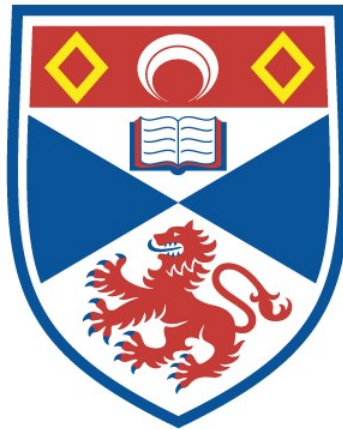


# **MULTIFRACTAL ZETA FUNCTIONS**

**Vuksan Mijović**

**A Thesis Submitted for the Degree of PhD  
at the  
University of St Andrews**



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# Multifractal Zeta Functions

Vuksan Mijović



University of  
St Andrews

This thesis is submitted in partial fulfilment for the degree of  
PhD at the  
University of St Andrews

March 2017

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## Abstract

Multifractals have during the past 20 – 25 years been the focus of enormous attention in the mathematical literature. Loosely speaking there are two main ingredients in multifractal analysis: the multifractal spectra and the Renyi dimensions. One of the main goals in multifractal analysis is to understand these two ingredients and their relationship with each other. Motivated by the powerful techniques provided by the use of the Artin-Mazur zeta-functions in number theory and the use of the Ruelle zeta-functions in dynamical systems, Lapidus and collaborators (see books by Lapidus & van Frankenhuysen [32, 33] and the references therein) have introduced and pioneered use of zeta-functions in fractal geometry. Inspired by this development, within the past 7 – 8 years several authors have paralleled this development by introducing zeta-functions into multifractal geometry. Our result inspired by this work will be given in section 2.2.2. There we introduce geometric multifractal zeta-functions providing precise information of very general classes of multifractal spectra, including, for example, the multifractal spectra of self-conformal measures and the multifractal spectra of ergodic Birkhoff averages of continuous functions. Results in that section are based on paper [37].

Dynamical zeta-functions has been introduced and developed by Ruelle [63, 64] and others, (see, for example, the surveys and books [3, 54, 55] and the references therein). It has been a major challenge to introduce and develop a natural and meaningful theory of dynamical multifractal zeta-functions paralleling existing theory of dynamical zeta functions. In particular, in the setting of self-conformal constructions, Olsen [49] introduced a family of dynamical multifractal zeta-functions designed to provide precise information of very general classes of multifractal spectra, including, for example, the multifractal spectra of self-conformal measures and the multifractal spectra of ergodic Birkhoff averages of continuous functions. However, recently it has been recognised that while self-conformal constructions provide a useful and important framework for studying fractal and multifractal geometry, the more general notion of graph-directed self-conformal constructions provide a substantially more flexible and useful framework, see, for example, [36] for an elaboration of this. In recognition of this viewpoint, in section 2.3.11 we provide main definitions of the multifractal pressure and the multifractal dynamical zeta-functions and we state our main results. This section is based on paper [38].

Setting we are working unifies various different multifractal spectra including fine multifractal spectra of self conformal measures or Birkhoff averages of continuous function. It was introduced by Olsen in [43]. In section 2.1 we propose answer to problem of defining Renyi spectra in more general settings and provide slight improvement of result regrading multifractal spectra in the case of Subshift of finite type.



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# Chapter 1

## Preliminaries

### 1.1 Fractals and multifractal preliminaries

In this section, we will introduce notation and some fundamental concepts regarding fractals. First, we will introduce a concept of fractal dimension, and then we will state some elementary facts about multifractal analysis. For the more detailed account of fractal in general and proofs that are missing reader could look at [19] or [59].

#### 1.1.1 Dimension Theory

The word dimension is one we use in everyday speech. We know that the line is 1-dimensional, a square is 2-dimensional and a cube is 3-dimensional. A more abstract example is  $\mathbb{R}^n$  which has a dimension equal to  $n$ . However, some examples are far less intuitive. For example, a point is 0-dimensional, finitely many points remain 0-dimensional. But what about a sequence of points, or set composed of uncountable many points that still have length 0? An example of that would be Cantor set, which is constructed in the following way. Let us start with an interval. Next let us remove the middle third interval from it. Next let us do the same to the remaining intervals. The Cantor set is what we would get after repeating the described procedure infinitely many times. Figure 1.1 shows a couple of steps of construction. Or what about the curve that has infinite length but is located inside the bounded region like Koch snowflake. A few steps in the construction of Koch snowflake are shown in figure 1.2.



Figure 1.1: Cantor set source [https://en.wikipedia.org/wiki/Cantor\\_set#/media/File:Cantor\\_set\\_in\\_seven\\_iterations.svg](https://en.wikipedia.org/wiki/Cantor_set#/media/File:Cantor_set_in_seven_iterations.svg)

These examples suggest that there might be room for refinement of the term dimension, especially if one is interested in sets like Cantor set and Koch snowflake or other "strange" sets. Those strange sets are called fractals. There is no universally accepted definition of fractal, there are typical properties like irregularity, but usually, there are some fine structures like some kind of self-similarity. The name fractal is due to Mandelbrot, after the Latin word fractus, which means broken. Besides their interesting structure, fractals come up as models to describe various physical phenomena. Dimension

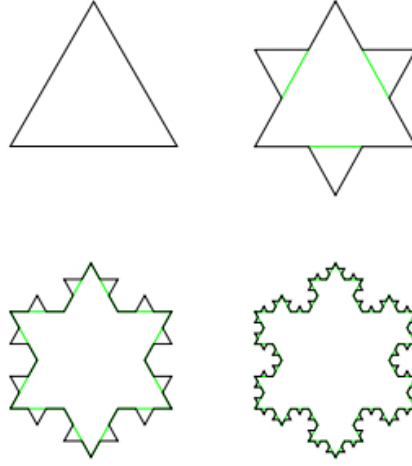


Figure 1.2: Koch snowflake source [https://en.wikipedia.org/wiki/Koch\\_snowflake#/media/File:KochFlake.svg](https://en.wikipedia.org/wiki/Koch_snowflake#/media/File:KochFlake.svg)

is one of the most prominent concepts in the fractal geometry because it allows for classification of fractals. There are various notions of dimension. In this section, we will introduce two, perhaps the most important, Hausdorff dimension and Box dimension. A notable property of these dimensions is that dimension does not have to be an integer. In the case of fractals most often it is not an integer but some positive real number.

We will be mainly interested in fractals as subsets of  $\mathbb{R}^n$ , but as we will see it is often useful/easier to work in different settings, i.e. in shift space which we will introduce in the section 1.2. To cover both cases, and because dimensions that we will introduce are metric property, the setting we will work in will be a complete metric space  $X$ , with the property that each closed bounded subset is compact. Let us denote metric on  $X$  as  $d$ .

### Hausdorff dimension

Let us introduce Hausdorff dimension, which is based on Hausdorff's paper [27] published in 1919. Roughly, the idea behind it is the following. If  $n$ -dimensional Lebesgue measure of the set is some positive real number, then the set is  $n$ -dimensional. So assume we can construct reasonable generalisation of  $n$ -dimensional Lebesgue measure, where instead of the integer parameter  $n$  we have a real non-negative parameter  $s$ . That generalisation is called Hausdorff  $s$  measure. If the Hausdorff  $s$  measure of a set is a positive real number then the set has dimension  $s$ . Roughly that is the idea, but note that situation is a bit more complicated than that. For example, the real line has a length (Lebesgue 1-dimensional measure) equal to  $\infty$ , yet the real line is 1-dimensional. To provide a complete answer to the problem of defining dimension of a set, let us be precise.

Now we will continue with the construction of Hausdorff  $s$  measure. Let us first give a couple of technical definitions. For any set  $F \subset X$  we will denote its diameter by  $|F|$  or  $\text{diam} F$ . Recall  $|F| = \sup_{x,y \in F} d(x,y)$ .

**Definition 1.1.1** ( $\delta$  - cover). Let  $F \subset X$ . Let  $\delta > 0$ . Let  $\{U_i\}_i$  be countable collection of open sets. We will call collection  $\{U_i\}_i$   $\delta$ -cover of  $F$ , if for each  $i$  we have  $U_i \subset X$ ,  $|U_i| < \delta$ , and  $F \subset \cup_i U_i$ .

Now we will define outer measure that will give rise to Hausdorff  $s$  measure. Let  $s, \delta > 0$ . Then let us define

$$\mathcal{H}_\delta^s(F) = \inf \left\{ \sum_i |U_i|^s \mid \{U_i\}_i \text{ is } \delta \text{ cover of set } F \right\}.$$

$\mathcal{H}_\delta^s(F)$  is non decreasing with respect to  $\delta$ , thus the following quantity is well defined.

$$\mathcal{H}^s(F) = \lim_{\delta \rightarrow 0} \mathcal{H}_\delta^s(F). \quad (1.1)$$

$\mathcal{H}^s$  is an outer measure on  $X$ . It is called *Hausdorff  $s$ -dimensional outer measure*. This outer measure defines Hausdorff  $s$ -dimensional measure.

Let  $X = \mathbb{R}^d$ , and let  $n$  be an integer, then  $\mathcal{H}^n$  is multiple of  $n$ -dimensional Lebesgue measure on Borel sets in  $\mathbb{R}^d$ . Hence  $\mathcal{H}^s$  is a generalization of Lebesgue  $n$  measure. We should note that definition of  $s$ -dimensional Hausdorff measure is similar to definition of Lebesgue measure. A major difference is that in definition 1.1.1 we allow  $s$  to have non-integer values.

Now that we have constructed reasonable extension of Lebesgue  $n$ -dimensional measure, we will continue by defining Hausdorff dimension and investigate its properties. The following interesting observation is what we need to define Hausdorff dimension.

**Proposition 1.1.2.** *Let us fix a set  $F \subset X$ . Then  $\mathcal{H}^s(F)$  as a function of  $s$  is equal to  $+\infty$  up to some number  $s_0$ . At  $s_0$  it is infinite or finite (may be zero), and 0 for all numbers greater than  $s_0$ .*

Now we can state the following definition.

**Definition 1.1.3.** (*Hausdorff dimension*) *Let us define*

$$\dim_{\mathcal{H}}(F) = \inf \{s \mid s \in [0, +\infty), \mathcal{H}^s(F) = 0\}. \quad (1.2)$$

*Then the number  $\dim_{\mathcal{H}}(F)$  is called the Hausdorff dimension of  $F$ .*

We will use convention that  $\dim_{\mathcal{H}}(\emptyset) = -\infty$ . Note that since  $\mathcal{H}^n$  is multiple of  $n$ -dimensional Lebesgue measure in cases of sets like lines, polygons or 3D bodies Hausdorff dimension have the value which we expect.

Let us list properties of Hausdorff dimension that we will use in the rest of the text.

- Monotonicity: Let  $A, B \subset X$  and  $A \subset B$ . Then  $\dim_{\mathcal{H}} A \leq \dim_{\mathcal{H}} B$ .
- Countable stability: Let  $\{A_i\}_i$  be countable family of subsets of  $X$ .  $\dim_{\mathcal{H}} \cup_i A_i = \sup_i \dim_{\mathcal{H}} A_i$ .
- Let  $(Y, d_y)$  be metric space and let  $f : X \rightarrow Y$  be Lipschitz map, i.e. there is constant  $L$  such that for each  $x, y \in X$  we have  $d_y(f(x), f(y)) \leq Ld(x, y)$ . Then  $\dim_{\mathcal{H}} f(A) \leq \dim_{\mathcal{H}} A$ .

Note that due to countable stability, we can say that Hausdorff dimension of sequence is 0 as a union of countable many points (it is easy to see that point is 0-dimensional).

Now when we have defined and investigated some properties of Hausdorff dimension, it is natural to ask about how to calculate it? This can be a very hard question. Some kind of upper bound is often easier to find. We can do so by finding some  $\delta$  cover for each  $\delta$ . Then we could make an upper bound estimate of  $\mathcal{H}_\delta^s(F)$ , hence make an upper bound estimate on  $\mathcal{H}^s(F)$ . Then from definition of Hausdorff dimension we have that if we can estimate  $\mathcal{H}^s(F)$  to be finite that give us upper estimate  $\dim_{\mathcal{H}} F \leq s$ . Lower estimates on the dimension are usually more involved to find. Here we will introduce one method that is particularly well suited for our case. It is called the Mass distribution principle, which is a simple yet powerful. The idea is that if there is measure which is "smaller" than Hausdorff  $s$  measure then dimension is greater or equal  $s$ . Formally:

**Proposition 1.1.4.** *Let  $s > 0$ . And let  $F \subset X$ . If there exists  $C > 0$ , and measure  $\mu$  such that  $\mu(F) > 0$ , and for any  $U \subset X$  we have  $|U|^s \geq C\mu(U)$  then we have*

$$\dim_H F \geq s.$$

The usefulness of this result is tied up with our ability to find measures that could be used in it. Sometimes one could find right measures by using ideas from thermodynamic formalism.

### Box Dimension

In order to illustrate the idea behind Box dimension, let us move for a moment to  $\mathbb{R}^3$ . Let us have an interval of length one. Then it could be covered by  $\lceil r^{-1} \rceil$  cubes, or, in layman's terms, boxes, which have edge equal to  $r$ . Let us have a look at a square which has edge equal to 1. Then it could be covered by  $\lceil r^{-2} \rceil$  cubes which have edge equal to  $r$ . If we have look at cube with edge equal to 1. Then it could be covered by  $\lceil r^{-3} \rceil$  cubes which have edge equal to  $r$ . A similar formula is true for more general sets as well. So data suggests that a  $d$ -dimensional set could be covered by roughly  $Cr^{-d}$  cubes with the edge equal to  $r$  for some constant  $C$ , where  $d$  is the dimension of set. Hence the idea is that the dimension of a set  $F$  is presented by  $\lim_{r \rightarrow 0} \frac{\log N_r(F)}{-\log r}$ , where  $N_r(F)$  is the number of boxes, with edge equal to  $r$ , needed to cover the set  $F$ . Of course, this limit does not have to exist, which we will take into account. This procedure naturally generalises to  $\mathbb{R}^n$ . In setting we are working in, i.e. metric space,  $X$  with a metric such that every bounded closed subset of  $X$  is compact, we do not have a definition of a cube, so we will work with sets of diameter equal to  $r$ . As we will see in  $\mathbb{R}^d$ , it will not make a difference. To be precise:

**Definition 1.1.5.** (*Box Dimension*)

Let  $F \subset X$  be bounded. Let us restrict to  $\delta$  - covers such that diameter of any set in the cover is equal to  $\delta$ . Denote  $N_\delta(F)$  to be the smallest number of sets in such  $\delta$  cover of  $F$ . Lower box dimension is defined by

$$\underline{\dim}_B(F) = \liminf_{\delta \rightarrow 0} \frac{\log N_\delta(F)}{-\log \delta}.$$

Upper box dimension is defined by

$$\overline{\dim}_B(F) = \limsup_{\delta \rightarrow 0} \frac{\log N_\delta(F)}{-\log \delta}.$$

When  $\underline{\dim}_B(F) = \overline{\dim}_B(F)$  then define  $\dim_B(F) = \underline{\dim}_B(F)$  and say that  $\dim_B(F)$  is called box dimension of set  $F$ .

Note that we require  $F$  to be compact in order to ensure that  $N_\delta(F)$  is finite. We will also use convention that  $\log 0 = -\infty$ , i.e. the empty set will have Box dimension equal to  $-\infty$ .

Let  $X = \mathbb{R}^n$ . Then we can calculate the box dimension by taking  $n$ -dimensional cubes of sides equal to  $\delta$  instead of sets with a diameter equal to  $\delta$ . Or we can go to simplify this even one step further. We can fix mesh of  $n$ -dimensional cubes  $[k_1\delta, (k_1+1)\delta] \times [k_2\delta, (k_2+1)\delta] \dots \times [k_n\delta, (k_n+1)\delta]$  for  $k_1, k_2, \dots, k_n \in \mathbb{Z}$ . Then we could use these cubes to calculate Box dimension.

Now is a good place to compare Box dimension to Hausdorff dimension. It is easy to see that the following claim holds.

**Proposition 1.1.6.** *Let  $F \subset X$  be bounded then  $\dim_H F \leq \underline{\dim}_B F$ .*

Remember that in the case of Hausdorff dimension, due to countable stability, the dimension of a sequence of numbers is always 0. Box dimension is only finitely stable. In case of sequence of numbers Box dimension depends on sequence, and it is not necessarily 0. For example, the box dimension of set

$\{\frac{1}{n}\}_n$  is  $\frac{1}{2}$ . Let us note that Hausdorff dimension of the rational numbers in segment  $[0, 1]$  is 0, and its Box dimension is 1. Those examples show properties of Box dimensions that are often considered as its weaknesses. However, there is a huge class of sets having the same Box and Hausdorff dimension. The main advantage of Box dimension over Hausdorff dimension is that Box dimension is easier to calculate, or at least estimate, which is important for applications.

### 1.1.2 Multifractal Analysis

Multifractal analysis has been in focus in mathematical literature in the last 20 – 25 years. For a more detailed overview see [47]. The idea behind it is the following. The Hausdorff dimension of a set does not describe that set entirely. So in order to investigate the structure of a fractal, it is reasonable to perform a more detailed analysis. We will see two different approaches. The fine approach where we define fine and coarse multifractal spectra and the coarse approach where we will define Renyi spectra. In the end of the section, we will mention the connection of these two, seemingly completely different approaches, so-called multifractal formalism. Here we will treat the general setting i.e. a complete metric space  $(X, d)$ , such that every closed bounded subset of  $X$  is compact. Here we will only try to illustrate ideas, and in later sections we will provide more details as appropriate.

#### Fine multifractal spectra

In order to perform more detailed analysis, it is reasonable to try to do so by investigating subsets of the fractal. Then we can calculate the Hausdorff dimension of subsets of interest, and the collection of these data we will call multifractal spectra of that fractal. Let us go through the following example. Let  $\mu$  be Borel measure with highly varying intensity and denote its support by  $K = \text{supp}\mu$ . That kind of measure appears in the modeling of different natural phenomena like spatial distribution of rain or dissipation of energy in highly turbulent flows. We will not be precise here regarding the term of measure of highly varying intensity, but hope the natural phenomena it models gives the reader an idea of what we are thinking of. We will be more precise in the latter sections. We are particularly interested in a case of  $K$  being fractal. For every point  $x \in K$  we can define *local dimension of the measure  $\mu$  at point  $x$*  by

$$\dim_{loc,\mu}(x) = \lim_{r \rightarrow 0} \frac{\log \mu(B(x, r))}{\log r} \text{ in the case that limit exists.}$$

We can single out from  $K$  subsets  $E_\alpha$  defined by

$$E_\alpha = \{x \in K \mid \dim_{loc,\mu}(x) = \alpha\}.$$

Now we just need to collect Hausdorff dimensions of these sets to define fine multifractal spectra. I.e. definition is following.

