathcal{P}^p(F_u) < \infty$, then for a typical $\omega \in \Omega$, we have $\mathcal{P}^p(F_\omega) = 0$.

Note that in Theorem 6.5 we do not require any separation conditions. Part (1) follows from Theorem 6.2 without the additional assumptions given above. We will prove Theorem 6.5 (2) in Section 6.3.6. Similar to above, condition (6.2) seems a little contrived at first sight but what it really means is that, for some $v \in \Omega$, we can give a simple upper bound for the packing dimension of F_v which is strictly smaller than the supremal packing dimension, p.

With the previous two theorems in mind, one might be tempted to think that something much more general is true. Namely, that for $s \geqslant 0$, we might have

(1) If $\inf_{u\in\Omega} \mathcal{H}^s(F_u) > 0$, then for a typical $\omega \in \Omega$, we have $\mathcal{H}^s(F_\omega) = \sup_{u\in\Omega} \mathcal{H}^s(F_u)$;

(2) If $\sup_{u\in\Omega} \mathcal{P}^s(F_u) < \infty$, then for a typical $\omega \in \Omega$, we have $\mathcal{P}^s(F_\omega) = \inf_{u\in\Omega} \mathcal{P}^s(F_u)$.

However, this is false. We will demonstrate this by constructing two simple examples in Section 6.4.1. This 'bad behaviour' of the typical packing and Hausdorff measures disappears to a certain extent if the mappings in the RIFS are similarities. This idea will be developed in the following section.

6.2.2 Results in the self-similar setting

In this section we extend the results of the previous section in the self-similar setting. It turns out that for random self-similar sets we can obtain more precise information and, furthermore, many of the strange phenomena which we observe in the general setting no longer occur. The first example of this is that, given the UOSC, the dimensions of F_ω are bounded by the dimensions of the attractors of the deterministic IFSs. This allows us to get our hands on the extremal quantities, see Theorem 6.6. Unfortunately, this rather neat property does not always hold in the general situation. In Section 6.4.3 we will give an example of a RIFS satisfying the UOSC for which the infimal (and thus typical) Hausdorff dimension is strictly less than the minimum Hausdorff dimension of the attractors of the deterministic IFSs. Secondly, given the UOSC and certain measure separation, we can compute the exact value of the typical Hausdorff and packing measure, see Theorem 6.7, which we are unable to do in the general situation.

Throughout this section let $\underline{\mathbb{I}}$ be a RIFS consisting of finitely many deterministic IFSs of similarity mappings on $\mathbb{R}^n$. For each $i \in D$, let s_i be the solution of

$$\sum_{j \in \mathcal{I}_i} \mathrm{Lip}(S_{i,j})^{s_i} = 1$$

and write $s_{\min} = \min_{i \in D} s_i$ and $s_{\max} = \max_{i \in D} s_i$.

Theorem 6.6. *Assume the UOSC is satisfied. Then*

(1) $0 < \sup_{\omega \in \Omega} \mathcal{P}^{s_{\max}}(F_\omega) < \infty$;

(2) $\sup_{\omega \in \Omega} \dim_{\mathrm{P}} F_\omega = \sup_{\omega \in \Omega} \overline{\dim}_{\mathrm{B}} F_\omega = s_{\max}$;

(3) $0 < \inf_{\omega \in \Omega} \mathcal{H}^{s_{\min}}(F_\omega) < \infty$;

(4) $\inf_{\omega \in \Omega} \dim_{\mathrm{H}} F_\omega = \inf_{\omega \in \Omega} \underline{\dim}_{\mathrm{B}} F_\omega = s_{\min}$.

We will prove Theorem 6.6 parts (1) and (3) in Section 6.3.7. Part (2) follows from part (1) and part (4) follows from part (3). Given certain measure separation, we can also compute the exact packing and Hausdorff measure for typical F_ω. Write $\mathcal{H}_{\min} = \inf_{\omega \in \Omega} \mathcal{H}^{s_{\min}}(F_\omega)$ and $\mathcal{P}_{\max} = \sup_{\omega \in \Omega} \mathcal{P}^{s_{\max}}(F_\omega)$.

Theorem 6.7. *Assume that $\underline{\mathbb{I}}$ satisfies the UOSC and the $\mathcal{P}^{s_{\min}}$-MSC. Then*

(1) If $s_{\min} = s_{\max} = s$, then for a typical $\omega \in \Omega$,

$$\dim_{\mathrm{H}} F_\omega = \dim_{\mathrm{P}} F_\omega = s$$

and

$$0 < \mathcal{H}^s(F_\omega) = \mathcal{H}_{\min} \leqslant \mathcal{P}_{\max} = \mathcal{P}^s(F_\omega) < \infty;$$

(2) If $s_{\min} < s_{\max}$, then for a typical $\omega \in \Omega$,

$$\dim_{\mathrm{H}} F_\omega = s_{\min} < s_{\max} = \dim_{\mathrm{P}} F_\omega,$$

$$\mathcal{H}^{s_{\min}}(F_\omega) = \infty$$

and

$$\mathcal{P}^{s_{\max}}(F_\omega) = 0.$$

We will prove Theorem 6.7 (1) in Section 6.3.8. Note that part (2) follows immediately from Theorems 6.4 and 6.5. In Section 6.5.1 we construct a simple example where we can apply Theorem 6.7.

It is worth noting here that it is possible to give easily checkable sufficient conditions for the $\mathcal{P}^{s_{\min}}$-MSC to hold. In particular, if we say that $\underline{\mathbb{I}}$ satisfies the uniform strong open set condition (USOSC) if the UOSC is satisfied and the open set U can be chosen such that, for every $\omega \in \Omega$, we have $U \cap F_\omega \neq \emptyset$, then we can use an argument similar to that used by Lalley in [L, Section 6], to show that the $\mathcal{P}^{s_{\min}}$-MSC is satisfied. Unfortunately, the USOSC is not equivalent to the UOSC as in the deterministic case, see [Sc1].

We can also obtain a partial result concerning packing measure without assuming any separation conditions.

Theorem 6.8. *Each deterministic IFS, $\mathbb{I}_i \in \mathbb{I}$, has an attractor with dimension d_i and similarity dimension $s_i \geqslant d_i$. Assume that $s_{\min} < \max_i d_i$. Write $p = \sup_{u \in \Omega} \dim_{\mathrm{P}} F_u$. Then, for a typical $\omega \in \Omega$, $\dim_{\mathrm{P}} F_\omega = p$, but $\mathcal{P}^p(F_\omega) = 0$.*

Proof. This follows immediately from Theorem 6.5. □

6.3 Proofs

Throughout this section let $G : (0, \infty) \to (0, \infty)$ be a gauge function.

6.3.1 Preliminary observations

In this section we will gather together some simple preliminary results and observations which will be used in the subsequent sections without being mentioned explicitly. The proofs are elementary (or classical) and are omitted.

Lemma 6.9 (scaling properties)**.** *Let $\phi : X \to X$ be a bi-Lipschitz map and $F \subseteq X$. Then*

$$D^-(G, \mathrm{Lip}^-(\phi))\, \mathcal{H}^G(F) \leqslant \mathcal{H}^G(\phi(F)) \leqslant D^+(G, \mathrm{Lip}^+(\phi))\, \mathcal{H}^G(F),$$

$$D^-(G, \mathrm{Lip}^-(\phi))\, \mathcal{P}_0^G(F) \leqslant \mathcal{P}_0^G(\phi(F)) \leqslant D^+(G, \mathrm{Lip}^+(\phi))\, \mathcal{P}_0^G(F)$$

and

$$D^-(G, \mathrm{Lip}^-(\phi))\, \mathcal{P}^G(F) \leqslant \mathcal{P}^G(\phi(F)) \leqslant D^+(G, \mathrm{Lip}^+(\phi))\, \mathcal{P}^G(F).$$

In particular, using the standard gauge,

$$\mathrm{Lip}^-(\phi)^s\, \mathcal{H}^s(F) \leqslant \mathcal{H}^s(\phi(F)) \leqslant \mathrm{Lip}^+(\phi)^s\, \mathcal{H}^s(F),$$

$$\mathrm{Lip}^-(\phi)^s\, \mathcal{P}_0^s(F) \leqslant \mathcal{P}_0^s(\phi(F)) \leqslant \mathrm{Lip}^+(\phi)^s\, \mathcal{P}_0^s(F)$$

and

$$\mathrm{Lip}^-(\phi)^s\, \mathcal{P}^s(F) \leqslant \mathcal{P}^s(\phi(F)) \leqslant \mathrm{Lip}^+(\phi)^s\, \mathcal{P}^s(F).$$

Lemma 6.9, says that if the gauge is doubling, then mapping a set under a bi-Lipschitz map only changes the measure by a constant. Clearly, if ϕ is bi-Lipschitz, then $\dim \phi(F) = \dim F$, where dim can be any of the four dimensions used here. We can also deduce that, for all $\omega \in \Omega$, the upper box dimension and packing dimension coincide.

Lemma 6.10 (packing and upper box dimension)**.** *For all $\omega \in \Omega$, $\dim_{\mathrm{P}} F_\omega = \overline{\dim}_{\mathrm{B}} F_\omega$.*

To prove this, simply note that all balls centered in F_ω contain a bi-Lipschitz image of $F_{(\omega_k, \omega_{k+1}, \dots)}$ for some sufficiently large k and, furthermore, F_ω can be written as a finite union of bi-Lipschitz images of $F_{(\omega_k, \omega_{k+1}, \dots)}$ and since upper box dimension is finitely stable, $\overline{\dim}_{\mathrm{B}} F_{(\omega_k, \omega_{k+1}, \dots)} = \overline{\dim}_{\mathrm{B}} F_\omega$ and the result follows. See the discussion on sufficient conditions for the equality of packing and upper box dimension given in Section 1.2.4 and, in particular, Proposition 1.1.

The following lemma will allow us to approximate F_ω in $\mathcal{K}(X)$ by approximating ω in Ω, which will be of vital importance in the subsequent proofs.

Lemma 6.11 (continuity properties)**.** *The map $\Psi : \big(\Omega, d_\Omega\big) \to \big(\mathcal{K}(X), d_{\mathcal{H}}\big)$ defined by $\Psi(\omega) = F_\omega$ is continuous.*

Finally, we will state a version of the mass distribution principle which we use to estimate the Hausdorff and packing measures of random self-similar sets in Section 6.3.7.

Proposition 6.12 (mass distribution principle). *Let μ be a Borel probability measure supported on a Borel set $F \subset \mathbb{R}^n$ and let $\lambda \in (0,\infty)$. Then*

(1) If $\limsup_{r\to 0} \mu\big(B(x,r)\big)\, r^{-s} \leqslant \lambda$ for all $x \in F$, then $\mathcal{H}^s(F) \geqslant \lambda^{-1}$;

(2) If $\liminf_{r\to 0} \mu\big(B(x,r)\big)\, r^{-s} \geqslant \lambda$ for all $x \in F$, then $\mathcal{P}^s(F) \leqslant \lambda^{-1}\, 2^s$.

For a proof of this, see [F7, Proposition 2.2] or [Mat, Theorems 6.9 and 6.11].

