DIRECTED GRAPH ITERATED FUNCTION SYSTEMS

Graeme Boore

**A Thesis Submitted for the Degree of PhD
at the
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Directed graph iterated function systems

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at the University of St Andrews

Graeme Boore

October 4, 2011

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I, Graeme Boore, hereby certify that this thesis, which is approximately 59,000 words in length, has been written by me, that it is the record of work carried out by me and that it has not been submitted in any previous application for a higher degree.

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Abstract

This thesis concerns an active research area within fractal geometry.

In the first part, in Chapters 2 and 3, for directed graph iterated function systems (IFSs) defined on $\mathbb{R}$, we prove that a class of 2-vertex directed graph IFSs have attractors that cannot be the attractors of standard (1-vertex directed graph) IFSs, with or without separation conditions. We also calculate their exact Hausdorff measure. Thus we are able to identify a new class of attractors for which the exact Hausdorff measure is known.

We give a constructive algorithm for calculating the set of gap lengths of any attractor as a finite union of cosets of finitely generated semigroups of positive real numbers. The generators of these semigroups are contracting similarity ratios of simple cycles in the directed graph. The algorithm works for any IFS defined on $\mathbb{R}$ with no limit on the number of vertices in the directed graph, provided a separation condition holds.

The second part, in Chapter 4, applies to directed graph IFSs defined on $\mathbb{R}^n$. We obtain an explicit calculable value for the power law behaviour as $r \to 0^+$, of the qth packing moment of μ_u, the self-similar measure at a vertex u, for the non-lattice case, with a corresponding limit for the lattice case. We do this

(i) for any $q \in \mathbb{R}$ if the strong separation condition holds,

(ii) for $q \geqslant 0$ if the weaker open set condition holds and a specified non-negative matrix associated with the system is irreducible.

In the non-lattice case this enables the rate of convergence of the packing L^q-spectrum of μ_u to be determined. We also show, for (ii) but allowing $q \in \mathbb{R}$, that the upper multifractal q box-dimension with respect to μ_u, of the set consisting of all the intersections of the components of F_u, is strictly less than the multifractal q Hausdorff dimension with respect to μ_u of F_u.

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Introduction and background theory

1.1 Introduction

In all that follows we are going to be concerned with that large class of fractals that are the attractors of directed graph iterated function systems or IFSs.

The first part of this thesis, in Chapters 2 and 3, was motivated by asking the question, "Do we really get anything new with a directed graph IFS as opposed to a standard IFS?" A standard IFS can always be represented as a 1-vertex directed graph IFS so the question is really, "Do we get anything new with a directed graph IFS with more than one vertex as opposed to a 1-vertex directed graph IFS?". In answering this it is natural to start with some of the simplest directed graph IFSs defined on $\mathbb{R}$ which have directed graphs consisting of 2 vertices and just 4 edges. We are able to show that the attractor at one of the vertices of a large class of such 2-vertex directed graph IFSs is not the attractor of any standard (1-vertex directed graph) IFS whatsoever, with or without separation conditions, overlapping or not. This result is proved in Theorems 3.5.8 and 3.5.9 of Chapter 3. Using Theorem 3.4.7 we calculate the exact Hausdorff measure of these attractors and so we extend the class of attractors for which the exact Hausdorff measure is known.

In answering our question in the affirmative we produce some mathematics that is interesting in its own right. For example Proposition 2.3.4 of Chapter 2 gives an algorithm for calculating the gap lengths of any attractor of any directed graph IFS defined on $\mathbb{R}$ for which the convex strong separation condition (CSSC) holds. Corollary 2.3.5 gives an expression for the set of gap lengths as a finite union of cosets of finitely generated semigroups of positive real numbers, where the generators of the semigroups are contracting similarity ratios of simple cycles in the graph. Of interest is Theorem 3.4.7 of Chapter 3 because it gives sufficient conditions for the calculation of the Hausdorff measure of both of the attractors of a class of 2-vertex directed graph IFSs. This adds to the work of Ayer and Strichartz [AS99] and Marion [Mar86].

The second part in Chapter 4 extends the work of Olsen [Ols02b], from the standard (1-vertex) setting to a general directed graph setting in n-dimensional Euclidean space. We obtain an explicit calculable value for the power law behaviour as $r \to 0^+$, of the qth packing moment of μ_u on F_u, for the non-lattice case, with a corresponding limit for the lattice case. We do this

(i) for any $q \in \mathbb{R}$ if the strong separation condition holds,

(ii) for $q \geqslant 0$ if the weaker open set condition holds and the matrix $\mathbf{B}(q, \gamma, l)$, defined in Subsection 4.2.6, is irreducible.

In the non-lattice case this enables the rate of convergence of the packing L^q-spectrum to be determined.

We also show, for (ii) but allowing $q \in \mathbb{R}$, that the upper multifractal q box-dimension with respect to μ_u of the set consisting of all the intersections of the components of F_u is strictly less than the multifractal q Hausdorff dimension with respect to μ_u of F_u.

The rest of this chapter is concerned with the basic definitions and notation that we will need, together with the background theory that provides the starting point for the work that follows.

1.2 Definitions and notation

Standard definitions are assumed wherever they are not explicitly stated, so for example we take as given, the basic set theory definitions and notation, the definitions of the set of positive integers $\mathbb{N}$, the set of positive real numbers $\mathbb{R}^+$, the set of real numbers $\mathbb{R}$, the definition of a metric space and so on. Many definitions are included as much for their notation as for their meaning. A selection of textbooks that could be consulted by the reader for definitions not given in the text are [All91], [Apo78], [Edg00], [Fal03], and [Mad88]. When we make a definition we will put the object being defined in italics.

1.2.1 Some basic definitions and notation

In these definitions (X, d) is a metric space.

The *empty set*, is the set with no elements, $\emptyset = \{\ \} = \{x : x \in X, x \neq x\}$.

We use the notation $\#A$ for the number of elements in a (finite) set A.

For $A, B \subset X$ the *set difference* or *complement of B with respect to A* is defined as $A \setminus B = \{x : x \in A, x \notin B\}$.

Let $x \in X$, then the *closed ball* of centre x and radius $r > 0$, is defined as

$$B(x, r) = \{y : y \in X,\ d(x, y) \leqslant r\}.$$

The *open ball* of centre x and radius $r > 0$, is defined as

$$S(x, r) = \{y : y \in X,\ d(x, y) < r\}.$$

A set $A \subset X$ is *open* if and only if for every $x \in A$ there exists $r > 0$ such that $S(x, r) \subset A$.

A set $A \subset X$ is *closed* if and only if $X \setminus A$ is open.

A non-empty set $A \subset \mathbb{R}$ is *bounded above* if there exists a real number $b \in \mathbb{R}$, such that $a \leqslant b$, for all $a \in A$. Any such number b is called an *upper bound* for A.

The completeness axiom for $\mathbb{R}$ ensures that any non-empty set of real numbers, $A \subset \mathbb{R}$, that is bounded above, has a *least upper bound* or *supremum* which we denote by $\sup A$. Specifically for a non-empty set $A \subset \mathbb{R}$, that is bounded above, there exists $\sup A \in \mathbb{R}$, such that

$$\begin{aligned} &\text{(i)} \quad a \leqslant \sup A, \text{ for all } a \in A, \\ \text{and } &\text{(ii)} \quad \text{if } b \text{ is an upper bound for } A, \text{ then} \sup A \leqslant b. \end{aligned}$$

If A is not bounded above then we put $\sup A = +\infty$ and if $A = \emptyset$ we put $\sup A = -\infty$. The *greatest lower bound* or *infimum* is defined in a similar way and is denoted by $\inf A$. In those cases where it is clear $\sup A \in A$ or $\inf A \in A$, we may use the notation $\max A$ or $\min A$ in place of $\sup A$ or $\inf A$.

We use the conventional notation for a *real interval*, so for $a, b \in \mathbb{R}$, $[a, b] = \{x : x \in \mathbb{R}, a \leqslant x \leqslant b\}$, $(a, b) = \{x : x \in \mathbb{R}, a < x < b\}$, $[a, +\infty) = \{x : x \in \mathbb{R}, a \leqslant x\}$ and so on.

The *diameter* of a non-empty set $A \subset X$, is defined as

$$|A| = \sup \{d(x, y) : x, y \in A\},$$

and we put $|\emptyset| = 0$.

The *distance between a point $x \in X$ and a non-empty set $A \subset X$*, is defined as

$$\operatorname{dist}(x, A) = \inf \{d(x, a) : a \in A\}.$$

The *distance between non-empty sets $A, B \subset X$*, is defined as

$$\operatorname{dist}(A, B) = \inf \{d(a, b) : a \in A, \, b \in B\}.$$

Let a, b be any objects, then the *ordered pair* (a, b) can be defined as the set $\big\{\{a\}, \{a, b\}\big\}$. It follows that $(a, b) = (u, v)$ if and only if $a = u$ and $b = v$. In general we may define an *ordered n-tuple* $(a_1, a_2, \ldots, a_n)$ in a similar way, which has the property $(a_1, a_2, \ldots, a_n) = (b_1, b_2, \ldots, b_n)$ if and only if $a_i = b_i$ for $1 \leqslant i \leqslant n$.

Let A, B be given sets, then $A \times B = \{(a, b) : a \in A, \, b \in B\}$ is the *Cartesian product* of A and B. The *Cartesian n-fold product* is defined in a similar way with elements that are ordered n-tuples. An example is n-dimensional Euclidean space, this is the Cartesian n-fold product of $\mathbb{R}$, which we denote by

$$\mathbb{R}^n = \underbrace{\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}}_{n \text{ times}} = \{(x_1, x_2, \ldots, x_n) : x_i \in \mathbb{R}, \, 1 \leqslant i \leqslant n\}.$$

As $\mathbb{R}^n$ is a real linear space, for the *vector translation* of a set $A \subset \mathbb{R}^n$ by a vector $t \in \mathbb{R}^n$, we use the notation

$$A + t = \{a + t : a \in A\}$$

and for the *scalar multiple* of a set $A \subset \mathbb{R}^n$ by $k \in \mathbb{R}$,

$$kA = \{ka : a \in A\}.$$

We will often use a notation of the form $(A_c)_{c \in B}$ and $(A)_{c \in B}$, when B is a finite set of n elements, as this is just a convenient way of writing down ordered n-tuples.

That is, if B is ordered as $B = (b_1, b_2, \ldots, b_n)$, then $(A_c)_{c\in B}$ and $(A)_{c\in B}$ are the ordered n-tuples

$$(A_c)_{c\in B} = (A_{b_1}, A_{b_2}, \ldots, A_{b_n}),$$
$$(A)_{c\in B} = (A, A, \ldots, A).$$

A *relation* is a set of ordered pairs.

A *function, map, mapping* or *transformation* f is a relation, such that if $(a, b) \in f$ and $(a, c) \in f$ then $b = c$. If $(a, b) \in f$ it is usual to write the uniquely determined b as $b = f(a)$. If A and B are any sets, we use the notation $f : A \to B$ to mean $f \subset A \times B$ and that f is a function from the set A to the set B, that is to each point $a \in A$ is associated a unique point $f(a) \in B$. In this case the *domain* of f is A and the *range* or *image* of f is a subset of B. We use the notation $f(A)$ for the image of f, that is

$$f(A) = \{f(a) : a \in A\} \subset B.$$

A function $f : A \to B$ is an *injection* if $(a_1, b) \in f$ and $(a_2, b) \in f$ implies $a_1 = a_2$, f is a *surjection* if $f(A) = B$, and f is a *bijection* if f is both an injection and a surjection.

We use the symbol $\circ$ to indicate the *composition of functions*, so if $f : A \to B$, $g : B \to C$, then $g \circ f : A \to C$, with $(g \circ f)(a) = g(f(a))$ for each $a \in A$.

A *sequence* is an ordered list of elements or symbols from a specified set A. Formally we define an *infinite sequence*, $(a_1, a_2, a_3, \ldots)$, as a function $a : \mathbb{N} \to A$, using the notation $A^{\mathbb{N}}$ for the set of all such infinite sequences. We define a *finite sequence* $(a_1, a_2, a_3, \ldots, a_n)$, as a function $a : \{1, 2, \ldots, n\} \to A$, using the notation A^n for the set of all such finite sequences of length n. It is conventional to write $a(i) = a_i$, for $i \in \mathbb{N}$. We use round brackets to distinguish sequences which are ordered from the curly brackets used for unordered sets. A *finite string* is a finite sequence the only difference being one of representation. For example if $S = \{a, b, c\}$ then (a, b, b, a, c, b), represents a finite sequence of elements from S, and *abbacb* represents the mathematically equivalent finite string of symbols from S. An *infinite string* is an infinite sequence and we may use any of the notations $\mathbf{a} = (a_n) = (a_1, a_2, a_3, \ldots) = a_1a_2a_3\ldots$. We use $\mathbf{a}|_k$ to indicate the restriction of a sequence to its first k terms, so $\mathbf{a}|_k = a_1a_2\cdots a_k$.

A *cylinder set* $[b_1b_2\cdots b_k]$ in a set $B \subset A^{\mathbb{N}}$ is defined as

$$[b_1b_2\cdots b_k] = \{\mathbf{a} : \mathbf{a} \in B,\ \mathbf{a}|_k = b_1b_2\cdots b_k\},$$

that is, the set of all sequences in B that start with the same prefix $b_1b_2\cdots b_k$.

For a union of the sets $\{A_\alpha : \alpha \in I\}$ we will use the notation

$$\bigcup_{\alpha\in I} A_\alpha = \{x : x \in A_\alpha \text{ for at least one } \alpha \in I\}.$$

If $I = \mathbb{N}$, $\{A_i\}$ may be used as an abbreviation of $\{A_i : i \in \mathbb{N}\}$ and we may write the union as

$$\bigcup_{i=1}^{\infty} A_i = \{x : x \in A_i \text{ for at least one } i \in \mathbb{N}\}.$$

We will use similar notation for an intersection of sets.

Any collection of open sets, $\{A_\alpha : \alpha \in I\}$, such that $A \subset \bigcup_{\alpha \in I} A_\alpha$, is called an *open cover* of A and $\{A_\alpha : \alpha \in I\}$ contains a *finite subcover* if $A \subset \bigcup_{\beta \in J} A_\beta$, where $J \subset I$ is a finite indexing set.

A set $A \subset X$ is *compact* if and only if any every open cover of A contains a finite subcover. In n-dimensional Euclidean space $(\mathbb{R}^n, |\ \ |)$, a set is compact if and only if it is closed and bounded.

The *interior* of a set $A \subset X$, denoted by A°, is the largest open set contained in A, that is

$$A^\circ = \bigcup_{G \in \mathcal{G}} G,$$

where $\mathcal{G} = \{G : G \subset A,\ G \text{ is open }\}$.

The *closure* of a set $A \subset X$, denoted by $\overline{A}$, is the smallest closed set containing A, that is

$$\overline{A} = \bigcap_{F \in \mathcal{F}} F,$$

where $\mathcal{F} = \{F : A \subset F,\ F \text{ is closed }\}$.

A set $A \subset \mathbb{R}^n$ is *convex* if and only if, whenever $x, y \in A$,

$$\lambda x + (1 - \lambda) y \in A$$

for all $\lambda \in [0, 1]$.

The *convex hull* of a set $A \subset \mathbb{R}^n$, denoted by $C(A)$, is the smallest convex set containing A, that is

$$C(A) = \bigcap_{C \in \mathcal{C}(A)} C.$$

where $\mathcal{C}(A) = \{C : A \subset C \subset \mathbb{R}^n,\ C \text{ is convex }\}$.

We often need to use the lower and upper limits of functions of the form $f : \mathbb{R}^+ \to \mathbb{R}$, as $x \to 0^+$, see [Fal03]. For an example of the use of these limits see the definition of the box counting dimension in Subsection 1.2.4. The *lower limit* of f as $x \to 0^+$ is defined as

$$\underline{\lim}_{x \to 0^+} f(x) = \lim_{r \to 0^+} (\inf \{f(x) : 0 < x < r\}).$$

The *upper limit* of f as $x \to 0^+$ is defined as

$$\overline{\lim}_{x \to 0^+} f(x) = \lim_{r \to 0^+} (\sup \{f(x) : 0 < x < r\}).$$

These limits always exist as extended real numbers in $\mathbb{R} \cup \{-\infty, +\infty\}$ with

$$\underline{\lim}_{x \to 0^+} f(x) \leqslant \overline{\lim}_{x \to 0^+} f(x).$$

Also $\lim_{x \to 0^+} f(x)$ exists as an extended real number if and only if

$$\underline{\lim}_{x \to 0^+} f(x) = \overline{\lim}_{x \to 0^+} f(x) = \lim_{x \to 0^+} f(x)$$

1.2.2 Measure theory

We remind the reader that (X, d) is a metric space, although much of measure theory holds in more general settings.

A set $\mathbb{A}$, whose elements are subsets of a set X, is an *algebra* if

(a) $\emptyset, X \in \mathbb{A}$.

(b) If $B \in \mathbb{A}$ then $X \setminus B \in \mathbb{A}$.

(c) If $B_i \in \mathbb{A}$ for $1 \leqslant i \leqslant n$, then $\bigcup_{i=1}^{n} B_i \in \mathbb{A}$.

A set $\mathbb{A}^*$, whose elements are subsets of a set X, is a *σ-algebra* if

(a) $\emptyset, X \in \mathbb{A}$.

(b) If $B \in \mathbb{A}$ then $X \setminus B \in \mathbb{A}$.

(c) If $B_i \in \mathbb{A}^*$ for $i \in \mathbb{N}$, then $\bigcup_{i=1}^{\infty} B_i \in \mathbb{A}^*$.

A *measure μ on an algebra* $\mathbb{A}$, also called a *premeasure*, is a non-negative extended real-valued function, $\mu : \mathbb{A} \to [0, +\infty]$, where $\mathbb{A}$ is an algebra of subsets of a set X, such that $\mu(\emptyset) = 0$, and if (B_i) is a countable sequence of disjoint sets in $\mathbb{A}$, with $\bigcup_{i=1}^{\infty} B_i \in \mathbb{A}$, then

$$\mu\left(\bigcup_{i=1}^{\infty} B_i\right) = \sum_{i=1}^{\infty} \mu(B_i).$$

A *measure* μ, is a non-negative extended real-valued function, $\mu : \mathbb{A}^* \to [0, +\infty]$, where $\mathbb{A}^*$ is a σ-algebra of subsets of a set X, such that $\mu(\emptyset) = 0$, and μ is *countably additive* in the sense that if (B_i) is a countable sequence of disjoint sets in $\mathbb{A}^*$, then

$$\mu\left(\bigcup_{i=1}^{\infty} B_i\right) = \sum_{i=1}^{\infty} \mu(B_i).$$

Countable additivity implies that μ is an increasing set function, that is if $B, C \in \mathbb{A}^*$ and $B \subset C$, then $\mu(B) \leqslant \mu(C)$. If $\mu(X) = 1$ then μ is a *probability measure.*

An *outer measure* ν, is a non-negative extended real-valued function, $\nu : \mathcal{P}(X) \to [0, +\infty]$, where $\mathcal{P}(X)$ is the set of all subsets of a set X, such that $\nu(\emptyset) = 0$, if $B, C \in \mathcal{P}(X)$ and $B \subset C$, then $\nu(B) \leqslant \nu(C)$, and ν is *countably subadditive* in the sense that if (B_i) is any countable sequence of sets in $\mathcal{P}(X)$ then

$$\nu\left(\bigcup_{i=1}^{\infty} B_i\right) \leqslant \sum_{i=1}^{\infty} \nu(B_i).$$

A subset $A \subset X$ is *ν-measurable* if

$$\nu(B) = \nu(B \cap A) + \nu(B \setminus A),$$

for all subsets $B \subset X$. Let $\mathbb{M}$ be the set of all ν-measurable sets then $\mathbb{M}$ is a σ-algebra and the restriction of ν to $\mathbb{M}$ is a measure, see Theorem 1.2, [Fal85].

Let U be the set of all open sets in X, then the set of all subsets of X, $\mathcal{P}(X)$, is a σ-algebra that contains U. The intersection of all the σ-algebras containing U is also a σ-algebra, it is the smallest σ-algebra containing U and is called the *Borel σ-algebra*, its elements are known as the *Borel sets.* Equivalently the set of Borel sets of X can be defined as the smallest σ-algebra containing the closed subsets of X.

An (outer) measure ν is a *Borel measure* if all Borel sets are ν-measurable.

An (outer) measure ν is a *Borel regular measure* if it is a Borel measure and if for every $A \subset X$ there is a Borel set B such that $A \subset B$ and $\nu(A) = \nu(B)$.

Let μ be any measure defined on the Borel sets in $\mathbb{R}^n$, then the *support* of μ is the smallest closed set $\mathrm{supp}\mu$ such that $\mu(\mathbb{R}^n \setminus \mathrm{supp}\mu) = 0$.

1.2.3 Hausdorff measure

In the definition of Hausdorff measure that follows, the symbol $|\ |$ is used for the diameter of a set. We also use $|\ |$ to indicate the length of a sequence and the Euclidean metric in $\mathbb{R}^n$. Hopefully it will always be clear from the context, and from a simple examination of the arguments it is applied to, what is the intended meaning of $|\ |$.

$\{U_i\}$ is a *δ-cover* of A, if $\{U_i\}$ is a countable (or finite) collection of sets of diameter at most δ that cover A, that is $A \subset \bigcup_{i=1}^{\infty} U_i$ and $0 \leqslant |U_i| \leqslant \delta$ for each $i \in \mathbb{N}$.

Let $s \geqslant 0$, we write $\mathcal{H}^s_\infty(A)$ for the *Hausdorff s-content* of a set A, with

$$\mathcal{H}^s_\infty(A) = \inf \left\{ \sum_{i=1}^{\infty} |U_i|^s : \{U_i\} \text{ is a cover of } A \right\},$$

where there is no retriction on the diameters of the covers.

For any $\delta > 0$, we define

$$\mathcal{H}^s_\delta(A) = \inf \left\{ \sum_{i=1}^{\infty} |U_i|^s : \{U_i\} \text{ is a } \delta\text{-cover of } A \right\}.$$

$\mathcal{H}^s_\delta$ is an outer measure on $\mathbb{R}^n$, see [Fal85], as it is an increasing set function and it is subadditive. It is not a Borel measure though because it is non-additive on Borel sets, for an example see [Mat99], Chapter 4, Exercise 1.

As δ decreases $\mathcal{H}^s_\delta(A)$ increases and so approaches a limit as $\delta \to 0^+$, which we write as

$$\mathcal{H}^s(A) = \lim_{\delta \to 0^+} \mathcal{H}^s_\delta(A).$$

$\mathcal{H}^s$ is an outer measure on $\mathbb{R}^n$ and the restriction of $\mathcal{H}^s$ to the σ-algebra of $\mathcal{H}^s$-measurable sets is called the *s-dimensional Hausdorff measure*. For a proof that $\mathcal{H}^s$ is a Borel regular measure see Corollary 4.5, [Mat99].

We follow [Mat99] and define $0^0 = 1$ and $|\emptyset|^s = 0$ whenever they appear in the above definitions. Since we defined $|\emptyset| = 0$ in Subsection 1.2.1, $|\emptyset|^s = 0$ anyway for $s > 0$, however confusion may arise for $s = 0$. We require $|\emptyset|^0 = 0$ so that $\mathcal{H}^0(\emptyset) = 0$, and we require $0^0 = 1$ so that $\mathcal{H}^0$ is the counting measure with

$$\mathcal{H}^0(A) = \text{ number of points in the set } A.$$

1.2.4 Hausdorff and box-counting dimension

It can be shown, see [Fal03], using the definitions of $\mathcal{H}^s_\delta$ and $\mathcal{H}^s$ in Subsection 1.2.3, that for $A \subset \mathbb{R}^n$, $\mathcal{H}^s(A) < \infty$ implies $\mathcal{H}^t(A) = 0$, for $t > s \geqslant 0$. The contrapositive of this statement is that $\mathcal{H}^t(A) > 0$ implies $\mathcal{H}^s(A) = \infty$, for $t > s \geqslant 0$. This means that there is a critical value of t at which $\mathcal{H}^t(A)$ jumps from ∞ to 0. This critical value is known as the Hausdorff dimension of A, and is denoted by $\dim_{\mathrm{H}} A$, where $0 \leqslant \dim_{\mathrm{H}} A \leqslant n$, and

$$\mathcal{H}^t(A) = \begin{cases} \infty & \text{if } 0 \leqslant t < \dim_{\mathrm{H}} A, \\ 0 & \text{if } t > \dim_{\mathrm{H}} A. \end{cases}$$

Formally the *Hausdorff dimension* of any set $A \subset \mathbb{R}^n$ is defined as

$$\dim_{\mathrm{H}} A = \inf \left\{ t : t \geqslant 0, \, \mathcal{H}^t(A) = 0 \right\} = \sup \left\{ t : t \geqslant 0, \, \mathcal{H}^t(A) = \infty \right\},$$

where the supremum of the empty set is taken to be 0. We will usually use the letter s to denote the Hausdorff dimension of a set A. For $s = \dim_{\mathrm{H}} A$, $\mathcal{H}^s(A)$ may be zero, infinite or may satisfy $0 < \mathcal{H}^s(A) < \infty$.

For $s = \dim_{\mathrm{H}} A$, A is an *s-set* if A is a Borel set and $0 < \mathcal{H}^s(A) < \infty$.

Let A be any bounded subset of $\mathbb{R}^n$ and let $N_\delta(A)$ be the smallest number of sets of diameter at most δ which can cover A. The *lower* and *upper box-counting dimensions* of A, see [Fal03], are defined as

$$\underline{\dim}_{\mathrm{B}} A = \varliminf_{\delta \to 0^+} \frac{\ln N_\delta(A)}{-\ln \delta},$$
$$\overline{\dim}_{\mathrm{B}} A = \varlimsup_{\delta \to 0^+} \frac{\ln N_\delta(A)}{-\ln \delta}.$$

If these limits are equal then the *box-counting dimension* of A is

$$\dim_{\mathrm{B}} A = \lim_{\delta \to 0^+} \frac{\ln N_\delta(A)}{-\ln \delta}.$$

1.2.5 The Hausdorff metric

Let A be a non-empty subset of $\mathbb{R}^n$, then for $r > 0$, the *closed r-neighbourhood* of A is defined as

$$A(r) = \{x : x \in \mathbb{R}^n, \, \mathrm{dist}(x, A) \leqslant r\},$$

see Subsection 1.2.1 for the definition of the distance function.

Let $K(\mathbb{R}^n)$ denote the set consisting of all the non-empty compact subsets of $\mathbb{R}^n$, then the *Hausdorff metric* is defined, for any $A, B \in K(\mathbb{R}^n)$, as

$$d_{\mathrm{H}}(A, B) = \inf \{r : A \subset B(r), \, B \subset A(r)\}. \tag{1.2.1}$$

An equivalent definition of the Hausdorff metric, is given by

$$d_{\mathrm{H}}(A, B) = \max \{\mathrm{dist}(a, B), \, \mathrm{dist}(b, A) : a \in A, \, b \in B\},$$

see [Mat99].

It can be shown that $(K(\mathbb{R}^n), d_{\mathrm{H}})$ is a complete metric space, see for example Theorem 2.5.3 [Edg00].

For a finite set V, containing $\#V$ elements, we will use the following notation for the $\#V$-fold Cartesian product of $K(\mathbb{R}^n)$,

$$(K(\mathbb{R}^n))^{\#V} = \underbrace{K(\mathbb{R}^n) \times K(\mathbb{R}^n) \times \cdots \times K(\mathbb{R}^n)}_{\#V \text{ times}}.$$

For any ordered $\#V$-tuples $(A_u)_{u\in V}, (B_u)_{u\in V} \in (K(\mathbb{R}^n))^{\#V}$, we may define a metric D_{H} as

$$D_{\mathrm{H}}\big((A_u)_{u\in V}, (B_u)_{u\in V}\big) = \max\{d_{\mathrm{H}}(A_u, B_u) : u \in V\}. \tag{1.2.2}$$

Using the completeness of the space $(K(\mathbb{R}^n), d_{\mathrm{H}})$, it is a routine matter to show that the product space $((K(\mathbb{R}^n))^{\#V}, D_{\mathrm{H}})$ is also a complete metric space.

1.2.6 Lipschitz functions, contraction mappings, similarities and the Euclidean metric

Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f : X \to Y$ is a *Lipschitz function* if there is a non-negative real number M, such that

$$d_Y(f(x), f(y)) \leqslant M d_X(x, y), \text{ for all } x, y \in X.$$

The least such number M is the *Lipschitz constant* for f.

Let (X, d) be a metric space. A function $f : X \to X$ is a *contraction mapping* if there is a real number c, $0 \leqslant c < 1$, such that

$$d(f(x), f(y)) \leqslant c d(x, y), \text{ for all } x, y \in X,$$

Any such real number c is a *contraction ratio* for f.

If $0 < c$ and

$$d(f(x), f(y)) = c d(x, y), \text{ for all } x, y \in X,$$

then f is a *similarity* and c is the *similarity ratio*.

In this thesis we are mainly concerned with n-dimensional Euclidean space, $(\mathbb{R}^n, |\ \ |)$, which is a complete space with respect to the Euclidean metric. For $x, y \in \mathbb{R}^n$, the *Euclidean metric* $|\ \ |$, is defined as

$$|x - y| = \left(\sum_{i=1}^{n} (x_i - y_i)^2\right)^{\frac{1}{2}}.$$

So a similarity $S : \mathbb{R}^n \to \mathbb{R}^n$ satisfies

$$|S(x) - S(y)| = r_S\,|x - y|, \text{ for all } x, y \in \mathbb{R}^n,$$

where $0 < r_S$, is the similarity ratio of S. If $0 < r_S < 1$ then S is a *contracting similarity* and r_S is its *contracting similarity ratio*.

1.2.7 Directed graphs

A *vertex* is a node or point. If u and v represent (not necessarily distinct) vertices then a *(directed) edge*, e, from u to v is a directed arc or arrow that connects u to v. The initial and terminal vertex functions i and t (defined below) give the initial (start) and terminal (end) vertices of an edge, so for an edge e from u to v, $i(e) = u$ and $t(e) = v$. A finite *(directed) path*, $\mathbf{e}$, which is of length $|\mathbf{e}| = k \in \mathbb{N}$, is a finite string of consecutive edges, and may be written as $\mathbf{e} = e_1 \cdots e_k$ for some edges e_i, $1 \leqslant i \leqslant k$, where $t(e_j) = i(e_{j+1})$ for $1 \leqslant j \leqslant k-1$. The *vertices of a path* are the initial and terminal vertices of its constituent edges, so the set of vertices of a path $\mathbf{e} = e_1 \cdots e_k$ is the set $\{i(e_1), t(e_i) : 1 \leqslant i \leqslant k\}$. We define the *vertex list* of $\mathbf{e}$ as the finite string of consecutive vertices, $v_1 v_2 v_3 \cdots v_{k+1} = i(e_1)t(e_1)t(e_2)\cdots t(e_k)$, which shows the order in which the path $\mathbf{e}$ visits its vertices.

A *directed graph*, $\big(V, E^*, i, t\big)$, consists of the set of all vertices V and the set of all finite (directed) paths E^*, together with the initial and terminal vertex functions $i : E^* \to V$ and $t : E^* \to V$. E^1 denotes the set of all (directed) edges in the graph, that is the set of all paths of length one, with $E^1 \subset E^*$. V and E^1 are always assumed to be finite sets.

The *initial* and *terminal vertex* functions are defined as follows. Let $\mathbf{e} \in E^*$ be any finite path, then we may write $\mathbf{e} = e_1 \cdots e_k$ for some edges $e_i \in E^1$, $1 \leqslant i \leqslant k$. The initial vertex of $\mathbf{e}$ is the initial vertex of its first edge, so $i(\mathbf{e}) = i(e_1)$ and similarly for the terminal vertex $t(\mathbf{e}) = t(e_k)$. If we wish to indicate the initial vertex of a path $\mathbf{l}$, as v say, we may use the notation $\mathbf{l}_v$.

For $\mathbf{f}, \mathbf{g} \in E^*$, $\mathbf{f}$ is a *subpath* of $\mathbf{g}$ if and only if $\mathbf{g} = \mathbf{sft}$ for some $\mathbf{s}, \mathbf{t} \in E^*$, where we assume the empty path is an element of E^*. We use the notation $\mathbf{f} \subset \mathbf{g}$ to indicate that $\mathbf{f}$ is a subpath of $\mathbf{g}$.

For $\mathbf{f}, \mathbf{g} \in E^*$, $\mathbf{f}$ is *not a subpath* of $\mathbf{g}$ if and only if $\mathbf{g} \neq \mathbf{sft}$ for all $\mathbf{s}, \mathbf{t} \in E^*$, where we assume the empty path is an element of E^*. We use the notation $\mathbf{f} \not\subset \mathbf{g}$ to indicate that $\mathbf{f}$ is not a subpath of $\mathbf{g}$.

A *simple path* visits no vertex more than once, so a path $\mathbf{e} = e_1 \cdots e_k \in E^*$ is simple if its vertex list contains exactly $k+1$ different vertices. For simple paths we use the notation $\mathbf{p}$.

A *cycle* is a path which has the same initial and terminal vertices, so the path $\mathbf{e} \in E^*$ is a cycle if $i(\mathbf{e}) = t(\mathbf{e})$. A cycle is independent of its initial and terminal vertex, that is if $\mathbf{e} = e_1 e_2 \cdots e_k$ is a cycle then we can also write $\mathbf{e} = e_k e_1 e_2 \cdots e_{k-1}$.

A *simple cycle* is a cycle which visits no vertex more than once apart from the initial and terminal vertices which are the same, so if $\mathbf{e} = e_1 \cdots e_k \in E^*$ is a simple cycle then $i(\mathbf{e}) = t(\mathbf{e})$ and its vertex list contains exactly k different vertices. For simple cycles we reserve the lower-case bold letter $\mathbf{c}$. A simple cycle of length one is often called a *loop*.

We write E^k for the set of all paths of length k, E^k_u for the set of all paths of length k starting at the vertex u and E^k_{uv} for the set of all paths of length k starting at the vertex u and finishing at v. Similarly E^*_u denotes the set of all finite paths starting at the vertex u and E^*_{uv} the set of all finite paths from the vertex u to v. E^*_{uu} denotes the set of finite paths or cycles which start and end at u. We may write $E^{\leqslant k}_{uv}$ for the set of all paths, from u to v, of length less than or equal to k. For the set of all infinite paths we write $E^{\mathbb{N}}$, the set of all infinite paths with initial vertex

u is indicated by $E_u^{\mathbb{N}}$. The restriction of an infinite path $\mathbf{e} \in E^{\mathbb{N}}$ to its first k edges is denoted by $\mathbf{e}|_k$. We use D_{uv}^* for the set of all finite simple paths from the vertex u to the vertex v, with $D_{uv}^* \subset E_{uv}^*$.

1.2.8 Directed graph iterated function systems

$\big(V, E^*, i, t, r, ((X_v, d_v))_{v \in V}, (S_e)_{e \in E^1}\big)$ denotes a *directed graph IFS*, where IFS stands for iterated function system, and $\big(V, E^*, i, t, r, p, ((X_v, d_v))_{v \in V}, (S_e)_{e \in E^1}\big)$ denotes a *directed graph IFS with probabilities.* $\big(V, E^*, i, t\big)$ is the directed graph of any such IFS (see Subsection 1.2.7) and we always assume the directed graph is *strongly connected*, so there is at least one path connecting any two vertices. We also assume that each vertex in the directed graph has at least two edges leaving it, this is to avoid self-similar sets that consist of just single point sets, and attractors that are just scalar copies of those at other vertices (see [EM92]). The functions $r : E^* \to (0, 1)$ and $p : E^* \to (0, 1)$ assign contraction ratios and probabilities to the finite paths in the graph. To each vertex $v \in V$, is associated a complete metric space (X_v, d_v) and to each directed edge $e \in E^1$ is assigned a contraction $S_e : X_{t(e)} \to X_{i(e)}$ which has the contraction ratio given by the function $r(e) = r_e$. We follow the convention already established in the literature, see [Edg00] or [EM92], that a similarity maps in the opposite direction to the direction of the edge it is associated with in the graph.

The *probability function* $p : E^* \to (0, 1)$, where for an edge $e \in E^1$ we write $p(e) = p_e$, is such that

$$\sum_{e \in E_u^1} p_e = \sum_{v \in V} \Bigg(\sum_{e \in E_{uv}^1} p_e \Bigg) = 1, \tag{1.2.3}$$

for any vertex $u \in V$. That is the probability weights across all the edges leaving a vertex always sum to one. For a path $\mathbf{e} = e_1 e_2 \cdots e_k \in E^*$ we define $p(\mathbf{e}) = p_{\mathbf{e}} = p_{e_1} p_{e_2} \cdots p_{e_k}$. Similarly for the *contraction ratio function* $r : E^* \to (0, 1)$, the contraction ratio along a path $\mathbf{e} = e_1 e_2 \cdots e_k \in E^*$ is defined as $r(\mathbf{e}) = r_{\mathbf{e}} = r_{e_1} r_{e_2} \cdots r_{e_k}$. The ratio $r_{\mathbf{e}}$ is the ratio for the contraction $S_{\mathbf{e}} : X_{t(\mathbf{e})} \to X_{i(\mathbf{e})}$ along the path $\mathbf{e}$, where $S_{\mathbf{e}} = S_{e_1} \circ S_{e_2} \circ \cdots \circ S_{e_k}$.

For a simple cycle $\mathbf{c}$ we reserve a capital C to represent the contraction ratio along $\mathbf{c}$, so we may write $r(\mathbf{c}) = r_{\mathbf{c}} = C$.

In this thesis we are only going to be concerned with directed graph IFSs defined on n-dimensional Euclidean space, with $((X_v, d_v))_{v \in V} = ((\mathbb{R}^n, |\ \ |))_{v \in V}$ and where $(S_e)_{e \in E^1}$ are contracting similarities and not just contractions. The reader may assume then, whenever the notation $\big(V, E^*, i, t, r, ((\mathbb{R}^n, |\ \ |))_{v \in V}, (S_e)_{e \in E^1}\big)$ or $\big(V, E^*, i, t, r, p, ((\mathbb{R}^n, |\ \ |))_{v \in V}, (S_e)_{e \in E^1}\big)$ appears in the text, that for each $e \in E^1$, $S_e : \mathbb{R}^n \to \mathbb{R}^n$ is a contracting similarity with r_e, $0 < r_e < 1$ its contracting similarity ratio.

We will often write *k-vertex IFS* as a shortening of k-vertex directed graph IFS.

1.2.9 The spectral radius of the matrix $\mathbf{A}(t)$

Let $\mathbf{v} = (v_1, v_2, \ldots, v_n)^T$ and $\mathbf{w} = (w_1, w_2, \ldots, w_n)^T$ be two real n-dimensional (column) vectors. We define $\mathbf{v} \leqslant \mathbf{w}$ as follows

$$\mathbf{v} \leqslant \mathbf{w} \text{ if and only if } v_i \leqslant w_i \text{ for all i, } 1 \leqslant i \leqslant n,$$

and similarly

$$\mathbf{v} < \mathbf{w} \text{ if and only if } v_i < w_i \text{ for all i, } 1 \leqslant i \leqslant n,$$

We denote the zero vector as $\mathbf{0} = (0, 0, \ldots, 0)^T$. A vector $\mathbf{v}$ is a *positive vector* if $\mathbf{0} < \mathbf{v}$ and is a *non-negative vector* if $\mathbf{0} \leqslant \mathbf{v}$. Using M_{ij} for the ijth entry of a matrix $\mathbf{M}$, these definitions can be extended to the set of real $m \times m$ matrices in the obvious way. A *non-negative matrix* $\mathbf{M}$, has $0 \leqslant M_{ij}$ for all of its entries, that is for all $1 \leqslant i, j \leqslant m$. For two non-negative matrices $\mathbf{C}$, $\mathbf{D}$, $\mathbf{C} \leqslant \mathbf{D}$ means $C_{ij} \leqslant D_{ij}$ for all $1 \leqslant i, j \leqslant m$ and $\mathbf{C} = \mathbf{D}$ means $C_{ij} = D_{ij}$ for all $1 \leqslant i, j \leqslant m$.

A non-negative $m \times m$ matrix $\mathbf{M}$ is *irreducible* if, for each $1 \leqslant i, j \leqslant m$, there exists $n = n(i, j) \in \mathbb{N}$, which may depend on i and j, such that the ijth entry of $\mathbf{M}^n$ is positive, that is

$$M^n_{ij} > 0.$$

If m is the number of vertices in the directed graph of a directed graph IFS, that is $m = \#V$, let $\mathbf{A}(t)$ denote the real $m \times m$ matrix whose uvth entry is

$$A_{uv}(t) = \sum_{e \in E^1_{uv}} r_e^t,$$

where $t \geqslant 0$. Clearly as r_e, $e \in E^1$, are positive similarity ratios, the matrix $\mathbf{A}(t)$ is non-negative. Since the graph is strongly connected there is at least one path $\mathbf{e} \in E^*_{uv}$ from any vertex u to any vertex v, and for such $\mathbf{e}$, if $n = n(u, v) = |\mathbf{e}|$ then it follows that the uvth entry of $\mathbf{A}^n(t)$ is

$$A^n_{uv}(t) = \sum_{\mathbf{e} \in E^n_{uv}} r_{\mathbf{e}}^t > 0,$$

and so $\mathbf{A}(t)$ is irreducible. The Perron-Frobenius Theorem, see Theorem 1.1, [Sen73], now ensures that

- $\mathbf{A}(t)$ has a real eigenvalue $r > 0$, such that $r \geqslant |\lambda|$ for any eigenvalue $\lambda \neq r$.
- r has strictly positive left and right eigenvectors, which are unique up to a scaling factor.

Let $\rho(\mathbf{A}(t)) = r$, then we call $\rho(\mathbf{A}(t))$ the *spectral radius of the matrix* $\mathbf{A}(t)$, and we may write

$$\rho(\mathbf{A}(t)) = \max\{|\lambda| : \lambda \text{ is an eigenvalue of } \mathbf{A}(t)\}. \tag{1.2.4}$$

In general we will use s to denote the unique non-negative number that is the solution of

$$\rho(\mathbf{A}(t)) = 1.$$

For a proof of the existence and uniqueness of s see [MW88] or [Edg00]. We denote by $\mathbf{h}$ the unique, up to scaling, positive eigenvector such that

$$\mathbf{A}(s)\mathbf{h} = \rho(\mathbf{A}(s))\mathbf{h} = \mathbf{h}. \tag{1.2.5}$$

1.2.10 Separation conditions

A directed graph IFS, $\left(V, E^*, i, t, r, ((\mathbb{R}^n, |\ \ |))_{v\in V}, (S_e)_{e\in E^1}\right)$, as defined in Subsection 1.2.8, determines a unique list of non-empty compact sets $(F_u)_{u\in V} \in (K(\mathbb{R}^n))^{\#V}$. This is a fundamental property of directed graph IFSs and is proved in Theorem 1.3.4 which follows in Section 1.3. The sets $(F_u)_{u\in V}$ are often referred to as the unique list of *attractors* or *self-similar sets* of the system. In order to obtain meaningful results about such attractors it is often necessary to restrict the directed graph IFS to a system which satisfies one or more of the separation conditions that are defined below.

Since an attractor F_u is a compact set it follows that the convex hull $C(F_u)$ is also compact for each $u \in V$. We write the $\#V$-fold Cartesian product of $\mathbb{R}^n$ as

$$(\mathbb{R}^n)^{\#V} = \underbrace{\mathbb{R}^n \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{\#V \text{ times}}.$$

- The *open set condition* (OSC) is satisfied if and only if there exist non-empty bounded open sets $(U_u)_{u\in V} \subset (\mathbb{R}^n)^{\#V}$, with for each $u \in V$,

  $$S_e(U_{t(e)}) \subset U_u \text{ for all } e \in E_u^1$$

  and

  $$S_e(U_{t(e)}) \cap S_f(U_{t(f)}) = \emptyset \text{ for all } e, f \in E_u^1, \text{ with } e \neq f.$$

  See [Hut81], [Fal03] or [Edg00].

- The *strong open set condition* (SOSC) is satisfied if and only if the OSC is satisfied for non-empty bounded open sets $(U_u)_{u\in V} \subset (\mathbb{R}^n)^{\#V}$, where for each $u \in V$,

  $$F_u \cap U_u \neq \emptyset.$$

  See [Wan97].

- The *convex open set condition* (COSC) is satisfied if and only if the OSC is satisfied for non-empty bounded open sets $(U_u)_{u\in V} \subset (\mathbb{R}^n)^{\#V}$, where these sets are also convex. See [FW09].

- The *strong separation condition* (SSC) is satisfied if and only if for each $u \in V$,

  $$S_e(F_{t(e)}) \cap S_f(F_{t(f)}) = \emptyset \text{ for all } e, f \in E_u^1, \text{ with } e \neq f.$$

- The *convex strong separation condition* (CSSC) is satisfied if and only if for each $u \in V$,

  $$S_e(C(F_{t(e)})) \cap S_f(C(F_{t(f)})) = \emptyset \text{ for all } e, f \in E_u^1, \text{ with } e \neq f.$$

If the COSC holds then it can be shown that the OSC is satisfied by the convex open sets $(C(F_u)^\circ)_{u\in V}$, provided $C(F_u)^\circ \neq \emptyset$ for each $u \in V$. If however $C(F_u)^\circ = \emptyset$ for some $u \in V$ then we may reduce the dimension n, of the parent space $\mathbb{R}^n$, in which the system is constructed. As an example the Cantor set, C, can be generated as a subset of $\mathbb{R}^2$ by using the similarities $S_1(x) = \frac{1}{3}x$, $S_2(x) = \frac{1}{3}x + (\frac{2}{3}, 0)$, for $x \in \mathbb{R}^2$, where $(\frac{2}{3}, 0)$ is a point in $\mathbb{R}^2$. Now $C(\mathrm{C})^\circ = ([0,1] \times \{0\})^\circ = \emptyset$ in $\mathbb{R}^2$, whereas $C(\mathrm{C})^\circ = (0,1)$, an open interval in $\mathbb{R}$. This is because whether a set is open or not depends on the parent space being referred to.

For 1-vertex directed graph IFSs the OSC was shown to be equivalent to the SOSC in [Sch94]. For general directed graph IFSs this equivalence is given by the following theorem which is proved in [Wan97].

Theorem 1.2.1. *Let $\big(V, E^*, i, t, r, ((\mathbb{R}^n, |\ \ |))_{v\in V}, (S_e)_{e\in E^1}\big)$ be a directed graph IFS, let $(F_u)_{u\in V}$ be the unique invariant list of non-empty compact sets as given in Theorem 1.3.4. The following statements are equivalent.*
(a) *The OSC holds.*
(b) *The SOSC holds.*
(c) $\mathcal{H}^s(F_u) > 0$, *for each $u \in V$, where $s = \dim_{\mathrm{H}} F_u$.*

Proof. That (b) implies (a) follows from the definitions. That (a) implies (c) is given in Theorem 1.3.7. That (c) implies (b) is given in [Wan97]. □

In Chapters 2 and 3 we will be mainly concerned with systems in $\mathbb{R}$ for which the CSSC holds. In Chapter 4 we consider systems in $\mathbb{R}^n$ for which the OSC/SOSC holds.

1.3 Background theory

In this section we give the main results that form the starting point for the mathematics in the chapters that follow.

A proof of the Contraction Mapping Theorem, also known as the Banach fixed-point principle, may be found in [Edg00], [Apo78] or [Mad88].

Theorem 1.3.1 (Contraction Mapping Theorem)**.** *Let (X, d) be a complete metric space, and let $f : X \to X$ be a contraction mapping. Then there is a unique element $x_f \in X$, such that*

$$f(x_f) = x_f.$$

Also, for each $x \in X$, the sequence of iterates

$$x,\ f(x),\ f^2(x),\ f^3(x),\ \ldots$$

converges to the unique fixed point x_f. That is for any $x \in X$,

$$d(f^k(x), x_f) \to 0, \ \text{as } k \to \infty.$$

1.3.1 Self-similar sets

In this subsection $((K(\mathbb{R}^n))^{\#V}, D_{\mathrm{H}})$ is the complete metric space defined in Subsection 1.2.5, where each component $K(\mathbb{R}^n)$ is the set of non-empty compact subsets of $\mathbb{R}^n$, and D_{H} is the metric defined in Equation (1.2.2), derived from the Hausdorff metric d_{H}.

Our next lemma is Proposition 2.5.6 of [Edg00].

Lemma 1.3.2. *Let (A_m) be a sequence of non-empty compact sets with $A_m \in K(\mathbb{R}^n)$, for each $m \in \mathbb{N} \cup \{0\}$, and suppose they decrease, that is*

$$A_0 \supset A_1 \supset A_2 \supset \cdots,$$

then (A_m) converges, with respect to the Hausdorff metric d_{H}, to the non-empty compact set $A \in K(\mathbb{R}^n)$, where $A = \bigcap_{m=0}^{\infty} A_m$.

In the next lemma we use the notation $(A_{0,u})_{u\in V} \supset (A_{1,u})_{u\in V}$ to mean $A_{0,u} \supset A_{1,u}$ componentwise for each $u \in V$ and so on.

Lemma 1.3.3. *Let $((A_{m,u})_{u\in V})$ be a sequence of non-empty compact sets with $(A_{m,u})_{u\in V} \in (K(\mathbb{R}^n))^{\#V}$, for each $m \in \mathbb{N} \cup \{0\}$, and suppose they decrease, that is*

$$(A_{0,u})_{u\in V} \supset (A_{1,u})_{u\in V} \supset (A_{2,u})_{u\in V} \supset \cdots,$$

then $((A_{m,u})_{u\in V})$ converges, with respect to the Hausdorff metric D_{H}, to the non-empty compact set $(A_u)_{u\in V} \in (K(\mathbb{R}^n))^{\#V}$, where

$$(A_u)_{u\in V} = \bigcap_{m=0}^{\infty} (A_{m,u})_{u\in V} = \left(\bigcap_{m=0}^{\infty} A_{m,u} \right)_{u\in V}.$$

Proof. Lemma 1.3.2 proves that if (A_n), $n \in \mathbb{N} \cup \{0\}$, is a decreasing sequence of non-empty compact sets in $K(\mathbb{R}^n)$ then it converges with respect to the Hausdorff metric d_{H}, to a non-empty compact set $A \in K(\mathbb{R}^n)$ where $A = \bigcap_{n=0}^{\infty} A_n$. So for each $u \in V$ the component sequence $(A_{n,u})$ converges to a non-empty compact set A_u with respect to the Hausdorff metric d_{H}, where $A_u = \bigcap_{n=0}^{\infty} A_{n,u}$ and the lemma now follows immediately from the definition of the metric D_{H}. □

In the special case of a 1-vertex directed graph IFS, the next theorem is the same as Theorem 9.1 of [Fal03]. As pointed out in Subsection 1.2.8, we are only going to be concerned with directed graph IFSs where $(S_e)_{e\in E^1}$ are contracting similarities, but we have taken care in the proof that follows, so that it is still valid if we take $(S_e)_{e\in E^1}$ to be more general contractions.

Theorem 1.3.4. *Let $\big(V, E^*, i, t, r, ((\mathbb{R}^n, |\ \ |))_{v\in V}, (S_e)_{e\in E^1}\big)$ be a directed graph IFS, and let the function $f : (K(\mathbb{R}^n))^{\#V} \to (K(\mathbb{R}^n))^{\#V}$, be given by*

$$f\big((A_u)_{u\in V}\big) = \left(\bigcup_{e\in E_u^1} S_e(A_{t(e)}) \right)_{u\in V}, \tag{1.3.1}$$

for each $(A_u)_{u\in V} \in (K(\mathbb{R}^n))^{\#V}$.

Then there exists a unique list of non-empty compact sets $(F_u)_{u\in V} \in (K(\mathbb{R}^n))^{\#V}$, *such that*

$$f\big((F_u)_{u\in V}\big) = \Bigg(\bigcup_{e\in E_u^1} S_e(F_{t(e)}) \Bigg)_{u\in V} = (F_u)_{u\in V}, \tag{1.3.2}$$

and where, for any $(A_u)_{u\in V} \in (K(\mathbb{R}^n))^{\#V}$,

$$D_{\mathrm{H}}\big(f^k\big((A_u)_{u\in V}\big), (F_u)_{u\in V}\big) \to 0, \ \text{as } k \to \infty. \tag{1.3.3}$$

For any $(B_u)_{u\in V} \in (K(\mathbb{R}^n))^{\#V}$, *such that* $f\big((B_u)_{u\in V}\big) \subset (B_u)_{u\in V}$, *where we put* $f^0\big((B_u)_{u\in V}\big) = (B_u)_{u\in V}$,

$$(F_u)_{u\in V} = \bigcap_{k=0}^{\infty} f^k\big((B_u)_{u\in V}\big). \tag{1.3.4}$$

Proof. First we state two results that will be needed. For any $A, B, C, D \in K(\mathbb{R}^n)$,

$$d_{\mathrm{H}}(A\cup B, C\cup D) \leqslant \max\{d_{\mathrm{H}}(A,C),\, d_{\mathrm{H}}(B,D)\}, \tag{1.3.5}$$

and also for any contracting similarity of the system, $S_e : \mathbb{R}^n \to \mathbb{R}^n$, $e \in E^1$, we have

$$d_{\mathrm{H}}(S_e(A), S_e(B)) \leqslant r_e d_{\mathrm{H}}(A,B). \tag{1.3.6}$$

The graph is strongly connected, so any vertex must be the terminal vertex of at least one edge and the set of vertices in the graph can be written as $V = \{t(e) : e \in E^1\}$, we use this fact in the inequalities that follow. For $(A_u)_{u\in V}, (B_u)_{u\in V} \in (K(\mathbb{R}^n))^{\#V}$,

$$\begin{aligned}
&D_{\mathrm{H}}\big(f\big((A_u)_{u\in V}\big), f\big((B_u)_{u\in V}\big)\big)\\
&\qquad = \max\Bigg\{ d_{\mathrm{H}}\Bigg(\bigcup_{e\in E_u^1} S_e(A_{t(e)}), \bigcup_{e\in E_u^1} S_e(B_{t(e)}) \Bigg) : u \in V \Bigg\} && \text{(by (1.3.1))}\\
&\qquad \leqslant \max\Big\{ d_{\mathrm{H}}(S_e(A_{t(e)}), S_e(B_{t(e)})) : u\in V,\ e \in E_u^1 \Big\} && \text{(by (1.3.5))}\\
&\qquad \leqslant \max\big\{r_e : e\in E^1\big\} \max\Big\{ d_{\mathrm{H}}(A_{t(e)}, B_{t(e)}) : e \in E^1 \Big\} && \text{(by (1.3.6))}\\
&\qquad = \max\big\{r_e : e\in E^1\big\} \max\Big\{ d_{\mathrm{H}}(A_u, B_u) : u \in V \Big\}\\
&\qquad = \max\big\{r_e : e\in E^1\big\} D_{\mathrm{H}}\big((A_u)_{u\in V}, (B_u)_{u\in V}\big).
\end{aligned}$$

This establishes that f is a contraction mapping with respect to the metric D_{H}, so Equations (1.3.2) and (1.3.3) follow immediately by Theorem 1.3.1, see Theorem 4.3.5 in [Edg00].

For Equation (1.3.4), we note that $\big(f^k\big((B_u)_{u\in V}\big)\big)$ is a decreasing sequence of non-empty compact sets in the metric space $((K(\mathbb{R}^n))^{\#V}, D_{\mathrm{H}})$ and so it will converge, with respect to the metric D_{H}, to a non-empty compact set $(C_u)_{u\in V} \in (K(\mathbb{R}^n))^{\#V}$ where

$$(C_u)_{u\in V} = \bigcap_{k=0}^{\infty} f^k\big((B_u)_{u\in V}\big),$$

by Lemma 1.3.3. That $D_{\mathrm{H}}\big((F_u)_{u\in V},(C_u)_{u\in V}\big)=0$ is given by the triangle inequality since

$$D_{\mathrm{H}}\big((F_u)_{u\in V},(C_u)_{u\in V}\big)\leqslant D_{\mathrm{H}}\big((F_u)_{u\in V},f^n\big((B_u)_{u\in V}\big)\big)+D_{\mathrm{H}}\big(f^n\big((B_u)_{u\in V},(C_u)_{u\in V}\big),$$

and the right hand side of this expression tends to 0 as $n\to\infty$. □

A metric space (X,d) is *sequentially compact* if and only if every sequence in X has a convergent subsequence, see [Mad88]. We use sequential compactness in the proof of the next lemma.

Lemma 1.3.5. *Let $(F_u)_{u\in V}\in(K(\mathbb{R}^n))^{\#V}$ be the unique invariant list of any directed graph IFS. For each $u\in V$, let the mapping, $\phi_u : E_u^{\mathbb{N}}\to F_u$, be defined for each infinite path $\mathbf{e}\in E_u^{\mathbb{N}}$ by*

$$\phi_u(\mathbf{e})=x,\ \textit{where}\ \{x\}=\bigcap_{k=1}^{\infty}S_{\mathbf{e}|_k}(F_{t(\mathbf{e}|_k)}). \tag{1.3.7}$$

Then ϕ_u is surjective. If the SSC is satisfied then ϕ_u is bijective.

Proof. For a sequence $\mathbf{e}\in E_u^{\mathbb{N}}$,

$$S_{\mathbf{e}|_k}(F_{t(\mathbf{e}|_k)})=(S_{e_1}\circ S_{e_2}\circ\cdots\circ S_{e_k})(F_{t(e_k)})=S_{e_1}(S_{e_2}(\cdots S_{e_k}(F_{t(e_k)})\cdots)),$$

and $\big(S_{\mathbf{e}|_k}(F_{t(\mathbf{e}|_k)})\big)$ is a sequence of non-empty compact sets that decreases with

$$F_u\supset S_{e_1}(F_{t(e_1)})\supset S_{e_1}(S_{e_2}(F_{t(e_2)}))\supset\cdots.$$

By Lemma 1.3.2, $\big(S_{\mathbf{e}|_k}(F_{t(\mathbf{e}|_k)})\big)$ converges to a non-empty compact set $A\in K(\mathbb{R}^n)$ with respect to d_{H}. Now let $\alpha=\max\{|F_u| : u\in V\}$ and $r_{\max}=\max\{r_e : e\in E^1\}$ then

$$\big|S_{\mathbf{e}|_k}(F_{t(\mathbf{e}|_k)})\big|=r_{\mathbf{e}|_k}\big|F_{t(\mathbf{e}|_k)}\big|\leqslant r_{\mathbf{e}|_k}\alpha\leqslant r_{\max}^k\alpha, \tag{1.3.8}$$

and clearly the sequence of diameters $\big(\big|S_{\mathbf{e}|_k}(F_{t(\mathbf{e}|_k)})\big|\big)$ must converge to 0 as $k\to\infty$. Any non-empty compact set of diameter 0 is a singleton set and so $A=\{x\}\subset F_u$. This establishes that the map ϕ_u is well-defined.

Now consider $E_u^{\mathbb{N}}$ as a metric space $(E_u^{\mathbb{N}},d_{1/2})$, with the metric $d_{1/2}$ defined, for $\mathbf{e},\mathbf{f}\in E_u^{\mathbb{N}}$ by

$$d_{1/2}(\mathbf{e},\mathbf{f})=\frac{1}{2^k},$$

where $k\in\mathbb{N}\cup\{0\}$, is the length of the longest common prefix of $\mathbf{e}$ and $\mathbf{f}$, see Proposition 2.1.8, [Edg00].

(a) *The metric space $(E_u^{\mathbb{N}},d_{1/2})$ is compact.*

A metric space is compact if and only if it is sequentially compact, see Theorem 21, Chapter 2, [Mad88]. We aim to show $(E_u^{\mathbb{N}},d_{1/2})$ is sequentially compact. Let $(\mathbf{a}_n)=(\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3,\ldots)$ be any sequence of paths in $E_u^{\mathbb{N}}$. If the sequence contains only finitely many different paths then there exists $m\in\mathbb{N}$ such that $\mathbf{a}_n=\mathbf{a}_m$ for all $n\geqslant m$ and $(\mathbf{a}_m,\mathbf{a}_{m+1},\mathbf{a}_{m+2},\ldots)$ is a convergent subsequence. Now suppose the sequence contains infinitely many different paths so that $A = \{\mathbf{a}_i : i \in \mathbb{N}\}$ is an infinite set. As

$$A = \bigcup_{e \in E_u^1} (A \cap [e])$$

it follows that $A \cap [e_1]$ must also be an infinite set for some edge $e_1 \in E_u^1$. Let $\mathbf{a}_{n_1}$ be the first path in the sequence $(\mathbf{a}_n)$ such that $\mathbf{a}_{n_1} \in A \cap [e_1]$. Because

$$A \cap [e_1] = \bigcup_{e \in E_{t(e_1)}^1} (A \cap [e_1 e])$$

there exists an edge $e_2 \in E_{t(e_1)}^1$ such that $A \cap [e_1 e_2]$ is an infinite set. Let $\mathbf{a}_{n_2}$ be the first path in the sequence $(\mathbf{a}_n)$, such that $\mathbf{a}_{n_2} \in A \cap [e_1 e_2]$ and $\mathbf{a}_{n_2} \neq \mathbf{a}_{n_1}$. Continuing in this way, for $k \in \mathbb{N}$, we may define $\mathbf{a}_{n_k}$ as the first path in $(\mathbf{a}_n)$ such that $\mathbf{a}_{n_k} \in A \cap [e_1 e_2 \cdots e_k]$, where $A \cap [e_1 e_2 \cdots e_k]$ is an infinite set, and $\mathbf{a}_{n_k} \neq \mathbf{a}_{n_j}$ for all $1 \leqslant j < k$. This defines a subsequence $(\mathbf{a}_{n_k})$ of $(\mathbf{a}_n)$ and a path $\mathbf{e} = e_1 e_2 \cdots \in E_u^{\mathbb{N}}$. For every $\varepsilon > 0$ there exists $m \in \mathbb{N}$ such that

$$d_{1/2}(\mathbf{a}_{n_k}, \mathbf{e}) \leqslant \frac{1}{2^m} < \varepsilon,$$

for all $k \geqslant m$, so $(\mathbf{a}_{n_k})$ converges to $\mathbf{e}$.

(b) ϕ_u *is continuous.*

Let $\mathbf{e} \in E_u^{\mathbb{N}}$ and let $\varepsilon > 0$ then by Equation (1.3.8) there exists $k \in \mathbb{N}$ such that

$$\left|S_{\mathbf{e}|_k}(F_{t(\mathbf{e}|_k)})\right| \leqslant r_{\max}^k \alpha < \varepsilon$$

Now let $\mathbf{f} \in E_u^{\mathbb{N}}$ with $d_{1/2}(\mathbf{e}, \mathbf{f}) < \frac{1}{2^k}$ then, from the definition of the metric $d_{1/2}$, it follows that $\mathbf{e}|_k = \mathbf{f}|_k$ so that $\phi_u(\mathbf{e}), \phi_u(\mathbf{f}) \in S_{\mathbf{e}|_k}(F_{t(\mathbf{e}|_k)})$ and

$$|\phi_u(\mathbf{e}) - \phi_u(\mathbf{f})| \leqslant \left|S_{\mathbf{e}|_k}(F_{t(\mathbf{e}|_k)})\right| < \varepsilon.$$

This establishes that ϕ_u is continuous.

(c) ϕ_u *is surjective.*

The continuous image of a compact set is compact, see Theorem 18, Chapter 2, [Mad88] or Theorem 2.3.15, [Edg00]. Parts (a) and (b) imply that $\phi_u(E_u^{\mathbb{N}})$ is a compact subset of F_u which is also clearly non-empty. Now consider $e \in E_u^1$ with $\mathbf{f} = e_1 e_2 \cdots \in E_{t(e)}^{\mathbb{N}}$ and $\mathbf{e} = e e_1 e_2 \cdots \in E_u^{\mathbb{N}}$ then it follows that $\phi_u(\mathbf{e}) = S_e(\phi_{t(e)}(\mathbf{f}))$, which is enough to show that

$$\phi_u(E_u^{\mathbb{N}}) = \bigcup_{e \in E_u^1} S_e(\phi_{t(e)}(E_{t(e)}^{\mathbb{N}})),$$

for each $u \in V$. This means that Equation (1.3.2) holds for the list of non-empty compact sets $(\phi_u(E_u^{\mathbb{N}}))_{u \in V}$ and so $(\phi_u(E_u^{\mathbb{N}}))_{u \in V} = (F_u)_{u \in V}$. Therefore ϕ_u is surjective.

Finally if the SSC is satisfied the union on the right hand side of Equation (1.3.1) is disjoint so F_u will be totally disconnected and ϕ_u will be injective. □

Lemma 1.3.6. *Let $(F_u)_{u\in V} \in (K(\mathbb{R}^n))^{\#V}$ be the unique invariant list of any directed graph IFS and suppose that the OSC is satisfied by the non-empty bounded open sets $(U_u)_{u\in V}$, then*

$$F_u \subset \overline{U}_u, \text{ for each } u \in V.$$

Proof. Clearly $(\overline{U}_u)_{u\in V} \in (K(\mathbb{R}^n))^{\#V}$ and for the function f as defined in Equation (1.3.1), the OSC ensures that $f\big((\overline{U}_u)_{u\in V}\big) \subset (\overline{U}_u)_{u\in V}$. By Theorem 1.3.4, Equation (1.3.4),

$$(F_u)_{u\in V} = \bigcap_{k=0}^{\infty} f^k\big((\overline{U}_u)_{u\in V}\big) \subset f^0\big((\overline{U}_u)_{u\in V}\big) = (\overline{U}_u)_{u\in V}.$$

□

A set A is an *s-set* if its s-dimensional Hausdorff measure is finite and positive. The next theorem states the dimension of the attractors of a directed graph IFS and shows that they are s-sets, provided the OSC holds. For a 1-vertex directed graph IFS, this theorem is the same as Theorem 9.3 of [Fal03].

Theorem 1.3.7. *Let $\big(V, E^*, i, t, r, ((\mathbb{R}^n, |\ \ |))_{v\in V}, (S_e)_{e\in E^1}\big)$ be a directed graph IFS and $(F_u)_{u\in V}$ the unique list of attractors of the system. Let $m = \#V$ and let $\mathbf{A}(t)$, $t \geqslant 0$, denote the $m \times m$ matrix whose uvth entry is*

$$A_{uv}(t) = \sum_{e\in E^1_{uv}} r_e^t.$$

Let $\rho\,(\mathbf{A}(t))$ be the spectral radius of $\mathbf{A}(t)$, and let s be the unique non-negative real number that is the solution of $\rho\,(\mathbf{A}(t)) = 1$.

If the OSC is satisfied then, for each $u \in V$,

$$s = \dim_{\mathrm{H}} F_u = \dim_{\mathrm{B}} F_u$$

and

$$0 < \mathcal{H}^s\,(F_u) < +\infty.$$

Notes on a proof. We prove $\mathcal{H}^s(F_u) < +\infty$ in full in part (a) that follows, but we only indicate an approach for a proof of $0 < \mathcal{H}^s(F_u)$ in part (b), and for a proof of $\overline{\dim}_{\mathrm{B}} F_u \leqslant s$ in part (c).

(a) $\mathcal{H}^s(F_u) < +\infty$, *for all $u \in V$.*

Consider any vertex $u \in V$ as fixed. Let $\mathbf{h}$ be the unique (up to scaling) positive eigenvector of Equation (1.2.5). If $m = \#V$ and the set of vertices is ordered as $V = (v_1, v_2, \ldots, v_m)$ then we use the notation $(h_v)_{v\in V}$ to represent the ordered m-tuple $(h_{v_1}, h_{v_2}, \ldots, h_{v_m})$, so the column eigenvector $\mathbf{h}$ can be written as

$$\mathbf{h} = (h_v)^T_{v\in V} = (h_{v_1}, h_{v_2}, \ldots, h_{v_m})^T.$$

Let $h_{\min} = \min\{h_v : v \in V\}$. It follows that, for any $k \in \mathbb{N}$

$$h_u = \big(\mathbf{A}(s)^k \mathbf{h}\big)_u = \sum_{\mathbf{e}\in E^k_u} r^s_{\mathbf{e}} h_{t(\mathbf{e})} \geqslant h_{\min} \sum_{\mathbf{e}\in E^k_u} r^s_{\mathbf{e}},$$

that is

$$\sum_{\mathbf{e}\in E_u^k} r_{\mathbf{e}}^s \leqslant \frac{h_u}{h_{\min}}, \text{ for all } k \in \mathbb{N}. \tag{1.3.9}$$

Iterating Equation (1.3.2) of Theorem 1.3.4, for any $k \in \mathbb{N}$, we obtain,

$$F_u = \bigcup_{\mathbf{e}\in E_u^k} S_{\mathbf{e}}(F_{t(\mathbf{e})}).$$

Let $\alpha = \max\{|F_v| : v \in V\}$ and $r_{\max} = \max\{r_e : e \in E^1\}$, then given any $\delta > 0$, we may choose k large enough so that

$$\left|S_{\mathbf{e}}(F_{t(\mathbf{e})})\right| = r_{\mathbf{e}}\left|F_{t(\mathbf{e})}\right| \leqslant r_{\max}^k \alpha \leqslant \delta,$$

for all paths $\mathbf{e} \in E_u^k$. For such k the sets $\left\{S_{\mathbf{e}}(F_{t(\mathbf{e})}) : \mathbf{e} \in E_u^k\right\}$ are a δ-cover of F_u with

$$\sum_{\mathbf{e}\in E_u^k} \left|S_{\mathbf{e}}(F_{t(\mathbf{e})})\right|^s = \sum_{\mathbf{e}\in E_u^k} r_{\mathbf{e}}^s \left|F_{t(\mathbf{e})}\right|^s \leqslant \alpha^s \sum_{\mathbf{e}\in E_u^k} r_{\mathbf{e}}^s \leqslant \alpha^s \frac{h_u}{h_{\min}},$$

where the last inequality uses Equation (1.3.9). This means that $\mathcal{H}_\delta^s(F_u) \leqslant \alpha^s \frac{h_u}{h_{\min}}$, for any $\delta > 0$, and so $\mathcal{H}^s(F_u) \leqslant \alpha^s \frac{h_u}{h_{\min}} < +\infty$.

(b) $0 < \mathcal{H}^s(F_u)$, *for all* $u \in V$.