**Definition 1.1.7** (Fine Multifractal spectra). *The function  $f_\mu$  defined by*

$$f_\mu(\alpha) = \dim_H E_\alpha$$

*is called the fine multifractal spectra of measure  $\mu$ .*

Note that  $f_\mu$  is defined on some subset of  $\mathbb{R}$ . At this point, it is not clear that this function has any nice properties, like some kind of continuity or its codomain being some regular set like interval. As we will see later, this will be the case in examples we are interested in. Further investigation shows that sets such that  $\dim_{loc,\mu}$  does not exist could have a very rich structure as well. That motivates the definition of generalised multifractal spectra. Before stating that definition let us introduce the following notation. For  $F : \mathbb{R}^+ \rightarrow \mathbb{R}$  denote  $\text{acc}_{r \rightarrow 0} F(r)$  to be set of accumulation points of the function  $F$  as  $r$  tends to 0. Then let us define:

**Definition 1.1.8** (Generalised Multifractal Spectra). *Define*

$$F_\mu(A) = \dim_H \left\{ x \mid \lim_{r \rightarrow 0} \frac{\log \mu(B(x, r))}{\log r} \subset A \right\}.$$

$F_\mu$  is called the generalised multifractal spectra of the measure  $\mu$ .

There are other types of multifractal spectra as well. In the example above we singled out from the set  $K$  sets with the same local dimension. But there are other reasonable ways to do so. Let us assume that we have dynamics defined on  $K$ , i.e. there is mapping  $T : K \rightarrow K$ . Let there be function  $f$  defined on  $K$ . One could consider sets  $B_\alpha$  defined by:

$$B_\alpha = \left\{ x \in K \mid \lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k x) = \alpha \right\}.$$

Then again by collecting Hausdorff dimension of these sets we have the following definition.

**Definition 1.1.9** (Birkhoff Multifractal spectra). *The function  $f_B$  defined by*

$$f_B(\alpha) = \dim_H B_\alpha$$

*is called Birkhoff multifractal spectra of the function  $f$ .*

### Coarse multifractal spectra

Coarse multifractal spectra follows a similar idea to the one of fine multifractal spectra. Fine multifractal spectra is associated to Hausdorff dimension, while coarse multifractal spectra is inspired by box counting. However, we could not exactly repeat the same argument, because when we single out set  $E_\alpha$  from  $K$ , as we did in the previous subsection, it is often dense in  $K$ . Recall Box dimension of set is equal to Box dimension of its closure hence  $\dim_B E_\alpha = \dim_B \bar{E}_\alpha = \dim_B K$ . So our spectra would be just the point  $\dim_B K$ . Therefore, we need to modify this idea. Due to the problem of Box dimension being too big we could decide to be more restrictive about "boxes" that we will count. So if we have an open set  $U$ , we will not check if there is a point from  $E_\alpha$  in it, because due to assumption that  $E_\alpha$  is dense in  $K$  it will be. But we will choose criteria that, at least morally, would estimate if there is enough points from  $E_\alpha$  in it. Recall  $E_\alpha$  is composed of points such that  $r^\alpha \sim \mu(B(x, r))$ . So such criterion could be that  $\alpha - \epsilon \leq \frac{\log \mu(U)}{\log |U|} \leq \alpha + \epsilon$ . One of reasons we need  $\epsilon$  in this criterion is that multifractal spectra often takes uncountable many values, and with  $\epsilon = 0$  we would cover just countably many. Now let us take a measure  $\mu$ , and fix a real number  $\alpha$  coarse multifractal spectra is defined as follows.

$$N_\delta(\alpha, \epsilon) = \sup \left\{ |I| \mid (B(x_i, \delta))_{i \in I} \text{ is a finite family of balls such that,} \right. \quad (1.3)$$

$$(\forall i \in I) x_i \in K, \quad (1.4)$$

$$(\forall i, j \in I, i \neq j) B(x_i, \delta) \cap B(x_j, \delta) = \emptyset \quad (1.5)$$

$$\left. \frac{\log \mu(B(x_i, \delta))}{\log \delta} \in [\alpha - \epsilon, \alpha + \epsilon] \right\}. \quad (1.6)$$

Then we can define coarse multifractal spectra as follows

**Definition 1.1.10** (Upper coarse multifractal spectra). *Let us define  $\bar{f}_\mu^c$  by*

$$\bar{f}_\mu^c(\alpha) = \lim_{\epsilon \rightarrow 0} \limsup_{\delta \rightarrow 0} \frac{\log N_\delta(\alpha, \epsilon)}{-\log \delta}.$$

$\bar{f}_\mu^c$  is called upper coarse multifractal spectra of measure  $\mu$ .

**Definition 1.1.11** (Lower coarse multifractal spectra). *Let us define  $f_{\mu}^c$  by*

$$f_{\mu}^c(\alpha) = \lim_{\epsilon \rightarrow 0} \liminf_{\delta \rightarrow 0} \frac{\log N_{\delta}(\alpha, \epsilon)}{-\log \delta}.$$

$f_{\mu}^c$  is called lower coarse multifractal spectra of measure  $\mu$ .

If  $\bar{f}_{\mu}^c(\alpha) = f_{\mu}^c(\alpha)$  then we define  $f_{\mu}^c(\alpha) = \bar{f}_{\mu}^c(\alpha)$ . Then  $f_{\mu}^c(\alpha)$  is called *coarse multifractal spectra* of measure  $\mu$ . Again if we have dynamics defined on  $K$ , i.e. we have defined map  $T : K \rightarrow K$  and a function  $f : K \rightarrow \mathbb{R}$ , like in the fine case, using the same procedure we can define coarse multifractal spectra for Birkhoff averages by defining

$$\begin{aligned} N_{\delta}(\alpha, \epsilon) = \sup \Big\{ & |I| \mid \{B(x_i, \delta)\}_{i \in I} \text{ is a finite family of balls such that,} \\ & (\forall i \in I) x_i \in K, \\ & (\forall i, j \in I, i \neq j) B(x_i, \delta) \cap B(x_j, \delta) = \emptyset \\ & \frac{1}{n} \sum_{k=0}^{n-1} f(T^k x_i) \in [\alpha - \epsilon, \alpha + \epsilon] \Big\} \end{aligned}$$

And then repeating definitions from above we define coarse multifractal spectra for Birkhoff averages. We will also introduce convention that supremum over empty set is  $-\infty$ .

### Renyi spectra

This approach is motivated by work of Renyi [60]. This could be seen as a generalization of Box dimension. Recall that box dimension is defined using the "counting function"  $N_{\delta}(F)$ . Remember  $N_{\delta}(F)$  counts the number of sets in the cover of  $F$ . So it effectively gives weight 1 to every set in cover. Now we will substitute 1 with some weights. So again let us take a Borel measure  $\mu$  on  $X$  and let  $\text{supp} \mu = K$ . Let  $F \subset X$ ,  $q \in \mathbb{R}$ ,  $\delta > 0$ . Then let us define  $q$ -th moment  $M_{\mu, \delta}(q, F)$  as

$$M_{\mu, \delta}(q, F) = \sup \left\{ \sum_{i \in I} \mu(B(x_i, \delta))^q \mid \{B(x_i, \delta)\}_{i \in I} \text{ is a finite family of balls such that,} \right. \quad (1.7)$$

$$(\forall i \in I) x_i \in F, \quad (1.8)$$

$$(\forall i, j \in I, i \neq j) B(x_i, \delta) \cap B(x_j, \delta) = \emptyset \Big\}. \quad (1.9)$$

Next let us define upper and lower Renyi spectra as  $\bar{\tau}(\cdot, F), \underline{\tau}(\cdot, F) : \mathbb{R} \rightarrow [-\infty, +\infty]$ .

$$\bar{\tau}(q, F) = \limsup_{\delta \rightarrow 0} \frac{\log M_{\mu, \delta}(q, F)}{-\log \delta}$$

$$\underline{\tau}(q, F) = \liminf_{\delta \rightarrow 0} \frac{\log M_{\mu, \delta}(q, F)}{-\log \delta}$$

If  $\underline{\tau} = \bar{\tau}$  we can define *Renyi spectra* as  $\tau = \underline{\tau}$ . Then when  $F = \text{supp} \mu$  we just omit set and use notation

$$M_{\mu, \delta}(q), \bar{\tau}(q), \underline{\tau}(q), \tau(q).$$

We notice that Renyi spectra associated to Birkhoff averages is missing here. The problem of defining Renyi spectra in case of Birkhoff averages will be tackled in section 2.1. In fact, we will provide an idea that hold in greater generality.

### Multifractal formalism

Based on remarkable insight paired with smart heuristics Halsey et al. in [26] showed that  $f_\mu$  and  $f_\mu^c$  can sometimes be calculated using Renyi dimension. Those kinds of statements are called multifractal formalism. Let us recall the definition of Legendre transform

**Definition 1.1.12** (Legendre transform). *Let  $f : \mathbb{R} \rightarrow \mathbb{R}$ .*

$$f^*(x) = \sup_{y \in \mathbb{R}} \{f(y) + yx\}.$$

Heuristic is that fine multifractal spectra is equal to coarse multifractal spectra and it is equal to Legendre transform of Renyi dimensions. i.e.

$$f_\mu = f_\mu^c = \tau^*.$$

As we will see, for measures we are interested in this will be true. However it often fails. It is easy to find measure supported on two non - overlapping Cantor sets with different ratios for which it will fail. For example of measure supported on Cantor set for which multifractal formalism does not hold, please see [67]. Refer to [47, 48] and [59] for additional information about multifractal formalism.

## 1.2 Subshifts of finite type

We will do a lot of work in settings of a certain type of subshift of finite type. As we will see, there are various theorems that show that in order to investigate multifractal spectra of graph oriented self-conformal iterated function systems we can limit ourselves on subshifts of finite type. On the other hand, Bowen made use of subshifts of finite type to approximate hyperbolic flows proving the Bowen's formula which connects Ruelle's work on thermodynamic formalism and dimension theory. In later sections method we will use to investigate our zeta function and multifractal pressure, that we will introduce, will be inspired by this result. In this section, we will first define subshifts of finite type, list some basic properties and notation, and introduce the notion of pressure. For more details one could look at [6], for the comprehensive account of thermodynamic formalism, reader should see Baladi's book [3], or Sarig's notes [65] for introduction or more comprehensive account [66].

First, we will define subshift of finite type and introduce notation that we will use. Then we will go to describe metric and topological properties, note some properties of measures on subshift of finite type, introduce pressure and introduce Gibbs measure.

**Definition 1.2.1** (Subshift of finite type). *Let  $\Sigma = \{1, 2, 3, \dots, N\}$  (or some other discrete finite set) and let  $A$  be  $N \times N$  binary (contains only 0's and 1's) matrix. Let*

$$\Sigma_A^{\mathbb{N}} = \{\mathbf{i} = i_1 i_2 i_3 \dots | A_{i_n i_{n+1}} = 1, n \in \mathbb{N}\}.$$

*Next define the shift map  $S : \Sigma_A^{\mathbb{N}} \rightarrow \Sigma_A^{\mathbb{N}}$  to be*

$$S(i_1 i_2 i_3 \dots) = i_2 i_3 \dots$$

*We call the pair  $(\Sigma_A^{\mathbb{N}}, S)$  a subshift of finite type. If matrix  $A$  contains only 1's then  $\Sigma_A^{\mathbb{N}}$  is called full shift.*

Let us introduce some useful notation. Let  $n \in \mathbb{N}$ . Then let us define

$$\Sigma_A^n = \{\mathbf{i} = i_1 i_2 i_3 \dots i_n | A_{i_k i_{k+1}} = 1, 1 \leq k \leq n-1\},$$

the family of finite strings of length  $n$ . Let us denote the family of all finite strings by

$$\Sigma_A^* = \bigcup_{n \in \mathbb{N}} \Sigma_A^n.$$



Then for string  $\mathbf{i} \in \Sigma_A^*$  or  $\mathbf{i} \in \Sigma_A^{\mathbb{N}}$  we will denote length of string by  $|\mathbf{i}|$ . For  $n \in \mathbb{N}_0$  and  $|\mathbf{i}| \geq n$  by  $\mathbf{i}|_n$  we will denote string of length  $n$  composed of first  $n$  letters of string  $\mathbf{i}$ . Now let us continue with describing metric properties of  $\Sigma_A^{\mathbb{N}}$ .

**Definition 1.2.2** (Metric on  $\Sigma_A^{\mathbb{N}}$ ). *Let us first choose  $\gamma \in (0, 1)$ . For any two strings  $\mathbf{i}, \mathbf{j} \in \Sigma_A^{\mathbb{N}}$ , where  $\mathbf{i} = i_1 i_2 i_3 \dots$  and  $\mathbf{j} = j_1 j_2 j_3 \dots$ , let us define*

$$s(\mathbf{i}, \mathbf{j}) = \begin{cases} +\infty, & \text{if } \mathbf{i} = \mathbf{j}, \\ \min \{n \mid n \in \mathbb{N}, i_n \neq j_n\} - 1, & \text{if } \mathbf{i} \neq \mathbf{j}. \end{cases}$$

Now metric  $d_\gamma$  is defined by

$$d_\gamma(\mathbf{i}, \mathbf{j}) = \gamma^{s(\mathbf{i}, \mathbf{j})}.$$

Often the exact value of  $\gamma$  is not important, so from now on let  $\gamma$  be some number from interval  $(0, 1)$ . Under this metric  $\Sigma_A^{\mathbb{N}}$  is compact metric space. It is easy to check that open balls with respect to metric  $d_\gamma$  are sets defined below:

**Definition 1.2.3** (Cylinders). *Let  $n \in \mathbb{N}$ . And let  $\mathbf{i} \in \Sigma_A^{\mathbb{N}}$  or  $\mathbf{i} \in \Sigma_A^*$ ,  $|\mathbf{i}| \geq n$ . Then  $n^{\text{th}}$  level cylinder noted as  $[\mathbf{i}]_n$  is defined by*

$$[\mathbf{i}]_n = \{\mathbf{j} \mid \mathbf{j} \in \Sigma_A^{\mathbb{N}}, \mathbf{j}|_n = \mathbf{i}|_n\}.$$

In addition we will use following notation. Let  $\mathbf{i} \in \Sigma_A^*$  then  $[\mathbf{i}] = [\mathbf{i}]_{|\mathbf{i}|}$ . Note that cylinders are not just open but closed sets as well. We want to limit ourselves to the certain class of subshifts of finite type. To do so we will now define following matrix properties.

**Definition 1.2.4** (Irreducibility). *We say that  $A$ , a positive  $N \times N$  matrix, is irreducible if for each  $i, j \in \mathbb{N}$  such that  $0 < i, j \leq N$  there is  $m \in \mathbb{N}$  such that  $A_{i,j}^m > 0$ .*

**Definition 1.2.5** (Aperiodicity). *We say that  $A$ , a positive  $N \times N$  matrix, is aperiodic if for each  $i \in \mathbb{N}$  such that  $0 < i \leq N$  great common divisor of all numbers  $m \in \mathbb{N}$  such that  $A^m > 0$  is equal to one.*

**Definition 1.2.6** (Prime Matrix). *We say that  $A$ , a positive  $N \times N$  matrix, is prime if there is  $M \in \mathbb{N}$  such that  $\forall n > M$  we have  $A^n > 0$ .*

It could be proved that for matrix being aperiodic and irreducible is equal to being prime (see [68]). From now we will assume that matrix associated with  $\Sigma_A^{\mathbb{N}}$  is irreducible and aperiodic.

Let us here list some elementary properties of probability measure on  $\Sigma_A^{\mathbb{N}}$ . First let us denote  $\mathcal{P}(\Sigma_A^{\mathbb{N}})$  to be space of all Borel probability measures on  $\Sigma_A^{\mathbb{N}}$ . Let us introduce metric  $d_L$ . It is called Wasserstein metric, and we will write it in form given by the Kantorovich-Rubinshtein theorem (see [29]). For each  $\mu, \nu \in \mathcal{P}(\Sigma_A^{\mathbb{N}})$  metric  $d_L$  is defined by

$$d_L(\mu, \nu) = \sup_{Lip(f) < 1} \left( \int_{\Sigma_A^{\mathbb{N}}} f d\mu - \int_{\Sigma_A^{\mathbb{N}}} f d\nu \right),$$

where for function  $f : \Sigma_A^{\mathbb{N}} \rightarrow \mathbb{R}$  we have  $Lip(f) = \sup_{\mathbf{i}, \mathbf{j} \in \Sigma_A^{\mathbb{N}}} \frac{|f(\mathbf{i}) - f(\mathbf{j})|}{d_\gamma(\mathbf{i}, \mathbf{j})}$  - i.e.  $Lip(f)$  is Lipschitz constant

of the function  $f$ . The space  $\mathcal{P}(\Sigma_A^{\mathbb{N}})$  is compact under this metric. Topology generated is called weak topology. This generates the following definition of limit, i.e. we say that  $\mu_n \rightarrow \mu$  if for every bounded, continuous function  $f : \Sigma_A^{\mathbb{N}} \rightarrow \mathbb{R}$  we have  $\lim_{n \rightarrow \infty} \int_{\Sigma_A^{\mathbb{N}}} f d\mu_n = \int_{\Sigma_A^{\mathbb{N}}} f d\mu$ . Let us mention that  $\mathcal{P}(\Sigma_A^{\mathbb{N}})$  is convex set as well.