6.3.2 Proof of Theorem 6.2 (1)

Suppose $\inf_{u\in\Omega} \mathcal{H}^G(F_u) = 0$. We will show that the set

$$H = \{\omega \in \Omega : \mathcal{H}^G(F_\omega) = 0\}$$

is residual. Writing $H_{m,n} = \{\omega \in \Omega : \mathcal{H}^G_{1/m}(F_\omega) < \frac{1}{n}\}$, we have

$$H = \bigcap_{m,n\in\mathbb{N}} H_{m,n},$$

so it suffices to prove that each $H_{m,n}$ is open and dense in (Ω, d_Ω). Fix $m, n \in \mathbb{N}$.

(i) $H_{m,n}$ is open.

Let $\omega \in H_{m,n}$. It follows that there exists a finite $(1/m)$-cover of F_ω by open sets, $\{U_i\}$, satisfying

$$\sum_i G(|U_i|) < \tfrac{1}{n}.$$

Let $\mathcal{U} = \partial\big(\cup_i U_i\big)$ be the boundary of the union of the covering sets, $\{U_i\}$, and let

$$\eta = \min_{x\in\mathcal{U}, y\in F_\omega} d(x,y)$$

which is strictly positive by the compactness of F_ω. Now choose $r > 0$ sufficiently small to ensure that if $u \in B(\omega, r)$, then $d_{\mathcal{H}}(F_\omega, F_u) < \eta/2$. Let $u \in B(\omega, r)$ and observe that $\{U_i\}$ is a $(1/m)$-cover for F_u giving that $\mathcal{H}^G_{1/m}(F_u) \leqslant \sum_i G(|U_i|) < \frac{1}{n}$. It follows that $B(\omega, r) \subseteq H_{m,n}$ and that $H_{m,n}$ is open.

(ii) $H_{m,n}$ is dense.

Let $\omega = (\omega_1, \omega_2, \dots) \in \Omega$ and $\varepsilon > 0$. Choose $k \in \mathbb{N}$ such that $2^{-k} < \varepsilon$ and choose $u = (u_1, u_2, \dots) \in \Omega$ such that

$$\mathcal{H}^G(F_u) < \frac{1/n}{|\mathcal{I}_{\omega_1}| \cdots |\mathcal{I}_{\omega_k}|}.$$

Let $v = (\omega_1, \dots, \omega_k, u_1, u_2, \dots)$. It follows that $d_\Omega(\omega, v) < \varepsilon$ and, since

$$F_v = \bigcup_{j_1\in\mathcal{I}_{\omega_1},\dots,j_k\in\mathcal{I}_{\omega_k}} S_{\omega_1,j_1} \circ \cdots \circ S_{\omega_k,j_k}(F_u),$$

it follows that

$$\begin{aligned}
\mathcal{H}^G_{1/m}(F_v) \;\leqslant\; \mathcal{H}^G(F_v) \;&=\; \mathcal{H}^G\Bigg(\bigcup_{j_1\in\mathcal{I}_{\omega_1},\dots,j_k\in\mathcal{I}_{\omega_k}} S_{\omega_1,j_1} \circ \cdots \circ S_{\omega_k,j_k}(F_u)\Bigg) \\
&\leqslant\; \sum_{j_1\in\mathcal{I}_{\omega_1},\dots,j_k\in\mathcal{I}_{\omega_k}} \mathcal{H}^G\big(F_u\big) \\
&\leqslant\; |\mathcal{I}_{\omega_1}| \cdots |\mathcal{I}_{\omega_k}|\, \mathcal{H}^G\big(F_u\big) \\
&<\; 1/n
\end{aligned}$$

and so $v \in H_{m,n}$, proving that $H_{m,n}$ is dense. □

6.3.3 Proof of Theorem 6.2 (2)

Assume that G is a doubling gauge and that $\sup_{u\in\Omega} \mathcal{P}^G(F_u) = \infty$. We will show that the set

$$P = \{\omega \in \Omega : \mathcal{P}^G(F_\omega) = \infty\}$$

is residual. The extra step in the definition of packing measure causes it to be more awkward to work with than Hausdorff measure. To circumvent these difficulties, we need the following two technical lemmas.

Lemma 6.13. *Suppose $F \subset X$ is such that for all open V which intersect F, $\mathcal{P}_0^G(F \cap V) = \infty$. Then $\mathcal{P}^G(F) = \infty$.*

Proof. Let $\{F_i\}_i$ be a countable sequence of sets such that $F \subset \cup_i F_i$. The Baire category Theorem implies that for some i and some open set V, $F \cap V \subseteq \overline{F_i}$ and hence, since packing pre-measure is stable under taking closures, $\mathcal{P}_0^G(F_i) = \mathcal{P}_0^G(\overline{F_i}) = \infty$. This means that, for every countable cover of F by closed sets, at least one of the closed sets must have infinite packing pre-measure, proving the result. $\square$

We will use Lemma 6.13 to prove the following Lemma, which will allow us to work with packing pre-measure instead of packing measure.

Lemma 6.14. *We have $P = \{\omega \in \Omega : \mathcal{P}_0^G(F_\omega) = \infty\}$.*

Proof. It is clear that $P \subseteq \{\omega \in \Omega : \mathcal{P}_0^G(F_\omega) = \infty\}$. We will now prove the opposite inclusion. Let $\omega \in \Omega$ be such that $\mathcal{P}_0^G(F_\omega) = \infty$ and let V be an open set which intersects F_ω. Choose k large enough to ensure that for some $i_1 \in \mathcal{I}_{\omega_1}, \dots, i_k \in \mathcal{I}_{\omega_1}$ we have

$$S_{\omega_1,i_1} \circ \cdots \circ S_{\omega_k,i_k}\big(F_{(\omega_{k+1},\omega_{k+2},\dots)}\big) \subseteq F \cap V.$$

Write $\phi = S_{\omega_1,i_1} \circ \cdots \circ S_{\omega_k,i_k}$ and $u = (\omega_{k+1}, \omega_{k+2}, \dots)$. Since packing pre-measure is finitely additive, we have

$$\begin{aligned}
\infty = \mathcal{P}_0^G(F_\omega) &= \mathcal{P}_0^G\left(\bigcup_{i_1\in\mathcal{I}_{\omega_1},\dots,i_k\in\mathcal{I}_{\omega_k}} S_{\omega_1,i_1} \circ \cdots \circ S_{\omega_k,i_k}(F_u)\right) \\
&\leqslant \sum_{i_1\in\mathcal{I}_{\omega_1},\dots,i_k\in\mathcal{I}_{\omega_k}} \mathcal{P}_0^G(F_u) \\
&\leqslant |\mathcal{I}_{\omega_1}| \cdots |\mathcal{I}_{\omega_k}|\ \mathcal{P}_0^G(F_u)
\end{aligned}$$

and therefore

$$\begin{aligned}
\mathcal{P}_0^G(F \cap V) &\geqslant \mathcal{P}_0^G(\phi(F_u)) \\
&\geqslant D^-\big(G, \mathrm{Lip}^-(\phi)\big)\, \mathcal{P}_0^G(F_u) \\
&= \infty.
\end{aligned}$$

Finally, by Lemma 6.13, we have that $\mathcal{P}^G(F_\omega) = \infty$ and hence $\omega \in P$. $\square$

Writing $P_{m,n} = \{\omega \in \Omega : \mathcal{P}_{0,\,1/m}^G(F_\omega) > n\}$, it follows from Lemma 6.14 that

$$P = \{\omega \in \Omega : \mathcal{P}_0^G(F_\omega) = \infty\} = \bigcap_{m,n\in\mathbb{N}} P_{m,n},$$

so it suffices to prove that each $P_{m,n}$ is open and dense in (Ω, d_Ω). Fix $m, n \in \mathbb{N}$.

(i) $P_{m,n}$ is open.

Let $\omega \in P_{m,n}$. It follows that there exists a finite centered $(1/m)$-packing of F_ω by closed balls, $\{U_i\}$, satisfying

$$\sum_i G(|U_i|) > n.$$

Let

$$\eta = \min_{i \neq j} \min_{x \in U_i, y \in U_j} d(x,y)$$

which is strictly positive since the sets U_i are closed. Now choose $r > 0$ sufficiently small to ensure that, if $u \in B(\omega, r)$, then $d_{\mathcal{H}}(F_\omega, F_u) < \eta/2$ and fix such a $u \in B(\omega, r)$. It follows that we can find a centered $(1/m)$-packing, $\{\tilde{U}_i\}$, of F_u, where $\tilde{U}_i$ is centered in F_u and has the same diameter as U_i. It follows that $\mathcal{P}^G_{0,\,1/m}(F_u) \geqslant \sum_i G(|U_i|) > n$ and therefore $B(\omega, r) \subseteq P_{m,n}$, proving that $P_{m,n}$ is open.

(ii) $P_{m,n}$ is dense.

Let $\omega = (\omega_1, \omega_2, \dots) \in \Omega$ and $\varepsilon > 0$. Choose $k \in \mathbb{N}$ such that $2^{-k} < \varepsilon$ and choose $u = (u_1, u_2, \dots) \in \Omega$ such that

$$\mathcal{P}^G_0(F_u) \geqslant \frac{n}{\max_{j_1 \in \mathcal{I}_{\omega_1}, \dots, j_k \in \mathcal{I}_{\omega_k}} D\big(G, \mathrm{Lip}^-\big(S_{\omega_1, j_1} \circ \dots \circ S_{\omega_k, j_k}\big)\big)}$$

Let $v = (\omega_1, \dots, \omega_k, u_1, u_2, \dots)$. It follows that $d_\Omega(\omega, v) < \varepsilon$ and, since

$$F_v = \bigcup_{j_1 \in \mathcal{I}_{\omega_1}, \dots, j_k \in \mathcal{I}_{\omega_k}} S_{\omega_1, j_1} \circ \dots \circ S_{\omega_k, j_k}(F_u),$$

it follows that

$$\begin{aligned}
\mathcal{P}^G_{0,\,1/m}(F_v) \geqslant \mathcal{P}^G_0(F_v) &= \mathcal{P}^G_0\Bigg(\bigcup_{j_1 \in \mathcal{I}_{\omega_1}, \dots, j_k \in \mathcal{I}_{\omega_k}} S_{\omega_1, j_1} \circ \dots \circ S_{\omega_k, j_k}(F_u)\Bigg) \\
&\geqslant \max_{j_1 \in \mathcal{I}_{\omega_1}, \dots, j_k \in \mathcal{I}_{\omega_k}} \mathcal{P}^G_0\big(S_{\omega_1, j_1} \circ \dots \circ S_{\omega_k, j_k}(F_u)\big) \\
&\geqslant \max_{j_1 \in \mathcal{I}_{\omega_1}, \dots, j_k \in \mathcal{I}_{\omega_k}} D\big(G, \mathrm{Lip}^-\big(S_{\omega_1, j_1} \circ \dots \circ S_{\omega_k, j_k}\big)\big)\, \mathcal{P}^G_0(F_u) \\
&\geqslant n
\end{aligned}$$

and so $v \in P_{m,n}$, proving that $P_{m,n}$ is dense. □

6.3.4 Proof of Theorem 6.2 (5)

It is well-known that lower box dimension is not finitely stable, see [F8, Chapter 3], i.e., it is not true in general that $\underline{\dim}_{\mathrm{B}} E \cup F \leqslant \max\{\underline{\dim}_{\mathrm{B}} E, \underline{\dim}_{\mathrm{B}} F\}$. To get around this problem in the following proof, we begin with a simple technical lemma.