Consider any vertex $u \in V$ as fixed. Let $p : E^* \to (0,1)$ be the probability function, see Subsection 1.2.8, defined by

$$p_{\mathbf{e}} = h_{i(\mathbf{e})}^{-1} r_{\mathbf{e}}^s h_{t(\mathbf{e})}.$$

Now

$$\sum_{e\in E_u^1} p_e = h_u^{-1} \sum_{e\in E_u^1} r_e^s h_{t(e)} = h_u^{-1}\big(\mathbf{A}(s)\mathbf{h}\big)_u = 1,$$

that is

$$\sum_{e\in E_u^1} p_e = 1. \tag{1.3.10}$$

Also for any finite path $\mathbf{e} \in E_u^k$ and an edge $f \in E_{t(\mathbf{e})}^1$, $\mathbf{e}f \in E_u^{k+1}$, and

$$\begin{aligned}\sum_{f\in E_{t(\mathbf{e})}^1} p_{\mathbf{e}f} &= h_u^{-1} \sum_{f\in E_{t(\mathbf{e})}^1} r_{\mathbf{e}f}^s h_{t(f)} = h_u^{-1} r_{\mathbf{e}}^s \sum_{f\in E_{t(\mathbf{e})}^1} r_f^s h_{t(f)} \\ &= h_u^{-1} r_{\mathbf{e}}^s \big(\mathbf{A}(s)\mathbf{h}\big)_{t(\mathbf{e})} = h_u^{-1} r_{\mathbf{e}}^s h_{t(\mathbf{e})} = p_{\mathbf{e}},\end{aligned}$$

that is

$$\sum_{f\in E_{t(\mathbf{e})}^1} p_{\mathbf{e}f} = p_{\mathbf{e}}. \tag{1.3.11}$$

Let $\mathbb{A}$ be the algebra of sets generated by the cylinder sets of the sequence space $E_u^{\mathbb{N}}$ and let μ_u be defined on any cylinder set $[\mathbf{e}|_k] = [e_1 e_2 \cdots e_k] \subset E_u^{\mathbb{N}}$ by

$$\mu_u([\mathbf{e}|_k]) = p_{\mathbf{e}|_k},$$

see Subsection 1.2.1 for the definition of a cylinder set and Subsection 1.2.2 for measure theory definitions. Now $(E_u^{\mathbb{N}}, d_{1/2})$ is a complete, compact metric space and the cylinder sets in $E_u^{\mathbb{N}}$ are both compact and open. This means that if $A_i \in \mathbb{A}$, $i \in \mathbb{N}$, are disjoint cylinders and $A = \bigcup_{i=1}^{\infty} A_i \in \mathbb{A}$, then $\{A_i\}$ is an open cover of A. But A is compact, being a finite union of compact sets, from the definition of the algebra $\mathbb{A}$. This means that $\{A_i\}$ must contain a finite subcover and so $A = \bigcup_{i=1}^{n} A_i \in \mathbb{A}$. That is $A \neq \bigcup_{i=1}^{\infty} A_i$ for any infinite collection of disjoint cylinders A_i. Using Equations (1.3.10) and (1.3.11), we can show that $\mu_u(A) = \sum_{i=1}^{n} \mu_u(A_i)$. Also

$$\mu_u(E_u^{\mathbb{N}}) = \sum_{e \in E_u^1} \mu_u([e]) = \sum_{e \in E_u^1} p_e = 1.$$

This means that μ_u is a finite premeasure on $\mathbb{A}$ and applying Hahn's Extension Theorem, see [Bar98], there exists a unique extension of μ_u to a measure on $\mathbb{A}^*$, the smallest σ-algebra containing $\mathbb{A}$.

We can use μ_u to define a probability measure $\widetilde{\mu}_u$ on F_u as follows. For any Borel subset $A \subset F_u$ let $\widetilde{\mu}_u$ be defined by

$$\widetilde{\mu}_u(A) = (\mu_u \circ \phi_u^{-1})(A),$$

where ϕ_u is the surjection of Lemma 1.3.5. We define $\widetilde{\mu}_u$ on any Borel set $A \subset \mathbb{R}^n$ by $\widetilde{\mu}_u(A) = \widetilde{\mu}_u(A \cap F_u)$. Also

$$\widetilde{\mu}_u(F_u) = (\mu_u \circ \phi_u^{-1})(F_u) = \mu_u(E_u^{\mathbb{N}}) = 1.$$

The rest of the proof now proceeds in an identical pattern to the proof of Theorem 9.3 given in [Fal03] using an application of the Mass Distribution Principle.

(c) $\overline{\dim}_{\mathrm{B}} F_u \leqslant s$, *for all* $u \in V$.

Again we omit a proof of this as one can be constructed following that given in Theorem 9.3, [Fal03].

Since

$$s = \dim_{\mathrm{H}} F_u \leqslant \underline{\dim}_{\mathrm{B}} F_u \leqslant \overline{\dim}_{\mathrm{B}} F_u \leqslant s,$$

we obtain

$$s = \dim_{\mathrm{H}} F_u = \dim_{\mathrm{B}} F_u, \text{ for all } u \in V.$$

Finally we draw the readers attention to another proof that $0 < \mathcal{H}^s(F_u) < +\infty$ which is given in the proof of Theorem 3, [MW88]. □

1.3.2 Weak convergence and self-similar measures

For the existence and uniqueness of self-similar measures we again apply the Contraction Mapping Theorem. First we describe the product space of complete spaces of measures on which a contraction is then defined. Following [Edg98], Section 2.5, we write $P(\mathbb{R}^n)$ for the set of all Borel probability measures on $\mathbb{R}^n$. Let V_γ denote the set of all functions $h : \mathbb{R}^n \to \mathbb{R}$ satisfying $|h(x) - h(y)| \leqslant |x - y|$, $|h(x)| \leqslant \gamma$, $(\gamma > 0)$, for all $x, y \in \mathbb{R}^n$, that is V_γ is the set of all Lipschitz functions $h : \mathbb{R}^n \to \mathbb{R}$

with Lipschitz constant $M \leqslant 1$, (see Subsection 1.2.6), that are bounded by $\gamma > 0$. For $\lambda, \nu \in P(\mathbb{R}^n)$ we define

$$\rho_\gamma(\lambda, \nu) = \sup\left\{\left|\int h(x)d\lambda(x) - \int h(x)d\nu(x)\right| : h \in V_\gamma\right\}$$

and it can be shown that ρ_γ is a metric on $P(\mathbb{R}^n)$, (see [Edg98], Proposition (2.5.14)). A sequence (λ_n) in $P(\mathbb{R}^n)$ converges setwise to $\lambda \in P(\mathbb{R}^n)$ if and only if $\lim_{n\to\infty} \lambda_n(A) = \lambda(A)$ for all Borel sets A. It turns out that setwise convergence is too strong to be very useful as the following simple example with $\mathbb{R}^n$, $n = 1$, illustrates. Putting $\lambda_n = \delta_{1/n}$, the Dirac point measure, it would seem reasonable to expect that (λ_n) converges to $\lambda = \delta_0$. However setwise convergence does not occur as $\lambda_n((-\infty, 0]) = 0$ for all n, whereas $\lambda((-\infty, 0]) = 1$. This is why we consider weak or narrow convergence which will allow us to claim convergence in cases like these. A sequence (λ_n) in $P(\mathbb{R}^n)$ *converges weakly* to $\lambda \in P(\mathbb{R}^n)$ if and only if $\lim_{n\to\infty} \int h(x)d\lambda_n(x) = \int h(x)d\lambda(x)$ for all bounded continuous functions $h : \mathbb{R}^n \to \mathbb{R}$, see [Edg98] Definition(2.5.7) where Edgar uses narrowly in place of weakly.

It can be shown, see [Edg98] Theorems(2.5.7) and (2.5.17), that the following statements are equivalent:

- $\lim_{n\to\infty} \rho_\gamma(\lambda_n, \lambda) = 0$, for each $\gamma > 0$.
- (λ_n) converges weakly to λ.
- $\lim_{n\to\infty} \lambda_n(E) = \lambda(E)$ for all Borel sets $E \subset \mathbb{R}^n$, with $\lambda(\partial E) = 0$. (∂E is the boundary of a set E).
- $\lim_{n\to\infty} \int h(x)d\lambda_n(x) = \int h(x)d\lambda(x)$ for all bounded Lipschitz functions $h : \mathbb{R}^n \to \mathbb{R}$.

$(P(\mathbb{R}^n), \rho_\gamma)$ is a complete metric space (see [Edg98], Theorem (2.5.25)), and if A is a compact subset of $\mathbb{R}^n$ then $P(A)$, the set of all Borel probability measures on A, is compact in the weak topology, see [Edg98] Corollary(2.5.29), which means $(P(A), \rho_\gamma)$ is also a complete metric space. In particular $(P(F_u), \rho_\gamma)$ is complete, for each $u \in V$, where $(F_u)_{u\in V}$ is the invariant list of self-similar sets given by Equation (1.3.2).

If the vertices of the directed graph are ordered as $V = (v_1, v_2, \ldots, v_m)$, where $m = \#V$, then we will use the following notation for the m-fold Cartesian product space

$$\prod_{u\in V} P(F_u) = P(F_{v_1}) \times P(F_{v_2}) \times \cdots \times P(F_{v_m}).$$

This product space $\left(\prod_{u\in V} P(F_u), D_\gamma\right)$ is also a complete space with respect to the metric D_γ, which is defined as the maximum of the coordinate metrics. That is for $(\lambda_u)_{u\in V}, (\nu_u)_{u\in V} \in \prod_{u\in V} P(F_u)$, the metric D_γ is defined as

$$D_\gamma\big((\lambda_u)_{u\in V}, (\nu_u)_{u\in V}\big) = \max\left\{\rho_\gamma(\lambda_u, \nu_u) : u \in V\right\}.$$

Let $(\lambda_u)_{u\in V} \in \prod_{u\in V} P(F_u)$ and let $g : \prod_{u\in V} P(F_u) \to \prod_{u\in V} P(F_u)$, be the map given by

$$g((\lambda_u(A_u))_{u\in V}) = \left(\sum_{e\in E_u^1} p_e \lambda_{t(e)}(S_e^{-1}(A_u)) \right)_{u\in V}, \tag{1.3.12}$$

for all Borel sets $(A_u)_{u\in V}$, where $p : E^* \to (0,1)$ is any probability function as described in Subsection 1.2.8. The argument given in the 1-vertex case in [Fal97], Theorem 2.8, can naturally be extended to show that g is a contraction which has a unique fixed point $(\mu_u)_{u\in V}$, of *self-similar Borel probability measures*, for which Equation (1.3.13) holds. It can be shown that $(\mathrm{supp}(\mu_u))_{u\in V}$ satisfies Equation (1.3.2) and this implies that $(\mathrm{supp}(\mu_u))_{u\in V} = (F_u)_{u\in V}$. These observations form the content of our next theorem which is Proposition 3 of [Wan97]. For 1-vertex IFSs, see Theorem 2.8 of [Fal97].

Theorem 1.3.8. *Let $\left(V, E^*, i, t, r, p, ((\mathbb{R}^n, |\ \ |))_{v\in V}, (S_e)_{e\in E^1}\right)$ be a directed graph IFS with probabilities and let $(F_u)_{u\in V}$ be the unique list of attractors of the system, then there exists a unique list of self-similar Borel probability measures $(\mu_u)_{u\in V}$ such that*

$$(\mu_u(A_u))_{u\in V} = \left(\sum_{e\in E_u^1} p_e \mu_{t(e)}\big(S_e^{-1}(A_u)\big) \right)_{u\in V}, \tag{1.3.13}$$

for all Borel sets $(A_u)_{u\in V}$, with $(\mathrm{supp}(\mu_u))_{u\in V} = (F_u)_{u\in V}$.

Proof. See Proposition 3 of [Wan97]. □

2

Gap lengths

2.1 Introduction

The material in this chapter was motivated by considering the question "Do we really get anything new with a directed graph IFS with more than 1 vertex as opposed to a standard (1-vertex) IFS?". To make this fairly vague question more concrete, we restrict the type of directed graphs under investigation and only consider directed graph IFSs on the real line of the form $\left(V, E^*, i, t, r, ((\mathbb{R}, |\ \ |))_{v\in V}, (S_e)_{e\in E^1}\right)$ for which the CSSC holds. The Cantor set is the attractor of such a system, and it is clear that it is also the attractor of infinitely many different sets of similarities.

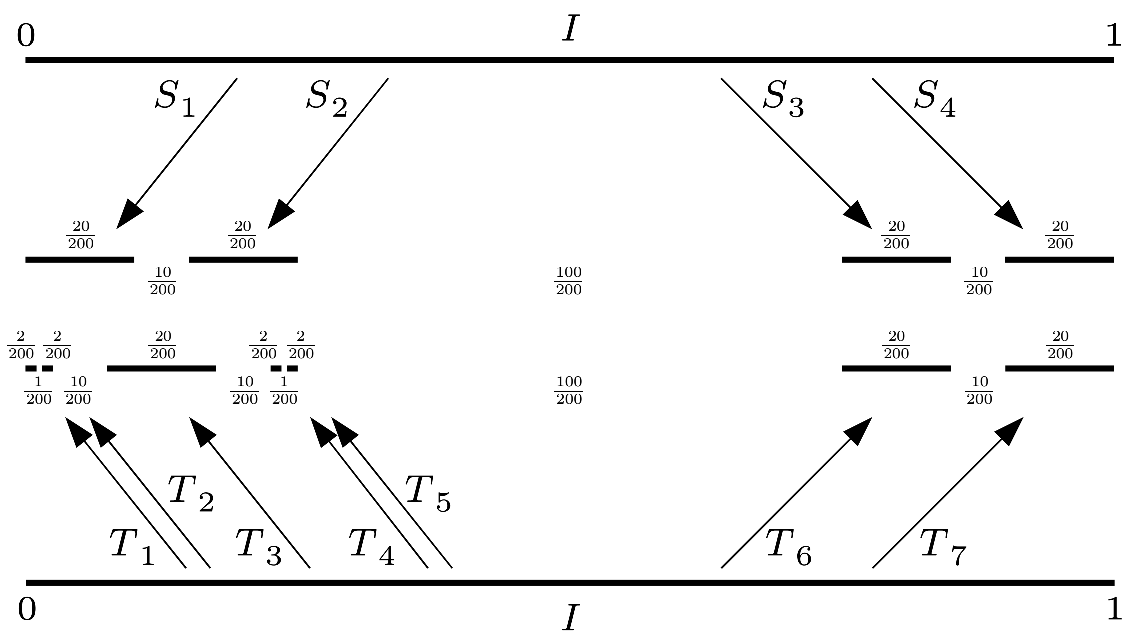


Figure. 2.1.1: Two 1-vertex IFSs with the same attractor, neither is an iteration of the other.

An interesting example of an attractor that is the attractor of two different sets of similarities, neither of which is an iteration of the other, is shown in Figure 2.1.1.

This is Example 6.2 in [FW09], scaled up to the unit interval. Here the two sets of similarities $S = \{S_1, S_2, S_3, S_4\}$ and $T = \{T_1, T_2, T_3, T_4, T_5, T_6, T_7\}$ have the same attractor but clearly T_3 is not a composition of any combination of similarities from S and S_1 is not a composition of any combination of similarities from T. It should be clear then that whilst any attractor is uniquely defined by a particular set of similarities, it may also be the attractor of (infinitely) many other different sets of similarities. Sets of similarities, on their own, are not enough to distinguish between one attractor and another. This means that in order to establish that an attractor of a 2-vertex system is not the attractor of a 1-vertex system we need to identify some property or characteristic of the attractor that is independent of the way it is produced, and then we need to show that no 1-vertex system can have this property. We may then give an affirmative answer to our question. It turns out that the set of gap lengths is a set that is unique to any attractor and is exactly the type of characteristic set that we are looking for.

The set of gap lengths of an attractor is well-defined for the attractor of any directed graph IFS on $\mathbb{R}$ for which the CSSC holds, it is discussed and defined in Section 2.2. When considering the set of gap lengths of an attractor similarities involving reflections are allowed. This means that two different attractors may have the same set of gap lengths. We point this out because, for all the specific examples of directed graph IFSs that are illustrated in this thesis, we do make the assumption that the similarities shown do not involve reflections. This assumption is useful because it enables us to completely define directed graph IFSs by the use of diagrams. The information contained, for example in Figure 2.1.1, explicitly defines all the similarities in S and T once we assume no reflections of the unit interval I occur anywhere in that diagram. We assume then that any similarity represented in a diagram in this thesis does not reflect.

As a summary we can also say that the maps

$$\text{sets of similarities} \;\rightarrow\; \text{attractors} \;\rightarrow\; \text{sets of gap lengths}$$

are not injective.

In Section 2.3, Proposition 2.3.4 and Corollary 2.3.5, we give a constructive proof that the gap lengths of any attractor can always be represented as a finite union of cosets of finitely generated semigroups. The path structure of the graphs is used to set up an algebraic substructure of the semigroup $(\mathbb{R}^+, \times)$ which arises naturally from consideration of the graphs and gap lengths of the IFSs. This explains the presence of the algebraic and graph-theoretic definitions given in Subsection 2.1.1. A basic 2-vertex example of a directed graph IFS is shown in Figure 2.4.1 and we calculate the unique set of gap lengths associated with one of its attractors in Section 2.4, using the algorithmic method of the proof of Proposition 2.3.4. In Section 2.5 we consider a more complicated system which we use to illustrate the main ideas in the proof of Proposition 2.3.4. Finally in Section 2.6, by considering the sets of gap lengths of their attractors, we show in Corollary 2.6.2, that the simple example of a 2-vertex IFS of Section 2.4 produces an attractor which cannot be the attractor of any standard (1-vertex) IFS satisfying the same separation condition, the CSSC. In Chapter 3 we go on to prove this even if the CSSC doesn't hold for the standard (1-vertex) IFSs. Corollary 2.6.2 is extended, using the same ideas, to a large family of directed graphs in Theorem 2.6.3. This gives a positive answer to our question, at least for the restricted class of directed graph IFSs considered in this chapter.

2.1.1 Definitions and notation

A *binary operation* $\star$, on a set S, is a function $\star : S \times S \to S$. Conventionally we write $\star((s,t)) = s \star t \in S$ as the image of the ordered pair $(s,t) \in S \times S$ under $\star$. From its definition a binary operation is always *closed* on its defining set, that is for all $s,t \in S$, $s \star t \in S$. If $s \star t = t \star s$ for all $s,t \in S$, we say that $\star$ is *commutative.* If $(s \star t) \star u = s \star (t \star u)$, for all $s,t,u \in S$ we say that $\star$ is *associative.* We call $e \in S$ an *identity element* for the binary operation on S if $s \star e = e \star s = s$, for all $s \in S$. Multiplication and addition on $\mathbb{R}$ are familiar examples of binary operations which are both commutative and associative whereas subtraction is just associative. For an example of a binary operation that is neither commutative nor associative consider $\star : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ defined by $n \star m = n^m$.

Let $U = \{u_1, u_2, \ldots, u_r\}$ be a set of positive real numbers, then U is a *multiplicatively rationally independent* set if, for any integers $m_i \in \mathbb{Z}$,

$$\sum_{i=1}^{r} m_i \ln u_i = 0,$$

implies $m_i = 0$ for all i, $1 \leqslant i \leqslant r$, or equivalently if

$$\prod_{i=1}^{r} u_i^{m_i} = 1,$$

then $m_i = 0$ for all i, $1 \leqslant i \leqslant r$. For convenience, we may write rational independence as a shortening of multiplicative rational independence.

A *semigroup* $(S, \star)$ consists of a non-empty set S, together with an associative binary operation $\star$ defined on S. Where the binary operation is obvious from the context we may write $st = s \star t$ for $s,t \in S$. If $(s_1, s_2, s_3, \ldots, s_n)$ is any finite sequence of elements of S, then every way of evaluating the product of these elements under $\star$, with the elements taken in the order of the sequence, leads to the same result, so that the product $s_1 \star s_2 \star s_3 \cdots \star s_n$ can be defined unambiguously. In other words we do not need brackets when evaluating products of elements of S under $\star$. This also means, for $n \in \mathbb{N}$ and $s \in S$, that we may let s^n represent the n-fold product, $s \star s \star \cdots \star s$, with, for all $i,j \in \mathbb{N}$, $s^i \star s^j = s^{i+j}$ and $(s^i)^j = s^{ij}$. (Proofs of these statements may be found in [All91]). As an example suppose $s,t,u \in S$, then we may write

$$(s\star(t\star u))\star(((u\star s)\star s)\star s) = (((s\star t)\star u)\star u)\star((s\star s)\star s) = s\star t\star u\star u\star s\star s\star s = stu^2s^3.$$

If $\star$ is commutative as well as associative then we can also write

$$s \star t \star u \star u \star s \star s \star s = stu^2s^3 = s^4tu^2.$$

Let $(S, \star)$ and $(T, \diamond)$ be any two semigroups then a *homomorphism* from $(S, \star)$ to $(T, \diamond)$ is a function $\phi : S \to T$ such that $\phi(s \star t) = \phi(s) \diamond \phi(t)$ for all $s,t \in S$.

Let $(S,\star)$ and $(T,\diamond)$ be any two semigroups then an *isomorphism* from $(S,\star)$ to $(T,\diamond)$ is a bijection $\phi : S \to T$ such that $\phi(s \star t) = \phi(s) \diamond \phi(t)$ for all $s,t \in S$.

Let $(S,\star)$ be a semigroup and suppose $T \subset S$ with $(T,\star)$ also being a semigroup under the same associative binary operation $\star$, then $(T,\star)$ is a *subsemigroup* of $(S,\star)$. We use the notation $(T,\star) \leqslant (S,\star)$ to mean $(T,\star)$ is a subsemigroup of $(S,\star)$.

Let $(T,\star) \leqslant (S,\star)$, then a *coset* of the subsemigroup $(T,\star)$ in $(S,\star)$, with *multiplier* $s \in S$, is the set $s \star T = \{s \star t : t \in T\}$. Where the binary operation is obvious from the context we may write $s \star T = sT$.

A *group* is a semigroup $(S,\star)$, with identity, where the semigroup also contains inverses, that is for each $s \in S$ there exists an inverse s^{-1} such that $s \star s^{-1} = s^{-1} \star s = \varepsilon$, where $\varepsilon \in S$ is the unique identity element.

Let T be a non-empty set, and let T^+ denote the infinite set of all finite strings or sequences of elements of T. Let ε denote the empty string, and let $T^* = T^+ \cup \{\varepsilon\}$. Let $\cdot$ be the operation of concatenation of strings defined for any strings $a_1a_2\cdots a_r,\ b_1b_2\cdots b_s \in T^*$ by $a_1a_2\cdots a_r \cdot b_1b_2\cdots b_s = a_1a_2\cdots a_rb_1b_2\cdots b_s \in T^*$. Clearly $\cdot$ is a closed, associative binary operation on T^*, with ε the unique identity element. $(T^+,\cdot)$ is called the *free semigroup* on T. $(T^*,\cdot)$ is the *free semigroup with identity,* on T, also known as the *free monoid*. If T is a finite set containing n elements, that is $T = \{t_1, t_2, \ldots, t_n\}$, then $(T^*,\cdot)$ is known as the *free semigroup with identity on n elements*, we also say that $(T^*,\cdot)$ is a *finitely generated free semigroup.* We write $T^* = \langle t_1, t_2, \ldots, t_n\rangle$ for a finitely generated semigroup and we may also write $T^* = \langle \varepsilon, t_1, t_2, \ldots, t_n\rangle$, when we wish to emphasise the presence of the identity.

We use the notation $(\mathbb{R}^+,\times)$ for the commutative semigroup of positive real numbers under multiplication. For $x_i \in \mathbb{R}^+$, $1 \leqslant i \leqslant n$, $\langle 1, x_1, x_2, \ldots, x_n\rangle$ is the finitely generated subsemigroup (with identity) of $(\mathbb{R}^+,\times)$, where

$$\langle 1, x_1, x_2, \ldots, x_n\rangle = \{x_1^{k_1}x_2^{k_2}\cdots x_n^{k_n} : k_i \in \mathbb{N}\cup\{0\},\ 1 \leqslant i \leqslant n\}$$

and for $y \in \mathbb{R}^+$ we write $y\,\langle 1, x_1, x_2, \ldots, x_n\rangle$ for a coset with $y\,\langle 1, x_1, x_2, \ldots, x_n\rangle = \{yx_1^{k_1}x_2^{k_2}\cdots x_j^{k_n} : k_i \in \mathbb{N}\cup\{0\},\ 1 \leqslant i \leqslant n\}$. We will use $\langle x_1, x_2, \ldots, x_n\rangle_{\text{group}} = \{x_1^{k_1}x_2^{k_2}\cdots x_j^{k_n} : k_i \in \mathbb{Z},\ 1 \leqslant i \leqslant n\}$ as the notation for a finitely generated group, the group operation again being multiplication.

Now consider $T = \{t_1, t_2, \ldots, t_n\}$, together with a function $g : T \to \mathbb{R}^+$. Let $\overline{T}^* \subset \mathbb{R}^+$ be defined as

$$\begin{aligned}\overline{T}^* &= \langle 1, g(t_1), g(t_2), \ldots, g(t_n)\rangle \\ &= \left\{g(t_1)^{k_1}g(t_2)^{k_2}\cdots g(t_n)^{k_n} : k_i \in \mathbb{N}\cup\{0\}\,,\ 1 \leqslant i \leqslant n\right\},\end{aligned}$$

then $(\overline{T}^*,\times) \leqslant (\mathbb{R}^+,\times)$.

For $T = \{t_1, t_2, \ldots, t_n\}$, together with a function $g : T \to \mathbb{R}^+$ and $(T^*,\cdot)$, $(\overline{T}^*,\times)$ as defined above, there is a *surjective homomorphism* $\psi : (T^*,\cdot) \to (\overline{T}^*,\times)$ defined for each finite string $a_1a_2\cdots a_r \in T^*$, by

$$\psi(a_1a_2\cdots a_r) = g(t_1)^{k_1}g(t_2)^{k_2}\cdots g(t_n)^{k_n}, \tag{2.1.1}$$

where, for each i, $1 \leqslant i \leqslant n$, k_i is the number of occurrences of the symbol t_i in the string $a_1a_2\cdots a_r$.

Let $S = \{s_1, s_2, \ldots, s_n\}$ be a finite set containing n distinct elements and let $S^\vartriangle \subset S^*$ be the set of all finite sequences or finite strings of elements of S which contain only distinct elements of S with no repetitions, together with the empty string ε. $S^\vartriangle$ is a finite set. Let $a_1a_2\cdots a_r \in S^\vartriangle$ and $b_1b_2\cdots b_s \in S^\vartriangle$ so $0 \leqslant r, s \leqslant n$ then the string $b_1b_2\cdots b_s$ may be concatenated onto the end of the string $a_1a_2\cdots a_r$ using the following procedure. For each j, $1 \leqslant j \leqslant s$, if $b_j = a_i$, for some i, $1 \leqslant i \leqslant r$, then we delete b_j from the string $b_1b_2\cdots b_s$. (To be clear what this involves, if we delete the symbol b from the string $abcd$ then we are left with the string acd). This process will produce a possibly empty string $b_{j_1}b_{j_2}\cdots b_{j_l}$, where $1 \leqslant j_1 < j_2 < \cdots < j_l \leqslant s$, and defines a binary operation $*$ on $S^\vartriangle$ as

$$a_1a_2\cdots a_r * b_1b_2\cdots b_s = a_1a_2\cdots a_r b_{j_1}b_{j_2}\cdots b_{j_l}.$$

The string on the right-hand side of the equation contains $t = r + l$ different symbols with $r \leqslant t \leqslant n$. The binary operation $*$ is associative on $S^\vartriangle$ so that $(S^\vartriangle, *)$ is a finite semigroup. As an example let $S = \{x, y, z\}$ then

$$S^\vartriangle = \{\varepsilon, x, y, z, xy, yx, xz, zx, yz, zy, xyz, yxz, xzy, zxy, yzx, zyx\},$$

and some examples of the operation $*$ acting on $S^\vartriangle$ are

$$\begin{aligned} xy * zxy &= xyz, \\ zxy * xy &= zxy, \\ (x * zx) * yz &= xzy, \\ x * (zx * yz) &= xzy. \end{aligned}$$

As the first two examples show $*$ is not commutative. (In general $*$ will only be commutative for the case when S contains a single element and $S = S^\vartriangle$).

Let $T = \{t_1, t_2, \ldots, t_n\}$. For any subset $U \subset T$, $(U^*, \cdot)$, $(T^*, \cdot)$ denote the corresponding finitely generated free semigroups with identity, as defined above, with $(U^*, \cdot) \leqslant (T^*, \cdot)$. We also have $(U^\vartriangle, *) \leqslant (T^\vartriangle, *)$. As sets $U^\vartriangle \subset T^\vartriangle \subset T^*$. Let $\imath : T^* \to T^*$ be the identity map, then for any subset $X \subset T$ and any string $a_1a_2\cdots a_n \in X \subset T^*$ we will write $\imath(a_1a_2\cdots a_r)$ whenever we wish to emphasise that the string $a_1a_2\cdots a_r$ is to be considered as an element of T^*. Let $\mathcal{T}$ be the set of cosets of finitely generated subsemigroups of $(T^*, \cdot)$, each one containing the identity, defined as follows,

$$\mathcal{T} = \{\imath(a_1a_2\cdots a_n)U^* : U \subset T,\ a_1a_2\cdots a_n \in U^\vartriangle \subset T^*\}. \tag{2.1.2}$$

Lemma 2.1.1. *For any non-empty subsets V, $W \subset T$, and for any strings $a_1a_2\cdots a_r \in V^\vartriangle$, $b_1b_2\cdots b_s \in W^\vartriangle$, $\imath(a_1a_2\cdots a_r)V^* = \imath(b_1b_2\cdots b_s)W^*$ if and only if $a_1a_2\cdots a_r = b_1b_2\cdots b_s$ and $V = W$.*

Proof. We assume $\imath(a_1a_2\cdots a_r)V^* = \imath(b_1b_2\cdots b_s)W^*$. As the identity $\varepsilon \in V^*$, and $\varepsilon \in W^*$, there exist strings $c_1c_2\cdots c_t \in V^*$, and $d_1d_2\cdots d_u \in W^*$ such that

$$\begin{aligned} a_1a_2\cdots a_r c_1c_2\cdots c_t &= b_1b_2\cdots b_s, \\ a_1a_2\cdots a_r &= b_1b_2\cdots b_s d_1d_2\cdots d_u, \end{aligned}$$

which means

$$a_1a_2\cdots a_r \;=\; a_1a_2\cdots a_rc_1c_2\cdots c_td_1d_2\cdots d_u,$$

hence $c_1c_2\cdots c_td_1d_2\cdots d_u = \varepsilon$, $c_1c_2\cdots c_t = \varepsilon$ and $a_1a_2\cdots a_r = b_1b_2\cdots b_s$. We can now write our assumption as $a_1a_2\cdots a_rV^* = a_1a_2\cdots a_rW^*$. The set of strings of length $r+1$ in the coset $a_1a_2\cdots a_rV^*$ is the set $\{a_1a_2\cdots a_rv_i : v_i \in V\} = a_1a_2\cdots a_rV$. Similarly the set of strings of length $r+1$ in the coset $a_1a_2\cdots a_rW^*$ is the set $a_1a_2\cdots a_rW$, so we must have $a_1a_2\cdots a_rV = a_1a_2\cdots a_rW$ and $V = W$. □

For any subsets $V, W \subset T$ it is clear that for $a_1a_2\cdots a_r \in V^\triangle$ and $b_1b_2\cdots b_s \in W^\triangle$, the product $a_1a_2\cdots a_r * b_1b_2\cdots b_s \in (V\cup W)^\triangle$. We may now define a binary operation $\circledcirc$ on $\mathcal{T}$ by

$$\imath(a_1a_2\cdots a_r)V^* \circledcirc \imath(b_1b_2\cdots b_s)W^* = \imath(a_1a_2\cdots a_r * b_1b_2\cdots b_s)(V\cup W)^*. \tag{2.1.3}$$

Lemma 2.1.1 ensures that the cosets in Equation (2.1.3) are represented uniquely which means that the operation $\circledcirc$ is well-defined. $(\mathcal{T},\circledcirc)$ is a non-commutative finite semigroup with identity, the identity element being the coset $\varepsilon\{\varepsilon\}^*$.

We now consider the case where $T = \{t_1, t_2, \ldots, t_n\}$ and we also have a function $g : T \to \mathbb{R}^+$. Any coset in $\mathcal{T}$ is just a collection of strings in T^*, and it will be mapped under the surjective homomorphism $\psi : (T^*, \cdot) \to (\overline{T}^*, \times)$ of Equation (2.1.1), to a coset of a finitely generated semigroup of positive real numbers. That is for $\imath(a_1a_2\cdots a_n)U^* \in \mathcal{T}$, $\imath(a_1a_2\cdots a_n)U^* \subset T^*$ and

$$\begin{aligned}\psi(\imath(a_1a_2\cdots a_n)U^*) &= \psi(\imath(a_1a_2\cdots a_n)) \times \psi(U^*)\\ &= g(a_1)g(a_2)\cdots g(a_n)\overline{U}^*,\end{aligned} \tag{2.1.4}$$

where $g(a_1)g(a_2)\cdots g(a_n)\overline{U}^* \subset \overline{T}^* \subset \mathbb{R}^+$. For instance suppose $T = \{t_1, t_2, t_3, t_4\}$ where $g(t_1), g(t_2), g(t_3), g(t_4)$ are positive real numbers then $t_1t_2\langle\varepsilon, t_1, t_2, t_4\rangle \in \mathcal{T}$, with

$$\psi(t_1t_2\langle\varepsilon, t_1, t_2, t_4\rangle) = g(t_1)g(t_2)\langle 1, g(t_1), g(t_2), g(t_4)\rangle,$$

and $g(t_1)g(t_2)\langle 1, g(t_1), g(t_2), g(t_4)\rangle \subset \langle 1, g(t_1), g(t_2), g(t_3), g(t_4)\rangle = \overline{T}^* \subset \mathbb{R}^+$.

We now define another function that we will need, $\varphi : T^* \to T^*$. Consider a finite string $a_1a_2\cdots a_r \in T^*$ of not necessarily distinct symbols where the exponent of each symbol in the string is 1. Strictly speaking we are considering the string as the function $a : \{1, 2, \ldots, r\} \to T$ so $a_i = a(i)$, $1 \leqslant i \leqslant r$. It is therefore entirely possible for instance that $a_1 = a_2$. Let $k_1 \in \mathbb{N}$ be the number of occurrences of a_1 in the string. We delete all occurrences of a_1 except for the first which we raise to the power k_1. The order of appearance from left to right of the other symbols in the string is not changed. This leaves a string of the form $a_1^{k_1}a_{j_2}\cdots$ for some $1 < j_2 \leqslant r$. If no such a_{j_2} exists then we stop. If a_{j_2} exists we apply the same process to a_{j_2} to obtain a string of the form $a_1^{k_1}a_{j_2}^{k_2}a_{j_3}\cdots$, where k_2 is the number of occurrences of the symbol a_{j_2} in the original string and $j_2 < j_3 \leqslant r$. If no such a_{j_3} exists then we stop. Continuing in this way the process must eventually terminate. This process defines the function φ with

$$\varphi(a_1a_2\cdots a_r) = a_1^{k_1}a_{j_2}^{k_2}\cdots a_{j_l}^{k_l}, \tag{2.1.5}$$

where k_1 is the number of occurrences of a_1 and k_s, $1 < s \leqslant l$, is the number of occurrences of a_{j_s} in the original string, $\sum_{j=1}^{l} k_j = r$, $1 < j_2 < \cdots < j_l \leqslant r$, and $\{1, j_2, \ldots, j_l\} \subset \{1, 2, \ldots, r\}$. In words the function φ collects up identical occurrences of a symbol into powers whilst preserving the order of the first appearance of the different symbols in the original string as we go from left to right. If $T = \{u, v, w, x, y, z\}$ some examples of this function in action are

$$\begin{array}{rcl} \varphi(uvwxyz) & = & uvwxyz, \\ \varphi(uzzwyvzvuzzzzuyvzv) & = & u^3z^7wy^2v^4, \\ \varphi(uvwxyzzyxwvu) & = & u^2v^2w^2x^2y^2z^2, \\ \varphi(zyxwvuuvwxyz) & = & z^2y^2x^2w^2v^2u^2 \end{array}$$

It is important to remember that the exponents used here are just a convenient shorthand for representing strings in T^*, and these examples could equally well be written as

$$\begin{array}{rcl} \varphi(uvwxyz) & = & uvwxyz, \\ \varphi(uzzwyvzvuzzzzuyvzv) & = & uuuzzzzzzzwyyvvvv, \\ \varphi(uvwxyzzyxwvu) & = & uuvvwwxxyyzz, \\ \varphi(zyxwvuuvwxyz) & = & zzyyxxwwvvuu. \end{array}$$

We note that $\varphi : T^* \to T^*$ is neither injective nor surjective.

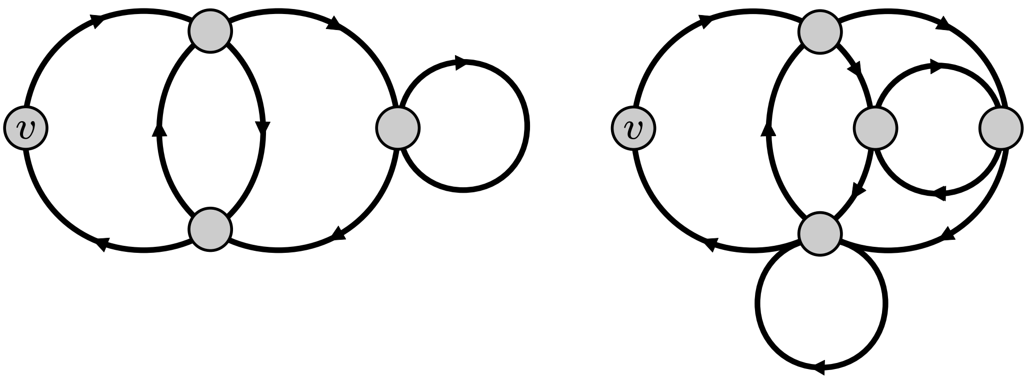
We now give some extra graph theoretic definitions that will be needed in this chapter. For the definitions of a (simple) path and a (simple) cycle see Subsection 1.2.7. We remind the reader here that the set of vertices of a path $\mathbf{e} = e_1 \cdots e_k \in E^*$ is the set $\{i(e_1), t(e_i) : 1 \leqslant i \leqslant k\}$, and its vertex list is $v_1v_2v_3 \cdots v_{k+1} = i(e_1)t(e_1)t(e_2) \cdots t(e_k)$, see Subsection 1.2.7. Let $\mathbf{e}, \mathbf{f} \in E^*$ be any two paths with $\mathbf{e} = e_1 \cdots e_k$ and $\mathbf{f} = f_1 \cdots f_j$, then $\mathbf{e}$ is *attached* to $\mathbf{f}$ if

$$\{i(e_1), t(e_i) : 1 \leqslant i \leqslant k\} \cap \{i(f_1), t(f_i) : 1 \leqslant i \leqslant j\} \neq \emptyset.$$

So two paths are attached if their vertex lists share a common vertex or vertices. More intuitively we can think of two paths being attached if they intersect at a vertex or vertices. We also say that a *path* $\mathbf{e}$ *is attached to a vertex* v if v is in the vertex list of $\mathbf{e}$.

A *chain* is a finite sequence (or finite string) of distinct simple cycles where each simple cycle in the sequence is attached only to its immediate predecessor and successor cycles and to no other cycles in the sequence. (It is a consequence of this definition that the vertex or vertices at which a cycle is attached to its predecessor cycle do not include any vertex or vertices at which it is attached to its successor cycle). A *chain attached to a vertex* v is a chain of distinct simple cycles such that the first cycle in the sequence is attached to the vertex v and thereafter no other cycle in the chain is attached to v. Hopefully Figure 2.1.2 makes the idea of a chain attached to a vertex clear.

The concept of a chain attached to a vertex extends naturally to that of a chain attached to a path. Let $\mathbf{e}$ denote a path. A *chain attached to a path* $\mathbf{e}$, is a finite



(a) (b)

Figure. 2.1.2: The directed graph in (a) contains just one chain of length 3 attached to v, whereas the directed graph in (b) does not contain any chain of length 3 attached to v.

sequence (or finite string) of distinct simple cycles forming a chain, as defined above, such that the first cycle in the chain is attached to the path $\mathbf{e}$ and thereafter no other cycle in the chain is attached to $\mathbf{e}$.

We will also need, (see Lemma 2.3.3), the idea of an *attached finite sequence of (distinct) simple cycles* where each simple cycle in the sequence is attached to its immediate predecessor and successor cycles, and may also be attached to other cycles in the sequence. A chain is an attached sequence of distinct simple cycles but the converse is not true in general. The length of a chain or an attached sequence of distinct simple cycles is the number of distinct simple cycles in the sequence. Figure 2.1.2(a) contains 38 attached sequences of distinct simple cycles of length 3, of which 8 are chains of length 3, and only one of these chains is attached to v. Figure 2.1.2(b) on the other hand, contains 688 attached sequences of distinct simple cycles of length 3, of which 16 are chains of length 3, and none of these chains are attached to v. The order of a sequence is being counted here, so that three distinct simple cycles all attached to each other give rise to 6 different attached sequences of distinct simple cycles of length 3. Similarly any chain automatically gives rise to just one other chain of the same length formed by taking the sequence of cycles in reverse order. There are 5 distinct simple cycles in the directed graph of Figure 2.1.2(a) and 10 distinct simple cycles in the directed graph of Figure 2.1.2(b). Assuming that these simple cycles are all attached to each other, gives upper bounds of $60 = \mathrm{P}_3^5$ and $720 = \mathrm{P}_3^{10}$, for the possible number of different attached sequences of distinct simple cycles of length 3, in Figures 2.1.2(a) and 2.1.2(b) respectively.

2.2 Gap lengths

In this section for systems on the real line, which satisfy the CSSC, we discuss, give a definition of, and provide two expressions for, the set of gap lengths of any attractor of such a system. We also prove that the attractors can be translated and scaled to become the attractors of a related IFS.

2.2.1 Definition of the set of gap lengths of an attractor

Any directed graph IFS, $\big(V, E^*, i, t, r, ((\mathbb{R}, |\ \ |))_{v\in V}, (S_e)_{e\in E^1}\big)$, for which the CSSC holds, has a unique list of attractors $(F_u)_{u\in V}$, such that

$$f\big((F_u)_{u\in V}\big) = \Big(\bigcup_{e\in E_u^1} S_e(F_{t(e)}) \Big)_{u\in V} = (F_u)_{u\in V}, \tag{2.2.1}$$

by Theorem 1.3.4, Equation (1.3.2), where $f : (K(\mathbb{R}^n))^{\#V} \to (K(\mathbb{R}^n))^{\#V}$, is the function defined in Equation (1.3.1). Let $(I_u)_{u\in V}$ be the closed intervals, $I_u = [a_u, b_u]$, such that $a_u, b_u \in F_u$ and $F_u \subset I_u$. Equivalently we could define $I_u = C(F_u)$, the convex hull of F_u. From this definition and Equation (2.2.1) it is clear that

$$\bigcup_{e\in E_u^1} S_e(I_{t(e)}) \subset I_u,$$

and so

$$f\left((I_u)_{u\in V}\right) \subset (I_u)_{u\in V}.$$

By Theorem 1.3.4, Equation (1.3.4) we obtain

$$(F_u)_{u\in V} = \bigcap_{k=0}^{\infty} f^k\big((I_u)_{u\in V}\big).$$

For convenience we now introduce the notation $(F_u^k)_{u\in V} = f^k\left((I_u)_{u\in V}\right)$ so we may write

$$(F_u)_{u\in V} = \bigcap_{k=0}^{\infty} (F_u^k)_{u\in V}. \tag{2.2.2}$$

and following [Fal03], we call $(F_u^k)_{u\in V}$ the *level-k intervals* and F_u^k the *level-k intervals at the vertex u* or the *k-th level intervals of the attractor F_u*. As Equation (2.2.2) confirms they provide increasingly good approximations to the attractors $(F_u)_{u\in V}$ as k increases. This is illustrated in Figure 2.2.1, which shows a 3 vertex directed graph and the corresponding level-k intervals, for $k = 0, 1, 2, 3$, and 4. The arrows between level-0 and level-1 represent the 6 similarities associated with the 6 (directed) edges of the graph. For example, the first arrow on the left shows the similarity that is associated with the edge from the vertex u to u, mapping I_u to an interval in F_u^1, the second arrow from the left shows the similarity associated with the edge from w to u, and maps I_u to an interval in F_w^1, and so on. We remind the reader that by convention similarities map in the opposite direction to the edges of the graph.

We are only considering directed graph IFSs for which the CSSC holds, so in general, if there are n edges leaving a vertex u, then the level-1 intervals F_u^1 at the vertex u will consist of n disjoint intervals which will have $n-1$ open intervals between them. That is $I_u \setminus F_u^1 = \bigcup_{i=1}^{n-1} J_i^1$, where each J_i^1 is an open interval. The lengths of intervals $I \subset \mathbb{R}$, are their diameters, so for any $a, b \in \mathbb{R}$, $a \leqslant b$, we have $|[a,b]| = |[a,b)| = |(a,b]| = |(a,b)| = b - a$. The *set of level-*1 *gap lengths at the vertex u* is defined as

$$G_u^1 = \left\{ \left|J_i^1\right| : J_i^1 \text{ is an open interval in } I_u \setminus F_u^1 = \bigcup_{i=1}^{n-1} J_i^1 \right\}.$$

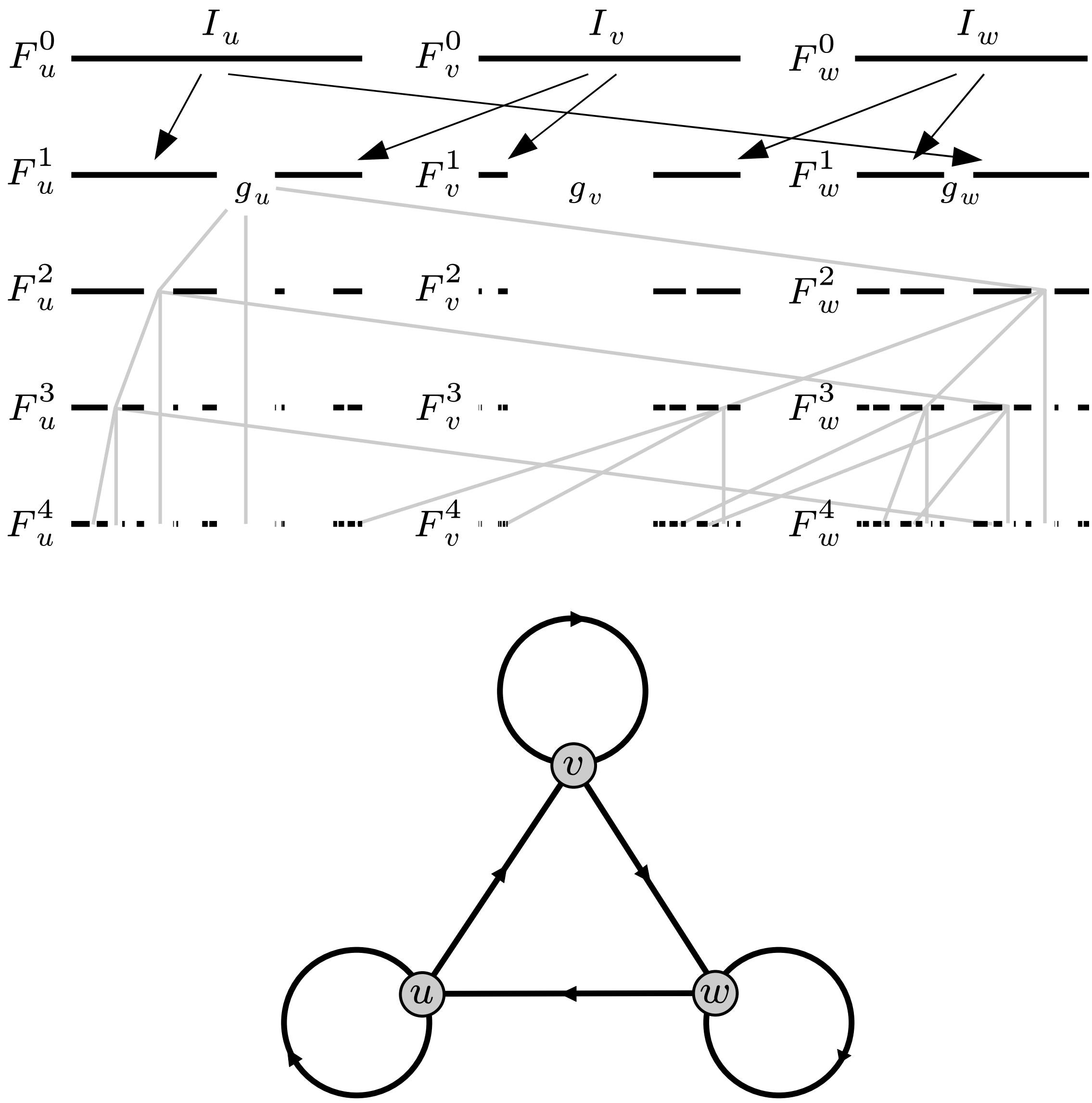


Figure. 2.2.1: A 3-vertex directed graph together with its level-k intervals for $0 \leqslant k \leqslant 4$.

In general $I_u \setminus F_u^k$ is a finite union of open intervals so $I_u \setminus F_u^k = \bigcup_{i \in H_k} J_i^k$, for some finite indexing set H_k. The *set of level-k gap lengths at the vertex u* is defined as

$$G_u^k = \left\{ \left| J_i^k \right| : J_i^k \text{ is an open interval in } I_u \setminus F_u^k = \bigcup_{i \in H_k} J_i^k \right\}.$$

In Figure 2.2.1, $G_u^1 = \{g_u\}$, $G_v^1 = \{g_v\}$, and $G_w^1 = \{g_w\}$ as shown. The grey lines in Figure 2.2.1 illustrate how scalar multiples of the level-1 gap length g_u appear in the subsequent levels, the scalar multiples being the products of the similarity ratios of the relevant similarities. Once a gap length appears in a level it remains there and this is indicated in Figure 2.2.1 by the grey lines remaining vertical.

From Equation (2.2.2) it follows that

$$I_u \setminus F_u = I_u \setminus \left(\bigcap_{k=0}^{\infty} F_u^k \right) = \bigcup_{k=0}^{\infty} I_u \setminus F_u^k.$$

Since $I_u \setminus F_u^k = \bigcup_{i \in H_k} J_i^k$, for some finite indexing set H_k and open intervals J_i^k it follows that $I_u \setminus F_u$ is a countable union of open intervals, which we may relabel as

$$I_u \setminus F_u = \bigcup_{i=1}^{\infty} J_i.$$

We define the uniquely determined *set of gap lengths of the attractor* F_u as

$$G_u = \bigcup_{n=1}^{\infty} G_u^n = \left\{ |J_i| : J_i \text{ is an open interval in } I_u \setminus F_u = \bigcup_{i=1}^{\infty} J_i \right\}. \tag{2.2.3}$$

We now give an alternative description of the set G_u. For each edge $e \in E^1$ let $R_e : \mathbb{R} \to \mathbb{R}$ be the map $R_e(x) = r_e x$, where r_e is the contracting similarity ratio of S_e. Let $\widetilde{f} : (K(\mathbb{R}^n))^{\#V} \to (K(\mathbb{R}^n))^{\#V}$, be defined by

$$\widetilde{f}\left((A_u)_{u \in V}\right) = \left(\bigcup_{e \in E_u^1} R_e(A_{t(e)}) \cup G_u^1 \right)_{u \in V},$$

for each $(A_u)_{u \in V} \in (K(\mathbb{R}^n))^{\#V}$. Here the sets of level-1 gap lengths, $(G_u^1)_{u \in V}$, which are called condensation sets in [Bar93] for standard (1-vertex) IFSs, are clearly non-empty and compact so $(G_u^1)_{u \in V} \in (K(\mathbb{R}^n))^{\#V}$. It can be shown that $\widetilde{f}$ is a contraction on $((K(\mathbb{R}^n))^{\#V}, D_{\mathrm{H}})$, which is a complete metric space, in much the same way that we showed f to be a contraction in the proof of Theorem 1.3.4, see also Theorem 9.1 of [Bar93] which gives a proof for 1-vertex systems. As

$$(G_u \cup \{0\})_{u \in V} = \left(\bigcup_{e \in E_u^1} R_e(G_{t(e)} \cup \{0\}) \cup G_u^1 \right)_{u \in V}, \tag{2.2.4}$$

the Contraction Mapping Theorem ensures that $(G_u \cup \{0\})_{u \in V}$ is the unique fixed point of $\widetilde{f}$. The invariance Equations (2.2.1) and (2.2.4) show clearly the close relationship between attractors and their sets of gap lengths.

2.2.2 Two expressions for G_u

In this section we write down two expressions for the unique set of gap lengths G_u in an attractor F_u of a directed graph IFS $\left(V, E^*, i, t, r, ((\mathbb{R}, |\ \ |))_{v \in V}, (S_e)_{e \in E^1}\right)$ which satisfies the CSSC. Both these expressions depend on the level-1 gap lengths $(G_v^1)_{v \in V}$, and the CSSC ensures that each of these sets is non-empty.

G_u in terms of similarity ratios

It is clear that once a gap length appears between level-k intervals it then remains in all subsequent levels, this is indicated in Figure 2.2.1 by the vertical lines. It follows that

$$\begin{aligned} G_u^2 &= G_u^1 \cup \bigcup_{\substack{v\in V\\ g_v\in G_v^1}} g_v \left\{r_e : e \in E_{uv}^1\right\} \\ &= \bigcup_{g_u\in G_u^1} g_u \left\{1, r_e : e \in E_{uu}^1\right\} \cup \bigcup_{\substack{v\in V\\ v\neq u\\ g_v\in G_v^1}} g_v \left\{r_e : e \in E_{uv}^1\right\}, \end{aligned}$$

and

$$\begin{aligned} G_u^3 &= G_u^2 \cup \bigcup_{\substack{v\in V\\ g_v\in G_v^1}} g_v \left\{r_{\mathbf{e}} : \mathbf{e} \in E_{uv}^2\right\} \\ &= G_u^1 \cup \bigcup_{\substack{v\in V\\ g_v\in G_v^1}} g_v \left\{r_e : e \in E_{uv}^1\right\} \cup \bigcup_{\substack{v\in V\\ g_v\in G_v^1}} g_v \left\{r_{\mathbf{e}} : \mathbf{e} \in E_{uv}^2\right\} \\ &= \bigcup_{g_u\in G_u^1} g_u \left\{1, r_{\mathbf{e}} : \mathbf{e} \in E_{uu}^{\leqslant 2}\right\} \cup \bigcup_{\substack{v\in V\\ v\neq u\\ g_v\in G_v^1}} g_v \left\{r_{\mathbf{e}} : \mathbf{e} \in E_{uv}^{\leqslant 2}\right\}, \end{aligned}$$

where $E_{uv}^{\leqslant 2}$ denotes the set of all paths from the vertex u to v of length less than or equal to 2. In general, the set of level-k gap lengths at the vertex u, is given by

$$G_u^k = \bigcup_{g_u\in G_u^1} g_u \left\{1, r_{\mathbf{e}} : \mathbf{e} \in E_{uu}^{\leqslant k-1}\right\} \cup \bigcup_{\substack{v\in V\\ v\neq u\\ g_v\in G_v^1}} g_v \left\{r_{\mathbf{e}} : \mathbf{e} \in E_{uv}^{\leqslant k-1}\right\}.$$

Finally from Equation (2.2.3) we obtain

$$G_u = \bigcup_{g_u\in G_u^1} g_u\left\{1, r_{\mathbf{e}} : \mathbf{e} \in E_{uu}^{*}\right\} \cup \bigcup_{\substack{v\in V\\ v\neq u\\ g_v\in G_v^1}} g_v\left\{r_{\mathbf{e}} : \mathbf{e} \in E_{uv}^{*}\right\}. \tag{2.2.5}$$

G_u in terms of arrays

Let M_m be the set of all $m \times m$ arrays whose entries are finite sets of real numbers. We may define a binary operation $\oslash$ on M_m by replacing normal multiplication and addition by pointwise set multiplication and set union. As an example, for $m = 2$,

$$\begin{pmatrix} S_1 & S_2 \\ S_3 & S_4 \end{pmatrix} \oslash \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix} = \begin{pmatrix} S_1T_1 \cup S_2T_3 & S_1T_2 \cup S_2T_4 \\ S_3T_1 \cup S_4T_3 & S_3T_2 \cup S_4T_4 \end{pmatrix}$$

where the pointwise multiplication of sets is given by

$$ST = \{st : s \in S,\, t \in T\}.$$

The binary operation $\oslash$ is associative and $(M_m, \oslash)$ is a semigroup with identity, the identity element being the $m \times m$ array $\mathbf{I}_m$ whose ijth entries are

$$I_{m_{ij}} = \begin{cases} \{1\} & \text{if } i = j, \\ \{0\} & \text{if } i \neq j. \end{cases}$$

For example in $(M_2, \oslash)$ the identity element is

$$\mathbf{I}_2 = \begin{pmatrix} \{1\} & \{0\} \\ \{0\} & \{1\} \end{pmatrix}.$$

Let $\mathbf{b} = (B_1, B_2, \dots B_m)^T$ be an $m \times 1$ column vector whose entries are finite sets of real numbers, then for $\mathbf{A} \in M_m$, we define $\mathbf{Ab}$ as

$$\mathbf{Ab} = \begin{pmatrix} A_{11} & \cdots & A_{1m} \\ \vdots & & \vdots \\ A_{m1} & \cdots & A_{mm} \end{pmatrix} \begin{pmatrix} B_1 \\ \vdots \\ B_m \end{pmatrix} = \begin{pmatrix} A_{11}B_1 \cup \cdots \cup A_{1m}B_m \\ \vdots \\ A_{m1}B_1 \cup \cdots \cup A_{mm}B_m \end{pmatrix}.$$

Consider a directed graph IFS with m vertices, that is $m = \#V$. Let the vertices be ordered as $V = (v_1, v_2, \dots, v_m)$. We define an array $\mathbf{M} \in M_m$, whose ijth entries are sets of similarity ratios defined as

$$M_{ij} = \left\{ r_e : e \in E^1_{v_i v_j} \right\}, \tag{2.2.6}$$

and a column vector, whose entries are the sets of level-1 gap lengths at each vertex, as

$$\mathbf{g} = \begin{pmatrix} G^1_{v_1} \\ \vdots \\ G^1_{v_m} \end{pmatrix}. \tag{2.2.7}$$

The level-k gap lengths at each vertex can now be given in array form by the formula

$$\begin{pmatrix} G^k_{v_1} \\ \vdots \\ G^k_{v_m} \end{pmatrix} = \bigcup_{j=0}^{k-1} \mathbf{M}^j \mathbf{g},$$

where $\mathbf{M}^j \in M_m$ indicates the j-fold product under $\oslash$ of $\mathbf{M}$, and $\mathbf{M}^0 = \mathbf{I}_m$. The sets of gap lengths of the attractors are also represented in array form by the following formula

$$\begin{pmatrix} G_{v_1} \\ \vdots \\ G_{v_m} \end{pmatrix} = \bigcup_{j=0}^{\infty} \mathbf{M}^j \mathbf{g}. \tag{2.2.8}$$

2.2.3 Translation and scaling of a directed graph IFS

Consider a directed graph IFS, $\left(V, E^*, i, t, r, ((\mathbb{R}, |\ \ |))_{v \in V}, (S_e)_{e \in E^1}\right)$, satisfying the CSSC, then by Equation (2.2.1), the attractors $(F_u)_{u \in V}$ are such that

$$(F_u)_{u \in V} = \Big(\bigcup_{e \in E^1_u} S_e(F_{t(e)}) \Big)_{u \in V}.$$

Each similarity, $S_e : \mathbb{R} \to \mathbb{R}$, may be written in the form $S_e(x) = r'_e x + c_e$, where the similarity ratio $r_e = |r'_e|$, $0 < r_e < 1$. We can define a new similarity $\widehat{S}_e : \mathbb{R} \to \mathbb{R}$, for each edge $e \in E^1$, as

$$\widehat{S}_e(x) = r'_e x + k(c_e + (1 - r'_e)t),$$

where $k, t \in \mathbb{R}$, $k > 0$. Let $(\widehat{F}_v)_{v \in V}$ denote the attractors of this new system. Now

$$\begin{aligned} k(S_e(x) + t) &= k(r'_e x + c_e + t) \\ &= r'_e(k(x + t)) + k(c_e + t - r'_e t) \\ &= r'_e(k(x + t)) + k(c_e + (1 - r'_e)t) \\ &= \widehat{S}_e(k(x + t)), \end{aligned}$$

so that

$$\begin{aligned} k(F_u + t) &= k\left(\bigcup_{e \in E^1_u} S_e(F_{t(e)}) + t \right) \\ &= \bigcup_{e \in E^1_u} k(S_e(F_{t(e)}) + t) \\ &= \bigcup_{e \in E^1_u} \widehat{S}_e(k(F_{t(e)} + t)). \end{aligned}$$

Therefore $(\widehat{F}_u)_{u \in V} = (k(F_u + t))_{u \in V}$. This shows that we may always translate and scale the attractor F_u, at a chosen vertex u, so that $I_u = [0, 1]$, $0, 1 \in F_u$, and $F_u \subset [0, 1]$. Equivalently we may always translate and scale so that $C(F_u) = [0, 1]$.

2.3 An expression for the set of gap lengths of an attractor

Here we prove a general result, given by Proposition 2.3.4 and Equation (2.3.7) of Corollary 2.3.5, that provides an expression for the set of gap lengths of an attractor in a directed graph IFS in terms of cosets of finitely generated semigroups. The proof we give is constructive and is illustrated by examples in Sections 2.4 and 2.5. First we provide some preliminary lemmas that will be used in the proof of Proposition 2.3.4.

Lemma 2.3.1. *Let* $\mathbf{g} \in E^*$ *be any path in a directed graph and let* $\mathbf{c} \in E^*$ *be any simple cycle. Suppose* $\mathbf{c}$ *is attached to* $\mathbf{g}$*, then we can always order the constituent edges of* $\mathbf{g}$ *and* $\mathbf{c}$ *to create a single new path* $\mathbf{f} \in E^*$ *which has the contracting similarity ratio*

$$r_{\mathbf{f}} = r_{\mathbf{g}} r_{\mathbf{c}},$$

where $i(\mathbf{f}) = i(\mathbf{g})$ *and* $t(\mathbf{f}) = t(\mathbf{g})$.

Proof. Starting at the initial vertex $i(\mathbf{g})$ we travel along the edges of $\mathbf{g}$ until we meet the first common vertex v which lies on both $\mathbf{g}$ and $\mathbf{c}$. We then travel all the way round the edges of $\mathbf{c}$ until we return to v. Finally we continue along all the remaining edges of $\mathbf{g}$ until we finish at the terminal vertex $t(\mathbf{g})$. Travelling along the edges of $\mathbf{g}$ and $\mathbf{c}$ in this order creates the required path $\mathbf{f} \in E^*$ with the required similarity ratio $r_{\mathbf{f}} = r_{\mathbf{g}} r_{\mathbf{c}}$ and clearly $i(\mathbf{f}) = i(\mathbf{g})$ and $t(\mathbf{f}) = t(\mathbf{g})$. □

Lemma 2.3.2. *Let* $\mathbf{f} \in E^*$ *be any path in a directed graph which is not a simple path, then the constituent edges of* $\mathbf{f}$ *can always be re-ordered to create a new path* $\mathbf{g} \in E^*$ *and a simple cycle* $\mathbf{c} \in E^*$*, which is attached to* $\mathbf{g}$*, such that*

$$r_{\mathbf{f}} = r_{\mathbf{g}} r_{\mathbf{c}},$$

where $i(\mathbf{f}) = i(\mathbf{g})$ *and* $t(\mathbf{f}) = t(\mathbf{g})$*.*

Proof. Since $\mathbf{f}$ is not a simple path, as we travel along it we must eventually visit at least one vertex twice, so there is a vertex w that is the first to appear twice in the vertex list of the path. For example if $xyzyx$ is the vertex list of a path then we regard y as the first vertex to appear twice and not x. Let $\mathbf{c} \in E^*_{ww}$ be the path traversed between the first two visits to w, then $\mathbf{c}$ is a simple cycle. We now let $\mathbf{g} \in E^*$ be the path formed by taking $\mathbf{f}$ and removing the edges of $\mathbf{c}$. It follows that $\mathbf{c}$ is a simple cycle attached to $\mathbf{g}$ at the vertex w with $r_{\mathbf{f}} = r_{\mathbf{g}} r_{\mathbf{c}}$ and clearly $i(\mathbf{f}) = i(\mathbf{g})$ and $t(\mathbf{f}) = t(\mathbf{g})$. □

We recall (see Subsection 2.1.1, the second paragraph after Figure 2.1.2) that a sequence of distinct simple cycles $\mathbf{c}_1 \cdots \mathbf{c}_k$ is attached if each cycle in the sequence is attached to its immediate predecessor and successor. A cycle in an attached finite sequence of (distinct) cycles may well be attached to other cycles in the sequence as well as its immediate predecessor and successor. A chain is therefore an attached sequence of distinct cycles but the converse is not true in general. This explains the significance of the next lemma.

Lemma 2.3.3. *Let* $k \in \mathbb{N}$*,* $k \geqslant 2$*, and suppose* $\mathbf{c}_1\mathbf{c}_2 \cdots \mathbf{c}_{k-1}\mathbf{c}_k$ *is an attached sequence of distinct simple cycles in a directed graph, then there exists a subsequence*

$$\{1, n_1, n_2, \ldots, n_j, k : 1 < n_1 < n_2 < \ldots < n_j < k\} \subset \{1, 2, \ldots, k\}$$

such that $\mathbf{c}_1\mathbf{c}_{n_1}\mathbf{c}_{n_2} \cdots \mathbf{c}_{n_j}\mathbf{c}_k$ *is a chain.*

Proof. We use the principle of strong mathematical induction.

Let $k \in \mathbb{N}$, $k \geqslant 2$, and let $P(k)$ be the following statement.

An attached sequence of distinct simple cycles $\mathbf{c}_1\mathbf{c}_2 \cdots \mathbf{c}_{k-1}\mathbf{c}_k$ *in a directed graph, contains a subsequence*

$$\{1, n_1, n_2, \ldots, n_j, k : 1 < n_1 < n_2 < \ldots < n_j < k\} \subset \{1, 2, \ldots, k\}$$

such that $\mathbf{c}_1\mathbf{c}_{n_1}\mathbf{c}_{n_2} \cdots \mathbf{c}_{n_j}\mathbf{c}_k$ *is a chain.*

Induction base.

P(2) is true. Two attached distinct simple cycles, $\mathbf{c}_1\mathbf{c}_2$, always form a chain.

Induction hypothesis.

Let $k \in \mathbb{N}$, $k \geqslant 2$, we assume $P(n)$ is true, for all $n \in \mathbb{N}$, $2 \leqslant n \leqslant k$.

Induction step.

Consider an attached sequence of distinct simple cycles $\mathbf{c}_1\mathbf{c}_2 \cdots \mathbf{c}_k\mathbf{c}_{k+1}$. If this sequence of cycles is a chain then $P(k+1)$ holds for this particular sequence, so we assume it is not a chain. This means there exists a least m, $3 \leqslant m \leqslant k+1$, such that $\mathbf{c}_1\mathbf{c}_2\cdots\mathbf{c}_{m-1}\mathbf{c}_m$ is not a chain but where $\mathbf{c}_1\mathbf{c}_2\cdots\mathbf{c}_{m-1}$ is a chain. It now follows from the definition of a chain that $\mathbf{c}_m$ is attached to at least one cycle $\mathbf{c}_i$ with $1 \leqslant i \leqslant m-2$. The sequence $\mathbf{c}_1\mathbf{c}_2\cdots\mathbf{c}_i\mathbf{c}_m$ is an attached sequence of distinct simple cycles of length at most $m-1$, and the attached sequence of distinct simple cycles $\mathbf{c}_1\mathbf{c}_2\cdots\mathbf{c}_i\mathbf{c}_m\mathbf{c}_{m+1}\cdots\mathbf{c}_{k+1}$ is now of length at most k. By the Induction Hypothesis $P(k+1)$ is true.

By the principle of strong mathematical induction $P(k)$ is true for all $k \in \mathbb{N}$, $2 \leqslant k$. □

Proposition 2.3.4. *Let* $\big(V, E^*, i, t, r, ((\mathbb{R}, |\ \ |))_{v\in V}, (S_e)_{e\in E^1}\big)$ *be a directed graph IFS which satisfies the CSSC, then the gap lengths of any attractor of such a system can always be represented as a finite union of cosets of finitely generated semigroups with identity. These semigroups are subsemigroups of* $(\mathbb{R}^+, \times)$*, and their generators are contracting similarity ratios of simple cycles in the graph.*

Proof. The simple cycles in the graph can be identified and labelled as

$$T = \{\mathbf{c}_i : i \in I\}$$

for some indexing set $I = \{1, 2, \ldots, m\}$. In what follows $\mathcal{T}$ is as defined in Equation (2.1.2), derived from the finite set $T = \{\mathbf{c}_i : i \in I\}$, with the binary operation $\circledcirc$ as given in Equation (2.1.3), so that $(\mathcal{T}, \circledcirc)$ is a finite semigroup. The map ψ is as defined in Equation (2.1.1), and maps strings in $T^* = \langle \varepsilon, \mathbf{c}_i : i \in I\rangle$ to positive real numbers, where the similarity ratio function $r : T \to \mathbb{R}^+$ is used in place of g. We consider a vertex $u \in V$ as fixed.

(a) *The definition of* $\mathcal{A}$*,* $\bigcup \mathcal{A}$ *and* $\psi(\bigcup \mathcal{A})$*, where* $(\mathcal{A}, \circledcirc) \leqslant (\mathcal{T}, \circledcirc)$*.*

- The set of chains of length 1 (simple cycles), attached to the vertex u, are identified and labelled as

  $$\{\mathbf{c}_{i_1} : i_1 \in I_1\}$$

  for some indexing set $I_1 \subset I$.

  The chains of length 1 generate the semigroup of strings

  $$A_1 = \langle \varepsilon, \mathbf{c}_\alpha : \alpha \in I_1\rangle\,,$$

  here we remind the reader that ε is the empty string.

- The set of chains of length 2, attached to the vertex u, are identified and labelled as

  $$\{\mathbf{c}_{i_1}\mathbf{c}_{i_2} : (i_1, i_2) \in I_2\}$$

  for some indexing set of ordered pairs $I_2 \subset I_1 \times I$.

  This set of chains of length 2 generates the cosets of semigroups of strings

  $$A_t = \mathbf{c}_{i_1}\,\langle \varepsilon, \mathbf{c}_\alpha, \mathbf{c}_\beta : \alpha \in I_1,\ (i_1, \beta) \in I_2\rangle$$

  for $t = 2, \ldots, t_2 + 1$ where $t_2 = \#\,\{i_1 : (i_1, i_2) \in I_2\}$ that is t_2 is the number of distinct elements in the set $\{i_1 : (i_1, i_2) \in I_2\}$.

- The set of chains of length 3, attached to the vertex u, are identified and labelled as
$$\{\mathbf{c}_{i_1}\mathbf{c}_{i_2}\mathbf{c}_{i_3} : (i_1, i_2, i_3) \in I_3\}$$
for some indexing set of ordered triples $I_3 \subset I_2 \times I \subset I_1 \times I \times I$.

  This set of chains of length 3 generates the cosets of semigroups of strings
$$A_t = \mathbf{c}_{i_1}\mathbf{c}_{i_2}\,\langle \varepsilon, \mathbf{c}_\alpha, \mathbf{c}_\beta, \mathbf{c}_\gamma : \alpha \in I_1,\ (i_1, \beta) \in I_2,\ (i_1, i_2, \gamma) \in I_3\rangle$$
for $t = t_2 + 2, \ldots, t_3 + t_2 + 1$ where $t_3 = \#\{(i_1, i_2) : (i_1, i_2, i_3) \in I_3\}$.