We will be interested in  $S$ -invariant measures. For measure  $\mu$  on  $\Sigma_A^{\mathbb{N}}$  we say that it is  $S$ -invariant or shift invariant, if for every measurable set  $A \subset \Sigma_A^{\mathbb{N}}$  we have  $\mu(S^{-1}(A)) = \mu(A)$ . Space of all probability shift invariant measures on  $\Sigma_A^{\mathbb{N}}$  we will denote with  $\mathcal{P}_S(\Sigma_A^{\mathbb{N}})$ .  $\mathcal{P}_S(\Sigma_A^{\mathbb{N}})$  is compact with respect to weak

topology. With  $h(\mu)$  we will denote entropy of shift invariant measure  $\mu$ . Recall entropy is defined the following way. Let us define  $\{\alpha_n\}_n$ , a sequence of partitions of  $\Sigma_A^{\mathbb{N}}$ , composed of cylinders of length  $n$ , i.e.  $\alpha_n = \{[i]_n | i \in \Sigma_A^{\mathbb{N}}\}$ . Then entropy of measure  $\mu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}})$  is defined as

$$h(\mu) = \lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{A \in \alpha_n} -\mu(A) \log \mu(A).$$

Now we will introduce notion of pressure. Let  $\phi$  be a real continuous function on  $\Sigma_A^{\mathbb{N}}$ . Let us with  $S_n \phi(i)$  denote  $\sum_{k=0}^{n-1} \phi(S^k i)$ . Then *pressure* of function  $\phi$  is defined as

$$P(\phi) = \lim_{n \rightarrow +\infty} \frac{1}{n} \log \sum_{\substack{i \in \Sigma_A^{\mathbb{N}} \\ S^n i = i}} \exp(S_n \phi(i)) = \lim_{n \rightarrow +\infty} \frac{1}{n} \log \sum_{i \in \Sigma_A^n} \sup_{j \in [i]} \exp(S_n \phi(j)). \quad (1.10)$$

Another way to express pressure of continuous function  $\phi$  is via the so-called variational principle (see [6])

$$P(\phi) = \sup_{\mu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}})} \left( h(\mu) + \int_{\Sigma_A^{\mathbb{N}}} \phi d\mu \right).$$

Space of all real continuous functions we will denote as  $C(\Sigma_A^{\mathbb{N}})$ . We say that  $\phi$  is Hölder continuous if there is  $\alpha > 0$  and constant  $C_\phi$  such that for each  $i, j \in \Sigma_A^{\mathbb{N}}$  we have

$$|\phi(i) - \phi(j)| \leq C_\phi d_\gamma(i, j)^\alpha$$

Set of all real Hölder continuous function with respect to metric  $d_\gamma$  we will denote as  $H(\Sigma_A^{\mathbb{N}})$ . Let us now fix  $\phi \in H(\Sigma_A^{\mathbb{N}})$ . Then (as could be seen in [55]) there is a unique shift invariant probability measure  $\mu_\phi$ , called Gibbs measure with potential  $\phi$ , for which there is a constant  $C$  such that for each  $i \in \Sigma_A^{\mathbb{N}}$  and each  $n \in \mathbb{N}$  we have

$$\frac{1}{C} \mu_\phi([i]_n) \leq \exp(S_n \phi(i) - nP(\phi)) \leq C \mu_\phi([i]_n).$$

Now when we defined Gibbs measure let us state following results about its relationship to pressure. Refer to [55] for proofs.

**Proposition 1.2.7.** *Let  $\phi \in H(\Sigma_A^{\mathbb{N}})$  and let  $\mu_\phi$  be Gibbs measure associated with potential  $\phi$ . Then*

$$P(\phi) = h(\mu_\phi) + \int_{\Sigma_A^{\mathbb{N}}} \phi d\mu_\phi.$$

**Proposition 1.2.8.** *Let  $\phi, \psi \in H(\Sigma_A^{\mathbb{N}})$  and let  $P(t) = P(\phi + t\psi)$ . Then  $P$  is differentiable at 0 and*

$$P'(0) = \int_{\Sigma_A^{\mathbb{N}}} \psi d\mu_\phi.$$

Let us, in the end of this subsection, introduce the following notation that we will be using throughout the text.

**Definition 1.2.9.** *Let  $i \in \Sigma_A^*$ , let  $\phi \in C(\Sigma_A^{\mathbb{N}})$ . Then let us define*

$$s_i(\phi) = \sup_{j \in [i]} \exp\left(\sum_{k=0}^{n-1} \phi(S^k j)\right).$$

*We will often write just  $s_1$  when choice of  $\phi$  is clear.*

## 1.3 Iterated function systems

We are mostly interested in fractals in  $\mathbb{R}^d$  that are generated via iterated function systems, which we will denote shorter IFS. In this section we will start with the simplest example of IFS first, self-similar IFS. Then we will gradually generalise settings. We will also provide an account of appropriate multifractal analysis.

### 1.3.1 Self-similar iterated function systems

The most basic, but nevertheless, important example of an iterated function system is a self-similar iterated function system.

Let  $S_1, S_2, \dots, S_N$  be similarities on  $\mathbb{R}^d$ , with similarity ratios  $r_1, r_2, \dots, r_N \in (0, 1)$ . It follows from [28] that there is a unique non-empty compact set  $K$  such that

$$K = \cup_{i=1}^N S_i K.$$

Set  $K$  is called *self-similar set* generated by IFS  $(S_1, S_2, \dots, S_N)$ . Let  $(p_1, p_2, \dots, p_N)$  be a probability vector. Then there is a unique probability measure  $\mu$  such that

$$\mu = \sum_i p_i \mu \circ S_i^{-1}.$$

Measure  $\mu$  is called *self-similar measure* generated by  $(S_1, S_2, \dots, S_N, p_1, p_2, \dots, p_N)$ .

**Definition 1.3.1** (Open set condition(OSC)). *We say that Open Set Condition is satisfied if there is non-empty open set  $U$  such that the following holds*

1. For each  $i, j$  such that  $i \neq j$  we have  $S_i U \cap S_j U = \emptyset$ .
2. For each  $i$  we have  $S_i U \subset U$ .

Let  $\Sigma = \{1, 2, 3, \dots, N\}$ . Then for  $\mathbf{i} \in \Sigma^*$ ,  $\mathbf{i} = i_1 i_2 \dots i_n$ , denote  $S_{\mathbf{i}} = S_{i_1} \circ S_{i_2} \circ \dots \circ S_{i_n}$ ,  $K_{\mathbf{i}} = S_{\mathbf{i}} K$ ,  $r_{\mathbf{i}} = r_{i_1} r_{i_2} \dots r_{i_n}$  and  $p_{\mathbf{i}} = p_{i_1} p_{i_2} \dots p_{i_n}$ .

Hausdorff and Box dimension of self-similar set  $K$  are equal(see [21]). If OSC is satisfied, as could be seen in [19], dimensions are equal to the real number  $s$  that is the solution of following equation

$$\sum_i r_i^s = 1.$$

Multifractal analysis could be performed as follows. Let  $\beta : \mathbb{R} \rightarrow \mathbb{R}$  be defined as

$$\sum_i r_i^{\beta(q)} p_i^q = 1.$$

If OSC is satisfied then from [16]

$$\beta^*(\alpha) = f_{\mu}(\alpha).$$

Recall  $\beta^*$  is Legendre transform of  $\beta$ .

### 1.3.2 Self-conformal sets and measures

Now we will consider more general settings of self-conformal measures. Let the following vector be given

$$(U, X, S_1, S_2, \dots, S_N, p_1, p_2, \dots, p_N),$$

where

1.  $U$  is open and connected subset of  $\mathbb{R}^d$ .
2.  $X$  is compact subset of  $U$  and  $\overline{\text{int}(X)} = X$ .
3. for each  $i$ ,  $S_i : U \rightarrow U$  is a contractive  $C^{1+\theta}$  diffeomorphism with  $\theta \in (0, 1)$  and  $S_i X \subset X$ .
4. The Conformality Condition: For each  $x \in U$ , we have that  $(DS_i)(x)$  is a contractive similarity map, i.e. there exists  $r_i(x) \in (0, 1)$  such that  $|(DS_i)(x)u - (DS_i)(x)v| = r_i(x)|u - v|$  for all  $u, v \in \mathbb{R}^d$ ; here  $(DS_i)(x)$  denotes the derivative of  $S_i$  at  $x$ .
5.  $(p_1, p_2, \dots, p_N)$  is a probability vector.

It follows from [28] that there exists a unique non-empty compact sets  $K$  called *self-conformal set*, such that

$$K = \cup_i S_i(K_i).$$

There is a unique probability measure  $\mu$ , called *self-conformal measure*, such that

$$\mu = \sum_i p_i \mu \circ S_i^{-1}.$$

We often assume the following technical condition.

**Definition 1.3.2** (Open Set Condition (OSC)). *We say that Open Set Condition is satisfied if there is non-empty open set  $O \subset X$  such that*

1. *For each  $i, j$  such that  $i \neq j$  we have  $S_i O \cap S_j O = \emptyset$ .*
2. *For each  $i$  we have  $S_i O \subset O$ .*

Let  $\Sigma = \{1, 2, 3, \dots, N\}$ . Then for  $\mathbf{i} \in \Sigma^*$ ,  $\mathbf{i} = i_1 i_2 \dots i_n$ , denote  $S_{\mathbf{i}} = S_{i_1} \circ S_{i_2} \circ \dots \circ S_{i_n}$ ,  $K_{\mathbf{i}} = S_{\mathbf{i}} K$  and  $p_{\mathbf{i}} = p_{i_1} p_{i_2} \dots p_{i_n}$ . The projection  $\pi : \Sigma^{\mathbb{N}} \rightarrow K$  is defined as

$$\pi(\mathbf{i}) = \cap_{n=1}^{+\infty} K_{\mathbf{i}|_n}.$$

Let  $\Lambda : \Sigma^{\mathbb{N}} \rightarrow K$  be defined as

$$\Lambda(\mathbf{i}) = \log |D(S_{i_1}(\pi S_{\mathbf{i}}))|.$$

**Theorem 1.3.3** (Bowen's Formula). *There is a unique real number  $s_0$  such that*

$$P(s_0 \Lambda) = 0.$$

*If OSC is satisfied we have  $s_0 = \dim_{\text{H}} K$ .*

Now we will proceed with describing multifractal properties of self-conformal measures.

### Multifractal spectra of self-conformal measures

Multifractal spectra of the self-conformal measure  $\mu$  is defined as

$$f_{\mu}(\alpha) = \dim_{\text{H}} \left\{ x \in K \mid \lim_{r \rightarrow 0} \frac{\log \mu(B(x, r))}{\log r} = \alpha \right\}. \quad (1.11)$$

Patzschke [56] managed to compute multifractal spectra  $f_{\mu}$  when OSC is satisfied, building on earlier results due to Arbeiter & Patzschke [2] and Cawley & Mauldin [10]. Let us define  $\Phi : \Sigma_A^{\mathbb{N}} \rightarrow \mathbb{R}$  as  $\Phi(\mathbf{i}) = \log p_{i_1}$  and let  $\beta : \mathbb{R} \rightarrow \mathbb{R}$  be defined as

$$P(\beta(q)\Lambda + q\Phi) = 0.$$

Then the following theorem holds

**Theorem 1.3.4** ([56]). *Let  $\mu$  be self-conformal measure and let  $\alpha \in \mathbb{R}$ . If the OSC is satisfied, then we have*

$$f_\mu(\alpha) = \beta^*(\alpha).$$

In general local dimension does not exist everywhere. Barreira and Schmeling [8](also Olsen & Winter [50, 51], Xiao, Wu & Gao [72] and Moran [39]) have shown that the set of divergence points, i.e. the set

$$\Delta_\mu = \left\{ x \in K \mid \frac{\log \mu(B(x, r))}{\log r} \text{ does not converge as } r \text{ tends to } 0 \right\}$$

typically has full Hausdorff dimension. Precisely let  $\mu$  be self-conformal measure and let OSC be satisfied. if  $\mu$  is proportional to  $\dim_H K$ -dimensional Hausdorff measure supported on  $K$ , where under two measures being proportional we mean that there are constants  $a, b > 0$  such that for any measurable set  $E$  or both measures of  $E$  are zero or quotient of measures of  $E$  is in  $[a, b]$ , then

$$\left\{ x \in K \mid \frac{\log \mu(B(x, r))}{\log r} \text{ does not converge as } r \text{ tends to } 0 \right\} = \emptyset.$$

Otherwise it has full dimension, i.e.

$$\dim_H \left\{ x \in K \mid \frac{\log \mu(B(x, r))}{\log r} \text{ does not converge as } r \text{ tends to } 0 \right\} = \dim_H K.$$

This implies that set  $\Delta_\mu$  possesses rich structure. In order to explore this more carefully Olsen & Winter in [50, 51] introduced various generalised multifractal spectra functions designed to 'see' different sets of divergence points. In order to define generalised multifractal spectra let us define the following. Let  $f : (0, +\infty) \rightarrow M$ ,  $M$  metric space then let us define  $\text{acc}_{x \rightarrow x_0} f(x)$  to be

$$\text{acc}_{x \rightarrow x_0} f(x) = \{y \mid y \in M, y \text{ accumulation point of } f \text{ when } x \rightarrow x_0\}.$$

Then Generalized Hausdorff multifractal spectra  $F_\mu$  of self-conformal measure  $\mu$  is defined by

$$F_\mu(C) = \dim_H \left\{ x \in K \mid \text{acc}_{r \rightarrow 0} \frac{\log \mu(B(x, r))}{\log r} \subset C \right\}, C \subset \mathbb{R}.$$

Note that generalized multifractal spectra is a genuine generalization of traditional Hausdorff multifractal spectra, namely if set  $C = \{\alpha\}$  then  $F_\mu(C) = f_\mu(\alpha)$ . For generalized multifractal spectra there is analogue of 1.3.4 as well. It was first obtained by Moran [39] and Olsen & Winter [50] and later in less restrictive settings by Li, Wu & Xiong [35].

**Theorem 1.3.5** ([35, 39, 50]). *Let  $\mu$  be a self-conformal measure and let  $C$  be a closed subset of  $\mathbb{R}$ . If the OSC is satisfied, then we have*

$$F_\mu(C) = \sup_{\alpha \in C} \beta^*(\alpha).$$

### Mixed multifractal spectra

Recently mixed multifractal spectra of self-conformal measures generates interest in literature, see [7, 39, 42, 43]. Mixed multifractal spectra is investigation of properties of finitely many measures at same time, instead considering just single measure. Let  $M \in \mathbb{N}$ . Let us have  $M$  probability vectors  $\mathbf{p}_m = (p_{m,1}, p_{m,2}, \dots, p_{m,N})$ . For the vector  $(V, X, (S_i)_{i=1, \dots, n}, (p_{m,i})_{i=1, \dots, n})$ , let  $\mu_m$  be the self-conformal measure associated with that vector. I.e.  $\mu_m$  is a unique probability measure which satisfies

$$\mu_m = \sum_i p_{m,i} \mu_m \circ S_i^{-1}.$$

Mixed multifractal spectra for the list  $\boldsymbol{\mu} = (\mu_1, \mu_2, \dots, \mu_M)$  is defined by

$$f_{\boldsymbol{\mu}}(\boldsymbol{\alpha}) = \dim_H \left\{ x \in K \mid \lim_{r \rightarrow 0} \left( \frac{\log \mu_1(B(x, r))}{\log r}, \frac{\log \mu_2(B(x, r))}{\log r}, \dots, \frac{\log \mu_M(B(x, r))}{\log r} \right) = \boldsymbol{\alpha} \right\},$$

where  $\boldsymbol{\alpha} \in \mathbb{R}^M$ . Of course, analogous to the case of a single self conformal measure, there is a generalised mixed multifractal spectra designed to 'see' different sets of divergence points. Namely the generalised mixed Hausdorff multifractal spectrum  $F_{\boldsymbol{\mu}}$  of the list of measures  $\boldsymbol{\mu} = (\mu_1, \mu_2, \dots, \mu_M)$  is defined by

$$F_{\boldsymbol{\mu}}(C) = \dim_H \left\{ x \in K \mid \text{acc}_{r \rightarrow 0} \left( \frac{\log \mu_1(B(x, r))}{\log r}, \frac{\log \mu_2(B(x, r))}{\log r}, \dots, \frac{\log \mu_M(B(x, r))}{\log r} \right) \subset C \right\},$$

for  $C \subset \mathbb{R}^M$ . Again note that this is a genuine extension of traditional mixed multifractal spectrum. Namely if we put set  $C$  to be singleton, i.e.  $C = \{\boldsymbol{\alpha}\}$  we get  $F_{\boldsymbol{\mu}}(C) = f_{\boldsymbol{\mu}}(\boldsymbol{\alpha})$ . Assuming that the OSC generalised multifractal spectra could be computed. In order to do so we introduce the following notation. Let us for  $\mathbf{i} \in \Sigma_A^{\mathbb{N}}$  define  $\Phi_m(\mathbf{i}) = \log p_{m, i_1}$ . Let  $\boldsymbol{\Phi} = (\Phi_1, \Phi_2, \dots, \Phi_M)$ . Let  $\beta : \mathbb{R}^M \rightarrow \mathbb{R}$  be defined by

$$P(\beta(q)\Lambda + \langle q, \boldsymbol{\Phi} \rangle) = 0.$$

Then for generalised mixed multifractal spectra  $F_{\boldsymbol{\mu}}$ , the following theorem holds.

**Theorem 1.3.6** ([39, 42]). *Let  $(\mu_1, \mu_2, \dots, \mu_M)$  be a list of self-conformal measures and let  $C \subset \mathbb{R}^M$  be closed set. If OSC is satisfied, then we have*

$$F_{\boldsymbol{\mu}}(C) = \sup_{\boldsymbol{\alpha} \in C} \beta^*(\boldsymbol{\alpha}).$$

In particular if  $\boldsymbol{\alpha} \in \mathbb{R}^M$  and OSC then

$$f_{\boldsymbol{\mu}}(\boldsymbol{\alpha}) = \beta^*(\boldsymbol{\alpha}).$$

Here  $\beta^*$  is Legendre transform of  $\beta$  which is defined as

$$\beta^*(\boldsymbol{\alpha}) = \inf_q \{ \beta(q) + \langle q, \boldsymbol{\alpha} \rangle \}.$$

### 1.3.3 Graph-directed self-conformal sets and graph-directed self-conformal measures

In this subsection we will first introduce notation for graph and associate shift of finite type with its infinite paths. Then we will go on and define generalisation of self-conformal sets and measures, i.e. graph-directed self-conformal sets and measures. This is genuine generalisation as could be seen at [36]. Then we will describe appropriate multifractal analysis.