Lemma 6.15. *Let $F \subset X$ be such that $\underline{\dim}_{\mathrm{B}} F = s$ and let $\{\phi_i\}_{i \in \mathcal{S}}$ be a finite collection of Lipschitz contractions. Then*

$$\underline{\dim}_{\mathrm{B}} \bigcup_{i \in \mathcal{S}} \phi_i(F) \leqslant s.$$

Proof. For all $\delta > 0$ we have

$$N_\delta\Bigg(\bigcup_{i \in \mathcal{S}} \phi_i(F)\Bigg) \leqslant \sum_{i \in \mathcal{S}} N_\delta\big(\phi_i(F)\big) \leqslant \sum_{i \in \mathcal{S}} N_{\delta/\mathrm{Lip}^+(\phi_i)}(F) \leqslant |\mathcal{S}| N_\delta(F).$$

Taking logs, dividing by $-\log \delta$ and computing the limes inferior completes the proof. □

We now turn to the proof of Theorem 6.2 (5). Let $b = \inf_{u\in\Omega} \underline{\dim}_{\mathrm{B}} F_u$. We will show that the set

$$B = \{\omega \in \Omega : \underline{\dim}_{\mathrm{B}} F_\omega \leqslant b\}$$

is residual, from which Theorem 6.2 (5) follows. Writing

$$B_n = \bigcup_{\delta\in(0,1/n)} \Big\{\omega \in \Omega : N_\delta(F_\omega) \leqslant \delta^{-(b+\frac{1}{n})}\Big\},$$

we have

$$B = \bigcap_{n\in\mathbb{N}} \bigcup_{\delta\in(0,1/n)} \Big\{\omega \in \Omega : \frac{\log N_\delta(F_\omega)}{-\log\delta} \leqslant b + \tfrac{1}{n}\Big\} = \bigcap_{n\in\mathbb{N}} B_n,$$

so it suffices to prove that each B_n is open and dense in (Ω, d_Ω). Fix $n \in \mathbb{N}$.

(i) B_n is open.

Let $\omega \in B_n$. It follows that for some $\delta < 1/n$ there exists a δ-cover of F_ω by fewer than $\delta^{-(b+\frac{1}{n})}$ open sets, $\{U_i\}$. Let $\mathcal{U} = \partial\big(\cup_i U_i\big)$ be the boundary of the union of the covering sets, $\{U_i\}$, and let

$$\eta = \min_{x\in\mathcal{U}, y\in F_\omega} d(x,y)$$

which is strictly positive by the compactness of F_ω. Now choose $r > 0$ sufficiently small to ensure that if $u \in B(\omega,r)$, then $d_{\mathcal{H}}(F_\omega, F_u) < \eta/2$. Let $u \in B(\omega,r)$ and observe that $\{U_i\}$ is a δ-cover for F_u giving that $N_\delta(F_u) \leqslant \delta^{-(b+\frac{1}{n})}$. It follows that $B(\omega,r) \subseteq B_n$ and therefore B_n is open.

(ii) B_n is dense.

Let $\omega = (\omega_1, \omega_2, \dots) \in \Omega$ and $\varepsilon > 0$. Let $u = (u_1, u_2, \dots) \in \Omega$ be such that $\underline{\dim}_{\mathrm{B}} F_u < b + 1/n$. Now choose $k \in \mathbb{N}$ such that $2^{-k} < \varepsilon$ and let $v = (\omega_1, \dots, \omega_k, u_1, u_2, \dots)$. It follows that $d_\Omega(v,\omega) < \varepsilon$ and, furthermore,

$$F_v = \bigcup_{j_1\in\mathcal{I}_{\omega_1},\dots,j_k\in\mathcal{I}_{\omega_k}} S_{\omega_1,j_1} \circ \cdots \circ S_{\omega_k,j_k}(F_u).$$

and since, for all $j_1 \in \mathcal{I}_{\omega_1}, \dots, j_k \in \mathcal{I}_{\omega_k}$ the map $S_{\omega_1,j_1} \circ \cdots \circ S_{\omega_k,j_k}$ is a Lipschitz contraction, it follows from Lemma 6.15 that $\underline{\dim}_{\mathrm{B}} F_v \leqslant \underline{\dim}_{\mathrm{B}} F_u < b + 1/n$ and so $v \in B_n$, proving that B_n is dense. □

6.3.5 Proof of Theorem 6.4 (2)

Write $h = \inf_{u\in\Omega} \dim_{\mathrm{H}} F_u$ and assume that $\inf_{u\in\Omega} \mathcal{H}^h(F_u) = \mathcal{H}_0 > 0$, $v = (v_1, v_2, \dots) \in \Omega$ satisfies condition (6.1) and that the RIFS satisfies the $\mathcal{H}^h$-MSC. We will show the set

$$M = \{\omega \in \Omega : \mathcal{H}^h(F_\omega) < \infty\}$$

is meagre, from which the result follows. Writing $M_n = \{\omega \in \Omega : \mathcal{H}^h(F_\omega) < n\}$, we have

$$M = \bigcup_{n\in\mathbb{N}} M_n,$$

so it suffices to show that each M_n is nowhere dense. Fix $n \in \mathbb{N}$, $\omega \in M_n$ and $r > 0$. Now choose $k \in \mathbb{N}$ such that $2^{-k} < r$. It follows that the open ball $B_l = B\big((\omega_1, \dots, \omega_k, v_1, v_2, \dots),\, 2^{-l}\big)$ is contained in $B(\omega, r)$ for all $l > k$. Let $u \in B_l$, and note that

$$u = (\omega_1, \dots, \omega_k, v_1, \dots, v_{l-k}, u_1, u_2, \dots)$$

for some $(u_1, u_2, \dots) \in \Omega$. Noting that the RIFS satisfies the $\mathcal{H}^h$-MSC and that Lip^- is supermultiplicative, we have

$$\mathcal{H}^h(F_u)$$

$$= \mathcal{H}^h\left(\bigcup_{i_1 \in \mathcal{I}_{\omega_1}, \dots, i_k \in \mathcal{I}_{\omega_k}} \bigcup_{j_1 \in \mathcal{I}_{v_1}, \dots, j_{l-k} \in \mathcal{I}_{v_{l-k}}} S_{\omega_1, i_1} \circ \dots \circ S_{\omega_k, i_k} \circ S_{v_1, j_1} \circ \dots \circ S_{v_{l-k}, j_{l-k}}\big(F_{(u_1, u_2, \dots)}\big) \right)$$

$$= \sum_{i_1 \in \mathcal{I}_{\omega_1}, \dots, i_k \in \mathcal{I}_{\omega_k}} \sum_{j_1 \in \mathcal{I}_{v_1}, \dots, j_{l-k} \in \mathcal{I}_{v_{l-k}}} \mathcal{H}^h\left(S_{\omega_1, i_1} \circ \dots \circ S_{\omega_k, i_k} \circ S_{v_1, j_1} \circ \dots \circ S_{v_{l-k}, j_{l-k}}\big(F_{(u_1, u_2, \dots)}\big) \right)$$

$$\geqslant \sum_{i_1 \in \mathcal{I}_{\omega_1}, \dots, i_k \in \mathcal{I}_{\omega_k}} \sum_{j_1 \in \mathcal{I}_{v_1}, \dots, j_{l-k} \in \mathcal{I}_{v_{l-k}}} \mathrm{Lip}^-(S_{\omega_1, i_1} \circ \dots \circ S_{\omega_k, i_k} \circ S_{v_1, j_1} \circ \dots \circ S_{v_{l-k}, j_{l-k}})^h \ \mathcal{H}^h\big(F_{(u_1, u_2, \dots)}\big)$$

$$\geqslant \mathcal{H}_0 \left(\sum_{i_1 \in \mathcal{I}_{\omega_1}, \dots, i_k \in \mathcal{I}_{\omega_k}} \mathrm{Lip}^-(S_{\omega_1, i_1} \circ \dots \circ S_{\omega_k, i_k})^h \right) \left(\sum_{j_1 \in \mathcal{I}_{v_1}, \dots, j_{l-k} \in \mathcal{I}_{v_{l-k}}} \mathrm{Lip}^-(S_{v_1, j_1} \circ \dots \circ S_{v_{l-k}, j_{l-k}})^h \right)$$

$$\to \infty$$

as $l \to \infty$. It follows that we may choose l large enough to ensure $B_l \subseteq B(\omega, r) \setminus M_n$ and so M_n is nowhere dense. □

6.3.6 Proof of Theorem 6.5 (2)

Write $p = \sup_{u \in \Omega} \dim_{\mathrm{P}} F_u$ and assume that $\sup_{u \in \Omega} \mathcal{P}^p(F_u) = \mathcal{P}_0 < \infty$ and that $v = (v_1, v_2, \dots) \in \Omega$ satisfies condition (6.2). We will show the set

$$N = \{\omega \in \Omega : \mathcal{P}^h(F_\omega) > 0\}$$

is meagre, from which the result follows. Writing $N_n = \{\omega \in \Omega : \mathcal{P}^p(F_\omega) > 1/n\}$, we have

$$N = \bigcup_{n \in \mathbb{N}} N_n,$$

so it suffices to show that each N_n is nowhere dense. Fix $n \in \mathbb{N}$, $\omega \in N_n$ and $r > 0$. Now choose $k \in \mathbb{N}$ such that $2^{-k} < r$. It follows that the open ball $B_l = B\big((\omega_1, \dots, \omega_k, v_1, v_2, \dots), 2^{-l}\big)$ is contained in $B(\omega, r)$ for all $l > k$. Let $u \in B_l$, and note that

$$u = (\omega_1, \dots, \omega_k, v_1, \dots, v_{l-k}, u_1, u_2, \dots)$$

for some $(u_1, u_2, \dots) \in \Omega$. Noting that Lip^+ is submultiplicative, we have

$$\mathcal{P}^p(F_u)$$

$$= \mathcal{P}^p\left(\bigcup_{i_1 \in \mathcal{I}_{\omega_1}, \dots, i_k \in \mathcal{I}_{\omega_k}} \bigcup_{j_1 \in \mathcal{I}_{v_1}, \dots, j_{l-k} \in \mathcal{I}_{v_{l-k}}} S_{\omega_1, i_1} \circ \dots \circ S_{\omega_k, i_k} \circ S_{v_1, j_1} \circ \dots \circ S_{v_{l-k}, j_{l-k}}\big(F_{(u_1, u_2, \dots)}\big) \right)$$

$$\leqslant \sum_{i_1 \in \mathcal{I}_{\omega_1}, \dots, i_k \in \mathcal{I}_{\omega_k}} \sum_{j_1 \in \mathcal{I}_{v_1}, \dots, j_{l-k} \in \mathcal{I}_{v_{l-k}}} \mathcal{P}^p\left(S_{\omega_1, i_1} \circ \dots \circ S_{\omega_k, i_k} \circ S_{v_1, j_1} \circ \dots \circ S_{v_{l-k}, j_{l-k}}\big(F_{(u_1, u_2, \dots)}\big) \right)$$

$$\leqslant \sum_{i_1 \in \mathcal{I}_{\omega_1}, \dots, i_k \in \mathcal{I}_{\omega_k}} \sum_{j_1 \in \mathcal{I}_{v_1}, \dots, j_{l-k} \in \mathcal{I}_{v_{l-k}}} \mathrm{Lip}^+(S_{\omega_1, i_1} \circ \cdots \circ S_{\omega_k, i_k} \circ S_{v_1, j_1} \circ \cdots \circ S_{v_{l-k}, j_{l-k}})^p \, \mathcal{P}^p\big(F_{(u_1, u_2, \dots)}\big)$$

$$\leqslant \mathcal{P}_0 \left(\sum_{i_1 \in \mathcal{I}_{\omega_1}, \dots, i_k \in \mathcal{I}_{\omega_k}} \mathrm{Lip}^+(S_{\omega_1, i_1} \circ \cdots \circ S_{\omega_k, i_k})^p \right) \left(\sum_{j_1 \in \mathcal{I}_{v_1}, \dots, j_{l-k} \in \mathcal{I}_{v_{l-k}}} \mathrm{Lip}^+(S_{v_1, j_1} \circ \cdots \circ S_{v_{l-k}, j_{l-k}})^p \right)$$

$$\to 0$$

as $l \to \infty$. It follows that we may choose l large enough to ensure $B_l \subseteq B(\omega, r) \setminus N_n$ and so N_n is nowhere dense. □

6.3.7 Proof of Theorem 6.6

The proof of Theorem 6.6 is a standard application of the mass distribution principle, Proposition 6.12. Similar arguments can be found in, for example, [F8, Chapter 9].