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

- The set of chains of length n, attached to the vertex u, are identified and labelled as
$$\{\mathbf{c}_{i_1}\mathbf{c}_{i_2}\cdots\mathbf{c}_{i_n} : (i_1, i_2, \ldots, i_n) \in I_n\}$$
for some indexing set of ordered n-tuples $I_n \subset I_{n-1} \times I \subset \cdots \subset I_1 \times I \times \cdots \times I$.

  This set of chains of length n generates the cosets of semigroups of strings
$$\begin{aligned} A_t = \mathbf{c}_{i_1}\mathbf{c}_{i_2}\cdots\mathbf{c}_{i_{n-1}}\langle \varepsilon, \mathbf{c}_\alpha, \mathbf{c}_\beta, \mathbf{c}_\gamma, \ldots, \mathbf{c}_\omega : \alpha \in I_1,\ (i_1, \beta) \in I_2,& \\ (i_1, i_2, \gamma) \in I_3, \ldots,\ (i_1, i_2, \ldots, i_{n-1}, \omega) \in I_n&\rangle \end{aligned}$$
for $t = 2 + \sum_{r=2}^{n-1} t_r, \ldots, 1 + \sum_{r=2}^{n} t_r$, where
$$t_n = \#\{(i_1, i_2, \ldots, i_{n-1}) : (i_1, i_2, \ldots, i_{n-1}, i_n) \in I_n\}.$$

This listing process must terminate as n, $n \leqslant m$, is simply the length of the longest chain(s) in the graph attached to the vertex u. Let $N = 1 + \sum_{r=2}^{n} t_r$ denote the number of elements in the list. Each of the cosets A_i is a collection of strings on the symbols $T = \{\mathbf{c}_i : i \in I\}$ that is $A_i \subset T^*$ for i, $1 \leqslant i \leqslant N$. $\mathcal{T}$ is as defined in Equation (2.1.2) so $A_i \in \mathcal{T}$ for i, $1 \leqslant i \leqslant N$. With the binary operation $\circledcirc$ as defined in Equation (2.1.3) it follows that $(\mathcal{A}, \circledcirc) \leqslant (\mathcal{T}, \circledcirc)$, where the finitely generated semigroup $\mathcal{A}$, is defined as
$$\mathcal{A} = \langle A_i : 1 \leqslant i \leqslant N\rangle.$$

It follows from the definition of the binary operation $\circledcirc$ in Equation (2.1.3), that the finite semigroup $\mathcal{A}$, can also be expressed as
$$\mathcal{A} = \left\{A_{j_1} \circledcirc A_{j_2} \circledcirc \cdots \circledcirc A_{j_k} : j_1 j_2 \cdots j_k \in \{1, 2, \ldots, N\}^{\vartriangle}\right\},$$
where we remind the reader that $\{1, 2, \ldots, N\}^{\vartriangle}$, defined in Subsection 2.1.1, is the set of all finite sequences of distinct elements from $\{1, 2, \ldots, N\}$ with no repetitions.

Let $\bigcup \mathcal{A}$ be defined as
$$\bigcup \mathcal{A} = \bigcup_{j_1 j_2 \cdots j_k \in \{1,2,\ldots,N\}^{\vartriangle}} A_{j_1} \circledcirc A_{j_2} \circledcirc \cdots \circledcirc A_{j_k},$$

then $\bigcup \mathcal{A} \subset T^*$. The contracting similarity ratio of each simple cycle in the graph provides us with a function $r : T \to (0,1)$, so taking ψ to be the function defined in Equation (2.1.1), using the similarity ratio function r in place of g, (also see Equation (2.1.4)), we now may define $\psi(\bigcup \mathcal{A})$ as

$$\psi\left(\bigcup \mathcal{A}\right) = \bigcup_{j_1 j_2 \cdots j_k \in \{1,2,\ldots,N\}^{\triangle}} \psi\left(A_{j_1} \circledcirc A_{j_2} \circledcirc \cdots \circledcirc A_{j_k}\right),$$

where $\psi(\bigcup \mathcal{A}) \subset \overline{T}^* = \langle 1, C_i : i \in I \rangle \subset \mathbb{R}^+$.

We prove, in parts (b) and (c) that follow, that

$$\psi\left(\bigcup \mathcal{A}\right) = \{1, r_{\mathbf{e}} : \mathbf{e} \in E^*_{uu}\}. \tag{2.3.1}$$

(b) $\psi\left(\bigcup \mathcal{A}\right) \subset \{1, r_{\mathbf{e}} : \mathbf{e} \in E^*_{uu}\}$.

The empty string $\varepsilon \in A_1$, so $\varepsilon \in \bigcup \mathcal{A}$ and $\psi(\varepsilon) = 1 \in \{1, r_{\mathbf{e}} : \mathbf{e} \in E^*_{uu}\}$. Now consider any non-empty string of simple cycles $\mathbf{c}_{i_1}\mathbf{c}_{i_2}\cdots\mathbf{c}_{i_l} \in \bigcup \mathcal{A}$ then

$$\mathbf{c}_{i_1}\mathbf{c}_{i_2}\cdots\mathbf{c}_{i_l} \in A_{j_1} \circledcirc A_{j_2} \circledcirc \cdots \circledcirc A_{j_k}$$

for some string $j_1 j_2 \cdots j_k \in \{1,2,\ldots,N\}^{\triangle}$. From the construction of each A_i and the definition of the binary operation $\circledcirc$ it is clear that each cycle $\mathbf{c}_{i_j}$ is either attached to the vertex u or to a cycle that appears on its left in the string $\mathbf{c}_{i_1}\mathbf{c}_{i_2}\cdots\mathbf{c}_{i_l}$. Applying Lemma 2.3.1 with $\mathbf{g} = \mathbf{c}_{i_1} \in E^*_{uu}$ and $\mathbf{c} = \mathbf{c}_{i_2}$ we obtain $\mathbf{f} \in E^*_{uu}$ with $r_{\mathbf{f}} = C_{i_1}C_{i_2}$. Now putting $\mathbf{g} = \mathbf{f}$ and $\mathbf{c} = \mathbf{c}_{i_3}$, applying Lemma 2.3.1 again produces a new cycle $\mathbf{f} \in E^*_{uu}$ with $r_{\mathbf{f}} = C_{i_1}C_{i_2}C_{i_3}$. Applying Lemma 2.3.1 repeatedly in this way we can produce a cycle $\mathbf{f} \in E^*_{uu}$ such that

$$r_{\mathbf{f}} = C_{i_1}C_{i_2}\cdots C_{i_l} = \psi(\mathbf{c}_{i_1}\mathbf{c}_{i_2}\cdots\mathbf{c}_{i_l}).$$

Hence $\psi(\bigcup \mathcal{A}) \subset \{1, r_{\mathbf{e}} : \mathbf{e} \in E^*_{uu}\}$.

(c) $\{1, r_{\mathbf{e}} : \mathbf{e} \in E^*_{uu}\} \subset \psi(\bigcup \mathcal{A})$.

An example illustrating the main ideas behind this proof is given in Figure 2.5.2 and the text that follows it.

We pointed out in part (b) that $1 \in \psi(\bigcup \mathcal{A})$ so we only need to consider the (non-empty) paths $\mathbf{e} \in E^*_{uu}$. If $\mathbf{e}$ is a simple cycle then it is a chain of length 1 attached to u, and so $\mathbf{e} \in A_1$ and $r_{\mathbf{e}} \in \psi(\bigcup \mathcal{A})$. So we assume $\mathbf{e}$ is not a simple cycle. Putting $\mathbf{f} = \mathbf{e} \in E^*_{uu}$ in Lemma 2.3.2 we obtain a path $\mathbf{g} \in E^*_{uu}$, attached to u, and a simple cycle in the graph $\mathbf{c}_{i_1}$, $i_1 \in I$, which is attached to $\mathbf{g}$ with $r_{\mathbf{e}} = r_{\mathbf{g}}C_{i_1}$. Either $\mathbf{g}$ is a simple cycle, or it is not. If it is a simple cycle we stop, if not, we put $\mathbf{f} = \mathbf{g}$ and apply Lemma 2.3.2 again to give a new path $\mathbf{g}$ and a new simple cycle in the graph $\mathbf{c}_{i_2}$, $i_2 \in I$, attached to $\mathbf{g}$, such that $r_{\mathbf{e}} = r_{\mathbf{g}}C_{i_2}C_{i_1}$. It is important to note here that the simple cycle $\mathbf{c}_{i_1}$ is attached to at least one of $\mathbf{g}$ or $\mathbf{c}_{i_2}$ and that $\mathbf{c}_{i_2}$ is attached to $\mathbf{g}$. This iterative process must eventually terminate when $\mathbf{g} \in E^*_{uu}$ becomes a simple cycle in the graph, which is attached to the vertex u. On termination, $r_{\mathbf{e}}$ will be expressed in the form

$$r_{\mathbf{e}} = C_{i_k}C_{i_{k-1}}\cdots C_{i_2}C_{i_1} = \psi(\mathbf{c}_{i_k}\mathbf{c}_{i_{k-1}}\cdots\mathbf{c}_{i_2}\mathbf{c}_{i_1}),$$

where simple cycles may be repeated in the string $\mathbf{c}_{i_k}\mathbf{c}_{i_{k-1}}\cdots\mathbf{c}_{i_2}\mathbf{c}_{i_1} \in T^*$, where $\mathbf{c}_{i_k}$ is attached to u, and where every other cycle in this string is either attached to the vertex u, or it is attached to at least one cycle appearing on its left. Applying the function φ, defined by Equation (2.1.5), we obtain

$$\varphi(\mathbf{c}_{i_k}\mathbf{c}_{i_{k-1}}\cdots\mathbf{c}_{i_2}\mathbf{c}_{i_1}) = \mathbf{c}_{j_1}^{k_1}\mathbf{c}_{j_2}^{k_2}\cdots\mathbf{c}_{j_l}^{k_l}, \tag{2.3.2}$$

where for convenience we have relabelled the indices, with $j_1 = i_k$. The function φ operates by collecting up identical occurrences of a symbol into powers whilst preserving the order of the first appearance of the different symbols in the original string as we go from left to right, see Equation (2.1.5), so it follows that the simple cycle $\mathbf{c}_{j_1}$ is attached to the vertex u and every other simple cycle in the string on the right-hand side of this equation is either attached to u, or it is attached to at least one simple cycle occuring to its left. In terms of similarity ratios

$$r_{\mathbf{e}} = \psi(\mathbf{c}_{i_k}\mathbf{c}_{i_{k-1}}\cdots\mathbf{c}_{i_2}\mathbf{c}_{i_1}) = \psi(\mathbf{c}_{j_1}^{k_1}\mathbf{c}_{j_2}^{k_2}\cdots\mathbf{c}_{j_l}^{k_l}) = C_{j_1}^{k_1}C_{j_2}^{k_2}\cdots C_{j_l}^{k_l}, \tag{2.3.3}$$

Now consider $\mathbf{c}_{j_l}$, if $\mathbf{c}_{j_l}$ is attached to u then $\mathbf{c}_{j_l} \in A_1$ and we put $A_{q_l} = A_1$. If $\mathbf{c}_{j_l}$ is not attached to u, then it is attached to $\mathbf{c}_{j_{a_1}}$ for some $a_1 \in \{1, 2, \cdots, l-1\}$. If $\mathbf{c}_{j_{a_1}}$ is not attached to u then it is attached to $\mathbf{c}_{j_{a_2}}$ for some $a_2 \in \{1, 2, \cdots, a_1 - 1\}$. We continue in this way until we arrive at the first cycle, $\mathbf{c}_{j_{a_s}}$ which is attached to u, at which point we stop. This process produces an attached sequence of distinct cycles, $\mathbf{c}_{j_{a_s}}\mathbf{c}_{j_{a_{s-1}}}\cdots\mathbf{c}_{j_{a_1}}\mathbf{c}_{j_l}$, where $\mathbf{c}_{j_{a_s}}$ is attached to u. By Lemma 2.3.3 there exists another subsequence, containing at least two terms,

$$\begin{aligned}&\{a_s, b_t, b_{t-1}, \ldots, b_1, l : a_s < b_t < b_{t-1} < \ldots < b_1 < l\} \\ &\qquad \subset \{a_s, a_{s-1}, \ldots, a_1, l : a_s < a_{s-1} < \cdots < a_1 < l\} \subset \{1, 2, \cdots, l-1, l\}\end{aligned}$$

such that $\mathbf{c}_{j_{a_s}}\mathbf{c}_{j_{b_t}}\mathbf{c}_{j_{b_{t-1}}}\cdots\mathbf{c}_{j_{b_1}}\mathbf{c}_{j_l}$ is a chain attached to the vertex u. Let A_{q_l} be the coset corresponding to this chain as given in the algorithm of part (a). So

$$A_{q_l} = \mathbf{c}_{j_{a_s}}\mathbf{c}_{j_{b_t}}\mathbf{c}_{j_{b_{t-1}}}\cdots\mathbf{c}_{j_{b_1}}\langle\varepsilon, \ldots, \mathbf{c}_{j_l}, \ldots\rangle,$$

where each simple cycle in the multiplier occurs on the right-hand side of Equation (2.3.2), and $\mathbf{c}_{j_l}$ is a generator of the semigroup. For each i, $1 < i \leqslant l$, we apply this process. It is clear from the definition of the operation $\circledcirc$, (see Equation (2.1.3) above), that $A_{q_1} \circledcirc A_{q_2} \circledcirc \cdots \circledcirc A_{q_l}$ will be of the form

$$A_{q_1} \circledcirc A_{q_2} \circledcirc \cdots \circledcirc A_{q_l} = \mathbf{c}_{j_{d_1}}\mathbf{c}_{j_{d_2}}\cdots\mathbf{c}_{j_{d_k}}\langle\varepsilon, \ldots, \mathbf{c}_{j_1}, \mathbf{c}_{j_2}, \ldots, \mathbf{c}_{j_{l-1}}, \mathbf{c}_{j_l}, \ldots\rangle.$$

for some, possibly empty, subsequence $(d_1, d_2, \ldots, d_k)$ of $(1, 2, \ldots, l-1)$, so each simple cycle in the multiplier occurs on the right-hand side of Equation (2.3.2). (If the subsequence $(d_1, d_2, \ldots, d_k)$ is empty, then $A_{q_l} \circledcirc \cdots \circledcirc A_{q_2} \circledcirc A_{q_1} = A_1$). Every simple cycle that occurs on the right-hand side of Equation (2.3.2) is also a generator of this semigroup. Now

$$\psi(A_{q_1} \circledcirc A_{q_2} \circledcirc \cdots \circledcirc A_{q_l}) = C_{j_{d_1}}C_{j_{d_2}}\cdots C_{j_{d_k}}\langle 1, \ldots, C_{j_1}, C_{j_2}, \ldots, C_{j_{l-1}}, C_{j_l}, \ldots\rangle,$$

and so from Equation (2.3.3)

$$r_{\mathbf{e}} = C_{j_1}^{k_1}C_{j_2}^{k_2}\cdots C_{j_l}^{k_l} \in \psi(A_{q_1} \circledcirc A_{q_2} \cdots \circledcirc A_{q_l}).$$

Therefore $\{1, r_{\mathbf{e}} : \mathbf{e} \in E^*_{uu}\} \subset \psi(\bigcup \mathcal{A})$ as was required to be shown.

(d) *The definition of* $\mathcal{B}^{\mathbf{p}}$, $\bigcup \mathcal{B}^{\mathbf{p}}$ *and* $\psi(\bigcup \mathcal{B}^{\mathbf{p}})$, *for each simple path* $\mathbf{p} \in D^*_{uv}$, $v \in V$, $v \neq u$, *where* $(\mathcal{B}^{\mathbf{p}}, \circledcirc) \leqslant (\mathcal{T}, \circledcirc)$.

Let $v \in V$, $v \neq u$ be fixed, and $\mathbf{p} \in D^*_{uv} \subset E^*_{uv}$ be any simple path from the vertex u to v which we also consider as fixed. Replacing the phrase, *attached to the vertex* u, by the phrase, *attached to the simple path* $\mathbf{p}$, in the algorithm for generating the cosets A_i, $1 \leqslant i \leqslant N$ in part (a), we obtain the following algorithm for generating a list of cosets of finitely generated string semigroups $B^{\mathbf{p}}_j$, $1 \leqslant j \leqslant M$.

- The set of chains of length 1, attached to the simple path $\mathbf{p}$, are identified and labelled as
$$\{\mathbf{c}_{i_1} : i_1 \in I^{\mathbf{p}}_1\}$$
for some indexing set $I^{\mathbf{p}}_1 \subset I$.

  These chains of length 1 generate the semigroup of strings
$$B^{\mathbf{p}}_1 = \langle \varepsilon, C_\alpha : \alpha \in I^{\mathbf{p}}_1 \rangle .$$

- The set of chains of length 2, attached to the simple path $\mathbf{p}$, are identified and labelled as
$$\{\mathbf{c}_{i_1}\mathbf{c}_{i_2} : (i_1, i_2) \in I^{\mathbf{p}}_2\}$$
for some indexing set of ordered pairs $I^{\mathbf{p}}_2 \subset I^{\mathbf{p}}_1 \times I$.

  This set of chains of length 2 generates the cosets of semigroups
$$B^{\mathbf{p}}_t = \mathbf{c}_{i_1} \langle \varepsilon, \mathbf{c}_\alpha, \mathbf{c}_\beta : \alpha \in I^{\mathbf{p}}_1,\ (i_1, \beta) \in I^{\mathbf{p}}_2 \rangle$$
for $t = 2, \ldots, t_2 + 1$ where $t_2 = \#\{i_1 : (i_1, i_2) \in I^{\mathbf{p}}_2\}$ that is t_2 is the number of distinct elements in the set $\{i_1 : (i_1, i_2) \in I^{\mathbf{p}}_2\}$.

- The set of chains of length 3, attached to the simple path $\mathbf{p}$, are identified and labelled as
$$\{\mathbf{c}_{i_1}\mathbf{c}_{i_2}\mathbf{c}_{i_3} : (i_1, i_2, i_3) \in I^{\mathbf{p}}_3\}$$
for some indexing set of ordered triples $I^{\mathbf{p}}_3 \subset I^{\mathbf{p}}_2 \times I \subset I^{\mathbf{p}}_1 \times I \times I$.

  This set of chains of length 3 generates the cosets of semigroups
$$B^{\mathbf{p}}_t = \mathbf{c}_{i_1}\mathbf{c}_{i_2} \langle \varepsilon, \mathbf{c}_\alpha, \mathbf{c}_\beta, \mathbf{c}_\gamma : \alpha \in I^{\mathbf{p}}_1,\ (i_1, \beta) \in I^{\mathbf{p}}_2,\ (i_1, i_2, \gamma) \in I^{\mathbf{p}}_3 \rangle$$
for $t = t_2 + 2, \ldots, t_3 + t_2 + 1$ where $t_3 = \#\{(i_1, i_2) : (i_1, i_2, i_3) \in I^{\mathbf{p}}_3\}$.

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

- The set of chains of length n, attached to the simple path $\mathbf{p}$, are identified and labelled as
$$\{\mathbf{c}_{i_1}\mathbf{c}_{i_2} \cdots \mathbf{c}_{i_n} : (i_1, i_2, \ldots, i_n) \in I^{\mathbf{p}}_n\}$$
for some indexing set of ordered n-tuples $I^{\mathbf{p}}_n \subset I^{\mathbf{p}}_{n-1} \times I \subset \cdots \subset I^{\mathbf{p}}_1 \times I \times \cdots \times I$.

This set of chains of length n generates the cosets of semigroups

$$B_t^{\mathbf{p}} = \mathbf{c}_{i_1}\mathbf{c}_{i_2}\cdots\mathbf{c}_{i_{n-1}}\langle\varepsilon,\mathbf{c}_\alpha,\mathbf{c}_\beta,\mathbf{c}_\gamma,\ldots,\mathbf{c}_\omega : \alpha\in I_1^{\mathbf{p}},\,(i_1,\beta)\in I_2^{\mathbf{p}},\\ (i_1,i_2,\gamma)\in I_3^{\mathbf{p}},\ldots,\,(i_1,i_2,\ldots,i_{n-1},\,\omega)\in I_n^{\mathbf{p}}\rangle$$

for $t = 2+\sum_{r=2}^{n-1} t_r,\ldots,1+\sum_{r=2}^{n} t_r$, where

$$t_n = \#\left\{(i_1,i_2,\ldots,i_{n-1}) : (i_1,i_2,\ldots,i_{n-1},i_n)\in I_n^{\mathbf{p}}\right\}.$$

This listing process must terminate as n, $n\leqslant m$, is simply the length of the longest chain(s) in the graph attached to the simple path $\mathbf{p}$. Let $M = 1+\sum_{r=2}^{n} t_r$ denote the number of elements in the list. Each of the cosets $B_i^{\mathbf{p}}$ is a collection of strings on the symbols $T=\{\mathbf{c}_i : i\in I\}$ that is $B_i^{\mathbf{p}}\subset T^*$ for i, $1\leqslant i\leqslant M$. Exactly as $\mathcal{A}$ was defined in part (a) above we define $(\mathcal{B}^{\mathbf{p}},\circledcirc)\leqslant(\mathcal{T},\circledcirc)$ as

$$\mathcal{B}^{\mathbf{p}} = \langle B_i^{\mathbf{p}} : 1\leqslant i\leqslant M\rangle\,.$$

Again, as in part (a), from the definition of the binary operation $\circledcirc$ in Equation (2.1.3), it follows that the finite semigroup $\mathcal{B}^{\mathbf{p}}$, can also be written as

$$\mathcal{B}^{\mathbf{p}} = \left\{B_{j_1}^{\mathbf{p}}\circledcirc B_{j_2}^{\mathbf{p}}\circledcirc\cdots\circledcirc B_{j_k}^{\mathbf{p}} : j_1j_2\cdots j_k\in\{1,2,\ldots,M\}^{\Delta}\right\}.$$

Let $\bigcup\mathcal{B}^{\mathbf{p}}$ be defined as

$$\bigcup\mathcal{B}^{\mathbf{p}} = \bigcup_{j_1j_2\cdots j_k\in\{1,2,\ldots,M\}^{\Delta}} B_{j_1}^{\mathbf{p}}\circledcirc B_{j_2}^{\mathbf{p}}\circledcirc\cdots\circledcirc B_{j_k}^{\mathbf{p}},$$

then $\bigcup\mathcal{B}^{\mathbf{p}}\subset T^*$. Taking ψ to be the function defined in Equation (2.1.1), using the function r in place of g, (also see Equation (2.1.4)), exactly as was done in part (a) above, we may now define $\psi(\bigcup\mathcal{B}^{\mathbf{p}})$ as

$$\psi\left(\bigcup\mathcal{B}^{\mathbf{p}}\right) = \bigcup_{j_1j_2\cdots j_k\in\{1,2,\ldots,M\}^{\Delta}} \psi\left(B_{j_1}^{\mathbf{p}}\circledcirc B_{j_2}^{\mathbf{p}}\circledcirc\cdots\circledcirc B_{j_k}^{\mathbf{p}}\right),$$

where $\psi(\bigcup\mathcal{B}^{\mathbf{p}})\subset\overline{T}^* = \langle 1, C_i : i\in I\rangle\subset\mathbb{R}^+$.

We show, in parts (e) and (f) that follow, that for a vertex v, $v\neq u$,

$$\bigcup_{\mathbf{p}\in D_{uv}^*} r_{\mathbf{p}}\psi\left(\bigcup\mathcal{B}^{\mathbf{p}}\right) = \{r_{\mathbf{e}} : \mathbf{e}\in E_{uv}^*\}\,. \tag{2.3.4}$$

The method used is almost identical to that given in parts (b) and (c).

(e) $\bigcup_{\mathbf{p}\in D_{uv}^*} r_{\mathbf{p}}\psi\left(\bigcup\mathcal{B}^{\mathbf{p}}\right)\subset\{r_{\mathbf{e}} : \mathbf{e}\in E_{uv}^*\}$.

Consider any simple path $\mathbf{p}\in D_{uv}^*\subset E_{uv}^*$ as fixed. For $\varepsilon\in B_1^{\mathbf{p}}$ clearly $r_{\mathbf{p}}\psi(\varepsilon) = r_{\mathbf{p}}\in\{r_{\mathbf{e}} : \mathbf{e}\in E_{uv}^*\}$. Now consider any non-empty string of simple cycles $\mathbf{c}_{i_1}\mathbf{c}_{i_2}\cdots\mathbf{c}_{i_l}\in\bigcup\mathcal{B}^{\mathbf{p}}$ then

$$\mathbf{c}_{i_1}\mathbf{c}_{i_2}\cdots\mathbf{c}_{i_l}\in B_{j_1}^{\mathbf{p}}\circledcirc B_{j_2}^{\mathbf{p}}\circledcirc\cdots\circledcirc B_{j_k}^{\mathbf{p}}$$

for some string $j_1 j_2 \cdots j_k \in \{1, 2, \dots, M\}^{\vartriangle}$. From the construction of each $B_i^{\mathbf{p}}$ and the definition of the binary operation $\circledcirc$ it is clear that each cycle $\mathbf{c}_{i_j}$ is either attached to the path $\mathbf{p}$ or to a cycle that appears on its left in the string $\mathbf{c}_{i_1}\mathbf{c}_{i_2}\cdots\mathbf{c}_{i_l}$. Applying Lemma 2.3.1 with $\mathbf{g} = \mathbf{p} \in E_{uv}^*$ and $\mathbf{c} = \mathbf{c}_{i_1}$ we obtain $\mathbf{f} \in E_{uv}^*$ with $r_{\mathbf{f}} = r_{\mathbf{p}} C_{i_1}$. Now putting $\mathbf{g} = \mathbf{f}$ and $\mathbf{c} = \mathbf{c}_{i_2}$, applying Lemma 2.3.1 again produces a new path $\mathbf{f} \in E_{uv}^*$ with $r_{\mathbf{f}} = r_{\mathbf{p}} C_{i_1} C_{i_2}$. Applying Lemma 2.3.1 repeatedly in this way we can produce a path $\mathbf{f} \in E_{uv}^*$ such that

$$r_{\mathbf{f}} = r_{\mathbf{p}} C_{i_1} C_{i_2} \cdots C_{i_l} = r_{\mathbf{p}} \psi(\mathbf{c}_{i_1}\mathbf{c}_{i_2}\cdots\mathbf{c}_{i_l}).$$

Hence $r_{\mathbf{p}}\psi(\bigcup \mathcal{B}^{\mathbf{p}}) \subset \{r_{\mathbf{e}} : \mathbf{e} \in E_{uv}^*\}$.

(f) $\{r_{\mathbf{e}} : \mathbf{e} \in E_{uv}^*\} \subset \bigcup_{\mathbf{p}\in D_{uv}^*} r_{\mathbf{p}}\psi\left(\bigcup \mathcal{B}^{\mathbf{p}}\right)$.

We note that for each $\mathbf{p} \in D_{uv}^*$, $\varepsilon \in B_1^{\mathbf{p}}$ and $1 = \psi(\varepsilon) \in \psi\left(\bigcup \mathcal{B}^{\mathbf{p}}\right)$, and so it follows that $\{r_{\mathbf{e}} : \mathbf{e} \in D_{uv}^* \subset E_{uv}^*\} \subset \bigcup_{\mathbf{p}\in D_{uv}^*} r_{\mathbf{p}}\psi\left(\bigcup \mathcal{B}^{\mathbf{p}}\right)$. So we consider $\mathbf{e} \in E_{uv}^* \setminus D_{uv}^*$, that is $\mathbf{e} \in E_{uv}^*$ where $\mathbf{e}$ is not a simple path. Repeated applications of Lemma 2.3.2 produce

$$r_{\mathbf{e}} = r_{\mathbf{p}} C_{i_k} C_{i_{k-1}} \cdots C_{i_2} C_{i_1} = r_{\mathbf{p}} \psi(\mathbf{c}_{i_k}\mathbf{c}_{i_{k-1}}\cdots\mathbf{c}_{i_2}\mathbf{c}_{i_1}),$$

for some simple path $\mathbf{p} \in D_{uv}^*$, where the simple cycles may be repeated in the sequence $\mathbf{c}_{i_k}\mathbf{c}_{i_{k-1}}\cdots\mathbf{c}_{i_2}\mathbf{c}_{i_1}$, where $\mathbf{c}_{i_k}$ is attached to the simple path $\mathbf{p}$, and where every other cycle is either attached to $\mathbf{p}$, or it is attached to at least one cycle appearing on its left. Applying the function φ, defined by Equation (2.1.5), we obtain

$$\varphi(\mathbf{c}_{i_k}\mathbf{c}_{i_{k-1}}\cdots\mathbf{c}_{i_2}\mathbf{c}_{i_1}) = \mathbf{c}_{j_1}^{k_1}\mathbf{c}_{j_2}^{k_2}\cdots\mathbf{c}_{j_l}^{k_l}, \tag{2.3.5}$$

where for convenience we have relabelled the indices, with $j_1 = i_k$. It follows from the way the function φ operates that the simple cycle $\mathbf{c}_{j_1}$ is attached to the simple path $\mathbf{p}$ and every other simple cycle in the string on the right-hand side of this equation is either attached to $\mathbf{p}$, or it is attached to at least one simple cycle occuring to its left. In terms of similarity ratios

$$r_{\mathbf{e}} = r_{\mathbf{p}}\psi(\mathbf{c}_{i_k}\mathbf{c}_{i_{k-1}}\cdots\mathbf{c}_{i_2}\mathbf{c}_{i_1}) = r_{\mathbf{p}}\psi(\mathbf{c}_{j_1}^{k_1}\mathbf{c}_{j_2}^{k_2}\cdots\mathbf{c}_{j_l}^{k_l}) = r_{\mathbf{p}} C_{j_1}^{k_1} C_{j_2}^{k_2}\cdots C_{j_l}^{k_l}, \tag{2.3.6}$$

By applying Lemma 2.3.3 in exactly the same way as was done in part (c) we identify cosets $B_{q_i}^{\mathbf{p}}$ such that

$$B_{q_1}^{\mathbf{p}} \circledcirc B_{q_2}^{\mathbf{p}} \circledcirc \cdots \circledcirc B_{q_l}^{\mathbf{p}} = \mathbf{c}_{j_{d_1}}\mathbf{c}_{j_{d_2}}\cdots\mathbf{c}_{j_{d_k}}\langle \varepsilon, \dots, \mathbf{c}_{j_1}, \mathbf{c}_{j_2}, \dots, \mathbf{c}_{j_{l-1}}, \mathbf{c}_{j_l}, \dots\rangle.$$

for some, possibly empty, subsequence $(d_1, d_2, \dots, d_k)$ of $(1, 2, \dots, l-1, l)$, where each simple cycle in the multiplier occurs on the right-hand side of Equation (2.3.5). (If the subsequence $(d_1, d_2, \dots, d_k)$ is empty, then $B_{q_1}^{\mathbf{p}} \circledcirc B_{q_2}^{\mathbf{p}} \circledcirc \cdots \circledcirc B_{q_l}^{\mathbf{p}} = B_1^{\mathbf{p}}$). Every simple cycle that occurs on the right-hand side of Equation (2.3.5) is also a generator of this semigroup. Now

$$\psi(B_{q_1}^{\mathbf{p}} \circledcirc B_{q_2}^{\mathbf{p}} \circledcirc \cdots \circledcirc B_{q_l}^{\mathbf{p}}) = C_{j_{d_1}} C_{j_{d_2}} \cdots C_{j_{d_k}} \langle 1, \dots, C_{j_1}, C_{j_2}, \dots, C_{j_{l-1}}, C_{j_l}, \dots\rangle,$$

and so from Equation (2.3.6)

$$r_{\mathbf{e}} = r_{\mathbf{p}} C_{j_1}^{k_1} C_{j_2}^{k_2} \cdots C_{j_l}^{k_l} \in r_{\mathbf{p}}\psi(B_{q_1}^{\mathbf{p}} \circledcirc B_{q_2}^{\mathbf{p}} \cdots \circledcirc B_{q_l}^{\mathbf{p}}).$$

Therefore $\{r_{\mathbf{e}} : \mathbf{e} \in E^*_{uv}\} \subset \bigcup_{\mathbf{p}\in D^*_{uv}} r_{\mathbf{p}}\psi\left(\bigcup \mathcal{B}^{\mathbf{p}}\right)$ as was required to be shown.

From Equation (2.2.5), the gap lengths of the attractor F_u at the vertex u are given by

$$G_u = \bigcup_{g_u \in G^1_u} g_u\left\{1, r_{\mathbf{e}} : \mathbf{e} \in E^*_{uu}\right\} \cup \bigcup_{\substack{v\in V\\ v\neq u\\ g_v\in G^1_v}} g_v\left\{r_{\mathbf{e}} : \mathbf{e} \in E^*_{uv}\right\}.$$

Using Equations (2.3.1) and (2.3.4), we obtain,

$$G_u = \bigcup_{g_u \in G^1_u} g_u\psi\left(\bigcup \mathcal{A}\right) \cup \bigcup_{\substack{v\in V\\ v\neq u\\ g_v\in G^1_v}} g_v\Bigg(\bigcup_{\mathbf{p}\in D^*_{uv}} r_{\mathbf{p}}\psi\left(\bigcup \mathcal{B}^{\mathbf{p}}\right)\Bigg).$$

This completes the proof because $\psi(\bigcup \mathcal{A})$ and $\psi\left(\bigcup \mathcal{B}^{\mathbf{p}}\right)$, for each $\mathbf{p} \in D^*_{uv}$, are finite unions of cosets of finitely generated semigroups, where the generators of the semigroups are similarity ratios of simple cycles in the directed graph. □

In the next corollary we collect together, for future reference, equations established in the proof of Proposition 2.3.4, that will be useful to us. Equations (2.3.10) and (2.3.11) follow from the algorithm in parts (a) and (d) in the proof of Proposition 2.3.4 as $\psi(\bigcup \mathcal{A}) \subset \overline{T}^* = \langle 1, C_i : i \in I\rangle$ and $\psi\left(\bigcup \mathcal{B}^{\mathbf{p}}\right) \subset \overline{T}^* = \langle 1, C_i : i \in I\rangle$.

Corollary 2.3.5. *Let $\big(V, E^*, i, t, r, ((\mathbb{R}, |\ \ |))_{v\in V}, (S_e)_{e\in E^1}\big)$ be a directed graph IFS satisfying the CSSC. Then the gap lengths of the attractor F_u, at a vertex u, can be expressed as*

$$G_u = \bigcup_{g_u \in G^1_u} g_u\psi\left(\bigcup \mathcal{A}\right) \cup \bigcup_{\substack{v\in V\\ v\neq u\\ g_v\in G^1_v}} g_v\Bigg(\bigcup_{\mathbf{p}\in D^*_{uv}} r_{\mathbf{p}}\psi\left(\bigcup \mathcal{B}^{\mathbf{p}}\right)\Bigg), \tag{2.3.7}$$

where

$$\psi\left(\bigcup \mathcal{A}\right) = \left\{1, r_{\mathbf{e}} : \mathbf{e} \in E^*_{uu}\right\}, \tag{2.3.8}$$

$$\bigcup_{\mathbf{p}\in D^*_{uv}} r_{\mathbf{p}}\psi\left(\bigcup \mathcal{B}^{\mathbf{p}}\right) = \left\{r_{\mathbf{e}} : \mathbf{e} \in E^*_{uv}\right\}, \quad \text{for } v \neq u. \tag{2.3.9}$$

$$\psi\left(\bigcup \mathcal{A}\right) \subset \langle 1, C_i : i \in I\rangle, \tag{2.3.10}$$

and

$$\psi\left(\bigcup \mathcal{B}^{\mathbf{p}}\right) \subset \langle 1, C_i : i \in I\rangle, \tag{2.3.11}$$

for simple paths $\mathbf{p} \in D^*_{uv}$, $v \in V$, $v \neq u$.

Corollary 2.3.6. *Let $\big(V, E^*, i, t, r, ((\mathbb{R}, |\ \ |))_{v\in V}, (S_e)_{e\in E^1}\big)$, be a standard (1-vertex) directed graph IFS, which satisfies the CSSC. Then the gap lengths of the attractor can be expressed as a finite union of cosets of a single finitely generated semigroup.*

Proof. For a 1-vertex directed graph with $V = \{w\}$, there are no simple paths, no chains attached to w of length greater than 1, and the only chains of length 1 attached to w are loops consisting of a single edge. It follows that $\psi\left(\bigcup \mathcal{A}\right) = \psi\left(A_1\right)$, and Equation (2.3.7) gives the gap lengths of the attractor as

$$G_w = \bigcup_{g_w \in G_w^1} g_w \psi\left(A_1\right),$$

where G_w^1 denotes the gap lengths at the single vertex w, in the first level iteration of I_w, as defined in Subsection 2.2.1. As $\psi\left(A_1\right)$ is just a finitely generated semigroup, G_w is a finite union of cosets of a single finitely generated semigroup. $\square$

2.4 An example of a 2-vertex IFS

The proof of Proposition 2.3.4 provides an algorithm for expressing the set of gap lengths of an attractor in terms of cosets of semigroups. In this section we illustrate

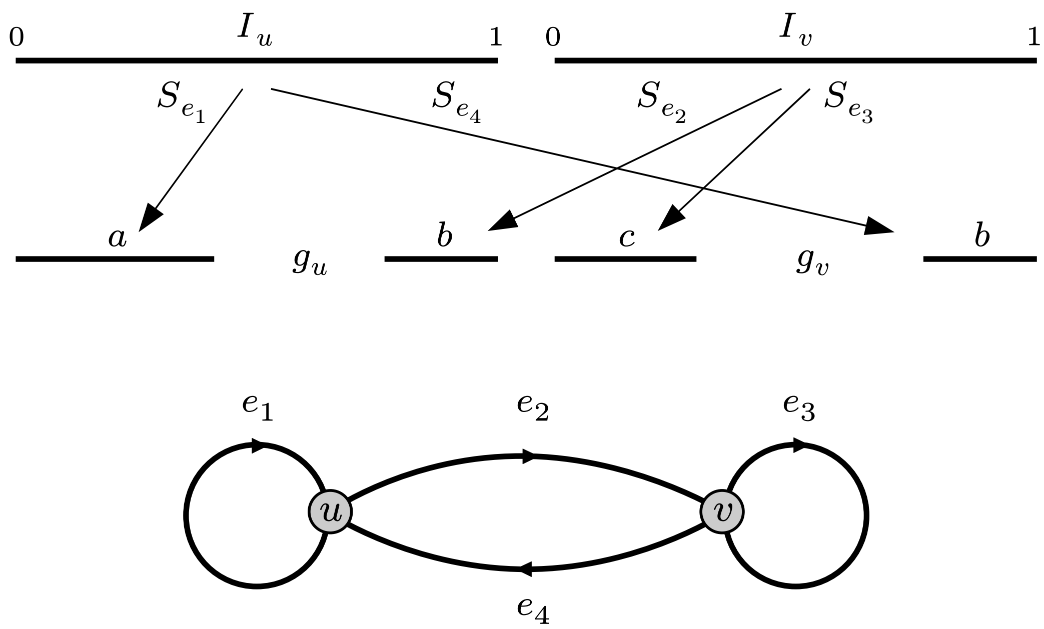


Figure. 2.4.1: A 2-vertex directed graph IFS.

the method by using it to calculate the gap lengths of the attractor F_u of the directed graph IFS illustrated in Figure 2.4.1 and for which the CSSC holds as g_u, g_v are assumed to be strictly positive.

For the graph in Figure 2.4.1 the simple cycles are

$$\mathbf{c}_1 = e_1, \quad \mathbf{c}_2 = e_2 e_4, \quad \text{and} \quad \mathbf{c}_3 = e_3.$$

The similarity ratios of the similarities along these simple cycles are

$$r(\mathbf{c}_1) = r_{e_1} = a, \quad r(\mathbf{c}_2) = r_{e_2} r_{e_4} = bb = b^2, \quad \text{and} \quad r(\mathbf{c}_3) = r_{e_3} = c.$$

Let

$$T = \{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3\}.$$

We now follow the algorithm in part (a) of the proof of Proposition 2.3.4.

- The set of chains of length 1 (simple cycles), attached to the vertex u, is

  $$\{\mathbf{c}_1, \mathbf{c}_2\}.$$

  The chains of length 1 generate the semigroup of strings

  $$A_1 = \langle \varepsilon, \mathbf{c}_1, \mathbf{c}_2 \rangle.$$

- The set of chains of length 2, attached to the vertex u, is

  $$\{\mathbf{c}_2\mathbf{c}_3\}$$

  This set generates the following coset of a semigroup of strings

  $$A_2 = \mathbf{c}_2 \langle \varepsilon, \mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3 \rangle.$$

- There are no chains of length 3 or more, attached to the vertex u, so we stop.

We calculate $(\mathcal{A}, \circledcirc)$ as

$$\mathcal{A} = \langle A_1, A_2 \rangle = \{A_1, A_2\},$$

since $A_1 \circledcirc A_1 = A_1$ and $A_1 \circledcirc A_2 = A_2 \circledcirc A_1 = A_2 \circledcirc A_2 = A_2$. This gives $\psi\left(\bigcup \mathcal{A}\right)$ as

$$\begin{aligned} \psi\left(\bigcup \mathcal{A}\right) &= \psi(A_1) \cup \psi(A_2) \\ &= \langle 1, r(\mathbf{c}_1), r(\mathbf{c}_2) \rangle \cup r(\mathbf{c}_2) \langle 1, r(\mathbf{c}_1), r(\mathbf{c}_2), r(\mathbf{c}_3) \rangle \\ &= \left\langle 1, a, b^2 \right\rangle \cup b^2 \left\langle 1, a, b^2, c \right\rangle, \end{aligned}$$

so

$$\psi\left(\bigcup \mathcal{A}\right) = \left\langle 1, a, b^2 \right\rangle \cup b^2 \left\langle 1, a, b^2, c \right\rangle. \tag{2.4.1}$$

There is only one simple path $\mathbf{p} = e_2$ from the vertex u to v, so we now follow the algorithm in part (d) of the proof of Proposition 2.3.4, just once for this simple path.

- The set of chains of length 1, attached to the simple path e_2, is

  $$\{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3\}.$$

  These chains of length 1 generate the semigroup of strings

  $$B_1^{e_2} = \langle \varepsilon, \mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3 \rangle.$$

- There are no chains of length 2 or more, attached to the simple path e_2, so we stop.

$(\mathcal{B}^{e_2}, \circledcirc)$ is

$$\mathcal{B}^{e_2} = \langle B_1^{e_2} \rangle = \{B_1^{e_2}\}\,.$$

So

$$\begin{aligned}\psi\left(\bigcup \mathcal{B}^{e_2}\right) &= \psi\left(B_1^{e_2}\right)\\ &= \langle 1, r(\mathbf{c}_1), r(\mathbf{c}_2), r(\mathbf{c}_3)\rangle\\ &= \left\langle 1, a, b^2, c\right\rangle.\end{aligned}$$

Finally Equation (2.3.7) gives the set of gap lengths of the attractor F_u as

$$\begin{aligned}G_u &= g_u\psi\left(\bigcup \mathcal{A}\right) \cup g_v r_{e_2}\psi\left(\bigcup \mathcal{B}^{e_2}\right)\\ &= g_u\left\langle 1, a, b^2\right\rangle \cup g_u b^2\left\langle 1, a, b^2, c\right\rangle \cup g_v b\left\langle 1, a, b^2, c\right\rangle\\ &= g_u\langle 1, a\rangle \cup g_u b^2\left\langle 1, a, b^2, c\right\rangle \cup g_v b\left\langle 1, a, b^2, c\right\rangle. \end{aligned} \tag{2.4.2}$$

Before going any further we use the array formula of Equation (2.2.8) as a check on this calculation. Putting $u = v_1$ and $v = v_2$ in Equations (2.2.6) and (2.2.7), means $\mathbf{M} \in M_2$ is given by

$$\mathbf{M} = \begin{pmatrix} \{a\} & \{b\} \\ \{b\} & \{c\} \end{pmatrix}.$$

and

$$\mathbf{g} = \begin{pmatrix} G_u^1 \\ G_v^1 \end{pmatrix} = \begin{pmatrix} \{g_u\} \\ \{g_v\} \end{pmatrix}.$$

From Equation (2.2.8) the sets of gap lengths G_u, G_v, of the attractors F_u, F_v, are given by

$$\begin{pmatrix} G_u \\ G_v \end{pmatrix} = \bigcup_{j=0}^{\infty} \mathbf{M}^j\mathbf{g}.$$

The first few arrays $\mathbf{M}^j$, for $j = 0, 1, 2, 3, 4$, are

$$\mathbf{M}^0 = \begin{pmatrix} \{1\} & \{0\} \\ \{0\} & \{1\} \end{pmatrix},$$

$$\mathbf{M} = \begin{pmatrix} \{a\} & \{b\} \\ \{b\} & \{c\} \end{pmatrix},$$

$$\mathbf{M}^2 = \begin{pmatrix} \{a^2, b^2\} & \{ab, bc\} \\ \{ab, bc\} & \{b^2, c^2\} \end{pmatrix},$$

$$\mathbf{M}^3 = \begin{pmatrix} \{a^3, ab^2, b^2c\} & \{a^2b, abc, bc^2, b^3\} \\ \{a^2b, abc, bc^2, b^3\} & \{ab^2, b^2c, c^3\} \end{pmatrix},$$

and

$$\mathbf{M}^4 = \begin{pmatrix} \{a^4, a^2b^2, ab^2c, b^2c^2, b^4\} & \{a^3b, a^2bc, ab^3, abc^2, b^3c, bc^3\} \\ \{a^3b, a^2bc, ab^3, abc^2, b^3c, bc^3\} & \{a^2b^2, ab^2c, b^2c^2, b^4, c^4\} \end{pmatrix}.$$

This gives the first few terms for G_u as

$$\begin{aligned}
G_u &= g_u\{1\} \cup g_v\{0\} \cup g_u\{a\} \cup g_v\{b\} \cup g_u\left\{a^2, b^2\right\} \cup g_v\{ab, bc\} \\
&\quad \cup g_u\left\{a^3, ab^2, b^2c\right\} \cup g_v\left\{a^2b, abc, bc^2, b^3\right\} \\
&\quad \cup g_u\left\{a^4, a^2b^2, ab^2c, b^2c^2, b^4\right\} \cup g_v\left\{a^3b, a^2bc, ab^3, abc^2, b^3c, bc^3\right\} \cup \cdots \\
&= g_u\left\{1, a, a^2, b^2, a^3, ab^2, b^2c, a^4, a^2b^2, ab^2c, b^2c^2, b^4, \ldots\right\} \cup \\
&\quad g_v\left\{b, ab, bc, a^2b, abc, bc^2, b^3, a^3b, a^2bc, ab^3, abc^2, b^3c, bc^3 \ldots\right\} \\
&= g_u\left\{1, a, a^2, a^3, a^4, \ldots\right\} \cup g_u b^2\left\{1, a, c, a^2, ac, c^2, b^2, \ldots\right\} \cup \\
&\quad g_v b\left\{1, a, c, a^2, ac, c^2, b^2, a^3, a^2c, ab^2, ac^2, b^2c, c^3, \ldots\right\},
\end{aligned}$$

and this shows how the cosets in $G_u = g_u\langle 1, a\rangle \cup g_u b^2\langle 1, a, b^2, c\rangle \cup g_v b\langle 1, a, b^2, c\rangle$, evolve with increasing powers of $\mathbf{M}$, that is with increasing path length.

2.5 A more complicated example

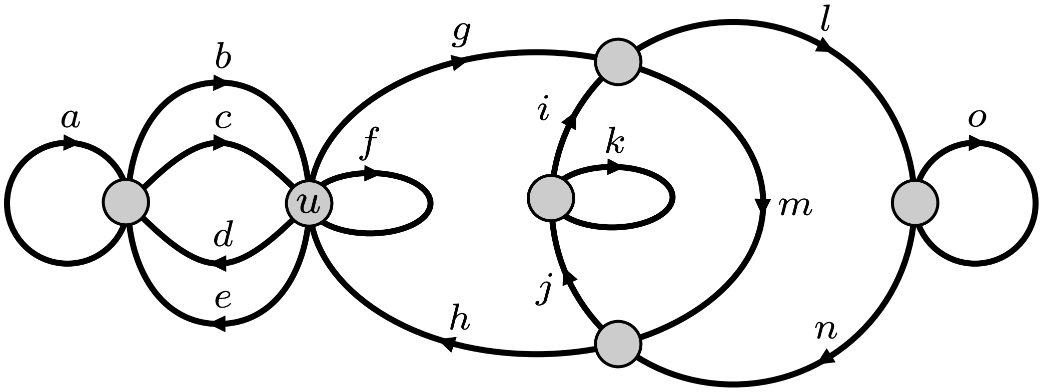


Figure. 2.5.1: A directed graph.

We now consider the more complicated directed graph shown in Figure 2.5.1. We will explicitly calculate an expression for $\psi\left(\bigcup \mathcal{A}\right)$, at the vertex u shown in Figure 2.5.1, by applying the algorithm in part (a) of the proof of Proposition 2.3.4.

There are 12 simple cycles in this graph, they are

$$\mathbf{c}_1 = dc,\ \mathbf{c}_2 = db, \mathbf{c}_3 = ec,\ \mathbf{c}_4 = eb,\ \mathbf{c}_5 = f,\ \mathbf{c}_6 = gmh,\ \mathbf{c}_7 = glnh,\ \mathbf{c}_8 = a,$$
$$\mathbf{c}_9 = imj,\ \mathbf{c}_{10} = ilnj,\ \mathbf{c}_{11} = k,\ \mathbf{c}_{12} = o.$$

Let

$$T = \{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_{12}\}.$$

- The set of chains (simple cycles) attached to the vertex u of length 1 is

$$\{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3, \mathbf{c}_4, \mathbf{c}_5, \mathbf{c}_6, \mathbf{c}_7\}\,.$$

For convenience we will use the notation

$$\mathbf{c} = \mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3, \mathbf{c}_4, \mathbf{c}_5, \mathbf{c}_6, \mathbf{c}_7$$

The chains of length 1 generate the semigroup of strings

$$A_1 = \langle \varepsilon, \mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3, \mathbf{c}_4, \mathbf{c}_5, \mathbf{c}_6, \mathbf{c}_7\rangle = \langle \varepsilon, \mathbf{c}\rangle\,.$$

- The set of chains of length 2, attached to the vertex u, is

$$\{\mathbf{c}_1\mathbf{c}_8, \mathbf{c}_2\mathbf{c}_8, \mathbf{c}_3\mathbf{c}_8, \mathbf{c}_4\mathbf{c}_8, \mathbf{c}_6\mathbf{c}_9, \mathbf{c}_6\mathbf{c}_{10}, \mathbf{c}_7\mathbf{c}_9, \mathbf{c}_7\mathbf{c}_{10}, \mathbf{c}_7\mathbf{c}_{12}\}\,.$$

This set of chains of length 2 generates the cosets of semigroups of strings

$$\begin{aligned}
A_2 &= \mathbf{c}_1 \langle \varepsilon, \mathbf{c}, \mathbf{c}_8\rangle\,,\\
A_3 &= \mathbf{c}_2 \langle \varepsilon, \mathbf{c}, \mathbf{c}_8\rangle\,,\\
A_4 &= \mathbf{c}_3 \langle \varepsilon, \mathbf{c}, \mathbf{c}_8\rangle\,,\\
A_5 &= \mathbf{c}_4 \langle \varepsilon, \mathbf{c}, \mathbf{c}_8\rangle\,,\\
A_6 &= \mathbf{c}_6 \langle \varepsilon, \mathbf{c}, \mathbf{c}_9, \mathbf{c}_{10}\rangle\,,\\
A_7 &= \mathbf{c}_7 \langle \varepsilon, \mathbf{c}, \mathbf{c}_9, \mathbf{c}_{10}, \mathbf{c}_{12}\rangle\,.
\end{aligned}$$

- The set of chains of length 3, attached to the vertex u, is

$$\{\mathbf{c}_6\mathbf{c}_{10}\mathbf{c}_{11}, \mathbf{c}_6\mathbf{c}_{10}\mathbf{c}_{12}, \mathbf{c}_6\mathbf{c}_9\mathbf{c}_{11}, \mathbf{c}_7\mathbf{c}_9\mathbf{c}_{11}, \mathbf{c}_7\mathbf{c}_{10}\mathbf{c}_{11}\}\,.$$

This set of chains of length 3 generates the cosets of semigroups of strings

$$\begin{aligned}
A_8 &= \mathbf{c}_6\mathbf{c}_{10} \langle \varepsilon, \mathbf{c}, \mathbf{c}_9, \mathbf{c}_{10}, \mathbf{c}_{11}, \mathbf{c}_{12}\rangle\,,\\
A_9 &= \mathbf{c}_6\mathbf{c}_9 \langle \varepsilon, \mathbf{c}, \mathbf{c}_9, \mathbf{c}_{10}, \mathbf{c}_{11}\rangle\,,\\
A_{10} &= \mathbf{c}_7\mathbf{c}_9 \langle \varepsilon, \mathbf{c}, \mathbf{c}_9, \mathbf{c}_{10}, \mathbf{c}_{11}, \mathbf{c}_{12}\rangle\,,\\
A_{11} &= \mathbf{c}_7\mathbf{c}_{10} \langle \varepsilon, \mathbf{c}, \mathbf{c}_9, \mathbf{c}_{10}, \mathbf{c}_{11}, \mathbf{c}_{12}\rangle\,.
\end{aligned}$$

- There are no chains of length 4 or more, attached to the vertex u, so we stop.

The finitely generated semigroup of cosets of string semigroups, $(\mathcal{A}, \circledcirc)$ is given by

$$\begin{aligned}
\mathcal{A} &= \langle A_1, A_2, \ldots, A_{11}\rangle\\
&= \left\{A_{j_1} \circledcirc A_{j_2} \circledcirc \cdots \circledcirc A_{j_k} : j_1 j_2 \cdots j_k \in \{1, 2, \ldots, 11\}^{\Delta}\right\}.
\end{aligned}$$

The similarity ratio of a simple cycle $\mathbf{c}_i$ is $r(\mathbf{c}_i) = r_{\mathbf{c}_i} = C_i$, and for convenience we will also use the notation

$$C = r(\mathbf{c}_1), r(\mathbf{c}_2), r(\mathbf{c}_3), r(\mathbf{c}_4), r(\mathbf{c}_5), r(\mathbf{c}_6), r(\mathbf{c}_7)$$

$$= C_1, C_2, C_3, C_4, C_5, C_6, C_7$$

so $\langle 1, C\rangle = \langle 1, C_1, C_2, C_3, C_4, C_5, C_6, C_7\rangle$.

We calculate $\psi(\bigcup \mathcal{A}) \subset \mathbb{R}^+$ as
$\psi\left(\bigcup \mathcal{A}\right) =$

$$\left.\begin{array}{ll} & \langle 1, C\rangle \\ \cup & C_1 \langle 1, C, C_8\rangle \\ \cup & C_2 \langle 1, C, C_8\rangle \\ \cup & C_3 \langle 1, C, C_8\rangle \\ \cup & C_4 \langle 1, C, C_8\rangle \\ \cup & C_6 \langle 1, C, C_9, C_{10}\rangle \\ \cup & C_7 \langle 1, C, C_9, C_{10}, C_{12}\rangle \\ \cup & C_6C_{10} \langle 1, C, C_9, C_{10}, C_{11}, C_{12}\rangle \\ \cup & C_6C_9 \langle 1, C, C_9, C_{10}, C_{11}\rangle \\ \cup & C_7C_9 \langle 1, C, C_9, C_{10}, C_{11}, C_{12}\rangle \\ \cup & C_7C_{10} \langle 1, C, C_9, C_{10}, C_{11}, C_{12}\rangle \end{array}\right\} \text{Single cosets, } \psi\left(A_i\right).$$

$$\left.\begin{array}{ll} \cup & C_1C_6 \langle 1, C, C_8, C_9, C_{10}\rangle \\ \cup & C_2C_6 \langle 1, C, C_8, C_9, C_{10}\rangle \\ \cup & C_3C_6 \langle 1, C, C_8, C_9, C_{10}\rangle \\ \cup & C_4C_6 \langle 1, C, C_8, C_9, C_{10}\rangle \\ \cup & C_1C_7 \langle 1, C, C_8, C_9, C_{10}, C_{12}\rangle \\ \cup & C_2C_7 \langle 1, C, C_8, C_9, C_{10}, C_{12}\rangle \\ \cup & C_3C_7 \langle 1, C, C_8, C_9, C_{10}, C_{12}\rangle \\ \cup & C_4C_7 \langle 1, C, C_8, C_9, C_{10}, C_{12}\rangle \\ \cup & C_6C_7 \langle 1, C, C_9, C_{10}, C_{12}\rangle \\ \cup & C_1(C_6C_{10}) \langle 1, C, C_8, C_9, C_{10}, C_{11}, C_{12}\rangle \\ \cup & C_2(C_6C_{10}) \langle 1, C, C_8, C_9, C_{10}, C_{11}, C_{12}\rangle \\ \cup & C_3(C_6C_{10}) \langle 1, C, C_8, C_9, C_{10}, C_{11}, C_{12}\rangle \\ \cup & C_4(C_6C_{10}) \langle 1, C, C_8, C_9, C_{10}, C_{11}, C_{12}\rangle \\ \cup & C_1(C_6C_9) \langle 1, C, C_8, C_9, C_{10}, C_{11}\rangle \\ \cup & C_2(C_6C_9) \langle 1, C, C_8, C_9, C_{10}, C_{11}\rangle \\ \cup & C_3(C_6C_9) \langle 1, C, C_8, C_9, C_{10}, C_{11}\rangle \\ \cup & C_4(C_6C_9) \langle 1, C, C_8, C_9, C_{10}, C_{11}\rangle \\ \cup & C_7(C_6C_9) \langle 1, C, C_9, C_{10}, C_{11}, C_{12}\rangle \\ \cup & C_1(C_7C_9) \langle 1, C, C_8, C_9, C_{10}, C_{11}, C_{12}\rangle \\ \cup & C_2(C_7C_9) \langle 1, C, C_8, C_9, C_{10}, C_{11}, C_{12}\rangle \\ \cup & C_3(C_7C_9) \langle 1, C, C_8, C_9, C_{10}, C_{11}, C_{12}\rangle \\ \cup & C_4(C_7C_9) \langle 1, C, C_8, C_9, C_{10}, C_{11}, C_{12}\rangle \\ \cup & C_1(C_7C_{10}) \langle 1, C, C_8, C_9, C_{10}, C_{11}, C_{12}\rangle \\ \cup & C_2(C_7C_{10}) \langle 1, C, C_8, C_9, C_{10}, C_{11}, C_{12}\rangle \\ \cup & C_3(C_7C_{10}) \langle 1, C, C_8, C_9, C_{10}, C_{11}, C_{12}\rangle \\ \cup & C_4(C_7C_{10}) \langle 1, C, C_8, C_9, C_{10}, C_{11}, C_{12}\rangle \end{array}\right\} \begin{array}{l}\text{Combinations of two} \\ \text{cosets, } \psi\left(A_{i_1} \circledcirc A_{i_2}\right).\end{array}$$

$$\left.\begin{array}{ll}\cup & C_1C_6C_7\,\langle 1, C, C_8, C_9, C_{10}, C_{12}\rangle \\ \cup & C_2C_6C_7\,\langle 1, C, C_8, C_9, C_{10}, C_{12}\rangle \\ \cup & C_3C_6C_7\,\langle 1, C, C_8, C_9, C_{10}, C_{12}\rangle \\ \cup & C_4C_6C_7\,\langle 1, C, C_8, C_9, C_{10}, C_{12}\rangle \\ \cup & C_7C_1(C_6C_9)\,\langle 1, C, C_8, C_9, C_{10}, C_{11}, C_{12}\rangle \\ \cup & C_7C_2(C_6C_9)\,\langle 1, C, C_8, C_9, C_{10}, C_{11}, C_{12}\rangle \\ \cup & C_7C_3(C_6C_9)\,\langle 1, C, C_8, C_9, C_{10}, C_{11}, C_{12}\rangle \\ \cup & C_7C_4(C_6C_9)\,\langle 1, C, C_8, C_9, C_{10}, C_{11}, C_{12}\rangle \end{array}\right\} \begin{array}{l}\text{Combinations of three} \\ \text{cosets, } \psi\,(A_{i_1} \circledcirc A_{i_2} \circledcirc A_{i_3}).\end{array}$$

$$\left.\begin{array}{ll}\cup & (C_6C_{10})C_1C_7\,\langle 1, C, C_8, C_9, C_{10}, C_{11}, C_{12}\rangle \\ \cup & (C_6C_{10})C_2C_7\,\langle 1, C, C_8, C_9, C_{10}, C_{11}, C_{12}\rangle \\ \cup & (C_6C_{10})C_3C_7\,\langle 1, C, C_8, C_9, C_{10}, C_{11}, C_{12}\rangle \\ \cup & (C_6C_{10})C_4C_7\,\langle 1, C, C_8, C_9, C_{10}, C_{11}, C_{12}\rangle \end{array}\right\} \begin{array}{l}\text{Combinations of four cosets,} \\ \psi\,(A_{i_1} \circledcirc A_{i_2} \circledcirc A_{i_3} \circledcirc A_{i_4}).\end{array}$$

The semigroup in the last group of combinations of four cosets, contains the similarity ratios of all the simple cycles in the graph as its generators, and this means that writing out further combinations in the above list is not going to give us anything new.

We now illustrate why it is the case that for a particular cycle in the graph $\mathbf{e} \in E^*_{uu}$ the similarity ratio $r_{\mathbf{e}} \in \psi(\bigcup \mathcal{A})$, this is just to give a concrete example of the central idea behind the method in part (c) of the proof of Proposition 2.3.4. As an example consider the cycle $dabglnjilnjkimh \in E^*_{uu}$, which is coloured grey

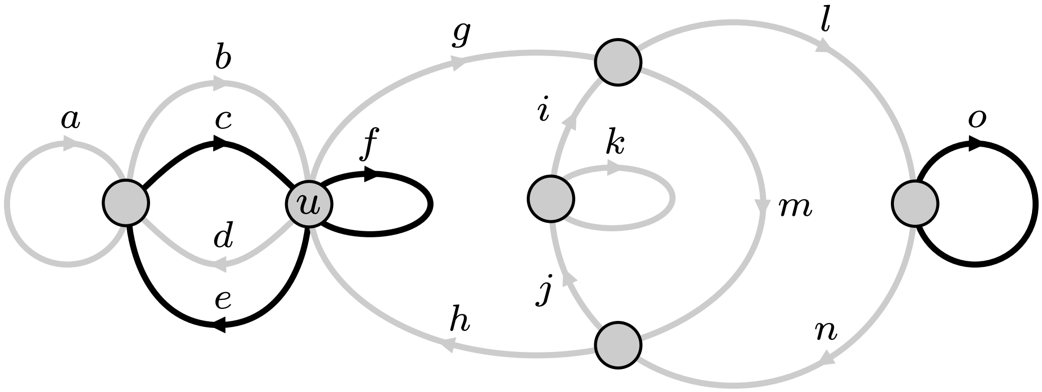


Figure. 2.5.2: The cycle $dabglnjilnjkimh$ starting and ending at the vertex u is shown in grey in this graph.

in Figure 2.5.2. As described in the proof of Lemma 2.3.2, we travel along the cycle until the first time a vertex is repeated. For our example the first vertex that is repeated is indicated by a pair of vertical bars, as follows,

$$d \mid a \mid bglnjilnjkimh.$$

The edge(s) between the repeated vertex are removed and labelled as one of the simple cycles of the graph listed above,

$$dbglnjilnjkimh, \quad \mathbf{c}_8 = a.$$

The process is repeated until no edges are left,

$$| \, db \, | \, glnjilnjkimh, \quad \mathbf{c}_8,$$

$$glnjilnjkimh, \quad \mathbf{c}_2 = db, \, \mathbf{c}_8,$$

$$g \, | \, lnji \, | \, lnjkimh, \quad \mathbf{c}_2, \, \mathbf{c}_8,$$

$$glnjkimh, \quad \mathbf{c}_{10} = lnji, \, \mathbf{c}_2, \, \mathbf{c}_8,$$

$$glnj \, | \, k \, | \, imh, \quad \mathbf{c}_{10}, \, \mathbf{c}_2, \, \mathbf{c}_8,$$

$$glnjimh, \quad \mathbf{c}_{11} = k, \, \mathbf{c}_{10}, \, \mathbf{c}_2, \, \mathbf{c}_8,$$

$$g \, | \, lnji \, | \, mh, \quad \mathbf{c}_{11}, \, \mathbf{c}_{10}, \, \mathbf{c}_2, \, \mathbf{c}_8,$$

$$gmh, \quad \mathbf{c}_{10} = lnji, \, \mathbf{c}_{11}, \, \mathbf{c}_{10}, \, \mathbf{c}_2, \, \mathbf{c}_8,$$

$$| \, gmh \, |, \quad \mathbf{c}_{10}, \, \mathbf{c}_{11}, \, \mathbf{c}_{10}, \, \mathbf{c}_2, \, \mathbf{c}_8,$$

$$\mathbf{c}_6 = gmh, \, \mathbf{c}_{10}, \, \mathbf{c}_{11}, \, \mathbf{c}_{10}, \, \mathbf{c}_2, \, \mathbf{c}_8.$$

The edges of the cycle $dabglnjilnjkimh$ have now been re-ordered into the simple cycles $\mathbf{c}_6\mathbf{c}_{10}\mathbf{c}_{11}\mathbf{c}_{10}\mathbf{c}_2\mathbf{c}_8$, where the first simple cycle $\mathbf{c}_6$ is attached to u, and where every other cycle in this list is either attached to u or is attached to a cycle appearing on its left. Applying the function φ, defined by Equation (2.1.5), we obtain

$$\varphi(\mathbf{c}_6\mathbf{c}_{10}\mathbf{c}_{11}\mathbf{c}_{10}\mathbf{c}_2\mathbf{c}_8) = \mathbf{c}_6\mathbf{c}_{10}^2\mathbf{c}_{11}\mathbf{c}_2\mathbf{c}_8.$$

Now $\mathbf{c}_2\mathbf{c}_8$, is a chain attached to the vertex u so the coset corresponding to $\mathbf{c}_8$ is A_3, and the coset corresponding to $\mathbf{c}_2$ is A_1. $\mathbf{c}_6\mathbf{c}_{10}\mathbf{c}_{11}$ is a chain attached to u, and this makes A_8 the corresponding coset for $\mathbf{c}_{11}$. The chain $\mathbf{c}_6\mathbf{c}_{10}$ gives A_6 as the coset for $\mathbf{c}_{10}$ and finally A_1 is the coset for $\mathbf{c}_6$, as $\mathbf{c}_6$ is attached to u. The product of cosets corresponding to the string of simple cycles $\mathbf{c}_6\mathbf{c}_{10}^2\mathbf{c}_{11}\mathbf{c}_2\mathbf{c}_8$ is therefore

$$\begin{aligned} A_1 \circledcirc A_6 \circledcirc A_8 \circledcirc A_1 \circledcirc A_3 &= A_8 \circledcirc A_3 \\ &= \mathbf{c}_6\mathbf{c}_{10}\mathbf{c}_2 \left\langle \varepsilon, \mathbf{c}, \mathbf{c}_8, \mathbf{c}_9, \mathbf{c}_{10}, \mathbf{c}_{11}, \mathbf{c}_{12} \right\rangle. \end{aligned}$$

In terms of similarity ratios it follows that

$$\begin{aligned} r_{dabglnjilnjkimh} &= r(\mathbf{c}_6)r(\mathbf{c}_{10})r(\mathbf{c}_{11})r(\mathbf{c}_{10})r(\mathbf{c}_2)r(\mathbf{c}_8) \\ &= C_6C_{10}C_{11}C_{10}C_2C_8 \\ &= C_6C_{10}^2C_{11}C_2C_8 \\ &\in C_6C_{10}C_2 \left\langle 1, C, C_8, C_9, C_{10}, C_{11}, C_{12} \right\rangle \\ &= \psi\left(A_8 \circledcirc A_3\right) \\ &\subset \psi\left(\bigcup \mathcal{A}\right), \end{aligned}$$

and so $r_{dabglnjilnjkimh} \in \psi\left(\bigcup \mathcal{A}\right)$, as we expected.

2.6 2-vertex systems are not the same as 1-vertex systems

We remind the reader that in this chapter we are only considering IFSs which satisfy the CSSC. Corollary 2.3.6 shows that a standard (1-vertex) IFS produces an attractor whose set of gap lengths can always be expressed as a finite union of cosets of a finitely generated semigroup. We now show that the example of a 2-vertex IFS illustrated in Figure 2.4.1 above, produces an attractor F_u at the vertex u, which cannot be the attractor of any standard (1-vertex) IFS. To do this it is clearly enough to show that the set of gap lengths G_u in F_u, given by Equation (2.4.2), cannot be represented by a finite union of cosets of a single finitely generated semigroup. That is we need to show

$$G_u = g_u \langle 1, a \rangle \cup g_u b^2 \left\langle 1, a, b^2, c \right\rangle \cup g_v b \left\langle 1, a, b^2, c \right\rangle \neq \bigcup_{j=1}^{m} h_j \langle 1, x_1, x_2, \ldots, x_n \rangle ,$$

for any positive real numbers h_j, $1 \leqslant j \leqslant m$, and x_k, $1 \leqslant k \leqslant n$, where we need to make the assumption that the set $\{g_u, g_v, a, b, c\}$ is multiplicatively rationally independent (see Subsection 2.1.1). For $\langle 1, a, b^2, c \rangle$ and $\langle g_u, g_v, a, b, c \rangle_{\text{group}}$ referred to in the next lemma and its proof, if $\{g_u, g_v, a, b, c\}$ is a multiplicatively rationally independent set then $\langle 1, a, b^2, c \rangle$ is a finitely generated free (commutative) semigroup with identity and $\langle g_u, g_v, a, b, c \rangle_{\text{group}}$ is a finitely generated free (commutative) group.

Lemma 2.6.1. *Let $\{g_u, g_v, a, b, c\} \subset \mathbb{R}^+$, be a multiplicatively rationally independent set, then*

$$g_u \langle 1, a \rangle \cup g_u b^2 \left\langle 1, a, b^2, c \right\rangle \cup g_v b \left\langle 1, a, b^2, c \right\rangle \neq \bigcup_{j=1}^{m} h_j \langle 1, x_1, x_2, \ldots, x_n \rangle ,$$

for any $h_j \in \mathbb{R}^+$, $1 \leqslant j \leqslant m$, and any $x_k \in \mathbb{R}^+$, $1 \leqslant k \leqslant n$.