#### Graph and Coding of graph with subshift of finite type

We will now start introducing notation for graphs. Let  $v_1, v_2, v_3, \dots, v_K$  be vertices and let us denote  $V = \{v_1, v_2, v_3, \dots, v_K\}$ . Let  $e_1, e_2, e_3, \dots, e_N$  be edges between vertices in  $V$ . Note that there could exist different edges that start and terminate in the same vertex. We will denote  $E = \{e_1, e_2, e_3, \dots, e_N\}$ . For edge  $e$  we will denote the initial point with  $i(e)$  and the terminal point with  $t(e)$ . Note that there could be edges with the same initial and terminal points. (Multi)Graph defined by these edges and vertices we will denote as  $G = (V, E)$ . For vertex  $v \in V$  we will denote

$$E_v = \{e \in E \mid i(e) = v\}.$$

For vertices  $v_1, v_2$  define

$$E_{v_1, v_2} = \{e \in E \mid i(e) = v_1, t(e) = v_2\}.$$

The family of all finite paths in graph which length is  $n$ , is denoted by

$$\Sigma_G^n = \{e_1 e_2 \cdots e_n | t(e_i) = i(e_{i+1}), i = 1, 2, \dots, n-1\}.$$

The family of all finite path in graph we will denote by

$$\Sigma_G^* = \cup_n \Sigma_G^n.$$

The family of infinite paths we will denote as

$$\Sigma_G^{\mathbb{N}} = \{e_1 e_2 e_3 \dots | t(e_n) = i(e_{n+1}), n \in \mathbb{N}\}.$$

Now note that  $\Sigma_G^{\mathbb{N}}$  is subshift of finite type  $\Sigma_A^{\mathbb{N}}$  where  $\Sigma = \{e_1, e_2, \dots, e_N\}$ ,  $N$  is the number of edges in  $E$ , and  $A$  is defined as follows

$$A_{ij} = \begin{cases} 1, & \text{if } t(e_i) = s(e_j), \\ 0, & \text{otherwise.} \end{cases} \quad (1.12)$$

Recall that we assume that  $A$  is aperiodic and irreducible. Note that from  $A$  being irreducible we have that  $G$  is strongly connected, i.e. there is a path between any two vertices in graph.

### Graph-directed self-conformal sets and measures

Now we will describe graph-directed self-conformal sets and measures. Let the following vector be given

$$(V, E, (U_v)_{v \in V}, (X_v)_{v \in V}, (S_e)_{e \in E}, (p_e)_{e \in E}),$$

where

1.  $(V, E)$  is graph with set of vertices  $V$  and edges  $E$ .
2. Each  $U_v$  is open and connected subset of  $\mathbb{R}^d$ .
3. Each  $X_v$  is compact subset of  $U_v$  and  $\overline{\text{int}(X_v)} = X_v$ .
4. for each  $e$ ,  $S_e : U_{t(e)} \rightarrow U_{i(e)}$  is a contractive  $C^{1+\theta}$  diffeomorphism with  $\theta \in (0, 1)$  and  $S_e X_{t(e)} \subset X_{i(e)}$ .
5. The Conformality Condition: For  $e$  and each  $x \in U_e$ , we have that  $(DS_e)(x)$  is a contractive similarity map, i.e. there exists  $r_e(x) \in (0, 1)$  such that  $|(DS_e)(x)u - (DS_e)(x)v| = r_e(x)|u - v|$  for all  $u, v \in \mathbb{R}^d$ ; here  $(DS_e)(x)$  denotes the derivative of  $S_e$  at  $x$ .
6. For each  $v \in V$ ,  $(p_e)_{e \in E_v}$  is a probability vector i.e.  $\sum_{w \in E_v} p_w = 1$ .

From [28] there exists a unique list of non-empty compact sets  $(K_v)_{v \in V}$ , called list of *graph-directed self-conformal sets*, such that

$$K_v = \cup_{w \in E_v} S_w(K_{t(w)}).$$

There is a unique list of probability measures  $(\mu_v)_{v \in V}$ , called list of *graph-directed self-conformal measures*, such that

$$\mu_v = \sum_{w \in E_v} p_w \mu_{t(w)} \circ S_e^{-1}.$$

We often assume the following technical condition.

**Definition 1.3.7** (Open Set Condition(OSC)). *We say that graph-directed self-conformal iterated function system satisfies Open Set Condition if there exists a list of non-empty open sets  $W_v \in V$  such that for each  $v \in V$  we have  $W_v \subset X_v$  and the following holds*

1. For each  $e_1, e_2$  such that  $e_1 \neq e_2$  we have  $S_{e_1}W_{t(e_1)} \cap S_{e_2}W_{t(e_2)} = \emptyset$ .
2. For each  $e$  we have  $S_e W_{t(e)} \subset W_{i(e)}$ .

Let  $\Sigma_A^{\mathbb{N}}$  be subshift of finite type which is associated with graph  $G = (E, V)$ . Let us introduce the following notation. For  $\mathbf{i} \in \Sigma_A^*$ ,  $\mathbf{i} = e_1 e_2 \dots e_n$ , let us define  $S_{\mathbf{i}} = S_{e_1} \dots S_{e_n}$ ,  $K_{\mathbf{i}} = S_{\mathbf{i}} K_{t(e_n)}$ ,  $p_{\mathbf{i}} = p_{e_1} \dots p_{e_n}$ . Projection  $\pi : \Sigma_A^{\mathbb{N}} \rightarrow \mathbb{R}^d$  is defined as follows

$$\pi(\mathbf{i}) = \cap_n K_{\mathbf{i}|_n}.$$

Let  $\Lambda : \Sigma_A^{\mathbb{N}} \rightarrow \mathbb{R}$  be defined by

$$\Lambda(\mathbf{i}) = \log |D(S_{e_1} \pi(S(\mathbf{i})))|.$$

**Theorem 1.3.8** (Bowen's Formula). *There is a unique real number  $s_0$  such that*

$$P(s_0 \Lambda) = 0.$$

*And if OSC is satisfied then for each  $v \in V$  we have  $s_0 = \dim_H K_v$ .*

### Multifractal spectra of graph-directed self-conformal measures

Let us have a list of graph-directed self-conformal measures  $(\mu_v)_{v \in V}$  associated with the vector  $(V, E, (U_v)_{v \in V}, (X_v)_{v \in V}, (S_e)_{e \in E}, (p_v)_{v \in V})$ . Multifractal spectrum of measure  $\mu_v$  is defined as

$$f_{\mu_v}(\alpha) = \dim_H \left\{ x \in K_v \mid \lim_{r \rightarrow 0} \frac{\log \mu_v B(x, r)}{\log r} = \alpha \right\}.$$

If the OSC is satisfied, then the multifractal spectrum  $f_{\mu_v}$  of  $\mu_v$  can be computed as follows, see Theorem 1.3.9 below. This result was first established by Edgar & Mauldin [16] in 1992 assuming that the maps  $(S_v)_{v \in V}$  were similarities and was subsequently extended to the conformal case by Cole [11, 12] building on earlier results due to Arbeiter & Patzschke [2] and Patzschke [56]. Let  $\Phi : \Sigma_A^{\mathbb{N}} \rightarrow \mathbb{R}$  be defined as  $\Phi(\mathbf{i}) = p_{e_1}$ , for  $\mathbf{i} = e_1 e_2 \dots$ , and let  $\beta : \mathbb{R} \rightarrow \mathbb{R}$  be defined by

$$P(\beta(q)\Lambda + q\Phi) = 0.$$

Then the following theorem holds

**Theorem 1.3.9** ([11, 56]). *Let  $(\mu_v)_v$  be a list of graph-directed self-conformal measures and let  $\alpha \in \mathbb{R}$ . If the OSC is satisfied, then we have*

$$f_{\mu_v}(\alpha) = \beta^*(\alpha),$$

*for all  $v \in V$ .*

In [50] Olsen & Winter defined and investigated, Generalized Hausdorff multifractal spectra  $F_{\mu_v}$  of  $\mu_v$  defined by

$$F_{\mu_v}(C) = \dim_H \left\{ x \in K_v \mid \text{acc}_{r \rightarrow 0} \frac{\log \mu_v(B(x, r))}{\log r} \subset C \right\}, C \subset \mathbb{R}$$

Note that generalized multifractal spectrum is a genuine generalization of the traditional Hausdorff multifractal spectra, namely if set  $C = \{\alpha\}$  then  $F_{\mu}(C) = f_{\mu}(\alpha)$ . There is an analogue of theorem 1.3.9 as well. The following theorem was first obtained by Moran [39] and Olsen & Winter and later in less restrictive settings by Li, Wu & Xiong [35].

**Theorem 1.3.10** ([35, 39, 50]). *Let  $(\mu_v)_v$  be a list of graph-directed self-conformal measures and let  $C$  be a closed subset of  $\mathbb{R}$ . If the OSC is satisfied, then we have*

$$F_{\mu_v}(C) = \sup_{\alpha \in C} \beta^*(\alpha).$$



### Multifractal Spectra of Ergodic Birkhoff averages

Multifractal analysis of Birkhoff averages received a lot of attention in the last 15-20 years, refer for example to [5, 22, 23, 24, 40, 43, 51]. The multifractal spectrum of  $F_f^{erg}(\alpha)$  of continuous function  $f : \Sigma_A^{\mathbb{N}} \rightarrow \mathbb{R}$  is defined by

$$F_f^{erg}(\alpha) = \dim_H \pi \left\{ \mathbf{i} \in \Sigma_A^{\mathbb{N}} \mid \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} f(S^k \mathbf{i}) = \alpha \right\},$$

For  $\alpha \in \mathbb{R}$ . One of the main problems in multifractal analysis of Birkhoff averages is the detailed study of the multifractal spectrum. Next theorem has been proved in various generality settings in [22, 23, 24, 40, 43, 51]. Before stating the theorem, let us introduce the following notation. If  $(x_n)_n$  is a sequence of points in a metric space  $X$ , then we write  $\text{acc}_n x_n$  for the set of accumulation points of the sequence  $(x_n)_n$ , i.e.

$$\text{acc}_n x_n = \{x \in X \mid x \text{ is an accumulation point of } (x_n)_n\}.$$

**Theorem 1.3.11** ([22, 23, 24, 40, 43, 51]). *Let  $f : \Sigma_A^{\mathbb{N}} \rightarrow \mathbb{R}$  be a continuous function. Let  $C$  be a closed subset of  $\mathbb{R}$ . If OSC is satisfied, then*

$$\dim_H \pi \left\{ \mathbf{i} \in \Sigma_A^{\mathbb{N}} \mid \text{acc}_{n \rightarrow +\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(S^k \mathbf{i}) \subset C \right\} = \sup_{\alpha \in C} \sup_{\substack{\mu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}}) \\ \int_{\Sigma_A^{\mathbb{N}}} f d\mu = \alpha}} - \frac{h(\mu)}{\int_{\Sigma_A^{\mathbb{N}}} \Lambda d\mu}.$$

In particular if OSC is satisfied and  $C = \{\alpha\}$  then

$$\dim_H \pi \left\{ \mathbf{i} \in \Sigma_A^{\mathbb{N}} \mid \lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(S^k \mathbf{i}) = \alpha \right\} = \sup_{\substack{\mu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}}) \\ \int_{\Sigma_A^{\mathbb{N}}} f d\mu = \alpha}} - \frac{h(\mu)}{\int_{\Sigma_A^{\mathbb{N}}} \Lambda d\mu}.$$

### Mixed Multifractal Spectra of Ergodic Birkhoff averages

Mixed multifractal spectra of ergodic Birkhoff averages investigates properties of finitely many real functions at same time instead of only one. Fix a positive integer  $M$ . The multifractal spectrum  $F_{\mathbf{f}}^{erg}$  of ergodic Birkhoff averages of a vector valued continuous function  $\mathbf{f} : \Sigma_A^{\mathbb{N}} \rightarrow \mathbb{R}^M$  is defined by

$$F_{\mathbf{f}}^{erg}(\alpha) = \dim_H \pi \left\{ \mathbf{i} \in \Sigma_A^{\mathbb{N}} \mid \lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{k=0}^{n-1} \mathbf{f}(S^k \mathbf{i}) = \alpha \right\}$$

for  $\alpha \in \mathbb{R}^M$ . We will use the following notation. Namely, if  $\mathbf{f} : \Sigma_A^{\mathbb{N}} \rightarrow \mathbb{R}^M$  is a continuous function with  $\mathbf{f} = (f_1, \dots, f_M)$ , then we will write

$$\int_{\Sigma_A^{\mathbb{N}}} \mathbf{f} d\mu = \left( \int_{\Sigma_A^{\mathbb{N}}} f_1 d\mu, \int_{\Sigma_A^{\mathbb{N}}} f_2 d\mu, \dots, \int_{\Sigma_A^{\mathbb{N}}} f_M d\mu \right)$$

for  $\mu \in \mathcal{P}(\Sigma_A^{\mathbb{N}})$ . We can now state following theorem.

**Theorem 1.3.12** ([22, 23, 24, 40, 41, 50]). *Let  $\mathbf{f} : \Sigma_A^{\mathbb{N}} \rightarrow \mathbb{R}^M$  be a continuous function. Let  $C$  be a closed subset of  $\mathbb{R}^M$ . If the OSC is satisfied, then*

$$\dim_H \pi \left\{ \mathbf{i} \in \Sigma_A^{\mathbb{N}} \mid \text{acc}_n \frac{1}{n} \sum_{k=0}^{n-1} \mathbf{f}(S^k \mathbf{i}) \subset C \right\} = \sup_{\substack{\mu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}}) \\ \int_{\Sigma_A^{\mathbb{N}}} \mathbf{f} d\mu \in C}} - \frac{h(\mu)}{\int_{\Sigma_A^{\mathbb{N}}} \Lambda d\mu}.$$

In particular if  $C = \alpha$  for  $\alpha \in \mathbb{R}^M$  then

$$\dim_{\text{H}} \pi \left\{ \mathbf{i} \in \Sigma_A^{\mathbb{N}} \mid \lim_n \frac{1}{n} \sum_{k=0}^{n-1} \mathbf{f}(S^k \mathbf{i}) = \alpha \right\} = \sup_{\substack{\mu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}}) \\ \int_{\Sigma_A^{\mathbb{N}}} \mathbf{f} d\mu = \alpha}} - \frac{h(\mu)}{\int_{\Sigma_A^{\mathbb{N}}} \Lambda d\mu}.$$

### 1.3.4 Gibbs measures on Subshift of Finite Type

As we have seen, the Subshifts of finite types are useful for investigation of IFS. In our examples we had set  $K$  and measure  $\mu$  generated by IFS. Then we defined projection  $\pi : \Sigma_A^{\mathbb{N}} \rightarrow K$ . In fact, in our examples measure  $\mu$  is a push forward of the so-called Markov measure on  $\Sigma_A^{\mathbb{N}}$ . Here we will discuss multifractal analysis of generalization of Markov measures, the so-called Gibbs measures that we defined in section 1.2.

Let us have a Subshift of finite type  $\Sigma_A^{\mathbb{N}}$ . Recall we assumed that matrix  $A$  is aperiodic and irreducible. Let us fix  $\Lambda \in H(\Sigma_A^{\mathbb{N}})$  and  $\Lambda < 0$ . Let metric  $d_{\Lambda}$  be defined as follows. For each  $\mathbf{i}, \mathbf{j} \in \Sigma_A^{\mathbb{N}}$  let  $n = s(\mathbf{j}, \mathbf{i})$ . Then

$$d_{\Lambda}(\mathbf{i}, \mathbf{j}) = \sup_{\mathbf{k} \in [\mathbf{i}]_n} \exp(S_n(\Lambda(\mathbf{k}))).$$

Let Hausdorff dimension and "boxes" required for dimension theory be defined with respect to the metric  $d_{\Lambda}$ . Note that for  $d_{\Lambda}$  and  $d_{\gamma}$  generates the same topology on  $\Sigma_A^{\mathbb{N}}$ .

Let us now fix  $\Phi \in H(\Sigma_A^{\mathbb{N}})$ . Let  $\mu_{\Phi}$  be Gibbs measure with potential  $\Phi$ .

Let us define  $\beta : \mathbb{R} \rightarrow \mathbb{R}$  by

$$P(\beta(q)\Lambda + q\Phi) = 0.$$

Then from [58] we have that if  $\mu_{\Phi} \neq \mu_{\Lambda}$ , where  $\mu_{\Lambda}$  is Gibbs measure with potential  $\Lambda$ , then

$$\beta^*(\alpha) = f_{\mu_{\Phi}}(\alpha).$$

# Chapter 2

## Results

### 2.1 Multifractal analysis on Subshift of Finite type

In this section, we will describe full Multifractal analysis on subshifts of finite type. In section 1.1.2 we have considered two cases, multifractal analysis of Birkhoff averages and multifractal measures. Here we will take on the approach that unifies these two cases and is more general as well. This framework is introduced by Olsen in [43].

$\Sigma_A^{\mathbb{N}}$  will play the role of set  $K$  from subsection 1.1.2. We will fix Hölder continuous function  $\Lambda : \Sigma_A^{\mathbb{N}} \rightarrow \mathbb{R}, \Lambda < 0$ . Then we will fix metric  $d_\Lambda$  with respect to which Hausdorff and Box dimension are defined. In this section we will denote  $s_{\mathbf{i}}(\Lambda)$  as  $s_{\mathbf{i}}$ .

In 1.1.2 we first singled out sets of points with the same local dimension or the same Birkhoff averages. Here for that purpose, we will define the following functions. Let  $U$  be continuous function, with respect to weak topology, that maps  $\mathcal{P}(\Sigma_A^{\mathbb{N}})$  to metric space  $X$ . And let us define the sequence of functions  $L_n : \Sigma_A^{\mathbb{N}} \rightarrow \mathcal{P}(\Sigma_A^{\mathbb{N}})$  as

$$L_n(\mathbf{i}) = \frac{1}{n} \sum_{k=0}^{n-1} \delta_{S^k \mathbf{i}}.$$

We will first define fine multifractal spectra in this setting. Then we will go on and define coarse multifractal spectra. Then we will propose a definition of Renyi dimension in this setting and present results regarding multifractal formalism.

#### Fine multifractal spectra

Define

$$E_\alpha = \left\{ \mathbf{i} \in \Sigma_A^{\mathbb{N}} \mid \lim_{n \rightarrow +\infty} U(L_n[\mathbf{i}|n]) = \alpha \right\}.$$

And let

$$f^{U,\Lambda}(\alpha) = \dim_{\text{H}} E_\alpha.$$

Let  $C \subset X$ . Then we can define

$$E_C = \left\{ \mathbf{i} \in \Sigma_A^{\mathbb{N}} \mid \text{acc}_{n \rightarrow +\infty} U(L_n[\mathbf{i}|n]) \subset C \right\}.$$

We define generalized multifractal spectra as

$$f^{U,\Lambda}(C) = \dim_{\text{H}} E_C.$$

### Coarse multifractal spectra

First let us note that box counting as idea in this setting is rather simple. That is due to the fact that for  $\mathbf{j} \in \Sigma_A^{\mathbb{N}}$  there is exactly one ball of radius  $r$ . I.e. let  $\mathbf{j} \in \Sigma_A^{\mathbb{N}}$  such that  $\mathbf{i} \in B(\mathbf{j}, r)$ . Let us prove that then  $B(\mathbf{i}, r) = B(\mathbf{j}, r)$ .