For each $i \in D$, let s_i be as in Section 6.2.2 and write $c = \min_{i \in D,\, j \in \mathcal{I}_i} \mathrm{Lip}(S_{i,j})$. We will now define a mass distribution on F_ω which will be used in the subsequent proofs. First define a measure, $\mu_\omega^{\mathrm{sym}}$, on the symbolic space, $\prod_{l=1}^\infty \mathcal{I}_{\omega_l}$, by

$$\mu_\omega^{\mathrm{sym}}\Big(\big\{(j_1, j_2, \dots) : j_1 = i_1, \dots, j_k = i_k\big\}\Big) = \mathrm{Lip}(S_{\omega_1, i_1})^{s_{\omega_1}} \cdots \mathrm{Lip}(S_{\omega_k, i_k})^{s_{\omega_k}}$$

for each $(i_1, \dots, i_k) \in \prod_{l=1}^k \mathcal{I}_{\omega_l}$. Now transfer $\mu_\omega^{\mathrm{sym}}$ to a Borel probability measure μ_ω, supported on F_ω, by

$$\mu_\omega(E) = \mu_\omega^{\mathrm{sym}}\left(\Big\{(i_1, i_2, \dots) \in \prod_{l=1}^\infty \mathcal{I}_{\omega_l} \;:\; \bigcap_k S_{\omega_1, i_1} \circ \cdots \circ S_{\omega_k, i_k}(X) \in E\Big\}\right)$$

for Borel sets $E \subseteq X$.

Proof of (1)

Since each deterministic IFS satisfies the OSC, it is clear that $\sup_{\omega \in \Omega} \mathcal{P}^{s_{\max}}(F_\omega) \geqslant \sup_{\omega \in \Omega} \mathcal{H}^{s_{\max}}(F_\omega) > 0$. We will now show that $\sup_{\omega \in \Omega} \mathcal{P}^{s_{\max}}(F_\omega) < \infty$. Fix $\omega = (\omega_1, \omega_2, \dots) \in \Omega$, let $x \in F_\omega$ and $r > 0$. Now let $l \in \mathbb{N}$ and $i_1 \in \mathcal{I}_{\omega_1}, \dots, i_l \in \mathcal{I}_{\omega_l}$ be such that

$$x \in S_{\omega_1, i_1} \circ \cdots \circ S_{\omega_l, i_l}(F_\omega)$$

and

$$\mathrm{Lip}(S_{\omega_1, i_1}) \cdots \mathrm{Lip}(S_{\omega_l, i_l})|X| < r \leqslant \mathrm{Lip}(S_{\omega_1, i_1}) \cdots \mathrm{Lip}(S_{\omega_{l-1}, i_{l-1}})|X|.$$

It follows that

$$\begin{aligned}
\mu_\omega(B(x, r))\, r^{-s_{\max}} &\geqslant \mu_\omega\Big(S_{\omega_1, i_1} \circ \cdots \circ S_{\omega_l, i_l}(F_\omega)\Big)\, r^{-s_{\max}} \\
&\geqslant \mathrm{Lip}(S_{\omega_1, i_1})^{s_{\omega_1}} \cdots \mathrm{Lip}(S_{\omega_l, i_l})^{s_{\omega_l}}\, r^{-s_{\max}} \\
&\geqslant \left(\frac{\mathrm{Lip}(S_{\omega_1, i_1}) \cdots \mathrm{Lip}(S_{\omega_l, i_l})}{r}\right)^{s_{\max}} \\
&\geqslant \left(\frac{r\, c\, |X|^{-1}}{r}\right)^{s_{\max}} \\
&= \big(c/|X|\big)^{s_{\max}}
\end{aligned}$$

and by Proposition 6.12 (2) it follows that $\mathcal{P}^{s_{\max}}(F_\omega) \leqslant \big(2\,|X|/c\big)^{s_{\max}} < \infty$ and, in particular,

$$0 < \sup_{\omega \in \Omega} \mathcal{P}^{s_{\max}}(F_\omega) < \infty$$

which completes the proof. $\square$

Proof of (3)

We will need the following lemma which appears as Lemma 9.2 in [F8].

Lemma 6.16. *Let $\{V_i\}$ be a collection of disjoint open subsets of $\mathbb{R}^n$ such that each V_i contains a ball of radius $a_1 r$ and is contained in a ball of radius $a_2 r$. Then any ball, B, of radius r intersects at most $(1+2a_2)^n a_1^{-n}$ of the closures $\overline{V}_i$.*

Let U be the open set used in the UOSC and let a_1, a_2 be such that U contains a ball of radius a_1 and is contained in a ball of radius a_2. Let $\mathcal{I}^*_\omega = \bigcup_{k\in\mathbb{N}} \prod_{l=1}^k \mathcal{I}_{\omega_l}$ and, for $r > 0$, let $\mathcal{I}^r_\omega$ be an r-stopping defined by

$$\mathcal{I}^r_\omega = \big\{(i_1, i_2, \dots, i_l) \in \mathcal{I}^*_\omega \ : \ \mathrm{Lip}(S_{\omega_1,i_1}) \cdots \mathrm{Lip}(S_{\omega_l,i_l}) \leqslant r < \mathrm{Lip}(S_{\omega_1,i_1}) \cdots \mathrm{Lip}(S_{\omega_{l-1},i_{l-1}})\big\}.$$

Note that

(1) $\big\{S_{\omega_1,i_1} \circ \cdots \circ S_{\omega_l,i_l}(U) : (i_1, i_2, \dots, i_l) \in \mathcal{I}^r_\omega\big\}$ is a collection of disjoint open subsets of $\mathbb{R}^n$;

(2) Each $S_{\omega_1,i_1} \circ \cdots \circ S_{\omega_l,i_l}(U)$ contains a ball of radius $c\,a_1 r$ and is contained in a ball of radius $a_2 r$;

(3) For each $(i_1, i_2, \dots, i_l) \in \mathcal{I}^r_\omega$, we have

$$S_{\omega_1,i_1} \circ \cdots \circ S_{\omega_l,i_l}(F_{(\omega_{l+1},\omega_{l+2},\dots)}) \subseteq S_{\omega_1,i_1} \circ \cdots \circ S_{\omega_l,i_l}(\overline{U}).$$

Since each deterministic IFS satisfies the OSC, it is clear that $\inf_{\omega\in\Omega} \mathcal{H}^{s_{\min}}(F_\omega) < \infty$. We will now show that $\inf_{\omega\in\Omega} \mathcal{H}^{s_{\min}}(F_\omega) > 0$. Fix $\omega = (\omega_1, \omega_2, \dots) \in \Omega$, let $x \in F_\omega$ and $r > 0$. It follows from (1)–(3) and Lemma 6.16 that

$$\begin{aligned}
\mu_\omega(B(x,r))\, r^{-s_{\min}} &= r^{-s_{\min}}\ \mu_\omega\big(B(x,r) \cap F\big) \\
&= r^{-s_{\min}}\ \mu^{\mathrm{sym}}_\omega\Bigg(\bigg\{(i_1, i_2, \dots) \in \prod_{l=1}^\infty \mathcal{I}_{\omega_l} \ : \ \bigcap_k S_{\omega_1,i_1} \circ \cdots \circ S_{\omega_k,i_k}(X) \in B(x,r) \cap F\bigg\}\Bigg) \\
&\leqslant r^{-s_{\min}}\ \mu^{\mathrm{sym}}_\omega\Bigg(\bigcup_{\substack{(i_1,i_2,\dots,i_l)\in\mathcal{I}^r_\omega: \\ B(x,r)\cap S_{\omega_1,i_1}\circ\cdots\circ S_{\omega_l,i_l}(\overline{U})\neq\emptyset}} \big\{(j_1, j_2, \dots) : j_1 = i_1, \dots, j_l = i_l\big\}\Bigg) \\
&\leqslant r^{-s_{\min}} \sum_{\substack{(i_1,i_2,\dots,i_l)\in\mathcal{I}^r_\omega: \\ B(x,r)\cap S_{\omega_1,i_1}\circ\cdots\circ S_{\omega_l,i_l}(\overline{U})\neq\emptyset}} \mathrm{Lip}(S_{\omega_1,i_1})^{s_{\omega_1}} \cdots \mathrm{Lip}(S_{\omega_l,i_l})^{s_{\omega_l}} \\
&\leqslant r^{-s_{\min}}\ \Big(\mathrm{Lip}(S_{\omega_1,i_1}) \cdots \mathrm{Lip}(S_{\omega_l,i_l})\Big)^{s_{\min}}\ (1+2a_2)^n (c\,a_1)^{-n} \\
&\leqslant (1+2a_2)^n (c\,a_1)^{-n} \\
&< \infty
\end{aligned}$$

and by Proposition 6.12 (1) it follows that $\mathcal{H}^{s_{\min}}(F_\omega) \geqslant (1+2a_2)^{-n}(c\,a_1)^n > 0$ and, in particular,

$$0 < \inf_{\omega\in\Omega} \mathcal{H}^{s_{\min}}(F_\omega) < \infty$$

which completes the proof. $\square$

6.3.8 Proof of Theorem 6.7 (1)

Write $\mathcal{H}_{\min} = \inf_{\omega\in\Omega} \mathcal{H}^{s_{\min}}(F_\omega)$ and $\mathcal{P}_{\max} = \sup_{\omega\in\Omega} \mathcal{P}^{s_{\max}}(F_\omega)$ and let $s = s_{\min} = s_{\max}$.

Hausdorff measure

We will show that the set
$$H = \{\omega \in \Omega : \mathcal{H}^s(F_\omega) = \mathcal{H}_{\min}\}$$
is residual. Writing $H_{m,n} = \{\omega \in \Omega : \mathcal{H}^s_{1/m}(F_\omega) < \mathcal{H}_{\min} + \frac{1}{n}\}$, we have

$$H = \bigcap_{m,n\in\mathbb{N}} H_{m,n},$$

so it suffices to prove that each $H_{m,n}$ is open and dense in (Ω, d_Ω). Fix $m, n \in \mathbb{N}$. It can be shown that $H_{m,n}$ is open using a similar approach to that used in the proof of Theorem 6.2 (1). We will now prove that $H_{m,n}$ is dense.