Proof. For a contradiction we assume there exist positive real numbers h_j, $1 \leqslant j \leqslant m$, and x_k, $1 \leqslant k \leqslant n$, such that

$$g_u \langle 1, a \rangle \cup g_u b^2 \left\langle 1, a, b^2, c \right\rangle \cup g_v b \left\langle 1, a, b^2, c \right\rangle = \bigcup_{j=1}^{m} h_j \langle 1, x_1, x_2, \ldots, x_n \rangle . \tag{2.6.1}$$

This can be written as

$$g_u A \cup g_v B = \bigcup_{j=1}^{m} h_j \langle 1, x_1, x_2, \ldots, x_n \rangle ,$$

where

$$A = \langle 1, a \rangle \cup b^2 \left\langle 1, a, b^2, c \right\rangle = \left\{ a^p b^{2q} c^r : p, q, r \in \mathbb{N} \cup \{0\} , \text{ if } q = 0 \text{ then } r = 0 \right\} ,$$

and

$$B = b \left\langle 1, a, b^2, c \right\rangle = \left\{ a^p b^{2q+1} c^r : p, q, r \in \mathbb{N} \cup \{0\} \right\} .$$

(a) $g_uA \cap g_vB = \emptyset$.
If $g_uA \cap g_vB \neq \emptyset$, then there exists $x \in g_uA \cap g_vB$ with

$$x = g_u^1 g_v^0 a^{p_1} b^{2q_1} c^{r_1} = g_u^0 g_v^1 a^{p_2} b^{2q_2+1} c^{r_2},$$

which, by the rational independence of the set $\{g_u, g_v, a, b, c\}$, implies $1 = 0$.

This means that for any h_j, $1 \leqslant j \leqslant m$, either $h_j \in g_uA$ or $h_j \in g_vB$ but not both, so we consider each case in turn in parts (b) and (c).

(b) *$h_j \in g_uA$ implies $h_j \langle 1, x_1, x_2, \ldots, x_n\rangle \subset g_uA$ and $\langle 1, x_1, x_2, \ldots, x_n\rangle \subset \langle 1, a, b^2, c\rangle$.*

Suppose $h_j \in g_uA$ then

$$h_j = g_u^1 a^{\alpha_1} b^{2\alpha_2} c^{\alpha_3}.$$

Let $x \in \langle 1, x_1, x_2, \ldots, x_n\rangle$. Assume $h_jx \in g_vB$ then

$$h_jx = g_u^1 a^{\alpha_1} b^{2\alpha_2} c^{\alpha_3} x = g_v^1 a^{\beta_1} b^{2\beta_2+1} c^{\beta_3},$$

and

$$x = g_u^{-1} g_v^1 a^{\beta_1-\alpha_1} b^{2(\beta_2-\alpha_2)+1} c^{\beta_3-\alpha_3},$$

where we are now considering $x \in \langle g_u, g_v, a, b, c\rangle_{\text{group}}$, the group operation being multiplication. Consider any $k \in \mathbb{N}$ with $k \geqslant 2$, then the exponent of g_v in h_jx^k is k. Now either $h_jx^k \in g_uA$ or $h_jx^k \in g_vB$. If $h_jx^k \in g_uA$ then by rational independence $k = 0$, which is a contradiction and if $h_jx^k \in g_vB$ then by rational independence $k = 1$ which is again a contradiction. Therefore $h_jx \notin g_vB$ and so $h_jx \in g_uA$.

We now write h_jx as

$$h_jx = g_u^1 a^{\alpha_1} b^{2\alpha_2} c^{\alpha_3} x = g_u^1 a^{\beta_1} b^{2\beta_2} c^{\beta_3},$$

with

$$x = a^{\beta_1-\alpha_1} b^{2(\beta_2-\alpha_2)} c^{\beta_3-\alpha_3},$$

where strictly speaking we are again considering $x \in \langle g_u, g_v, a, b, c\rangle_{\text{group}}$. For any $k \in \mathbb{N}$, the exponent of g_u in h_jx^k is 1 and so by rational independence, if $h_jx^k \in g_vB$, then $1 = 0$, which implies $h_jx^k \in g_uA$, and

$$h_jx^k = g_u^1 a^{\alpha_1+k(\beta_1-\alpha_1)} b^{2\alpha_2+2k(\beta_2-\alpha_2)} c^{\alpha_3+k(\beta_3-\alpha_3)} = g_u^1 a^{\delta_1(k)} b^{2\delta_2(k)} c^{\delta_3(k)},$$

where $\delta_1(k), \delta_2(k), \delta_3(k) \in \mathbb{N} \cup \{0\}$. Again by rational independence we may conclude that $\alpha_1+k(\beta_1-\alpha_1) \geqslant 0$, $2\alpha_2+2k(\beta_2-\alpha_2) \geqslant 0$, and $\alpha_3+k(\beta_3-\alpha_3) \geqslant 0$, for all $k \in \mathbb{N}$. Hence $\beta_1 - \alpha_1 \geqslant 0$, $\beta_2 - \alpha_2 \geqslant 0$, $\beta_3 - \alpha_3 \geqslant 0$ so that $x \in \langle 1, a, b^2, c\rangle$. In summary we have shown that $h_j \in g_uA$ implies $h_jx \in g_uA$ for all $x \in \langle 1, x_1, x_2, \ldots, x_n\rangle$, and $\langle 1, x_1, x_2, \ldots, x_n\rangle \subset \langle 1, a, b^2, c\rangle$.

(c) *$h_j \in g_vB$ implies $h_j \langle 1, x_1, x_2, \ldots, x_n\rangle \subset g_vB$ and $\langle 1, x_1, x_2, \ldots, x_n\rangle \subset \langle 1, a, b^2, c\rangle$.*

Suppose that $h_j \in g_vB$, then

$$h_j = g_v^1 a^{\alpha_1} b^{2\alpha_2+1} c^{\alpha_3}.$$

Let $x \in \langle 1, x_1, x_2, \ldots, x_n \rangle$. Assume $h_j x \in g_u A$ then

$$h_j x = g_v^1 a^{\alpha_1} b^{2\alpha_2+1} c^{\alpha_3} x = g_u^1 a^{\beta_1} b^{2\beta_2} c^{\beta_3},$$

and

$$x = g_u^1 g_v^{-1} a^{\beta_1-\alpha_1} b^{2(\beta_2-\alpha_2)-1} c^{\beta_3-\alpha_3},$$

where we are now considering $x \in \langle g_u, g_v, a, b, c\rangle_{\text{group}}$. Consider any $k \in \mathbb{N}$ with $k \geqslant 2$, then the exponent of g_u in $h_j x^k$ is k. Now either $h_j x^k \in g_u A$ or $h_j x^k \in g_v B$. If $h_j x^k \in g_u A$ then by rational independence $k = 1$, which is a contradiction and if $h_j x^k \in g_v B$ then by rational independence $k = 0$ which is again a contradiction. Therefore $h_j x \notin g_u A$ and so $h_j x \in g_v B$.

We now write $h_j x$ as

$$h_j x = g_v^1 a^{\alpha_1} b^{2\alpha_2+1} c^{\alpha_3} x = g_v^1 a^{\beta_1} b^{2\beta_2+1} c^{\beta_3},$$

with

$$x = a^{\beta_1-\alpha_1} b^{2(\beta_2-\alpha_2)} c^{\beta_3-\alpha_3},$$

where strictly speaking we are now considering $x \in \langle g_u, g_v, a, b, c\rangle_{\text{group}}$. For any $k \in \mathbb{N}$, the exponent of g_v in $h_j x^k$ is 1 and so by rational independence, if $h_j x^k \in g_u A$, then $1 = 0$, which implies $h_j x^k \in g_v B$, and

$$h_j x^k = g_v^1 a^{\alpha_1+k(\beta_1-\alpha_1)} b^{2\alpha_2+1+2k(\beta_2-\alpha_2)} c^{\alpha_3+k(\beta_3-\alpha_3)} = g_v^1 a^{\delta_1(k)} b^{2\delta_2(k)+1} c^{\delta_3(k)},$$

where $\delta_1(k), \delta_2(k), \delta_3(k) \in \mathbb{N} \cup \{0\}$. Again by rational independence we may conclude that $\alpha_1 + k(\beta_1 - \alpha_1) \geqslant 0$, $2\alpha_2 + 1 + 2k(\beta_2 - \alpha_2) \geqslant 0$, and $\alpha_3 + k(\beta_3 - \alpha_3) \geqslant 0$, for all $k \in \mathbb{N}$. Hence $\beta_1 - \alpha_1 \geqslant 0$, $\beta_2 - \alpha_2 \geqslant 0$, $\beta_3 - \alpha_3 \geqslant 0$ and $x \in \langle 1, a, b^2, c\rangle$. In summary we have shown that $h_j \in g_v B$ implies $h_j x \in g_v B$ for all $x \in \langle 1, x_1, x_2, \ldots, x_n\rangle$, and $\langle 1, x_1, x_2, \ldots, x_n\rangle \subset \langle 1, a, b^2, c\rangle$.

Relabelling the h_j if necessary, the results of parts (a), (b) and (c) imply that the set $\{h_1, h_2, \ldots, h_m\}$ must split into two non-empty subsets, $\{h_1, h_2, \ldots, h_r\} \subset g_u A$ and $\{h_{r+1}, h_{r+2}, \ldots, h_m\} \subset g_v B$, with

$$g_u A = \bigcup_{j=1}^{r} h_j \langle 1, x_1, x_2, \ldots, x_n\rangle, \tag{2.6.2}$$

and

$$g_v B = \bigcup_{j=r+1}^{m} h_j \langle 1, x_1, x_2, \ldots, x_n\rangle,$$

where $\langle 1, x_1, x_2, \ldots, x_n\rangle \subset \langle 1, a, b^2, c\rangle$.

(d) *At least one of the generators of the semigroup $\langle 1, x_1, x_2, \ldots, x_n\rangle$, is of the form $x_k = c^t$, for some x_k, $1 \leqslant k \leqslant n$, and some $t \in \mathbb{N}$.*

We recall that

$$g_u A = g_u \left\{ a^p b^{2q} c^r : p, q, r, \in \mathbb{N} \cup \{0\},\ \text{if } q = 0 \text{ then } r = 0 \right\},$$

so that $g_u b^2 c^m \in g_u A$ for all $m \in \mathbb{N}$. Considering $M \in \mathbb{N}$ as fixed, then from Equation (2.6.2)

$$g_u b^2 c^M = h_s x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}, \tag{2.6.3}$$

for some $h_s \in g_u A$, $1 \leqslant s \leqslant r$, and non-negative integers $i_k \in \mathbb{N} \cup \{0\}$, $1 \leqslant k \leqslant n$. For a contradiction we now assume that none of the x_k, $1 \leqslant k \leqslant n$, is of the form $x_k = c^t$, $t \in \mathbb{N}$. Rational independence and the fact that $\langle 1, x_1, x_2, \ldots, x_n \rangle \subset \langle 1, a, b^2, c \rangle$, then implies that $h_s = g_u c^p$ and $x_k = b^2 c^q$, for some k, $1 \leqslant k \leqslant n$, with $i_k = 1$ and $i_l = 0$ for all $l \neq k$, and where $p, q \in \mathbb{N} \cup \{0\}$, with $p + q = M$. That is Equation (2.6.3) reduces to

$$g_u b^2 c^M = h_s x_k.$$

Since we only have a finite number of generators in the semigroup $\langle 1, x_1, x_2, \ldots, x_n \rangle$ and a finite set of numbers $\{h_i : 1 \leqslant i \leqslant r\}$, we can only produce at most $r \times n$ distinct numbers of the form $g_u b^2 c^m$, on the right-hand side of Equation (2.6.2). Therefore

$$\left\{ g_u b^2 c^m : m \in \mathbb{N} \right\} \not\subset \bigcup_{j=1}^{r} h_j \langle 1, x_1, x_2, \ldots, x_n \rangle,$$

but

$$\left\{ g_u b^2 c^m : m \in \mathbb{N} \right\} \subset g_u A.$$

This contradiction of Equation (2.6.2) means our assumption is false and at least one of the generators x_k, $1 \leqslant k \leqslant n$, must be of the form $x_k = c^t$, for some $t \in \mathbb{N}$. (As a constructive example $h_1 = g_u b^2$, $h_2 = g_u b^2 c$ and $x_1 = c^2$ are enough to generate all the numbers in the set $\{g_u b^2 c^m : m \in \mathbb{N}\}$).

(e) $g_u a \notin \bigcup_{j=1}^{r} h_j \langle 1, x_1, x_2, \ldots, x_n \rangle$.

From the result of part (d), relabelling the x_k if necessary, so that $x_1 = c^t$, $t \in \mathbb{N}$, we may write Equation (2.6.2) as

$$\begin{aligned} g_u A &= g_u \langle 1, a \rangle \cup g_u b^2 \left\langle 1, a, b^2, c \right\rangle \\ &= \bigcup_{j=1}^{r} h_j \left\langle 1, c^t, x_2, \ldots, x_n \right\rangle, \end{aligned}$$

where $\langle 1, c^t, x_2, \ldots, x_n \rangle \subset \langle 1, a, b^2, c \rangle$. Now $g_u \langle 1, a \rangle \cap g_u b^2 \langle 1, a, b^2, c \rangle = \emptyset$, by the rational independence of the set $\{g_u, g_v, a, b, c\}$, so for each j, $1 \leqslant j \leqslant r$, either $h_j \in g_u \langle 1, a \rangle$ or $h_j \in g_u b^2 \langle 1, a, b^2, c \rangle$ but not both.

Suppose $h_j \in g_u \langle 1, a \rangle$, then $h_j = g_u a^k$ for some $k \in \mathbb{N} \cup \{0\}$. It follows, again by rational independence, that $h_j c^t = g_u a^k c^t \notin g_u \langle 1, a \rangle$ and $h_j c^t = g_u a^k c^t \notin g_u b^2 \langle 1, a, b^2, c \rangle$, that is $h_j c^t \notin g_u A$. This contradiction means $h_j \in g_u b^2 \langle 1, a, b^2, c \rangle$ for each j, $1 \leqslant j \leqslant r$, and so we may write h_j as

$$h_j = g_u b^2 a^{k_j} b^{2 l_j} c^{m_j},$$

for some $k_j, l_j, m_j \in \mathbb{N} \cup \{0\}$. The rational independence of the set $\{g_u, g_v, a, b, c\}$, together with the fact that $\langle 1, c^t, x_2, \ldots, x_n \rangle \subset \langle 1, a, b^2, c \rangle$, implies that

$$g_u a \notin \bigcup_{j=1}^{r} g_u b^2 a^{k_j} b^{2 l_j} c^{m_j} \left\langle 1, c^t, x_2, \ldots, x_n \right\rangle = \bigcup_{i=1}^{r} h_i \langle 1, x_1, x_2, \ldots, x_n \rangle.$$

As $g_u a \in g_u A$, this is again a contradiction of Equation (2.6.2). Therefore our original assumption is false, that is Equation (2.6.1) does not hold. □

Lemma 2.6.1 proves that the set of gap lengths G_u, as given by Equation (2.4.2), cannot be represented by a finite union of cosets of a single finitely generated semigroup. It follows, by Corollary 2.3.6, that the attractor F_u, of the 2-vertex directed graph IFS of Section 2.4, cannot be the attractor of any standard (1-vertex) IFS satisfying the CSSC. We state this formally as Corollary 2.6.2. We remind the reader that as stated in Subsection 1.2.8 the notation $\big(V, E^*, i, t, r, ((\mathbb{R}, |\ \ |))_{v\in V}, (S_e)_{e\in E^1}\big)$ represents a directed graph IFS, for which $(S_e)_{e\in E^1}$ are contracting similarities defined on $\mathbb{R}$, and not just contractions.

Corollary 2.6.2. *For the 2-vertex IFS of Figure 2.4.1, if the set $\{g_u, g_v, a, b, c\} \subset \mathbb{R}^+$ is a multiplicatively rationally independent set, then the attractor at the vertex u, F_u, is not the attractor of any standard (1-vertex) IFS, defined on $\mathbb{R}$, for which the CSSC holds.*

We now show in the next theorem, that any directed graph containing a particular subgraph, of the type shown in Figure 2.6.1, will produce an attractor which cannot be the attractor of any standard (1-vertex) IFS for which the CSSC holds. This generalises Corollary 2.6.2 and its proof is very similar to the proof of Lemma 2.6.1.

Theorem 2.6.3. *Let $\big(V, E^*, i, t, r, ((\mathbb{R}, |\ \ |))_{v\in V}, (S_e)_{e\in E^1}\big)$ be any directed graph IFS, satisfying the CSSC, whose directed graph contains three distinct simple cycles $\mathbf{c}_1$, $\mathbf{c}_2$, and $\mathbf{c}_3$, such that $\mathbf{c}_1$ is attached to a vertex u, $\mathbf{c}_2\mathbf{c}_3$ is a chain of length 2 attached to u and no chain in the graph, attached to u, contains both $\mathbf{c}_1$ and $\mathbf{c}_3$. Let $X_u \subset \mathbb{R}^+$, be the set of gap lengths and contracting similarity ratios*

$$X_u = \left\{g_w, C_i, r_{\mathbf{p}} : g_w \in G_w^1,\ w \in V,\ \mathbf{c}_i \in T,\ \mathbf{p} \in D_{uv}^*,\ v \in V,\ v \neq u\right\},$$

where G_w^1 is the set of level-1 gap lengths at the vertex $w \in V$, $T = \{\mathbf{c}_i : i \in I\}$, the set of all simple cycles in the graph, and $D_{uv}^* \subset E_{uv}^*$, is the set of all simple paths from the vertex u to the vertex v.

Suppose the set X_u is multiplicatively rationally independent, then the attractor at the vertex u, F_u, is not the attractor of any standard (1-vertex) IFS, defined on $\mathbb{R}$, for which the CSSC holds.

Proof. For convenience we restate Equation (2.3.7) which gives the gap lengths of the attractor F_u as

$$G_u = \bigcup_{g_u\in G_u^1} g_u\psi\left(\bigcup\mathcal{A}\right) \cup \bigcup_{\substack{v\in V\\ v\neq u\\ g_v\in G_v^1}} g_v\left(\bigcup_{\mathbf{p}\in D_{uv}^*} r_{\mathbf{p}}\psi\left(\bigcup\mathcal{B}^{\mathbf{p}}\right)\right). \tag{2.6.4}$$

From Equations (2.3.10) and (2.3.11) in Corollary 2.3.5,

$$\psi(\bigcup\mathcal{A}) \subset \langle 1, C_i : i \in I\rangle\,, \tag{2.6.5}$$

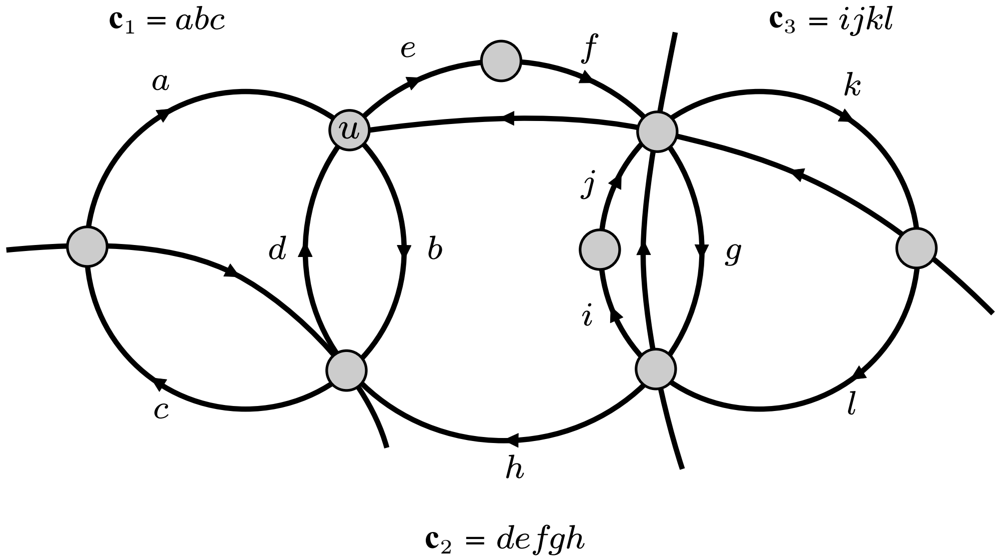


Figure. 2.6.1: An example of the type of subgraph of a directed graph, containing the three simple cycles $\mathbf{c}_1$, $\mathbf{c}_2$, and $\mathbf{c}_3$, which are necessary for Theorem 2.6.3. It is also assumed that there is no chain of simple cycles, attached to the vertex u, which contains both $\mathbf{c}_1$ and $\mathbf{c}_3$.

and

$$\psi\left(\bigcup \mathcal{B}^{\mathbf{p}}\right) \subset \langle 1, C_i : i \in I\rangle\,, \tag{2.6.6}$$

for simple paths $\mathbf{p} \in D^*_{uv}$, $v \in V$, $v \neq u$.

To prove the theorem we need to show that G_u cannot be represented as a finite union of cosets of a single finitely generated semigroup. For a contradiction we assume there exist positive real numbers h_j, $1 \leqslant j \leqslant m$, and x_k, $1 \leqslant k \leqslant n$, such that

$$G_u = \bigcup_{j=1}^{m} h_j \langle 1, x_1, x_2, \ldots, x_n\rangle\,. \tag{2.6.7}$$

(a) *The union on the right-hand side of Equation (2.6.4) is pairwise disjoint.*

This follows from the rational independence of the set X_u and Equations (2.6.5) and (2.6.6). For example if $g_u\psi(\bigcup \mathcal{A}) \cap g_v r_{\mathbf{p}}\psi\left(\bigcup \mathcal{B}^{\mathbf{p}}\right) \neq \emptyset$, then there exists $x \in g_u\psi(\bigcup \mathcal{A}) \cap g_v r_{\mathbf{p}}\psi\left(\bigcup \mathcal{B}^{\mathbf{p}}\right)$, with

$$x = g_u \prod_{i\in I} C_i^{m_i} = g_v r_{\mathbf{p}} \prod_{i\in I} C_i^{n_i},$$

which is impossible by the rational independence of the set X_u. In the same way, if $g_u, g_{u'} \in G_u^1$ with $g_u \neq g_{u'}$, and $g_u\psi(\bigcup \mathcal{A}) \cap g_{u'}\psi(\bigcup \mathcal{A}) \neq \emptyset$, then there exists $x \in g_u\psi(\bigcup \mathcal{A}) \cap g_{u'}\psi(\bigcup \mathcal{A})$, with

$$x = g_u \prod_{i\in I} C_i^{m_i} = g_{u'} \prod_{i\in I} C_i^{n_i},$$

which is again impossible by the rational independence of the set X_u. This argument can be applied for any pair of elements in the union on the right-hand side of Equation (2.6.4).

(b) $h_j \in g_u\psi(\bigcup \mathcal{A})$ *implies* $h_j\langle 1, x_1, x_2, \ldots, x_n\rangle \subset g_u\psi(\bigcup \mathcal{A})$ *and* $\langle 1, x_1, x_2, \ldots, x_n\rangle \subset \langle 1, C_i : i \in I\rangle$.

Suppose $h_j \in g_u\psi(\bigcup \mathcal{A})$ then, by Equation (2.6.5),

$$h_j = g_u^1 \prod_{i\in I} C_i^{m_i},$$

for some $m_i \in \mathbb{N} \cup \{0\}$. Let $x \in \langle 1, x_1, x_2, \ldots, x_n\rangle$. Suppose $h_j x \notin g_u\psi(\bigcup \mathcal{A})$ then from the disjoint union in Equation (2.6.4)

$$h_j x = g_u^1 \prod_{i\in I} C_i^{m_i} x = g_v^1 r_{\mathbf{p}}^{\alpha} \prod_{i\in I} C_i^{n_i},$$

for $\alpha \in \{0, 1\}$ and some $n_i \in \mathbb{N} \cup \{0\}$. Hence

$$x = g_u^{-1} g_v^1 r_{\mathbf{p}}^{\alpha} \prod_{i\in I} C_i^{n_i - m_i},$$

where we are now considering $x \in \langle X_u\rangle_{\text{group}}$. Consider any $k \in \mathbb{N}$ with $k \geqslant 2$, then the exponent of g_v in $h_j x^k$ is k. Now $h_j x^k$ lies in one of the elements of the disjoint union on the right-hand side of Equation (2.6.4) where the exponent of g_v is either 0 or 1. This is a contradiction of the rational independence of the set X_u. Therefore $h_j x \in g_u\psi(\bigcup \mathcal{A})$.

We now write $h_j x$ as

$$h_j x = \left(g_u^1 \prod_{i\in I} C_i^{m_i}\right) x = g_u^1 \prod_{i\in I} C_i^{n_i},$$

with

$$x = \prod_{i\in I} C_i^{n_i - m_i},$$

where strictly speaking we are again considering $x \in \langle X_u\rangle_{\text{group}}$. For any $k \in \mathbb{N}$, the exponent of g_u in $h_j x^k$ is 1 and so by rational independence, if $h_j x^k \notin g_u\psi(\bigcup \mathcal{A})$, then the exponent of g_u will be 0 which is impossible, this implies $h_j x^k \in g_u\psi(\bigcup \mathcal{A})$, and

$$h_j x^k = g_u^1 \prod_{i\in I} C_i^{m_i + k(n_i - m_i)} = g_u^1 \prod_{i\in I} C_i^{\delta_i(k)},$$

where $\delta_i(k) \in \mathbb{N} \cup \{0\}$. Again by rational independence we may conclude that $m_i + k(n_i - m_i) \geqslant 0$, for all $k \in \mathbb{N}$. Hence $n_i - m_i \geqslant 0$, so that $x \in \langle 1, C_i : i \in I\rangle$.

(c) $h_j \in g_v r_{\mathbf{p}}\psi\left(\bigcup \mathcal{B}^{\mathbf{p}}\right)$ *implies* $h_j \langle 1, x_1, x_2, \ldots, x_n\rangle \subset g_v r_{\mathbf{p}}\psi\left(\bigcup \mathcal{B}^{\mathbf{p}}\right)$ *and* $\langle 1, x_1, x_2, \ldots, x_n\rangle \subset \langle 1, C_i : i \in I\rangle$.

The proof uses an identical method to that given in part (b).

Consider one of the gap lengths $g_u \in G_u^1$ as fixed. Relabelling the h_j if necessary, parts (a), (b) and (c), imply the existence of a subset $\{h_1, h_2, \ldots, h_r\} \subset \{h_1, h_2, \ldots, h_m\}$ such that

$$g_u\psi\left(\bigcup \mathcal{A}\right) = \bigcup_{j=1}^{r} h_j \langle 1, x_1, x_2, \ldots, x_n\rangle, \tag{2.6.8}$$

where $\langle 1, x_1, x_2, \ldots, x_n \rangle \subset \langle 1, C_i : i \in I \rangle$.

(d) $g_u \langle 1, C_1 \rangle \cup g_u C_2 \langle 1, C_1, C_2, C_3 \rangle \subset g_u \psi (\bigcup \mathcal{A}) = \bigcup_{j=1}^r h_j \langle 1, x_1, x_2, \ldots, x_n \rangle$.
From the algorithm in part (a) of the proof of Proposition 2.3.4

$$\langle \varepsilon, \mathbf{c}_1, \mathbf{c}_2 \rangle \subset \langle \varepsilon, \mathbf{c}_1, \mathbf{c}_2, \ldots \rangle = A_1$$
$$\mathbf{c}_2 \langle \varepsilon, \mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3 \rangle \subset \mathbf{c}_2 \langle \varepsilon, \mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3, \ldots\ldots \rangle = A_k, \text{ for some } k.$$

The dots ... in the semigroups here indicate the possible presence of finitely many other simple cycles in the graph as generators. It follows that

$$g_u \psi(\langle \varepsilon, \mathbf{c}_1, \mathbf{c}_2 \rangle) \cup g_u \psi (\mathbf{c}_2 \langle \varepsilon, \mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3 \rangle) \subset g_u \psi \left(\bigcup \mathcal{A} \right).$$

Now

$$\begin{aligned} g_u \langle 1, C_1 \rangle \cup g_u C_2 \langle 1, C_1, C_2, C_3 \rangle &= g_u \langle 1, C_1, C_2 \rangle \cup g_u C_2 \langle 1, C_1, C_2, C_3 \rangle \\ &= g_u \psi(\langle \varepsilon, \mathbf{c}_1, \mathbf{c}_2 \rangle) \cup g_u \psi(\mathbf{c}_2 \langle \varepsilon, \mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3 \rangle) \\ &\subset g_u \psi \left(\bigcup \mathcal{A} \right) \\ &= \bigcup_{j=1}^r h_j \langle 1, x_1, x_2, \ldots, x_n \rangle, \end{aligned}$$

as required.

(e) *At least one of the generators of the semigroup $\langle 1, x_1, x_2, \ldots, x_n \rangle$, is of the form $x_k = C_3^t$, for some x_k, $1 \leqslant k \leqslant n$, and some $t \in \mathbb{N}$.*

From part (d), $\{g_u C_2 C_3^m : m \in \mathbb{N}\} \subset \bigcup_{j=1}^r h_j \langle 1, x_1, x_2, \ldots, x_n \rangle$. Considering $M \in \mathbb{N}$ as fixed,

$$g_u C_2 C_3^M = h_s x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n}, \tag{2.6.9}$$

for some h_s, $1 \leqslant s \leqslant r$, and non-negative integers $m_k \in \mathbb{N} \cup \{0\}$, $1 \leqslant k \leqslant n$. For a contradiction we now assume that none of the x_k, $1 \leqslant k \leqslant n$, is of the form $x_k = C_3^t$, $t \in \mathbb{N}$. From Equations (2.6.5) and (2.6.8), $h_s \in g_u \psi (\bigcup \mathcal{A}) \subset g_u \langle 1, C_i : i \in I \rangle$. Rational independence of the set X_u, and the fact that $\langle 1, x_1, x_2, \ldots, x_n \rangle \subset \langle 1, C_i : i \in I \rangle$, then implies that $h_s = g_u C_3^p$ and $x_k = C_2 C_3^q$ for some k, $1 \leqslant k \leqslant n$, with $m_k = 1$ and $m_l = 0$ for all $l \neq k$, where $p, q \in \mathbb{N} \cup \{0\}$, with $p + q = M$. That is Equation (2.6.9) reduces to

$$g_u C_2 C_3^M = h_s x_k.$$

Since we only have a finite number of generators in the semigroup $\langle 1, x_1, x_2, \ldots, x_n \rangle$ and a finite set of numbers $\{h_j : 1 \leqslant j \leqslant r\}$, we can only generate a finite, (at most $r \times n$), set of distinct numbers of the form $g_u b^2 c^m$, $m \in \mathbb{N}$, in the set

$$\bigcup_{j=1}^r h_j \langle 1, x_1, x_2, \ldots, x_n \rangle.$$

Therefore

$$\{g_u C_2 C_3^m : m \in \mathbb{N}\} \not\subset \bigcup_{j=1}^r h_j \langle 1, x_1, x_2, \ldots, x_n \rangle,$$

but from part (d)

$$\{g_u C_2 C_3^m : m \in \mathbb{N}\} \subset \bigcup_{j=1}^{r} h_j \left\langle 1, x_1, x_2, \ldots, x_n \right\rangle .$$

This contradiction means our assumption is false and at least one of the generators x_k, $1 \leqslant k \leqslant n$, must be of the form $x_k = C_3^t$, for some $t \in \mathbb{N}$.

(f) $g_u C_1 \notin g_u \psi \left(\bigcup \mathcal{A}\right) = \bigcup_{j=1}^{r} h_j \left\langle 1, x_1, x_2, \ldots, x_n \right\rangle$.

From the result of part (d), relabelling the x_k if necessary, so that $x_1 = C_3^t$, $t \in \mathbb{N}$, Equation (2.6.8) becomes

$$g_u \psi \left(\bigcup \mathcal{A}\right) = \bigcup_{j=1}^{r} h_j \left\langle 1, C_3^t, x_2, \ldots, x_n \right\rangle .$$

From Equation (2.6.5), $\psi(\bigcup \mathcal{A}) \subset \langle 1, C_i : i \in I \rangle$, so for each j, $1 \leqslant j \leqslant r$,

$$h_j \in g_u \psi(\bigcup \mathcal{A}) \subset g_u \left\langle 1, C_i : i \in I \right\rangle ,$$

and we may write h_j as

$$h_j = g_u \prod_{i \in I} C_i^{m_{j_i}},$$

for some $m_{j_i} \in \mathbb{N} \cup \{0\}$. Now

$$h_j C_3^t = \left(g_u \prod_{i \in I} C_i^{m_{j_i}} \right) C_3^t \in g_u \psi(\bigcup \mathcal{A}),$$

with

$$\left(\prod_{i \in I} C_i^{m_{j_i}} \right) C_3^t \in \psi(\bigcup \mathcal{A}).$$

By assumption the simple cycle $\mathbf{c}_3$ only occurs in chains attached to u which are of length 2 or more and which do not contain the simple cycle $\mathbf{c}_1$. From the way the algorithm works in part (a) of the proof of Proposition 2.3.4, and the rational independence of the set X_u, it follows that for any $y \in \psi\left(\bigcup \mathcal{A}\right) \subset \langle 1, C_i : i \in I \rangle$, where y contains C_3 as a factor, y must also contain as a factor $C_k C_3$ where $k \in I$, $k \neq 1$. This means we can write

$$h_j C_3^t = \left(g_u \prod_{i \in I} C_i^{m_{j_i}} \right) C_3^t = g_u C_{k_j} C_3^t \prod_{i \in I} C_i^{n_{j_i}},$$

where $n_{j_i} = m_{j_i}$ for $i \neq k_j$ and $n_{j_i} = m_{j_i} - 1 \geqslant 0$ for $i = k_j$, and so, for each j, $1 \leqslant j \leqslant r$,

$$h_j = g_u C_{k_j} \prod_{i \in I} C_i^{n_{j_i}},$$

for some $n_{j_i} \in \mathbb{N} \cup \{0\}$ and $k_j \in I$, $k_j \neq 1$. Since $C_{k_j} \neq C_1$ for each j, $1 \leqslant j \leqslant r$, the rational independence of the set X_u now implies that

$$g_u C_1 \notin \bigcup_{j=1}^{r} g_u C_{k_j} \prod_{i \in I} C_i^{n_{j_i}} \left\langle 1, C_3^t, x_2, \ldots, x_n \right\rangle$$

$$= \bigcup_{j=1}^{r} h_j \langle 1, x_1, x_2, \ldots, x_n \rangle$$
$$= g_u \psi \left(\bigcup \mathcal{A} \right).$$

This is a contradiction of part (d) and therefore Equation (2.6.7) does not hold and the proof is complete. □

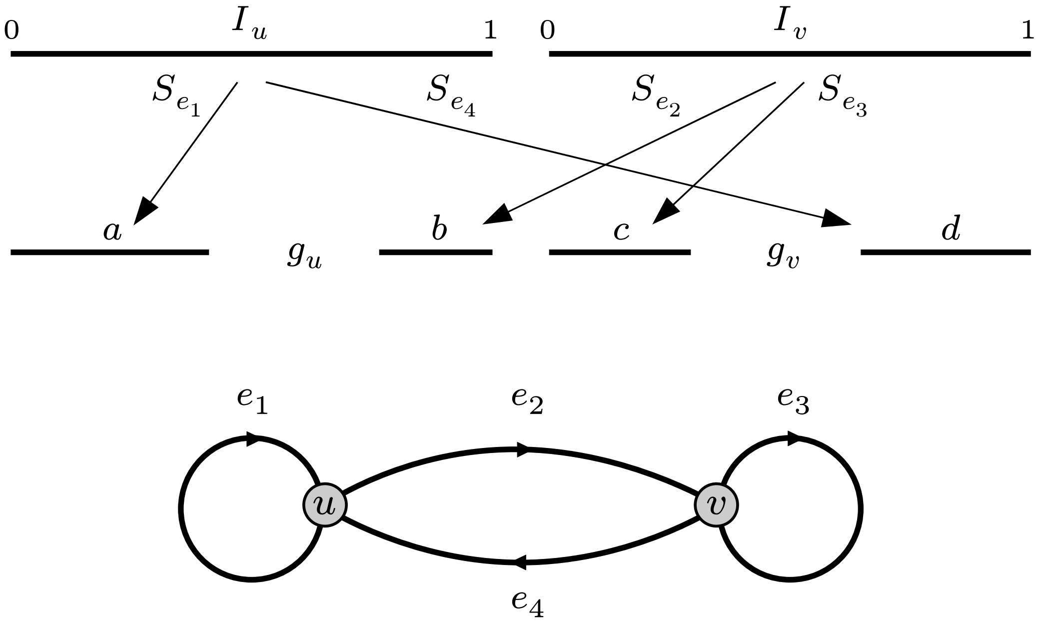


Figure. 2.6.2: The similarity ratio along e_4 in Figure 2.4.1 has been changed from $r_{e_4} = b$ to $r_{e_4} = d$.

For the directed graph IFS in Figure 2.4.1, the set $X_u = \{g_u, g_v, a, b^2, c, b\}$, and this is not a rationally independent set, so we cannot apply Theorem 2.6.3 in that case. In fact Lemma 2.6.1 assumes only that the set $\{g_u, g_v, a, b, c\}$ is rationally independent. This indicates that the assumption that X_u is rationally independent, is a strong assumption in Theorem 2.6.3. However if we were to change the similarity ratio along the edge e_4 in Figure 2.4.1 from b to d, with $r_{e_4} = d$, as shown in Figure 2.6.2, then the set $X_u = \{g_u, g_v, a, bd, c, b\}$. This is a rationally independent set, if the set $\{g_u, g_v, a, b, c, d\}$ is rationally independent, and so Theorem 2.6.3 yields the next corollary.

Corollary 2.6.4. *For the 2-vertex IFS of Figure 2.6.2, if the set $\{g_u, g_v, a, b, c, d\} \subset \mathbb{R}^+$ is a multiplicatively rationally independent set, then the attractor at the vertex u, F_u, is not the attractor of any standard (1-vertex) IFS, defined on $\mathbb{R}$, for which the CSSC holds.*

We conclude this section by investigating the connections between the set X_u of Theorem 2.6.3 and the set X, where X is defined as

$$X = \left\{ g_w, r_e : g_w \in G_w^1, w \in V, e \in E^1 \right\}, \tag{2.6.10}$$

E^1 being the set of all edges in the graph. This is useful because often X is a much easier set to deal with than X_u as regards rational independence. First we make some definitions that will be needed.

Let D_u^T be the set consisting of all the simple cycles and all simple paths starting from the vertex u, in any directed graph IFS, $\big(V, E^*, i, t, r, ((\mathbb{R}, |\ \ |))_{v \in V}, (S_e)_{e \in E^1}\big)$, satisfying the CSSC. That is, let

$$D_u^T = \{\mathbf{c}_i, \mathbf{p} : \mathbf{c}_i \in T,\ \mathbf{p} \in D_{uv}^*,\ v \in V,\ v \neq u\}\,. \tag{2.6.11}$$

Any such directed graph IFS, has a graph that is strongly connected with at least two edges leaving each vertex, see Subsection 1.2.8, so if the graph has n vertices the minimum number of edges in the graph is $2n$. This leads to the following definition. A *minimal graph* is a directed graph that has n vertices and $2n$ edges, with exactly two edges leaving each vertex.

We define a E^1-D_u^T matrix, $\mathbf{M}_u$ as follows. Let $s = \#E^1$ and $t = \#D_u^T$ with the enumerations $E^1 = \{e_1, e_2, \ldots, e_s\}$ and $D_u^T = \{\mathbf{d}_1, \mathbf{d}_2, \ldots, \mathbf{d}_t\}$. The $s \times t$ matrix $\mathbf{M}_u$, which we refer to as the E^1-D_u^T matrix, has ijth entry

$$M_{u_{ij}} = \begin{cases} 1 & \text{if } e_i \text{ is an edge in } \mathbf{d}_j, \\ 0 & \text{if } e_i \text{ is not an edge in } \mathbf{d}_j. \end{cases} \tag{2.6.12}$$

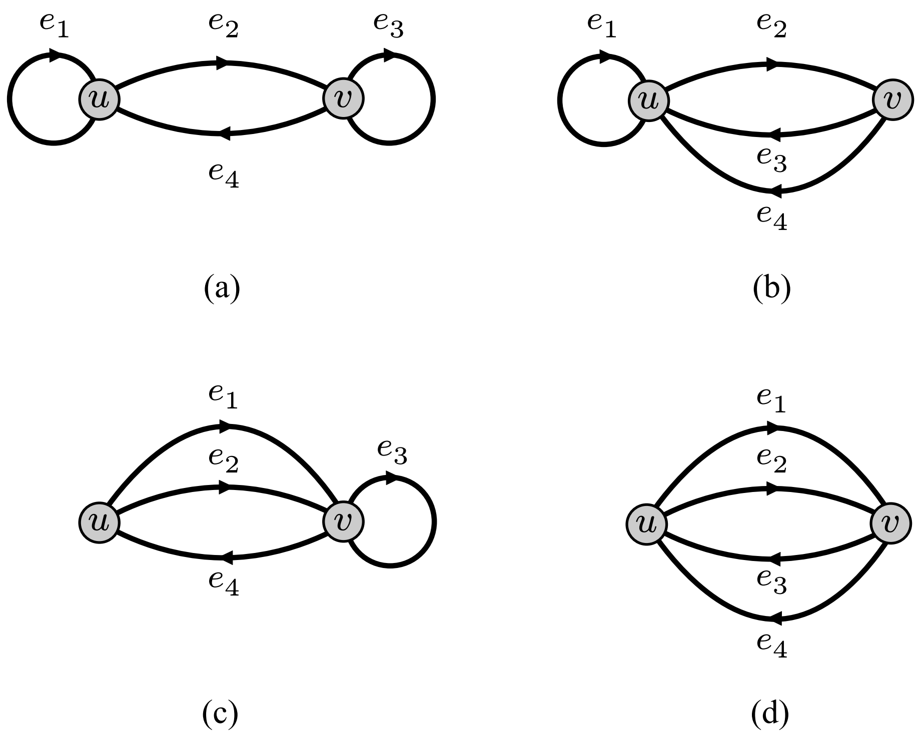


Figure. 2.6.3: The four minimal graphs for 2-vertex directed graph IFSs.

As an illustration of these definitions we consider those 2-vertex IFSs which have minimal graphs. There are only four possibilities for the associated graphs and these are illustrated in Figure 2.6.3. For graph (a) $\#D_u^T = 4$, for graph (b) $\#D_u^T = 4$, for graph (c) $\#D_u^T = 5$, and for graph (d) $\#D_u^T = 6$.

For graph (a) which is the same graph as our examples in Figures 2.4.1 and 2.6.2, we can write $D_u^T = \{\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3, \mathbf{d}_4\}$, where $\mathbf{d}_1 = e_1$, $\mathbf{d}_2 = e_2$, $\mathbf{d}_3 = e_3$ and $\mathbf{d}_4 = e_2e_4$. The 4×4 E^1-D_u^T matrix is then

$$\mathbf{M}_u = \begin{array}{c} \\ e_1 \\ e_2 \\ e_3 \\ e_4 \end{array} \begin{array}{c} \begin{array}{cccc} \mathbf{d}_1 & \mathbf{d}_2 & \mathbf{d}_3 & \mathbf{d}_4 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

We note that this is an invertible matrix.

For graph (b) $D_u^T = \{\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3, \mathbf{d}_4\}$, where $\mathbf{d}_1 = e_1$, $\mathbf{d}_2 = e_2$, $\mathbf{d}_3 = e_2e_3$ and $\mathbf{d}_4 = e_2e_4$. The 4×4 E^1-D_u^T matrix is

$$\mathbf{M}_u = \begin{array}{c} \\ e_1 \\ e_2 \\ e_3 \\ e_4 \end{array} \begin{array}{c} \begin{array}{cccc} \mathbf{d}_1 & \mathbf{d}_2 & \mathbf{d}_3 & \mathbf{d}_4 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

which is also invertible.

For graph (c), $D_u^T = \{\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3, \mathbf{d}_4, \mathbf{d}_5\}$, where $\mathbf{d}_1 = e_1$, $\mathbf{d}_2 = e_2$, $\mathbf{d}_3 = e_3$, $\mathbf{d}_4 = e_1e_4$, and $\mathbf{d}_5 = e_2e_4$. The 4×5 E^1-D_u^T matrix is

$$\mathbf{M}_u = \begin{array}{c} \\ e_1 \\ e_2 \\ e_3 \\ e_4 \end{array} \begin{array}{c} \begin{array}{ccccc} \mathbf{d}_1 & \mathbf{d}_2 & \mathbf{d}_3 & \mathbf{d}_4 & \mathbf{d}_5 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \end{array}$$

We note that the matrix is not square and also that we cannot assign similarity ratios to the graph and also keep the set $\{r_{\mathbf{d}_1}, r_{\mathbf{d}_2}, r_{\mathbf{d}_3}, r_{\mathbf{d}_4}, r_{\mathbf{d}_5}\}$ rationally independent since

$$\frac{r_{\mathbf{d}_4}}{r_{\mathbf{d}_1}} = \frac{r_{e_1e_4}}{r_{e_1}} = r_{e_4} = \frac{r_{e_2e_4}}{r_{e_2}} = \frac{r_{\mathbf{d}_5}}{r_{\mathbf{d}_2}}.$$

For graph (d), $D_u^T = \{\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3, \mathbf{d}_4, \mathbf{d}_5, \mathbf{d}_6\}$, where $\mathbf{d}_1 = e_1$, $\mathbf{d}_2 = e_2$, $\mathbf{d}_3 = e_1e_3$, $\mathbf{d}_4 = e_1e_4$, $\mathbf{d}_5 = e_2e_3$, and $\mathbf{d}_6 = e_2e_4$. The 4×6 E^1-D_u^T matrix is

$$\mathbf{M}_u = \begin{array}{c} \\ e_1 \\ e_2 \\ e_3 \\ e_4 \end{array} \begin{array}{c} \begin{array}{cccccc} \mathbf{d}_1 & \mathbf{d}_2 & \mathbf{d}_3 & \mathbf{d}_4 & \mathbf{d}_5 & \mathbf{d}_6 \end{array} \\ \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix} \end{array}$$

For this graph the E^1-D_u^T matrix is not square and again it is not possible to assign similarity ratios to the four edges of the graph in such a way that the set $\{r_{\mathbf{d}_1}, r_{\mathbf{d}_2}, r_{\mathbf{d}_3}, r_{\mathbf{d}_4}, r_{\mathbf{d}_5}, r_{\mathbf{d}_6}\}$ is rationally independent. This follows since

$$\frac{r_{\mathbf{d}_3}}{r_{\mathbf{d}_1}} = \frac{r_{e_1e_3}}{r_{e_1}} = r_{e_3} = \frac{r_{e_2e_3}}{r_{e_2}} = \frac{r_{\mathbf{d}_5}}{r_{\mathbf{d}_2}}.$$

We now establish the following lemma. X_u, X, D_u^T, and $\mathbf{M}_u$ are as defined in Theorem 2.6.3, Equations (2.6.10), (2.6.11) and (2.6.12) respectively.

Lemma 2.6.5. *For any directed graph IFS, $\big(V, E^*, i, t, r, ((\mathbb{R}, |\ \ |))_{v\in V}, (S_e)_{e\in E^1}\big)$, satisfying the CSSC, with u any vertex in the graph, if the E^1-D_u^T matrix $\mathbf{M}_u$ is invertible, then*

X is rationally independent if and only if X_u is rationally independent.

Proof. The fact that $\mathbf{M}_u$ is invertible means $\#D_u^T = \#E^1$ so we can enumerate E^1 and D_u^T as $E^1 = \{e_1, e_2, \ldots, e_s\}$ and $D_u^T = \{\mathbf{d}_1, \mathbf{d}_2, \ldots, \mathbf{d}_s\}$. Also we may enumerate the set of level-1 gap lengths as $\bigcup_{v\in V} G_v^1 = \{g_1, g_2, \ldots, g_r\}$. Then X and X_u are given by

$$X = \left\{g_i, r_{e_j} : 1 \leqslant i \leqslant r,\ 1 \leqslant j \leqslant s\right\}$$
$$X_u = \left\{g_i, r_{\mathbf{d}_j} : 1 \leqslant i \leqslant r,\ 1 \leqslant j \leqslant s\right\}.$$

Suppose the set X is rationally independent. We aim to show that this implies that X_u is rationally independent, so with this in mind suppose

$$\prod_{i=1}^{r} g_i^{\alpha_i} \prod_{j=1}^{s} r_{\mathbf{d}_j}^{\delta_j} = \prod_{i=1}^{r} g_i^{\alpha_i{}'} \prod_{j=1}^{s} r_{\mathbf{d}_j}^{\delta_j{}'}, \tag{2.6.13}$$

for some $\alpha_i, \alpha_i{}', \delta_j, \delta_i{}' \in \mathbb{Z}$. Now

$$\prod_{i=1}^{r} g_i^{\alpha_i} \prod_{j=1}^{s} r_{e_j}^{\beta_j} = \prod_{i=1}^{r} g_i^{\alpha_i} \prod_{j=1}^{s} r_{\mathbf{d}_j}^{\delta_j}$$

where the integers β_j are given by the matrix equation

$$\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_r \\ \beta_1 \\ \vdots \\ \beta_s \end{pmatrix} = \begin{pmatrix} \mathbf{I}_r & \mathbf{0} \\ & \\ \mathbf{0} & \mathbf{M}_u \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_r \\ \delta_1 \\ \vdots \\ \delta_s \end{pmatrix} \tag{2.6.14}$$

where $\mathbf{I}_r$ is the $r \times r$ identity matrix, $\mathbf{M}_u$ is the $s \times s$ matrix defined in Equation (2.6.12) and $\mathbf{0}$ indicates that all other entries in this $(r+s) \times (r+s)$ matrix are 0. In the same way

$$\prod_{i=1}^{r} g_i^{\alpha_i{}'} \prod_{j=1}^{s} r_{e_j}^{\beta_j{}'} = \prod_{i=1}^{r} g_i^{\alpha_i{}'} \prod_{j=1}^{s} r_{\mathbf{d}_j}^{\delta_j{}'}$$

where the integers $\beta_j{}'$ are given by the matrix equation

$$\begin{pmatrix} \alpha_1{}' \\ \vdots \\ \alpha_r{}' \\ \beta_1{}' \\ \vdots \\ \beta_s{}' \end{pmatrix} = \begin{pmatrix} \mathbf{I}_r & \mathbf{0} \\ & \\ \mathbf{0} & \mathbf{M}_u \end{pmatrix} \begin{pmatrix} \alpha_1{}' \\ \vdots \\ \alpha_r{}' \\ \delta_1{}' \\ \vdots \\ \delta_s{}' \end{pmatrix} \tag{2.6.15}$$

Equation (2.6.13) implies that

$$\prod_{i=1}^{r} g_i^{\alpha_i} \prod_{j=1}^{s} r_{e_j}^{\beta_j} = \prod_{i=1}^{r} g_i^{\alpha_i{}'} \prod_{j=1}^{s} r_{e_j}^{\beta_j{}'},$$

and the rational independence of the set X now gives $\alpha_i = \alpha_i{}'$ for $1 \leqslant i \leqslant r$, and $\beta_j = \beta_j{}'$ for $1 \leqslant j \leqslant s$. By Equations (2.6.14) and (2.6.15) we obtain

$$\begin{pmatrix} \mathbf{I}_r & \mathbf{0} \\ & \\ \mathbf{0} & \mathbf{M}_u \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_r \\ \delta_1 \\ \vdots \\ \delta_s \end{pmatrix} = \begin{pmatrix} \mathbf{I}_r & \mathbf{0} \\ & \\ \mathbf{0} & \mathbf{M}_u \end{pmatrix} \begin{pmatrix} \alpha_1{}' \\ \vdots \\ \alpha_r{}' \\ \delta_1{}' \\ \vdots \\ \delta_s{}' \end{pmatrix}$$

The invertibility of the matrix $\mathbf{M}_u$ ensures the invertibility of the $(r+s) \times (r+s)$ matrix in this equation and so we have shown that $\alpha_i = \alpha_i{}'$ for $1 \leqslant i \leqslant r$, and $\delta_j = \delta_j{}'$ for $1 \leqslant j \leqslant s$. This proves that X_u is rationally independent.

The proof of the converse follows the same route but using the inverse of the $(r+s) \times (r+s)$ matrix. $\square$

Since X is an easier set to deal with than X_u, this lemma is of practical use and may be applied on a case by case basis before applying Theorem 2.6.3 to a particular graph. For example, in graphs (a) and (b) of Figure 2.6.3, adding loops at the vertices or edges going from vertex v to vertex u will maintain equality in the equation $\#D_u^T = \#E^1$, with the E^1-D_u^T matrix remaining invertible. For any minimal graph with n vertices and $2n$ edges, with an invertible E^1-D_u^T matrix, edges can always be added in this way, maintaining the invertibility of the E^1-D_u^T matrix. As minimal graphs with invertible E^1-D_u^T matrices always exist for any n, this means that Theorem 2.6.3 can be applied to a large family of directed graph IFSs.

An example is shown in Figure 2.6.4 for $n = 4$, here the 8 black edges form a minimal graph and the 8×8 E^1-D_u^T matrix is invertible. The edges that have then been added are coloured in grey. For the whole graph $D_u^T = \{\mathbf{d}_i : 1 \leqslant i \leqslant 14\}$, where $\mathbf{d}_1 = e_1$, $\mathbf{d}_2 = e_1e_2$, $\mathbf{d}_3 = e_1e_2e_3$, $\mathbf{d}_4 = e_1e_2e_3e_4$, $\mathbf{d}_5 = e_5$, $\mathbf{d}_6 = e_6$, $\mathbf{d}_7 = e_7$, $\mathbf{d}_8 = e_8$, and adding the edges e_9, e_{10}, e_{11}, e_{12}, e_{13} and e_{14} has added $\mathbf{d}_9 = e_1e_2e_9$, $\mathbf{d}_{10} = e_{10}$, $\mathbf{d}_{11} = e_1e_{11}$, $\mathbf{d}_{12} = e_{12}$, $\mathbf{d}_{13} = e_2e_{13}$ and $\mathbf{d}_{14} = e_2e_3e_{14}$ to D_u^T, maintaining

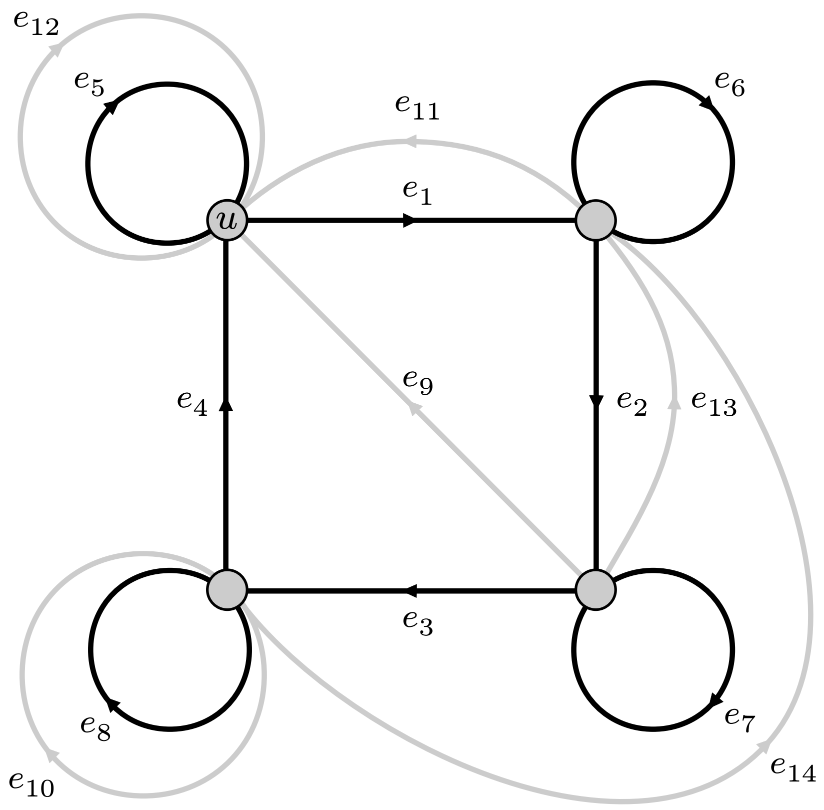


Figure. 2.6.4: The added edges shown in grey maintain the invertibility of the E^1-D_u^T matrix.

the equality $\#D_u^T = \#E^1$. The 14×14 E^1-D_u^T matrix for this graph is

$$\mathbf{M}_u = \begin{array}{cc} & \begin{array}{cccccccccccccc} \mathbf{d}_1 & \mathbf{d}_2 & \mathbf{d}_3 & \mathbf{d}_4 & \mathbf{d}_5 & \mathbf{d}_6 & \mathbf{d}_7 & \mathbf{d}_8 & \mathbf{d}_9 & \mathbf{d}_{10} & \mathbf{d}_{11} & \mathbf{d}_{12} & \mathbf{d}_{13} & \mathbf{d}_{14} \end{array} \\ \begin{array}{c} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \\ e_7 \\ e_8 \\ e_9 \\ e_{10} \\ e_{11} \\ e_{12} \\ e_{13} \\ e_{14} \end{array} & \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

which is clearly still invertible after the addition of the six edges. We point out here that adding an edge which is directed from the vertex u to one of the other three vertices will increase the number of elements in D_u^T by more than one, resulting in $\#D_u^T > \#E^1$.

As we have seen Lemma 2.6.5 ensures that Theorem 2.6.3 applies to a large family of directed graph IFSs, and the simple examples considered above suggest that it may well be possible to prove the following three statements for any directed graph IFS $\left(V, E^*, i, t, r, ((\mathbb{R}, |\ \ |))_{v\in V}, (S_e)_{e\in E^1}\right)$, satisfying the CSSC, with u any vertex in the graph.

(1) $\#D_u^T \geqslant \#E^1$.

(2) If $\#D_u^T = \#E^1$ then the E^1-D_u^T matrix is invertible.

(3) If $\#D_u^T > \#E^1$ then the set $\left\{r_{\mathbf{d}_i} : \mathbf{d}_i \in D_u^T\right\}$ cannot be made rationally independent.

If we restrict the directed graphs to be only those which contain a minimal graph as a subgraph, that is the family of graphs where each graph can be built up by adding edges to a minimal graph, then statement (1) can be proved. The proof uses induction on the number of vertices in the graph. We leave it as an area of possible future investigation as to whether proofs of, or counter-examples to, statements (1), (2) or (3) can be found.

2.7 Conclusion

We have shown in Proposition 2.3.4, that the gap lengths in any attractor of a system satisfying the CSSC, can always be represented as a finite union of cosets of a finite number of finitely generated semigroups with identity. What is more, the proof is constructive, providing a method of explicitly calculating these cosets of semigroups, as expressed in Equation (2.3.7). We then showed in Lemma 2.6.1, Section 2.6, that the particular 2-vertex directed-graph IFS, of Section 2.4, produces an attractor F_u that cannot be the attractor of any standard (1-vertex) IFS satisfying the CSSC. The same result was obtained for a large family of directed graph IFSs in Theorem 2.6.3 and Lemma 2.6.5.

We conclude that we can give an affirmative answer to our question of Section 2.1 at least for the restricted class of directed graph IFSs we have considered in this chapter.

3

Exact Hausdorff measure

3.1 Introduction

In Theorems 3.5.8 and 3.5.9, we prove that our 2-vertex IFS examples of Chapter 2, illustrated there in Figures 2.4.1 and 2.6.2 and reproduced in this chapter in Figures 3.5.3 and 3.5.2, produce attractors that cannot be the attractors of 1-vertex IFSs, overlapping or otherwise. We do this by considering the attractor, F_u, at the vertex u of one of our 2-vertex systems, then using arguments based on those given by Feng and Wang [FW09] we show that if $\mathcal{H}^s(F_u) = |I_u|^s$ then any 1-vertex IFS which has F_u as its attractor must also satisfy the CSSC, but this is impossible as we proved in Chapter 2. In Theorem 3.4.7 we give sufficient conditions to ensure that $\mathcal{H}^s(F_u) = |I_u|^s$ and so we extend the class of attractors for which the exact Hausdorff measure is known. In fact Theorem 3.4.7 is of interest in its own right because it provides sufficient conditions for the calculation of the exact Hausdorff measure of both of the attractors of a class of 2-vertex IFSs defined on $\mathbb{R}$. This adds to the work of Ayer and Strichartz [AS99] and Marion [Mar86]. The class of sets for which the exact Hausdorff measure is known is surprisingly small, see [ZF04] for a survey of recent results and open problems in this area. Another consequence of $\mathcal{H}^s(F_u) = |I_u|^s$ is given in Theorem 3.5.2, where we show that any similarity $S : F_u \to F_u$, with contracting similarity ratio r_S, $0 < r_S < 1$, must have a similarity ratio equivalent to the similarity ratio along a cycle in the graph, that is we prove that $r_S = r_{\mathbf{e}}$ for some cycle $\mathbf{e} \in E^*_{uu}$.

The next two sections lead up to a proof of Corollary 3.3.4 which gives a density result that we prove for general systems defined on $\mathbb{R}^n$, for which the OSC holds. We then return to our 2-vertex examples defined on $\mathbb{R}$, in Sections 3.4 and 3.5.

3.2 Restricted normalised Hausdorff measures are self-similar

For now consider a system, $\big(V, E^*, i, t, r, p, ((\mathbb{R}^n, |\ \ |))_{v\in V}, (S_e)_{e\in E^1}\big)$, which satisfies the OSC, so the conclusions of Theorem 1.3.7 all hold for the list of attractors $(F_u)_{u\in V}$. That is, there is a unique non-negative number s such that $\rho\left(\mathbf{A}(s)\right) = 1$, and for each $u \in V$, $s = \dim_{\mathrm{H}} F_u = \dim_{\mathrm{B}} F_u$, with $0 < \mathcal{H}^s\left(F_u\right) < +\infty$. As described in Subsection 1.2.9 we take $\mathbf{h}$ to be the positive eigenvector, which is unique up to a scaling factor, such that $\mathbf{A}(s)\mathbf{h} = \mathbf{h}$. We remind the reader that if $m = \#V$ and the set of vertices is ordered as $V = (v_1, v_2, \ldots, v_m)$ then we use the notation $(h_v)_{v\in V}$ to represent the ordered m-tuple $(h_{v_1}, h_{v_2}, \ldots, h_{v_m})$, so the positive column eigenvector $\mathbf{h}$ can be written as

$$\mathbf{h} = (h_v)^T_{v\in V} = (h_{v_1}, h_{v_2}, \ldots, h_{v_m})^T,$$

with $h_v > 0$ for all $v \in V$. In the same way $(\mathcal{H}^s(F_v))^T_{v\in V}$ represents a column vector in Lemma 3.2.2(a) that follows.

Exactly as was done in the proof of part (b) of Theorem 1.3.7, we define a probability function $p : E^* \to (0, 1)$, for each path $\mathbf{e} \in E^*$, as

$$p_{\mathbf{e}} = h^{-1}_{i(\mathbf{e})} r^s_{\mathbf{e}} h_{t(\mathbf{e})}. \tag{3.2.1}$$

Since

$$\sum_{e\in E^1_u} p_e = \sum_{e\in E^1_u} h^{-1}_u r^s_e h_{t(e)} = h^{-1}_u (\mathbf{A}(s)\mathbf{h})_u = h^{-1}_u h_u = 1,$$

at each vertex $u \in V$, it is clear that $p_e = h^{-1}_{i(e)} r^s_e h_{t(e)}$ defines a valid probability function for the graph, as described in Subsection 1.2.8.

By Theorem 1.3.8 there exists a unique list of self-similar Borel probability measures, $(\mu_u)_{u\in V}$, with $\operatorname{supp}(\mu_u) = F_u$, for each $u \in V$. For future reference, we now state the invariance equations satisfied by $(F_u)_{u\in V}$, and $(\mu_u)_{u\in V}$, as

$$(F_u)_{u\in V} = \left(\bigcup_{e\in E^1_u} S_e(F_{t(e)}) \right)_{u\in V}, \tag{3.2.2}$$

$$(\mu_u(A_u))_{u\in V} = \left(\sum_{e\in E^1_u} h^{-1}_u r^s_e h_{t(e)} \mu_{t(e)}(S^{-1}_e(A_u)) \right)_{u\in V}, \tag{3.2.3}$$

for all Borel sets $(A_u)_{u\in V} \subset (\mathbb{R}^n)^{\#V}$, (see Theorem 1.3.4, Equation (1.3.2), and Theorem 1.3.8, Equation (1.3.13)).

In Lemma 3.2.3 we show that the self-similar measures in Equation (3.2.3) are in fact restricted normalised Hausdorff measures, but first we prove in Lemma 3.2.2 that $(\mathcal{H}^s(F_v))^T_{v\in V}$ is the unique, (up to scaling), positive eigenvector of the matrix $\mathbf{A}(s)$.

Lemma 3.2.1. *Let* $\mathbf{M}$ *be a non-negative irreducible* $n \times n$ *matrix with* $\rho(\mathbf{M}) = 1$. *Suppose* $\mathbf{v} = (v_1, v_2, \ldots, v_n)^T$ *is a positive vector such that*

$$\mathbf{0} < \mathbf{v} \leqslant \mathbf{M}\mathbf{v},$$

then

$$\mathbf{v} = \mathbf{M}\mathbf{v}.$$

Proof. The transpose of $\mathbf{M}$ will also be a non-negative irreducible matrix and will also have $\rho(\mathbf{M}^T) = 1$, (see [Sen73]). Let $\mathbf{e}$ be the unique (up to scaling) positive eigenvector of $\mathbf{M}^T$, with $e_i > 0$ for $1 \leqslant i \leqslant n$. Taking the scalar product

$$\mathbf{v} \cdot \mathbf{e} \leqslant (\mathbf{M}\mathbf{v}) \cdot \mathbf{e} = \mathbf{v} \cdot (\mathbf{M}^T\mathbf{e}) = \mathbf{v} \cdot \mathbf{e},$$

so $\mathbf{v} \cdot \mathbf{e} = (\mathbf{M}\mathbf{v}) \cdot \mathbf{e}$, that is $\sum_{i=1}^n v_i e_i = \sum_{i=1}^n (\mathbf{M}\mathbf{v})_i e_i$, and since $v_i \leqslant (\mathbf{M}\mathbf{v})_i$, and $e_i > 0$ it follows that $v_i = (\mathbf{M}\mathbf{v})_i$ for all i, $1 \leqslant i \leqslant n$, which implies $\mathbf{v} = \mathbf{M}\mathbf{v}$. □

Lemma 3.2.2. (a) $(\mathcal{H}^s(F_v))^T_{v\in V}$ *is the unique (up to scaling) positive eigenvector of the matrix* $\mathbf{A}(s)$*, that is*

$$\mathbf{A}(s)(\mathcal{H}^s(F_v))^T_{v\in V} = (\mathcal{H}^s(F_v))^T_{v\in V}.$$

(b) *For* $e, f \in E^1_u$ *with* $e \neq f$,

$$\mathcal{H}^s(S_e(F_{t(e)}) \cap S_f(F_{t(f)})) = 0.$$

Proof. (a)

$$\begin{aligned}
\mathcal{H}^s(F_u) &= \mathcal{H}^s\Big(\bigcup_{e\in E^1_u} S_e(F_{t(e)}) \Big) && \text{(by (3.2.2))} \\
&\leqslant \sum_{e\in E^1_u} \mathcal{H}^s(S_e(F_{t(e)})) \\
&= \sum_{e\in E^1_u} r_e^s \mathcal{H}^s(F_{t(e)}) && \text{(by the scaling property)} \\
&= \Big(\mathbf{A}(s)(\mathcal{H}^s(F_v))^T_{v\in V}\Big)_u,
\end{aligned}$$

so $(\mathcal{H}^s(F_v))^T_{v\in V}$ is a positive vector for which

$$\mathbf{0} < (\mathcal{H}^s(F_v))^T_{v\in V} \leqslant \mathbf{A}(s)(\mathcal{H}^s(F_v))^T_{v\in V}.$$

The matrix $\mathbf{A}(s)$ is non-negative from its definition, it is also irreducible because the graph is strongly connected, and $\rho(\mathbf{A}(s)) = 1$. Applying Lemma 3.2.1 proves part (a).

(b) For a finite measure λ, if A_i, $i \in \mathbb{N}$, are λ-measurable sets, then $\lambda(\bigcup_{i=1}^\infty A_i) = \sum_{i=1}^\infty \lambda(A_i)$ if and only if $\lambda(A_i \cap A_j) = 0$, for all $i, j \in \mathbb{N}$, with $i \neq j$. Part (b) follows immediately using this property of the measure $\mathcal{H}^s$ on the $\mathcal{H}^s$-measurable sets $S_e(F_{t(e)})$, $e \in E^1_u$, as

$$\mathcal{H}^s(F_u) = \mathcal{H}^s\Big(\bigcup_{e\in E^1_u} S_e(F_{t(e)}) \Big) = \sum_{e\in E^1_u} \mathcal{H}^s(S_e(F_{t(e)})),$$

from part (a). □

Given the result of Lemma 3.2.2(a) we put $\mathbf{h} = (h_v)^T_{v\in V} = (\mathcal{H}^s(F_v))^T_{v\in V}$ to denote the eigenvector of $\mathbf{A}(s)$, using any of these notations for convenience from now on.

Lemma 3.2.3. *For each* $u \in V$,

$$\mu_u(A) = \frac{\mathcal{H}^s(F_u \cap A)}{\mathcal{H}^s(F_u)} = h_u^{-1}\mathcal{H}^s(F_u \cap A),$$

for all Borel sets $A \subset \mathbb{R}^n$.

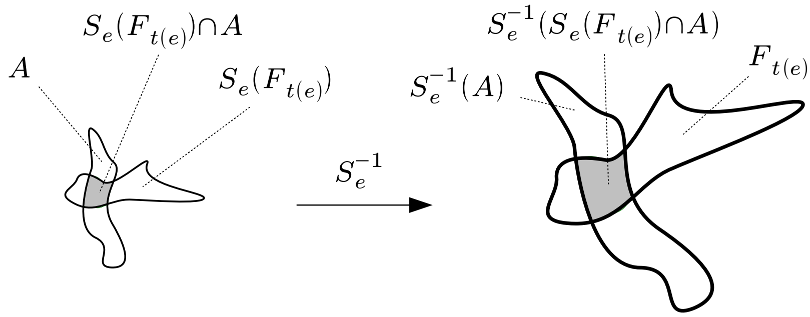


Figure. 3.2.1: An illustration of the map S_e^{-1}.