Note that all possible distances from  $\mathbf{j}$  to some other points are contained in set  $\{s_{\mathbf{j}|n} | n \in \mathbb{N}_0\}$ . Hence  $B(\mathbf{j}, r) = [\mathbf{j}|n_r]$ , where  $n_r = \inf\{n | n \in \mathbb{N}_0, s_{\mathbf{j}|n} < r\}$ . From  $\mathbf{i} \in B(\mathbf{j}, r)$  we have that  $\{\mathbf{j}|n_r = \mathbf{i}|n_r\}$ . For  $n < n_r$  we have  $s_{\mathbf{i}|n} = s_{\mathbf{j}|n} \geq r$ . Hence we conclude  $B(\mathbf{i}, r) = [\mathbf{i}|n_r]$ . Therefore our definition of  $N_\delta^{U,\Lambda}(\alpha, \epsilon)$  from 1.3 is reduced to

$$N_\delta^{U,\Lambda}(\alpha, \epsilon) = \# \{ \mathbf{i} \in \Sigma_A^* | s_{\mathbf{i}} \sim \delta, U(L_{|\mathbf{i}|}[\mathbf{i}]) \in B(\alpha, \epsilon) \}.$$

Where  $s_{\mathbf{i}} \sim \delta$  means that  $s_{\mathbf{i}} \leq \delta$  and  $s_{\mathbf{i}|(|\mathbf{i}|-1)} > \delta$ .

Or more generally, let  $C \subset X$  then let us define

$$N_\delta^{U,\Lambda}(C, \epsilon) = \# \{ \mathbf{i} \in \Sigma_A^* | s_{\mathbf{i}} \sim \delta, U(L_{|\mathbf{i}|}[\mathbf{i}]) \in B(C, \epsilon) \}.$$

Lower coarse spectra, and upper coarse spectra, are defined respectively as

$$\begin{aligned} \underline{f}_c^{U,\Lambda}(C) &= \lim_{\epsilon \rightarrow 0} \liminf_{\delta \rightarrow 0} \frac{\log N_\delta(C, \epsilon)}{-\log \delta}, \\ \bar{f}_c^{U,\Lambda}(C) &= \lim_{\epsilon \rightarrow 0} \limsup_{\delta \rightarrow 0} \frac{\log N_\delta(C, \epsilon)}{-\log \delta}. \end{aligned}$$

Corollary of 2.1.2, is that in fact under condition stated below  $\underline{f}_c^{U,\Lambda} = \bar{f}_c^{U,\Lambda}$ , i.e. coarse multifractal spectra is well defined.

**Definition 2.1.1.** *Let  $C \subset X$  be a closed set,  $\Lambda \in H(\Sigma_A^{\mathbb{N}}), \Lambda < 0$ . Then let us define generalized coarse multifractal spectra as*

$$f_c^{U,\Lambda}(C) = \lim_{\epsilon \rightarrow 0} \lim_{\delta \rightarrow 0} \frac{\log N_\delta(C, \epsilon)}{-\log \delta}.$$

Somewhat similar results about coarse multifractal spectra could be found in [43, 50], i.e. our result holds on subshift of finite type, and it hold for closed set  $C$  and continuous function  $U$ . Result in [43] holds under OSC (which is less restrictive than subshift of finite type) but  $U$  is restricted to be continuous affine. In [50] result holds under OSC and  $U$  is allowed to be continuous, but set  $C$  is a singleton.

**Theorem 2.1.2.** *Let  $\Lambda \in H(\Sigma_A^{\mathbb{N}}), \Lambda < 0$ . And let  $C \subset X$  be closed. Then we have that*

$$f^{U,\Lambda}(C) = f_c^{U,\Lambda}(C) = \sup_{\substack{\mu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}}) \\ U\mu \in C}} - \frac{h(\mu)}{\int_{\Sigma_A^{\mathbb{N}}} \Lambda d\mu}.$$

Proof is provided in section 3.2. It is a corollary of investigation of geometric zeta function.

### Renyi dimension

Here we will propose expression analogous to 1.7 for our more general settings. Inspiration is by coarse zeta function introduced by Olsen in [48]. Note that our setting, if we focus only on self-conformal measures, does not cover case with no separation condition. For existence of Renyi dimension for self conformal measures with no separation conditions refer to [57]. We will assume, in addition, that  $X$  is inner product space with inner product  $\langle \cdot, \cdot \rangle$ . Let us introduce the following notation.

**Definition 2.1.3.** For every  $\mathbf{i} = i_1 i_2 \dots i_n \in \Sigma_A^*$  let us choose some  $\mathbf{i}' = j_1 j_2 \dots j_m \in \Sigma_A^*$  such that  $A_{i_n, j_1} = 1$ , and  $A_{j_m, i_1} = 1$ . Then define  $\bar{\mathbf{i}} = \mathbf{i}(\mathbf{i}')\mathbf{i}(\mathbf{i}')\mathbf{i}(\mathbf{i}') \dots$ . That way we have defined mappings  $\bar{\cdot} : \Sigma_A^* \rightarrow \Sigma_A^{\mathbb{N}}$  and  $\cdot' : \Sigma_A^* \rightarrow \Sigma_A^*$ .

Note that  $\mathbf{i}'$  exists since  $A$  is irreducible, so there exists a path in associated graph that goes from  $i_n$  to  $i_1$ . Since  $A$  is prime, there is  $M \in \mathbb{N}$  such that  $A^M > 0$ . So we can, and we will, require  $|\mathbf{i}'| \leq M$ . Now we will define

$$M_\delta(q) = \sum_{s_{\mathbf{i}} \sim \delta} s_{\mathbf{i}}^{\langle q, U(L_{|\mathbf{i}|} \bar{\mathbf{i}}) \rangle}.$$

Recall that  $s_{\mathbf{i}} \sim \delta$  means that  $s_{\mathbf{i}} \leq \delta$  and  $s_{\mathbf{i}||(\mathbf{i}|-1)} > \delta$ . Upper and lower Renyi spectra are respectively defined as

$$\begin{aligned} \underline{\tau}^{U, \Lambda}(q) &= \liminf_{\delta \rightarrow 0} \frac{\log \sum_{s_{\mathbf{i}} \sim \delta} s_{\mathbf{i}}^{\langle q, U(L_{|\mathbf{i}|} \bar{\mathbf{i}}) \rangle}}{-\log \delta}, \\ \bar{\tau}^{U, \Lambda}(q) &= \limsup_{\delta \rightarrow 0} \frac{\log \sum_{s_{\mathbf{i}} \sim \delta} s_{\mathbf{i}}^{\langle q, U(L_{|\mathbf{i}|} \bar{\mathbf{i}}) \rangle}}{-\log \delta}. \end{aligned}$$

In fact under condition stated below we have  $\underline{\tau} = \bar{\tau}$  so the Renyi spectra is defined as

$$\tau^{U, \Lambda}(q) = \lim_{\delta \rightarrow 0} \frac{\log \sum_{s_{\mathbf{i}} \sim \delta} s_{\mathbf{i}}^{\langle q, U(L_{|\mathbf{i}|} \bar{\mathbf{i}}) \rangle}}{-\log \delta}.$$

We have the following result.

**Theorem 2.1.4.** Let  $X$  be inner product space with inner product  $\langle \cdot, \cdot \rangle$ . Let  $\Lambda \in H(\Sigma_A^{\mathbb{N}})$ ,  $\Lambda < 0$  then  $\tau^{U, \Lambda} : X \rightarrow \mathbb{R}$  defined below is well defined

$$\tau^{U, \Lambda}(q) = \lim_{r \rightarrow 0} \frac{\log \sum_{s_{\mathbf{i}} \sim r} s_{\mathbf{i}}^{\langle q, U(L_{|\mathbf{i}|} \bar{\mathbf{i}}) \rangle}}{-\log r}.$$

Additionally

$$\tau^{U, \Lambda}(q) = \sup_{\mu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}})} \left\{ -\frac{h(\mu)}{\int_{\Sigma_A^{\mathbb{N}}} \Lambda d\mu} - \langle q, U(\mu) \rangle \right\}.$$

Proof is provided in section 3.3. The above result shows that  $\tau^{U, \Lambda}$  is equal to abscissa of convergence of coarse multifractal zeta function Olsen introduced in [48], for which in some cases, for example self-conformal measures or ergodic Birkhoff averages of continuous functions, multifractal formalism holds.

## 2.2 Zeta Functions

Main objects of this section are geometric and dynamical zeta functions. In the first part we will talk about geometric zeta functions, while in the second part we will talk about dynamical zeta functions and multifractal pressure.

### 2.2.1 Geometric zeta function

Motivated by the powerful techniques provided by the use of the Artin-Mazur zeta-functions in number theory and the use of the Ruelle zeta-functions in dynamical systems, Lapidus and collaborators (see books by Lapidus & van Frankenhuysen [32, 33] and the references therein) have introduced and pioneered use of zeta-functions in fractal geometry. Inspired by this development, within the past 7–8

years several authors have paralleled this development by introducing zeta-functions into multifractal geometry. We will first present a special case of our result in settings of self-similar IFS and as appropriate compare our approach adopted in [37] to the one adopted by Rock & Lapidus in [30]. Then we will state our main results in full generality. After we state our main results, we will describe and compare our approach to the one introduced by Baker in [4].

### Geometric zeta function of self-similar measure

Here we will illustrate our ideas in a simple setting, we consider the following example involving self-similar measures. Recall we introduced self-similar IFS in 1.3.1. Let  $K$  be self-similar set, and  $\mu$  be self-similar measure associated with vector  $(S_1, \dots, S_N, p_1, \dots, p_N)$ . For  $\alpha \in \mathbb{R}$ , we are now attempting to introduce a "natural" self-similar multifractal zeta-function  $\zeta_\alpha^{\text{sim}}$  whose abscissa of convergence equals  $f_\mu(\alpha)$ . Abscissa of convergence of  $\zeta_\alpha^{\text{sim}}$ , which we will denote as  $\sigma_{ab}(\zeta_\alpha^{\text{sim}}(\cdot))$  is defined as

$$\sigma_{ab}(\zeta_\alpha^{\text{sim}}(\cdot)) = \inf\{s \in \mathbb{R} \mid \zeta_\alpha^{\text{sim}}(s) \text{ converges}\}.$$

We will start introduction of a "natural" multifractal zeta-function as follows. Namely, since  $f_\mu(\alpha)$  measures the size of the set of points  $x$  for which  $\lim_{\delta \rightarrow 0} \frac{\log \mu(B(x, \delta))}{\log \delta} = \alpha$  and since  $\frac{\log \mu(B(x, \delta))}{\log \delta}$  has the same form as  $\frac{\log p_i}{\log r_i}$ , it is natural to define the self-similar multifractal zeta-function as

$$\zeta_\alpha^{\text{sim}}(s) = \sum_{\substack{\mathbf{i} \in \Sigma^* \\ \frac{\log p_i}{\log r_i} = \alpha}} r_i^s,$$

for these complex numbers  $s$  for which the series converges absolutely. This approach has been adopted by Rock in [61], then additionally investigated by Rock & Lapidus in [30] and [17]. However it seems that ability of this approach to produce relevant multifractal information is limited. We will describe now what we think the problem is and suggest a solution. Later, after we state our main result in general settings, we will describe and compare our approach to one introduced by Baker in [4]. An easy and straightforward calculation (which we present below) shows that the abscissa of convergence  $\sigma_{ab}(\zeta_\alpha^{\text{sim}}(\cdot))$  of  $\zeta_\alpha^{\text{sim}}$  is less or equal to  $f_\mu(\alpha)$ , i.e.

$$\sigma_{ab}(\zeta_\alpha^{\text{sim}}(\cdot)) \leq f_\mu(\alpha). \quad (2.1)$$

Indeed, if  $\alpha \notin \left[ \min_i \frac{\log p_i}{\log r_i}, \max_i \frac{\log p_i}{\log r_i} \right]$  then it is easily seen that for all  $\mathbf{i} \in \Sigma^*$ , we have  $\frac{\log p_i}{\log r_i} \neq \alpha$ , hence  $\sigma_{ab}(\zeta_\alpha^{\text{sim}}(\cdot)) = -\infty$ . On the other hand, if  $\alpha \in \left[ \min_i \frac{\log p_i}{\log r_i}, \max_i \frac{\log p_i}{\log r_i} \right]$  then it follows from [10, 19, 56] that there is a (unique)  $q \in \mathbb{R}$  with  $f_\mu(\alpha) = \alpha q + \beta(q)$ . Hence, for each  $\epsilon > 0$ , we have

following

$$\begin{aligned}
\zeta_\alpha^{sim}(f_\mu(\alpha) + \epsilon) &= \sum_{\substack{\mathbf{i} \in \Sigma^* \\ \frac{\log p_{\mathbf{i}}}{\log r_{\mathbf{i}}} = \alpha}} r_{\mathbf{i}}^{f_\mu(\alpha) + \epsilon} \\
&= \sum_{\substack{\mathbf{i} \in \Sigma^* \\ \frac{\log p_{\mathbf{i}}}{\log r_{\mathbf{i}}} = \alpha}} r_{\mathbf{i}}^{\beta(q) + q\alpha + \epsilon} \\
&= \sum_{\substack{\mathbf{i} \in \Sigma^* \\ \frac{\log p_{\mathbf{i}}}{\log r_{\mathbf{i}}} = \alpha}} p_{\mathbf{i}}^q r_{\mathbf{i}}^{\beta(q) + \epsilon} \quad \text{using } \frac{\log p_{\mathbf{i}}}{\log r_{\mathbf{i}}} = \alpha \\
&\leq \sum_{n=1}^{+\infty} \left( \sum_{\mathbf{i} \in \Sigma} r_{\mathbf{i}}^{\beta(q) + \epsilon} p_{\mathbf{i}}^q \right)^n \\
&< \infty \quad \text{using } \sum_i p_i^q r_i^{\beta(q) + \epsilon} < 1.
\end{aligned}$$

Therefore we have that  $\sigma_{ab}(\zeta_\alpha^{sim}(\cdot)) \leq f_\mu(\alpha) + \epsilon$ . Letting  $\epsilon \rightarrow 0$  we have  $\sigma_{ab}(\zeta_\alpha^{sim}(\cdot)) \leq f_\mu(\alpha)$ . However, it is also clear that in general equality does not hold. Indeed, the set  $\left\{ \frac{\log p_{\mathbf{i}}}{\log r_{\mathbf{i}}} | \mathbf{i} \in \Sigma^* \right\}$  is clearly countable (because  $\Sigma^*$  is countable) and if  $\alpha \in \mathbb{R} \setminus \left\{ \frac{\log p_{\mathbf{i}}}{\log r_{\mathbf{i}}} | \mathbf{i} \in \Sigma^* \right\}$  then  $\sigma_{ab}(\zeta_\alpha^{sim}(\cdot)) = -\infty$  (because the series that defines  $\zeta_\alpha^{sim}(s)$  is obtained by summing over the empty set). Since it also follows from [10, 19, 56] that  $f_\mu(\alpha) > 0$  for all  $\alpha \in \left( \min_i \frac{\log p_i}{\log r_i}, \max_i \frac{\log p_i}{\log r_i} \right)$ , we therefore conclude that:  $\sigma_{ab}(\zeta_\alpha^{sim}(\cdot)) = -\infty < 0 < f_\mu$  for all except at most countably many  $\alpha \in \left( \min_i \frac{\log p_i}{\log r_i}, \max_i \frac{\log p_i}{\log r_i} \right)$ .

It follows from the above discussion that while the definition of  $\zeta_\alpha^{sim}$  is "natural", it does not encode sufficient information allowing us to recover the multifractal spectra  $f_\mu(\alpha)$ . The reason for the strict inequality in 2.1 is, of course, clear: even though there are no strings  $\mathbf{i} \in \Sigma^*$  for which the ratio  $\frac{\log p_{\mathbf{i}}}{\log r_{\mathbf{i}}}$  equals  $\alpha$  if  $\alpha \in \left( \min_i \frac{\log p_i}{\log r_i}, \max_i \frac{\log p_i}{\log r_i} \right) \setminus \left\{ \frac{\log p_{\mathbf{i}}}{\log r_{\mathbf{i}}} | \mathbf{i} \in \Sigma^* \right\}$ , there are nevertheless many sequences

$(\mathbf{i}_n)_n$  of strings  $\Sigma^*$  for which the ratios  $\frac{\log p_{\mathbf{i}_n}}{\log r_{\mathbf{i}_n}}$  converges to  $\alpha$ . In order to capture this, it is necessary

to ensure that these strings  $\mathbf{i}$  for which the ratio  $\frac{\log p_{\mathbf{i}_n}}{\log r_{\mathbf{i}_n}}$  is "close" to  $\alpha$  are also included in the series

defining the multifractal zeta-function. For this reason, we modify the definition of  $\zeta_\alpha^{sim}$  and introduce a self-similar multifractal zeta-function obtained by replacing the original small "target" set  $\{\alpha\}$  by a larger "target" set  $I$  (for example, we may choose the enlarged "target" set  $I$  to be a non-degenerate interval). In order to make this idea precise we proceed as follows. For a closed interval  $I$ , we define the self-similar multifractal zeta-function  $\zeta_I^{sim}$  by

$$\zeta_I^{sim}(s) = \sum_{\substack{\mathbf{i} \in \Sigma^* \\ \frac{\log p_{\mathbf{i}}}{\log r_{\mathbf{i}}} \in I}} r_{\mathbf{i}}^s$$

for these complex numbers  $s$  for which the series converges absolutely. Observe that if  $I = \{\alpha\}$ , then  $\zeta_I^{sim}(s) = \zeta_\alpha^{sim}(s)$ . We can now proceed in two equally natural ways. We can consider either a family of enlarged "target" sets shrinking to the original main "target"  $\alpha$  - this approach will be referred to

as the shrinking target approach, or, alternatively, we can consider a fixed enlarged "target" set and regard this as our original main "target"; this approach will be referred to as the fixed target approach. We now discuss these approaches in more detail.