Let $\omega = (\omega_1, \omega_2, \dots) \in \Omega$ and $\varepsilon > 0$. Choose $k \in \mathbb{N}$ such that $2^{-k} < \varepsilon$ and choose $u = (u_1, u_2, \dots) \in \Omega$ such that
$$\mathcal{H}^s(F_u) < \mathcal{H}_{\min} + \tfrac{1}{n}.$$
Let $v = (\omega_1, \dots, \omega_k, u_1, u_2, \dots)$. It follows that $d_\Omega(\omega, v) < \varepsilon$ and, furthermore,

$$\begin{aligned}
\mathcal{H}^s_{1/m}(F_v) \leqslant \mathcal{H}^s(F_v) &= \mathcal{H}^s\left(\bigcup_{j_1\in\mathcal{I}_{\omega_1},\dots,j_k\in\mathcal{I}_{\omega_k}} S_{\omega_1,j_1}\circ\cdots\circ S_{\omega_k,j_k}(F_u)\right)\\
&\leqslant \sum_{j_1\in\mathcal{I}_{\omega_1},\dots,j_k\in\mathcal{I}_{\omega_k}} \mathrm{Lip}(S_{\omega_1,j_1})^s\cdots\mathrm{Lip}(S_{\omega_k,j_k})^s\ \mathcal{H}^s\big(F_u\big)\\
&< \Big(\mathcal{H}_{\min} + \tfrac{1}{n}\Big) \sum_{j_1\in\mathcal{I}_{\omega_1},\dots,j_k\in\mathcal{I}_{\omega_k}} \mathrm{Lip}(S_{\omega_1,j_1})^s\cdots\mathrm{Lip}(S_{\omega_k,j_k})^s\\
&= \mathcal{H}_{\min} + \tfrac{1}{n}
\end{aligned}$$

where the final equality is due to the fact that s is a solution to the Hutchinson-Moran formula (1.1) for each deterministic IFS. It follows that $v \in H_{m,n}$, proving that $H_{m,n}$ is dense.

Packing measure

We will show that the set $P = \{\omega \in \Omega : \mathcal{P}^s(F_\omega) = \mathcal{P}_{\max}\}$ is residual. It was proved in [FHW] that if a compact set has finite packing pre-measure, then the packing measure and packing pre-measure coincide. Writing $P_{m,n} = \{\omega \in \Omega : \mathcal{P}^s_{0,\,1/m}(F_\omega) > \mathcal{P}_{\max} - \frac{1}{n}\}$, it follows that

$$P \supseteq \{\omega \in \Omega : \mathcal{P}^s_0(F_\omega) = \mathcal{P}_{\max}\} = \bigcap_{m,n\in\mathbb{N}} P_{m,n},$$

so it suffices to prove that each $P_{m,n}$ is open and dense. Fix $m, n \in \mathbb{N}$. It can be shown using a similar approach to that used in the proof of Theorem 6.2 (2) that $P_{m,n}$ is open. We will now show that it is also dense.

Let $\omega = (\omega_1, \omega_2, \dots) \in \Omega$ and $\varepsilon > 0$. Choose $k \in \mathbb{N}$ such that $2^{-k} < \varepsilon$ and choose $u = (u_1, u_2, \dots) \in \Omega$ such that $\mathcal{P}^s(F_u) > \mathcal{P}_{\max} - \frac{1}{n}$. Let $v = (\omega_1, \dots, \omega_k, u_1, u_2, \dots)$. It follows that $d_\Omega(\omega, v) < \varepsilon$ and, furthermore,

$$
\begin{aligned}
\mathcal{P}^s_{0,1/m}(F_v) \geqslant \mathcal{P}^s(F_v) &= \mathcal{P}^s\left(\bigcup_{j_1 \in \mathcal{I}_{\omega_1}, \dots, j_k \in \mathcal{I}_{\omega_k}} S_{\omega_1, j_1} \circ \dots \circ S_{\omega_k, j_k}\big(F_u\big) \right) \\
&= \sum_{j_1 \in \mathcal{I}_{\omega_1}, \dots, j_k \in \mathcal{I}_{\omega_k}} \mathrm{Lip}(S_{\omega_1, j_1})^s \cdots \mathrm{Lip}(S_{\omega_k, j_k})^s \ \mathcal{P}^s\big(F_u\big) \\
&> \left(\mathcal{P}_{\max} - \tfrac{1}{n}\right) \sum_{j_1 \in \mathcal{I}_{\omega_1}, \dots, j_k \in \mathcal{I}_{\omega_k}} \mathrm{Lip}(S_{\omega_1, j_1})^s \cdots \mathrm{Lip}(S_{\omega_k, j_k})^s \\
&= \mathcal{P}_{\max} - \tfrac{1}{n}
\end{aligned}
$$

where the final equality is due to the fact that s is a solution to to the Hutchinson-Moran formula (1.1) for each deterministic IFS. It follows that $u \in P_{m,n}$, proving that $P_{m,n}$ is dense. □

6.4 Self-affine examples with interesting properties

In this section we provide a number of self-affine examples designed to illustrate some of the key points made in Section 6.2.

6.4.1 Typical Hausdorff measure

In this section we give a simple example which shows that the Hausdorff measure can typically be positive and finite even if the supremal Hausdorff measure is infinite. The existence of such an example is slightly surprising in view of Theorem 6.4 and the behaviour observed in the self-similar setting, see Theorem 6.7.

Let $\mathbb{\underline{I}} = \{\mathbb{I}_1, \mathbb{I}_2\}$ be a RIFS where $\mathbb{I}_1$ and $\mathbb{I}_2$ are IFSs of orientation-preserving affine self-maps on $[0,1]^2$ corresponding to the figure below.

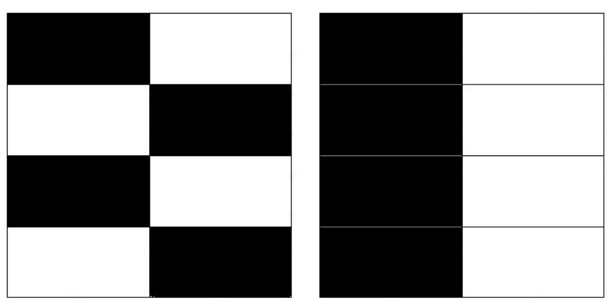

Figure 19: The defining pattern for a random Sierpiński carpet with $N = 2$, $m_1 = m_2 = 2$ and $n_1 = n_2 = 4$.

It is clear that $\inf_{\omega \in \Omega} \dim_{\mathrm{H}} F_\omega = 1$ and $\inf_{\omega \in \Omega} \mathcal{H}^1(F_\omega) = 1 < \infty = \sup_{\omega \in \Omega} \mathcal{H}^1(F_\omega)$. It follows from Theorem 6.2 that the typical Hausdorff dimension is 1. We will now show that the typical Hausdorff measure is also infimal and, in particular, positive and finite. We will show that the set $H = \{\omega \in \Omega : \mathcal{H}^1(F_\omega) = 1\}$ is a dense G_δ set and thus residual. It can be shown that H is G_δ using a very similar approach to that used in the proof of Theorem 6.2 (1). It remains to show that H is dense.

Let $\omega = (\omega_1, \omega_2, \dots) \in \Omega$ and $\varepsilon > 0$. Choose $k \in \mathbb{N}$ such that $2^{-k} < \varepsilon$ and let $v = (\omega_1, \dots, \omega_k, 2, 2, \dots)$. It follows that $d_\Omega(\omega, v) < \varepsilon$ and, furthermore, since $F_{(2,2,\dots)} = \{0\} \times [0,1]$, we have

$$
F_v = \bigcup_{j_1 \in \mathcal{I}_{\omega_1}, \dots, j_k \in \mathcal{I}_{\omega_k}} S_{\omega_1, j_1} \circ \dots \circ S_{\omega_k, j_k}\big(\{0\} \times [0,1]\big)
$$

and, since the vertical component of every map in $\mathbb{I}$ is a similarity with contraction ratio 1/4 and both deterministic IFSs consist of 4 maps, we have

$$\mathcal{H}^1(F_v) \leqslant \sum_{j_1 \in \mathcal{I}_{\omega_1}, \dots, j_k \in \mathcal{I}_{\omega_k}} \mathcal{H}^1\left(S_{\omega_1, j_1} \circ \cdots \circ S_{\omega_k, j_k}\big(\{0\} \times [0,1]\big)\right) = 4^k \, 4^{-k} \, \mathcal{H}^1\big(\{0\} \times [0,1]\big) = 1$$

and so $v \in H$, proving that H is dense.

6.4.2 Typical packing measure

In this section we give a simple example which shows that the packing measure can typically be positive and finite even if the infimal packing measure is zero. The existence of such an example is slightly surprising in view of Theorem 6.5 and the behaviour observed in the self-similar setting, see Theorem 6.7.

Let $\mathbb{I} = \{\mathbb{I}_1, \mathbb{I}_2\}$ be a RIFS where $\mathbb{I}_1$ and $\mathbb{I}_2$ are IFSs of orientation-preserving affine self-maps on $[0,1]^2$ corresponding to the figure below.

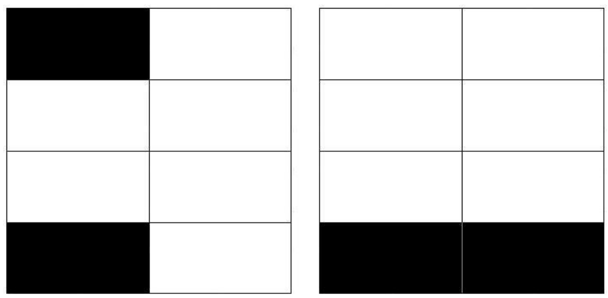

Figure 20: The defining pattern for a random Sierpiński carpet with $N = 2$, $m_1 = m_2 = 2$ and $n_1 = n_2 = 4$.

We claim that $\inf_{\omega \in \Omega} \mathcal{P}^1(F_\omega) = 0 < 1 \leqslant \sup_{\omega \in \Omega} \mathcal{P}^1(F_\omega) \leqslant 4$ and it follows that $\sup_{\omega \in \Omega} \dim_{\mathrm{P}} F_\omega = 1$. The only inequality which is not obvious is $\sup_{\omega \in \Omega} \mathcal{P}^1(F_\omega) \leqslant 4$, which we will now prove. Fix $\omega \in \Omega$ and define a mass distribution, μ_ω, on F_ω by assigning each level k rectangle mass 2^{-k} in a similar way to the construction of the measures in Section 6.3.7. It is easy to see that for all $x \in F_\omega$ we have $\liminf_{r \to 0} \mu(B(x,r))\, r^{-1} \geqslant 1/2$, and it follows from Proposition 6.12 (2) that $\mathcal{P}^1(F_\omega) \leqslant 4$. Theorem 6.2 gives that the typical packing dimension is 1. We will now show that the typical packing measure is greater than or equal to 1 and, in particular, positive and finite. We will show that the set $P = \{\omega \in \Omega : \mathcal{P}^1(F_\omega) \geqslant 1\}$ is a dense G_δ set and thus residual. It follows from the main result in [FHW] and Lemma 6.14 that $P = \{\omega \in \Omega : \mathcal{P}^1_0(F_\omega) \geqslant 1\}$ and it can thus be shown that P is G_δ using a very similar approach to that used in the proof of Theorem 6.2 (2). It remains to show that P is dense.