Proof. First we note that

$$\mathcal{H}^s(F_{t(e)} \cap S_e^{-1}(A)) = \mathcal{H}^s(S_e^{-1}(S_e(F_{t(e)}) \cap A)) = (r_e^s)^{-1}\mathcal{H}^s(S_e(F_{t(e)}) \cap A). \quad (3.2.4)$$

This is clear from the scaling property of the measure and is illustrated in Figure 3.2.1 in $\mathbb{R}^2$. Also as $S_e(F_{t(e)}) \cap A \subset S_e(F_{t(e)})$, for all Borel sets $A \subset \mathbb{R}^n$, using Lemma 3.2.2 (b) we obtain,

$$\mathcal{H}^s((S_e(F_{t(e)}) \cap A) \cap (S_f(F_{t(f)}) \cap A)) \leqslant \mathcal{H}^s(S_e(F_{t(e)}) \cap S_f(F_{t(f)})) = 0,$$

for $e, f \in E_u^1$ with $e \neq f$, so that

$$\sum_{e \in E_u^1} \mathcal{H}^s(S_e(F_{t(e)}) \cap A) = \mathcal{H}^s\Big(\Big(\bigcup_{e \in E_u^1} S_e(F_{t(e)}) \Big) \cap A \Big). \quad (3.2.5)$$

Substituting the normalised restricted measures $(h_v^{-1}\mathcal{H}^s(F_v \cap A))_{v \in V}$ into the right hand side of Equation (3.2.3) gives

$$\begin{aligned}
&\sum_{e \in E_u^1} h_u^{-1} r_e^s h_{t(e)} \Big(h_{t(e)}^{-1} \mathcal{H}^s(F_{t(e)} \cap S_e^{-1}(A)) \Big) \\
&\qquad = \sum_{e \in E_u^1} h_u^{-1} r_e^s h_{t(e)} \Big(h_{t(e)}^{-1} (r_e^s)^{-1} \mathcal{H}^s(S_e(F_{t(e)}) \cap A) \Big) \quad \text{(by (3.2.4))} \\
&\qquad = h_u^{-1} \sum_{e \in E_u^1} \mathcal{H}^s\big(S_e(F_{t(e)}) \cap A\big) \\
&\qquad = h_u^{-1}\mathcal{H}^s\Big(\Big(\bigcup_{e \in E_u^1} S_e(F_{t(e)}) \Big) \cap A \Big) \quad \text{(by (3.2.5))} \\
&\qquad = h_u^{-1}\mathcal{H}^s(F_u \cap A) \quad \text{(by (3.2.2))}.
\end{aligned}$$

Therefore Equation (3.2.3) holds for the list of measures $(h_u^{-1}\mathcal{H}^s(F_u \cap A_u))_{u \in V}$, where $(A_u)_{u \in V} \subset (\mathbb{R}^n)^{\#V}$ are any Borel sets, and so

$$(\mu_u(A_u))_{u \in V} = (h_u^{-1}\mathcal{H}^s(F_u \cap A_u))_{u \in V}. \qquad \square$$

3.3 A density result

Our aim in this section is to prove Corollary 3.3.4. The definition of an s-straight set is introduced as it provides a useful intermediate step in the argument.

As defined in [Del02], a set $B \subset \mathbb{R}^n$ is *s-straight*, if

$$\mathcal{H}^s_\infty(B) = \mathcal{H}^s(B) < +\infty.$$

We point out that it is always the case that $\mathcal{H}^s_\infty(B) \leqslant \mathcal{H}^s(B)$, so if $\mathcal{H}^s(B) = 0$ then B is s-straight.

Lemma 3.3.1. *If $B \subset \mathbb{R}^n$ is s-straight then $\mathcal{H}^s(A) \leqslant |A|^s$, for all $\mathcal{H}^s$-measurable subsets $A \subset B$.*

Proof. For a contradiction we assume there is a $\mathcal{H}^s$-measurable subset $A \subset B$ such that

$$0 < |A|^s < \mathcal{H}^s(A) - \varepsilon, \tag{3.3.1}$$

for some $\varepsilon > 0$. Now

$$\mathcal{H}^s(B \setminus A) = \mathcal{H}^s(B) - \mathcal{H}^s(A), \tag{3.3.2}$$

and we may find a cover $\{U_i\}$ of $B \setminus A$ with

$$\sum_{i=1}^{\infty} |U_i|^s \leqslant \mathcal{H}^s(B \setminus A) + \frac{\varepsilon}{2}. \tag{3.3.3}$$

It follows that $B \subset A \bigcup (\bigcup_{i=1}^{\infty} U_i)$, so

$$\begin{aligned}
\mathcal{H}^s_\infty(B) &\leqslant |A|^s + \sum_{i=1}^{\infty} |U_i|^s && \text{(from the definition of } \mathcal{H}^s_\infty) \\
&\leqslant |A|^s + \mathcal{H}^s(B \setminus A) + \frac{\varepsilon}{2} && \text{(by (3.3.3))} \\
&< \mathcal{H}^s(A) - \varepsilon + \mathcal{H}^s(B \setminus A) + \frac{\varepsilon}{2} && \text{(by assumption (3.3.1))} \\
&= \mathcal{H}^s(B) - \frac{\varepsilon}{2} && \text{(by (3.3.2))},
\end{aligned}$$

which means $\mathcal{H}^s_\infty(B) \neq \mathcal{H}^s(B)$ which is a contradiction as B is s-straight. □

Lemma 3.3.2. *Let $A \subset \mathbb{R}^n$ and let $S : \mathbb{R}^n \to \mathbb{R}^n$ be a similarity with similarity ratio $\lambda > 0$, then*

$$\mathcal{H}^s_\infty(S(A)) = \lambda^s \mathcal{H}^s_\infty(A).$$

Proof. (See [Fal03]). Let $\{U_i\}$ be any cover of A, then $\{S(U_i)\}$ is a cover of $S(A)$ and

$$\sum_{i=1}^{\infty} |S(U_i)|^s = \lambda^s \sum_{i=1}^{\infty} |U_i|^s,$$

so $\mathcal{H}^s_\infty(S(A)) \leqslant \lambda^s \mathcal{H}^s_\infty(A)$. Now S^{-1} is a similarity with ratio $\frac{1}{\lambda}$ so the same argument applies to give $\mathcal{H}^s_\infty(S^{-1}(S(A))) \leqslant \frac{1}{\lambda^s}\mathcal{H}^s_\infty(S(A))$, which means $\mathcal{H}^s_\infty(S(A)) \geqslant \lambda^s \mathcal{H}^s_\infty(A)$, which proves the result. □

Lemma 3.3.3. *F_u is s-straight, for all $u \in V$.*

Proof. We aim to prove $\mathcal{H}^s_\infty(F_u) = \mathcal{H}^s(F_u)$. This is enough to show F_u is s-straight since $0 < \mathcal{H}^s(F_u) < \infty$. For a contradiction suppose for some $u \in V$,

$$0 \leqslant \mathcal{H}^s_\infty(F_u) < \mathcal{H}^s(F_u). \tag{3.3.4}$$

Now consider a vertex $v \in V$, $v \neq u$. As the graph is strongly connected we can always find a path $\mathbf{e}$ from the vertex v to u, and suppose such a path has length m. That is $|\mathbf{e}| = m$, $i(\mathbf{e}) = v$, and $t(e) = u$. By iterating Equation (3.2.2) above, we may write

$$F_v = \bigcup_{e \in E_v^1} S_e(F_{t(e)}) = \bigcup_{\mathbf{e} \in E_v^m} S_{\mathbf{e}}(F_{t(\mathbf{e})}),$$

where E_v^m is the set of paths of length m which have initial vertex v. It follows that

$$\begin{aligned}
\mathcal{H}^s_\infty(F_v) &\leqslant \sum_{\mathbf{e} \in E_v^m} \mathcal{H}^s_\infty(S_{\mathbf{e}}(F_{t(\mathbf{e})})) && (\mathcal{H}^s_\infty \text{ is an outer measure}) \\
&= \sum_{\mathbf{e} \in E_v^m} r_{\mathbf{e}}^s \mathcal{H}^s_\infty(F_{t(\mathbf{e})}) && (\text{by Lemma 3.3.2}) \\
&< \sum_{\mathbf{e} \in E_v^m} r_{\mathbf{e}}^s \mathcal{H}^s(F_{t(\mathbf{e})}) && (\text{by (3.3.4)}).
\end{aligned}$$

By Lemma 3.2.2(a) we obtain

$$\mathcal{H}^s_\infty(F_v) < \sum_{\mathbf{e} \in E_v^m} r_{\mathbf{e}}^s \mathcal{H}^s(F_{t(\mathbf{e})}) = \left(\mathbf{A}(s)^m (\mathcal{H}^s(F_w))^T_{w \in V}\right)_v = \mathcal{H}^s(F_v).$$

This argument may be repeated for any vertex, so our assumption in Equation (3.3.4) implies

$$0 \leqslant \mathcal{H}^s_\infty(F_v) < \mathcal{H}^s(F_v), \tag{3.3.5}$$

for all $v \in V$.

For reasons that will become apparent shortly, let $\varepsilon > 0$ be given by,

$$\varepsilon = \min\left\{ \frac{h_{\max}}{2},\ \min\left\{ \frac{\mathcal{H}^s(F_v) - \mathcal{H}^s_\infty(F_v)}{2} : v \in V \right\} \right\}, \tag{3.3.6}$$

where $h_{\max} > 0$ is defined as $h_{\max} = \max\{h_v : v \in V\}$. Using (3.3.6), for each $v \in V$, we may choose some cover $\{U_{v,i}\}$ of F_v, with no diameter restriction, such that

$$\sum_{i=1}^\infty |U_{v,i}|^s < \mathcal{H}^s_\infty(F_v) + \varepsilon \leqslant \mathcal{H}^s(F_v) - \varepsilon.$$

That is, for each $v \in V$, $F_v \subset \bigcup_{i=1}^\infty U_{v,i}$, with

$$\sum_{i=1}^\infty |U_{v,i}|^s < \mathcal{H}^s(F_v) - \varepsilon. \tag{3.3.7}$$

Let $\alpha = \sup\{|U_{v,i}| : v \in V,\ i \in \mathbb{N}\}$, then α is finite by (3.3.7). Let $r_{\max} = \max\{r_e : e \in E^1\}$. For a given $\delta > 0$, we may choose $k \in \mathbb{N}$ such that

$$\left|S_{\mathbf{e}}(U_{t(\mathbf{e}),i})\right| = r_{\mathbf{e}}\left|U_{t(\mathbf{e}),i}\right| \leqslant r_{\max}^k \alpha < \delta,$$

for all paths $\mathbf{e} \in E_u^k$, and all $u \in V$. For such k,

$$F_u = \bigcup_{\mathbf{e}\in E_u^k} S_{\mathbf{e}}(F_{t(\mathbf{e})}) \subset \bigcup_{\mathbf{e}\in E_u^k} S_{\mathbf{e}}\Big(\bigcup_{i=1}^{\infty} U_{t(\mathbf{e}),i}\Big) = \bigcup_{\mathbf{e}\in E_u^k}\bigcup_{i=1}^{\infty} S_{\mathbf{e}}(U_{t(\mathbf{e}),i})$$

where the right hand side here is a δ-cover of F_u, and

$$\begin{aligned}\sum_{\mathbf{e}\in E_u^k}\sum_{i=1}^{\infty}\left|S_{\mathbf{e}}(U_{t(\mathbf{e}),i})\right|^s &= \sum_{\mathbf{e}\in E_u^k}\sum_{i=1}^{\infty} r_{\mathbf{e}}^s\left|U_{t(\mathbf{e}),i}\right|^s \\ &< \sum_{\mathbf{e}\in E_u^k} r_{\mathbf{e}}^s\big(\mathcal{H}^s(F_{t(\mathbf{e})}) - \varepsilon\big) \qquad \text{by (3.3.7).}\end{aligned}$$

By Lemma 3.2.2 (a), $\sum_{\mathbf{e}\in E_u^k} h_u^{-1} r_{\mathbf{e}}^s h_{t(\mathbf{e})} = h_u^{-1}\big(\mathbf{A}(s)^k\mathbf{h}\big)_u = 1$, so

$$\sum_{\mathbf{e}\in E_u^k} r_{\mathbf{e}}^s \mathcal{H}^s(F_{t(\mathbf{e})}) = \sum_{\mathbf{e}\in E_u^k} r_{\mathbf{e}}^s h_{t(\mathbf{e})} = h_u,$$

and

$$\sum_{\mathbf{e}\in E_u^k} r_{\mathbf{e}}^s \geqslant \frac{h_u}{h_{\max}},$$

which implies

$$\mathcal{H}_\delta^s(F_u) \leqslant \sum_{\mathbf{e}\in E_u^k}\sum_{i=1}^{\infty}\left|S_{\mathbf{e}}(U_{t(\mathbf{e}),i})\right|^s < \sum_{\mathbf{e}\in E_u^k} r_{\mathbf{e}}^s\big(\mathcal{H}^s(F_{t(\mathbf{e})}) - \varepsilon\big) \leqslant h_u\Big(1 - \frac{\varepsilon}{h_{\max}}\Big).$$

From the choice of ε in (3.3.6), $0 < \varepsilon \leqslant \frac{h_{\max}}{2}$, which ensures $\frac{1}{2} \leqslant \Big(1 - \frac{\varepsilon}{h_{\max}}\Big) < 1$ and as this argument holds for any δ we may conclude that

$$\mathcal{H}^s(F_u) \leqslant h_u\Big(1 - \frac{\varepsilon}{h_{\max}}\Big) < h_u = \mathcal{H}^s(F_u),$$

which is the required contradiction. □

Corollary 3.3.4.

(a) $\mathcal{H}^s(A) \leqslant |A|^s$ *for all* $\mathcal{H}^s$*-measurable subsets* $A \subset F_u$,

(b) $\sup\Big\{\dfrac{\mathcal{H}^s(A)}{|A|^s} : A \text{ is } \mathcal{H}^s\text{-measurable, } A \subset F_u,\ |A| > 0\Big\} = 1.$

Proof. (a) is an immediate consequence of Lemma 3.3.1 and Lemma 3.3.3. It also holds for $A = \emptyset$ as $0 = \mathcal{H}^s(\emptyset) \leqslant |\emptyset|^s = 0$, see Subsection 1.2.3.

(b) Let

$$\alpha = \sup\left\{ \frac{\mathcal{H}^s(A)}{|A|^s} : A \text{ is } \mathcal{H}^s\text{-measurable, } A \subset F_u,\ |A| > 0 \right\},$$

then from part (a), $\alpha \leqslant 1$. It remains to show that $\alpha \geqslant 1$.

Given $\varepsilon > 0$ we can find a cover $\{U_i\}$ of F_u, such that

$$\sum_{i=1}^{\infty} |U_i|^s < \mathcal{H}^s_\infty(F_u) + \varepsilon = \mathcal{H}^s(F_u) + \varepsilon,$$

by Lemma 3.3.3. Each set U_i is contained in a closed set of the same diameter, so we may consider the cover to consist of closed sets which are $\mathcal{H}^s$-measurable. Also $F_u \cap U_i$ is a Borel set and so is $\mathcal{H}^s$-measurable, for each $i \in \mathbb{N}$. As $F_u \subset \bigcup_{i=1}^{\infty} U_i$, we obtain,

$$\begin{aligned}
\mathcal{H}^s(F_u) &= \mathcal{H}^s\Big(F_u \cap \bigcup_{i=1}^{\infty} U_i \Big) = \mathcal{H}^s\Big(\bigcup_{i=1}^{\infty} (F_u \cap U_i) \Big) \\
&\leqslant \sum_{i=1}^{\infty} \mathcal{H}^s\big(F_u \cap U_i\big) \leqslant \sum_{i=1}^{\infty} \alpha\, |F_u \cap U_i|^s \\
&\leqslant \alpha \sum_{i=1}^{\infty} |U_i|^s < \alpha\big(\mathcal{H}^s(F_u) + \varepsilon\big).
\end{aligned}$$

This argument holds for any $\varepsilon > 0$, so we conclude that $\mathcal{H}^s(F_u) \leqslant \alpha \mathcal{H}^s(F_u)$, and this, as $0 < \mathcal{H}^s(F_u) < +\infty$, implies that $\alpha \geqslant 1$. □

3.4 The calculation of exact Hausdorff measure

We now return to our familiar example of a 2-vertex directed graph IFS in $(\mathbb{R}, |\ \ |)$, illustrated in Figure 3.4.1. We recall that I_u, I_v, are the smallest closed intervals containing the attractors F_u, F_v, so $\{a_u, b_u\} \subset F_u \subset I_u = [a_u, b_u]$, with $|F_u| = |I_u| = b_u - a_u$, and similarly at the vertex v. We remind the reader that we are assuming that all similarities represented in diagrams, like the similarities $S_{e_1}, S_{e_2}, S_{e_3}$ and S_{e_4} of Figure 3.4.1, do not reflect and so reflections are not considered in what follows. We consider the gap lengths g_u, g_v, to be strictly positive, which means the CSSC is satisfied, so all the results of the preceding sections apply. This time we do not assume $I_u = I_v = [0,1]$. For the rest of this section a, g_u, b, c, g_v, d, $|I_u| = a + g_u + b$, $|I_v| = c + g_v + d$, are as illustrated in Figure 3.4.1, and $s = \dim_{\mathrm{H}} F_u = \dim_{\mathrm{H}} F_v$, denotes the Hausdorff dimension of the attractors of the system. The contracting similarity ratios of the similarities are given by

$$\begin{aligned}
r_{e_1} &= \frac{|S_{e_1}(I_u)|}{|I_u|} = \frac{a}{|I_u|}, \quad r_{e_2} = \frac{|S_{e_2}(I_v)|}{|I_v|} = \frac{b}{|I_v|}, \\
r_{e_3} &= \frac{|S_{e_3}(I_v)|}{|I_v|} = \frac{c}{|I_v|}, \quad r_{e_4} = \frac{|S_{e_4}(I_u)|}{|I_u|} = \frac{d}{|I_u|}.
\end{aligned} \tag{3.4.1}$$

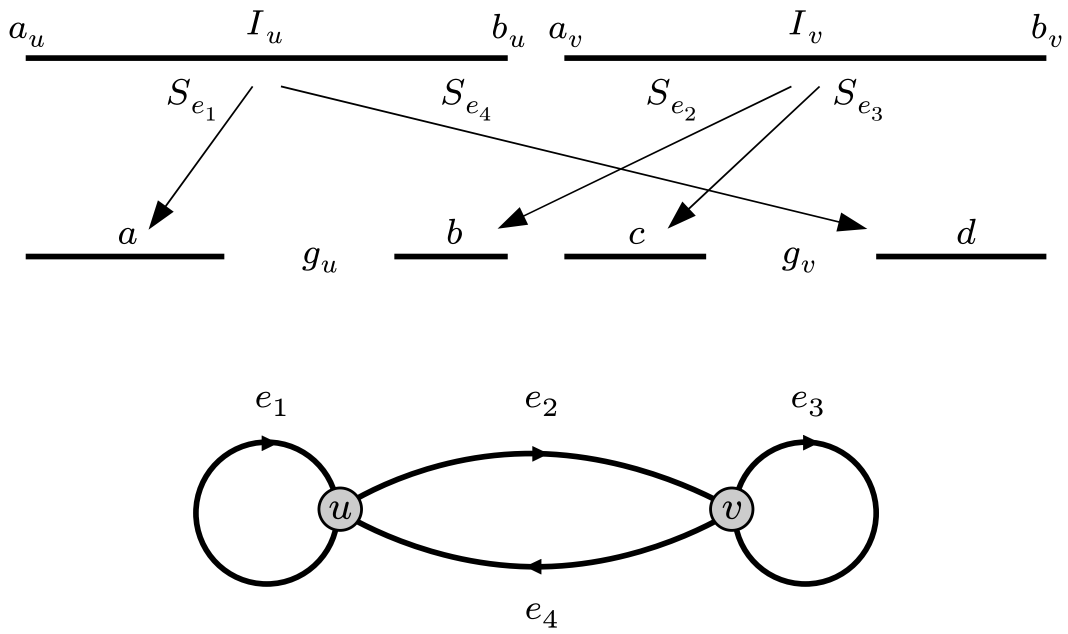


Figure. 3.4.1: A 2-vertex directed graph IFS in $\mathbb{R}$, the similarities S_{e_1}, S_{e_2}, S_{e_3} and S_{e_4} do not reflect.

We reserve the letter J to denote a closed interval in all that follows and we also assume that $|J| > 0$. The *density of an interval* $J \subset I_u$ is defined as

$$d_u(J) = \frac{\mu_u(J)}{|J|^s} = \frac{\mathcal{H}^s(F_u \cap J)}{\mathcal{H}^s(F_u)\,|J|^s},$$

and for $J \subset I_v$, as

$$d_v(J) = \frac{\mu_v(J)}{|J|^s} = \frac{\mathcal{H}^s(F_v \cap J)}{\mathcal{H}^s(F_v)\,|J|^s}.$$

The *maximum density* for the intervals of F_u is the constant

$$\sup\big\{d_u(J) : J \subset I_u\big\},$$

and for the intervals of F_v is

$$\sup\big\{d_v(J) : J \subset I_v\big\}.$$

We now produce a series of technical lemmas which lead up to Theorem 3.4.7, this is an important theorem which gives sufficient conditions for the Hausdorff measure to be calculated. Our next lemma is an immediate consequence of Corollary 3.3.4 of the preceding section.

Lemma 3.4.1. *For the 2-vertex directed graph IFS of Figure 3.4.1,*

$$\sup\big\{d_u(J) : J \subset I_u\big\} = \frac{1}{\mathcal{H}^s(F_u)} \geqslant \frac{1}{|I_u|^s}, \quad \sup\big\{d_v(J) : J \subset I_v\big\} = \frac{1}{\mathcal{H}^s(F_v)} \geqslant \frac{1}{|I_v|^s}.$$

Proof. It is clear that

$$\sup\left\{\frac{\mathcal{H}^s(F_u \cap J)}{|J|^s} : J \subset I_u\right\} \leqslant \sup\left\{\frac{\mathcal{H}^s(A)}{|A|^s} : A \text{ is } \mathcal{H}^s\text{-measurable, } A \subset F_u\right\},$$

and given any $\mathcal{H}^s$-measurable subset $A \subset F_u$, we may find a closed interval $J \subset I_u$ such that $A \subset F_u \cap J$, with $|A| = |J|$, so the opposite inequality also holds. Corollary 3.3.4(b) now implies

$$\sup\left\{\frac{\mathcal{H}^s(F_u \cap J)}{|J|^s} : J \subset I_u\right\} = 1,$$

and so

$$\sup\{d_u(J) : J \subset I_u\} = \frac{1}{\mathcal{H}^s(F_u)} \geqslant \frac{1}{|I_u|^s},$$

by Corollary 3.3.4(a). □

In the next lemma we collect together some useful densities for future reference and it is convenient to use the eigenvector notation $\mathbf{h} = (h_v)_{v \in V}^T = (\mathcal{H}^s(F_v))_{v \in V}^T$, with $h_u = \mathcal{H}^s(F_u)$ and $h_v = \mathcal{H}^s(F_v)$.

Lemma 3.4.2. *For the* 2*-vertex directed graph IFS of Figure 3.4.1,*

(a) $d_u(I_u) = d_u(S_{e_1}(I_u)) = \dfrac{1}{|I_u|^s}$, (b) $d_u(S_{e_2}(I_v)) = \dfrac{h_v}{h_u}\dfrac{1}{|I_v|^s}$,

(c) $d_v(I_v) = d_v(S_{e_3}(I_v)) = \dfrac{1}{|I_v|^s}$, (d) $d_v(S_{e_4}(I_u)) = \dfrac{h_u}{h_v}\dfrac{1}{|I_u|^s}$,

(e) $J \subset S_{e_1}(I_u)$, $d_u(S_{e_1}^{-1}(J)) = d_u(J)$, (f) $J \subset S_{e_2}(I_v)$, $\dfrac{h_v}{h_u} d_v(S_{e_2}^{-1}(J)) = d_u(J)$,

(g) $J \subset S_{e_3}(I_v)$, $d_v(S_{e_3}^{-1}(J)) = d_v(J)$, (h) $J \subset S_{e_4}(I_u)$, $\dfrac{h_u}{h_v} d_u(S_{e_4}^{-1}(J)) = d_v(J)$.

Proof. We prove (h), the others can be proved in much the same way.

$$\begin{aligned}\frac{h_u}{h_v} d_u(S_{e_4}^{-1}(J)) &= \frac{\mathcal{H}^s(F_u)}{\mathcal{H}^s(F_v)} \frac{\mathcal{H}^s\big(F_u \cap S_{e_4}^{-1}(J)\big)}{\mathcal{H}^s(F_u)\left|S_{e_4}^{-1}(J)\right|^s} = \frac{\mathcal{H}^s\big(S_{e_4}^{-1}(S_{e_4}(F_u) \cap J)\big)}{\mathcal{H}^s(F_v)\left|S_{e_4}^{-1}(J)\right|^s} \\ &= \frac{r_{e_4}^{-s}\mathcal{H}^s\big(S_{e_4}(F_u) \cap J)\big)}{\mathcal{H}^s(F_v) r_{e_4}^{-s}\left|J\right|^s} = \frac{\mathcal{H}^s(F_v \cap J))}{\mathcal{H}^s(F_v)\left|J\right|^s} = d_v(J).\end{aligned}$$ □

The arguments that follow are based on those given by Ayer and Strichartz in [AS99], particularly Lemmas 2.1, 3.1, 4.1 and Theorem 4.2 of that paper. See also [Mar86], Theorem 7.1. Our next lemma is Lemma 3.1 of [AS99].

Lemma 3.4.3. *For constants $\alpha, \beta, \gamma, \delta > 0$, $0 < s < 1$, and the variable $x \geqslant 0$, let f be the function*

$$f(x) = \frac{\alpha + \beta x^s}{(\gamma + \delta x)^s},$$

then f attains its maximum value of

$$\left(\left(\frac{\alpha}{\gamma^s}\right)^{\frac{1}{1-s}} + \left(\frac{\beta}{\delta^s}\right)^{\frac{1}{1-s}}\right)^{1-s}$$

at

$$x_{\max} = \left(\frac{\gamma\beta}{\delta\alpha}\right)^{\frac{1}{1-s}}.$$

The function f is strictly increasing on $0 \leqslant x < x_{\max}$, and strictly decreasing on $x > x_{\max}$.

Proof.

$$f'(x) = s\frac{\left(\frac{\gamma\beta}{x^{1-s}} - \delta\alpha\right)}{(\gamma+\delta x)^{1+s}},$$

so $f'(x) = 0$ at $x = x_{\max}$. For $x < x_{\max}$, $f'(x) > 0$ and for $x > x_{\max}$, $f'(x) < 0$.

The maximum value is given by

$$\begin{aligned} f(x_{\max}) &= \frac{\alpha + \beta\left(\frac{\gamma\beta}{\delta\alpha}\right)^{\frac{s}{1-s}}}{\left(\gamma + \delta\left(\frac{\gamma\beta}{\delta\alpha}\right)^{\frac{1}{1-s}}\right)^s} \\ &= \frac{\alpha(\delta\alpha)^{\frac{s}{1-s}} + \beta(\gamma\beta)^{\frac{s}{1-s}}}{\left(\gamma(\delta\alpha)^{\frac{1}{1-s}} + \delta(\gamma\beta)^{\frac{1}{1-s}}\right)^s} \\ &= \frac{\alpha^{\frac{1}{1-s}}\delta^{\frac{s}{1-s}} + \beta^{\frac{1}{1-s}}\gamma^{\frac{s}{1-s}}}{\gamma^{\frac{s}{1-s}}\delta^{\frac{s}{1-s}}\left(\left(\frac{\alpha}{\gamma^s}\right)^{\frac{1}{1-s}} + \left(\frac{\beta}{\delta^s}\right)^{\frac{1}{1-s}}\right)^s} \\ &= \frac{\left(\frac{\alpha}{\gamma^s}\right)^{\frac{1}{1-s}} + \left(\frac{\beta}{\delta^s}\right)^{\frac{1}{1-s}}}{\left(\left(\frac{\alpha}{\gamma^s}\right)^{\frac{1}{1-s}} + \left(\frac{\beta}{\delta^s}\right)^{\frac{1}{1-s}}\right)^s}. \end{aligned}$$

□

The value of $\frac{h_v}{h_u}$ can be calculated using Lemma 3.2.2(a), which states that

$$\mathbf{A}(s)(\mathcal{H}^s(F_v))^T_{v\in V} = (\mathcal{H}^s(F_v))^T_{v\in V},$$

so

$$\begin{pmatrix} r^s_{e_1} & r^s_{e_2} \\ r^s_{e_4} & r^s_{e_3} \end{pmatrix}\begin{pmatrix} h_u \\ h_v \end{pmatrix} = \begin{pmatrix} h_u \\ h_v \end{pmatrix} \tag{3.4.2}$$

and this implies

$$\frac{h_v}{h_u} = \frac{1 - r^s_{e_1}}{r^s_{e_2}} = \frac{1 - \frac{a^s}{|I_u|^s}}{\frac{b^s}{|I_v|^s}} = \frac{|I_v|^s - \frac{|I_v|^s}{|I_u|^s}a^s}{b^s}, \tag{3.4.3}$$

using the equations of (3.4.1).

In the next lemma there is a good reason for the choice of functions f_u and f_v. If we were instead to use

$$l_u(x,y) = \frac{x^s + y^s}{(x + g_u + y)^s},$$

and

$$l_v(x,y) = \frac{x^s + y^s}{(x + g_v + y)^s},$$

then, in order to obtain $l_u(a,b), l_v(c,d) \leqslant 1$, we would require $\frac{h_u|I_v|^s}{h_v|I_u|^s} = 1$ and this is a much more restrictive condition than (1). Also it is not obvious how this condition could be checked.

Lemma 3.4.4. *For the 2-vertex directed graph IFS of Figure 3.4.1, let*

$$P = \{(x,y) : 0 \leqslant x \leqslant a,\ 0 \leqslant y \leqslant b\} \setminus \{(a,b)\},$$

$$Q = \{(x,y) : 0 \leqslant x \leqslant c,\ 0 \leqslant y \leqslant d\} \setminus \{(c,d)\},$$

$$f_u(x,y) = \frac{x^s + \frac{h_v}{h_u} y^s}{(x+g_u+y)^s},$$

and

$$f_v(x,y) = \frac{x^s + \frac{h_u}{h_v} y^s}{(x+g_v+y)^s}.$$

Suppose the following three conditions hold,

$$\begin{aligned} &(1) \quad |I_u| = |I_v|\,, \\ &(2) \quad \frac{h_v}{h_u} \leqslant 1, \\ &(3) \quad \frac{(a+g_u)(|I_u|^s - a^s)}{ba^s} \geqslant 1, \end{aligned}$$

then

$$\begin{aligned} &(a) \quad f_u(a,b) = f_v(c,d) = 1, \\ &(b) \quad f_u(x,y) < 1, \textit{for all } (x,y) \in P, \\ &(c) \quad f_v(x,y) < 1, \textit{for all } (x,y) \in Q. \end{aligned}$$

Proof.

$$\begin{aligned} (a) \quad f_u(a,b) &= \frac{a^s + \frac{h_v}{h_u} b^s}{(a+g_u+b)^s} \\ &= \frac{|I_u|^s r_{e_1}^s + \frac{h_v}{h_u} |I_v|^s r_{e_2}^s}{|I_u|^s} && \text{(by (3.4.1))} \\ &= \frac{1}{h_u}\left(r_{e_1}^s h_u + \frac{|I_v|^s}{|I_u|^s} r_{e_2}^s h_v \right) \\ &= \frac{1}{h_u}\left(r_{e_1}^s h_u + r_{e_2}^s h_v \right) && \text{(by (1))} \\ &= 1 && \text{(by (3.4.2))}. \end{aligned}$$

$$\begin{aligned} f_v(c,d) &= \frac{c^s + \frac{h_u}{h_v} d^s}{(c+g_v+d)^s} \\ &= \frac{|I_v|^s r_{e_3}^s + \frac{h_u}{h_v} |I_u|^s r_{e_4}^s}{|I_v|^s} && \text{(by (3.4.1))} \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{h_v}\left(\frac{|I_u|^s}{|I_v|^s} r_{e_4}^s h_u + r_{e_3}^s h_v\right) \\
&= \frac{1}{h_v}\left(r_{e_4}^s h_u + r_{e_3}^s h_v\right) && \text{(by (1))} \\
&= 1 && \text{(by (3.4.2))}.
\end{aligned}$$

In parts (b) and (c), as $f_u(0,0) = f_v(0,0) = 0 < 1$, this point is not considered.

(b) Consider

$$f_u(a,y) = \frac{a^s + \frac{h_v}{h_u} y^s}{(a + g_u + y)^s}.$$

Putting $\alpha = a^s$, $\beta = \frac{h_v}{h_u} = \frac{|I_u|^s - a^s}{b^s}$, by Equation (3.4.3) and (1), $\gamma = a + g_u$ and $\delta = 1$, then Lemma 3.4.3 shows that the maximum value of $f_u(a,y)$ occurs at the point

$$y_{\max} = \left(\frac{\gamma\beta}{\delta\alpha}\right)^{\frac{1}{1-s}} = \left(\frac{(a+g_u)(|I_u|^s - a^s)}{b^s a^s}\right)^{\frac{1}{1-s}},$$

and so

$$\frac{y_{\max}}{b} = \left(\frac{(a+g_u)(|I_u|^s - a^s)}{b a^s}\right)^{\frac{1}{1-s}} \geqslant 1,$$

by (3). It follows that $f_u(a,y) < 1$ for $(a,y) \in P$, since $f_u(a,b) = 1$ and $f_u(a,y)$ strictly increases up to $y = y_{\max} \geqslant b$, by Lemma 3.4.3.

Let $\overline{y}$, $0 < \overline{y} \leqslant b$, be fixed. Consider

$$f_u(x,\overline{y}) = \frac{x^s + \frac{h_v}{h_u}\overline{y}^s}{(x + g_u + \overline{y})^s},$$

which is now just a function of x. Putting $\alpha = \frac{h_v}{h_u}\overline{y}^s$, $\beta = 1$, $\gamma = g_u + \overline{y}$ and $\delta = 1$, Lemma 3.4.3 gives the maximum value of $f_u(x,\overline{y})$ to be

$$\left(\left(\frac{\frac{h_v}{h_u}\overline{y}^s}{(g_u + \overline{y})^s}\right)^{\frac{1}{1-s}} + 1\right)^{1-s} > 1,$$

since $\frac{h_v}{h_u}\overline{y}^s > 0$, and $g_u + \overline{y} > 0$. Further since $\lim_{x\to\infty} f_u(x,\overline{y}) = 1$, and as $f_u(x,\overline{y})$ is strictly decreasing for $x > x_{\max}$, it follows, since $f_u(a,\overline{y}) \leqslant 1$, that $a < x_{\max}$. Hence $f_u(x,\overline{y})$ will be strictly increasing for $0 \leqslant x \leqslant a$, and so $f_u(x,\overline{y}) < 1$ for all $(x,\overline{y}) \in P$.

(c) Consider

$$f_v(c,y) = \frac{c^s + \frac{h_u}{h_v} y^s}{(c + g_v + y)^s}.$$

Putting $\alpha = c^s$, $\beta = \frac{h_u}{h_v} \geqslant 1$ by (2), $\gamma = c + g_v$ and $\delta = 1$ in Lemma 3.4.3, the maximum value of $f_v(c,y)$ will be strictly greater than 1, since $\beta \geqslant 1$, $\delta = 1$ and $\alpha, \gamma > 0$. Also $\lim_{y\to\infty} f_v(c,y) = \frac{h_u}{h_v} \geqslant 1$, by (2) and as $f_v(c,y)$ is strictly decreasing for $y > y_{\max}$ this implies that $d < y_{\max}$ since $f_v(c,d) = 1$. So $f_v(c,y)$ will be strictly increasing for $0 \leqslant y \leqslant d$, hence $f_v(c,y) < 1$ for all $(c,y) \in Q$.

Let $\overline{y}$, $0 < \overline{y} \leqslant d$, be fixed. Consider

$$f_v(x, \overline{y}) = \frac{x^s + \frac{h_u}{h_v}\overline{y}^s}{(x + g_v + \overline{y})^s},$$

which is now just a function of x. Putting $\alpha = \frac{h_u}{h_v}\overline{y}^s$, $\beta = 1$, $\gamma = g_v + \overline{y}$ and $\delta = 1$, Lemma 3.4.3 gives the maximum value of $f_v(x, \overline{y})$ to be strictly greater than 1, since $\beta = \delta = 1$ and $\alpha, \gamma > 0$. Further since $\lim_{x\to\infty} f_v(x, \overline{y}) = 1$, and as $f_v(x, \overline{y})$ is strictly decreasing for $x > x_{\max}$, it follows since $f_v(c, \overline{y}) \leqslant 1$, that $c < x_{\max}$. Hence $f_v(x, \overline{y})$ will be strictly increasing for $0 \leqslant x \leqslant c$, and so $f_v(x, \overline{y}) < 1$ for all $(x, \overline{y}) \in Q$. □

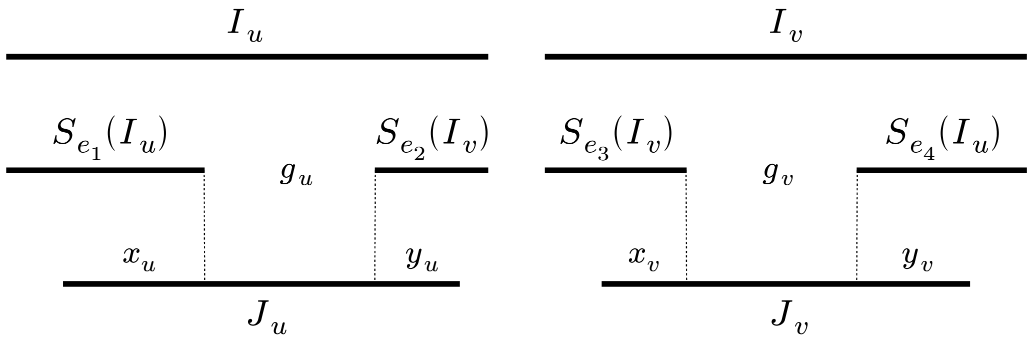


Figure. 3.4.2: The intervals J_u and J_v.

The next two lemmas give important results which we will apply in the proof of Theorem 3.4.7 which follows immediately after.

Lemma 3.4.5. *For the 2-vertex directed graph IFS of Figure 3.4.1, let $J_u = [u_l, u_r] \subset I_u$ be an interval where $u_l \in S_{e_1}(I_u)$ and $u_r \in S_{e_2}(I_v)$, with $d_u(J_u) > 0$. Let $J_v = [v_l, v_r] \subset I_v$ where $v_l \in S_{e_3}(I_v)$ and $v_r \in S_{e_4}(I_u)$, with $d_v(J_v) > 0$. Suppose also that the conditions of Lemma 3.4.4 hold.*

(a) *If $J_u \neq I_u$, then*

$$d_u(J_u) < \max\left\{ d_u(I_u \cap S_{e_1}^{-1}(J_u)),\ d_v(I_v \cap S_{e_2}^{-1}(J_u)) \right\}.$$

(b) *If $J_v \neq I_v$, then*

$$d_v(J_v) < \max\left\{ d_v(I_v \cap S_{e_3}^{-1}(J_v)),\ d_u(I_u \cap S_{e_4}^{-1}(J_v)) \right\}.$$

Proof. Typical examples of the intervals J_u and J_v are illustrated in Figure 3.4.2 where they span the gaps between level-1 intervals. The lengths x_u, y_u, x_v, y_v, also shown in Figure 3.4.2, are defined as

$$x_u = |S_{e_1}(I_u) \cap J_u|\,,\ y_u = |S_{e_2}(I_v) \cap J_u|\,,\ x_v = |S_{e_3}(I_v) \cap J_v|\,,\ y_v = |S_{e_4}(I_u) \cap J_v|\,.$$

As we are assuming $d_u(J_u) > 0$, at least one of x_u or y_u, or both, will be strictly positive, and similarly for x_v and y_v.

(a) $$d_u(J_u) = \frac{\mathcal{H}^s(F_u \cap J_u)}{\mathcal{H}^s(F_u)\,|J_u|^s}$$

$$
\begin{aligned}
&= \frac{\mathcal{H}^s(F_u \cap (S_{e_1}(I_u) \cap J_u)) + \mathcal{H}^s(F_u \cap (S_{e_2}(I_v) \cap J_u))}{\mathcal{H}^s(F_u)\,|J_u|^s} \\
&= \frac{|S_{e_1}(I_u) \cap J_u|^s\, d_u(S_{e_1}(I_u) \cap J_u) + |S_{e_2}(I_v) \cap J_u|^s\, d_u(S_{e_2}(I_v) \cap J_u)}{|J_u|^s} \\
&= \frac{x_u^s d_u(S_{e_1}(I_u) \cap J_u) + y_u^s d_u(S_{e_2}(I_v) \cap J_u)}{(x_u + g_u + y_u)^s}.
\end{aligned}
$$

Applying Lemma 3.4.2(e), (f), and Lemma 3.4.4(b), we obtain,

$$
\begin{aligned}
d_u(J_u) &= \frac{x_u^s d_u(I_u \cap S_{e_1}^{-1}(J_u)) + \frac{h_v}{h_u} y_u^s d_v(I_v \cap S_{e_2}^{-1}(J_u))}{(x_u + g_u + y_u)^s} \\
&= \frac{x_u^s d_u(I_u \cap S_{e_1}^{-1}(J_u)) + \frac{h_v}{h_u} y_u^s d_v(I_v \cap S_{e_2}^{-1}(J_u))}{(x_u + g_u + y_u)^s} \\
&\leqslant \left(\frac{x_u^s + \frac{h_v}{h_u} y_u^s}{(x_u + g_u + y_u)^s} \right) \max\left\{ d_u(I_u \cap S_{e_1}^{-1}(J_u)),\ d_v(I_v \cap S_{e_2}^{-1}(J_u)) \right\} \\
&= f_u(x_u, y_u) \max\left\{ d_u(I_u \cap S_{e_1}^{-1}(J_u)),\ d_v(I_v \cap S_{e_2}^{-1}(J_u)) \right\} \\
&< \max\left\{ d_u(I_u \cap S_{e_1}^{-1}(J_u)),\ d_v(I_v \cap S_{e_2}^{-1}(J_u)) \right\}.
\end{aligned}
$$

(b)
$$
\begin{aligned}
d_v(J_v) &= \frac{\mathcal{H}^s(F_v \cap J_v)}{\mathcal{H}^s(F_v)\,|J_v|^s} \\
&= \frac{\mathcal{H}^s(F_v \cap (S_{e_3}(I_v) \cap J_v)) + \mathcal{H}^s(F_u \cap (S_{e_4}(I_u) \cap J_v))}{\mathcal{H}^s(F_v)\,|J_v|^s} \\
&= \frac{|S_{e_3}(I_v) \cap J_v|^s\, d_v(S_{e_3}(I_v) \cap J_v) + |S_{e_4}(I_u) \cap J_v|^s\, d_v(S_{e_4}(I_u) \cap J_v)}{|J_v|^s} \\
&= \frac{x_v^s d_v(S_{e_3}(I_v) \cap J_v) + y_v^s d_v(S_{e_4}(I_u) \cap J_v)}{(x_v + g_v + y_v)^s}.
\end{aligned}
$$

Applying Lemma 3.4.2(g), (h), and Lemma 3.4.4(c), we obtain,

$$
\begin{aligned}
d_v(J_v) &= \frac{x_v^s d_v(I_v \cap S_{e_3}^{-1}(J_v)) + \frac{h_u}{h_v} y_v^s d_u(I_u \cap S_{e_4}^{-1}(J_v))}{(x_u + g_u + y_u)^s} \\
&= \frac{x_v^s d_v(I_v \cap S_{e_3}^{-1}(J_v)) + \frac{h_u}{h_v} y_v^s d_u(I_u \cap S_{e_4}^{-1}(J_4))}{(x_v + g_v + y_v)^s} \\
&\leqslant \left(\frac{x_v^s + \frac{h_u}{h_v} y_v^s}{(x_v + g_v + y_v)^s} \right) \max\left\{ d_v(I_v \cap S_{e_3}^{-1}(J_v)),\ d_u(I_u \cap S_{e_4}^{-1}(J_v)) \right\} \\
&= f_v(x_v, y_v) \max\left\{ d_v(I_v \cap S_{e_3}^{-1}(J_v)),\ d_u(I_u \cap S_{e_4}^{-1}(J_v)) \right\} \\
&< \max\left\{ d_v(I_v \cap S_{e_3}^{-1}(J_v)),\ d_u(I_u \cap S_{e_4}^{-1}(J_v)) \right\}.
\end{aligned}
$$
□

We now consider $\sup\{d_u(J) : S_{e_1}(I_u) \subset J \subset I_u\}$. As shown in Figure 3.4.1, $I_u = [a_u, b_u]$, and $S_{e_1}(I_u) = [a_u, a_u + a]$, so

$$
\sup\{d_u(J) : S_{e_1}(I_u) \subset J \subset I_u\} = \sup\left\{ \frac{\mathcal{H}^s(F_u \cap [a_u, x])}{\mathcal{H}^s(F_u)(x - a_u)^s} : x \in [a_u + a, b_u] \right\}.
$$

The function $\frac{\mathcal{H}^s(F_u\cap[a_u,x])}{\mathcal{H}^s(F_u)(x-a_u)^s}$ is a continuous function of x on the compact interval $[a_u+a, b_u]$, where $a>0$, so it is bounded and attains its bound for at least one $x_0 \in [a_u+a, b_u]$. For such x_0, we may define an interval $L_u = [a_u, x_0]$, $S_{e_1}(I_u) \subset L_u \subset I_u$, which satisfies,

$$d_u(L_u) = \sup\{d_u(J) : S_{e_1}(I_u) \subset J \subset I_u\}. \tag{3.4.4}$$

Similarly intervals L_v, R_u, R_v, exist which satisfy the following equations,

$$d_v(L_v) = \sup\{d_v(J) : S_{e_3}(I_v) \subset J \subset I_v\}, \tag{3.4.5}$$
$$d_u(R_u) = \sup\{d_u(J) : S_{e_2}(I_v) \subset J \subset I_u\}, \tag{3.4.6}$$
$$d_v(R_v) = \sup\{d_u(J) : S_{e_4}(I_u) \subset J \subset I_v\}. \tag{3.4.7}$$

Some candidates for the intervals L_u, L_v, R_u, R_v are illustrated in Figure 3.4.3.

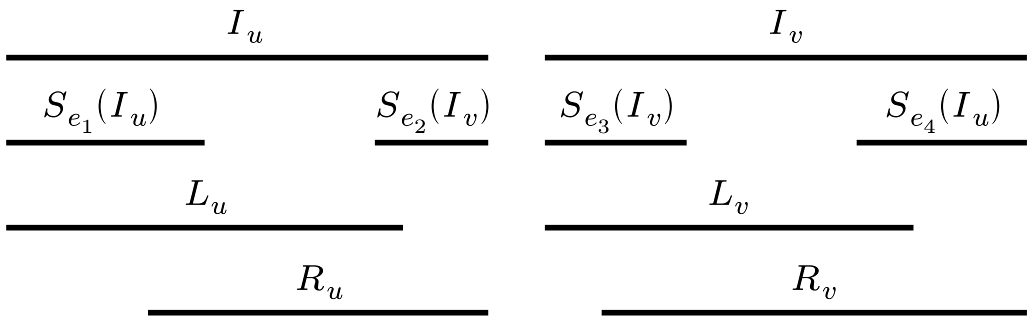


Figure. 3.4.3: Some possibilities for the intervals L_u, L_v, R_u, and R_v.

Lemma 3.4.6. *For the 2-vertex directed graph IFS of Figure 3.4.1, let the intervals L_u, L_v, R_u, and R_v be as defined in Equations (3.4.4), (3.4.5), (3.4.6), and (3.4.7), and suppose the conditions of Lemma 3.4.4 hold.*

Then

$$\text{(a) } d_u(L_u) = \frac{1}{|I_u|^s}, \qquad \text{(b) } d_v(L_v) = \frac{1}{|I_u|^s},$$
$$\text{(c) } d_u(R_u) = \frac{1}{|I_u|^s}, \qquad \text{(d) } d_v(R_v) = \frac{h_u}{h_v}\frac{1}{|I_u|^s}.$$

Proof. (a) As stated in Lemma 3.4.2(a), $d_u(I_u) = d_u(S_{e_1}(I_u)) = \frac{1}{|I_u|^s}$, which implies, from the definition of L_u in Equation (3.4.4), that $d_u(L_u) \geqslant \frac{1}{|I_u|^s}$. For a contradiction we assume $d_u(L_u) > \frac{1}{|I_u|^s}$. Clearly $S_{e_1}(I_u) \subsetneqq L_u \subsetneqq I_u$. Also if the right hand endpoint of the interval L_u were to lie in the gap between the intervals $S_{e_1}(I_u)$ and $S_{e_2}(I_v)$ then $d_u(S_{e_1}(I_u)) > d_u(L_u)$ which contradicts our assumption, so the right hand endpoint of L_u lies in $S_{e_2}(I_v)$. This is the situation illustrated in Figure 3.4.4.

Applying Lemma 3.4.5(a), we obtain

$$d_u(L_u) < d_v(I_v \cap S_{e_2}^{-1}(L_u)),$$

since $d_u(I_u \cap S_{e_1}^{-1}(L_u)) = d_u(I_u) = \frac{1}{|I_u|^s} < d_u(L_u)$. If necessary, by repeatedly applying the expanding similarity $S_{e_3}^{-1}$ to the interval $I_v \cap S_{e_2}^{-1}(L_u)$, we must eventually

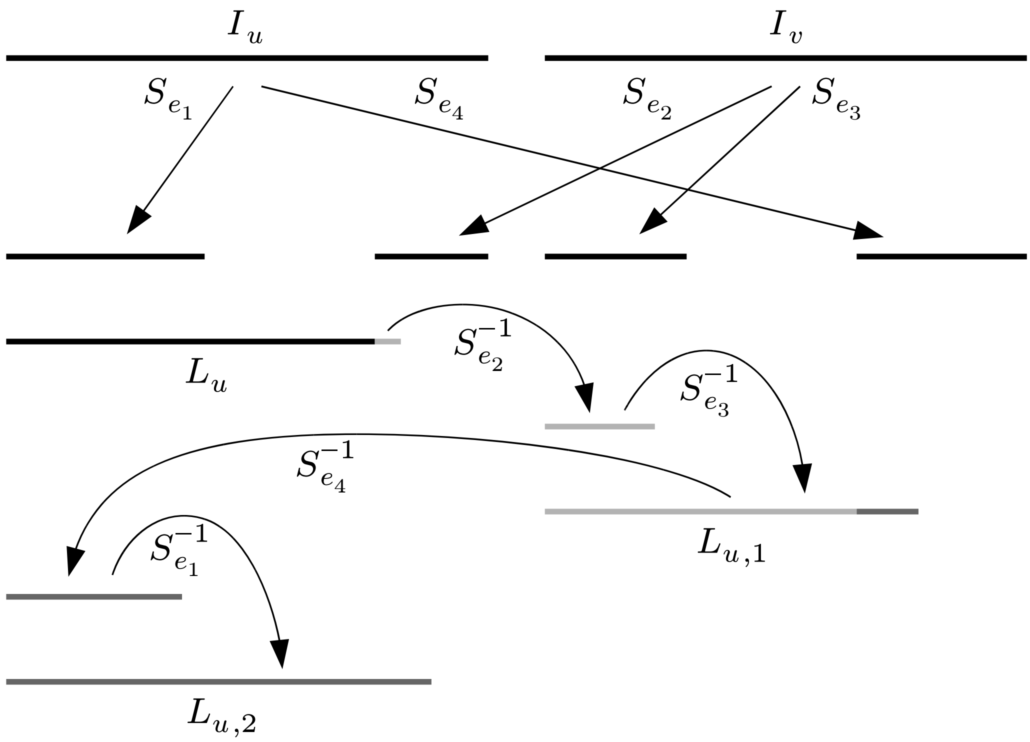


Figure. 3.4.4: The intervals L_u, $L_{u,1}$, and $L_{u,2}$.

arrive at an interval $L_{u,1}$, which is not contained in the interval $S_{e_3}(I_v)$, where $L_{u,1} = S_{e_3}^{-m}(I_v \cap S_{e_2}^{-1}(L_u))$, for some $m \geqslant 0$. By Lemma 3.4.2(g) $d_v(S_{e_3}^{-m}(I_v \cap S_{e_2}^{-1}(L_u))) = d_v(I_v \cap S_{e_2}^{-1}(L_u))$ so

$$d_u(L_u) < d_v(L_{u,1}). \tag{3.4.8}$$

and so $d_v(L_{u,1}) > \frac{1}{|I_u|^s}$. Again the right hand endpoint of $L_{u,1}$ cannot lie in the gap between the intervals $S_{e_3}(I_v)$ and $S_{e_4}(I_u)$ for then, $d_v(S_{e_3}(I_v)) > d_v(L_{u,1}) > \frac{1}{|I_u|^s}$. This is impossible because $d_v(S_{e_3}(I_v)) = \frac{1}{|I_u|^s}$, by Lemma 3.4.2(c) and condition (1) of Lemma 3.4.4. Similarly since $d_v(I_v) = \frac{1}{|I_u|^s}$, again by Lemma 3.4.2(c) and condition (1) of Lemma 3.4.4, we cannot have $L_{u,1} = I_v$. Therefore $S_{e_3}(I_v) \subsetneqq L_{u,1} \subsetneqq I_v$. The situation is shown in Figure 3.4.4 for $m = 1$.

Now we may apply Lemma 3.4.5(b), to obtain

$$d_v(L_{u,1}) < d_u(I_u \cap S_{e_4}^{-1}(L_{u,1})),$$

as $d_v(I_v \cap S_{e_3}^{-1}(L_{u,1})) = d_v(I_v) = \frac{1}{|I_u|^s} < d_v(L_{u,1})$. If necessary, by repeatedly applying the expanding similarity $S_{e_1}^{-1}$ to the interval $I_u \cap S_{e_4}^{-1}(L_{u,1})$, we must eventually arrive at an interval $L_{u,2}$, with $S_{e_1}(I_u) \subset L_{u,2}$, where $L_{u,2} = S_{e_1}^{-n}(I_u \cap S_{e_4}^{-1}(L_{u,1}))$, for some $n \geqslant 0$. The situation is illustrated in Figure 3.4.4 for $n = 1$. By Lemma 3.4.2(e) $d_u(L_{u,2}) = d_u(I_u \cap S_{e_4}^{-1}(L_{u,1}))$ so

$$d_v(L_{u,1}) < d_u(L_{u,2}). \tag{3.4.9}$$

From the definition of the interval L_u in Equation (3.4.4), $d_u(L_{u,2}) \leqslant d_u(L_u)$, which together with Equations (3.4.8) and (3.4.9) gives

$$d_u(L_u) < d_v(L_{u,1}) < d_u(L_{u,2}) \leqslant d_u(L_u).$$

This contradiction completes the proof of part (a).

(b) The proof is symmetrically identical to that given in part (a).

We have given a fully detailed proof of part (a), so now for clarity we no longer include all the details, such as endpoints of intervals not lying in gaps etc., where the inclusion of such details would be repetitive.

(c) From Lemma 3.4.2(a), (b), $d_u(I_u) = \frac{1}{|I_u|^s}$, $d_u(S_{e_2}(I_v)) = \frac{h_v}{h_u}\frac{1}{|I_v|^s}$ and by conditions (1) and (2) of Lemma 3.4.4, it follows that $d_u(S_{e_2}(I_v)) = \frac{h_v}{h_u}\frac{1}{|I_u|^s} \leqslant \frac{1}{|I_u|^s}$, which means it must be the case that $d_u(R_u) \geqslant \frac{1}{|I_u|^s}$. For a contradiction we assume that $d_u(R_u) > \frac{1}{|I_u|^s}$. This immediately implies $S_{e_2}(I_v) \subsetneqq R_u \subsetneqq I_u$. Lemma 3.4.5(a) gives

$$d_u(R_u) < d_u(I_u \cap S_{e_1}^{-1}(R_u)).$$

We now expand the interval $I_u \cap S_{e_1}^{-1}(R_u)$ under the action of the expanding similarities $S_{e_2}^{-1}$ and $S_{e_4}^{-1}$, if necessary, until we obtain an interval $R_{u,1}$ which is not contained in either $S_{e_2}(I_v)$ or $S_{e_4}(I_u)$. There are just two possibilities,

$$\text{(i)} \quad S_{e_2}(I_v) \subset R_{u,1} = \big((S_{e_4}^{-1} \circ S_{e_2}^{-1})^m\big)\big(I_u \cap S_{e_1}^{-1}(R_u)\big) \subset I_u,$$
$$\text{(ii)} \quad S_{e_4}(I_u) \subset R_{u,1} = \big(S_{e_2}^{-1} \circ (S_{e_4}^{-1} \circ S_{e_2}^{-1})^n\big)\big(I_u \cap S_{e_1}^{-1}(R_u)\big) \subset I_v,$$

for some $m, n \geqslant 0$. For an interval $J \subset (S_{e_2} \circ S_{e_4})(I_u)$, we note that by Lemma 3.4.2(h), (f),

$$d_u((S_{e_4}^{-1} \circ S_{e_2}^{-1})(J)) = \frac{h_v}{h_u} d_v(S_{e_2}^{-1}(J)) = \frac{h_v}{h_u}\frac{h_u}{h_v} d_u(J),$$

that is

$$d_u((S_{e_4}^{-1} \circ S_{e_2}^{-1})(J)) = d_u(J). \tag{3.4.10}$$

For (i), Equation (3.4.10) implies that

$$d_u(R_u) < d_u(I_u \cap S_{e_1}^{-1}(R_u)) = d_u(R_{u,1}) \leqslant d_u(R_u),$$

so only (ii) can hold. Again by Equation (3.4.10) and Lemma 3.4.2(f), we see that for (ii),

$$\frac{h_v}{h_u} d_v(R_{u,1}) = d_u(I_u \cap S_{e_1}^{-1}(R_u)),$$

and this means that

$$d_u(R_u) < d_u(I_u \cap S_{e_1}^{-1}(R_u)) = \frac{h_v}{h_u} d_v(R_{u,1}) \leqslant \frac{h_v}{h_u} d_v(R_v),$$

from the definition of R_v in Equation (3.4.7).

So we have shown that if $d_u(R_u) > \frac{1}{|I_u|^s}$ then

$$d_u(R_u) < \frac{h_v}{h_u} d_v(R_v). \tag{3.4.11}$$

We pause now in our proof of part (c), which we will continue, after first giving a proof of part (d).

(d) From Lemma 3.4.2(c), (d), $d_v(I_v) = \frac{1}{|I_v|^s}$, $d_v(S_{e_4}(I_u)) = \frac{h_u}{h_v}\frac{1}{|I_u|^s}$ and by conditions (1) and (2) of Lemma 3.4.4, it follows that $d_v(I_v) = \frac{1}{|I_u|^s}$ and $d_v(S_{e_4}(I_u)) = \frac{h_u}{h_v}\frac{1}{|I_u|^s} \geqslant \frac{1}{|I_u|^s}$, which means it must be the case that $d_v(R_v) \geqslant \frac{h_u}{h_v}\frac{1}{|I_u|^s}$. For a contradiction we assume that $d_v(R_v) > \frac{h_u}{h_v}\frac{1}{|I_u|^s}$. This immediately implies $S_{e_4}(I_u) \subsetneqq R_v \subsetneqq I_v$. Lemma 3.4.5(b) gives

$$d_v(R_v) < d_v(I_v \cap S_{e_3}^{-1}(R_v)).$$

We now expand the interval $I_v \cap S_{e_3}^{-1}(R_v)$ under the action of the expanding similarities $S_{e_4}^{-1}$ and $S_{e_2}^{-1}$, if necessary, until we obtain an interval $R_{v,1}$ which is not contained in either $S_{e_4}(I_u)$ or $S_{e_2}(I_v)$. There are just two possibilities

$$\begin{aligned} &\text{(i)} \quad S_{e_4}(I_u) \subset R_{v,1} = \big((S_{e_2}^{-1} \circ S_{e_4}^{-1})^i\big)\big(I_v \cap S_{e_3}^{-1}(R_v)\big) \subset I_v, \\ &\text{(ii)} \quad S_{e_2}(I_v) \subset R_{v,1} = \big(S_{e_4}^{-1} \circ (S_{e_2}^{-1} \circ S_{e_4}^{-1})^j\big)\big(I_v \cap S_{e_3}^{-1}(R_v)\big) \subset I_u, \end{aligned}$$

for some $i, j \geqslant 0$. For an interval $J \subset (S_{e_4} \circ S_{e_2})(I_v)$, we note that by Lemma 3.4.2(f), (h),

$$d_v((S_{e_2}^{-1} \circ S_{e_4}^{-1})(J)) = \frac{h_u}{h_v} d_u(S_{e_4}^{-1}(J)) = \frac{h_u}{h_v}\frac{h_v}{h_u} d_v(J),$$

that is

$$d_v((S_{e_2}^{-1} \circ S_{e_4}^{-1})(J)) = d_v(J). \tag{3.4.12}$$

For (i), Equation (3.4.12) implies that

$$d_v(R_v) < d_v(I_v \cap S_{e_3}^{-1}(R_v)) = d_v(R_{v,1}) \leqslant d_v(R_v),$$

so only (ii) can hold. Again by Equation (3.4.12) and Lemma 3.4.2(h), we see that for (ii),

$$\frac{h_u}{h_v} d_u(R_{v,1}) = d_v(I_v \cap S_{e_3}^{-1}(R_v)),$$

and this means that

$$d_v(R_v) < d_v(I_v \cap S_{e_3}^{-1}(R_v)) = \frac{h_u}{h_v} d_u(R_{v,1}) \leqslant \frac{h_u}{h_v} d_u(R_u), \tag{3.4.13}$$

from the definition of R_u in Equation (3.4.6). Now our assumption is that $d_v(R_v) > \frac{h_u}{h_v}\frac{1}{|I_u|^s}$, and Equation (3.4.13) implies that $d_u(R_u) > \frac{1}{|I_u|^s}$. This means that the argument already given in part (c) is valid, because in part (c), the assumption was made that $d_u(R_u) > \frac{1}{|I_u|^s}$. Using Equations (3.4.11) and (3.4.13) we obtain

$$d_v(R_v) < \frac{h_u}{h_v} d_u(R_u) < \frac{h_u}{h_v}\frac{h_v}{h_u} d_v(R_v) = d_v(R_v).$$

This contradiction completes the proof of part (d).

(c) (continued) Equation (3.4.11) and part (d) now combine to give

$$d_u(R_u) < \frac{h_v}{h_u} d_v(R_v) = \frac{h_v}{h_u}\frac{h_u}{h_v}\frac{1}{|I_u|^s} = \frac{1}{|I_u|^s}.$$

This contradiction completes the proof of part (c). □

The next theorem gives sufficient conditions for the calculation of the Hausdorff measure of both of the attractors of the 2-vertex IFS of Figure 3.4.1.

Theorem 3.4.7. *For the 2-vertex directed graph IFS of Figure 3.4.1, where* $s = \dim_{\mathrm{H}} F_u = \dim_{\mathrm{H}} F_v$*, suppose that the following conditions hold,*

$$\begin{aligned} &(1) \quad |I_u| = |I_v|\,, \\ &(2) \quad \frac{h_v}{h_u} \leqslant 1, \\ &(3) \quad \frac{(a+g_u)(|I_u|^s - a^s)}{ba^s} \geqslant 1. \end{aligned}$$

Then

$$\mathcal{H}^s(F_u) = |I_u|^s \quad \textit{and} \quad \mathcal{H}^s(F_v) = |I_u|^s \left(\frac{1 - r_{e_1}^s}{r_{e_2}^s} \right).$$

Proof. For any interval $J \subset I_u$, with $d_u(J) > 0$, we aim to show that $d_u(J) \leqslant \frac{1}{|I_u|^s}$, then, by Lemma 3.4.1, the maximum density will satisfy

$$\sup\big\{ d_u(J) : J \subset I_u \big\} = \frac{1}{\mathcal{H}^s(F_u)} = \frac{1}{|I_u|^s}.$$

By Lemma 3.4.2(e), for any interval $J \subset I_u$, $d_u(J) = d_u(S_{e_1}^{-1}(S_{e_1}(J))) = d_u(S_{e_1}(J))$, so it is enough to prove $d_u(J) \leqslant \frac{1}{|I_u|^s}$ for any interval J contained in a first level interval.

Let $J \subset I_u$ be any interval contained in one of the level-1 intervals $S_{e_1}(I_u)$ or $S_{e_2}(I_v)$ with $d_u(J) > 0$. Operating on J with the expanding similarities $S_{e_1}^{-1}$, $S_{e_2}^{-1}$, $S_{e_3}^{-1}$, $S_{e_4}^{-1}$, as necessary, we must eventually arrive at an interval $J_u \subset I_u$ or $J_v \subset I_v$, which is not contained in any level-1 interval.

For the moment we make the assumption that the endpoints of J_u and J_v lie in level-1 intervals so that they span the gaps between the level-1 intervals as illustrated in Figure 3.4.2.

For $J_u \subset I_u$ the maps $S_{e_2}^{-1}$ and $S_{e_4}^{-1}$ must be applied an equal number of times to the interval J, and so the scaling factors of $\frac{h_v}{h_u}$ and $\frac{h_u}{h_v}$ in Lemma 3.4.2(f) and (h) will cancel each other out. This means, by Lemma 3.4.2(e), (f), (g) and (h), that

$$d_u(J) = d_u(J_u). \tag{3.4.14}$$

If $J_u = I_u$, then $d_u(J_u) = \frac{1}{|I_u|^s}$, by Lemma 3.4.2(a), so we may assume $J_u \neq I_u$. Applying Lemma 3.4.5(a) gives

$$d_u(J) = d_u(J_u) < \max\big\{\, d_u(I_u \cap S_{e_1}^{-1}(J_u)),\; d_v(I_v \cap S_{e_2}^{-1}(J_u)) \,\big\}\,. \tag{3.4.15}$$

For $J_v \subset I_v$, the map $S_{e_2}^{-1}$ must have been applied exactly one more time to the interval J than the map $S_{e_4}^{-1}$, so a factor of $\frac{h_v}{h_u}$ will occur by Lemma 3.4.2(f). This means, by Lemma 3.4.2(e), (f), (g) and (h), that

$$d_u(J) = \frac{h_v}{h_u} d_v(J_v). \tag{3.4.16}$$

If $J_v = I_v$, then $\frac{h_v}{h_u} d_v(J_v) = \frac{h_v}{h_u}\frac{1}{|I_u|^s} \leqslant \frac{1}{|I_u|^s}$, by Lemma 3.4.2(c) and condition (2), so we may assume $J_v \neq I_v$. Applying Lemma 3.4.5(b) gives

$$d_u(J) = \frac{h_v}{h_u} d_v(J_v) < \frac{h_v}{h_u} \max\left\{\, d_v(I_v \cap S_{e_3}^{-1}(J_v)),\ d_u(I_u \cap S_{e_4}^{-1}(J_v))\,\right\}. \qquad (3.4.17)$$

We now determine upper bounds for the densities (a) $d_u(I_u \cap S_{e_1}^{-1}(J_u))$, (b) $d_v(I_v \cap S_{e_2}^{-1}(J_u))$, (c) $d_v(I_v \cap S_{e_3}^{-1}(J_v))$, and (d) $d_u(I_u \cap S_{e_4}^{-1}(J_v))$, considering each in turn.

(a) $d_u(I_u \cap S_{e_1}^{-1}(J_u)) \leqslant \frac{1}{|I_u|^s}$.

Expanding the interval $I_u \cap S_{e_1}^{-1}(J_u)$, if necessary, we obtain an interval $J_{u,1}$, not contained in any level-1 interval, where one of the following two possibilities hold,

$$\begin{aligned}
&\text{(i)} \quad S_{e_2}(I_v) \subset J_{u,1} = \big((S_{e_4}^{-1} \circ S_{e_2}^{-1})^m\big)\big(I_u \cap S_{e_1}^{-1}(J_u)\big) \subset I_u,\\
&\text{(ii)} \quad S_{e_4}(I_u) \subset J_{u,1} = \big(S_{e_2}^{-1} \circ (S_{e_4}^{-1} \circ S_{e_2}^{-1})^n\big)\big(I_u \cap S_{e_1}^{-1}(J_u)\big) \subset I_v,
\end{aligned}$$

for $m, n \geqslant 0$. For (i), using Lemma 3.4.2(f) and (h), and Lemma 3.4.6(c), we obtain,

$$d_u(I_u \cap S_{e_1}^{-1}(J_u)) = d_u(J_{u,1}) \leqslant d_u(R_u) = \frac{1}{|I_u|^s}.$$

For (ii), using Lemma 3.4.2(f) and (h), and Lemma 3.4.6(d), we obtain,

$$d_u(I_u \cap S_{e_1}^{-1}(J_u)) = \frac{h_v}{h_u} d_v(J_{u,1}) \leqslant \frac{h_v}{h_u} d_v(R_v) = \frac{h_v}{h_u}\frac{h_u}{h_v}\frac{1}{|I_u|^s} = \frac{1}{|I_u|^s},$$

In both cases

$$d_u(I_u \cap S_{e_1}^{-1}(J_u)) \leqslant \frac{1}{|I_u|^s}.$$

(b) $d_v(I_v \cap S_{e_2}^{-1}(J_u)) \leqslant \frac{1}{|I_u|^s}$.

Expanding the interval $I_v \cap S_{e_2}^{-1}(J_u)$, if necessary, we obtain an interval $J_{u,1} \subset I_v$, not contained in any level-1 interval, with

$$S_{e_3}(I_v) \subset J_{u,1} = \big((S_{e_3}^{-1})^m\big)\big(I_v \cap S_{e_2}^{-1}(J_u)\big) \subset I_v,$$

for $m \geqslant 0$. By Lemma 3.4.2(g) and Lemma 3.4.6(b),

$$d_v(I_v \cap S_{e_2}^{-1}(J_u)) = d_v(J_{u,1}) \leqslant d_v(L_v) = \frac{1}{|I_u|^s}.$$

(c) $d_v(I_v \cap S_{e_3}^{-1}(J_v)) \leqslant \frac{h_u}{h_v}\frac{1}{|I_u|^s}$.

Expanding the interval $I_v \cap S_{e_3}^{-1}(J_v)$, if necessary, we obtain an interval $J_{v,1}$, not contained in any level-1 interval, where one of the following two possibilities hold,

$$\begin{aligned}
&\text{(i)} \quad S_{e_4}(I_u) \subset J_{v,1} = \big((S_{e_2}^{-1} \circ S_{e_4}^{-1})^m\big)\big(I_v \cap S_{e_3}^{-1}(J_v)\big) \subset I_v,\\
&\text{(ii)} \quad S_{e_2}(I_v) \subset J_{v,1} = \big(S_{e_4}^{-1} \circ (S_{e_2}^{-1} \circ S_{e_4}^{-1})^m\big)\big(I_v \cap S_{e_3}^{-1}(J_v)\big) \subset I_u,
\end{aligned}$$

for $m, n \geqslant 0$. For (i), using Lemma 3.4.2(f) and (h), and Lemma 3.4.6(d), we obtain,

$$d_v(I_v \cap S_{e_3}^{-1}(J_v)) = d_v(J_{v,1}) \leqslant d_v(R_v) = \frac{h_u}{h_v}\frac{1}{|I_u|^s}.$$

For (ii), using Lemma 3.4.2(f) and (h), and Lemma 3.4.6(c), we obtain,

$$d_v(I_v \cap S_{e_3}^{-1}(J_v)) = \frac{h_u}{h_v} d_u(J_{v,1}) \leqslant \frac{h_u}{h_v} d_u(R_u) = \frac{h_u}{h_v} \frac{1}{|I_u|^s}.$$

In both cases

$$d_v(I_v \cap S_{e_3}^{-1}(J_v)) \leqslant \frac{h_u}{h_v} \frac{1}{|I_u|^s}.$$

(d) $d_u(I_u \cap S_{e_4}^{-1}(J_v)) \leqslant \frac{1}{|I_u|^s}$.