### The shrinking target approach

For a given (small) "target"  $\{\alpha\}$ , we consider the following family  $[\alpha - r, \alpha + r]$ ,  $r > 0$  of enlarged "target" sets  $[\alpha - r, \alpha + r]$  shrinking to the original main "target"  $\{\alpha\}$  as  $r \rightarrow 0$ , and attempt to relate the limiting behavior of the abscissa convergence of  $\zeta_{[\alpha-r, \alpha+r]}^{\text{sim}}$  to the multifractal spectrum  $f_\mu(\alpha)$  at  $\alpha$ . In order to make this idea formal we proceed as follows. For each  $\alpha \in \mathbb{R}$  and for each  $r > 0$ , we define the zeta-function

$$\zeta_{[\alpha-r, \alpha+r]}^{\text{sim}}(s) = \sum_{\substack{\mathbf{i} \in \Sigma^* \\ \frac{\log p_{\mathbf{i}}}{\log r_{\mathbf{i}}} \in [\alpha-r, \alpha+r]}} r_{\mathbf{i}}^s$$

The next result, which is an application of one of our main results i.e. theorem 2.2.4, shows that the multifractal zeta-functions  $\zeta_{[r-\alpha, r+\alpha]}^{\text{sim}}$  encode sufficient information allowing us to recover the multifractal spectra  $f_\mu(\alpha)$  by letting  $r \rightarrow 0$ .

**Theorem 2.2.1** (Shrinking targets). *Let  $\mu$  be self-similar measure. Let the OSC be satisfied. For  $\alpha \in \mathbb{R}$  and  $r > 0$ , we have*

$$\lim_{r \rightarrow 0} \sigma_{ab}(\zeta_{[\alpha-r, \alpha+r]}^{\text{sim}}(\cdot)) = f_\mu(\alpha),$$

where  $\sigma_{ab}(\zeta_{[\alpha-r, \alpha+r]}^{\text{sim}}(\cdot))$  denotes the abscissa of convergence of the zeta-function  $\zeta_{[\alpha-r, \alpha+r]}^{\text{sim}}$ .

### The fixed target approach

Alternatively we can keep the enlarged "target" set  $I$  fixed and attempt to relate the abscissa of convergence of the multifractal zeta-function  $\zeta_I^{\text{sim}}$  associated with the enlarged "target" set  $I$  to the values of the multifractal spectrum  $f_\mu(\alpha)$  for  $\alpha \in I$ . Of course, inequality 2.1 shows that if the "target" set  $I$  is "too small", then this is not possible. However, if the enlarged "target" set  $I$  satisfies a mild non-degeneracy condition, namely condition 2.2, guaranteeing that  $I$  is sufficiently 'big', then the next result, which is also an application of one of our main results (see 2.2.6), shows that this is possible. More precisely the result shows that if the enlarged "target" set  $I$  satisfies condition 2.2, then the multifractal zeta-function  $\zeta_I^{\text{sim}}$  associated with the enlarged "target" set  $I$  encode sufficient information allowing us to recover the supremum  $\sup_{\alpha \in I} f_\mu(\alpha)$  and  $\sup_{\alpha \in I} f_\mu^c(\alpha)$  of the multifractal spectra  $f_\mu(\alpha)$  and  $f_\mu^c(\alpha)$  for  $\alpha \in I$ .

**Theorem 2.2.2** (Fixed targets). *Let  $\mu$  be self-similar measure. Let the OSC be satisfied. For a closed interval  $I$  if*

$$\text{int} I \cap \left( \min_i \frac{\log p_i}{\log r_i}, \max_i \frac{\log p_i}{\log r_i} \right) \neq \emptyset \quad (2.2)$$

(where  $\text{int} I$  denotes the interior of  $I$ ), then we have

$$\sigma_{ab}(\zeta_I^{\text{sim}}(\cdot)) = \sup_{\alpha \in I} f_\mu(\alpha) = \sup_{\alpha \in I} f_\mu^c(\alpha).$$

Where  $\sigma_{ab}(\zeta_I^{\text{sim}}(\cdot))$  denotes the abscissa of convergence of the zeta-function  $\zeta_I^{\text{sim}}$ .



### 2.2.2 Multifractal Geometric Zeta function

Now we will state our results in full generality. First let us introduce the following notation. Let  $(X, d)$  be metric space and  $F \subset X$  then  $B(F, r)$  will be defined as

$$B(F, r) = \{x \in X | d(x, F) < r\}$$

and  $B[F, r]$  will be defined as

$$B[F, r] = \{x \in X | d(x, F) \leq r\}.$$

Let us continue by introducing our zeta function in full generality.

**Definition 2.2.3** (Multifractal Geometric Zeta function). *Let  $X$  be a metric space. Let  $C \subset X$ , and let  $U : \mathcal{P}(\Sigma_A^{\mathbb{N}}) \rightarrow X$  be continuous with respect to weak topology. Let  $\Lambda \in H(\Sigma_A^{\mathbb{N}}), \Lambda < 0$ . Let us denote  $s_i = s_i(\Lambda)$ . Then geometric zeta function is defined as follows:*

$$\zeta_C^{geo}(s) = \sum_{\substack{i \in \Sigma_A^* \\ UL_n[i] \subset C}} s_i^s.$$

For every complex number  $s$  for which it converges.

Abscissa of convergence is defined as  $\sigma_{ab}(\zeta_C^{U, \Lambda}(\cdot)) = \inf \{t \mid t \in \mathbb{R}, \zeta_C^{geo}(t) \text{ converge}\}.$

#### The shrinking target approach result

For a given "target"  $C$ , we consider the following family  $B(C, r), r > 0$  of enlarged "target" sets  $B(C, r)$  shrinking to the original main "target"  $C$  as  $r \rightarrow 0$ , and attempt to relate the limiting behavior of the abscissa of convergence of the zeta-function  $\zeta_{B(C, r)}^{U, \Lambda}$  to the coarse multifractal spectrum  $f^{U, \Lambda}(C)$  and other multifractal quantities. Our first main result, i.e. the theorem below, shows that this is possible. More precisely, the theorem below shows that the abscissa of convergence of the zeta-function  $\zeta_{B(C, r)}^{U, \Lambda}$  converges as  $r \rightarrow 0$ , and that this limit is equal to the coarse multifractal spectrum of  $C$ . We also show that the limit can be obtained by a variational principle involving the supremum of the entropy of all shift invariant Borel probability measures  $\mu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}})$  with  $U\mu \in C$ . In section 2.3 we show that in many important cases the limit  $\lim_{r \rightarrow 0} \sigma_{ab}(\zeta_{B(C, r)}^{U, \Lambda}(\cdot))$  is equal to the traditional multifractal spectra.

**Theorem 2.2.4** (The main theorem for geometric zeta function - Shrinking targets). *Let  $X$  be metric space and let  $U : \mathcal{P}(\Sigma_A^{\mathbb{N}}) \rightarrow X$ , be continuous with respect to the weak topology and  $\Lambda \in H(\Sigma_A^{\mathbb{N}}), \Lambda < 0$ . Let  $C \subset X$  be closed set. Then*

$$\lim_{r \rightarrow 0} \sigma_{ab}(\zeta_{B(C, r)}^{U, \Lambda}(\cdot)) = f^{U, \Lambda}(C).$$

Or equivalently,

$$\lim_{r \rightarrow 0} \sigma_{ab}(\zeta_{B(C, r)}^{U, \Lambda}(\cdot)) = \sup_{\substack{\mu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}}) \\ U\mu \in C}} - \frac{h(\mu)}{\int_{\Sigma_A^{\mathbb{N}}} \Lambda d\mu}.$$

To prove the theorem 2.2.4, it is enough to prove the following:

$$\bar{f}_c^{U, \Lambda}(C) \leq \liminf_{r \rightarrow 0} \sigma_{ab}(\zeta_{B(C, r)}^{U, \Lambda}(\cdot)) \tag{2.3}$$

$$\limsup_{r \rightarrow 0} \sigma_{ab}(\zeta_{B(C, r)}^{U, \Lambda}(\cdot)) \leq \sup_{\substack{\mu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}}) \\ U\mu \in C}} - \frac{h(\mu)}{\int_{\Sigma_A^{\mathbb{N}}} \Lambda d\mu} \tag{2.4}$$

$$\sup_{\substack{\mu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}}) \\ U\mu \in C}} - \frac{h(\mu)}{\int_{\Sigma_A^{\mathbb{N}}} \Lambda d\mu} \leq \underline{f}^{U, \Lambda}(C) \tag{2.5}$$

$$f^{U, \Lambda}(C) \leq \underline{f}_c^{U, \Lambda}(C) \tag{2.6}$$

Inequality 2.3 is proved directly. Proof of 2.4 is based on Large deviations. Proof of inequality 2.5 is based on use of ergodic theory. Proof of 2.6 follows, after wee bit of work, from definitions. All proofs are provided in section 3.2.

### The Fixed target approach

Alternatively, instead of choosing a family of "target" sets that shrinks to the given "target"  $C$ , we can keep the given "target" set  $C$  fixed and attempt to relate the abscissa of convergence of the multifractal zeta function  $\zeta_C^{U,\Lambda}$  associated with the "target" set  $C$  to the values of the multifractal spectrum  $f^{U,\Lambda}(C)$ . Of course, the example in 2.2.1 shows that if the "target" set  $C$  is "too small", then this is not possible. However, if the coarse multifractal spectrum  $f^{U,\Lambda}$  satisfies a continuity condition at  $C$  guaranteeing that the interior of  $C$  is 'sufficiently big', then our second main result, i.e. theorem below, shows that this is possible. More precisely, it shows that if the coarse multifractal spectrum  $f^{U,\Lambda}$  is inner continuous at  $C$  (the definition of inner continuity will be given below), then the abscissa of convergence of the zeta-function  $\zeta_C^{U,\Lambda}$  equals the coarse multifractal spectrum of  $C$ . In analogy with Theorem 2.2.4 we also show that the abscissa of convergence of  $\zeta_C^{U,\Lambda}$  can be obtained by a variational principle involving the supremum of the entropy of all shift invariant Borel probability measures  $\mu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}})$  with  $\mu \in C$ . However, before stating the theorem below, we first define the continuity condition that the coarse multifractal spectrum  $f^{U,\Lambda}$  is required to satisfy.

Let us introduce following notation for set  $C \subset X$

$$I(C, r) = \{x \in C | d(\partial C, x) > r\},$$

and

$$I[C, r] = \{x \in C | d(\partial C, x) \geq r\}.$$

**Definition 2.2.5** (Inner continuity). *Let  $P(X)$  be set of subsets of  $X$ . Then function  $\Phi : P(X) \rightarrow \mathbb{R}$  is called inner continuous at  $C$  if*

$$\lim_{r \rightarrow 0} \Phi(I(C, r)) = \Phi(C).$$

Let us now state the theorem.

**Theorem 2.2.6** (The main theorem for geometric zeta function - Fixed targets). *Fix  $M \in \mathbb{N}$ . Let  $U : \mathcal{P}_S(\Sigma_A^{\mathbb{N}}) \rightarrow \mathbb{R}^M$  be continuous with respect to weak topology. Let  $\Lambda \in H(\Sigma_A^{\mathbb{N}})$ ,  $\Lambda < 0$ . Let  $C \subset \mathbb{R}^M$  be a closed set, and assume that  $f^{U,\Lambda}$  is inner continuous at  $C$ . Then*

$$\sigma_{ab}(\zeta_C^{U,\Lambda}(\cdot)) = f^{U,\Lambda}(C),$$

or equivalently

$$\sigma_{ab}(\zeta_C^{U,\Lambda}(\cdot)) = \sup_{\substack{\mu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}}) \\ U\mu \in C}} - \frac{h(\mu)}{\int_{\Sigma_A^{\mathbb{N}}} \Lambda d\mu}.$$

This theorem follows from the theorem 2.2.4. Proof is provided in subsection 3.2.1.

### Baker's multifractal zeta function

Baker in [4] introduced a multifractal zeta function on cookie cutter sets and a Gibbs measure supported on it. Here we will state his results and describe his approach, and then compare it to our results and approach to multifractal zeta function.

Let us first define a cookie cutter set and introduce notation. Let  $I_1, I_2, \dots, I_N$  be disjoint subintervals of  $[0, 1]$ . Let  $T : \cup_{k=1}^N I_k \rightarrow [0, 1]$  be  $C^{1+\theta}$ ,  $\theta \in (0, 1)$ ,  $|T'(x)| > 1$  on all intervals  $I_1, I_2, \dots, I_N$  and for each interval  $I_k$  we have  $T(I_k) = [0, 1]$ . Define  $K = \cap_{k=1}^{+\infty} T^{-k}[0, 1]$ . Let  $\Sigma = \{1, 2, 3, \dots, N\}$ . Define projection  $\pi : \Sigma^{\mathbb{N}} \rightarrow K$  by

$$\pi(\mathbf{i}) = \cap_{n=1}^{+\infty} T^{-n} I_{i_n}.$$

For  $\mathbf{i} \in \Sigma^{\mathbb{N}}$  define  $\Lambda : \Sigma^{\mathbb{N}} \rightarrow \mathbb{R}$  by  $\Lambda(\mathbf{i}) = -\log |T'(\pi(\mathbf{i}))|$ . Let us fix a metric  $d_{\Lambda}$  on  $\Sigma^{\mathbb{N}}$ .  $\pi$  is bi-Lipschitz as mapping from  $(\Sigma^*, d_{\Lambda})$  to  $K$ , so from a dimensional point of view these two are equivalent. We will skip the definition of Gibbs measure and pressure and being cohomologous in this setting since it is analogous to setting of Subshift of finite type. Let us fix  $\phi : K \rightarrow \mathbb{R}$ . And let  $\mu_{\phi}$  be Gibbs measure with potential  $\phi$ . Let  $I^n$  denote set of intervals that builds up set  $T^{-n}(\cup_{k=1}^N I_k)$ . For  $x \in K$  let us with  $I^n(x)$  denote the interval  $I$  such that  $I \in I^n$  and  $x \in I$ .

We will fix a unique invariant ergodic Gibbs probability measure  $\mu_{\phi}$  supported on cookie cutter set with Hölder continuous potential  $\phi$ . Then Baker's multifractal zeta function is defined as follows.

$$\zeta_{\alpha}^{\mu_{\phi}}(s) = \sum_{n=1}^{+\infty} \sum_{\substack{I \in I^n \\ \frac{\mu_{\phi}(I)}{|I|^{\alpha}} \in [a, b]}} |I|^s.$$

Where  $a, b$  are two fixed positive numbers. Let us have a closer look at  $I \in I^n, \frac{\mu_{\phi}(I)}{|I|^{\alpha}} \in [a, b]$ . Let  $x \in I$ . There is a constant  $C$  such that

$$\begin{aligned} \frac{\mu_{\phi}(I)}{|I|^{\alpha}} &\leq C \frac{\exp(S_n \phi(x))}{\exp(\alpha S_n \Lambda(x))} \\ &= C \exp(S_n \phi(x) - \alpha S_n \Lambda(x)) \end{aligned}$$

and

$$\begin{aligned} \frac{\mu_{\phi}(I)}{|I|^{\alpha}} &\geq \frac{1}{C} \frac{\exp(S_n \phi(x))}{\exp(\alpha S_n \Lambda(x))} \\ &= \frac{1}{C} \exp(S_n \phi(x) - \alpha S_n \Lambda(x)). \end{aligned}$$

So from  $I \in I^n, \frac{\mu_{\phi}(I)}{|I|^{\alpha}} \in [a, b]$  we have that  $\exp(S_n(\phi - \alpha\Lambda)(x)) \in [\frac{a}{C}, Cb] \Rightarrow S_n(\phi - \alpha\Lambda)(x) \in [a', b']$ .

So we have that  $\frac{1}{n} S_n(\phi - \alpha\Lambda)(x) \in [\frac{a'}{n}, \frac{b'}{n}]$ . Using that and due to Birkhoff's ergodic theorem we have that  $\mu_{\phi}$  almost surely,

$$\lim_{n \rightarrow +\infty} \frac{1}{n} S_n(\phi - \alpha\Lambda)(x) = \int_{\Sigma^{\mathbb{N}}} \phi - \alpha\Lambda d\mu_{\phi} = \lim_{n \rightarrow +\infty} \left[ \frac{a'}{n}, \frac{b'}{n} \right] = 0$$

The discussion we have just had should demystify following technical definitions. Define

$$\mathcal{I}_{\alpha} = \left\{ \int_{\Sigma^{\mathbb{N}}} \phi - \alpha\Lambda d\nu \mid \nu \text{ is } T \text{ invariant} \right\}. \quad (2.7)$$

and define

$$\mathcal{R} = \{\alpha \mid 0 \in \mathcal{I}_{\alpha}\}.$$

Easy result is that if  $\alpha \notin \mathcal{R}$ , then  $\zeta_{\alpha}^{\mu_{\phi}}(s)$  is entire. Let us state the following cohomological condition on which the main result i.e. theorem 2.2.8 will depend.

**Definition 2.2.7.** (Condition A) We say that  $\phi$  satisfy Condition A if there is no  $t \in \mathcal{R}$  such that  $\phi - t\Lambda$  is cohomologous to zero.

Let us now state the main result for Baker's multifractal zeta function.

**Theorem 2.2.8** ([4]). If  $\mu_{\phi}$  satisfies Condition A, and  $\alpha$  is such that  $0 \in \text{int}(\mathcal{I}_{\alpha})$  then for  $s \in \mathbb{R}$

$$\limsup_{s \rightarrow f_{\mu_{\phi}}(\alpha)} \zeta_{\alpha}^{\mu_{\phi}}(s)(s - f_{\mu_{\phi}}(\alpha))^{\frac{1}{2}} < \infty,$$

and if  $b - a$  is sufficiently large the abscissa of convergence is  $f_{\mu_\phi}(\alpha)$  and

$$\liminf_{s \rightarrow f_{\mu_\phi}(\alpha)} \zeta_\alpha^{\mu_\phi}(s)(s - f_{\mu_\phi}(\alpha))^{\frac{1}{2}} > 0.$$

Let us now compare Baker's and our approach to multifractal zeta functions. Both zeta functions are defined as sum over intervals that encode multifractal information. In Baker's case, heuristically, it is done by avoiding to sum over "wrong" ones, i.e. let  $\lim_{n \rightarrow +\infty} \frac{\log \mu_\phi(I^n(x))}{\log |I^n(x)|} = \beta \neq \alpha$  then it is easy to see that  $\frac{\mu_\phi(I^n(x))}{|I^n(x)|^\alpha} \rightarrow 0$  or  $+\infty$ . So if we fix positive numbers  $a$  and  $b$ , we should avoid these. This is different from the approach we adopted since we collect intervals for which  $|I|^{\alpha+\epsilon} \leq \mu_\phi(I) \leq |I|^{\alpha-\epsilon}$ , i.e. we are summing over more intervals than necessary and then let  $\epsilon$  to tend to 0. Although having to take the limit when  $\epsilon$  tends to 0 looks less elegant, we are able to tackle more different multifractal spectra which, especially non-linear case like relative Birkhoff averages with exponent (see 2.3), seems not to be in the reach of Baker's approach. Also, we are able to provide information about generalised multifractal spectra and, under mild conditions, able to omit  $\epsilon$  from our approach. However, Baker's approach does provide some information about suitable pole of zeta function. That is due to fact that he can use thermodynamic formalism, especially spectral properties of appropriate Ruelle operator. Our results are inspired by thermodynamic formalism, however we were unsuccessful in finding multifractal analogue of Ruelle operator with good properties in our settings. Nevertheless, we introduced multifractal pressure that we will describe in 2.2.3.