Let $\omega = (\omega_1, \omega_2, \dots) \in \Omega$ and $\varepsilon > 0$. Choose $k \in \mathbb{N}$ such that $2^{-k} < \varepsilon$ and let $v = (\omega_1, \dots, \omega_k, 2, 2, \dots)$. It follows that $d_\Omega(\omega, v) < \varepsilon$ and, furthermore, since $F_{(2,2,\dots)} = [0,1] \times \{0\}$, we have

$$F_v = \bigcup_{j_1 \in \mathcal{I}_{\omega_1}, \dots, j_k \in \mathcal{I}_{\omega_k}} S_{\omega_1, j_1} \circ \cdots \circ S_{\omega_k, j_k}\big([0,1] \times \{0\}\big)$$

and, since the horizontal component of every map in $\mathbb{I}$ is a similarity with contraction ratio 1/2 and both deterministic IFSs consist of 2 maps, we have

$$\mathcal{P}^1_0(F_v) \;=\; \mathcal{P}^1(F_v) = \sum_{j_1 \in \mathcal{I}_{\omega_1}, \dots, j_k \in \mathcal{I}_{\omega_k}} \mathcal{P}^1\left(S_{\omega_1, j_1} \circ \cdots \circ S_{\omega_k, j_k}\big([0,1] \times \{0\}\big)\right) = 2^k \, 2^{-k} \, \mathcal{P}^1\big([0,1] \times \{0\}\big) = 1$$

and so $u \in P$, proving that P is dense.

Remark 6.17. *We believe that a more delicate application of the mass distribution principle will yield that, in fact,* $\sup_{\omega \in \Omega} \mathcal{P}^1(F_\omega) = 1$, *but since the important thing for our purposes is that the typical value is positive and finite, we omit further calculation.*

6.4.3 Dimension outside range

In this section we give a simple example which shows that in the non-conformal setting the dimension of the random attractor need not be bounded below by the minimum dimension of the deterministic attractors. This is in stark contrast to Theorem 6.6, concerning random self-similar sets. Furthermore, $\inf_{u \in \Omega} \dim_{\mathrm{H}} F_u$ is not attained by any finite combination of the determinsitic IFSs. Let $\underline{\mathbb{I}} = \{\mathbb{I}_1, \mathbb{I}_2\}$ be a RIFS where $\mathbb{I}_1$ and $\mathbb{I}_2$ are IFSs of orientation-preserving affine self-maps on $[0,1]^2$ corresponding to the figure below.

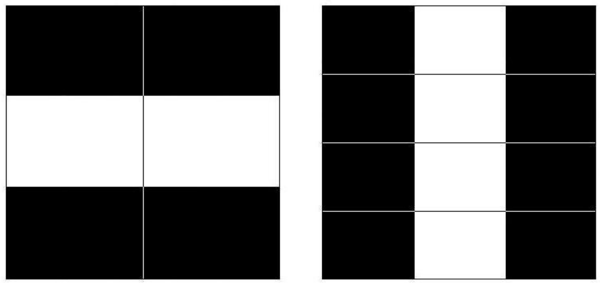

Figure 21: The defining pattern for a random Sierpiński carpet with $N = 2$, $m_1 = 2$, $n_1 = 3$, $m_2 = 3$ and $n_2 = 4$.

The results of [Be1, McM] give that for both deterministic attractors the Hausdorff, box and packing dimensions are all equal to $1 + \log 2/\log 3 \approx 1.63$. For $p \in [0,1]$, associate a probability vector $(p,\, 1-p)$ with this system. By the result of [FO], given here as Theorem 1.9, the almost sure Hausdorff dimension of F_ω is given by

$$\begin{aligned}
\dim_{\mathrm{H}} F_\omega \quad &= \quad \frac{p}{\log 2^p 3^{1-p}} \log\left(2^{\log 2^p 3^{1-p}/\log 3^p 4^{1-p}} + 2^{\log 2^p 3^{1-p}/\log 3^p 4^{1-p}}\right) \\
&\qquad + \frac{1-p}{\log 2^p 3^{1-p}} \log\left(4^{\log 2^p 3^{1-p}/\log 3^p 4^{1-p}} + 4^{\log 2^p 3^{1-p}/\log 3^p 4^{1-p}}\right) \\
&= \quad \frac{\log 2}{\log 2^p 3^{1-p}} + (2-p)\,\frac{\log 2}{\log 3^p 4^{1-p}}.
\end{aligned}$$

In fact, since each deterministic IFS has *uniform vertical fibres*, it follows from results in [GuLi2] that the above formula also gives the almost sure box and packing dimensions of F_ω. Plotting this as a function of p, we obtain

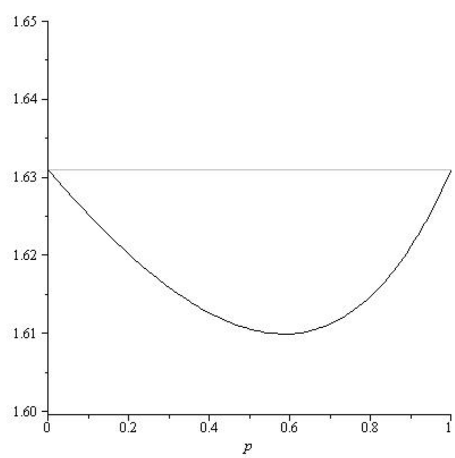

Figure 22: A graph of the almost sure Hausdorff dimension as a function of p. The grey line shows the dimension of the deterministic attractors.

Notice the nonlinear dependence on p and the fact that for $p \in (0,1)$ the almost sure dimension is lower than the minimum dimension of the two deterministic attractors. In particular, the dimension of F_ω is not bounded below by the minimum Hausdorff dimension of the deterministic attractors, despite the fact that the UOSC is satisfied. As such, it is not at all clear what the infimal (and thus typical) Hausdorff dimension is. This is in stark contrast to the self-similar setting, see Theorem 6.6 (4). It is natural to ask if the infimal dimension is attained by an attractor of a deterministic IFS given by a finite combination of the original deterministic IFSs, $\mathbb{I}_1$ and $\mathbb{I}_2$. We will argue now that it is not. Finite combinations of $\mathbb{I}_1, \mathbb{I}_2$ give deterministic IFSs with attractors equal to F_ω for some 'rational' $\omega \in \Omega$, i.e., some ω which consists of a finite word over D repeated infinitely often. Fix such a finite combination, let N_1 be the number of times we have used $\mathbb{I}_1$ and let N_2 be the number of times we have used $\mathbb{I}_2$. It is clear, and in fact it follows from the results in [GuLi2], that the Hausdorff dimension of the attractor is equal to the almost sure Hausdorff dimension of the attractor corresponding to $p = N_1/(N_1 + N_2) \in \mathbb{Q}$. However, elementary optimisation reveals that the minimum almost sure Hausdorff dimension (seen as the minimum of the graph above) is attained by $p = 2 - \sqrt{2} \notin \mathbb{Q}$.

6.5 Some fun examples

6.5.1 Typical measure not positive and finite

In this section we will give a straightforward example which has the interesting property that, although the Hausdorff and packing measures of the attractors of the deterministic IFSs in the appropriate dimension are positive and finite, the typical Hausdorff and packing measures are infinity and zero, respectively.

Let $S_1, S_2, S_3 : [0,1] \to [0,1]$ be defined by

$$S_1(x) = x/3, \qquad S_2(x) = x/3 + 1/3, \qquad \text{and} \qquad S_3(x) = x/3 + 2/3.$$

Let $\underline{\mathbb{I}}$ be the RIFS consisting of the two deterministic IFSs, $\{S_1, S_3\}$ and $\{S_1, S_2, S_3\}$. The attractors for these systems are the middle 1/3 Cantor set, $C_{1/3}$, and the unit interval, $[0,1]$, respectively. Also, since the first IFS is contained in the second, for all $\omega \in \Omega$,

$$C_{1/3} \subseteq F_\omega \subseteq [0,1]$$

from which it follows that dimensions are bounded between $s = \frac{\log 2}{\log 3}$ and 1 and that

$$\inf_{u \in \Omega} \mathcal{H}^s(F_u) = \mathcal{H}^s(C_{1/3}) = 1$$

and

$$\sup_{u \in \Omega} \mathcal{P}^1(F_u) = \mathcal{P}^1([0,1]) = 1.$$

It follows from Theorem 6.7 that, for a typical $\omega \in \Omega$, the set F_ω has Hausdorff and lower box dimension equal to $\frac{\log 2}{\log 3}$ and packing and upper box dimension equal to 1 but $\frac{\log 2}{\log 3}$-dimensional Hausdorff measure equal to ∞ and 1-dimensional packing measure equal to 0. It is clear that the $\mathcal{P}^{\log 2/\log 3}$-MSC is satisfied.

6.5.2 A nonlinear example: random cookie cutters

Although the previous examples illustrate some of the key phenomena we wish to discuss, they have all been based on RIFSs consisting of translate linear (affine) maps. Of course, Theorems 6.2, 6.4 and 6.5 apply in far more general circumstances than this. In this section we construct a more complicated example using nonlinear maps to which we can apply Theorems 6.4 and 6.5 to deduce that neither the typical Hausdorff nor packing measures are positive and finite in the appropriate dimensions.

Let $f_1, f_2 : [0,1] \to \mathbb{R}$ be defined by

$$f_1(x) = -5x(x-1) \qquad \text{and} \qquad f_2(x) = 9(x - 1/6)(x - 5/6)$$

respectively. We will now construct a RIFS using the inverse branches of f_1 and f_2.

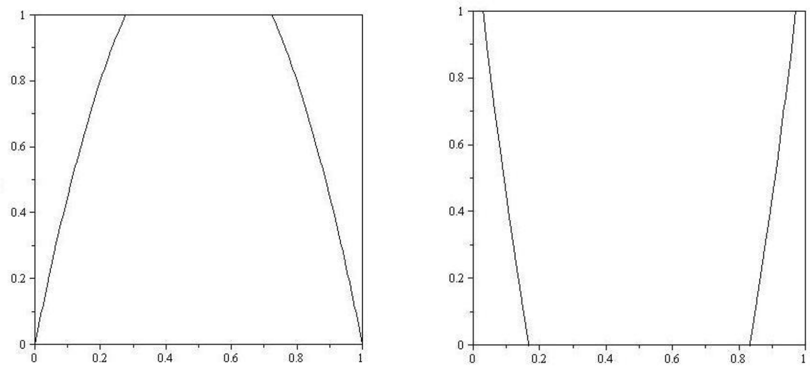

Figure 23: Graphs of the maps f_1 (left) and f_2 (right) restricted to the unit square.