Expanding the interval $I_u \cap S_{e_4}^{-1}(J_v)$, if necessary, we obtain an interval $J_{v,1} \subset I_u$, not contained in any level-1 interval, with

$$S_{e_1}(I_u) \subset J_{v,1} = \big((S_{e_1}^{-1})^m\big)\big(I_u \cap S_{e_4}^{-1}(J_v)\big) \subset I_u,$$

for $m \geqslant 0$. By Lemma 3.4.2(e) and Lemma 3.4.6(a),

$$d_u(I_u \cap S_{e_4}^{-1}(J_v)) = d_u(J_{v,1}) \leqslant d_u(L_u) = \frac{1}{|I_u|^s}.$$

This completes the proof of parts (a), (b), (c) and (d). Putting the results of parts (a) and (b) into Equation (3.4.15) gives

$$d_u(J) = d_u(J_u) < \max\left\{ d_u(I_u \cap S_{e_1}^{-1}(J_u)),\ d_v(I_v \cap S_{e_2}^{-1}(J_u)) \right\} \leqslant \frac{1}{|I_u|^s}.$$

Putting the results of parts (c) and (d) into Equation (3.4.17), remembering that by condition (2), $\frac{h_u}{h_v} \geqslant 1$, gives

$$\begin{aligned} d_u(J) = \frac{h_v}{h_u} d_v(J_v) &< \frac{h_v}{h_u} \max\left\{ d_v(I_v \cap S_{e_3}^{-1}(J_v)),\ d_u(I_u \cap S_{e_4}^{-1}(J_v)) \right\} \\ &\leqslant \frac{h_v}{h_u}\frac{h_u}{h_v}\frac{1}{|I_u|^s} = \frac{1}{|I_u|^s}. \end{aligned}$$

It only remains now to consider those situations where J is expanded into an interval $J_u \subset I_u$ or $J_v \subset I_v$ but where one of the endpoints of J_u or J_v lies in a gap between level-1 intervals. The situation with both endpoints of J_u and J_v being in gaps cannot happen as we are assuming $d_u(J) > 0$.

Suppose the right-hand endpoint of J_u lies in the gap between $S_{e_1}(I_u)$ and $S_{e_2}(I_v)$ then by Equation (3.4.14), Lemma 3.4.2(e) and part (a)

$$d_u(J) = d_u(J_u) < d_u(S_{e_1}(I_u) \cap J_u) = d_u(I_u \cap S_{e_1}^{-1}(J_u)) \leqslant \frac{1}{|I_u|^s}.$$

If the left-hand endpoint of J_u lies in the gap between $S_{e_1}(I_u)$ and $S_{e_2}(I_v)$ then by Equation (3.4.14), Lemma 3.4.2(f), part (b) and (2)

$$d_u(J) = d_u(J_u) < d_u(S_{e_2}(I_v) \cap J_u) = \frac{h_v}{h_u} d_v(I_v \cap S_{e_2}^{-1}(J_u)) \leqslant \frac{1}{|I_u|^s}.$$

If the right-hand endpoint of J_v lies in the gap between $S_{e_3}(I_v)$ and $S_{e_4}(I_u)$ then by Equation (3.4.16), Lemma 3.4.2(g) and part (c)

$$d_u(J) = \frac{h_v}{h_u} d_v(J_v) < \frac{h_v}{h_u} d_v(S_{e_3}(I_v) \cap J_v) = \frac{h_v}{h_u} d_v(I_v \cap S_{e_3}^{-1}(J_v)) \leqslant \frac{h_v}{h_u}\frac{h_u}{h_v}\frac{1}{|I_u|^s} = \frac{1}{|I_u|^s}.$$

Finally if the left-hand endpoint of J_v lies in the gap between $S_{e_3}(I_v)$ and $S_{e_4}(I_u)$ then by Equation (3.4.16), Lemma 3.4.2(h), and part (d)

$$d_u(J) = \frac{h_v}{h_u} d_v(J_v) < \frac{h_v}{h_u} d_v(S_{e_4}(I_u) \cap J_v) = \frac{h_v}{h_u}\frac{h_u}{h_v} d_u(I_u \cap S_{e_4}^{-1}(J_v)) \leqslant \frac{1}{|I_u|^s}.$$

Therefore in all cases

$$d_u(J) \leqslant \frac{1}{|I_u|^s}.$$

This completes the proof that $\mathcal{H}^s(F_u) = |I_u|^s$. The expression for $\mathcal{H}^s(F_v)$ now follows immediately by Equation (3.4.3). $\square$

```
> restart;
> a:=11/23: g_u:=5/23: b:=7/23:  c:=1/73: g_v:=71/73: d:=1/73:
> r_e_1:=a/(a+g_u+b): r_e_2:=b/(c+g_v+d): r_e_3:=c/(c+g_v+d):
  r_e_4:=d/(a+g_u+b):
> f:= t -> (r_e_1^t-1)*(r_e_3^t-1)-r_e_2^t*r_e_4^t:
> eqn := f(t)=0: s:=fsolve( eqn, t, 0..1 );
                              s := 0.3301952122
> evalf((1-r_e_1^s)/r_e_2^s);
                                0.3201592325
> evalf((a+g_u)*((a+g_u+b)^s-a^s)/(b*a^s));
                                0.6303364488
```

Figure. 3.4.5: Maple output 1.

We now give a few examples to show that it is a routine matter to check whether or not the three conditions of Theorem 3.4.7 all hold. In the calculations that follow we use Maple to compute the value of s for which the matrix $\mathbf{A}(s)$ has eigenvalue 1. In each of the Maple outputs this value of s can be seen to be unique for s in the range $0 \leqslant s \leqslant 1$, and so this is the Hausdorff dimension, see Theorem 6.9.6, [Edg00].

Suppose we are given the following parameters for the system,

$$a = \frac{11}{23},\ g_u = \frac{5}{23},\ b = \frac{7}{23},\ c = \frac{1}{73},\ g_v = \frac{71}{73}, d = \frac{1}{73}. \tag{3.4.18}$$

Here $|I_u| = |I_v| = 1$, so condition (1) is satisfied. As the Maple output in Figure 3.4.5 shows, $s = 0.3301952122$, and

$$\frac{h_v}{h_u} = \frac{1 - r_{e_1}^s}{r_{e_2}^s} = 0.3201592325 < 1,$$

```
> restart;
> a:=11/23: g_u:=5/23: b:=7/23:  c:=13/73: g_v:=53/73: d:=7/73
> r_e_1:=a/(a+g_u+b): r_e_2:=b/(c+g_v+d): r_e_3:=c/(c+g_v+d):
  r_e_4:=d/(a+g_u+b):
> f:= t -> (r_e_1^t-1)*(r_e_3^t-1)-r_e_2^t*r_e_4^t:
> eqn := f(t)=0: s:=fsolve( eqn, t, 0..1 );
                          s := 0.4934118279
> evalf((1-r_e_1^s)/r_e_2^s);
                           0.5486642748
> evalf((a+g_u)*((a+g_u+b)^s-a^s)/(b*a^s));
                           1.003400992
```

Figure. 3.4.6: Maple output 2.

so condition (2) is also satisfied. However condition (3) of Theorem 3.4.7 does not hold, as

$$\frac{(a+g_u)(|I_u|^s - a^s)}{ba^s} = 0.6303364488 < 1.$$

Adjusting the parameters c, g_v, and d to

$$a = \frac{11}{23},\ g_u = \frac{5}{23},\ b = \frac{7}{23},\ c = \frac{13}{73},\ g_v = \frac{53}{73}, d = \frac{7}{73},$$

and recalculating, we find that all three conditions of Lemma 3.4.4 now hold, as the Maple output of Figure 3.4.6 confirms. The Hausdorff dimension has increased to $s = 0.4934118279$. As before $|I_u| = |I_v| = 1$, so (1) is satisfied.

$$\frac{h_v}{h_u} = \frac{1 - r_{e_1}^s}{r_{e_2}^s} = 0.5486642748 < 1,$$

so condition (2) also holds, and condition (3) is now satisfied because

$$\frac{(a+g_u)(|I_u|^s - a^s)}{ba^s} = 1.003400992 > 1.$$

Finally Figure 3.4.7 shows a plot of $\frac{(a+g_u)(|I_u|^s-a^s)}{ba^s}$ as the dimension s varies from 0 to 1. Here the parameters a, g_u, and b, as given in the equations of (3.4.18) are kept fixed, so we consider s to vary as the parameters c, g_v, and d are adjusted. It is clear from the graph that once $\frac{(a+g_u)(|I_u|^s-a^s)}{ba^s} \geqslant 1$, it will remain that way for greater dimensions.

It is important to note though, that increasing the dimension of the system, by adjusting only the parameters c, g_v, and d, eventually results in condition (2) failing. To see this consider the following parameters

$$a = \frac{11}{23},\ g_u = \frac{5}{23},\ b = \frac{7}{23},\ c = \frac{43}{73},\ g_v = \frac{7}{73}, d = \frac{23}{73},$$

which give a Hausdorff dimension of $s = 0.7990855723$. For these parameters, conditions (1) and (3) hold but (2) fails because $\frac{h_v}{h_u} = 1.152194154$.

Overall, however, this brief analysis does confirm that conditions (1), (2), and (3) will hold for a wide range of values of the parameters of the system.

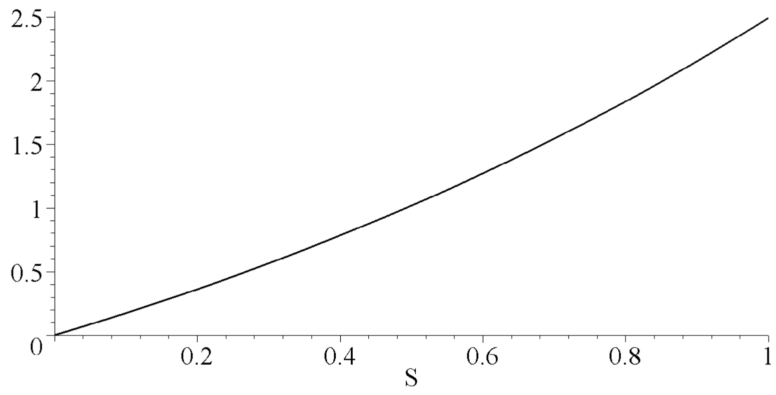


Figure. 3.4.7: A plot of $\frac{(a+g_u)(|I_u|^s - a^s)}{ba^s}$.

3.5 Some important consequences of $\mathcal{H}^s(F_u) = |I_u|^s$

Theorem 3.4.7 of the preceding section establishes that for a large class of 2-vertex systems, of the type illustrated in Figure 3.4.1, we can ensure by a suitable choice of parameters that $\mathcal{H}^s(F_u) = |I_u|^s$. Before proving Theorems 3.5.8 and 3.5.9 we first give some important consequences of $\mathcal{H}^s(F_u) = |I_u|^s$ in Lemmas 3.5.1 and 3.5.7. The arguments we use are based on those employed by Feng and Wang in [FW09].

To illustrate the significance of Lemma 3.5.1, consider the 1-vertex directed graph IFS defined on $\mathbb{R}$ and shown in Figure 3.5.1, where the similarities are

$$S_1(x) = \frac{1}{3}x, \quad S_2(x) = \frac{1}{27}x + \frac{4}{27}, \quad S_3(x) = \frac{1}{3}x + \frac{2}{3}.$$

This is a modification of the Cantor set, C, which is the attractor of the IFS defined by S_1 and S_3, and for which $\mathcal{H}^s(C) = 1$. The attractor F is the unique non-empty compact set satisfying

$$F = \bigcup_{i=1}^{3} S_i(F).$$

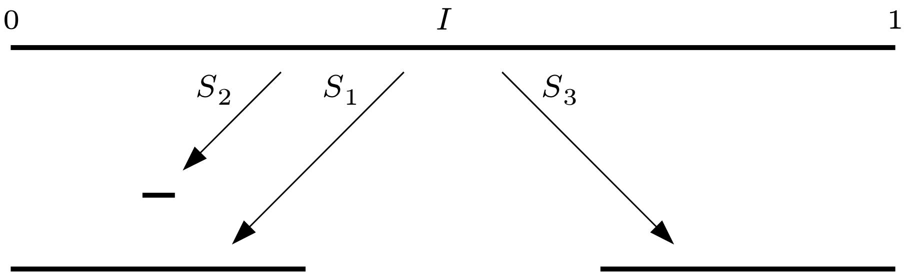


Figure. 3.5.1: A 1-vertex directed graph IFS in $\mathbb{R}$.

The OSC is satisfied for this system, this can be verified by taking the open set as $U = \left(0, \frac{1}{3}\right) \cup \left(\frac{2}{3}, 1\right)$, but the COSC does not hold. Actually the SSC is satisfied but the CSSC is not. In particular $S_2(I) \subset S_1(I)$ but $S_1(F) \cap S_2(F) = \emptyset$. So $S_2(I) \subset S_1(I)$ does not imply $S_2(F) \subset S_1(F)$. In fact

$$S_1(F) \subsetneqq F \cap S_1(I), \tag{3.5.1}$$

since $F \cap S_1(I) = S_1(F) \cup S_2(F)$. Equation (3.5.1) implies that $\mathcal{H}^s(F) \neq |I|^s$, as is shown in our next lemma, and so $\mathcal{H}^s(F) < 1$, by Lemmas 3.3.1 and 3.3.3.

Lemma 3.5.1. *Let $\left(V, E^*, i, t, r, (\mathbb{R}, |\ \ |)_{v\in V}, (S_e)_{e\in E^1}\right)$ be any directed graph IFS for which the OSC holds. For the attractor F_u at a vertex u, let $s = \dim_{\mathrm{H}} F_u$ and $\{a_u, b_u\} \subset F_u \subset I_u = [a_u, b_u]$. Let $S : \mathbb{R} \to \mathbb{R}$ be any similarity with contracting similarity ratio r_S, $0 < r_S < 1$, and let $S(I_u) = [S(a_u), S(b_u)] = [a_S, b_S]$.*

If $S(F_u) \subset F_u$ and $\mathcal{H}^s(F_u) = |I_u|^s$ then

$$\begin{aligned} &\text{(a)} \quad \mathcal{H}^s(S(F_u)) = \mathcal{H}^s(F_u \cap S(I_u)) = (b_S - a_S)^s, \\ &\text{(b)} \quad S(F_u) = F_u \cap S(I_u). \end{aligned}$$

Proof. (a) Clearly $S(F_u) \subset F_u \cap S(I_u)$, so

$$\begin{aligned} (b_S - a_S)^s &\geqslant \mathcal{H}^s(F_u \cap [a_S, b_S]) && \text{(by Corollary 3.3.4(a))} \\ &= \mathcal{H}^s(F_u \cap S(I_u)) \\ &\geqslant \mathcal{H}^s(S(F_u)) \\ &= r_S^s \mathcal{H}^s(F_u) && \text{(by the scaling property of the measure)} \\ &= \frac{(b_S - a_S)^s}{|I_u|^s} \mathcal{H}^s(F_u) \\ &= (b_S - a_S)^s && \text{(as } \mathcal{H}^s(F_u) = |I_u|^s\text{)}. \end{aligned}$$

(b) As $S(F_u) \subset F_u \cap S(I_u)$, we assume for a contradiction that $S(F_u) \subsetneqq F_u \cap S(I_u)$, so there exists a point $x \in F_u \cap S(I_u)$, such that $x \notin S(F_u)$. As $S(F_u)$ is compact, $\mathrm{dist}(x, S(F_u)) > 0$. The map, $\phi_u : E_u^{\mathbb{N}} \to F_u$, given by

$$\phi_u(\mathbf{e}) = y, \ \{y\} = \bigcap_{k=1}^{\infty} S_{\mathbf{e}|_k}(F_{t(\mathbf{e}|_k)})$$

is surjective, by Lemma 1.3.5, so there exists an infinite path in the directed graph, $\mathbf{e} \in E_u^{\mathbb{N}}$, with

$$\{x\} = \bigcap_{k=1}^{\infty} S_{\mathbf{e}|_k}(F_{t(\mathbf{e}|_k)}).$$

Now $\left(S_{\mathbf{e}|_k}(F_{t(\mathbf{e}|_k)})\right)$, is a decreasing sequence of non-empty compact subsets of F_u, whose diameters tend to zero as k tends to infinity, and so there exists $m \in \mathbb{N}$, such that $x \in S_{\mathbf{e}|_m}(F_{t(\mathbf{e}|_m)}) \subset F_u$, and

$$\left|S_{\mathbf{e}|_m}(F_{t(\mathbf{e}|_m)})\right| < \mathrm{dist}(x, S(F_u)).$$

It follows that $S_{\mathbf{e}|_m}(F_{t(\mathbf{e}|_m)}) \cap S(F_u) = \emptyset$. Also $x \in [a_S, b_S]$, and as $a_S, b_S \in S(F_u)$, $\mathrm{dist}(x, S(F_u)) \leqslant x - a_S$ and $\mathrm{dist}(x, S(F_u)) \leqslant b_S - x$ which means $S_{\mathbf{e}|_m}(F_{t(\mathbf{e}|_m)}) \subset [a_S, b_S] = S(I_u)$. In summary,

$$S_{\mathbf{e}|_m}(F_{t(\mathbf{e}|_m)}) \cup S(F_u) \subset F_u \cap S(I_u), \tag{3.5.2}$$

where the union on the left hand side is disjoint.

Using the scaling property of Hausdorff measure gives

$$\mathcal{H}^s(S_{\mathbf{e}|_m}(F_{t(\mathbf{e}|_m)})) = r^s_{\mathbf{e}|_m}\mathcal{H}^s(F_{t(\mathbf{e}|_m)}) > 0, \tag{3.5.3}$$

by Theorem 1.3.7.

We now derive a contradiction as follows

$$\begin{aligned}
(b_S - a_S)^s &= \mathcal{H}^s(F_u \cap S(I_u)) && \text{(by part(a))}\\
&\geqslant \mathcal{H}^s(S_{\mathbf{e}|_m}(F_{t(\mathbf{e}|_m)}) \cup S(F_u)) && \text{(by Equation (3.5.2))}\\
&= \mathcal{H}^s(S_{\mathbf{e}|_m}(F_{t(\mathbf{e}|_m)})) + \mathcal{H}^s(S(F_u)) && \text{(the union is disjoint)}\\
&> \mathcal{H}^s(S(F_u)) && \text{(by Equation (3.5.3))}\\
&= (b_S - a_S)^s && \text{(by part (a)).}
\end{aligned}$$

□

We will use Lemma 3.5.1 in our next theorem, which also uses definitions and equations which we dealt with earlier in Chapter 2, but which we restate here for ease of reference. Let $\big(V, E^*, i, t, r, (\mathbb{R}, |\ \ |)_{v\in V}, (S_e)_{e\in E^1}\big)$, be any directed graph IFS satisfying the CSSC. The set of gap lengths and contracting similarity ratios, $X_u \subset \mathbb{R}^+$, is defined in Theorem 2.6.3, as

$$X_u = \left\{g_w, C_i, r_{\mathbf{p}} : g_w \in G^1_w,\ w \in V,\ \mathbf{c}_i \in T,\ \mathbf{p} \in D^*_{uv},\ v \in V,\ v \neq u\right\}, \tag{3.5.4}$$

where G^1_w is the set of level-1 gap lengths at the vertex $w \in V$, $T = \{\mathbf{c}_i : i \in I\}$, the set of all simple cycles in the graph, and $D^*_{uv} \subset E^*_{uv}$, is the set of all simple paths from the vertex u to the vertex v. From Corollary 2.3.5, Equations (2.3.7), (2.3.8), (2.3.9), (2.3.10) and (2.3.11), the set of gap lengths at the vertex u is given by

$$G_u = \bigcup_{g_u \in G^1_u} g_u\psi\left(\bigcup\mathcal{A}\right) \ \cup \bigcup_{\substack{v\in V\\ v\neq u\\ g_v\in G^1_v}} g_v\left(\bigcup_{\mathbf{p}\in D^*_{uv}} r_{\mathbf{p}}\psi\left(\bigcup\mathcal{B}^{\mathbf{p}}\right)\right), \tag{3.5.5}$$

where

$$\psi\left(\bigcup\mathcal{A}\right) = \{1, r_{\mathbf{e}} : \mathbf{e} \in E^*_{uu}\}, \tag{3.5.6}$$

and

$$\bigcup_{\mathbf{p}\in D^*_{uv}} r_{\mathbf{p}}\psi\left(\bigcup\mathcal{B}^{\mathbf{p}}\right) = \{r_{\mathbf{e}} : \mathbf{e} \in E^*_{uv}\}, \quad \text{for } v \neq u, \tag{3.5.7}$$

with

$$\psi\left(\bigcup\mathcal{A}\right) \subset \langle 1, C_i : i \in I\rangle, \tag{3.5.8}$$

and

$$\psi\left(\bigcup\mathcal{B}^{\mathbf{p}}\right) \subset \langle 1, C_i : i \in I\rangle. \tag{3.5.9}$$

We remind the reader that $\langle 1, C_i : i \in I\rangle$ is the semigroup whose generators are the contracting similarity ratios of the simple cycles in the graph, that is $C_i = r_{\mathbf{c}_i} = r(\mathbf{c}_i)$ where $\mathbf{c}_i \in T$. As we saw in the proof of Theorem 2.6.3, if the set X_u is multiplicatively rationally independent then the union on the right hand side of Equation (3.5.5) is pairwise disjoint.

Theorem 3.5.2. *Let $\left(V, E^*, i, t, r, (\mathbb{R}, |\ \ |)_{v\in V}, (S_e)_{e\in E^1}\right)$ be any directed graph IFS for which the CSSC holds. For the attractor F_u at the vertex u, let $s = \dim_{\mathrm{H}} F_u$ and $\{a_u, b_u\} \subset F_u \subset I_u = [a_u, b_u]$. Suppose $\mathcal{H}^s(F_u) = |I_u|^s$ and suppose that the set $X_u \subset \mathbb{R}^+$, defined in Equation (3.5.4), is multiplicatively rationally independent. Let $\mathcal{S}$ denote the set of all similarities $S : \mathbb{R} \to \mathbb{R}$, where S satisfies $S(F_u) \subset F_u$ and has the contracting similarity ratio r_S, $0 < r_S < 1$. That is let*

$$\mathcal{S} = \{S : S : \mathbb{R} \to \mathbb{R} \text{ is a similarity, } S(F_u) \subset F_u \text{ and } 0 < r_S < 1\}$$

and let

$$R_{\mathcal{S}} = \{r_S : S \in \mathcal{S}\} .$$

Then

$$R_{\mathcal{S}} = \{r_{\mathbf{e}} : \mathbf{e} \in E^*_{uu}\} .$$

Proof. As $\mathcal{H}^s(F_u) = |I_u|^s$, for any $S \in \mathcal{S}$, Lemma 3.5.1(b) implies that $S(F_u) = F_u \cap S(I_u)$. It follows that any gap length in the set $S(F_u)$ must be a gap length in F_u. Now the set of gap lengths in $S(F_u)$ is just the set $r_S G_u$, and as S^k is also a similarity satisfying $S^k(F_u) \subset F_u$, we have established that,

$$r_S^k G_u \subset G_u, \quad \text{for all } k \in \mathbb{N}. \tag{3.5.10}$$

Let $g_u \in G_u^1$, be any level-1 gap length at the vertex u, then by (3.5.10), $r_S g_u \in G_u$ and since the union on the right-hand side of Equation (3.5.5) is pairwise disjoint, there are just three possibilities,

$$\begin{aligned}
&\text{(i)} \quad r_S g_u \in g_u \psi\left(\bigcup \mathcal{A}\right),\\
&\text{(ii)} \quad r_S g_u \in g_u' \psi\left(\bigcup \mathcal{A}\right), \text{ for some } g_u' \in G_u^1,\ g_u' \neq g_u,\\
&\text{(iii)} \quad r_S g_u \in g_v r_{\mathbf{p}} \psi\left(\bigcup \mathcal{B}^{\mathbf{p}}\right), \text{ for some } g_v \in G_v^1,\ \mathbf{p} \in D^*_{uv},\ v \neq u.
\end{aligned}$$

We first show that the situation in (ii) and (iii) cannot happen. If (ii) holds then by Equation (3.5.8),

$$r_S g_u = g_u' \prod_{i\in I} C_i^{m_i}, \text{ for some } m_i \in \mathbb{N} \cup \{0\}, \ i \in I,$$

and so, using (3.5.10),

$$r_S^k g_u = \left(g_u^{-1} g_u' \prod_{i\in I} C_i^{m_i}\right)^k g_u \in G_u, \text{ for all } k \in \mathbb{N}.$$

Hence for any $k \geqslant 2$, we have an expression for the gap length $r_S^k g_u$, written as a product of elements of X_u, for which the exponent of g_u' is $k \geqslant 2$. This is impossible by the rational independence of the set X_u, since Equations (3.5.5), (3.5.8) and (3.5.9) imply that the exponent of g_u' can only be 0 or 1 in such an expression.

If (iii) holds then by (3.5.9) we may express $r_S g_u$ as

$$r_S g_u = g_v r_{\mathbf{p}} \prod_{i\in I} C_i^{n_i}, \text{ for some } n_i \in \mathbb{N} \cup \{0\}, \ i \in I,$$

and so, using (3.5.10),

$$r_S^k g_u = \left(g_u^{-1} g_v r_{\mathbf{p}} \prod_{i \in I} C_i^{n_i} \right)^k g_u \in G_u, \text{ for all } k \in \mathbb{N}.$$

This time for $k \geqslant 2$ the exponent of g_v is $k \geqslant 2$, when it can only be 0 or 1 for a gap length written in terms of the elements of X_u, by the rational independence of X_u.

So the only possibility is (i), and this gives, $r_S \in \psi\left(\bigcup \mathcal{A}\right) = \{1, r_{\mathbf{e}} : \mathbf{e} \in E_{uu}^*\}$, by (3.5.6), and as $0 < r_S < 1$, $r_S \in \{r_{\mathbf{e}} : \mathbf{e} \in E_{uu}^*\}$, so that $R_{\mathcal{S}} \subset \{r_{\mathbf{e}} : \mathbf{e} \in E_{uu}^*\}$. Clearly for any finite path $\mathbf{e} \in E_{uu}^*$, $S_{\mathbf{e}} \in \mathcal{S}$, so we have $\{r_{\mathbf{e}} : \mathbf{e} \in E_{uu}^*\} \subset R_{\mathcal{S}}$. Therefore $R_{\mathcal{S}} = \{r_{\mathbf{e}} : \mathbf{e} \in E_{uu}^*\}$. □

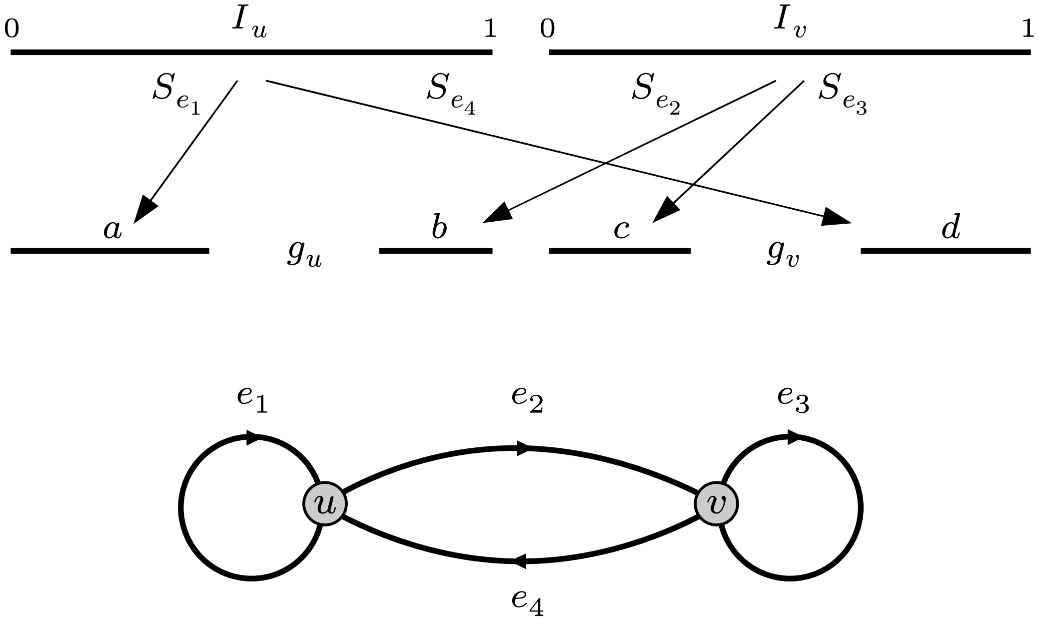


Figure. 3.5.2: A 2-vertex IFS in $\mathbb{R}$, the similarities S_{e_1}, S_{e_2}, S_{e_3} and S_{e_4} do not reflect.

Corollary 3.5.3. *Consider the 2-vertex IFS, as shown in Figure 3.5.2, for which the CSSC holds. For the attractor F_u at the vertex u, let $s = \dim_{\mathrm{H}} F_u$. Suppose $\mathcal{H}^s(F_u) = |I_u|^s = 1$, and suppose that the set $X_u = \{g_u, g_v, a, bd, c, b\} \subset \mathbb{R}^+$, as defined in Equation (3.5.4), is multiplicatively rationally independent. Let*

$$\mathcal{S} = \{S : S : \mathbb{R} \to \mathbb{R} \text{ is a similarity, } S(F_u) \subset F_u \text{ and } 0 < r_S < 1\}$$

and

$$R_{\mathcal{S}} = \{r_S : S \in \mathcal{S}\}.$$

Then

$$R_{\mathcal{S}} = \{r_{\mathbf{e}} : \mathbf{e} \in E_{uu}^*\} = \langle a \rangle \cup bd \langle 1, a, bd, c \rangle.$$

Proof. This follows immediately by Theorem 3.5.2 and Equation (3.5.6) as

$$\psi\left(\bigcup\mathcal{A}\right) = \langle 1, a, bd\rangle \cup bd\,\langle 1, a, bd, c\rangle = \langle 1, a\rangle \cup bd\,\langle 1, a, bd, c\rangle\,,$$

where we have applied the algorithm given in part (a) of the proof of Proposition 2.3.4. □

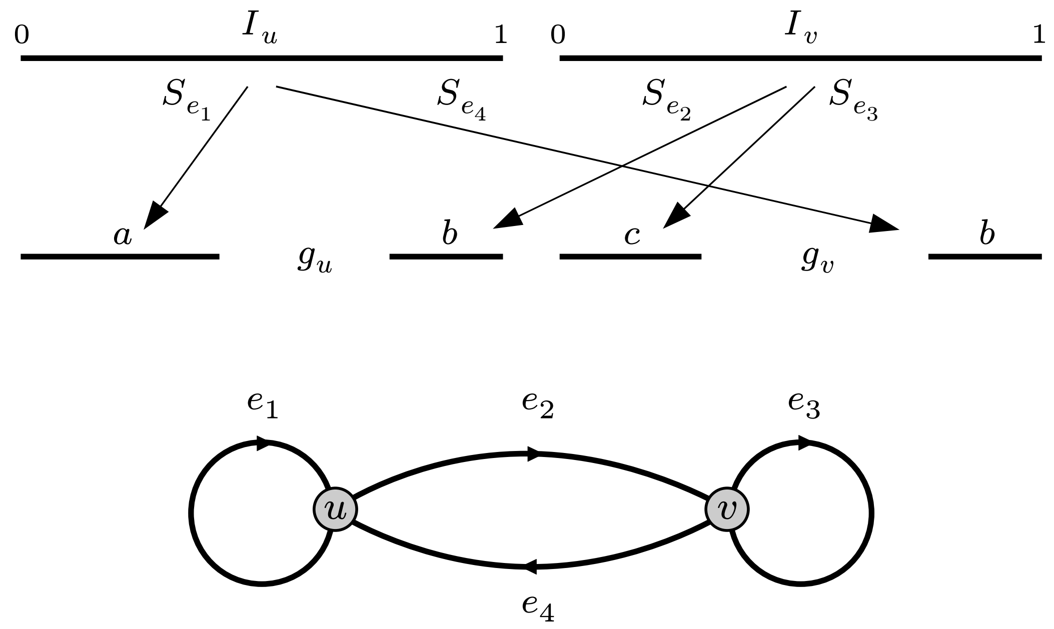


Figure. 3.5.3: A second example of a 2-vertex IFS in $\mathbb{R}$, the similarities S_{e_1}, S_{e_2}, S_{e_3} and S_{e_4} do not reflect.

Corollary 3.5.4. *Consider the* 2*-vertex IFS, as shown in Figure 3.5.3, for which the CSSC holds. For the attractor F_u at the vertex u, let $s = \dim_{\mathrm{H}} F_u$. Suppose $\mathcal{H}^s(F_u) = |I_u|^s = 1$, and suppose that the set $\{g_u, g_v, a, b, c\} \subset \mathbb{R}^+$, is multiplicatively rationally independent. Let*

$$\mathcal{S} = \{S : S : \mathbb{R} \to \mathbb{R} \text{ is a similarity, } S(F_u) \subset F_u \text{ and } 0 < r_S < 1\}$$

and

$$R_{\mathcal{S}} = \{r_S : S \in \mathcal{S}\}\,.$$

Then

$$R_{\mathcal{S}} = \{r_{\mathbf{e}} : \mathbf{e} \in E^*_{uu}\} = \langle a\rangle \cup b^2\left\langle 1, a, b^2, c\right\rangle.$$

Proof. As we saw in Equation (2.4.2), the gap lengths in the attractor F_u are

$$\begin{aligned} G_u &= g_u\psi\left(\bigcup\mathcal{A}\right) \quad \bigcup \quad g_v r_{e_4}\psi\left(\bigcup\mathcal{B}^{e_2}\right) \\ &= g_u\left(\left\langle 1, a, b^2\right\rangle \cup b^2\left\langle 1, a, b^2, c\right\rangle\right) \cup g_v b\left\langle 1, a, b^2, c\right\rangle \\ &= g_u\left(\langle 1, a\rangle \cup b^2\left\langle 1, a, b^2, c\right\rangle\right) \cup g_v b\left\langle 1, a, b^2, c\right\rangle. \end{aligned}$$

By Equation (3.5.6)

$$\psi\left(\bigcup \mathcal{A}\right) = \{1, r_{\mathbf{e}} : \mathbf{e} \in E^*_{uu}\} = \langle 1, a\rangle \cup b^2 \left\langle 1, a, b^2, c\right\rangle,$$

and

$$\{r_{\mathbf{e}} : \mathbf{e} \in E^*_{uu}\} = \langle a\rangle \cup b^2 \left\langle 1, a, b^2, c\right\rangle.$$

The proof now follows the same structure as the proof of Theorem 3.5.2. If S is any contracting similarity such that $S(F_u) \subset F_u$, then by Lemma 3.5.1, $S(F_u) = F_u \cap S(I_u)$, and as we argued in the proof of Theorem 3.5.2, this is enough to ensure that

$$r_S^k G_u \subset G_u, \text{ for all } k \in \mathbb{N},$$

where r_S is the contracting similarity ratio of S. Now $r_S g_u \in G_u$ so suppose, for a contradiction, that $r_S g_u \in g_v b\,\langle 1, a, b^2, c\rangle$. We may express r_S as $r_S = g_u^{-1} g_v b a^i b^{2j} c^k$ for some $i, j, k \in \mathbb{N}\cup\{0\}$. As $r_S^m G_u \subset G_u$ it follows that $r_S^m g_u = (g_u^{-1} g_v b a^i b^{2j} c^k)^m g_u \in G_u$. For $m \geqslant 2$ the exponent of g_v in this expression is greater than 1, but this is impossible by the multiplicative rational independence of the set $\{g_u, g_v, a, b, c\}$. This contradiction implies that $r_S g_u \in g_u(\langle 1, a\rangle \cup b^2\langle 1, a, b^2, c\rangle)$ and so $r_S \in \langle 1, a\rangle \cup b^2\langle 1, a, b^2, c\rangle$, and as $0 < r_S < 1$, $r_S \in \langle a\rangle \cup b^2\langle 1, a, b^2, c\rangle$, which proves

$$R_{\mathcal{S}} \subset \langle a\rangle \cup b^2 \left\langle 1, a, b^2, c\right\rangle = \{r_{\mathbf{e}} : \mathbf{e} \in E^*_{uu}\}.$$

For any finite path $\mathbf{e} \in E^*_{uu}$, $S_{\mathbf{e}} \in \mathcal{S}$, so we have $\{r_{\mathbf{e}} : \mathbf{e} \in E^*_{uu}\} \subset R_{\mathcal{S}}$. Therefore $R_{\mathcal{S}} = \{r_{\mathbf{e}} : \mathbf{e} \in E^*_{uu}\}$. □

The next two lemmas provide us with useful inequalities.

Lemma 3.5.5. *Let $a, b > 0$, and $0 < p < 1$. Then*

$$(a+b)^p < a^p + b^p.$$

Proof. Consider the function f defined by

$$f(t) = 1 + t^p - \big(1+t\big)^p, \tag{3.5.11}$$

with derivative

$$f'(t) = p\left(\frac{1}{t^{1-p}} - \frac{1}{(1+t)^{1-p}}\right).$$

For $t > 0$, $f'(t) > 0$, and as $f(0) = 0$, this implies $f(t) > 0$ for $t > 0$. Putting $t = \frac{b}{a} > 0$ in (3.5.11) gives $a^p + b^p - (a+b)^p > 0$. □

Lemma 3.5.6. *For $x, z > 0$, $y \geqslant 0$ and $0 < p < 1$,*

$$(x+y+z)^p < (x+y)^p + (y+z)^p - y^p.$$

Proof. For $y = 0$ the inequality reduces to $(x+z)^p < x^p + z^p$, which is true by Lemma 3.5.5, so we may assume $x, y, z > 0$. Let

$$f(t_1, t_2) = (1+t_1)^p + (1+t_2)^p - (1+t_1+t_2)^p - 1, \tag{3.5.12}$$

then

$$\frac{\partial f}{\partial t_1} = p\left(\frac{1}{(1+t_1)^{1-p}} - \frac{1}{(1+t_1+t_2)^{1-p}}\right),$$

so $\frac{\partial f}{\partial t_1} > 0$ for $t_1, t_2 > 0$. Similarly $\frac{\partial f}{\partial t_2} > 0$ for $t_1, t_2 > 0$. We have that $f(t_1, t_2)$ is continuous for $t_1, t_2 \geqslant 0$, $f(0,0) = 0$, and $\frac{\partial f}{\partial t_1}, \frac{\partial f}{\partial t_2} > 0$ for $t_1, t_2 > 0$, from which we may deduce that $f(t_1, t_2) > 0$ for $t_1, t_2 > 0$. Now putting $t_1 = \frac{x}{y}$ and $t_2 = \frac{z}{y}$ in (3.5.12) produces the required inequality. □

We saw in the 1-vertex example of Figure 3.5.1 above, that $S_2(I) \subset S_1(I)$ does not generally imply $S_2(F) \subset S_1(F)$, and this explains the significance of the next lemma. This lemma is presented by Feng and Wang as a claim in the proof of Theorem 4.1 of [FW09].

Lemma 3.5.7. *Let* $\big(V, E^*, i, t, r, (\mathbb{R}, |\ \ |)_{v \in V}, (S_e)_{e \in E^1}\big)$ *be any directed graph IFS for which the OSC holds. For the attractor* F_u *at the vertex* u*, let* $s = \dim_{\mathrm{H}} F_u$ *and* $\{a_u, b_u\} \subset F_u \subset I_u = [a_u, b_u]$*. Let* $S : \mathbb{R} \to \mathbb{R}$ *and* $T : \mathbb{R} \to \mathbb{R}$ *be any two distinct similarities with contracting similarity ratios* $0 < r_S, r_T < 1$.

If $S(F_u) \subset F_u$, $T(F_u) \subset F_u$ *and* $\mathcal{H}^s(F_u) = |I_u|^s$*, then exactly one of the following three statements occurs*

(a) $S(I_u) \cap T(I_u) = \emptyset$, *which implies* $S(F_u) \cap T(F_u) = \emptyset$,

(b) $S(I_u) \subset T(I_u)$, *which implies* $S(F_u) \subset T(F_u)$,

(c) $T(I_u) \subset S(I_u)$, *which implies* $T(F_u) \subset S(F_u)$.

Proof. There are just five possibilities for the intervals $S(I_u) = [a_S, b_S]$, $T(I_u) = [a_T, b_T]$,

(a) $[a_S, b_S] \cap [a_T, b_T] = \emptyset$,

(b) $[a_S, b_S] \subset [a_T, b_T]$,

(c) $[a_T, b_T] \subset [a_S, b_S]$,

(d) $a_S < a_T \leqslant b_S < b_T$,

(e) $a_T < a_S \leqslant b_T < b_S$.

First we prove that the situation in (d) cannot happen.

$$\begin{aligned}
(b_T - a_S)^s &\geqslant \mathcal{H}^s(F_u \cap [a_S, b_T]) && \text{(by Corollary 3.3.4(a))}\\
&= \mathcal{H}^s(F_u \cap [a_S, b_S]) + \mathcal{H}^s(F_u \cap [a_T, b_T])\\
&\qquad - \mathcal{H}^s(F_u \cap [a_T, b_S]) && \text{(a property of the measure)}\\
&= \mathcal{H}^s(F_u \cap S(I_u)) + \mathcal{H}^s(F_u \cap T(I_u))\\
&\qquad - \mathcal{H}^s(F_u \cap [a_T, b_S]),\\
&\geqslant (b_S - a_S)^s + (b_T - a_T)^s - (b_S - a_T)^s && \text{(by Lemma 3.5.1(a)}\\
& && \text{and Corollary 3.3.4(a))}\\
&> (b_T - a_S)^s.
\end{aligned}$$

The last inequality is obtained by putting $x = a_T - a_S > 0$, $y = b_S - a_T \geqslant 0$, and $z = b_T - b_S > 0$ in Lemma 3.5.6. This contradiction shows that (d) cannot occur and a similar argument can clearly be constructed to prove that (e) cannot happen either. Since $S \neq T$, exactly one of (a), (b), or (c) must occur. It only remains to prove the implications in the statement of the lemma.

That $S(I_u) \cap T(I_u) = \emptyset$ implies $S(F_u) \cap T(F_u) = \emptyset$ follows immediately as $S(F_u) \subset S(I_u)$ and $T(F_u) \subset T(I_u)$.

To see that $S(I_u) \subset T(I_u)$ implies $S(F_u) \subset T(F_u)$ we apply Lemma 3.5.1(b), to obtain

$$S(F_u) = F_u \cap S(I_u) \subset F_u \cap T(I_u) = T(F_u).$$

Similarly, that $T(I_u) \subset S(I_u)$ implies $T(F_u) \subset S(F_u)$ also follows immediately by Lemma 3.5.1(b), since

$$T(F_u) = F_u \cap T(I_u) \subset F_u \cap S(I_u) = S(F_u).$$ □

Theorem 3.5.8. *For the* 2*-vertex IFS of Figure 3.5.2, suppose conditions (1), (2) and (3), of Theorem 3.4.7 all hold, so that $\mathcal{H}^s(F_u) = |I_u|^s = 1$, and suppose also that the set $\{g_u, g_v, a, b, c, d\} \subset \mathbb{R}^+$ is multiplicatively rationally independent.*

*Then the attractor at the vertex u, F_u, is not the attractor of any standard (*1*-vertex) IFS, defined on $\mathbb{R}$, with or without separation conditions.*

Proof. For a contradiction we suppose F_u is the attractor of a 1-vertex IFS, so F_u will satisfy an invariance equation of the form

$$F_u = \bigcup_{i=1}^{n} S_i(F_u), \tag{3.5.13}$$

for some $n \geqslant 2$. If $S_j(I_u) \cap S_k(I_u) \neq \emptyset$ for any $j \neq k$, $1 \leqslant j, k \leqslant n$, then by Lemma 3.5.7, either $S_j(I_u) \subset S_k(I_u)$, with $S_j(F_u) \subset S_k(F_u)$, or $S_k(I_u) \subset S_j(I_u)$, with $S_k(F_u) \subset S_j(F_u)$. Without loss of generality suppose $S_j(F_u) \subset S_k(F_u)$, then we may rewrite Equation (3.5.13) as

$$F_u = \bigcup_{\substack{i=1 \\ i \neq j}}^{n} S_i(F_u).$$

We may continue in this way, if necessary, relabelling and reducing the number of similarities n in Equation (3.5.13) to m, $2 \leqslant m \leqslant n$, with

$$F_u = \bigcup_{i=1}^{m} S_i(F_u),$$

where $S_j(I_u) \cap S_k(I_u) = \emptyset$ for any $j \neq k$, $1 \leqslant j, k \leqslant m$. That is F_u is the attractor of a 1-vertex IFS that satisfies the CSSC. Because the set $\{g_u, g_v, a, b, c, d\}$ is multiplicatively rationally independent no such 1-vertex IFS exists by Corollary 2.6.4. This is the required contradiction. □

Theorem 3.5.9. *For the* 2*-vertex IFS of Figure 3.5.3, suppose conditions (1), (2) and (3), of Theorem 3.4.7 all hold, so that $\mathcal{H}^s(F_u) = |I_u|^s = 1$, and suppose also that the set $\{g_u, g_v, a, b, c\} \subset \mathbb{R}^+$ is multiplicatively rationally independent.*

*Then the attractor at the vertex u, F_u, is not the attractor of any standard (*1*-vertex) IFS, defined on $\mathbb{R}$, with or without separation conditions.*

Proof. The proof is the same as that given for Theorem 3.5.8, except we apply Corollary 2.6.2 in place of Corollary 2.6.4. □

We now give a specific example for which we apply Theorem 3.5.9. Consider

```
> restart;
> a:=1/4: g_u:=5/12: b:=1/3:  c:=1/7: g_v:=11/21: d:=1/3:
> r_e_1:=a/(a+g_u+b): r_e_2:=b/(c+g_v+d): r_e_3:=c/(c+g_v+d):
  r_e_4:=d/(a+g_u+b):
> f:= t -> (r_e_1^t-1)*(r_e_3^t-1)-r_e_2^t*r_e_4^t:
> eqn := f(t)=0: s:=fsolve( eqn, t, 0..1 );
                          s := 0.5147069928
> evalf((1-r_e_1^s)/r_e_2^s);
                            0.8978943038
> evalf((a+g_u)*((a+g_u+b)^s-a^s)/(b*a^s));
                            2.082389923
```

Figure. 3.5.4: Maple output 3.

the following parameters for the directed graph IFS of Figure 3.5.3, which are also illustrated in Figure 3.5.5,

$$a = \frac{1}{4},\ g_u = \frac{5}{12},\ b = \frac{1}{3},\ c = \frac{1}{7},\ g_v = \frac{11}{21}.$$

From the Maple output 3, in Figure 3.5.4, where we have put $b = d = \frac{1}{3}$, the Hausdorff dimension is $s = 0.5147069928$, and $|I_u| = |I_v| = 1$, with

$$\frac{h_v}{h_u} = \frac{1 - r_{e_1}^s}{r_{e_2}^s} = 0.8978943038 < 1.$$

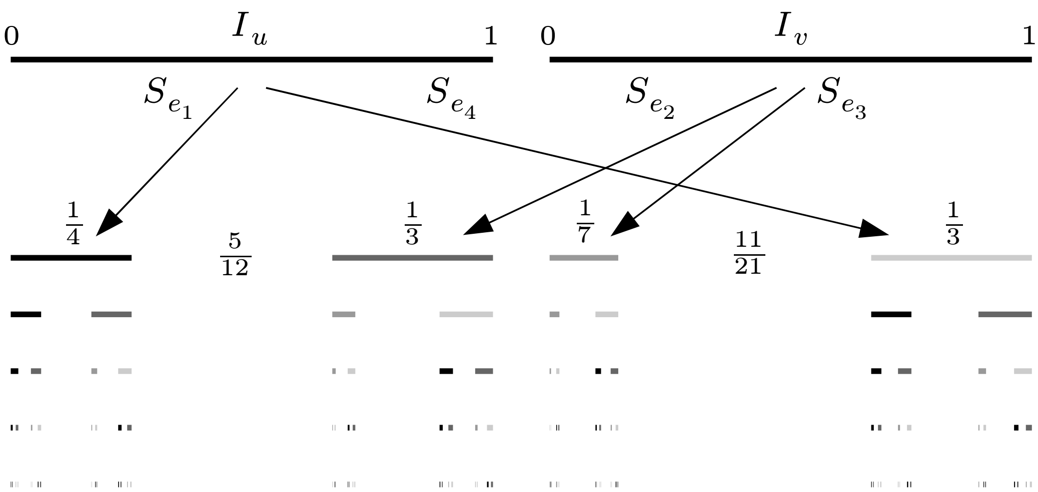


Figure. 3.5.5: Level-k intervals for $0 \leqslant k \leqslant 5$.

Also

$$\frac{(a + g_u)(|I_u|^s - a^s)}{ba^s} = 2.082389923 > 1,$$

so conditions (1), (2) and (3) of Theorem 3.4.7 all hold, which means $\mathcal{H}^s(F_u) = |I_u|^s = 1$ and $\mathcal{H}^s(F_v) = 0.8978943038$.

The set

$$\{g_u, g_v, a, b, c\} = \left\{\frac{5}{12}, \frac{11}{21}, \frac{1}{4}, \frac{1}{3}, \frac{1}{7}\right\},$$

is multiplicatively rationally independent. Theorem 3.5.9 now ensures that the attractor F_u, at the vertex u, cannot be the attractor of any standard (1-vertex) IFS. Figure 3.5.5 illustrates the level-k intervals, for $0 \leqslant k \leqslant 5$, for this particular example.

3.6 Conclusion

The work of Chapters 2 and 3 gives a rigorous mathematical argument which proves that we really do get something new with directed graph IFSs of more than one vertex. In Theorems 3.5.8 and 3.5.9, we have established the existence of a class of 2-vertex IFSs that have attractors that cannot be the attractors of standard (1-vertex) IFSs, with or without separation conditions. The attractors of these 2-vertex IFSs are of interest not only because we are able to compute their Hausdorff measure, but also because they give us information about properties not shared by 1-vertex IFSs. Also, because they are the attractors of such simple 2-vertex IFSs, it seems likely that most directed graph IFSs produce genuinely new fractals, with many 3-vertex IFSs having attractors that cannot be the attractors of 1 or 2-vertex IFSs and so on.

An obvious question to ask now is what about directed graph IFSs in $(\mathbb{R}^n, |\ \ |)$, with $n \geqslant 2$? It is not easy to see what the higher dimensional analogue of the set of gap lengths may be, but trying to find some other characteristic set or property of attractors in higher dimensions could be an interesting area for future investigation. As we have seen in this chapter, the calculation of Hausdorff measure is another, related area, where it may also be worth thinking about how the work of [Mar86], [AS99] and Theorem 3.4.7, might be extended to systems in higher dimensions.

4

An application of the Renewal Theorem

4.1 Introduction

In this final chapter we apply the Renewal Theorem to directed graph IFSs with probabilities, $\big(V, E^*, i, t, r, p, ((\mathbb{R}^n, |\ \ |))_{v\in V}, (S_e)_{e\in E^1}\big)$, as defined in Subsection 1.2.8. Theorem 1.3.4 ensures the existence of a unique list of non-empty compact sets $(F_u)_{u\in V}$ such that

$$(F_u)_{u\in V} = \left(\bigcup_{e\in E_u^1} S_e(F_{t(e)}) \right)_{u\in V}, \tag{4.1.1}$$

and Theorem 1.3.8, implies the existence of a unique list of self-similar Borel probability measures $(\mu_u)_{u\in V}$ such that

$$(\mu_u(A_u))_{u\in V} = \left(\sum_{e\in E_u^1} p_e \mu_{t(e)}\big(S_e^{-1}(A_u)\big) \right)_{u\in V}, \tag{4.1.2}$$

for all Borel sets $(A_u)_{u\in V} \subset (\mathbb{R}^n)_{u\in V}$, and where $\mathrm{supp}\mu_u = F_u$ for each $u \in V$.

The work of this chapter is in an area of fractal geometry know as multifractal analysis. A single measure μ may give rise to a whole range of fractals, each fractal determined by its points having the same local intensity of the measure. Such measures have become known as multifractal measures and are an important area of research in fractal geometry, see [Fal03] for more information. Our main object of study is the multifractal q box-dimension, also called the packing L^q-spectrum, as defined in Equation (4.2.2). It is a (multifractal) measure-theoretic generalisation of the box-counting dimension, and indeed for the specific case with $q = 0$ in Equation (4.2.2), that is what it is.

In Theorem 4.3.1 we obtain an explicit calculable value for the power law behaviour as $r \to 0^+$, of the qth packing moment of μ_u, for the non-lattice case, with a corresponding limit for the lattice case. We do this

(i) for any $q \in \mathbb{R}$ if the strong separation condition holds,

(ii) for $q \geqslant 0$ if the weaker open set condition holds, where we also assume that the non-negative matrix $\mathbf{B}(q, \gamma, l)$, as defined in Subsection 4.2.6, is irreducible.

In the non-lattice case this enables the rate of convergence of the packing L^q-spectrum to be determined. This extends a result in [Ols02b] from standard (1-vertex) IFSs to general directed graph IFSs. The method of proof is an application of the Renewal Theorem for a system of renewal equations, as stated by Crump [Cru70], and is similar to the method used by Lalley [Lal88] and Olsen [Ols02b]. As is the case in both of those papers the hard work is in the OSC situation of (ii), see Section 4.7, where we need to prove that a function is directly Riemann integrable in order to apply the Renewal Theorem.

We also show in Theorem 4.3.2, for the situation in (ii) but allowing $q \in \mathbb{R}$, that the upper multifractal q box-dimension with respect to μ_u of the set consisting of all the intersections of the components of F_u is strictly less than the multifractal q Hausdorff dimension with respect to μ_u of F_u. This extends a result in [Ols02a] from standard (1-vertex) IFSs to general directed graph IFSs with probabilities.

4.2 Definitions and notation

In this section we give the definitions and notation that are not covered in Section 1.2 and that are needed for the work that follows.

4.2.1 Directly Riemann integrable functions

Let $z(t)$ be a function, $z : [0, \infty) \to \mathbb{R}$. For a fixed $h > 0$, let m_n and M_n denote the minimum and maximum respectively of $z(t)$, in the interval $[(n-1)h, nh]$. The function $z(t)$ is *directly Riemann integrable* whenever the sums $h\sum_{n=1}^{\infty} m_n$ and $h\sum_{n=1}^{\infty} M_n$ converge absolutely to the same limit I as $h \to 0^+$, in which case $I = \int z(t)dt$, see [Fel66].

The following sufficient condition for direct Riemann integrability is used in Olsen [Ols02b] and Lalley [Lal88].

If $z(t)$ is Riemann integrable on all compact subintervals of $[0, \infty)$ and there exist $c_1, c_2 > 0$ such that

$$|z(t)| \leqslant c_1 e^{-c_2 t} \tag{4.2.1}$$

for all $t \in [0, \infty)$, then $z(t)$ is directly Riemann integrable, see [Ols02b].

4.2.2 The qth packing moment and the packing L^q-spectrum

Let $A \subset \mathbb{R}^n$ be a bounded subset of $\mathbb{R}^n$. For $r > 0$, a subset $D \subset A$ is *an r-separated subset of A* if $|x - y| > 2r$ for all $x, y \in D$ with $x \neq y$. This means that

$$B(x, r) \cap B(y, r) = \emptyset$$

for all $x, y \in D$ with $x \neq y$.

For $u \in V$, $q \in \mathbb{R}$, $r > 0$, and $A \subset F_u$, we define the *qth packing moment of μ_u on A at scale r* as

$$M_u^q(A, r) = \sup\left\{\sum_{x \in D} \mu_u(B(x, r))^q : D \text{ is an } r\text{-separated subset of } A\right\}.$$

The *packing L^q-spectrum on $A \subset F_u$ of μ_u*, is defined by the following limit, if it exists,

$$\lim_{r \to 0^+} \frac{\ln(M_u^q(A,r))}{-\ln r}.$$

4.2.3 Multifractal dimensions

The definitions in this subsection are taken from [Ols02a], they are only used in the statement of Theorem 4.3.2 and its proof in Section 4.8.

The *lower and upper multifractal q box-dimension of a set $A \subset F_u$ with respect to μ_u* are defined as

$$\underline{\dim}^q_{\mu_u,\mathrm{B}} A = \lim_{\delta \to 0^+} \inf_{0<r<\delta} \frac{\ln M_u^q(A,r)}{-\ln r},$$
$$\overline{\dim}^q_{\mu_u,\mathrm{B}} A = \lim_{\delta \to 0^+} \sup_{0<r<\delta} \frac{\ln M_u^q(A,r)}{-\ln r},$$

respectively. When these limits are equal the *multifractal q box-dimension of a set $A \subset F_u$ with respect to μ_u* is given by

$$\dim^q_{\mu_u,\mathrm{B}} A = \lim_{r \to 0^+} \frac{\ln(M_u^q(A,r))}{-\ln r}. \tag{4.2.2}$$

$\dim^q_{\mu_u,\mathrm{B}} A$ is also known as the packing L^q-spectrum on $A \subset F_u$ of μ_u, as defined in Subsection 4.2.2.

We now define the *multifractal q Hausdorff dimension of F_u with respect to μ_u* of a set $A \subset F_u$, for which we use the notation $\dim^q_{\mu_u,\mathrm{H}} A$. Let $A \subset F_u \subset \mathbb{R}^n$ and $\delta > 0$. A countable family of closed balls $\{B(x_i,r_i)\}$ in $\mathbb{R}^n$ is a centred δ-covering of A, if $A \subset \bigcup_{i=1}^\infty B(x_i,r_i)$ and $x_i \in A$, $0 < r_i < \delta$, for all $i \in \mathbb{N}$.

For $q, t \in \mathbb{R}$,

$$\begin{aligned}
\widehat{\mathcal{H}}^{q,t}_{\mu_u,\delta}(A) &= \inf \Big\{ \sum_{i=1}^{\infty} \mu_u(B(x_i,r_i))^q (2r_i)^t : \\
&\qquad\qquad \{B(x_i,r_i)\} \text{ is a centred } \delta\text{-covering of } A \Big\}, \qquad A \neq \emptyset, \\
\widehat{\mathcal{H}}^{q,t}_{\mu_u,\delta}(\emptyset) &= 0, \\
\widehat{\mathcal{H}}^{q,t}_{\mu_u}(A) &= \sup_{\delta>0} \widehat{\mathcal{H}}^{q,t}_{\mu_u,\delta}(A), \\
\mathcal{H}^{q,t}_{\mu_u}(A) &= \sup_{B \subset A} \widehat{\mathcal{H}}^{q,t}_{\mu_u}(B).
\end{aligned}$$

$\mathcal{H}^{q,t}_{\mu_u}(A)$ is a Borel measure and $\dim^q_{\mu_u,\mathrm{H}} A \in [-\infty,+\infty]$ is the unique number for which

$$\mathcal{H}^{q,t}_{\mu_u}(A) = \begin{cases} +\infty & \text{if } t < \dim^q_{\mu_u,\mathrm{H}}(A), \\ 0 & \text{if } \dim^q_{\mu_u,\mathrm{H}}(A) < t. \end{cases}$$

4.2.4 A lattice matrix of measures

Let $\mathcal{M}^+(\mathbb{R})$ denote the space of all positive Borel measures on $\mathbb{R}$ and let

$$\boldsymbol{\mu} = \begin{pmatrix} \mu_{11} & \cdots & \mu_{1m} \\ \vdots & & \vdots \\ \mu_{m1} & \cdots & \mu_{mm} \end{pmatrix}$$

be an $m \times m$ matrix of measures with $\mu_{ij} \in \mathcal{M}^+(\mathbb{R})$ for all i, j.

In this subsection we use the notation $\langle A \rangle_{\text{group},+}$ for the additive commutative group generated by the elements of a set $A \subset \mathbb{R}$.

A measure ν is *arithmetic with span* $\kappa > 0$ if and only if $\operatorname{supp}\nu \subset \kappa\mathbb{Z}$, where κ is the largest such number. This means that $\langle \operatorname{supp}\nu \rangle_{\text{group},+} = \kappa\mathbb{Z}$.

ν is a *lattice measure with span* $\kappa > 0$ if and only if there exist real numbers c, and $\kappa > 0$, such that $\operatorname{supp}\nu \subset c + \kappa\mathbb{Z}$, where κ is taken to be the largest such number.

Following Definition 3.1 in Crump's paper, [Cru70], we say that the matrix $\boldsymbol{\mu}$ is a *lattice matrix* if and only if the following conditions are met:

(a) Each μ_{ii} is arithmetic with span $\lambda_{ii} > 0$. That is $\langle \operatorname{supp}\mu_{ii} \rangle_{\text{group},+} = \lambda_{ii}\mathbb{Z}$.

(b) Each μ_{ij}, $i \neq j$, is a lattice measure with span $\lambda_{ij} > 0$. That is there exist real numbers b_{ij} and $\lambda_{ij} > 0$ such that $\operatorname{supp}\mu_{ij} \subset b_{ij} + \lambda_{ij}\mathbb{Z}$ where λ_{ij} is taken to be the largest such number.

(c) Each λ_{ij} is an integer multiple of some number, the largest such number we shall call λ. That is $\langle \{\lambda_{ij} : 1 \leqslant i, j \leqslant m\} \rangle_{\text{group},+} = \lambda\mathbb{Z}$.

(d) If $a_{ij} \in \operatorname{supp}\mu_{ij}$, $a_{jk} \in \operatorname{supp}\mu_{jk}$ and $a_{ik} \in \operatorname{supp}\mu_{ik}$ then $a_{ij} + a_{jk} = a_{ik} + n\lambda$, for some $n \in \mathbb{Z}$ (where n may depend on i, j and k). That is for all i, j, k we have $\operatorname{supp}\mu_{ij} + \operatorname{supp}\mu_{jk} - \operatorname{supp}\mu_{ik} \subset \lambda\mathbb{Z}$.

The unique number λ is called the *span* of $\boldsymbol{\mu}$.

Associated with the directed graph IFS with probabilities is a square $m \times m$ matrix of finite measures $\mathbf{P} = (P_{uv})$, where $m = \#V$ is the number of vertices in the graph, and the uvth entry $P_{uv} \in \mathcal{M}^+(\mathbb{R})$ is defined as

$$P_{uv} = \sum_{e \in E^1_{uv}} p_e^q r_e^{\beta(q)} \delta_{\ln(1/r_e)}$$

Here $\delta_{\ln(1/r_e)}$ is the Dirac measure defined, for $x \in \mathbb{R}$, as

$$\delta_x(B) = \begin{cases} 1 & \text{if } x \in B, \\ 0 & \text{if } x \notin B, \end{cases}$$

for all Borel sets $B \subset \mathbb{R}$. We define the function $\beta : \mathbb{R} \to \mathbb{R}$ in the next section.

4.2.5 The matrix $\mathbf{A}(q,\beta)$

Corresponding to the matrix of measures $\mathbf{P}$, defined above, is the $m\times m$ non-negative matrix, $\mathbf{A}(q,\beta) = (A_{uv})$, where $m = \#V$ is the number of vertices in the graph, whose entries are given by evaluating each measure P_{uv} over $\mathbb{R}$ as follows:

$$A_{uv}(q,\beta) = P_{uv}(\mathbb{R}) = \sum_{e\in E^1_{uv}} p_e^q r_e^{\beta(q)}.$$

We will use the notation $\mathbf{P}(\mathbb{R}) = \mathbf{A}(q,\beta)$. Because the directed graph is strongly-connected, the non-negative matrix $\mathbf{A}(q,\beta)$ will be irreducible for $q,\beta(q)\in\mathbb{R}$. The spectral radius of $\mathbf{A}(q,\beta)$ is given by $\rho(\mathbf{A}(q,\beta))$ where

$$\rho(\mathbf{A}(q,\beta)) = \max\left\{|\lambda| : \lambda \text{ is an eigenvalue of } \mathbf{A}(q,\beta)\right\}.$$

See Subsection 1.2.9, which describes the particular situation with $q = 0$, for the matrix $\mathbf{A}(t)$, where $t = \beta(0)$. It can be shown, using the Perron-Frobenius Theorem, see [EM92], that for a given $q\in\mathbb{R}$ there exists a unique value of $\beta(q)$ such that $\rho(\mathbf{A}(q,\beta)) = 1$, and this defines the function, $\beta:\mathbb{R}\to\mathbb{R}$, implicitly as a function of q. We require $\rho(\mathbf{A}(q,\beta)) = 1$ in the application of the Renewal Theorem as given in [Cru70].

We will use the notation $\mathbf{A}(q,\beta,n)$ for the nth power of the matrix $\mathbf{A}(q,\beta)$, so

$$\mathbf{A}(q,\beta,n) = \left(\mathbf{A}(q,\beta)\right)^n,$$

where the uvth entry is

$$A_{uv}(q,\beta,n) = \sum_{\mathbf{e}\in E^n_{uv}} p_{\mathbf{e}}^q r_{\mathbf{e}}^{\beta(q)}.$$

4.2.6 The matrix $\mathbf{B}(q,\gamma,l)$

In this subsection, as in Subsections 4.2.4 and 4.2.5, $m = \#V$ is the number of vertices in the graph. Because the OSC holds, by Theorem 1.2.1 of Subsection 1.2.10, we may take $(U_u)_{u\in V}\subset(\mathbb{R}^n)^{\#V} = (\mathbb{R}^n)^m$ to be a list of non-empty bounded open sets which satisfy the SOSC. For each vertex $u\in V$, we may choose a point $x_u\in F_u\cap U_u$ and a radius $r_u>0$ such that $B(x_u,r_u)\subset U_u$, where $B(x_u,r_u)$ is the closed ball of centre x_u and radius r_u. The map $\phi_u : E_u^{\mathbb{N}}\to F_u$ given in Lemma 1.3.5, of Chapter 1 is surjective so there exists an infinite path $\mathbf{e}_u\in E_u^{\mathbb{N}}$ with

$$\{x_u\} = \bigcap_{k=1}^{\infty} S_{\mathbf{e}_u|_k}(F_{t(\mathbf{e}_u|_k)})$$

Now $\left(S_{\mathbf{e}_u|_k}(F_{t(\mathbf{e}_u|_k)})\right)$ is a decreasing sequence of non-empty compact sets whose diameters tend to zero as k tends to infinity and so there exists $N(u)\in\mathbb{N}$ such that

$$\left|S_{\mathbf{e}_u|_n}(F_{t(\mathbf{e}_u|_n)})\right| < r_u$$

for all $n > N(u)$. Let $r_{\min} = \min\{r_u : u\in V\}$ and let $N_{\max} = \max\{N(u) : u\in V\}$. Let $l > N_{\max}$ be chosen and for each $u\in V$ let $\mathbf{l}_u = \mathbf{e}_u|_l$ then it follows that

$S_{\mathbf{l}_u}(F_{t(\mathbf{l}_u)}) \subset U_u$. This means that we can always create a family of paths $(\mathbf{l}_u)_{u\in V}$, all of the same length $l = |\mathbf{l}_u|$, where $l \in \mathbb{N}$ may be chosen as large as we like, such that

$$S_{\mathbf{l}_u}(F_{t(\mathbf{l}_u)}) \subset U_u, \tag{4.2.3}$$

for each $u \in V$. We now define a $m \times m$ matrix of non-negative real numbers, $\mathbf{B}(q, \gamma, l)$, whose uvth entry is given by

$$B_{uv}(q, \gamma, l) = \sum_{\substack{\mathbf{e}\in E^l_{uv} \\ \mathbf{e}\neq \mathbf{l}_u}} p^q_{\mathbf{e}} r^{\gamma(q)}_{\mathbf{e}}. \tag{4.2.4}$$

We always assume that the non-negative matrix $\mathbf{B}(q, \gamma, l)$ is irreducible, see Subsection 4.3.1. It can be shown that for a given $q \in \mathbb{R}$ there exists a unique value of $\gamma(q)$ such that the spectral radius $\rho(\mathbf{B}(q, \gamma, l)) = 1$, and this defines the function, $\gamma : \mathbb{R} \to \mathbb{R}$, implicitly as a function of q. A proof can be constructed along the lines of that given for $\beta(q)$ and $\mathbf{A}(q, \beta)$ in [EM92], using the Perron-Frobenius Theorem.

Also the Perron-Frobenius Theorem, see [Sen73], ensures the existence of a strictly positive right eigenvector $\mathbf{b}$ for the matrix $\mathbf{B}(q, \gamma, l)$ with eigenvalue 1, so that

$$\mathbf{B}(q, \gamma, l)\mathbf{b} = \mathbf{b}. \tag{4.2.5}$$

4.3 Two theorems

Theorem 4.3.1. *Let* $\big(V, E^*, i, t, r, p, ((\mathbb{R}^n, |\ \ |))_{v\in V}, (S_e)_{e\in E^1}\big)$ *be a directed graph IFS with probabilities, and suppose that either*

(i) *$q \in \mathbb{R}$ and the SSC holds,*

or (ii) *$q \geqslant 0$, the OSC holds, and the non-negative matrix $\mathbf{B}(q, \gamma, l)$, as defined in Subsection 4.2.6, is irreducible with $\rho(\mathbf{B}(q, \gamma, l)) = 1$.*

Let $\mathbf{P}$ be the matrix of measures, as defined in Subsection 4.2.4, let $\mathbf{A}(q, \beta)$ be the corresponding non-negative real matrix, as defined in Subsection 4.2.5, with $\rho(\mathbf{A}(q, \beta)) = 1$, and let $u \in V$.

(a) *If $\mathbf{P}$ is a lattice matrix with span $\lambda > 0$, there exists a periodic positive function, $f_u : \mathbb{R} \to \mathbb{R}^+$, which has period λ, (that is $f_u(t) = f_u(t + n\lambda)$ for any $n \in \mathbb{Z}$), such that*

$$\lim_{n\to+\infty} \frac{M^q_u(F_u, e^{-(t+n\lambda)})}{e^{-(t+n\lambda)(-\beta(q))}} = f_u(t),$$

for each $t \geqslant \max\{\ln(r_e^{-1}) : e \in E^1\}$.

It follows that

$$\lim_{n\to+\infty} \frac{\ln(M^q_u(F_u, e^{-(t+n\lambda)}))}{(t + n\lambda)} = \beta(q),$$

for each $t \geqslant \max\{\ln(r_e^{-1}) : e \in E^1\}$, and the rate of convergence, as $n \to +\infty$, is

$$\frac{\ln(M^q_u(F_u, e^{-(t+n\lambda)}))}{(t + n\lambda)} = \beta(q) + O\left(\frac{1}{n}\right).$$

(b) *If* $\mathbf{P}$ *is not a lattice matrix, there exists a constant* $C_u > 0$*, such that*

$$\lim_{r\to 0^+} \frac{M_u^q(F_u, r)}{r^{-\beta(q)}} = C_u.$$

It follows that the packing L^q*-spectrum on* F_u *of* μ_u *is given by*

$$\lim_{r\to 0^+} \frac{\ln(M_u^q(F_u, r))}{-\ln r} = \beta(q),$$

and the rate of convergence, as $r \to 0^+$*, is*

$$\frac{\ln(M_u^q(F_u, r))}{-\ln r} = \beta(q) + O\left(\frac{1}{-\ln r}\right).$$

Theorem 4.3.2. *Let* $\left(V, E^*, i, t, r, p, ((\mathbb{R}^n, |\ \ |))_{v\in V}, (S_e)_{e\in E^1}\right)$ *be a directed graph IFS with probabilities, suppose that the OSC holds, and the non-negative matrix* $\mathbf{B}(q, \gamma, l)$*, as defined in Subsection 4.2.6, is irreducible.*

Then, for $q \in \mathbb{R}$*,*

$$\overline{\dim}^q_{\mu_u,\mathrm{B}}\left(\bigcup_{e\in E_u^1}\bigcup_{\substack{f\in E_u^1\\ f\neq e}}\left(S_e(F_{t(e)}) \cap S_f(F_{t(f)})\right)\right) < \dim^q_{\mu_u,\mathrm{H}} F_u.$$

4.3.1 The irreducibility of the matrix $\mathbf{B}(q, \gamma, l)$

In the statements of Theorem 4.3.1 and Theorem 4.3.2 it is assumed that the matrix $\mathbf{B}(q, \gamma, l)$ is irreducible. As this is a genuine assumption it would need to be verified on a case by case basis. This is an area which it would be interesting to investigate further but for now we make do with a few facts about matrices that may be helpful in any specific cases.