### 2.2.3 Dynamical zeta function

In addition to the distinctively geometric approaches in [4, 34, 37, 46], it has been a major challenge to introduce and develop a natural and meaningful theory of dynamical multifractal zeta-functions paralleling the existing powerful theory of dynamical zeta-functions introduced and developed by Ruelle [63, 64] and others, see, for example, the surveys and books [3, 54, 55] and the references therein. In particular, in the setting of self-conformal constructions, Olsen in [49] introduced a family of dynamical multifractal zeta-functions designed to provide precise information of very general classes of multifractal spectra, including, for example, the multifractal spectra of self-conformal measures and the multifractal spectra of ergodic Birkhoff averages of continuous functions. However, recently it has been recognised that while self-conformal constructions provide a useful and important framework for studying fractal and multifractal geometry, the more general notion of graph-directed self-conformal constructions provide a substantially more flexible and useful framework, see, for example, [36] for an elaboration of this. In recognition of this viewpoint, the purpose of this section is to develop a dynamical theory of multifractal zeta functions in the setting of graph-directed self-conformal constructions. We will use the following notation. Namely, if  $(a_n)_n$  is a sequence of complex numbers and if  $f$  is the power series defined by  $f(z) = \sum_n a_n z^n$  for  $z \in \mathbb{C}$ , then we will denote the radius of convergence of  $f$  by  $\sigma_{rad}(f)$ , i.e. we write  $\sigma_{rad}(f) =$  'the radius of convergence of  $f$ '. Our definitions and results are motivated by the notion of pressure from the thermodynamic formalism and the dynamical zeta-functions introduced by Ruelle [63, 64]; see, also [3, 54, 55]. In addition, Bowen's formula expressing the Hausdorff dimension of a self-conformal set in terms of the pressure (or the dynamical zeta function) plays a leitmotif in our work. Because of this we now recall the definition of pressure and dynamical zeta function, and the statement of Bowen's formula. The dynamical zeta-function of  $\phi$  is defined by

$$\zeta^{\text{dyn}}(\phi, z) = \sum_{n=1}^{+\infty} \frac{z^n}{n} \sum_{\mathbf{i} \in f i x_n} \exp(S_n \phi(\mathbf{i})).$$

for these complex numbers  $z$  for which the series converge, see [55]. Where  $fix_n = \{\mathbf{i} | \mathbf{i} \in \Sigma_A^{\mathbb{N}}, S^n(\mathbf{i}) = \mathbf{i}\}$ . Recall pressure of continuous function  $\phi : \Sigma_A^{\mathbb{N}} \rightarrow \mathbb{R}$  is defined as

$$P(\phi) = \lim_{n \rightarrow +\infty} \frac{1}{n} \log \sum_{\mathbf{i} \in fix_n} \exp(S_n \phi(\mathbf{i})) = \lim_{n \rightarrow +\infty} \frac{1}{n} \log \sum_{\mathbf{i} \in \Sigma_A^n} \sup_{\mathbf{j} \in [\mathbf{i}]} \exp(S_n \phi(\mathbf{j})).$$

We now list two easily established and well-known properties of the pressure  $P(\phi)$  and of the radius of convergence  $\sigma_{rad}(\zeta^{\text{dyn}}(\phi, \cdot))$  of the power-series  $\zeta^{\text{dyn}}(\phi, \cdot)$ . While both results are well-known and easily proved (see, for example [6, 20]), we have decided to list them since they play an important part in the discussion of our results.

**Theorem 2.2.9** (see, for example, [6, 20]). *Fix a continuous function  $\phi : \Sigma_A^{\mathbb{N}} \rightarrow \mathbb{R}$ . Then we have*

$$-\log \sigma_{rad}(\zeta^{\text{dyn}}(\phi, \cdot)) = P(\phi).$$

**Theorem 2.2.10** (see, for example, [6, 20]). *Fix a continuous function  $\phi : \Sigma_A^{\mathbb{N}} \rightarrow \mathbb{R}$  with  $\phi < 0$ . Then the function  $t \rightarrow P(t\phi)$ , where  $t \in \mathbb{R}$ , is continuous, strictly decreasing and convex with  $\lim_{t \rightarrow -\infty} P(t\phi) = +\infty$  and  $\lim_{t \rightarrow \infty} P(t\phi) = -\infty$ . In particular, there is a unique real number  $s_0$  such that*

$$P(s_0\phi) = 0;$$

*alternatively,  $s_0$  is the unique real number such that*

$$\sigma_{rad}(\zeta^{\text{dyn}}(s_0\phi, \cdot)) = 1.$$

.

The main importance of the pressure (for the purpose of this exposition) is that it provides a beautiful formula for the Hausdorff dimension of a graph-directed self-conformal set satisfying the OSC.

**Theorem 2.2.11** (see, for example, [6, 20]). *Let graph-directed IFS be given by vector*

$$(V, E, (U_v)_{v \in V}, (X_v)_{v \in V}, (S_e)_{e \in E}, (p_e)_{e \in E}).$$

*Let  $\Lambda : \Sigma_A^{\mathbb{N}} \rightarrow \mathbb{R}$  be defined by*

$$\Lambda(\mathbf{i}) = \log |D(S_{e_1} \pi(S(\mathbf{i}))|.$$

*There is a unique real number  $s_0$  such that*

$$P(s_0\Lambda) = 0.$$

*And if OSC is satisfied then for each  $v \in V$  we have  $s_0 = \dim_{\text{H}} K_v$ .*

We will now define multifractal dynamical zeta function. Let  $X$  be metric space. Let  $C \subset X$ ,  $U : \mathcal{P}(\Sigma_A^{\mathbb{N}}) \rightarrow X$  be continuous with respect to weak topology. And let  $\phi : \Sigma_A^{\mathbb{N}} \rightarrow \mathbb{R}$  be continuous then

$$\zeta_C^{\text{dyn}, U}(\phi, z) = \sum_{n=1}^{+\infty} \frac{z^n}{n} \sum_{\substack{\mathbf{i} \in \Sigma_A^n \\ UL_n[\mathbf{i}] \subset C}} s_{\mathbf{i}}(\phi).$$

Multifractal pressure will be defined as

**Definition 2.2.12.** *Let  $X$  be metric space. Let  $C \subset X$ , let  $U : \mathcal{P}(\Sigma_A^{\mathbb{N}}) \rightarrow X$  be continuous. And let  $\phi : \Sigma_A^{\mathbb{N}} \rightarrow \mathbb{R}$  be continuous then let us define*

$$P_C^U(\phi) = \limsup_{n \rightarrow +\infty} \frac{1}{n} \log \sum_{\substack{\mathbf{i} \in \Sigma_A^n \\ UL_n[\mathbf{i}] \subset C}} s_{\mathbf{i}}(\phi).$$

$P_C^U(\phi)$  is called multifractal pressure.

$$-\log \sigma_{rad}(\zeta_C^{\text{dyn}, U}(\phi, \cdot)) = P_C^U(\phi). \quad (2.8)$$

Proof is simple but for the sake of completeness it is provided in section 3.1.

Like in case of Geometric multifractal zeta functions, we will adopt two approaches. Shrinking target and fixed target approach.

### Main results for shrinking target approach

**Theorem 2.2.13** (The shrinking target variational principle for the multifractal pressure). *Let  $X$  be a metric space and let  $U : \mathcal{P}(\Sigma_A^{\mathbb{N}}) \rightarrow X$  be continuous with respect to the weak topology. Let  $C \subset X$  be a closed subset of  $X$ . Fix a continuous function  $\phi : \Sigma_A^{\mathbb{N}} \rightarrow \mathbb{R}$ . We have*

$$\lim_{r \rightarrow 0} P_{B(C, r)}^U(\phi) = \sup_{\substack{\mu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}}) \\ U\mu \in C}} \left\{ h(\mu) + \int_{\Sigma_A^{\mathbb{N}}} \phi d\mu \right\}. \quad (2.9)$$

Alternatively we have

$$\lim_{r \rightarrow 0} -\log \sigma_{rad}(\zeta_{B(C, r)}^{\text{dyn}, U}(\phi, \cdot)) = \sup_{\substack{\mu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}}) \\ U\mu \in C}} \left\{ h(\mu) + \int_{\Sigma_A^{\mathbb{N}}} \phi d\mu \right\}. \quad (2.10)$$

Proof of equality 2.9 is provided in section 3.1. It is based on techniques from large deviation theory. Equality 2.10 follows directly from equalities 2.9 and 2.8.

Observe that if we let  $C = X$ , then the multifractal pressure equals the usual pressure, i.e.  $P_{B(C, r)}^U(\phi) = P(\phi)$  and the variational principle in above theorem therefore simplifies to the usual variational principle, namely,

$$P(\phi) = \sup_{\mu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}})} \left\{ h(\mu) + \int_{\Sigma_A^{\mathbb{N}}} \phi d\mu \right\}.$$

Important corollary of the theorem above is the following analogue of Bowen's formula

**Theorem 2.2.14** (The shrinking target multifractal Bowen's formula). *Let  $X$  be a metric space and let  $U : \mathcal{P}_S(\Sigma_A^{\mathbb{N}}) \rightarrow X$  be continuous with respect to the weak topology. Let  $C \subset X$  be a closed subset of  $X$  such that  $C \cap U(\mathcal{P}_S(\Sigma_A^{\mathbb{N}})) \neq \emptyset$ . Fix a continuous function  $\phi : \Sigma_A^{\mathbb{N}} \rightarrow \mathbb{R}$  with  $\phi < 0$  and let  $\mathcal{F}(C)$  be the unique real number such that*

$$\lim_{r \rightarrow 0} P_{B(C, r)}^U(\mathcal{F}(C)\phi, \cdot) = 0.$$

Alternatively,  $\mathcal{F}(C)$  is the unique real number such that

$$\lim_{r \rightarrow 0} \sigma_{rad}(\zeta_{B(C, r)}^{\text{dyn}, U}(\mathcal{F}(C)\phi, \cdot)) = 1.$$

Then

$$\mathcal{F}(C) = \sup_{\substack{\mu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}}) \\ U\mu \in C}} - \frac{h(\mu)}{\int_{\Sigma_A^{\mathbb{N}}} \phi d\mu}.$$

Proof.

There is a unique real number  $s_0$  such that

$$\lim_{r \rightarrow 0} P_{B(C,r)}^U(s_0 \phi) = \sup_{\substack{\mu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}}) \\ U\mu \in C}} \left\{ h(\mu) + \int_{\Sigma_A^{\mathbb{N}}} s_0 \phi d\mu \right\} = 0. \quad (2.11)$$

We will provide proof of that later as proposition 3.1.9. Now the theorem follows directly from theorem 2.2.13, equality 2.8 and proposition 3.1.9. ■

For the sake of uniformity if  $C \cap U(\mathcal{P}_S(\Sigma_A^{\mathbb{N}})) = \emptyset$  we set  $\mathcal{F}(C)$  from the theorem above to be  $-\infty$ .

### Main results for fixed target approach

If the set  $C$  is "too small", then it follows from the discussion similar to that about geometric multifractal zeta function that we, in general, cannot expect any meaningful results in the fixed target setting. However, if the set  $C$  satisfies a non-degeneracy condition guaranteeing that it is not "too small", then meaningful results can be obtained in the fixed target setting. This is the content of Theorem and Corollary below.

**Theorem 2.2.15** (The fixed target variational principle for the multifractal pressure.). *Let  $X$  be a normed vector space. Let  $\Gamma : \mathcal{P}_S(\Sigma_A^{\mathbb{N}}) \rightarrow X$  be continuous and affine and let  $\Delta : \mathcal{P}(\Sigma_A^{\mathbb{N}}) \rightarrow X$  continuous and affine with  $\Delta(\mu) \neq 0$  for all  $\mu \in \mathcal{P}(\Sigma_A^{\mathbb{N}})$ . Define  $U : \mathcal{P}(\Sigma_A^{\mathbb{N}}) \rightarrow X$  by  $U = \frac{\Gamma}{\Delta}$ . Let  $C$  be a closed and convex subset of  $X$  and assume that  $\text{int}C \cap U(\mathcal{P}(\Sigma_A^{\mathbb{N}})) \neq \emptyset$  then*

1. *We have*

$$P_C^U(\phi) = \sup_{\substack{\mu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}}) \\ U\mu \in C}} \left\{ h(\mu) + \int_{\Sigma_A^{\mathbb{N}}} \phi d\mu \right\}. \quad (2.12)$$

2. *We have*

$$-\log \sigma_{rad}(\zeta_C^{\text{dyn}, U}(\phi; \cdot)) = \sup_{\substack{\mu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}}) \\ U\mu \in C}} \left\{ h(\mu) + \int_{\Sigma_A^{\mathbb{N}}} \phi d\mu \right\} = \sup_{\substack{\mu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}}) \\ U\mu \in \text{int}C}} \left\{ h(\mu) + \int_{\Sigma_A^{\mathbb{N}}} \phi d\mu \right\}.$$

Proof.

Equality 2.12 is proven as proposition 3.1.16. It makes use of variation principle for multifractal pressure for shrinking targets. The rest follows from equality 2.8. ■

Again, we observe that if we let  $C = X$  in, then the multifractal pressure equals

the usual pressure, i.e.  $P_C^U(\phi) = P(\phi)$ , and the variational principle in the theorem above therefore is simplified to the usual variational principle, namely,

$$P(\phi) = \sup_{\mu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}})} \left\{ h(\mu) + \int_{\Sigma_A^{\mathbb{N}}} \phi d\mu \right\}.$$

Important corollary of the theorem above is the following analogue of Bowen's formula.

**Theorem 2.2.16** (The fixed target multifractal Bowen's formula). *Let  $X$  be a normed vector space. Let  $\Gamma : \mathcal{P}(\Sigma_A^{\mathbb{N}}) \rightarrow X$  be continuous and affine and let  $\Delta : \mathcal{P}(\Sigma_A^{\mathbb{N}}) \rightarrow X$  be continuous and affine with  $\Delta(\mu) \neq 0$  for all  $\mu \in \mathcal{P}(\Sigma_A^{\mathbb{N}})$ . Define  $U : \mathcal{P}(\Sigma_A^{\mathbb{N}}) \rightarrow X$  by  $U = \frac{\Gamma}{\Delta}$ . Let  $C$  be a closed and convex subset*

of  $X$  and assume that  $\text{int}C \cap U(\mathcal{P}(\Sigma_A^{\mathbb{N}})) \neq \emptyset$ . Let  $\phi : \Sigma_A^{\mathbb{N}} \rightarrow X$  be continuous with  $\phi < 0$ . Let  $f(C)$  be the unique real number such that

$$P_C^U(\mathcal{F}(C)\phi) = 0,$$

alternatively,  $\mathcal{F}(C)$  is the unique real number such that

$$\sigma_{\text{rad}}(\zeta_C^{\text{dyn}, U}(\mathcal{F}(C)\phi, \cdot)) = 1.$$

Then

$$\mathcal{F}(C) = \sup_{\substack{\mu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}}) \\ U\mu \in C}} - \frac{h(\mu)}{\int_{\Sigma_A^{\mathbb{N}}} \phi d\mu}.$$

Proof.

There is a unique real number  $s_0$  such that

$$\lim_{r \rightarrow 0} P_{B(C, r)}^U(s_0\phi) = \sup_{\substack{\mu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}}) \\ U\mu \in C}} \left\{ h(\mu) + \int_{\Sigma_A^{\mathbb{N}}} s_0\phi d\mu \right\} = 0. \quad (2.13)$$

We will provide proof of that later as proposition 3.1.9. Now the theorem follows directly from theorem 2.2.15, equality 2.8 and proposition 3.1.9. ■

For the sake of uniformity if  $C \cap U(\mathcal{P}_S(\Sigma_A^{\mathbb{N}})) = \emptyset$  we set  $\mathcal{F}(C)$  from the theorem above to be  $-\infty$ .

## 2.3 Applications

Here we present, as application of our main result, multifractal zeta function for all of spectra we described so far. I.e. we present application to

1. Multifractal zeta functions for Self-similar measure.
2. Multifractal zeta functions for Self-conformal measure.
3. Multifractal zeta functions for Self-conformal graph-directed measures.
4. Multifractal zeta functions for mixed multifractal spectra of Self-conformal measures.
5. Multifractal zeta functions for Gibbs measure.
6. Multifractal zeta functions for ergodic averages of continuous functions.
7. Multifractal zeta functions for mixed ergodic averages of continuous functions.
8. Multifractal zeta functions for mixed relative ergodic averages of continuous functions.
9. Multifractal zeta functions for mixed relative ergodic averages with exponent of continuous functions.

### Multifractal zeta functions for multifractal spectra of self-similar measures

Due to an important role of self-similar measures in measure theory, it is instructive to note this special case of the theorems above.