Observe that f_1 maps each of the intervals $X_{1,1} = \big[0,\, \frac{1}{2} - \frac{1}{10}\sqrt{5}\big]$ and $X_{1,2} = \big[\frac{1}{2} + \frac{1}{10}\sqrt{5},\, 1\big]$ bijectively onto $[0,1]$ and furthermore f_1' is continuous with

$$2 \leqslant |f_1'(x)| \leqslant 5 \tag{6.3}$$

for $x \in X_{1,1} \cup X_{1,2}$. Similarly, f_2 maps each of the intervals $X_{2,1} = \big[\frac{1}{2} - \frac{1}{3}\sqrt{2},\, \frac{1}{6}\big]$ and $X_{2,2} = \big[\frac{5}{6},\, \frac{1}{2} + \frac{1}{3}\sqrt{2}\big]$ bijectively onto $[0,1]$, f_2' is continuous and

$$6 \leqslant |f_2'(x)| \leqslant 9 \tag{6.4}$$

for $x \in X_{2,1} \cup X_{2,2}$. The dynamical properties of f_1 and f_2 are interesting in their own right, but we will be particularly interested in the sets

$$F_1 = \bigcap_{k\geqslant 0} f_1^{-k}\big([0,1]\big) \qquad \text{and} \qquad F_2 = \bigcap_{k\geqslant 0} f_2^{-k}\big([0,1]\big)$$

which are the *dynamical repellers* for the maps f_1 and f_2 respectively. Repellers of this type are often called *cookie cutters* and the Hausdorff and packing dimension can be computed via the thermodynamical formalism, see for example [F7, Chapters 4–5], or [R]. We can view F_1 and F_2 as attractors of deterministic IFSs consisting of the inverse branches of f_1 and f_2. In particular, the inverse branches of f_1 are given by

$$S_{1,1}(x) = \tfrac{1}{2} - \tfrac{1}{2}\sqrt{1 - \tfrac{4}{5}x} \qquad \text{and} \qquad S_{1,2}(x) = \tfrac{1}{2} + \tfrac{1}{2}\sqrt{1 - \tfrac{4}{5}x}$$

and the inverse branches of f_2 are given by

$$S_{2,1}(x) = \tfrac{1}{2} - \tfrac{1}{3}\sqrt{1+x} \qquad \text{and} \qquad S_{2,2}(x) = \tfrac{1}{2} + \tfrac{1}{3}\sqrt{1+x}.$$

Let $\mathbb{I}$ be the RIFS consisting of $\mathbb{I}_1 = \{S_{1,1}, S_{1,2}\}$ and $\mathbb{I}_2 = \{S_{2,1}, S_{2,2}\}$. Here F_1 corresponds to the choice $(1,1,\dots) \in \Omega$ and F_2 corresponds to the choice $(2,2,\dots) \in \Omega$. For an arbitrary $\omega = (\omega_1, \omega_2, \dots) \in \Omega$, we obtain a *random cookie cutter*

$$F_\omega \;=\; \bigcap_{k\geqslant 0} f_{\omega_1}^{-1} \circ \dots \circ f_{\omega_k}^{-1}\big([0,1]\big) \;=\; \bigcap_{k\geqslant 0}\ \bigcup_{i_1,\dots,i_k \in \{1,2\}} S_{\omega_1,i_1} \circ \dots \circ S_{\omega_k,i_k}\big([0,1]\big).$$

Write $h = \inf_{u\in\Omega} \dim_{\mathrm{H}} F_u$ and $p = \sup_{u\in\Omega} \dim_{\mathrm{P}} F_u$. It follows from (6.3–6.4), the fact that f_1', f_2' are continuous and the mean value theorem that, for $i = 1,2$,

$$1/5 \leqslant \mathrm{Lip}^-(S_{1,i}) \leqslant \mathrm{Lip}^+(S_{1,i}) \leqslant 1/2 \qquad \text{and} \qquad 1/9 \leqslant \mathrm{Lip}^-(S_{2,i}) \leqslant \mathrm{Lip}^+(S_{2,i}) \leqslant 1/6$$

and applying standard estimates for the dimension gives

$$h \leqslant \dim_{\mathrm{H}} F_2 \leqslant \frac{\log 2}{\log 6} < \frac{\log 2}{\log 5} \leqslant \dim_{\mathrm{P}} F_1 \leqslant p,$$

see [F8, Propositions 9.6–9.7]. Furthermore,

$$\sum_{i_1,\dots,i_k\in\{1,2\}} \mathrm{Lip}^-(S_{1,i_1}\circ\cdots\circ S_{1,i_k})^h \geqslant \left(2\cdot 5^{-h}\right)^k \to \infty$$

and

$$\sum_{i_1,\dots,i_k\in\{1,2\}} \mathrm{Lip}^+(S_{2,j_1}\circ\cdots\circ S_{2,j_k})^p \leqslant \left(2\cdot 6^{-p}\right)^k \to 0$$

as $k\to\infty$. It follows from Theorem 6.2, 6.4 and 6.5 that, for a typical $\omega\in\Omega$, $\dim_{\mathrm{H}} F_\omega = h < p = \dim_{\mathrm{P}} F_\omega$, but

$$\mathcal{H}^h(F_\omega) = \begin{cases} 0 & \text{if } \inf_{u\in\Omega}\ \mathcal{H}^h(F_u)=0 \\ \infty & \text{if } \inf_{u\in\Omega}\ \mathcal{H}^h(F_u)>0 \end{cases}$$

and

$$\mathcal{P}^p(F_\omega) = \begin{cases} 0 & \text{if } \sup_{u\in\Omega}\ \mathcal{P}^p(F_u)<\infty \\ \infty & \text{if } \sup_{u\in\Omega}\ \mathcal{P}^p(F_u)=\infty \end{cases}$$

In particular, for a typical $\omega\in\Omega$, the random cookie cutter F_ω is 'dimensionless' in the sense that neither the s-dimensional Hausdorff measure nor the s-dimensional packing measure are positive and finite for any $s\geqslant 0$.

6.5.3 Pictorial examples

In this section we give some pictorial examples of attractors of RIFSs to illustrate some of the rich and complicated structures we can expect to see. Although our results apply in both examples, we do not perform any calculations.

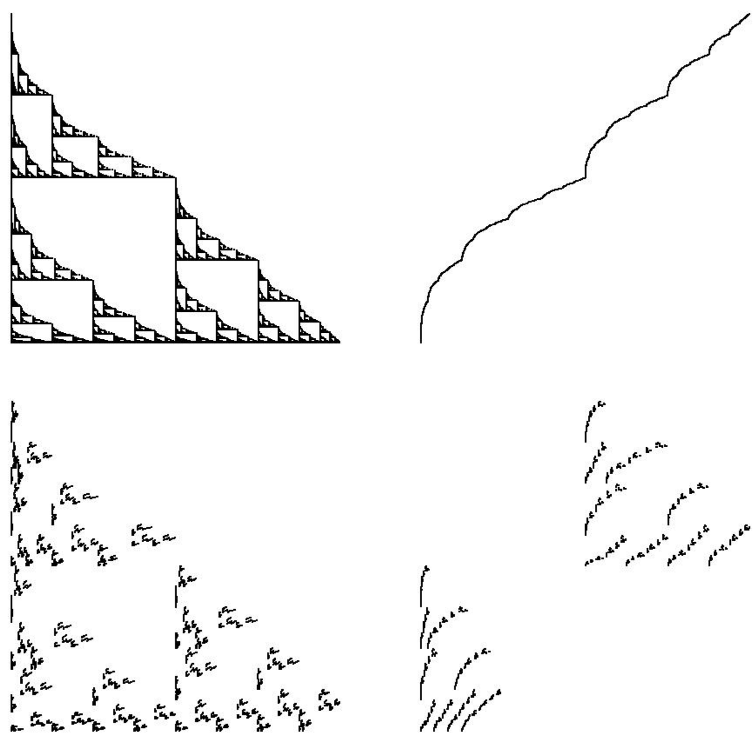

Figure 24: The attractors of deterministic IFSs $\mathbb{I}_1$ (top-left) and $\mathbb{I}_2$ (top-right) along with two random attractors of $\mathbb{I}=\{\mathbb{I}_1,\mathbb{I}_2\}$ corresponding to $\omega=(1,1,1,2,1,2,\dots)$ (bottom-left) and $\omega=(2,1,1,2,2,2,\dots)$ (bottom-right).

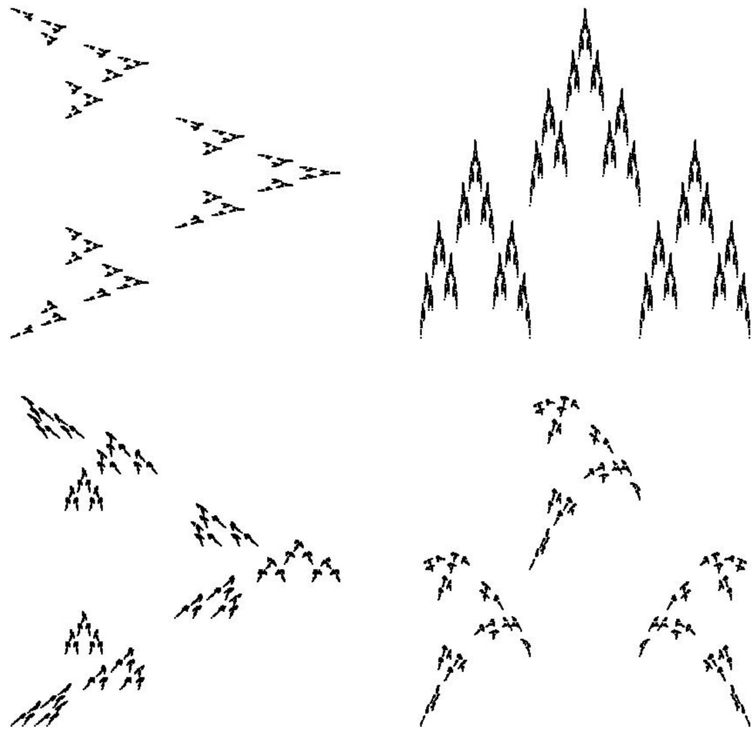

Figure 25: The attractors of deterministic IFSs $\mathbb{I}'_1$ (top-left) and $\mathbb{I}'_2$ (top-right) along with two random attractors of $\mathbb{I}' = \{\mathbb{I}'_1, \mathbb{I}'_2\}$ corresponding to $\omega = (1, 1, 2, 2, 1, 2, \dots)$ (bottom-left) and $\omega = (2, 1, 1, 2, 1, 1, \dots)$ (bottom-right).

6.6 Discussion

In this section we collect and discuss some of the questions raised by the results in this chapter.

(1) *Is the typical measure always extremal?* We have shown that the typical dimensions behave rather well in that the typical Hausdorff and lower box dimensions are always infimal and the typical packing and upper box dimensions are always supremal. The typical Hausdorff and packing measures behave rather worse, and our examples show that they can both be either infimal or supremal. However, we have not proved that they are *always extremal*.

(2) *Computing the extremal dimensions.* Theorem 6.2 tells us that the typical dimensions are extremal in very general circumstances. However, it gives no indication of how one might compute the extremal dimensions. This may be a very difficult problem, and the example in Section 6.4.3 sheds some light on that difficulty. Given a RIFS, can we say anything non-trivial about the extremal dimensions in general? Theorem 6.6 tells us how to compute the extremal dimensions in the self-similar setting, assuming the UOSC.