From [Min88], if $\mathbf{C}$ is a non-negative, irreducible matrix, with index of primitivity h, where

$$h = \#\left\{\lambda : \lambda \text{ is an eigenvalue of } \mathbf{C} \text{ with } |\lambda| = \rho(\mathbf{C})\right\},$$

then

$$\mathbf{C}^k \text{ is irreducible if and only if } (k, h) = 1.$$

A matrix is primitive if $h = 1$ and so if $\mathbf{A}(q, \beta, 1)$ is primitive then $\mathbf{A}(q, \beta, k)$ is irreducible for all $k \in \mathbb{N}$.

We also note that if $\mathbf{A}(q, \beta, 1)$ has at least one positive diagonal element then it is primitive.

As noted in Subsection 4.2.6 the length l of the paths $(\mathbf{l}_v)_{v\in V}$, in Equation (4.2.3) can be chosen to be as large as we like, which means we can always ensure that $(l, h) = 1$ so that $\mathbf{A}(q, \beta, l)$ is irreducible. The matrix $\mathbf{B}(q, \gamma, l)$ is very closely related to $\mathbf{A}(q, \beta, l)$ so this may be of help in determining the irreducibility of $\mathbf{B}(q, \gamma, l)$.

Finally we note that our definition of a directed graph IFS given in Subsection 1.2.8 assumes that each vertex in the directed graph has at least two edges leaving it.

4.4 The Renewal Theorem

Before stating the Renewal Theorem for a system of renewal equations, we give the standard version from [Fel66], which is the theorem used by Lalley in [Lal88], and which applies to a single renewal equation. A single renewal equation is obtained by putting $m = 1$ in Equations (4.4.4) below, which gives,

$$M(t) = \int_0^t M(t-u)dF(u) \,+\, z(t). \tag{4.4.1}$$

where $z(t)$ is bounded on every finite interval and vanishes for $t < 0$ and F is a finite Borel probability measure, which Feller calls a distribution, concentrated on $[0, \infty]$ and with $F((-\infty, 0]) = 0$. As in Subsection 4.2.4, F is arithmetic with span $\lambda > 0$ if supp$F \subset \lambda\mathbb{Z}$ where λ is the largest such number. Let

$$\mu = \int_0^\infty y dF(y)$$

where $\mu \leqslant \infty$. When $\mu = \infty$ Feller interprets μ^{-1} as 0.

The following theorem is given in [Fel66], Theorem 2, Chapter 11.

Theorem 4.4.1. *Let $M(t)$ be as in Equation (4.4.1), and suppose $z(t)$ is directly Riemann integrable.*

(a) *If F is arithmetic with span λ then*

$$M(t + n\lambda) \,\longrightarrow\, \frac{\lambda}{\mu}\sum_{l=-\infty}^{\infty} z(t + \lambda l) \tag{4.4.2}$$

as $n \longrightarrow \infty$.

(b) *If F is not arithmetic then*

$$M(t) \,\longrightarrow\, \frac{1}{\mu}\int_0^\infty z(u)du \tag{4.4.3}$$

as $t \longrightarrow \infty$.

In what follows we will be applying the Renewal Theorem (Theorem 4.4.2) for a system of renewal-type equations as given in Crump's paper [Cru70]. A system of renewal equations is of the form

$$M_i(t) = \int_0^t \sum_{j=1}^m M_j(t-u)dF_{ij}(u) \,+\, z_i(t), \qquad i = 1, 2\cdots, m \tag{4.4.4}$$

where each $z_i(t)$ is bounded on every finite interval and vanishes for $t < 0$ and each F_{ij} is a finite Borel measure with $F_{ij}((-\infty, 0)) = 0$. If we let $\mathbf{F} = (F_{ij})$, $\mathbf{Z}^{\mathrm{T}}(t) = (z_1(t), \cdots, z_m(t))$ and $\mathbf{M}^{\mathrm{T}}(t) = (M_1(t), \cdots, M_m(t))$ we can put Equation (4.4.4) in the compact form

$$\mathbf{M}(t) = \mathbf{F} * \mathbf{M}(t) \,+\, \mathbf{Z}(t) \tag{4.4.5}$$

where $*$ behaves in exactly the same way as matrix multiplication except we convolve elements instead of multiplying them so that

$$(F_{ij} * M_j)(t) = \int_0^t M_j(t-u)dF_{ij}(u).$$

We use the notation $\mathbf{F}(B)$ to mean the matrix of real numbers $(F_{ij}(B))$ where $B \subset \mathbb{R}$ is a Borel set. There are three conditions given by Crump that need to be satisfied:

(i) The largest eigenvalue of the matrix $\mathbf{F}((-\infty, 0])$ is less than 1.

(ii) The matrix $\mathbf{F}(\mathbb{R})$ has all non-negative entries.

(iii) For at least one pair i, j, the finite measure F_{ij} is not concentrated at the origin.

We follow Crump's notation putting $G_{ij}(\alpha) = \int e^{-\alpha u} dF_{ij}(u)$, and $\mathbf{G}(\alpha) = (G_{ij}(\alpha))$. The determinant of $\mathbf{I}-\mathbf{G}(\alpha)$ is denoted by $\Delta(\alpha)$, where $\mathbf{I}$ is the $m \times m$ identity matrix and $(c_{ij}(\alpha))$ is the adjoint matrix of $\mathbf{I}-\mathbf{G}(\alpha)$. We put $\alpha_{ij} = c_{ij}(0)/\Delta'(0)$ whenever $\int (F_{ij}(\infty) - F_{ij}(t))dt < \infty$ for each i and j, (here Δ' is the derivative). If at least one of these integrals is infinite we put $\alpha_{ij} = 0$ for each i and j. In theory the values of α_{ij} can be calculated for a particular system which means the limits in the Renewal Theorem, Theorem 4.4.2 below, can be determined explicitly.

The statement of the Renewal Theorem that follows, which is the one we shall be using, is given in [Cru70], Theorem 3.1 (ii).

Theorem 4.4.2 (The Renewal Theorem)**.** *Suppose the spectral radius $\rho(\mathbf{F}(\mathbb{R})) = 1$. Let the vector $\mathbf{M}(t)$ be as in (4.4.5) and suppose each $z_i(t)$ in $\mathbf{Z}(t)$ is directly Riemann integrable.*

(a) *If $\mathbf{F}$ is a lattice matrix then for each i*

$$M_i(t+n\lambda) \to \sum_{j=1}^{m} \lambda\alpha_{ij} \sum_{l=-\infty}^{\infty} z_j(t+\lambda l)$$

as $n \to \infty$.

(b) *If $\mathbf{F}$ is not a lattice matrix then for each i*

$$M_i(t) \to \sum_{j=1}^{m} \alpha_{ij} \int_0^\infty z_j(u)du$$

as $t \to \infty$.

4.5 A system of renewal equations

In this section we derive a system of renewal equations of the form given in (4.4.5). Consider any $q \in \mathbb{R}$ to be fixed. For each vertex $u \in V$, let $L_u : (0, \infty) \to \mathbb{R}$ be the error function defined, for $r > 0$, by

$$L_u(r) = M_u^q(F_u, r) - \sum_{e \in E_u^1} p_e^q M_{t(e)}^q(F_{t(e)}, r_e^{-1} r). \tag{4.5.1}$$

Let $\mathbf{P} = (P_{uv})$ be the $m \times m$ matrix of finite measures as defined in Subsection 4.2.4 and let $\mathbf{A}(q,\beta) = (A_{uv})$ be the $m \times m$ matrix of real numbers as defined in Section 4.2.5, where $m = \#V$.

For each $u \in V$, let $H_u : [0,\infty) \to [0,\infty)$, be the function defined by

$$H_u(t) = e^{-t\beta(q)} M_u^q(F_u, e^{-t}),$$

and let $h_u : [0,\infty) \to [0,\infty)$, be the function defined by

$$h_u(t) = e^{-t\beta(q)} L_u(e^{-t}), \tag{4.5.2}$$

which means, by Equation (4.5.1), that for $t \geqslant 0$,

$$h_u(t) = e^{-t\beta(q)} \left(M_u^q(F_u, e^{-t}) - \sum_{e \in E_u^1} p_e^q M_{t(e)}^q (F_{t(e)}, r_e^{-1} e^{-t}) \right). \tag{4.5.3}$$

We note that

$$H_u(t - \ln(r_e^{-1})) \ = \ (r_e^{-1} e^{-t})^{\beta(q)} M_u^q(F_u, r_e^{-1} e^{-t}). \tag{4.5.4}$$

For each $u \in V$,

$$\begin{aligned}
H_u(t) &= e^{-t\beta(q)} M_u^q(F_u, e^{-t}) \\
&= e^{-t\beta(q)} \left(\sum_{e \in E_u^1} p_e^q M_{t(e)}^q (F_{t(e)}, r_e^{-1} e^{-t}) + L_u(e^{-t}) \right) && \text{(by (4.5.1))} \\
&= \sum_{e \in E_u^1} p_e^q r_e^{\beta(q)} (r_e^{-1} e^{-t})^{\beta(q)} M_{t(e)}^q (F_{t(e)}, e^{-t} r_e^{-1}) + h_u(t) && \text{(by (4.5.2))} \\
&= \sum_{e \in E_u^1} p_e^q r_e^{\beta(q)} H_{t(e)}(t - \ln(r_e^{-1})) + h_u(t) && \text{(by (4.5.4))} \\
&= \sum_{v \in V} \left(\sum_{e \in E_{uv}^1} p_e^q r_e^{\beta(q)} H_v(t - \ln(r_e^{-1})) \right) + h_u(t) \\
&= \sum_{v \in V} \left(\int_0^t H_v(t-x) dP_{uv}(x) \right) + h_u(t) \\
&= \int_0^t \sum_{v \in V} H_v(t-x) dP_{uv}(x) + h_u(t),
\end{aligned}$$

for large enough $t \geqslant \max\{\ln(r_e^{-1}) \ : \ e \in E^1\}$. The penultimate equality follows from the definition of the finite measure P_{uv} in Subsection 4.2.4.

We now have a system of renewal equations (see Equation (4.4.4) in Section 4.4), where for each $u \in V$

$$H_u(t) = \int_0^t \sum_{v \in V} H_v(t-x) dP_{uv}(x) + h_u(t). \tag{4.5.5}$$

In compact form, (see Equation (4.4.5) in Section 2.2),

$$\mathbf{H}(t) = \mathbf{P} \, * \, \mathbf{H}(t) \, + \, \mathbf{h}(t).$$

It is clear that Crump's conditions (i) and (ii) of Section 4.4 are satisfied by the matrices $\mathbf{P}(-\infty, 0])$ and $\mathbf{P}(\mathbb{R})$ respectively and that (iii) also holds. Also if $\mathbf{A}(q,\beta)$ is the matrix of Subsection 4.2.5, then $\rho(\mathbf{P}(\mathbb{R})) = \rho(\mathbf{A}(q,\beta)) = 1$.

If we assume for the moment that the functions $(h_u)_{u\in V}$ are directly Riemann integrable then we may apply the Renewal Theorem, Theorem 4.4.2, to obtain the following.

By Theorem 4.4.2(a), if $\mathbf{P}$ is a lattice matrix with span $\lambda > 0$, then

$$\lim_{n\to+\infty} H_u(t+n\lambda) = \lim_{n\to+\infty} \frac{M_u^q(F_u, e^{-(t+n\lambda)})}{e^{-(t+n\lambda)(-\beta(q))}} = \sum_{v\in V} \lambda\alpha_{uv} \sum_{l=-\infty}^{\infty} h_v(t+\lambda l) = f_u(t),$$

for each $t \geqslant \max\{\ln(r_e^{-1}) : e \in E^1\}$. Here $h_v(t) = 0$ for $t < 0$, for each $v \in V$, and $f_u : \mathbb{R} \to \mathbb{R}^+$ is a positive periodic function with $f_u(t) = f_u(t+n\lambda)$ for any $n \in \mathbb{Z}$. Taking logarithms of both sides of this equation gives

$$\lim_{n\to+\infty} \frac{\ln(M_u^q(F_u, e^{-(t+n\lambda)}))}{t+n\lambda} = \beta(q).$$

For the rate of convergence of the last limit, it is convenient to define a function $g : \mathbb{N} \to \mathbb{R}^+$, by

$$g(n) = \frac{M_u^q(F_u, e^{-(t+n\lambda)})}{f_u(t)e^{-(t+n\lambda)(-\beta(q))}},$$

where $t \geqslant \max\{\ln(r_e^{-1}) : e \in E^1\}$ is considered as fixed, $\lim_{n\to+\infty} g(n) = 1$, and $f_u(t) > 0$. This gives

$$\begin{aligned}
\left|\frac{\ln(M_u^q(F_u, e^{-(t+n\lambda)}))}{(t+n\lambda)} - \beta(q)\right| &= \left|\frac{\ln(f_u(t)e^{-(t+n\lambda)(-\beta(q))}g(n))}{(t+n\lambda)} - \beta(q)\right| \\
&= \left|\frac{\ln(f_u(t)g(n))}{(t+n\lambda)} + \beta(q) - \beta(q)\right| \\
&= \left|\frac{\ln(f_u(t)g(n))}{(t+n\lambda)}\right| \\
&\leqslant \frac{K}{n},
\end{aligned}$$

for large enough n, and some constant $K > 0$. Therefore

$$\frac{\ln(M_u^q(F_u, e^{-(t+n\lambda)}))}{(t+n\lambda)} = \beta(q) + O\left(\frac{1}{n}\right),$$

as $n \to +\infty$.

By Theorem 4.4.2(b), if $\mathbf{P}$ is not a lattice matrix then

$$\lim_{t\to+\infty} H_u(t) = \lim_{t\to+\infty} \frac{M_u^q(F_u, e^{-t})}{e^{-t(-\beta(q))}} = \lim_{r\to0^+} \frac{M_u^q(F_u, r)}{r^{-\beta(q)}} = \sum_{v\in V} \alpha_{uv} \int_0^\infty h_v(x)dx = C_u.$$

The constant C_u is positive. Taking logarithms of both sides of this equation gives

$$\lim_{r\to0^+} \frac{\ln(M_u^q(F_u, r))}{-\ln r} = \beta(q).$$

For the rate of convergence of this last limit, let $g : (0, +\infty) \to \mathbb{R}^+$, be defined as

$$g(r) = \frac{M_u^q(F_u, r)}{C_u r^{-\beta(q)}},$$

where $\lim_{r \to 0^+} g(r) = 1$. We now obtain

$$\begin{aligned}\left|\frac{\ln(M_u^q(F_u, r))}{-\ln r} - \beta(q)\right| &= \left|\frac{\ln(C_u r^{-\beta(q)} g(r))}{-\ln r} - \beta(q)\right| \\ &= \left|\frac{\ln(C_u g(r))}{-\ln r} + \beta(q) - \beta(q)\right| \\ &= \left|\frac{\ln(C_u g(r))}{-\ln r}\right| \\ &\leqslant \frac{K}{-\ln r},\end{aligned}$$

for small enough r, and some constant $K > 0$. Therefore

$$\frac{\ln(M_u^q(F_u, r))}{-\ln r} = \beta(q) + O\left(\frac{1}{-\ln r}\right),$$

as $r \to 0^+$.

This means that the proof of Theorem 4.3.1 will be complete once we have shown that the functions $(h_u)_{u \in V}$, given by Equation (4.5.3), are directly Riemann integrable in both of the following cases,

(i) $q \in \mathbb{R}$ and the SSC holds,

(ii) $q \geqslant 0$, the OSC holds, and the non-negative matrix $\mathbf{B}(q, \gamma, l)$, as defined in Subsection 4.2.6, is irreducible with $\rho(\mathbf{B}(q, \gamma, l)) = 1$.

We do this for (i) in Section 4.6 and for (ii) in Section 4.7.

4.6 Proof of Theorem 4.3.1 (i) - SSC

In this section we prove that the functions $(h_u)_{u \in V}$, as given in Equation (4.5.3), are directly Riemann integrable for

(i) $q \in \mathbb{R}$ *and the SSC holds.*

The argument extends that given in Section 5 of [Ols02b] for a standard (1-vertex) IFS with probabilities to a general directed graph IFS with probabilities.

Consider $u \in V$ as fixed and let

$$\delta = \frac{1}{2} \min\left\{\operatorname{dist}(S_e(F_{t(e)}), S_f(F_{t(f)})) : e, f \in E_u^1,\ e \neq f\right\}, \tag{4.6.1}$$

see Subsection 1.2.1 for the definiton of the distance between two sets. By the SSC, for $e, f \in E_u^1$, $e \neq f$, $S_e(F_{t(e)}) \cap S_f(F_{t(f)}) = \emptyset$, and as $S_e(F_{t(e)})$, $S_f(F_{t(f)})$ are non-empty compact subsets of F_u, this implies $\delta > 0$.

We will use D to indicate an r-separated subset of F_u and for each edge $e \in E_u^1$ D_e indicates an r-separated subset of $S_e(F_{t(e)}) \subset F_u$.

Lemma 4.6.1. *Let $q \in \mathbb{R}$, let $u \in V$ be fixed and let $r \in (0, \delta)$. Then*

(a)
$$M_u^q(F_u, r) = \sum_{e \in E_u^1} M_u^q(S_e(F_{t(e)}), r).$$

(b) *For each $e \in E_u^1$,*
$$M_u^q(S_e(F_{t(e)}), r) \;=\; p_e^q M_{t(e)}^q(F_{t(e)}, r_e^{-1} r).$$

(c)
$$M_u^q(F_u, r) = \sum_{e \in E_u^1} p_e^q M_{t(e)}^q(F_{t(e)}, r_e^{-1} r).$$

Proof. Part (c) follows immediately from parts (a) and (b).

(a) Let D be an r-separated subset of F_u, then by Equation (4.1.1),
$$D = \bigcup_{e \in E_u^1} (D \cap S_e(F_{t(e)})) = \bigcup_{e \in E_u^1} D_e,$$
where the union is disjoint by the SSC and each $D_e = D \cap S_e(F_{t(e)})$ is an r-separated subset of $S_e(F_{t(e)})$.

For each $e \in E_u^1$ let D'_e be an r-separated subset of $S_e(F_{t(e)})$. Then for $e, f \in E_u^1$, $e \neq f$, it follows by the SSC and the definition of δ, that for $0 < r < \delta$,
$$\bigcup_{x \in D'_e} B(x, r) \bigcap \bigcup_{y \in D'_f} B(y, r) = \emptyset.$$
This means that
$$D' = \bigcup_{e \in E_u^1} D'_e,$$
will be an r-separated subset of F_u, where the union is disjoint.

From these observations we obtain
$$\sum_{x \in D} \mu_u(B(x, r))^q = \sum_{e \in E_u^1} \left(\sum_{x \in D_e} \mu_u(B(x, r))^q \right) \leqslant \sum_{e \in E_u^1} M_u^q(S_e(F_{t(e)}), r),$$
and
$$M_u^q(F_u, r) \geqslant \sum_{x \in D'} \mu_u(B(x, r))^q = \sum_{e \in E_u^1} \left(\sum_{x \in D'_e} \mu_u(B(x, r))^q \right).$$
Taking the supremum over all r-separated subset D in the first inequality, and all r-separated subsets D'_e in the second inequality, gives the required result.

(b) Let D_e be an r-separated subset of $S_e(F_{t(e)})$ for $e \in E_u^1$. As S_e is a similarity with contracting similarity ratio r_e so S_e^{-1} is a similarity with an expanding similarity ratio of r_e^{-1} and for any $x \in D_e$,
$$S_e^{-1}(B(x, r)) = B(S_e^{-1}(x), r_e^{-1} r). \tag{4.6.2}$$

As $\operatorname{supp}\mu_{t(f)} = F_{t(f)}$, $0 < r < \delta$, and the SSC is satisfied, it is clear that for all $f \in E_u^1$ with $f \neq e$,

$$\mu_{t(f)}(S_f^{-1}(B(x,r))) = 0, \text{ for all } x \in D_e. \tag{4.6.3}$$

This means that

$$\begin{aligned}
\sum_{x\in D_e} \mu_u(B(x,r))^q &= \sum_{x\in D_e}\Bigg(\sum_{f\in E_u^1} p_f\mu_{t(f)}\big(S_f^{-1}(B(x,r))\big)\Bigg)^q && \text{(by Equation (4.1.2))}\\
&= \sum_{x\in D_e}\big(p_e\mu_{t(e)}\big(S_e^{-1}(B(x,r))\big)\big)^q && \text{(by (4.6.3))}\\
&= p_e^q\sum_{x\in D_e}\mu_{t(e)}\big(B(S_e^{-1}(x), r_e^{-1}r)\big)^q && \text{(by (4.6.2))}\\
&= p_e^q\sum_{x\in S_e^{-1}(D_e)}\mu_{t(e)}\big(B(x, r_e^{-1}r)\big)^q,
\end{aligned}$$

that is

$$\sum_{x\in D_e}\mu_u(B(x,r))^q = p_e^q\sum_{x\in S_e^{-1}(D_e)}\mu_{t(e)}\big(B(x,r_e^{-1}r)\big)^q. \tag{4.6.4}$$

From Equation (4.6.2) $S_e^{-1}(D_e)$ is an $r_e^{-1}r$-separated subset of $F_{t(e)}$, so that

$$\sum_{x\in D_e}\mu_u(B(x,r))^q = p_e^q\sum_{x\in S_e^{-1}(D_e)}\mu_{t(e)}\big(B(x,r_e^{-1}r)\big)^q \leqslant p_e^q M_{t(e)}^q(F_{t(e)}, r_e^{-1}r).$$

Similarly if D is any $r_e^{-1}r$-separated subset of $F_{t(e)}$ then $D_e = S_e(D)$ will be an r-separated subset of $S_e(F_{t(e)})$, so that

$$M_u^q(S_e(F_{t(e)}), r) \geqslant \sum_{x\in D_e}\mu_u(B(x,r))^q \ = \ p_e^q\sum_{x\in D}\mu_{t(e)}\big(B(x,r_e^{-1}r)\big)^q,$$

where the last equality is obtained by putting $D_e = S_e(D)$ in Equation (4.6.4). Taking the supremum over any r-separated subset D_e of $S_e(F_{t(e)})$ in the first inequality, and over any $r_e^{-1}r$-separated subset D of $F_{t(e)}$ in the second inequality completes the proof. $\square$

As defined in Equations (4.5.2) and (4.5.3), the function h_u is given by

$$h_u(t) = e^{-t\beta(q)}L_u(e^{-t}) = e^{-t\beta(q)}\Bigg(M_u^q(F_u, e^{-t}) - \sum_{e\in E_u^1} p_e^q M_{t(e)}^q(F_{t(e)}, r_e^{-1}e^{-t})\Bigg).$$

Without loss of generality we now assume $\delta < 1$.

(a) $h_u(t)$ *is Riemann integrable on any compact interval* $[a,b] \subset [0,+\infty)$.

For $q > 0$ the function $M_u^q(F_u, z)$, considered as a function of z, is increasing and for $q \leqslant 0$ it is decreasing, on any compact interval $[c,d] \subset (0,1]$. So $M_u^q(F_u, e^{-t})$ and $M_v^q(F_v, e^{-t}r_e^{-1})$, as functions of t, are also monotone and therefore Riemann integrable on any compact interval $[a,b] \subset [0,+\infty)$. This means $L_u(e^{-t})$ and hence $h_u(t)$ are Riemann integrable on any compact interval $[a,b] \subset [0,+\infty)$.

(b) $h_u(t)$ *is bounded for* $t \in [0, -\ln \delta]$.

If $t \in [0, -\ln \delta]$ then $e^{-t} \in [\delta, 1]$, and it follows from the definition of the qth packing moment, see Subsection 4.2.2, that $M_u^q(F_u, e^{-t})$ is bounded on the interval $[0, -\ln \delta]$ and so is $M_{t(e)}^q(F_{t(e)}, r_e^{-1}e^{-t})$, for each $e \in E_u^1$. Hence $h_u(t)$ is bounded on the interval $[0, -\ln \delta]$.

(c) $h_u(t) = 0$ *for* $t \in (-\ln \delta, +\infty)$.

For $t \in (-\ln \delta, +\infty)$, that is for $e^{-t} \in (0, \delta)$, Lemma 4.6.1(c) implies that,

$$L_u(e^{-t}) = M_u^q(F_u, e^{-t}) - \sum_{e \in E_u^1} p_e^q M_{t(e)}^q(F_{t(e)}, r_e^{-1}e^{-t}) = 0,$$

see Equation (4.5.1). This proves $h_u(t) = 0$ for $t \in (-\ln \delta, +\infty)$.

From statements (b) and (c) it is clear that we can always find positive constants c_1 and c_2, so that Equation (4.2.1) of Subsection 4.2.1 is true. Taken together with statement (a), this is enough to prove that $h_u(t)$ is directly Riemann integrable.

4.7 Proof of Theorem 4.3.1 (ii) - OSC

The argument given in this section extends that given in Section 2 of [Ols02b], for a standard (1-vertex) IFS with probabilities, to a general directed graph IFS with probabilities.

We prove that the functions $(h_u)_{u \in V}$, as given in Equation (4.5.3), are directly Riemann integrable for

(ii) $q \geqslant 0$, *the OSC holds, and the non-negative matrix* $\mathbf{B}(q, \gamma, l)$, *as defined in Subsection 4.2.6, is irreducible with* $\rho(\mathbf{B}(q, \gamma, l)) = 1$.

For the OSC, statement (c) of Section 4.6 may not be true, that is it may not be the case that $h_u(t) = 0$ for $t \in (-\ln \delta, +\infty)$. Indeed we may have a situation where $S_e(F_{t(e)}) \cap S_f(F_{t(f)}) \neq \emptyset$ for $e, f \in E_u^1$, $f \neq e$, so that $\delta = 0$, where δ is the minimum distance betweeen components as defined in Equation (4.6.1). Instead to show $h_u(t)$ is directly Riemann integrable we will show, that for some small δ, $0 < \delta < 1$,

(a) $h_u(t)$ *is Riemann integrable on any compact interval* $[a, b] \subset [0, +\infty)$.

(b) $h_u(t)$ *is bounded for* $t \in [0, -\ln \delta]$.

(c′) *There exist positive constants* d_1 *and* d_2 *such that* $|h_u(t)| \leqslant d_1 e^{-d_2 t}$, *for* $t \in (-\ln \delta, +\infty)$.

Statements (a) and (b) hold by exactly the same arguments as those given in Section 4.6. Statements (b) and (c′) imply that there exist positive constants c_1 and c_2, so that Equation (4.2.1) of Subsection 4.2.1 is true. It is clear then that to show $h_u(t)$ is directly Riemann integrable we need only to prove statement (c′) which we do in two parts.

The first part is to show, see Lemma 4.7.5, that for any δ, $0 < \delta < 1$, for $t \in (-\ln \delta, +\infty)$,

$$|h_u(t)| \leqslant e^{-t\beta(q)} \sum_{e \in E_u^1} \Bigg(\sum_{\substack{f \in E_u^1 \\ f \neq e}} Q_{e,f}^q(e^{-t}) \Bigg), \tag{4.7.1}$$

where $Q^q_{e,f}(r)$ is defined in Equation (4.7.8). In fact it is in the proof of this inequality that we require $q \geqslant 0$, see the proof of Lemma 4.7.3(b) below.

The second part is to show, see Theorem 4.7.19, that for a suitable choice of δ, $0 < \delta < 1$, for $r \in (0, \delta)$,

$$Q^q_{e,f}(r) \leqslant C_{e,f}(q) r^{-\gamma(q)}, \tag{4.7.2}$$

for some positive $C_{e,f}(q)$. We are able to prove this inequality for $q \in \mathbb{R}$ and it is this inequality that is used in the proof of Theorem 4.3.2, as given in Section 4.8.

Now as $t \in (-\ln\delta, +\infty)$ if and only if $e^{-t} \in (0, \delta)$, inequalities (4.7.1) and (4.7.2) combine to give

$$|h_u(t)| \leqslant d_1 e^{-t(\beta(q)-\gamma(q))},$$

for some positive constant d_1 (which depends on q). As $\beta(q) > \gamma(q)$ by Lemma 4.7.1 below, putting $d_2 = \beta(q) - \gamma(q) > 0$ completes the proof of statement (c′).

It remains then to prove inequalities (4.7.1) and (4.7.2), which we do in Lemma 4.7.5 and Theorem 4.7.19. We have broken the proofs of these two results down into a sequence of small steps in the form of a sequence of lemmas that now follow.

Lemma 4.7.1. *Let* $\big(V, E^*, i, t, r, p, ((\mathbb{R}^n, |\ \ |))_{v\in V}, (S_e)_{e\in E^1}\big)$ *be a directed graph IFS with probabilities which satisfies the OSC. Suppose that the non-negative matrix* $\mathbf{B}(q,\gamma,l)$*, as defined in Subsection 4.2.6, is irreducible. Let* $q \in \mathbb{R}$ *and let* $\beta(q)$,$\gamma(q) \in \mathbb{R}$ *be the unique numbers for which* $\rho(\mathbf{A}(q,\beta)) = \rho(\mathbf{B}(q,\gamma,l)) = 1$*, as described in Subsections 4.2.5 and 4.2.6.*

Then

$$\gamma(q) < \beta(q).$$

Proof. We can replace $\beta(q)$ by $\gamma(q)$ in the definition of $\mathbf{A}(q,\beta,l)$, so that the uvth entry of the matrix $\mathbf{A}(q,\gamma,l)$ is

$$A_{uv}(q,\gamma,l) = \sum_{\mathbf{e}\in E^l_{uv}} p^q_{\mathbf{e}} r^\gamma_{\mathbf{e}}.$$

Along each row u of the matrices $\mathbf{B}(q,\gamma,l)$ and $\mathbf{A}(q,\gamma,l)$

$$B_{uv}(q,\gamma,l) = \sum_{\substack{\mathbf{e}\in E^l_{uv}\\ \mathbf{e}\neq \mathbf{l}_u}} p^q_{\mathbf{e}} r^\gamma_{\mathbf{e}} = \sum_{\mathbf{e}\in E^l_{uv}} p^q_{\mathbf{e}} r^\gamma_{\mathbf{e}} = A_{uv}(q,\gamma,l),$$

except at the single entry with $v = t(\mathbf{l}_u)$ where there is a strict inequality

$$B_{ut(\mathbf{l}_u)}(q,\gamma,l) = \sum_{\substack{\mathbf{e}\in E^l_{ut(\mathbf{l}_u)}\\ \mathbf{e}\neq \mathbf{l}_u}} p^q_{\mathbf{e}} r^\gamma_{\mathbf{e}} < \sum_{\mathbf{e}\in E^l_{ut(\mathbf{l}_u)}} p^q_{\mathbf{e}} r^\gamma_{\mathbf{e}} = A_{ut(\mathbf{l}_u)}(q,\gamma,l).$$

It follows that

$$\mathbf{B}(q,\gamma,l) \neq \mathbf{A}(q,\gamma,l). \tag{4.7.3}$$

and

$$\mathbf{B}(q,\gamma,l) \leqslant \mathbf{A}(q,\gamma,l). \tag{4.7.4}$$

Let M_m denote the set of real $m \times m$ matrices. For $\mathbf{M} \in M_m$, $\rho(\mathbf{M})$ is the spectral radius of $\mathbf{M}$ as defined in Equation (1.2.4). The *spectral radius formula* states that

$$\rho(\mathbf{M}) = \lim_{k\to\infty} \left\|\mathbf{M}^k\right\|^{\frac{1}{k}}, \tag{4.7.5}$$

see [Sen73] or [Mad88]. All norms are equivalent on finite dimensional spaces so it doesn't matter which norm we use, but for our purposes it is convenient to give a specific norm, defined for a matrix $\mathbf{M} \in M_m$, with ijth entry $M_{ij} \in \mathbb{R}$, by

$$\|\mathbf{M}\| = \max\left\{\sum_{j=1}^{m} |M_{ij}| : 1 \leqslant i \leqslant m\right\}. \tag{4.7.6}$$

We now prove statements (a), (b), and (c) that follow.

(a) *For non-negative matrices* $\mathbf{C}, \mathbf{D}$, *if* $\mathbf{C} \leqslant \mathbf{D}$ *then* $\rho(\mathbf{C}) \leqslant \rho(\mathbf{D})$.

Since $\mathbf{0} \leqslant \mathbf{C} \leqslant \mathbf{D}$, it follows that $\mathbf{0} \leqslant \mathbf{C}^n \leqslant \mathbf{D}^n$, for any $n \in \mathbb{N}$, and from the specific definition of the norm in (4.7.6) this means that $0 \leqslant \|\mathbf{C}^n\| \leqslant \|\mathbf{D}^n\|$. Equation (4.7.5) now implies $\rho(\mathbf{C}) \leqslant \rho(\mathbf{D})$.

(b) *For* $k \in \mathbb{N}$ *and a non-negative matrix* $\mathbf{C}$, $\rho(\mathbf{C}^k) = \rho(\mathbf{C})^k$.

Let $n \in \mathbb{N}$. If λ is an eigenvalue of $\mathbf{C}$ then λ^n is an eigenvalue of $\mathbf{C}^n$. It follows from the definition of the spectral radius in Equation (1.2.4) that $\rho(\mathbf{C}^n) \geqslant \rho(\mathbf{C})^n$. The norm defined in Equation (4.7.6) is submultiplicative, see [Mad88], so $\|\mathbf{C}^n\| \leqslant \|\mathbf{C}\|^n$. It follows that $\left\|(\mathbf{C}^n)^k\right\|^{\frac{1}{k}} \leqslant (\left\|\mathbf{C}^k\right\|^{\frac{1}{k}})^n$ for any $k \in \mathbb{N}$ and Equation 4.7.5 implies $\rho(\mathbf{C}^n) \leqslant \rho(\mathbf{C})^n$.

(c) *For non-negative matrices* $\mathbf{C}, \mathbf{D} \in M_m$, *if* $\rho(\mathbf{C}) = \rho(\mathbf{D}) = 1$, *and* $\mathbf{C} \leqslant \mathbf{D}$, *then* $\mathbf{C} = \mathbf{D}$.

By the Perron-Frobenius Theorem, $\mathbf{C}$ has a unique (up to scaling) positive eigenvector $\mathbf{c}$, with eigenvalue $\rho(\mathbf{C}) = 1$, see Subsection 1.2.9. It follows that

$$\mathbf{0} \leqslant (\mathbf{D} - \mathbf{C})\mathbf{c} = \mathbf{D}\mathbf{c} - \mathbf{c}, \tag{4.7.7}$$

which means

$$\mathbf{c} \leqslant \mathbf{D}\mathbf{c}.$$

Applying Lemma 3.2.1 of Chapter 3, gives $\mathbf{c} = \mathbf{D}\mathbf{c}$, and so by Equation (4.7.7),

$$(\mathbf{D} - \mathbf{C})\mathbf{c} = \mathbf{D}\mathbf{c} - \mathbf{c} = \mathbf{0},$$

and $\mathbf{D} = \mathbf{C}$.

Inequality (4.7.4), together with statements (a) and (b), means

$$1 = \rho(\mathbf{B}(q, \gamma, l)) \leqslant \rho(\mathbf{A}(q, \gamma, l)) = (\rho(\mathbf{A}(q, \gamma, 1)))^l,$$

and so $\rho(\mathbf{A}(q, \gamma, 1)) \geqslant 1$. This implies that $\gamma \leqslant \beta$ because $\rho(\mathbf{A}(q, \gamma, 1))$ (strictly) decreases as γ increases and $\rho(\mathbf{A}(q, \beta, 1)) = 1$, (see [EM92], Section 3, for the details). If $\gamma = \beta$ then

$$1 \;=\; \rho(\mathbf{B}(q, \gamma, l)) \;= \rho(\mathbf{A}(q, \gamma, l)).$$

and so by statement (c),

$$\mathbf{B}(q, \gamma, l)) = \mathbf{A}(q, \gamma, l).$$

This contradicts Equation (4.7.3). Therefore $\gamma < \beta$. $\square$

For a set $A \subset \mathbb{R}^n$, $A(r)$ is the closed r-neighbourhood of A, as defined in Subsection 1.2.5, so for an edge $f \in E^1_u$, $S_f(F_{t(f)})(r)$ is the closed r-neighbourhood of $S_f(F_{t(f)})$. For edges $e, f \in E^1_u$, $f \neq e$, we define $Q^q_{e,f}(r)$, for $r > 0$, as

$$Q^q_{e,f}(r) = M^q_u\big(S_e(F_{t(e)}) \cap S_f(F_{t(f)})(r),\, r\big). \tag{4.7.8}$$

From now on we consider the vertex $u \in V$ to be fixed and we will use the following notation in the preliminary Lemmas 4.7.2 and 4.7.3 that lead up to the important Lemma 4.7.5.

As in Section 4.6, D indicates an r-separated subset of F_u and for each edge $e \in E^1_u$, we will use D_e to indicate an r-separated subset of $S_e(F_{t(e)}) \subset F_u$.

Given an r-separated subset D_e, of $S_e(F_{t(e)})$, for each $f \in E^1_u$, with $f \neq e$, let

$$H_{e,f} = D_e \cap S_f(F_{t(f)})(r), \tag{4.7.9}$$

and

$$H_e = \bigcup_{\substack{f \in E^1_u \\ f \neq e}} H_{e,f}, \tag{4.7.10}$$

where this union is not necessarily disjoint.

Let

$$G_e = D_e \setminus \bigcup_{\substack{f \in E^1_u \\ f \neq e}} S_f(F_{t(f)})(r) = D_e \setminus H_e, \tag{4.7.11}$$

so that

$$D_e = G_e \cup H_e, \tag{4.7.12}$$

and this union is disjoint.

Let

$$G = \bigcup_{e \in E^1_u} G_e, \tag{4.7.13}$$

then G is always an r-separated subset of F_u, and this union is disjoint.

Figure 4.7.1 illustrates these definitions, where points in $\mathbb{R}^2$ are represented as basic grey shapes. The grey squares are points in G_e and $H_e = H_{e,f} \cup H_{e,g}$ consists of the three points represented by the grey triangle, diamond and circle, the union is not disjoint here because the diamond belongs to both $H_{e,f}$ and $H_{e,g}$. However it is clear from the diagram that $D_e = G_e \cup H_e$ is a disjoint union.

Lemma 4.7.2. *Let $q \in \mathbb{R}$ and $r > 0$. Then*

(a)

$$M^q_u(F_u, r) \leqslant \sum_{e \in E^1_u} M^q_u(S_e(F_{t(e)}), r).$$

(b) *For each $e \in E^1_u$,*

$$M^q_u(S_e(F_{t(e)}), r) \leqslant p^q_e M^q_{t(e)}(F_{t(e)}, r_e^{-1} r) + \sum_{\substack{f \in E^1_u \\ f \neq e}} Q^q_{e,f}(r).$$

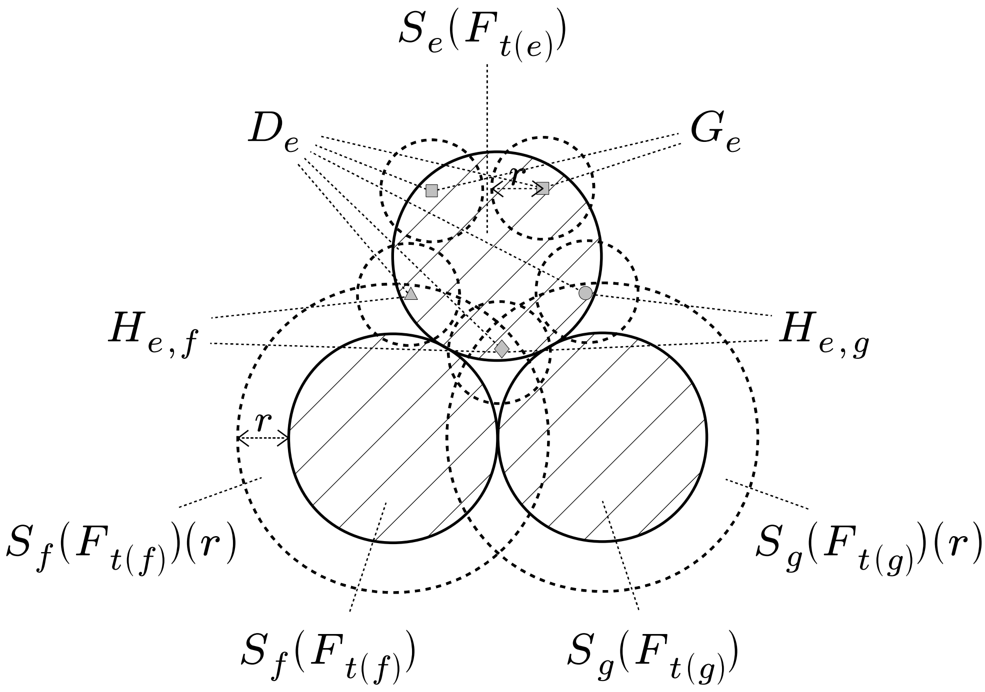


Figure. 4.7.1: A representation in $\mathbb{R}^2$ of r-separated subsets D_e, G_e, $H_{e,f}$ and $H_{e,g}$.

(c)

$$M_u^q(F_u, r) - \sum_{e \in E_u^1} p_e^q M_{t(e)}^q(F_{t(e)}, r_e^{-1} r) \leqslant \sum_{e \in E_u^1} \left(\sum_{\substack{f \in E_u^1 \\ f \neq e}} Q_{e,f}^q(r) \right).$$

Proof. Part (c) is an immediate consequence of parts (a) and (b).

(a) Let D be an r-separated subset of F_u then, by Equation (4.1.1),

$$D = \bigcup_{e \in E_u^1} (D \cap S_e(F_{t(e)})) = \bigcup_{e \in E_u^1} D_e,$$

where this union is not necessarily disjoint and each D_e is an r-separated subset of $S_e(F_{t(e)})$. It follows that

$$\sum_{x \in D} \mu_u(B(x,r))^q \leqslant \sum_{e \in E_u^1} \left(\sum_{x \in D_e} \mu_u(B(x,r))^q \right) \leqslant \sum_{e \in E_u^1} M_u^q(S_e(F_{t(e)}), r),$$

where the second inequality follows directly from the definition of $M_u^q(S_e(F_{t(e)}), r)$. Taking the supremum over all r-separated subsets D of F_u gives the required result, which should be compared with Lemma 4.6.1(a).

(b) Let D_e be an r-separated subset of $S_e(F_{t(e)})$, then from Equations (4.7.9), (4.7.10), (4.7.11), (4.7.12), and (4.7.13), it follows that,

$$\sum_{x \in D_e} \mu_u(B(x,r))^q$$

$$
\begin{aligned}
&= \sum_{x\in G_e} \mu_u(B(x,r))^q + \sum_{x\in H_e} \mu_u(B(x,r))^q \\
&\leqslant \sum_{x\in G_e} \mu_u(B(x,r))^q + \sum_{\substack{f\in E_u^1\\ f\neq e}} \left(\sum_{x\in H_{e,f}} \mu_u(B(x,r))^q \right) \\
&= \sum_{x\in G_e} \left(\sum_{f\in E_u^1} p_f \mu_{t(f)}\big(S_f^{-1}(B(x,r))\big) \right)^q + \sum_{\substack{f\in E_u^1\\ f\neq e}} \left(\sum_{x\in H_{e,f}} \mu_u(B(x,r))^q \right) \\
&= \sum_{x\in G_e} p_e^q \mu_{t(e)}\big(S_e^{-1}(B(x,r))\big)^q + \sum_{\substack{f\in E_u^1\\ f\neq e}} \left(\sum_{x\in H_{e,f}} \mu_u(B(x,r))^q \right).
\end{aligned}
$$

Here we have the first inequality because the union may not be disjoint in Equation (4.7.10). We then apply Equation (4.1.2), and the last equality follows since $\mu_{t(f)}(S_f^{-1}(B(x,r))) = 0$ for $f \in E_u^1$ with $f \neq e$. Specifically, any point $x \in G_e$, is at least a distance r from $S_f(F_{t(f)})$ for $f \neq e$, from the definition of G_e in Equation (4.7.11). For such x, $S_f^{-1}(B(x,r)) \cap F_{t(f)} = \emptyset$, and as $\mathrm{supp}\mu_{t(f)} = F_{t(f)}$, this implies $\mu_{t(f)}(S_f^{-1}(B(x,r))) = 0$.

We note that as $H_{e,f}$ is an r-separated subset of $S_e(F_{t(e)}) \cap S_f(F_{t(f)})(r)$,

$$
\sum_{x\in H_{e,f}} \mu_u(B(x,r))^q \leqslant Q_{e,f}^q(r),
$$

and as $S_e^{-1}(B(x,r)) = B(S_e^{-1}(x), r_e^{-1}r)$ we obtain

$$
\begin{aligned}
\sum_{x\in D_e} \mu_u(B(x,r))^q &\leqslant\ p_e^q \sum_{x\in G_e} \mu_{t(e)}\big(B(S_e^{-1}(x), r_e^{-1}r)\big)^q + \sum_{\substack{f\in E_u^1\\ f\neq e}} Q_{e,f}^q(r) \\
&=\ p_e^q \sum_{x\in S_e^{-1}(G_e)} \mu_{t(e)}(B(x, r_e^{-1}r))^q + \sum_{\substack{f\in E_u^1\\ f\neq e}} Q_{e,f}^q(r) \\
&\leqslant\ p_e^q M_{t(e)}^q(F_{t(e)}, r_e^{-1}r) + \sum_{\substack{f\in E_u^1\\ f\neq e}} Q_{e,f}^q(r),
\end{aligned}
$$

where this last inequality holds since $S_e^{-1}(G_e)$ is an $r_e^{-1}r$-separated subset of $F_{t(e)}$. As D_e is any r-separated subset of $S_e(F_{t(e)})$, this proves part (b). □

Lemma 4.7.3. *Let $r > 0$.*

(a) *For $q \in \mathbb{R}$,*

$$
-\sum_{e\in E_u^1} \left(\sum_{\substack{f\in E_u^1\\ f\neq e}} Q_{e,f}^q(r) \right) \leqslant M_u^q(F_u, r) - \sum_{e\in E_u^1} M_u^q(S_e(F_{t(e)}), r).
$$

(b) *For $q \geqslant 0$, and each $e \in E_u^1$,*

$$
M_u^q(S_e(F_{t(e)}), r) \geqslant p_e^q M_{t(e)}^q(F_{t(e)}, r_e^{-1}r).
$$

(c) *For* $q \geqslant 0$,

$$-\sum_{e\in E_u^1}\left(\sum_{\substack{f\in E_u^1\\ f\neq e}} Q_{e,f}^q(r)\right) \leqslant M_u^q(F_u,r) - \sum_{e\in E_u^1} p_e^q M_{t(e)}^q(F_{t(e)}, r_e^{-1}r).$$

Proof. Part (c) is an immediate consequence of parts (a) and (b).

(a) Let D be an r-separated subset of F_u and for each $e \in E_u^1$, let

$$D_e = D \cap S_e(F_{t(e)}),$$

so that D_e is an r-separated subset of $S_e(F_{t(e)})$. From Equations (4.7.9), (4.7.10), (4.7.11), (4.7.12), and (4.7.13), it follows that,

$$\begin{aligned}
M_u^q(F_u,r) &\geqslant \sum_{x\in G}\mu_u(B(x,r))^q\\
&= \sum_{e\in E_u^1}\left(\sum_{x\in G_e}\mu_u(B(x,r))^q\right)\\
&= \sum_{e\in E_u^1}\left(\sum_{x\in D_e\setminus H_e}\mu_u(B(x,r))^q\right)\\
&= \sum_{e\in E_u^1}\left(\sum_{x\in D_e}\mu_u(B(x,r))^q - \sum_{x\in H_e}\mu_u(B(x,r))^q\right)\\
&\geqslant \sum_{e\in E_u^1}\left(\sum_{x\in D_e}\mu_u(B(x,r))^q - \sum_{\substack{f\in E_u^1\\ f\neq e}}\left(\sum_{x\in H_{e,f}}\mu_u(B(x,r))^q\right)\right),
\end{aligned}$$

where the last line is an inequality because the union in Equation (4.7.10) is not necessarily disjoint. Finally it follows from the definition of $Q_{e,f}^q(r)$ and $M_u^q(S_e(F_{t(e)}),r)$ that

$$M_u^q(F_u,r) \geqslant \sum_{e\in E_u^1}\left(M_u^q(S_e(F_{t(e)}),r) - \sum_{\substack{f\in E_u^1\\ f\neq e}} Q_{e,f}^q(r)\right),$$

which proves part (a).

(b) Let D be an $r_e^{-1}r$-separated subset of $F_{t(e)}$ and let $D_e = S_e(D)$ so that as usual D_e is an r-separated subset of $S_e(F_{t(e)})$.

From Equations (4.7.9), (4.7.10), (4.7.11), (4.7.12), (4.7.13), and using the same arguments given in detail in the proof of Lemma 4.7.2(b), we obtain,

$$\begin{aligned}
M_u^q(S_e(F_{t(e)}),r) &\geqslant \sum_{x\in D_e}\mu_u(B(x,r))^q\\
&= \sum_{x\in G_e}\mu_u(B(x,r))^q + \sum_{x\in H_e}\mu_u(B(x,r))^q\\
&= \sum_{x\in G_e}\left(\sum_{f\in E_u^1} p_f\mu_{t(f)}\big(S_f^{-1}(B(x,r))\big)\right)^q + \sum_{x\in H_e}\mu_u(B(x,r))^q
\end{aligned}$$

$$= \sum_{x \in G_e} p_e^q \mu_{t(e)}(S_e^{-1}(B(x,r)))^q + \sum_{x \in H_e} \mu_u(B(x,r))^q.$$

By Equation (4.1.2), if $q \geqslant 0$, then the following inequality must hold,

$$\mu_u(B(x,r))^q = \Bigg(\sum_{f \in E_u^1} p_f \mu_{t(f)}\big(S_f^{-1}(B(x,r))\big) \Bigg)^q \geqslant p_e^q \mu_{t(e)}(S_e^{-1}(B(x,r)))^q,$$

that is,

$$\mu_u(B(x,r))^q - p_e^q \mu_{t(e)}(S_e^{-1}(B(x,r)))^q \geqslant 0. \tag{4.7.14}$$

Using this inequality gives

$$\begin{aligned}
&M_u^q(S_e(F_{t(e)}), r) \\
&\geqslant \sum_{x \in G_e \cup H_e} p_e^q \mu_{t(e)}(S_e^{-1}(B(x,r)))^q + \sum_{x \in H_e} \big(\mu_u(B(x,r))^q - p_e^q \mu_{t(e)}(S_e^{-1}(B(x,r)))^q\big) \\
&\geqslant \sum_{x \in G_e \cup H_e} p_e^q \mu_{t(e)}(S_e^{-1}(B(x,r)))^q \qquad \text{(by Inequality (4.7.14))} \\
&= p_e^q \sum_{x \in D_e} \mu_{t(e)}(S_e^{-1}(B(x,r)))^q \\
&= p_e^q \sum_{x \in S_e^{-1}(D_e)} \mu_{t(e)}(B(x, r_e^{-1} r))^q \\
&= p_e^q \sum_{x \in D} \mu_{t(e)}(B(x, r_e^{-1} r))^q.
\end{aligned}$$

As D is any $r_e^{-1}r$-separated subset of $F_{t(e)}$ this completes the proof of part (b). Compare this with Lemma 4.6.1(b). $\square$

Lemma 4.7.4. *Let $q \geqslant 0$ and $r > 0$, then*

$$\left| M_u^q(F_u, r) - \sum_{e \in E_u^1} p_e^q M_{t(e)}^q(F_{t(e)}, r_e^{-1} r) \right| \leqslant \sum_{e \in E_u^1} \Bigg(\sum_{\substack{f \in E_u^1 \\ f \neq e}} Q_{e,f}^q(r) \Bigg).$$

Proof. This is established by Lemma 4.7.2(c) and Lemma 4.7.3(c). It's worth comparing this with Lemma 4.6.1(c). $\square$

Our next lemma proves Inequality (4.7.1).

Lemma 4.7.5. *Let $q \geqslant 0$ and $t \in [0, +\infty)$, then for the functions $(h_u)_{u \in V}$, as defined in Equation (4.5.3),*

$$|h_u(t)| \leqslant e^{-t\beta(q)} \sum_{e \in E_u^1} \Bigg(\sum_{\substack{f \in E_u^1 \\ f \neq e}} Q_{e,f}^q(e^{-t}) \Bigg).$$

Proof. $t \in [0, +\infty)$ if and only if $e^{-t} \in (0, 1]$, so Equation (4.5.3) together with Lemma 4.7.4 imply

$$\begin{aligned} |h_u(t)| &= e^{-t\beta(q)} \left| M_u^q(F_u, e^{-t}) - \sum_{e \in E_u^1} p_e^q M_{t(e)}^q(F_{t(e)}, r_e^{-1} e^{-t}) \right| \\ &\leqslant e^{-t\beta(q)} \sum_{e \in E_u^1} \Bigg(\sum_{\substack{f \in E_u^1 \\ f \neq e}} Q_{e,f}^q(e^{-t}) \Bigg). \end{aligned}$$ □

The remainder of this section is concerned with proving Inequality (4.7.2). We need to find a bound on $Q_{e,f}^q(r)$ for $r \in (0, \delta)$, for some suitable small δ, $0 < \delta < 1$. As usual $H_{e,f}$ indicates an r-separated subset of $S_e(F_{t(e)}) \cap S_f(F_{t(f)})(r)$, where the edges $e, f \in E_u^1$, are taken as fixed with $e \neq f$. We aim to find a bound for $\sum_{x \in H_{e,f}} \mu_u(B(x,r))^q$ and hence $Q_{e,f}^q(r)$. First we give some preliminary results that will be needed.

In the proof of part (b) of the next lemma we will need the idea of a limit point. Let $A \subset X$ where (X, d) is a metric space and let $x \in X$, where x is not necessarily in A. Then x is a *limit point* of A if and only if every open ball $S(x, r)$, $r > 0$, contains a point of A different from x, see [Mad88]. We use the notation A' for the set of limit points of A. Equivalently, $x \in A'$ if and only if $S(x, r) \cap (A \setminus \{x\}) \neq \emptyset$, for every $r > 0$. The closure of A can be written as $\overline{A} = A \cup A'$, see [Mad88].

Lemma 4.7.6. *Let* $\big(V, E^*, i, t, r, ((\mathbb{R}^n, |\ \ |))_{v \in V}, (S_e)_{e \in E^1}\big)$ *be a directed graph IFS which satisfies the OSC, let* $(U_u)_{u \in V}$ *be the non-empty bounded open sets of the OSC, and let* $\mathbf{f}, \mathbf{g} \in E_u^*$ *be any two paths which are not subpaths of each other, that is* $\mathbf{f} \not\subset \mathbf{g}$ *and* $\mathbf{g} \not\subset \mathbf{f}$, *(see Subsection 1.2.7).*

Then

$$\begin{aligned} &\text{(a)} \quad S_{\mathbf{f}}(U_{t(\mathbf{f})}) \cap S_{\mathbf{g}}(U_{t(\mathbf{g})}) = \emptyset, \\ &\text{(b)} \quad S_{\mathbf{f}}(F_{t(\mathbf{f})}) \cap S_{\mathbf{g}}(U_{t(\mathbf{g})}) = \emptyset. \end{aligned}$$

Proof. (a) Since $\mathbf{f} \not\subset \mathbf{g}$ and $\mathbf{g} \not\subset \mathbf{f}$, there is a least $k \in \mathbb{N}$ such that $f_k \neq g_k$, where f_k, g_k are the kth edges of the paths $\mathbf{f}$, $\mathbf{g}$. By the OSC $S_{f_k}(U_{t(f_k)}) \cap S_{g_k}(U_{t(g_k)}) = \emptyset$ and so $S_{\mathbf{f}|_k}(U_{t(\mathbf{f}|_k)}) \cap S_{\mathbf{g}|_k}(U_{t(\mathbf{g}|_k)}) = \emptyset$. Now we can put $\mathbf{f} = \mathbf{f}|_k \mathbf{f}'$ and $\mathbf{g} = \mathbf{g}|_k \mathbf{g}'$ for (possibly empty) paths $\mathbf{f}', \mathbf{g}' \in E^*$. Again by the OSC, for non-empty $\mathbf{f}', \mathbf{g}'$, $S_{\mathbf{f}'}(U_{t(\mathbf{f}')}) \subset U_{i(\mathbf{f}')} = U_{t(\mathbf{f}|_k)}$ and $S_{\mathbf{g}'}(U_{t(\mathbf{g}')}) \subset U_{i(\mathbf{g}')} = U_{t(\mathbf{g}|_k)}$, which proves (a).

(b) For any two open sets $A, B \subset \mathbb{R}^n$ with $A \cap B = \emptyset$, it is also the case that $\overline{A} \cap B = \emptyset$. For a contradiction we suppose there is a point $y \in \overline{A} \cap B$ and as $A \cap B = \emptyset$, it follows that $y \in A'$. As B is open there is an open ball $S(y, r) \subset B$, for some $r > 0$, but $y \in A'$ implies there is a point $z \in S(y, r) \cap (A \setminus \{y\})$. This means $z \in A \cap B$ which is the required contradiction.

So by part (a) $S_{\mathbf{f}}(\overline{U}_{t(\mathbf{f})}) \cap S_{\mathbf{g}}(U_{t(\mathbf{g})}) = \emptyset$. By Lemma 1.3.6, $F_{t(\mathbf{f})} \subset \overline{U}_{t(\mathbf{f})}$, which proves (b). □

Lemma 4.7.7. *Let* $\big(V, E^*, i, t, r, p, ((\mathbb{R}^n, |\ \ |))_{v \in V}, (S_e)_{e \in E^1}\big)$ *be a directed graph IFS with probabilities which satisfies the OSC. Then for any finite path* $\mathbf{e} \in E_u^*$,

$$\mu_u(S_{\mathbf{e}}(F_{t(\mathbf{e})})) = p_{\mathbf{e}}.$$

Proof. A proof for the 1-vertex case is given in [Gra95].

For any $k \in \mathbb{N}$, Equation (4.1.2) may be iterated k times to obtain

$$\mu_u(A_u) = \sum_{\mathbf{f} \in E_u^k} p_{\mathbf{f}} \mu_{t(\mathbf{f})}\big(S_{\mathbf{f}}^{-1}(A_u)\big). \tag{4.7.15}$$

The first step in the proof is to prove that for each $v \in V$,

$$\mu_v(F_v \setminus U_v) = 0.$$

Exactly as was argued in Subsection 4.2.6, see Equation (4.2.3), by Theorem 1.2.1 of Subsection 1.2.10, we may take $(U_v)_{v \in V}$ to be the non-empty bounded open sets for which the SOSC holds and there exists a family of paths $(\mathbf{l}_v)_{v \in V}$, all of the same length $l = |\mathbf{l}_v|$, where $l \in \mathbb{N}$ may be chosen as large as we like, such that for each $v \in V$,

$$S_{\mathbf{l}_v}(F_{t(\mathbf{l}_v)}) \subset U_v. \tag{4.7.16}$$

It is clear that $\mu_v(U_v) > 0$, for each $v \in V$, since by Equations (4.7.15) and (4.7.16) we obtain

$$0 < p_{\mathbf{l}_v} = p_{\mathbf{l}_v} \mu_{t(\mathbf{l}_v)}(F_{t(\mathbf{l}_v)}) \leqslant \sum_{\mathbf{f} \in E_v^l} p_{\mathbf{f}} \mu_{t(\mathbf{f})}\big(S_{\mathbf{f}}^{-1}(S_{\mathbf{l}_v}(F_{t(\mathbf{l}_v)}))\big) = \mu_v(S_{\mathbf{l}_v}(F_{t(\mathbf{l}_v)})) \leqslant \mu_v(U_v).$$

We may now choose a vertex u such that the open set U_u is of minimal measure with

$$\mu_u(U_u) \leqslant \mu_v(U_v) \tag{4.7.17}$$

for all $v \in V$. From the definition of the OSC,

$$\bigcup_{\mathbf{f} \in E_u^l} S_{\mathbf{f}}(U_{t(\mathbf{f})}) \subset U_u, \tag{4.7.18}$$

where the union on the left hand side is disjoint by Lemma 4.7.6(a). This implies

$$\begin{aligned}
\mu_u(U_u) &\geqslant \mu_u\Bigg(\bigcup_{\mathbf{f} \in E_u^l} S_{\mathbf{f}}(U_{t(\mathbf{f})}) \Bigg) \\
&= \sum_{\mathbf{f} \in E_u^l} \mu_u\big(S_{\mathbf{f}}(U_{t(\mathbf{f})})\big) && \text{(the union is disjoint)} \\
&= \sum_{\mathbf{f} \in E_u^l} \Bigg(\sum_{\mathbf{g} \in E_u^l} p_{\mathbf{g}} \mu_{t(\mathbf{g})}\big(S_{\mathbf{g}}^{-1}(S_{\mathbf{f}}(U_{t(\mathbf{f})}))\big) \Bigg) && \text{(by Equation (4.7.15))} \\
&\geqslant \sum_{\mathbf{f} \in E_u^l} p_{\mathbf{f}} \mu_{t(\mathbf{f})}(U_{t(\mathbf{f})}) \\
&\geqslant \mu_u(U_u) \sum_{\mathbf{f} \in E_u^l} p_{\mathbf{f}} && \text{(by Inequality (4.7.17))} \\
&= \mu_u(U_u) && \text{(by Equation (1.2.3))},
\end{aligned}$$

and so

$$\mu_u(U_u) = \mu_u\Bigg(\bigcup_{\mathbf{f}\in E_u^l} S_{\mathbf{f}}(U_{t(\mathbf{f})}) \Bigg). \tag{4.7.19}$$

Now

$$\begin{aligned}
&\mu_u\big(S_{\mathbf{l}_u}(F_{t(\mathbf{l}_u)}) \setminus S_{\mathbf{l}_u}(U_{t(\mathbf{l}_u)})\big) \\
&\qquad = \mu_u\Bigg(S_{\mathbf{l}_u}(F_{t(\mathbf{l}_u)}) \setminus \Bigg(\bigcup_{\mathbf{f}\in E_u^l} S_{\mathbf{f}}(U_{t(\mathbf{f})}) \Bigg)\Bigg) && \text{(by Lemma 4.7.6(b))} \\
&\qquad \leqslant \mu_u\Bigg(U_u \setminus \Bigg(\bigcup_{\mathbf{f}\in E_u^l} S_{\mathbf{f}}(U_{t(\mathbf{f})}) \Bigg)\Bigg) && \text{(by (4.7.16))} \\
&\qquad = \mu_u(U_u) - \mu_u\Bigg(\bigcup_{\mathbf{f}\in E_u^l} S_{\mathbf{f}}(U_{t(\mathbf{f})}) \Bigg) && \text{(by (4.7.18))} \\
&\qquad = 0 && \text{(by (4.7.19))}.
\end{aligned}$$

That is

$$\begin{aligned}
0 &= \mu_u\big(S_{\mathbf{l}_u}(F_{t(\mathbf{l}_u)}) \setminus S_{\mathbf{l}_u}(U_{t(\mathbf{l}_u)})\big) \\
&= \mu_u\big(S_{\mathbf{l}_u}(F_{t(\mathbf{l}_u)} \setminus U_{t(\mathbf{l}_u)})\big) \\
&= \sum_{\mathbf{f}\in E_u^l} p_{\mathbf{f}}\mu_{t(\mathbf{f})}\big(S_{\mathbf{f}}^{-1}(S_{\mathbf{l}_u}(F_{t(\mathbf{l}_u)} \setminus U_{t(\mathbf{l}_u)}))\big) && \text{(by (4.7.15))} \\
&\geqslant p_{\mathbf{l}_u}\mu_{t(\mathbf{l}_u)}(F_{t(\mathbf{l}_u)} \setminus U_{t(\mathbf{l}_u)}).
\end{aligned}$$

This proves $\mu_{t(\mathbf{l}_u)}(F_{t(\mathbf{l}_u)} \setminus U_{t(\mathbf{l}_u)}) = 0$. The path $\mathbf{l}_u$ can always be extended to a path $\mathbf{l}'_u \in E_u^{l'}$, for some $l' > l$, with $t(\mathbf{l}'_u) = v$ for any vertex $v \in V$. This is because the graph is strongly connected, so a path $\mathbf{g}$ can always be found with $i(\mathbf{g}) = t(\mathbf{l}_u)$ and $t(\mathbf{g}) = v$, for any vertex $v \in V$. So we may put $\mathbf{l}'_u = \mathbf{l}_u\mathbf{g}$ and Equation (4.7.16) becomes

$$S_{\mathbf{l}'_u}(F_{t(\mathbf{l}'_u)}) \subset U_u.$$

Repeating the argument above with $\mathbf{l}_u$ replaced by $\mathbf{l}'_u$ and l replaced by l', proves that $\mu_{t(\mathbf{l}'_u)}(F_{t(\mathbf{l}'_u)} \setminus U_{t(\mathbf{l}'_u)}))) = 0$, and as $t(\mathbf{l}'_u) = v$, this proves that for each $v \in V$,

$$\mu_v(F_v \setminus U_v) = 0. \tag{4.7.20}$$

Consider any finite path $\mathbf{e} \in E^*$, then $\mathbf{e} \in E_v^n$ for some vertex v and $n = |\mathbf{e}|$. Let $\mathbf{f}, \mathbf{g} \in E_v^n$, $\mathbf{f} \neq \mathbf{g}$. By Lemma 4.7.6(b), $S_{\mathbf{f}}(U_{t(\mathbf{f})}) \cap S_{\mathbf{g}}(F_{t(\mathbf{g})}) = \emptyset$, so that

$$S_{\mathbf{f}}^{-1}\big(S_{\mathbf{f}}(U_{t(\mathbf{f})}) \cap S_{\mathbf{g}}(F_{t(\mathbf{g})})\big) = U_{t(\mathbf{f})} \cap S_{\mathbf{f}}^{-1}\big(S_{\mathbf{g}}(F_{t(\mathbf{g})})\big) = \emptyset. \tag{4.7.21}$$

This means that, since $\operatorname{supp}\mu_{t(\mathbf{f})} = F_{t(\mathbf{f})}$,

$$\begin{aligned}
&\mu_{t(\mathbf{f})}\big(S_{\mathbf{f}}^{-1}\big(S_{\mathbf{g}}(F_{t(\mathbf{g})})\big)\big) = \mu_{t(\mathbf{f})}\big(F_{t(\mathbf{f})} \cap S_{\mathbf{f}}^{-1}\big(S_{\mathbf{g}}(F_{t(\mathbf{g})})\big)\big) \\
&\qquad = \mu_{t(\mathbf{f})}\big(U_{t(\mathbf{f})} \cap S_{\mathbf{f}}^{-1}\big(S_{\mathbf{g}}(F_{t(\mathbf{g})})\big)\big) + \mu_{t(\mathbf{f})}\big(\big(F_{t(\mathbf{f})} \setminus U_{t(\mathbf{f})}\big) \cap S_{\mathbf{f}}^{-1}\big(S_{\mathbf{g}}(F_{t(\mathbf{g})})\big)\big) \\
&\qquad = 0,
\end{aligned}$$

by (4.7.21) and (4.7.20). That is for $\mathbf{f}, \mathbf{g} \in E_v^n$, $\mathbf{f} \neq \mathbf{g}$,

$$\mu_{t(\mathbf{f})}\big(S_{\mathbf{f}}^{-1}\big(S_{\mathbf{g}}(F_{t(\mathbf{g})})\big)\big) = 0. \tag{4.7.22}$$

For any $\mathbf{e} \in E^*$, $\mathbf{e} \in E_v^n$ for some vertex v with $n = |\mathbf{e}|$, and so by Equation (4.7.15) we obtain

$$\begin{aligned}
\mu_v(S_{\mathbf{e}}(F_{t(\mathbf{e})})) &= \sum_{\mathbf{f} \in E_v^n} p_{\mathbf{f}} \mu_{t(\mathbf{f})}\big(S_{\mathbf{f}}^{-1}(S_{\mathbf{e}}(F_{t(\mathbf{e})}))\big) \\
&= p_{\mathbf{e}} \mu_{t(\mathbf{e})}(F_{t(\mathbf{e})}) && \text{(by (4.7.22))} \\
&= p_{\mathbf{e}} && \text{(as } \mu_{t(\mathbf{e})}(F_{t(\mathbf{e})}) = 1\text{).}
\end{aligned}$$
□

The statement of the next lemma is a simple adaptation of a lemma in [Hut81], and of Lemma 9.2 in [Fal03].

Lemma 4.7.8. *Let $r, c_1, c_2 > 0$, and let $\{V_i\}$ be subsets of $\mathbb{R}^n$. Suppose each set V_i contains a closed ball B_i of radius $c_1 r$ and is contained in a closed ball of radius $c_2 r$, and that $\{B_i\}$ is a disjoint set. Then for any $x \in \mathbb{R}^n$,*

$$\#\left\{i : \overline{V}_i \cap B(x, r) \neq \emptyset\right\} \leqslant \left(\frac{1 + 2c_2}{c_1}\right)^n.$$

Lemma 4.7.9. *Let $p \in \mathbb{R}$, let $a_i \geqslant 0$ for $1 \leqslant i \leqslant m$, and let $m \leqslant C$. Then*

$$\left(\sum_{i=1}^m a_i\right)^p \leqslant \max\left\{1, C^{p-1}\right\} \sum_{i=1}^m a_i^p.$$

Proof. Minkowski's inequality, for $0 < p < 1$, can be used to show

$$\left(\sum_{i=1}^m a_i\right)^p \leqslant \sum_{i=1}^m a_i^p,$$

and as this also clearly holds for $p \leqslant 0$ and $p = 1$, the inequality holds for $p \leqslant 1$.

For $1 < p$, Hölder's inequality can be used to show

$$\left(\sum_{i=1}^m a_i\right)^p \leqslant m^{p-1} \sum_{i=1}^m a_i^p \leqslant C^{p-1} \sum_{i=1}^m a_i^p.$$
□

We take $(\mathbf{l}_v)_{v \in V}$ to be the fixed list of paths used in the definition of the non-negative matrix $\mathbf{B}(q, \gamma, l)$, in Subsection 4.2.6. The paths $(\mathbf{l}_v)_{v \in V}$, all of the same length $l = |\mathbf{l}_v|$, are such that

$$S_{\mathbf{l}_v}(F_{t(\mathbf{l}_v)}) \subset U_v, \tag{4.7.23}$$

for each $v \in V$, where $(U_v)_{v \in V}$ is the list of non-empty bounded open sets which satisfy the SOSC as described in Subsection 4.2.6.