(3) *The bi-Lipschitz requirement.* Throughout this chapter we assumed that all of our maps are bi-Lipschitz. It is easily seen, however, that not all of our proofs require this. In fact, Theorem 6.2 parts (1), (3) and (5) go through assuming that the maps are simply contractions. Also, a slightly weaker version of Theorem 6.5 can be proved, which states that if there exists $v \in \Omega$ satisfying conditon (6.2) and $\sup_{u \in \Omega} \mathcal{P}^p(F_u) < \infty$, then for a typical $\omega \in \Omega$, we have $\mathcal{P}^p(F_\omega) = 0$.

(4) *Strengthening of Theorem 6.6.* In view of the non-conformal example given in Section 6.4, it seems that the validity of the bounds given in Theorem 6.6 depend on two things: conformality; and separation properties. It seems likely that one could prove an analogous result using conformal mappings instead of similarities and replacing each s_i with the solution of Bowen's formula corre-

sponding to the IFS, $\mathbb{I}_i$. What could be a more interesting question is whether or not the UOSC condition is required in the self-similar case.

(5) *Doubling gauges.* At first sight it is somewhat curious that in Theorem 6.2 we require that the gauge is doubling for the result concerning packing measure, but can use arbitrary gauges for Hausdorff measure. In fact, it is not uncommon that doubling gauges play an important role when studying packing measure, see, for example, [JP, WW].

(6) *Dimension outside range.* The example in Section 6.4.3 shows that the dimensions can be strictly less than the minimum of the dimensions of the attractors of the deterministic IFSs. We have not, however, proved that the dimensions can be bigger than the maximum of the dimensions of the attractors of the deterministic IFSs

(7) *Separation properties in the self-similar case.* In Theorems 6.6 and 6.7 we assumed various separation properties. In fact, some parts of these theorems go through assuming slightly weaker conditions. For example, in Theorem 6.7 (1) we require only the $\mathcal{H}^{s_{\min}}$-MSC to prove that the typical Hausdorff measure is infimal and positive and finite. We choose to state these theorems using the stronger separation properties in order to simplify exposition and not shroud the key ideas.

(8) *More randomness.* It is possible to introduce more randomness into our construction. In particular, one might relax the requirement that at the kth level of the construction we use the same IFS within each kth level iterate of X. In this case our sequence space, Ω, would be replaced by a space of infinite rooted trees. We believe that although this is a significantly more general construction, the topological properties of Ω would not change significantly and most of our arguments should generalise without too much difficulty. One might also consider the intermediate levels of randomness given by V-variable fractals introduced in [BHS2] and discussed in detail in [Ba2].

(9) *Typical versus almost sure.* An interesting consequence of Theorem 6.2 is that our topological approach gives drastically different results to the probabilistic (or measure theoretic) approach. For example, compare Theorem 1.8 with our result, Theorem 6.7. A similar comparison has cropped up in a wide variety of situations with, roughly speaking, the topological approach favouring divergence and the probabilistic approach favouring converegence. Indeed, our results on dimension are of this nature. A similar phenomenon has arisen in, for example: dimensions of measures [H, O4]; dimensions of graphs of continuous functions [FH, MW1, HP]; and frequency properties of expansions of real numbers [S]. These references are given as a sample of some of the situations where a contrast between topological and probabilistic approaches have been observed and are by no means a complete list. For example, generic dimensions of measures and graphs of continuous functions have been studied extensively and, for a more complete survey, the reader is referred to [O4] and [FH] and the references therein. Concerning our results on the typical Hausdorff and packing measures of random self-similar sets, a recent result of Balka, Farkas, Fraser and Hyde [BFFH] has unearthed a remarkably similar phenomenon in a completely different context. Fix a compact metric space X and consider the Banach space of continuous functions from X into $\mathbb{R}^n$ for some $n \in \mathbb{N}$, denoted by $C_n(X)$. They consider the dimension and measure of the image $f(X)$ for typical $f \in C_n(X)$ and obtain the following result.

Theorem 6.18 ([BFFH])**.** *Let* $\dim_{\mathrm{T}} X$ *denote the topological dimension of* X *(which never exceeds the Hausdorff dimension and is always an integer). We have the following dichotomy:*

(1) If $n \leqslant \dim_{\mathrm{T}} X$*, then for a typical* $f \in C_n(X)$*, we have*

$$\dim_{\mathrm{P}} f(X) = \dim_{\mathrm{H}} f(X) = n$$

and

$$0 \ < \ \mathcal{H}^n(f(X)) = \mathcal{P}^n(f(X)) \ < \ \infty.$$

(2) If $n > \dim_{\mathrm{T}} X$*, then for a typical* $f \in C_n(X)$*, we have*

$$\dim_{\mathrm{H}} f(X) = \dim_{\mathrm{T}} X < n = \dim_{\mathrm{P}} f(X)$$

$$\mathcal{H}^{\dim_{\mathrm{T}} X}(f(X)) = \infty$$

and

$$\mathcal{P}^n(f(X)) = 0$$

and, moreover, the measure $\mathcal{H}^{\dim_{\mathrm{T}} X}|_{f(X)}$ *is not* σ*-finite.*

This result should be compared to our Theorem 6.7, which gives a similar dichotomy where, in one situation, the typical Hausdorff and packing measures are positive and finite; and, in the other, the typical Hausdorff measure in the typical Hausdorff dimension is infinite and the typical packing measure in the typical packing dimension is zero.

(10) *Choice of topological space.* Baire category theory can be used in much more general spaces than just complete metric spaces. In fact, all one needs is a *Baire topological space*, i.e., a topological space where the intersection of any countable collection of open dense sets is dense. In Section 6.1.1 we introduced a topology on Ω to allow us to examine the size of subsets of Ω using Baire category. Of course we could have formulated our analysis in terms of the set $\Lambda = \{F_\omega : \omega \in \Omega\}$ equipped with the topology induced by the Hausdorff metric. We note here that these two approaches are essentially equivalent. Define an equivalence relation, R, on Ω by $\omega \, R \, u \Leftrightarrow F_\omega = F_u$ and let $q : \Omega \to \Omega/R$ be the quotient map, where Ω/R is equipped with the quotient topology. Let $\Psi : \Omega \to \mathcal{K}(K)$ be defined by $\Psi(\omega) = F_\omega$ and $\hat{\Psi} : \Omega/R \to \mathcal{K}(K)$ be defined by $\hat{\Psi}([\omega]) = F_\omega$ and observe that Ψ is continuous by Lemma 6.11 and that $\hat{\Psi}$ is clearly well-defined. The following diagram commutes

$$\begin{tikzcd}
\Omega \arrow[r, two heads, "q"] \arrow[dr, "\Psi"'] & \Omega/R \arrow[d, tail, two heads, "\hat{\Psi}"] \\
 & \Lambda
\end{tikzcd}$$

and furthermore, $\hat{\Psi}$ is a homeomorphism. It is easy to see that Ω/R, and hence Λ, are Baire and that images of residual subsets of Ω under q are residual in Ω/R. It follows that all of our results could be phrased as 'for a typical set $F_\omega \in \Lambda$...' instead of 'for a typical $\omega \in \Omega$...'.

List of symbols and abbreviations

$\alpha_1(A) \geqslant \ldots \geqslant \alpha_n(A)$	the singular values of a linear map A on $\mathbb{R}^n$
$B(x,r)$	a metric ball centered at x with radius r which can be taken to be open or closed
COSC	condensation open set condition
CRE	covering regularity exponent
$D^-(G,c)$	lower doubling constant for a gauge function G and a positive real number c
$D^+(G,c)$	upper doubling constant for a gauge function G and a positive real number c
$\dim$	an unspecified dimension
$\dim_{\mathrm{A}}$	Assouad dimension
$\underline{\dim}_{\mathrm{B}}$	lower box dimension
$\overline{\dim}_{\mathrm{B}}$	upper box dimension
$\dim_{\mathrm{B}}$	box dimension
$\dim_{\mathrm{H}}$	Hausdorff dimension
$\dim_{\mathrm{L}}$	lower dimension
$\dim_{\mathrm{P}}$	packing dimension
$\dim_{\mathrm{T}}$	topological dimension
$d_{\mathcal{H}}$	Hausdorff metric
d_{Ω}	standard metric on the symbolic space Ω
F_C	inhomogeneous attractor with condensation C
F_ω	random attractor corresponding to ω
G	gauge function
$\mathcal{H}^G_\delta$	δ-approximate Hausdorff measure in the gauge G
$\mathcal{H}^G$	Hausdorff measure in the gauge G
$\mathcal{H}^s$	Hausdorff measure in the standard gauge $x \mapsto x^s$
$\mathcal{I}$	a finite index set for an iterated function system (IFS)
$\mathcal{I}^*$	the set of all finite sequences with entries in $\mathcal{I}$
$\mathcal{I}^{\mathbb{N}}$	the set of all infinite sequences with entries in $\mathcal{I}$
$\underline{\mathbb{I}}$	a random iterated functions system (RIFS)
$\mathbb{I}$	an iterated functions system (IFS)
IFS	iterated function system
$\mathcal{K}(X)$	set of all non-empty compact subsets of X
$\mathrm{Lip}^-(T)$	lower Lipschitz constant of a map T
$\mathrm{Lip}^+(T)$	upper Lipschitz constant of a map T
$\mathrm{Lip}(T)$	Lipschitz constant of a similarity map T
$\mathcal{L}^n$	n-dimensional Lebesgue measure
MSC	measure separation condition

$M_\delta(F)$	maximum number of sets in a centered δ-packing of a set F
$\mathbb{N}$	the natural numbers (which do not include 0)
$\mathbb{N}_0$	the natural numbers union $\{0\}$
N_δ	one of the standard covering or packing functions at scale δ
$\mathcal{O}$	the orbital set for an inhomogeneous attractor
Ω	the set of all infinite sequences with entries in a finite digit set D
OSC	open set condition
$p_t(C)$	t-covering regularity exponent of a compact set C
$p_{t,\delta}(C)$	(t,δ)-covering regularity exponent of a compact set C
$\mathcal{P}_0^G$	packing pre-measure in the gauge G
$\mathcal{P}_{0,\delta}^G$	δ-approximate packing pre-measure in the gauge G
$\mathcal{P}_0^s$	packing pre-measure in the standard gauge $x \mapsto x^s$
$\mathcal{P}_{0,\delta}^s$	δ-approximate packing pre-measure in the standard gauge $x \mapsto x^s$
$\mathcal{P}^s$	packing measure in the standard gauge $x \mapsto x^s$
$\mathcal{P}^G$	packing measure in the gauge G
$\phi^s(A)$	singular value function of a linear map $A : \mathbb{R}^n \to \mathbb{R}^n$
$\psi^s(A)$	modified singular value function of a linear map $A : \mathbb{R}^2 \to \mathbb{R}^2$
$\mathbb{P}$	a probability measure on Ω induced by a probability vector $\underline{p}$
$\mathbb{Q}$	the rational numbers
$Q(\boldsymbol{i}, r)$	approximate cube centered at $\Pi(\boldsymbol{i})$ with radius r
$\mathbb{R}$	the real numbers
RIFS	random iterated function system
ROSC	rectangular open set condition
SOSC	strong open set condition
(X, d)	a compact metric space
$\mathbb{Z}$	the integers

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