The compactness of $S_{\mathbf{l}_v}(F_{t(\mathbf{l}_v)})$ also means that the distance from $S_{\mathbf{l}_v}(F_{t(\mathbf{l}_v)})$ to the closed set $\mathbb{R}^n \setminus U_v$ is positive, and this leads to an associated list of positive constants $(c_v)_{v \in V}$ where

$$c_v = \operatorname{dist}(S_{\mathbf{l}_v}(F_{t(\mathbf{l}_v)}), \mathbb{R}^n \setminus U_v) > 0. \tag{4.7.24}$$

By Lemma 4.7.6(b), for all finite paths $\mathbf{e}, \mathbf{g} \in E_u^*$, with

$$\mathbf{e} = e_1 \cdots e_k, \ \mathbf{g} = g_1 \cdots g_j, \ \text{and } e_1 \neq g_1,$$

$$S_{\mathbf{e}}(F_{t(\mathbf{e})}) \cap S_{\mathbf{g}}(U_{t(\mathbf{g})}) = \emptyset. \tag{4.7.25}$$

Lemma 4.7.10. *Let $(\mathbf{l}_v)_{v \in V}$ be the list of paths defined in Equation (4.7.23) and Subsection 4.2.6, and let $(c_v)_{v \in V}$ be the associated list of positive constants defined in Equation (4.7.24). Let $\mathbf{e}, \mathbf{g} \in E_u^*$ be any finite paths with*

$$\mathbf{e} = e_1 \cdots e_k, \ \mathbf{g} = g_1 \cdots g_j, \ \text{and } e_1 \neq g_1,$$

and suppose

$$\operatorname{dist}(S_{\mathbf{e}}(F_{t(\mathbf{e})}), S_{\mathbf{g}}(F_{t(\mathbf{g})})) \leqslant c_w r_{\mathbf{g}}$$

for some vertex $w \in V$.

Then $\mathbf{l}_w$ is not a subpath of $g_2 \cdots g_j$, that is $\mathbf{l}_w \not\subset g_2 \cdots g_j$.

Proof. For a contradiction we assume $\mathbf{l}_w$ is a subpath of $g_2 \cdots g_j$ so that $\mathbf{g} = \mathbf{s}\mathbf{l}_w\mathbf{t}$ where $\mathbf{s} \neq \emptyset$, $i(\mathbf{s}) = u$, $t(\mathbf{s}) = i(\mathbf{l}_w) = w$ and $t(\mathbf{l}_w) = i(\mathbf{t})$. Clearly $S_{\mathbf{g}}(F_{t(\mathbf{g})}) = S_{\mathbf{s}\mathbf{l}_w\mathbf{t}}(F_{t(\mathbf{t})}) \subset S_{\mathbf{s}\mathbf{l}_w}(F_{t(\mathbf{l}_w)}) \subset S_{\mathbf{s}}(U_w)$, by (4.7.23). This implies that

$$\begin{aligned}
\operatorname{dist}(S_{\mathbf{g}}(F_{t(\mathbf{g})}), \mathbb{R}^n \setminus S_{\mathbf{s}}(U_w)) &\geqslant \operatorname{dist}(S_{\mathbf{s}\mathbf{l}_w}(F_{t(\mathbf{l}_w)}), \mathbb{R}^n \setminus S_{\mathbf{s}}(U_w)) \\
&= r_{\mathbf{s}} c_w && \text{(by (4.7.24))} \\
&> r_{\mathbf{g}} c_w && \text{(as } \mathbf{s} \subset \mathbf{g}\text{)}.
\end{aligned}$$

By assumption $s_1 = g_1 \neq e_1$ and also $t(\mathbf{s}) = w$ so that $S_{\mathbf{e}}(F_{t(\mathbf{e})}) \cap S_{\mathbf{s}}(U_w) = \emptyset$ by (4.7.25). This means that $S_{\mathbf{e}}(F_{t(\mathbf{e})}) \subset \mathbb{R}^n \setminus S_{\mathbf{s}}(U_w)$ and so

$$\begin{aligned}
\operatorname{dist}(S_{\mathbf{g}}(F_{t(\mathbf{g})}), S_{\mathbf{e}}(F_{t(\mathbf{e})})) &\geqslant \operatorname{dist}(S_{\mathbf{g}}(F_{t(\mathbf{g})}), \mathbb{R}^n \setminus S_{\mathbf{s}}(U_w)) \\
&> c_w r_{\mathbf{g}}.
\end{aligned}$$

This is the required contradiction. □

We remind the reader that $r_{\min} = \min\{r_e : e \in E^1\}$ and we define $d_{\max} = \max\{|F_u| : u \in V\}$, with $r_{\max}$ and $d_{\min}$ defined similarly. Also let $c_{\min} = \min\{c_u : u \in V\}$ where the positive constants $(c_u)_{u \in V}$ are as given in Equation (4.7.24).

For $\mathbf{e} \in E_u^*$, $\mathbf{e}|_{|\mathbf{e}|-1}$ is the finite path obtained by deleting the last edge of $\mathbf{e}$. For $r > 0$ let

$$E_u^*(r) = \left\{ \mathbf{e} \in E_u^* : r_{\mathbf{e}} \left|F_{t(\mathbf{e})}\right| < r \leqslant r_{\mathbf{e}|_{|\mathbf{e}|-1}} \left|F_{t(\mathbf{e}|_{|\mathbf{e}|-1})}\right| \right\} \tag{4.7.26}$$

We make the following observations about the set of finite paths $E_u^*(r)$.

- For paths $\mathbf{e} \in E_u^*(r)$, the sets $S_{\mathbf{e}}(F_{t(\mathbf{e})}) \subset F_u$ are all roughly of diameter r since $\left|S_{\mathbf{e}}(F_{t(\mathbf{e})})\right| = r_{\mathbf{e}} \left|F_{t(\mathbf{e})}\right|$.
- It can be shown, using Lemma 1.3.5, that

$$F_u = \bigcup_{\mathbf{e} \in E_u^*(r)} S_{\mathbf{e}}(F_{t(\mathbf{e})}).$$

- If $(U_v)_{v\in V}$ are the open sets of the OSC, then the sets $\{S_{\mathbf{e}}(U_{t(\mathbf{e})}) : \mathbf{e} \in E_u^*(r)\}$ are disjoint open sets. This follows from the definition of the OSC and Lemma 4.7.6(a), using the fact that $\mathbf{e}, \mathbf{g} \in E_u^*(r)$, $\mathbf{e} \neq \mathbf{g}$, implies $\mathbf{e} \not\subset \mathbf{g}$ and $\mathbf{g} \not\subset \mathbf{e}$.

We may always choose $N \in \mathbb{N}$ large enough so that

$$\frac{2d_{\max}}{c_{\min}} \leqslant \frac{1}{r_{\max}^{N-1}}, \tag{4.7.27}$$

and for such N we now define δ as

$$\delta = r_{\min}^{N+l+1} d_{\min}. \tag{4.7.28}$$

Because we are not restricted in our choice of the length l, of the paths $(\mathbf{l}_v)_{v\in V}$, it is clear that from now on we may assume $0 < \delta < 1$ for any given system.

We remind the reader that for a given r, $H_{e,f}$ is an r-separated subset of $S_e(F_{t(e)}) \cap S_f(F_{t(f)})(r)$, where the edges $e, f \in E_u^1$, are taken as fixed with $e \neq f$.

Lemma 4.7.11. *Let $r \in (0, \delta)$, let $x \in H_{e,f}$, let $(\mathbf{l}_v)_{v\in V}$ be the list of paths defined in Equation (4.7.23) and Subsection 4.2.6, let N be as defined in Equation (4.7.27), and let $\mathbf{e} = e_1 \dots e_{|\mathbf{e}|} \in E_u^*(r)$ be such that* $\operatorname{dist}(x, S_{\mathbf{e}}(F_{t(\mathbf{e})})) \leqslant r$.

Then

$$\mathbf{l}_v \not\subset e_2 \dots e_{|\mathbf{e}|-N},$$

for all $v \in V$.

Proof. For $r \in (0, \delta)$, considered fixed, and a path $\mathbf{e} \in E_u^*(r)$, if $|\mathbf{e}| < N + l + 1$, then

$$r < \delta = r_{\min}^{N+l+1} d_{\min} < r_{\min}^{|\mathbf{e}|} d_{\min} \leqslant r_{\mathbf{e}} \left|F_{t(\mathbf{e})}\right| < r,$$

and this contradiction ensures $|\mathbf{e}| \geqslant N + l + 1$. Let $\mathbf{e} \in E_u^*(r)$ be written as $\mathbf{e} = e_1 \dots\ e_{|\mathbf{e}|}$. Either $e_1 = e$ or $e_1 \neq e$, and so we consider these two cases in turn.

(a) $e_1 = e$.

In this case $e_1 \neq f$. Since $S_{\mathbf{e}}(F_{t(\mathbf{e})}) \subset S_{\mathbf{e}|_{|\mathbf{e}|-N}}(F_{t(\mathbf{e}|_{|\mathbf{e}|-N})})$ it follows that

$$\operatorname{dist}(x, S_{\mathbf{e}|_{|\mathbf{e}|-N}}(F_{t(\mathbf{e}|_{|\mathbf{e}|-N})})) \leqslant \operatorname{dist}(x, S_{\mathbf{e}}(F_{t(\mathbf{e})})) \leqslant r.$$

As $x \in H_{e,f}$, $x \in S_f(F_{t(f)})(r)$ and from the definition of the closed r-neighbourhood

$$\operatorname{dist}(x, S_f(F_{t(f)})) \leqslant r.$$

Hence

$$\begin{aligned}
\operatorname{dist}(S_f(F_{t(f)}),&S_{\mathbf{e}|_{|\mathbf{e}|-N}}(F_{t(\mathbf{e}|_{|\mathbf{e}|-N})})) \\
&\leqslant \operatorname{dist}(x, S_f(F_{t(f)})) + \operatorname{dist}(x, S_{\mathbf{e}|_{|\mathbf{e}|-N}}(F_{t(\mathbf{e}|_{|\mathbf{e}|-N})})) \\
&\leqslant 2r \\
&\leqslant \frac{c_{\min}}{r_{\max}^{N-1} d_{\max}} r && \text{(by (4.7.27))} \\
&\leqslant \frac{c_{\min}}{r_{\max}^{N-1}} \left(r_{\mathbf{e}|_{|\mathbf{e}|-1}} \frac{\left|F_{t(\mathbf{e}|_{|\mathbf{e}|-1})}\right|}{d_{\max}} \right) && \text{(as } \mathbf{e} \in E_u^*(r))
\end{aligned}$$

$$\begin{aligned} &\leqslant c_{\min} r_{\mathbf{e}|_{|\mathbf{e}|-N}} \\ &\leqslant c_v r_{\mathbf{e}|_{|\mathbf{e}|-N}}, \end{aligned}$$

for all $v \in V$.

Applying Lemma 4.7.10 it follows that $\mathbf{l}_v \not\subset e_2 \dots e_{|\mathbf{e}|-N}$ for all $v \in V$.

(b) $e_1 \neq e$.

In this case the argument is almost identical to that given in part (a), but using $S_e(F_{t(e)})$ in place of $S_f(F_{t(f)})$, where we have $\mathrm{dist}(x, S_e(F_{t(e)})) = 0 \leqslant r$. □

Lemma 4.7.12. *Let $r \in (0, \delta)$ and let $x \in H_{e,f}$, then there exists a path $\mathbf{e}_x \in E_u^*(r)$, which depends on x, such that*

$$\text{(a)} \qquad x \in S_{\mathbf{e}_x}(F_{t(\mathbf{e}_x)}) \subset F_u \cap B(x,r) \subset \bigcup_{\substack{\mathbf{e} \in E_u^*(r) \\ \mathrm{dist}(x, S_{\mathbf{e}}(F_{t(\mathbf{e})})) \leqslant r}} S_{\mathbf{e}}(F_{t(\mathbf{e})}),$$

$$\text{(b)} \qquad \mu_u(B(x,r))^q \leqslant \begin{cases} \mu_u(S_{\mathbf{e}_x}(F_{t(\mathbf{e}_x)}))^q, & \text{if } q \leqslant 0, \\ \left(\displaystyle\sum_{\substack{\mathbf{e} \in E_u^*(r) \\ \mathrm{dist}(x, S_{\mathbf{e}}(F_{t(\mathbf{e})})) \leqslant r}} \mu_u(S_{\mathbf{e}}(F_{t(\mathbf{e})})) \right)^q, & \text{if } q > 0. \end{cases}$$

Proof. As $\mathrm{supp}\mu_u = F_u$, $\mu_u(B(x,r)) = \mu_u(F_u \cap B(x,r))$, part (b) is an immediate consequence of part (a).

$H_{e,f}$ is an r-separated subset of $S_e(F_{t(e)}) \cap S_f(F_{t(f)})(r)$, so $x \in F_u$ and the map $\phi_u : E_u^{\mathbb{N}} \to F_u$ given in Lemma 1.3.5, of Chapter 1 ensures the existence of an infinite path $\mathbf{e} \in E_u^{\mathbb{N}}$ with

$$\{x\} = \bigcap_{k=1}^{\infty} S_{\mathbf{e}|_k}(F_{t(\mathbf{e}|_k)})$$

Now $\left(S_{\mathbf{e}|_k}(F_{t(\mathbf{e}|_k)})\right)$ is a decreasing sequence of non-empty compact sets whose diameters tend to zero as k tends to infinity and so there exists $n \in \mathbb{N}$ such that

$$\left|S_{\mathbf{e}|_n}(F_{t(\mathbf{e}|_n)})\right| = r_{\mathbf{e}|_n} \left|F_{t(\mathbf{e}|_n)}\right| < r \leqslant r_{\mathbf{e}|_{n-1}} \left|F_{t(\mathbf{e}|_{n-1})}\right| = \left|S_{\mathbf{e}|_{n-1}}(F_{t(\mathbf{e}|_{n-1})})\right|.$$

Putting $\mathbf{e}_x = \mathbf{e}|_n$, $\mathbf{e}_x \in E_u^*(r)$ and

$$x \in S_{\mathbf{e}_x}(F_{t(\mathbf{e}_x)}) \subset F_u \cap B(x,r).$$

By the same argument for any $y \in F_u \cap B(x,r)$ there exists a path $\mathbf{e}_y \in E_u^*(r)$ such that $y \in S_{\mathbf{e}_y}(F_{t(\mathbf{e}_y)}) \subset F_u \cap B(y,r)$. Since $y \in B(x,r)$ it follows that

$$\mathrm{dist}(x, S_{\mathbf{e}_y}(F_{t(\mathbf{e}_y)})) \leqslant r$$

so that

$$F_u \cap B(x,r) \subset \bigcup_{\substack{\mathbf{e} \in E_u^*(r) \\ \mathrm{dist}(x, S_{\mathbf{e}}(F_{t(\mathbf{e})})) \leqslant r}} S_{\mathbf{e}}(F_{t(\mathbf{e})}).$$ □

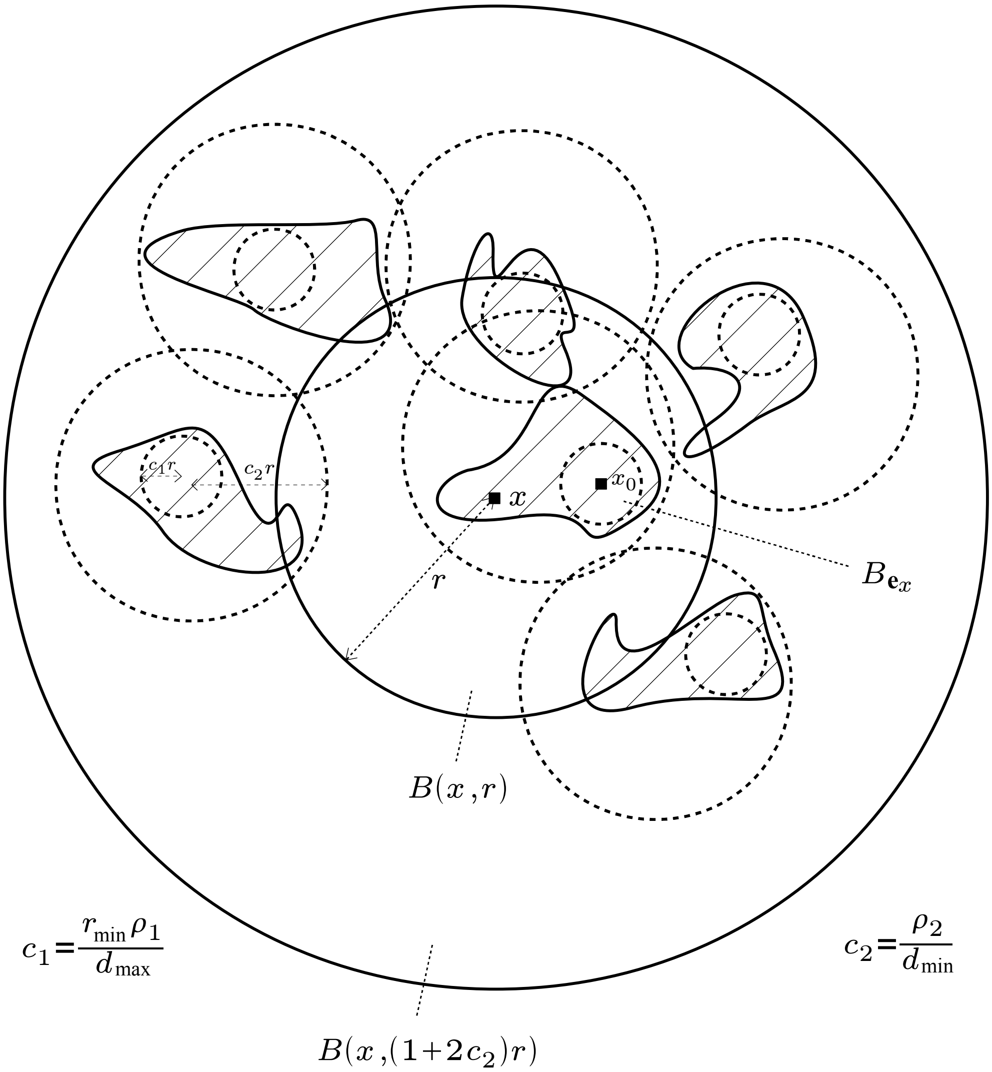


Figure. 4.7.2: The components $\{S_{\mathbf{e}}(\overline{U}_{t(\mathbf{e})}) : \mathbf{e} \in E_u^*(r),\ \mathrm{dist}(x, S_{\mathbf{e}}(F_{t(\mathbf{e})})) \leqslant r\}$, illustrated as shaded areas in $\mathbb{R}^2$. Each component contains a closed ball of radius $c_1 r$ and is contained in a closed ball of radius $c_2 r$.

Lemma 4.7.13. *Let $q \in \mathbb{R}$. Then there exists a positive number $C_2(q)$, such that for all $r \in (0, \delta)$ and all $x \in H_{e,f}$,*

$$\mu_u(B(x,r))^q \leqslant C_2(q) \sum_{\substack{\mathbf{e} \in E_u^*(r) \\ \mathrm{dist}(x, S_{\mathbf{e}}(F_{t(\mathbf{e})})) \leqslant r}} \mu_u(S_{\mathbf{e}}(F_{t(\mathbf{e})}))^q.$$

Proof. The sets $\{S_{\mathbf{e}}(U_{t(\mathbf{e})}) : \mathbf{e} \in E_u^*(r)\}$ are disjoint open sets, where $(U_v)_{v \in V}$ are the open sets of the SOSC. We may assume each U_v contains a closed ball of radius ρ_1 and is contained in a closed ball of radius ρ_2, so that $S_{\mathbf{e}}(\overline{U}_{t(\mathbf{e})})$ contains a closed ball of radius $r_{\mathbf{e}}\rho_1$ and is contained in a closed ball of radius $r_{\mathbf{e}}\rho_2$. For any $\mathbf{e} \in E_u^*(r)$

$$r_{\mathbf{e}} \left|F_{t(\mathbf{e})}\right| < r \leqslant r_{\mathbf{e}|_{|\mathbf{e}|-1}} \left|F_{t(\mathbf{e}|_{|\mathbf{e}|-1})}\right|,$$

so that
$$\frac{r_{\min}}{d_{\max}} r \leqslant r_{\mathbf{e}} < \frac{r}{d_{\min}}.$$
This means that, for each $\mathbf{e} \in E_u^*(r)$, $S_{\mathbf{e}}(\overline{U}_{t(\mathbf{e})})$ contains a closed ball of radius $\frac{r_{\min}\rho_1}{d_{\max}} r = c_1 r$, which we label as $B_{\mathbf{e}}$, and is contained in a closed ball of radius $\frac{\rho_2}{d_{\min}} r = c_2 r$. The set $\{B_{\mathbf{e}} : \mathbf{e} \in E_u^*(r)\}$ is a disjoint set because $\left\{S_{\mathbf{e}}(U_{t(\mathbf{e})}) : \mathbf{e} \in E_u^*(r)\right\}$ is disjoint. The situation is illustrated schematically in Figure 4.7.2, in $\mathbb{R}^2$, where we have indicated the closed ball $B_{\mathbf{e}_x}$ contained in the component $S_{\mathbf{e}_x}(\overline{U}_{t(\mathbf{e}_x)})$. We now obtain

$$\begin{aligned}
\#\{\mathbf{e} : \mathbf{e} \in &E_u^*(r),\, \mathrm{dist}(x, S_{\mathbf{e}}(F_{t(\mathbf{e})})) \leqslant r\} \\
&= \#\left\{\mathbf{e} : \mathbf{e} \in E_u^*(r),\, S_{\mathbf{e}}(F_{t(\mathbf{e})}) \cap B(x,r) \neq \emptyset\right\} \\
&\leqslant \#\left\{\mathbf{e} : \mathbf{e} \in E_u^*(r),\, S_{\mathbf{e}}(\overline{U}_{t(\mathbf{e})}) \cap B(x,r) \neq \emptyset\right\} \quad \text{(by Lemma 1.3.6)} \\
&\leqslant \left(\frac{1 + \frac{2\rho_2}{d_{\min}}}{\frac{r_{\min}\rho_1}{d_{\max}}} \right)^n = C_1 \quad \text{(by Lemma 4.7.8).}
\end{aligned}$$

Here we have used Lemma 4.7.8 with $c_1 = \frac{r_{\min}\rho_1}{d_{\max}}$ and $c_2 = \frac{\rho_2}{d_{\min}}$. Applying Lemma 4.7.9 gives

$$\left(\sum_{\substack{\mathbf{e} \in E_u^*(r) \\ \mathrm{dist}(x, S_{\mathbf{e}}(F_{t(\mathbf{e})})) \leqslant r}} \mu_u(S_{\mathbf{e}}(F_{t(\mathbf{e})})) \right)^q \leqslant C_2(q) \sum_{\substack{\mathbf{e} \in E_u^*(r) \\ \mathrm{dist}(x, S_{\mathbf{e}}(F_{t(\mathbf{e})})) \leqslant r}} \mu_u(S_{\mathbf{e}}(F_{t(\mathbf{e})}))^q,$$

where $C_2(q) = \max\left\{1, C_1^{q-1}\right\}$. As $\mathbf{e}_x \in E_u^*(r)$ and $\mathrm{dist}(x, S_{\mathbf{e}_x}(F_{t(\mathbf{e}_x)})) = 0$ we also have
$$\mu_u(S_{\mathbf{e}_x}(F_{t(\mathbf{e}_x)}))^q \leqslant C_2(q) \sum_{\substack{\mathbf{e} \in E_u^*(r) \\ \mathrm{dist}(x, S_{\mathbf{e}}(F_{t(\mathbf{e})})) \leqslant r}} \mu_u(S_{\mathbf{e}}(F_{t(\mathbf{e})}))^q.$$

The result now follows by Lemma 4.7.12(b). □

In the next lemma we use Lemma 4.7.8 a second time to obtain a bound for $\sum_{x \in H_{e,f}} \mu_u(B(x,r))^q$.

Lemma 4.7.14. *Let $q \in \mathbb{R}$, let $r \in (0, \delta)$, and let $(\mathbf{l}_v)_{v \in V}$ be the list of paths defined in Equation (4.7.23) and Subsection 4.2.6.*

Then
$$\sum_{x \in H_{e,f}} \mu_u(B(x,r))^q \leqslant C_2(q) C_3 \sum_{\substack{\mathbf{e} \in E_u^*(r) \\ \forall v \in V:\ \mathbf{l}_v \not\subset e_2 \ldots e_{|\mathbf{e}|-N}}} p_{\mathbf{e}}^q.$$

Proof. By Lemma 4.7.12(a), given any $y \in H_{e,f}$, we can find a path $\mathbf{e}_y \in E_u^*(r)$ such that

$$\begin{aligned}
y \in S_{\mathbf{e}_y}(F_{t(\mathbf{e}_y)}) \subset F_u \cap B(y,r) \subset &\bigcup_{\substack{\mathbf{e} \in E_u^*(r) \\ \mathrm{dist}(y, S_{\mathbf{e}}(F_{t(\mathbf{e})})) \leqslant r}} S_{\mathbf{e}}(F_{t(\mathbf{e})}) \\
\subset &\bigcup_{\substack{\mathbf{e} \in E_u^*(r) \\ \mathrm{dist}(y, S_{\mathbf{e}}(F_{t(\mathbf{e})})) \leqslant r}} S_{\mathbf{e}}(\overline{U}_{t(\mathbf{e})}),
\end{aligned} \tag{4.7.29}$$

where $(U_v)_{v\in V}$ are the open sets of the SOSC and we have used Lemma 1.3.6. For $y \in H_{e,f}$ it is convenient to use the notation

$$\overline{U}(y) = \bigcup_{\substack{\mathbf{e}\in E_u^*(r) \\ \operatorname{dist}(y,S_{\mathbf{e}}(F_{t(\mathbf{e})}))\leqslant r}} S_{\mathbf{e}}(\overline{U}_{t(\mathbf{e})}).$$

As before we are assuming the open sets, $(U_v)_{v\in V}$, each contain a closed ball of radius ρ_1 and are contained in a closed ball of radius ρ_2. As explained in the proof of Lemma 4.7.13 this means that for each $\mathbf{e} \in E_u^*(r)$, $S_{\mathbf{e}}(\overline{U}_{t(\mathbf{e})})$ contains a closed ball, $B_{\mathbf{e}}$, of radius $\frac{r_{\min}\rho_1}{d_{\max}}r$ and is contained in a closed ball of radius $\frac{\rho_2}{d_{\min}}r$, where the set $\{B_{\mathbf{e}} : \mathbf{e} \in E_u^*(r)\}$ is disjoint.

Now consider $x, y \in H_{e,f}$ with $x \neq y$. The paths $\mathbf{e}_x, \mathbf{e}_y \in E_u^*(r)$, established in Equation (4.7.29) by Lemma 4.7.12(a), cannot be the same since $\mathbf{e}_x = \mathbf{e}_y$ means $x \in B(y, r)$ which is impossible as $H_{e,f}$ is r-separated. So $\mathbf{e}_x \neq \mathbf{e}_y$ for $x, y \in H_{e,f}$, $x \neq y$. This means that for each of the sets

$$\Big\{\overline{U}(x) : x \in H_{e,f}\Big\},$$

we may choose a single closed ball $B_x = B_{\mathbf{e}_x} \subset S_{\mathbf{e}_x}(\overline{U}_{t(\mathbf{e}_x)}) \subset \overline{U}(x)$, where B_x is of radius $\frac{r_{\min}\rho_1}{d_{\max}}r = c_1 r$, with $B_x = B(x_0, \frac{r_{\min}\rho_1}{d_{\max}}r)$ for some point $x_0 \in S_{\mathbf{e}_x}(\overline{U}_{t(\mathbf{e}_x)})$. The closed ball $B_x = B_{\mathbf{e}_x}$ is indicated in both Figures 4.7.2 and 4.7.3. The set $\{B_x : x \in H_{e,f}\}$ is then a disjoint set of closed balls.

It is also the case that $\big|S_{\mathbf{e}}(\overline{U}_{t(\mathbf{e})})\big| \leqslant \frac{2\rho_2}{d_{\min}}r$, because $S_{\mathbf{e}}(\overline{U}_{t(\mathbf{e})})$ is contained in a closed ball of radius $\frac{\rho_2}{d_{\min}}r$, so that for each $x \in H_{e,f}$,

$$\overline{U}(x) \subset B\Big(x, \Big(1 + \frac{2\rho_2}{d_{\min}}\Big)r\Big).$$

That is, for each $x \in H_{e,f}$, $\overline{U}(x)$ is contained in a closed ball of radius $\big(1 + \frac{2\rho_2}{d_{\min}}\big)r = c_2 r$. The situation is illustrated in Figure 4.7.3 in $\mathbb{R}^2$, for two sets $\overline{U}(x)$ and $\overline{U}(y)$.

We are now in a position to apply Lemma 4.7.8 again. Let $x \in H_{e,f}$ be fixed and let $\mathbf{g} \in E_u^*(r)$ be a path for which $\operatorname{dist}(x, S_{\mathbf{g}}(F_{t(\mathbf{g})})) \leqslant r$, which we also consider to be fixed. Let $N(x, \mathbf{g}) \in \mathbb{N}$, be the number of times the term $\mu_u(S_{\mathbf{g}}(F_{t(\mathbf{g})}))^q$ is counted in the sum

$$\sum_{y\in H_{e,f}} \sum_{\substack{\mathbf{e}\in E_u^*(r) \\ \operatorname{dist}(y,S_{\mathbf{e}}(F_{t(\mathbf{e})}))\leqslant r}} \mu_u(S_{\mathbf{e}}(F_{t(\mathbf{e})}))^q.$$

Then

$$\begin{aligned} N(x, \mathbf{g}) &= \#\left\{y : y \in H_{e,f} \text{ and } \operatorname{dist}(y, S_{\mathbf{g}}(F_{t(\mathbf{g})})) \leqslant r\right\} \\ &\leqslant \#\left\{ y : y \in H_{e,f} \text{ and } \bigcup_{\substack{\mathbf{e}\in E_u^*(r) \\ \operatorname{dist}(y,S_{\mathbf{e}}(F_{t(\mathbf{e})}))\leqslant r}} S_{\mathbf{e}}(F_{t(\mathbf{e})}) \quad \bigcap B(x, r) \neq \emptyset \right\} \\ &\leqslant \#\left\{y : y \in H_{e,f} \text{ and } \overline{U}(y) \cap B(x, r) \neq \emptyset\right\} \qquad \text{(by (4.7.29))} \end{aligned}$$

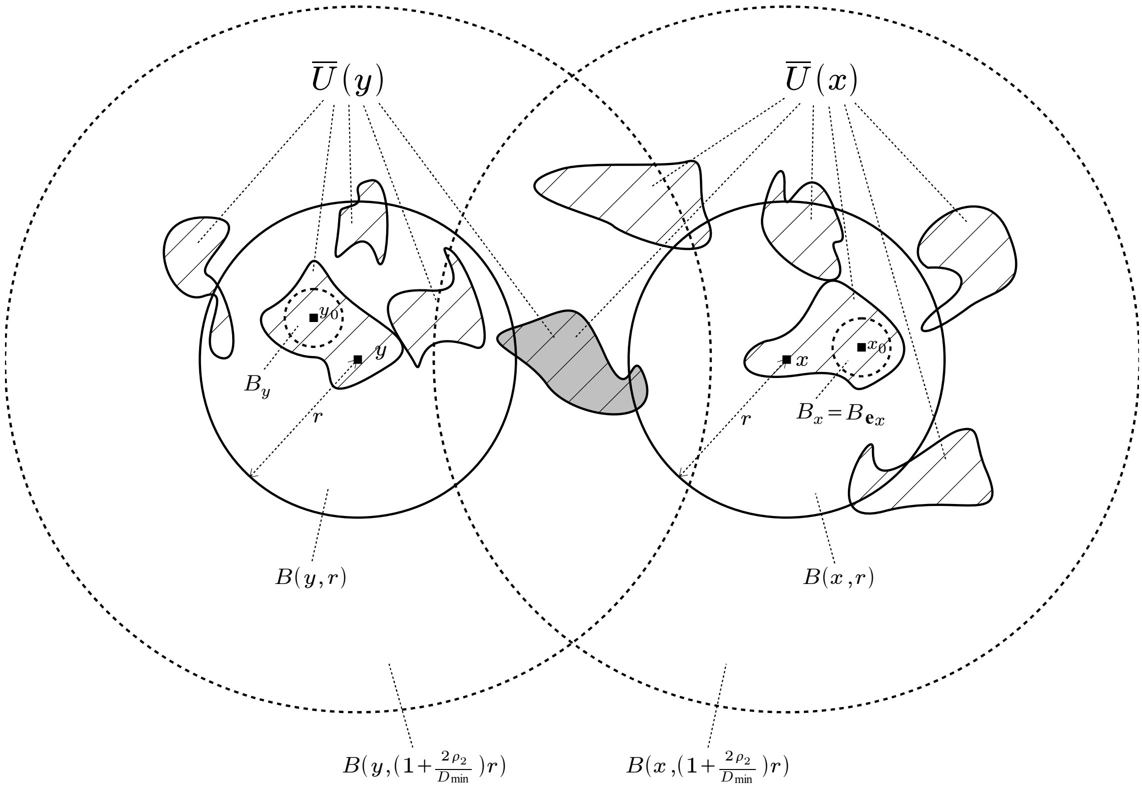


Figure. 4.7.3: Two sets of components $\overline{U}(x)$ and $\overline{U}(y)$, illustrated in $\mathbb{R}^2$. The grey component belongs to both $\overline{U}(x)$ and $\overline{U}(y)$.

$$\leqslant \left(\frac{1 + 2\left(1 + \frac{2\rho_2}{d_{\min}}\right)}{\frac{r_{\min}\rho_1}{d_{\max}}} \right)^n = C_3 \qquad \text{(by Lemma 4.7.8).}$$

Here we have applied Lemma 4.7.8 with $c_1 = \frac{r_{\min}\rho_1}{d_{\max}}$ and $c_2 = 1 + \frac{2\rho_2}{d_{\min}}$. Using this result it is clear that for each distinct path $\mathbf{e}$ in the sum

$$\sum_{x \in H_{e,f}} \sum_{\substack{\mathbf{e} \in E_u^*(r) \\ \operatorname{dist}(x, S_\mathbf{e}(F_{t(\mathbf{e})})) \leqslant r}} \mu_u(S_\mathbf{e}(F_{t(\mathbf{e})}))^q,$$

the term $\mu_u(S_\mathbf{e}(F_{t(\mathbf{e})}))^q$ is counted at most C_3 times. As an example, if $\mathbf{e}'$ is the path corresponding to the component $S_{\mathbf{e}'}(\overline{U}_{t(\mathbf{e}')})$, coloured grey in Figure 4.7.3, then $\operatorname{dist}(x, S_{\mathbf{e}'}(F_{t(\mathbf{e}')})) \leqslant r$ and $\operatorname{dist}(y, S_{\mathbf{e}'}(F_{t(\mathbf{e}')})) \leqslant r$, so that $\mu_u(S_{\mathbf{e}'}(F_{t(\mathbf{e}')}))^q$ would be counted at least twice in this sum, for $x, y \in H_{e,f}$.

This implies that

$$\begin{aligned}
\sum_{x \in H_{e,f}} \mu_u(B(x,r))^q &\leqslant C_2(q) \sum_{x \in H_{e,f}} \sum_{\substack{\mathbf{e} \in E_u^*(r) \\ \operatorname{dist}(x, S_\mathbf{e}(F_{t(\mathbf{e})})) \leqslant r}} \mu_u(S_\mathbf{e}(F_{t(\mathbf{e})}))^q \quad \text{(by Lemma 4.7.13)} \\
&\leqslant C_2(q) C_3 \sum_{\substack{\mathbf{e} \in E_u^*(r) \\ \forall v \in V:\ \mathbf{l}_v \not\subset e_2 \ldots e_{|\mathbf{e}|-N}}} \mu_u(S_\mathbf{e}(F_{t(\mathbf{e})}))^q \quad \text{(by Lemma 4.7.11)}
\end{aligned}$$

$$\leqslant C_2(q)C_3 \sum_{\substack{\mathbf{e}\in E_u^*(r)\\ \forall v\in V:\ \mathbf{l}_v \not\subset e_2\dots e_{|\mathbf{e}|-N}}} p_{\mathbf{e}}^q \qquad \text{(by Lemma 4.7.7).}$$

□

We now define, for $r \in (0,+\infty)$, two related vectors $(G_w(r))_{w\in V}$ and $(\mathcal{G}_w(r))_{w\in V}$. For $q \in \mathbb{R}$, let $\gamma = \gamma(q) \in \mathbb{R}$ be the unique number such that $\rho(\mathbf{B}(q,\gamma,l)) = 1$ for the matrix $\mathbf{B}(q,\gamma,l))$, as defined in Subsection 4.2.6, which we assume is irreducible. Let $(\mathbf{l}_v)_{v\in V}$ be the list of paths defined in Equation (4.7.23) and Subsection 4.2.6, let N be as chosen in Inequality (4.7.27), let $r \in (0,+\infty)$, and let

$$\alpha = \frac{1}{r_{\max}^N d_{\max}} \quad \text{and} \quad \beta = \frac{1}{r_{\min}^{N+1} d_{\min}}.$$

For each $w \in V$, let

$$G_w(r) = \sum_{\substack{\mathbf{g}\in E_w^*\\ \alpha r \leqslant r_{\mathbf{g}} < \beta r\\ \forall v\in V:\ \mathbf{l}_v \not\subset \mathbf{g}}} p_{\mathbf{g}}^q, \tag{4.7.30}$$

and

$$\mathcal{G}_w(r) = r^{\gamma(q)} G_w(r). \tag{4.7.31}$$

We point out here that for small r, with $r \in (0,\delta)$,

$$G_w(r) = \sum_{\substack{\mathbf{g}\in E_w^*\\ \alpha r \leqslant r_{\mathbf{g}} < \beta r\\ \forall v\in V:\ \mathbf{l}_v \not\subset \mathbf{g}}} p_{\mathbf{g}}^q = \sum_{\substack{\mathbf{g}\in E_w^*\\ |\mathbf{g}|\geqslant l\\ \alpha r \leqslant r_{\mathbf{g}} < \beta r\\ \forall v\in V:\ \mathbf{l}_v \not\subset \mathbf{g}}} p_{\mathbf{g}}^q. \tag{4.7.32}$$

This is because $0 < r < \delta = r_{\min}^{N+l+1} d_{\min}$, so if $|\mathbf{g}| < l$ and $r_{\mathbf{g}} < \frac{r}{r_{\min}^{N+1} d_{\min}} = \beta r$ then

$$r_{\mathbf{g}} < \frac{r}{r_{\min}^{N+1} d_{\min}} < r_{\min}^{l} < r_{\min}^{|\mathbf{g}|} \leqslant r_{\mathbf{g}},$$

and this contradiction ensures $|\mathbf{g}| \geqslant l$.

Lemma 4.7.15. *Let $r \in (0,\delta)$, and let $G_w(r)$ be as defined in Equation (4.7.30), for each $w \in V$.*

Then

$$\sum_{x\in H_{e,f}} \mu_u(B(x,r))^q \leqslant C_2(q)C_3C_4(q) \sum_{w\in V} G_w(r).$$

Proof. As we showed in the proof of Lemma 4.7.11, for $r \in (0,\delta)$, $\mathbf{e} \in E_u^*(r)$ implies $|\mathbf{e}| \geqslant N+l+1$, so $\mathbf{e}$ can always be written as $\mathbf{e} = \mathbf{sgt}$, for some paths $\mathbf{s},\mathbf{g},\mathbf{t}$, with $|\mathbf{s}| = 1$, $|\mathbf{g}| \geqslant l$ and $|\mathbf{t}| = N$. From the definition of $E_u^*(r)$, for $\mathbf{e} = \mathbf{sgt} \in E_u^*(r)$,

$$r_{\min}^{N+1} r_{\mathbf{g}} d_{\min} \leqslant r_{\mathbf{sgt}} \left|F_{t(\mathbf{sgt})}\right| < r \leqslant r_{\mathbf{sgt}|_{|\mathbf{sgt}|-1}} \left|F_{t(\mathbf{sgt}|_{|\mathbf{sgt}|-1})}\right| \leqslant r_{\max}^N r_{\mathbf{g}} d_{\max},$$

and so

$$\alpha r = \frac{r}{r_{\max}^N d_{\max}} \leqslant r_{\mathbf{g}} < \frac{r}{r_{\min}^{N+1} d_{\min}} = \beta r. \tag{4.7.33}$$

This gives

$$
\begin{aligned}
\sum_{x\in H_{e,f}} \mu_u(B(x,r))^q &\leqslant C_2(q)C_3 \sum_{\substack{\mathbf{e}\in E_u^*(r)\\ \forall v\in V:\ \mathbf{l}_v \not\subset e_2\ldots e_{|\mathbf{e}|-N}}} p_{\mathbf{e}}^q && \text{(by Lemma 4.7.14)}\\
&= C_2(q)C_3 \sum_{\substack{\mathbf{e}=\mathbf{sgt}\in E_u^*(r)\\ |\mathbf{s}|=1,\,|\mathbf{t}|=N,\,|\mathbf{g}|\geqslant l\\ \forall v\in V:\ \mathbf{l}_v \not\subset e_2\ldots e_{|\mathbf{e}|-N}=\mathbf{g}}} p_{\mathbf{sgt}}^q\\
&\leqslant C_2(q)C_3 \sum_{\mathbf{s}\in E_u^1}\sum_{\mathbf{t}\in E^N} \sum_{\substack{\mathbf{g}\in E^*\\ |\mathbf{g}|\geqslant l\\ \alpha r\leqslant r_{\mathbf{g}}<\beta r\\ \forall v\in V:\ \mathbf{l}_v\not\subset \mathbf{g}}} p_{\mathbf{s}}^q p_{\mathbf{g}}^q p_{\mathbf{t}}^q && \text{(by (4.7.33))}\\
&\leqslant C_2(q)C_3C_4(q) \sum_{\substack{\mathbf{g}\in E^*\\ |\mathbf{g}|\geqslant l\\ \alpha r\leqslant r_{\mathbf{g}}<\beta r\\ \forall v\in V:\ \mathbf{l}_v\not\subset \mathbf{g}}} p_{\mathbf{g}}^q\\
&= C_2(q)C_3C_4(q) \sum_{w\in V} \sum_{\substack{\mathbf{g}\in E_w^*\\ |\mathbf{g}|\geqslant l\\ \alpha r\leqslant r_{\mathbf{g}}<\beta r\\ \forall v\in V:\ \mathbf{l}_v\not\subset \mathbf{g}}} p_{\mathbf{g}}^q\\
&= C_2(q)C_3C_4(q) \sum_{w\in V} G_w(r) && \text{(by (4.7.32)).}
\end{aligned}
$$

The positive constant $C_4(q)$, which depends on q, is given by

$$C_4(q) = k^{1+N}(\max\{p_e^q : e \in E^1\})^{1+N},$$

k being the maximum number of edges leaving any vertex in the directed graph. □

Lemma 4.7.16. *Let $r \in (0,\delta)$, and let $\mathcal{G}_w(r)$ be as defined in Equation (4.7.31), for each $w \in V$.*

Then

$$\mathcal{G}_w(r) \leqslant \sum_{z\in V} \sum_{\substack{\mathbf{s}\in E_{wz}^l\\ \mathbf{s}\neq \mathbf{l}_w}} p_{\mathbf{s}}^q r_{\mathbf{s}}^\gamma \mathcal{G}_z\left(\frac{r}{r_{\mathbf{s}}}\right).$$

Proof. It is clear from the definition of a subpath in Subsection 1.2.7 of Chapter 1, that

$$
\begin{aligned}
&\big\{\mathbf{st} : \mathbf{s}\in E_{wz}^l, \mathbf{t}\in E_z^* \text{ and } \forall v\in V,\ \mathbf{l}_v \not\subset \mathbf{st}\big\}\\
&\qquad\subset \big\{\mathbf{st} : \mathbf{s}\in E_{wz}^l,\ \mathbf{t}\in E_z^*,\ \mathbf{s}\neq \mathbf{l}_w \text{ and } \forall v\in V,\ \mathbf{l}_v\not\subset \mathbf{t}\big\},
\end{aligned}
\tag{4.7.34}
$$

and this implies

$$G_w(r) = \sum_{\substack{\mathbf{g}\in E_w^*\\ \alpha r\leqslant r_{\mathbf{g}}<\beta r\\ \forall v\in V:\ \mathbf{l}_v\not\subset \mathbf{g}}} p_{\mathbf{g}}^q$$

$$
\begin{aligned}
&= \sum_{\substack{\mathbf{g}\in E_w^* \\ |\mathbf{g}|\geqslant l \\ \alpha r\leqslant r_{\mathbf{g}}<\beta r \\ \forall v\in V:\ \mathbf{l}_v\not\subset \mathbf{g}}} p_{\mathbf{g}}^q && \text{(by (4.7.32), as } r\in(0,\delta))\\
&= \sum_{\substack{\mathbf{s}\in E_w^l \\ \mathbf{t}\in E_{t(\mathbf{s})}^* \\ \alpha r\leqslant r_{\mathbf{st}}<\beta r \\ \forall v\in V:\ \mathbf{l}_v\not\subset \mathbf{st}}} p_{\mathbf{st}}^q \\
&= \sum_{z\in V} \sum_{\substack{\mathbf{s}\in E_{wz}^l \\ \mathbf{t}\in E_z^* \\ \alpha r\leqslant r_{\mathbf{st}}<\beta r \\ \forall v\in V:\ \mathbf{l}_v\not\subset \mathbf{st}}} p_{\mathbf{st}}^q \\
&\leqslant \sum_{z\in V} \sum_{\substack{\mathbf{s}\in E_{wz}^l \\ \mathbf{s}\neq \mathbf{l}_w \\ \mathbf{t}\in E_z^* \\ \alpha r\leqslant r_{\mathbf{st}}<\beta r \\ \forall v\in V:\ \mathbf{l}_v\not\subset \mathbf{t}}} p_{\mathbf{st}}^q && \text{(by (4.7.34))}\\
&= \sum_{z\in V} \sum_{\substack{\mathbf{s}\in E_{wz}^l \\ \mathbf{s}\neq \mathbf{l}_w}} p_{\mathbf{s}}^q \sum_{\substack{\mathbf{t}\in E_z^* \\ \alpha \frac{r}{r_{\mathbf{s}}}\leqslant r_{\mathbf{t}}<\beta \frac{r}{r_{\mathbf{s}}} \\ \forall v\in V:\ \mathbf{l}_v\not\subset \mathbf{t}}} p_{\mathbf{t}}^q \\
&= \sum_{z\in V} \sum_{\substack{\mathbf{s}\in E_{wz}^l \\ \mathbf{s}\neq \mathbf{l}_w}} p_{\mathbf{s}}^q\, G_z\left(\frac{r}{r_{\mathbf{s}}}\right).
\end{aligned}
$$

We have used the convention here that $E_{t(\mathbf{s})}^*, E_z^*$ include the empty path which is summed over but doesn't contribute to the sum.

As $\mathcal{G}_w(r) = r^\gamma G_w(r)$, we obtain,

$$
\begin{aligned}
\mathcal{G}_w(r) &\leqslant \sum_{z\in V} \sum_{\substack{\mathbf{s}\in E_{wz}^l \\ \mathbf{s}\neq \mathbf{l}_w}} p_{\mathbf{s}}^q r_{\mathbf{s}}^\gamma \left(\frac{r}{r_{\mathbf{s}}}\right)^\gamma G_z\left(\frac{r}{r_{\mathbf{s}}}\right)\\
&= \sum_{z\in V} \sum_{\substack{\mathbf{s}\in E_{wz}^l \\ \mathbf{s}\neq \mathbf{l}_w}} p_{\mathbf{s}}^q r_{\mathbf{s}}^\gamma\, \mathcal{G}_z\left(\frac{r}{r_{\mathbf{s}}}\right). \qquad \square
\end{aligned}
$$

In the next two lemmas we use the notations

$$\boldsymbol{\mathcal{G}}(r) = (\mathcal{G}_w(r))_{w\in V},$$

and

$$\sup_{a\leqslant r} \boldsymbol{\mathcal{G}}(r) = \left(\sup_{a\leqslant r} \mathcal{G}_w(r)\right)_{w\in V}.$$

Lemma 4.7.17. *Let* $\mathbf{b}$ *be the positive right eigenvector of* $\mathbf{B}(q,\gamma,l)$*, with eigenvalue* $\rho(\mathbf{B}(q,\gamma,l)) = 1$*, as given in Equation (4.2.5). Let* $a \in (0,\delta)$ *and let* $\mathcal{G}_w(r)$ *be as defined in Equation (4.7.31), for each* $w \in V$*.*

Then

$$\sup_{a\leqslant r}\boldsymbol{\mathcal{G}}(r)\leqslant C_a(q)\mathbf{b},$$

for some positive $C_a(q)$.

Proof. For each $q\in\mathbb{R}$, $\gamma(q)$ is uniquely defined as the real number for which $\rho(\mathbf{B}(q,\gamma,l))=1$ and $\mathbf{b}$ is the associated positive eigenvector with eigenvalue 1, as given in Equation (4.2.5). To prove the lemma it is enough to show, for each $w\in V$, that

$$\sup_{a\leqslant r}\mathcal{G}_w(r)<+\infty.$$

As the eigenvector $\mathbf{b}>0$, a positive number $C_a(q)$ can then be determined. We remind the reader that

$$\alpha=\frac{1}{r_{\max}^N d_{\max}}\quad\text{and}\quad\beta=\frac{1}{r_{\min}^{N+1}d_{\min}}.$$

If $\gamma(q)\geqslant 0$ then $\alpha r\leqslant r_{\mathbf{g}}$ implies

$$r^\gamma\leqslant\left(\frac{r_{\mathbf{g}}}{\alpha}\right)^\gamma=\left(r_{\max}^N d_{\max}r_{\mathbf{g}}\right)^\gamma.$$

If $\gamma(q)<0$ then $r_{\mathbf{g}}<\beta r$ implies

$$r^\gamma<\left(\frac{r_{\mathbf{g}}}{\beta}\right)^\gamma=\left(r_{\min}^{N+1}d_{\min}r_{\mathbf{g}}\right)^\gamma.$$

So for $r\in[a,\infty)$, and each $w\in V$,

$$\begin{aligned}0\;\leqslant\;\mathcal{G}_w(r)\;&=\;r^\gamma G_w(r)=\sum_{\substack{\mathbf{g}\in E_w^*\\ \alpha r\leqslant r_{\mathbf{g}}<\beta r\\ \forall v\in V:\ \mathbf{l}_v\not\subset\mathbf{g}}}p_{\mathbf{g}}^q r^\gamma\\ &\leqslant\max\left\{(r_{\max}^N d_{\max})^\gamma,(r_{\min}^{N+1}d_{\min})^\gamma\right\}\sum_{\substack{\mathbf{g}\in E_w^*\\ \alpha a\leqslant r_{\mathbf{g}}\\ \forall v\in V:\ \mathbf{l}_v\not\subset\mathbf{g}}}p_{\mathbf{g}}^q r_{\mathbf{g}}^\gamma<\;+\infty.\end{aligned}$$

The strict inequality holds as there are only a finite number of paths $\mathbf{g}\in E_w^*$ with $\alpha a\leqslant r_{\mathbf{g}}$. □

Lemma 4.7.18. *Let* $\mathbf{b}$ *be the positive right eigenvector of* $\mathbf{B}(q,\gamma,l)$, *with eigenvalue* $\rho(\mathbf{B}(q,\gamma,l))=1$, *as given in Equation (4.2.5). Let* $a\in(0,\delta)$ *and let* $\mathcal{G}_w(r)$ *be as defined in Equation (4.7.31), for each* $w\in V$.

Then

$$\sup_{0<r}\boldsymbol{\mathcal{G}}(r)\leqslant C_a(q)\mathbf{b},$$

for some positive $C_a(q)$.

Proof. As in Lemma 4.7.17, let $a \in (0, \delta)$ be fixed, and let $\Delta = r_{\max}^l < 1$, then $\Delta \geqslant r_{\mathbf{s}}$ for all paths $\mathbf{s} \in E^l$, and $a\Delta < a < \delta$. Consider $r \in [a\Delta, \delta)$, with $a\Delta \leqslant r < \delta$, then

$$a \leqslant a\frac{\Delta}{r_{\mathbf{s}}} \leqslant \sigma < \frac{\delta}{r_{\mathbf{s}}}, \tag{4.7.35}$$

where $\sigma = \frac{r}{r_{\mathbf{s}}}$. It follows that

$$\begin{aligned}
\sup_{a\Delta \leqslant r < \delta} \mathcal{G}_w(r) &\leqslant \sup_{a\Delta \leqslant r < \delta} \left\{ \sum_{z \in V} \sum_{\substack{\mathbf{s} \in E_{wz}^l \\ \mathbf{s} \neq \mathbf{l}_w}} p_{\mathbf{s}}^q r_{\mathbf{s}}^\gamma \mathcal{G}_z\left(\frac{r}{r_{\mathbf{s}}}\right) \right\} && \text{(by Lemma 4.7.16)} \\
&= \sup_{a\frac{\Delta}{r_{\mathbf{s}}} \leqslant \sigma < \frac{\delta}{r_{\mathbf{s}}}} \left\{ \sum_{z \in V} \sum_{\substack{\mathbf{s} \in E_{wz}^l \\ \mathbf{s} \neq \mathbf{l}_w}} p_{\mathbf{s}}^q r_{\mathbf{s}}^\gamma \mathcal{G}_z(\sigma) \right\} \\
&\leqslant \sup_{a \leqslant \sigma} \left\{ \sum_{z \in V} \sum_{\substack{\mathbf{s} \in E_{wz}^l \\ \mathbf{s} \neq \mathbf{l}_w}} p_{\mathbf{s}}^q r_{\mathbf{s}}^\gamma \mathcal{G}_z(\sigma) \right\} && \text{(by Inequality (4.7.35))} \\
&\leqslant \sum_{z \in V} \sum_{\substack{\mathbf{s} \in E_{wz}^l \\ \mathbf{s} \neq \mathbf{l}_w}} p_{\mathbf{s}}^q r_{\mathbf{s}}^\gamma \sup_{a \leqslant \sigma} \left\{ \mathcal{G}_z(\sigma) \right\} \\
&= \sum_{z \in V} \sum_{\substack{\mathbf{s} \in E_{wz}^l \\ \mathbf{s} \neq \mathbf{l}_w}} p_{\mathbf{s}}^q r_{\mathbf{s}}^\gamma \sup_{a \leqslant r} \left\{ \mathcal{G}_z(r) \right\}.
\end{aligned}$$

In matrix form, using the notation $\sup \boldsymbol{\mathcal{G}}(r) = (\sup \mathcal{G}_w(r))_{w \in V}$, this is

$$\sup_{a\Delta \leqslant r < \delta} \boldsymbol{\mathcal{G}}(r) \leqslant \mathbf{B}(q, \gamma, l) \sup_{a \leqslant r} \boldsymbol{\mathcal{G}}(r).$$

By Lemma 4.7.17, we can find a positive number $C_a(q)$ such that

$$\sup_{a \leqslant r} \boldsymbol{\mathcal{G}}(r) \leqslant C_a(q)\mathbf{b}, \tag{4.7.36}$$

and this means that

$$\sup_{a\Delta \leqslant r < \delta} \boldsymbol{\mathcal{G}}(r) \leqslant \mathbf{B}(q, \gamma, l) \sup_{a \leqslant r} \boldsymbol{\mathcal{G}}(r) \leqslant \mathbf{B}(q, \gamma, l) C_a(q)\mathbf{b} = C_a(q)\mathbf{b}, \tag{4.7.37}$$

the last equality holds as $\mathbf{b}$ is a positive eigenvector of $\mathbf{B}(q, \gamma, l)$, with eigenvalue 1. Since $a\Delta < a < \delta$, inequalities (4.7.36) and (4.7.37) together imply

$$\sup_{a\Delta \leqslant r} \boldsymbol{\mathcal{G}}(r) \leqslant C_a(q)\mathbf{b}. \tag{4.7.38}$$

Now let $n \in \mathbb{N}$ and let $P(n)$ be the following statement

$$\sup_{a\Delta^n \leqslant r} \boldsymbol{\mathcal{G}}(r) \leqslant C_a(q)\mathbf{b}.$$

We use induction to prove that $P(n)$ is true for all $n \in \mathbb{N}$.

Induction base.
$P(1)$ is true by Inequality (4.7.38).

Induction hypothesis.
Let $k \in \mathbb{N}$ and suppose $P(k)$ is true, that is suppose

$$\sup_{a\Delta^k \leqslant r} \boldsymbol{\mathcal{G}}(r) \leqslant C_a(q)\mathbf{b}. \tag{4.7.39}$$

Induction step.
Let $r \in [a\Delta^{k+1}, \delta)$ then as in Inequality (4.7.35) above it follows that

$$a\Delta^k \leqslant a\frac{\Delta^{k+1}}{r_{\mathbf{s}}} \leqslant \sigma < \frac{\delta}{r_{\mathbf{s}}}, \tag{4.7.40}$$

where $\sigma = \frac{r}{r_{\mathbf{s}}}$. We now obtain

$$\begin{aligned}
\sup_{a\Delta^{k+1} \leqslant r < \delta} \mathcal{G}_w(r) &\leqslant \sup_{a\Delta^{k+1} \leqslant r < \delta} \left\{ \sum_{z \in V} \sum_{\substack{\mathbf{s} \in E^l_{wz} \\ \mathbf{s} \neq \mathbf{l}_w}} p^q_{\mathbf{s}} r^\gamma_{\mathbf{s}} \mathcal{G}_z\left(\frac{r}{r_{\mathbf{s}}}\right) \right\} \quad \text{(by Lemma 4.7.16)} \\
&= \sup_{a\frac{\Delta^{k+1}}{r_{\mathbf{s}}} \leqslant \sigma < \frac{\delta}{r_{\mathbf{s}}}} \left\{ \sum_{z \in V} \sum_{\substack{\mathbf{s} \in E^l_{wz} \\ \mathbf{s} \neq \mathbf{l}_w}} p^q_{\mathbf{s}} r^\gamma_{\mathbf{s}} \mathcal{G}_z(\sigma) \right\} \\
&\leqslant \sup_{a\Delta^k \leqslant \sigma} \left\{ \sum_{z \in V} \sum_{\substack{\mathbf{s} \in E^l_{wz} \\ \mathbf{s} \neq \mathbf{l}_w}} p^q_{\mathbf{s}} r^\gamma_{\mathbf{s}} \mathcal{G}_z(\sigma) \right\} \quad \text{(by Inequality (4.7.40))} \\
&\leqslant \sum_{z \in V} \sum_{\substack{\mathbf{s} \in E^l_{wz} \\ \mathbf{s} \neq \mathbf{l}_w}} p^q_{\mathbf{s}} r^\gamma_{\mathbf{s}} \sup_{a\Delta^k \leqslant \sigma} \left\{ \mathcal{G}_z(\sigma) \right\} \\
&= \sum_{z \in V} \sum_{\substack{\mathbf{s} \in E^l_{wz} \\ \mathbf{s} \neq \mathbf{l}_w}} p^q_{\mathbf{s}} r^\gamma_{\mathbf{s}} \sup_{a\Delta^k \leqslant r} \left\{ \mathcal{G}_z(r) \right\}.
\end{aligned}$$

In matrix form, using the notation $\sup \boldsymbol{\mathcal{G}}(r) = (\sup \mathcal{G}_w(r))_{w \in V}$, this is

$$\sup_{a\Delta^{k+1} \leqslant r < \delta} \boldsymbol{\mathcal{G}}(r) \leqslant \mathbf{B}(q, \gamma, l) \sup_{a\Delta^k \leqslant r} \boldsymbol{\mathcal{G}}(r).$$

By the induction hypothesis, Inequality (4.7.39),

$$\sup_{a\Delta^{k+1} \leqslant r < \delta} \boldsymbol{\mathcal{G}}(r) \leqslant \mathbf{B}(q, \gamma, l) \sup_{a\Delta^k \leqslant r} \boldsymbol{\mathcal{G}}(r) \leqslant \mathbf{B}(q, \gamma, l) C_a(q)\mathbf{b} = C_a(q)\mathbf{b}. \tag{4.7.41}$$

As $a\Delta^{k+1} < a\Delta^k < \delta$, inequalities (4.7.39) and (4.7.41) together imply

$$\sup_{a\Delta^{k+1} \leqslant r} \boldsymbol{\mathcal{G}}(r) \leqslant C_a(q)\mathbf{b},$$

which proves that $P(k)$ implies $P(k+1)$. This completes the induction step.

By the principal of mathematical induction $P(n)$ is true for all $n \in \mathbb{N}$, that is

$$\sup_{a\Delta^n \leqslant r} \boldsymbol{\mathcal{G}}(r) \leqslant \ C_a(q)\mathbf{b}$$

for all $n \in \mathbb{N}$. Therefore

$$\sup_{0<r} \boldsymbol{\mathcal{G}}(r) \leqslant \ C_a(q)\mathbf{b}. \qquad \square$$

Theorem 4.7.19. *Let $\big(V, E^*, i, t, r, p, ((\mathbb{R}^n, |\ \ |))_{v\in V}, (S_e)_{e\in E^1}\big)$ be a directed graph IFS with probabilities, suppose that the OSC holds, and the non-negative matrix $\mathbf{B}(q, \gamma, l)$, as defined in Subsection 4.2.6, is irreducible. Let $q \in \mathbb{R}$ and let $\gamma = \gamma(q) \in \mathbb{R}$, be the unique number such that $\rho(\mathbf{B}(q, \gamma, l)) = 1$. Let $r \in (0, \delta)$, where δ is as defined in Equation (4.7.28), and let $e, f \in E^1_u$, $e \neq f$.*

Then

$$Q^q_{e,f}(r) \leqslant C_{e,f}(q) r^{-\gamma(q)},$$

for some positive $C_{e,f}(q)$.

Proof. For $r \in (0, \delta)$,

$$\begin{aligned}
\sum_{x\in H_{e,f}} \mu_u(B(x,r))^q &\leqslant C_2(q)C_3C_4(q)\sum_{w\in V} G_w(r) && \text{(by Lemma 4.7.15)}\\
&= C_2(q)C_3C_4(q)\sum_{w\in V} r^{-\gamma}\mathcal{G}_w(r) && \text{(by (4.7.31))}\\
&\leqslant C_2(q)C_3C_4(q)C_5(q)r^{-\gamma},
\end{aligned}$$

where the last inequality follows by Lemma 4.7.18, putting

$$C_5(q) = mC_a(q)\max\big\{b_v : v \in V\big\},$$

where m is the number of vertices in the graph, and the positive eigenvector of the matrix $\mathbf{B}(q, \gamma, l)$ is $\mathbf{b} = (b_v)^T_{v\in V}$. This means that for $e, f \in E^1_u$, $e \neq f$,

$$\begin{aligned}
Q^q_{e,f}(r) &= M^q_u(S_e(F_{t(e)}) \cap S_f(F_{t(f)}(r), r)\\
&= \sup\Big\{ \sum_{x\in H_{e,f}} \mu_u(B(x,r))^q : H_{e,f} \ \text{ is an}\\
&\qquad\qquad r\text{-separated subset of } S_e(F_{t(e)}) \cap S_f(F_{t(f)}(r) \Big\}\\
&\leqslant C_2(q)C_3C_4(q)C_5(q)r^{-\gamma}\\
&= C_{e,f}(q)r^{-\gamma}. \qquad \square
\end{aligned}$$

4.8 Proof of Theorem 4.3.2

$S_f(F_{t(f)})(r)$ is the closed r-neighbourhood of $S_f(F_{t(f)})$, as defined in Subsection 1.2.5. For edges $e \in E^1_{uv}$, $f \in E^1_{uw}$, $f \neq e$, $Q^q_{e,f}(r)$ is defined, for $r > 0$, as

$$Q^q_{e,f}(r) = M^q_u\big(S_e(F_{t(e)}) \cap S_f(F_{t(f)})(r),\, r\big),$$

see Equation (4.7.8). For $q \in \mathbb{R}$, and for δ, $0 < \delta < 1$, as given by Equation (4.7.28), by Theorem 4.7.19, of Section 4.7, for $r \in (0, \delta)$,

$$Q^q_{e,f}(r) \leqslant C_{e,f}(q) r^{-\gamma},$$

for some positive $C_{e,f}(q)$. This means that

$$\begin{aligned}
M^q_u\Bigg(\bigcup_{e\in E^1_u} \bigcup_{\substack{f\in E^1_u\\ f\neq e}} \big(S_e(F_{t(e)}) \cap S_f(F_{t(f)}) \big),\ r \Bigg)
&\leqslant M^q_u\Bigg(\bigcup_{e\in E^1_u} \bigcup_{\substack{f\in E^1_u\\ f\neq e}} \big(S_e(F_{t(e)}) \cap S_f(F_{t(f)})(r) \big),\ r \Bigg)\\
&\leqslant \sum_{e\in E^1_u} \sum_{\substack{f\in E^1_u\\ f\neq e}} Q^q_{e,f}(r)\\
&\leqslant C(q) r^{-\gamma},
\end{aligned}$$

for some positive $C(q)$. Therefore, from the definition of the upper multifractal q box dimension of F_u with respect to μ_u, given in Subsection 4.2.3,

$$\begin{aligned}
&\overline{\dim}^q_{\mu_u,\mathrm{B}}\Bigg(\bigcup_{e\in E^1_u} \bigcup_{\substack{f\in E^1_u\\ f\neq e}} \big(S_e(F_{t(e)}) \cap S_f(F_{t(f)}) \big) \Bigg)\\
&= \lim_{\delta'\to 0^+} \sup_{0<r<\delta'} \frac{\ln M^q_u\Bigg(\bigcup_{e\in E^1_u} \bigcup_{\substack{f\in E^1_u\\ f\neq e}} \big(S_e(F_{t(e)}) \cap S_f(F_{t(f)}) \big),\ r \Bigg)}{-\ln r}\\
&\leqslant \lim_{\delta'\to 0^+} \sup_{0<r<\delta'} \frac{\ln C(q)}{-\ln r} + \gamma\\
&= \gamma\\
&< \beta,
\end{aligned}$$

where $\gamma < \beta$ by Lemma 4.7.1. It can be shown, see Theorems 2.6.1 and 2.6.2 of [Ols94], that the multifractal q Hausdorff dimension of F_u with respect to μ_u satisfies $\dim^q_{\mu_u,\mathrm{H}} F_u = \beta(q)$, for each vertex $u \in V$, where $\beta(q)$ is as defined in Subsection 4.2.5, and this completes the proof.

4.9 Conclusion

After the long and technical proof of Theorem 4.7.19 in the penultimate section, and with the aid of hindsight, it is reasonable to give Theorem 4.7.19 the same importance as the two theorems of Section 4.3 which depend on it. The work of this chapter has also highlighted the significance of the matrix $\mathbf{B}(q, \gamma, l)$ and has raised an interesting question regarding its irreducibility. However we must leave open here, the question as to what conditions are sufficient to imply the irreducibility of this matrix, as this point marks the end of this thesis.

References

[All91] R. Allenby, *Rings, Fields and Groups*, Butterworth-Heinemann, Oxford, 1991.

[Apo78] T. M. Apostol, *Mathematical Analysis*, Addison-Wesley, Massachusetts, 1978.

[AS99] E. Ayer and R. S. Strichartz, *Exact Hausdorff measure and intervals of maximum density for Cantor sets*, Trans. Amer. Math. Soc. **351** (1999), 3725–3741.

[Bar93] M. F. Barnsley, *Fractals Everywhere*, Academic Press, San Diego, 1993.

[Bar98] R. G. Bartle, *The Elements of Integration and Lebesgue measure*, Springer-Verlag, New York, 1998.

[Cru70] K. S. Crump, *On systems of renewal equations*, Journal of Mathematical Analysis and Applications **30** (1970), 425–434.

[Del02] R. Delaware, *Every set of finite Hausdorff measure is a countable union of sets whose Hausdorff measure and content coincide*, Proc. Amer. Math. Soc. **131** (2002), 2537–2542.

[Edg98] G. A. Edgar, *Integral, Probability, and Fractal Measures*, Springer-Verlag, New York, 1998.

[Edg00] ———, *Measure, Topology, and Fractal Geometry*, Springer-Verlag, New York, 2000.

[EM92] G. A. Edgar and R. D. Mauldin, *Multifractal decompositions of digraph recursive fractals*, Proc. London Math. Soc. (3) **65** (1992), 604–628.

[Fal85] K. J. Falconer, *The Geometry of Fractal Sets*, Cambridge University Press, Cambridge, 1985.

[Fal97] ———, *Techniques in Fractal Geometry*, John Wiley, Chichester, 1997.

[Fal03] ———, *Fractal Geometry, Mathematical Foundations and Applications*, John Wiley, Chichester, 2nd Ed. 2003.

[Fel66] W. Feller, *An Introduction to Probability Theory and its Applications, vol II*, John Wiley, New York, 1966.

[FW09] De-J. Feng and Y. Wang, *On the structures of generating iterated function systems of Cantor sets*, Adv. Math. **222** (2009), 1964–1981.

[Gra95] S. Graf, *On Bandt's tangential distribution for self-similar measures*, Monatsh. Math. **120** (1995), 223–246.

[Hut81] J. Hutchinson, *Fractals and self-similarity*, Indiana Univ. Math. J. **30** (1981), 713–747.

[Lal88] S. Lalley, *The packing and covering functions of some self-similar fractals*, Indiana Univ. Math. J. **37** (1988), 699–710.

[Mad88] I. J. Maddox, *Elements of Functional Analysis*, Cambridge University Press, Cambridge, 1988.

[Mar86] J. Marion, *Mesure de Hausdorff d'un fractal `a similitude interne*, Ann. sc. Qubec. **10** (1986), 51–84.

[Mat99] P. Mattila, *Geometry of Sets and Measures in Euclidean Spaces*, Cambridge University Press, Cambridge, 1999.

[Min88] H. Minc, *Non-negative Matrices*, John Wiley, New York, 1988.

[MW88] R. D. Mauldin and S. C. Williams, *Hausdorff dimension in graph directed constructions*, Trans. Amer. Math. Soc. **309** (1988), 811–829.

[Ols94] L. Olsen, *Random Geometrically Graph Directed Self-similar Multifractals*, Pitman Research Notes in Mathematics Series, Volume 307, Longman Scientific & Technical, 1994.

[Ols02a] ———, *Dimensions of overlaps of self-similar fractals and self-similar multifractals*, Indiana Univ. Math. J. **51 6** (2002), 1461–1477.

[Ols02b] ———, *Empirical multifractal moment measures and moment scaling functions of self-similar multifractals*, Math. Proc. Camb. Phil. Soc. **133** (2002), 459–485.

[Sch94] A. Schief, *Separation properties for self-similar sets*, Proc. Amer. Math. Soc. **122** (1994), 111–115.

[Sen73] E. Seneta, *Non-negative Matrices*, George Allen & Unwin Ltd, London, 1973.

[Wan97] J. Wang, *The open set conditions for graph directed self-similar sets*, Random Comput. Dynam. **5 4** (1997), 283–305.

[ZF04] Z. Zhou and L. Feng, *Twelve open problems on the exact value of the Hausdorff measure and on topological entropy: a brief survey of recent results*, IOP. Nonlinearity. **17** (2004), 493–502.

Notation

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