**Theorem 2.3.1** (Geometric multifractal zeta functions for Self-similar measure). *Let  $S_1, S_2, \dots, S_N$  be similarities on  $\mathbb{R}^d$ , with similarity ratios  $r_1, r_2, \dots, r_N \in (0, 1)$ . Let  $(p_1, p_2, \dots, p_n)$  be probability vector. Let  $\mu$  be the self-similar measure associated with the list*

$$(S_1, S_2, \dots, S_N, p_1, p_2, \dots, p_N).$$

*I.e.  $\mu$  is a unique probability measure such that  $\mu = \sum_{i=1}^N p_i \mu_i \circ S_i^{-1}$ .*

*For set  $C \subset \mathbb{R}$  let us define zeta function*

$$\zeta_C^{\text{sim}}(s) = \sum_{\substack{\mathbf{i} \in \Sigma^* \\ \frac{\log p_{\mathbf{i}}}{\log r_{\mathbf{i}}} \in C}} r_{\mathbf{i}}^s.$$

*Define  $\beta : \mathbb{R} \rightarrow \mathbb{R}$  by*

$$\sum_{i=1}^N r_i^{\beta(q)} p_i^q = 1$$

*for  $q \in \mathbb{R}$ . Now in addition assume that  $C$  is a closed set. Then the following holds:*

1. *For  $r > 0$ ,*

$$\lim_{r \rightarrow 0} \sigma_{ab}(\zeta_{B(C,r)}^{\text{sim}}(\cdot)) = \sup_{\alpha \in C} \beta^*(\alpha).$$

*Specifically if  $C = \{\alpha\}$*

$$\lim_{r \rightarrow 0} \sigma_{ab}(\zeta_{B(\alpha,r)}^{\text{sim}}(\cdot)) = \beta^*(\alpha).$$

*Additionally if the OSC is satisfied then*

$$\begin{aligned} \lim_{r \rightarrow 0} \sigma_{ab}(\zeta_{B(C,r)}^{\text{sim}}(\cdot)) &= \sup_{\alpha \in C} \dim_H \left\{ x \in K \mid \lim_{r \rightarrow 0} \frac{\log \mu(B(x, r))}{\log r} = \alpha \right\} \\ &= \dim_H \left\{ x \in K \mid \text{acc}_{r \rightarrow 0} \frac{\log \mu(B(x, r))}{\log r} \subset C \right\}. \end{aligned}$$

*Specifically if  $C = \{\alpha\}$*

$$\lim_{r \rightarrow 0} \sigma_{ab}(\zeta_{B(\alpha,r)}^{\text{sim}}(\cdot)) = \dim_H \left\{ x \in K \mid \lim_{r \rightarrow 0} \frac{\log \mu(B(x, r))}{\log r} = \alpha \right\}.$$

2. *If  $C$  is a closed interval and  $\text{int}C \cap -\beta'(\mathbb{R}) \neq \emptyset$  then we have*

$$\sigma_{ab}(\zeta_C^{\text{sim}}(\cdot)) = \sup_{\alpha \in C} \beta^*(\alpha).$$

*Additionally if the OSC is satisfied then we have*

$$\begin{aligned} \sigma_{ab}(\zeta_C^{\text{sim}}(\cdot)) &= \sup_{\alpha \in C} \dim_H \left\{ x \in K \mid \lim_{r \rightarrow 0} \frac{\log \mu(B(x, r))}{\log r} = \alpha \right\} \\ &= \dim_H \left\{ x \in K \mid \text{acc}_{r \rightarrow 0} \frac{\log \mu(B(x, r))}{\log r} \subset C \right\}. \end{aligned}$$

Proof will be provided in more general settings, i.e. look at proof of theorem 2.3.7.

**Theorem 2.3.2** (Dynamical multifractal zeta functions for Self-similar measure). *Let  $S_1, S_2, \dots, S_N$  be similarities on  $\mathbb{R}^d$ , with similarity ratios  $r_1, r_2, \dots, r_N \in (0, 1)$ . Let  $(p_1, p_2, \dots, p_n)$  be probability vector. Let  $\mu$  be the self-similar measure associated with the list*

$$(S_1, S_2, \dots, S_N, p_1, p_2, \dots, p_N).$$

*I.e.  $\mu$  is a unique probability measure such that  $\mu = \sum_{i=1}^N p_i \mu_i \circ S_i^{-1}$ .*

*For set  $C \subset \mathbb{R}$  let us define zeta function*

$$\zeta_C^{\text{dyn-sim}}(s, z) = \sum_{n=1}^{+\infty} \frac{z^n}{n} \sum_{\substack{\mathbf{i} \in \Sigma^n \\ \frac{\log p_{\mathbf{i}}}{\log r_{\mathbf{i}}} \in C}} r_{\mathbf{i}}^s.$$

*Define  $\beta : \mathbb{R} \rightarrow \mathbb{R}$  by*

$$\sum_{i=1}^N r_i^{\beta(q)} p_i^q = 1$$

*for  $q \in \mathbb{R}$ . Now in addition assume that  $C$  is a closed set. Then the following holds:  
There is a unique real number  $\mathcal{F}(C)$  such that*

$$\lim_{r \rightarrow 0} \sigma_{\text{rad}}(\zeta_{B(C, r)}^{\text{dyn-sim}}(\mathcal{F}(C), \cdot)) = 1.$$

*Then,*

$$\mathcal{F}(C) = \sup_{\alpha \in C} \beta^*(\alpha).$$

*Specifically if  $C = \{\alpha\}$  there is a unique real number  $\mathcal{F}(\alpha)$  such that*

$$\lim_{r \rightarrow 0} \sigma_{\text{rad}}(\zeta_{B(\alpha, r)}^{\text{dyn-sim}}(\mathcal{F}(\alpha), \cdot)) = 1.$$

*Then*

$$\mathcal{F}(\alpha) = \beta^*(\alpha).$$

*Additionally if the OSC is satisfied we have that*

$$\begin{aligned} \mathcal{F}(C) &= \sup_{\alpha \in C} \dim_{\text{H}} \left\{ x \in K \mid \lim_{r \rightarrow 0} \frac{\log \mu(B(x, r))}{\log r} = \alpha \right\} \\ &= \dim_{\text{H}} \left\{ x \in K \mid \text{acc}_{r \rightarrow 0} \frac{\log \mu(B(x, r))}{\log r} \subset C \right\}. \end{aligned}$$

*Specifically if  $C = \{\alpha\}$*

$$\mathcal{F}(C) = \dim_{\text{H}} \left\{ x \in K \mid \lim_{r \rightarrow 0} \frac{\log \mu(B(x, r))}{\log r} = \alpha \right\}.$$

Proof will be provided in more general settings, i.e. look at the proof of theorem 2.3.11.

### Multifractal zeta functions for Self-conformal measure

**Theorem 2.3.3** (Geometric multifractal zeta function for Self-conformal measure). *Let  $\mu$  be the self-conformal measure associated with the list*

$$(U, X, S_1, S_2, \dots, S_N, p_1, p_2, \dots, p_N).$$

I.e.  $\mu$  is a unique probability measure such that  $\mu = \sum_{i=1}^N p_i \mu_i \circ S_i^{-1}$ .

For  $\mathbf{i} \in \Sigma^*$  define

$$s_{\mathbf{i}} = \sup_{\mathbf{j} \in \Sigma^{\mathbb{N}}} |D(S_{\mathbf{i}}\pi(\mathbf{j}))|.$$

For set  $C \subset \mathbb{R}$  let us define zeta function

$$\zeta_C^{\text{con}}(s) = \sum_{\substack{\mathbf{i} \in \Sigma^* \\ \frac{\log p_{\mathbf{i}}}{\log |K_{\mathbf{i}}|} \in C}} s_{\mathbf{i}}^s.$$

For  $\mathbf{i} \in \Sigma^{\mathbb{N}}$ , define  $\Lambda(\mathbf{i}) = \log |D(S_{i_1}\pi(\mathbf{S}\mathbf{i}))|$ ,  $\Phi(\mathbf{i}) = \log p_{i_1}$ , and define  $\beta : \mathbb{R} \rightarrow \mathbb{R}$  by

$$P(\beta(q)\Lambda + q\Phi) = 0$$

for  $q \in \mathbb{R}$ . Now in addition assume that  $C$  is a closed set. Then the following holds:

1. For  $r > 0$ ,

$$\lim_{r \rightarrow 0} \sigma_{ab}(\zeta_{B(C,r)}^{\text{con}}(\cdot)) = \sup_{\alpha \in C} \beta^*(\alpha).$$

Specifically if  $C = \{\alpha\}$

$$\lim_{r \rightarrow 0} \sigma_{ab}(\zeta_{B(\alpha,r)}^{\text{con}}(\cdot)) = \beta^*(\alpha).$$

Additionally if the OSC is satisfied then

$$\begin{aligned} \lim_{r \rightarrow 0} \sigma_{ab}(\zeta_{B(C,r)}^{\text{con}}(\cdot)) &= \sup_{\alpha \in C} \dim_{\text{H}} \left\{ x \in K \mid \lim_{r \rightarrow 0} \frac{\log \mu(B(x,r))}{\log r} = \alpha \right\} \\ &= \dim_{\text{H}} \left\{ x \in K \mid \text{acc}_{r \rightarrow 0} \frac{\log \mu(B(x,r))}{\log r} \subset C \right\}. \end{aligned}$$

Specifically if  $C = \{\alpha\}$

$$\lim_{r \rightarrow 0} \sigma_{ab}(\zeta_{B(\alpha,r)}^{\text{con}}(\cdot)) = \dim_{\text{H}} \left\{ x \in K \mid \lim_{r \rightarrow 0} \frac{\log \mu(B(x,r))}{\log r} = \alpha \right\}.$$

2. If  $C$  is a closed interval and  $\text{int}C \cap -\beta'(\mathbb{R}) \neq \emptyset$  then we have

$$\sigma_{ab}(\zeta_C^{\text{con}}(\cdot)) = \sup_{\alpha \in C} \beta^*(\alpha).$$

Additionally if the OSC is satisfied then we have

$$\begin{aligned} \sigma_{ab}(\zeta_C^{\text{con}}(\cdot)) &= \sup_{\alpha \in C} \dim_{\text{H}} \left\{ x \in K \mid \lim_{r \rightarrow 0} \frac{\log \mu(B(x,r))}{\log r} = \alpha \right\} \\ &= \dim_{\text{H}} \left\{ x \in K \mid \text{acc}_{r \rightarrow 0} \frac{\log \mu(B(x,r))}{\log r} \subset C \right\}. \end{aligned}$$

Proof will be provided in more general settings, i.e. look at the proof of theorem 2.3.7.

**Theorem 2.3.4** (Dynamical multifractal zeta function for Self-conformal measure). *Let  $\mu$  be the self-conformal measure associated with the list*

$$(U, X, S_1, S_2, \dots, S_N, p_1, p_2, \dots, p_N).$$

I.e.  $\mu$  is a unique probability measure such that  $\mu = \sum_{i=1}^N p_i \mu_i \circ S_i^{-1}$ .

For  $\mathbf{i} \in \Sigma^*$  define

$$s_{\mathbf{i}} = \sup_{\mathbf{j} \in \Sigma^{\mathbb{N}}} |D(S_{\mathbf{i}}\pi(\mathbf{j}))|.$$

For set  $C \subset \mathbb{R}$  let us define zeta function

$$\zeta_C^{\text{dyn-con}}(s, z) = \sum_{n=1}^{+\infty} \frac{z^n}{n} \sum_{\substack{\mathbf{i} \in \Sigma^n \\ \frac{\log p_{\mathbf{i}}}{\log |K_{\mathbf{i}}|} \in C}} s_{\mathbf{i}}^s.$$

For  $\mathbf{i} \in \Sigma_A^{\mathbb{N}}$ , define  $\Lambda(\mathbf{i}) = \log |D(S_{i_1}\pi(\mathbf{S}\mathbf{i}))|$ ,  $\Phi(\mathbf{i}) = \log p_{i_1}$ , and define  $\beta : \mathbb{R} \rightarrow \mathbb{R}$  by

$$P(\beta(q)\Lambda + q\Phi) = 0$$

for  $q \in \mathbb{R}$ . Now in addition assume that  $C$  is a closed set. Then the following holds:  
There is a unique real number  $\mathcal{F}(C)$  such that

$$\lim_{r \rightarrow 0} \sigma_{\text{rad}}(\zeta_{B(C,r)}^{\text{dyn-con}}(\mathcal{F}(C), \cdot)) = 1.$$

Then,

$$\mathcal{F}(C) = \sup_{\alpha \in C} \beta^*(\alpha).$$

Specifically if  $C = \{\alpha\}$  there is a unique real number  $\mathcal{F}(\alpha)$  such that

$$\lim_{r \rightarrow 0} \sigma_{\text{rad}}(\zeta_{B(\alpha,r)}^{\text{dyn-con}}(\mathcal{F}(\alpha), \cdot)) = 1.$$

Then

$$\mathcal{F}(\alpha) = \beta^*(\alpha).$$

Additionally if the OSC is satisfied we have that

$$\begin{aligned} \mathcal{F}(C) &= \sup_{\alpha \in C} \dim_{\text{H}} \left\{ x \in K \mid \lim_{r \rightarrow 0} \frac{\log \mu(B(x, r))}{\log r} = \alpha \right\} \\ &= \dim_{\text{H}} \left\{ x \in K \mid \text{acc}_{r \rightarrow 0} \frac{\log \mu(B(x, r))}{\log r} \subset C \right\}. \end{aligned}$$

Specifically if  $C = \{\alpha\}$

$$\mathcal{F}(\alpha) = \dim_{\text{H}} \left\{ x \in K \mid \lim_{r \rightarrow 0} \frac{\log \mu(B(x, r))}{\log r} = \alpha \right\}.$$

Proof will be provided in more general settings, i.e. look at the proof of theorem 2.3.11.

### Multifractal zeta functions for Self-conformal graph-directed measure

**Theorem 2.3.5** (Geometric multifractal zeta function for Self-conformal graph-directed measure).

Let  $(\mu_v)_{v \in V}$  be the list of self-conformal measures associated with the list

$$(V, E, (U_v)_{v \in V}, (X_v)_{v \in V}, (S_e)_{e \in E}, (p_e)_{e \in E}).$$

I.e. for each  $v \in V$ ,  $\mu_v$  is a unique probability measure such that  $\mu_v = \sum_{t(e)=v} p_e \mu_{i(e)} \circ S_e^{-1}$ .

For  $\mathbf{i} \in \Sigma_A^*$  define

$$s_{\mathbf{i}} = \sup_{\mathbf{j} \in \Sigma_A^{\mathbb{N}}} |D(S_{\mathbf{i}}\pi(\mathbf{j}))|.$$

For set  $C \subset \mathbb{R}$  let us define zeta function

$$\zeta_C^{\text{con}}(s) = \sum_{\substack{\mathbf{i} \in \Sigma_A^* \\ \frac{\log p_{\mathbf{i}}}{\log |K_{\mathbf{i}}|} \in C}} s_{\mathbf{i}}^s.$$

For  $\mathbf{i} \in \Sigma_A^{\mathbb{N}}$ , define  $\Lambda(\mathbf{i}) = \log |D(S_{i_1} \pi(\mathbf{Si}))|$ ,  $\Phi(\mathbf{i}) = \log p_{i_1}$ , and define  $\beta : \mathbb{R} \rightarrow \mathbb{R}$  by

$$P(\beta(q)\Lambda + q\Phi) = 0$$

for  $q \in \mathbb{R}$ . Now in addition assume that  $C$  is a closed set. Then the following holds:

1. For  $r > 0$ ,

$$\lim_{r \rightarrow 0} \sigma_{ab}(\zeta_{B(C,r)}^{\text{con}}(\cdot)) = \sup_{\alpha \in C} \beta^*(\alpha).$$

Specifically if  $C = \{\alpha\}$

$$\lim_{r \rightarrow 0} \sigma_{ab}(\zeta_{B(\alpha,r)}^{\text{con}}(\cdot)) = \beta^*(\alpha).$$

Additionally if the OSC is satisfied then for each  $v \in V$  we have

$$\begin{aligned} \lim_{r \rightarrow 0} \sigma_{ab}(\zeta_{B(C,r)}^{\text{con}}(\cdot)) &= \sup_{\alpha \in C} \dim_{\text{H}} \left\{ x \in K_v \mid \lim_{r \rightarrow 0} \frac{\log \mu_v(B(x,r))}{\log r} = \alpha \right\} \\ &= \dim_{\text{H}} \left\{ x \in K_v \mid \text{acc}_{r \rightarrow 0} \frac{\log \mu_v(B(x,r))}{\log r} \subset C \right\}. \end{aligned}$$

Specifically if  $C = \{\alpha\}$  then for each  $v \in V$  we have

$$\lim_{r \rightarrow 0} \sigma_{ab}(\zeta_{B(\alpha,r)}^{\text{con}}(\cdot)) = \dim_{\text{H}} \left\{ x \in K_v \mid \lim_{r \rightarrow 0} \frac{\log \mu_v(B(x,r))}{\log r} = \alpha \right\}.$$

2. If  $C$  is a closed interval and  $\text{int}C \cap -\beta'(\mathbb{R}) \neq \emptyset$  then we have

$$\sigma_{ab}(\zeta_C^{\text{con}}(\cdot)) = \sup_{\alpha \in C} \beta^*(\alpha).$$

Additionally if the OSC is satisfied then for each  $v \in V$  we have

$$\begin{aligned} \sigma_{ab}(\zeta_C^{\text{con}}(\cdot)) &= \sup_{\alpha \in C} \dim_{\text{H}} \left\{ x \in K_v \mid \lim_{r \rightarrow 0} \frac{\log \mu_v(B(x,r))}{\log r} = \alpha \right\} \\ &= \dim_{\text{H}} \left\{ x \in K_v \mid \text{acc}_{r \rightarrow 0} \frac{\log \mu_v(B(x,r))}{\log r} \subset C \right\}. \end{aligned}$$

Proof is analogous to proof of 2.3.7 and hence omitted.

**Theorem 2.3.6** (Dynamical multifractal zeta function for Self-conformal graph-directed measure).  
Let  $(\mu_v)_{v \in V}$  be the list of self-conformal measures associated with the list

$$(V, E, (U_v)_{v \in V}, (X_v)_{v \in V}, (S_e)_{e \in E}, (p_e)_{e \in E}).$$

I.e. for each  $v \in V$ ,  $\mu_v$  is a unique probability measure such that  $\mu_v = \sum_{t(e)=v} p_e \mu_{i(e)} \circ S_e^{-1}$ .

For  $\mathbf{i} \in \Sigma_A^*$  define

$$s_{\mathbf{i}} = \sup_{\mathbf{j} \in \Sigma_A^{\mathbb{N}}} |D(S_{\mathbf{i}} \pi(\mathbf{j}))|.$$

For set  $C \subset \mathbb{R}$  let us define zeta function

$$\zeta_C^{\text{dyn-con}}(s, z) = \sum_{n=1}^{+\infty} \frac{z^n}{n} \sum_{\substack{\mathbf{i} \in \Sigma_A^{\mathbb{N}} \\ \frac{\log p_{\mathbf{i}}}{\log |K_{\mathbf{i}}|} \in C}} s_{\mathbf{i}}^s.$$

For  $\mathbf{i} \in \Sigma_A^{\mathbb{N}}$ , define  $\Lambda(\mathbf{i}) = \log |D(S_{i_1} \pi(S\mathbf{i}))|$ ,  $\Phi(\mathbf{i}) = \log p_{i_1}$ , and define  $\beta : \mathbb{R} \rightarrow \mathbb{R}$  by

$$P(\beta(q)\Lambda + q\Phi) = 0$$

for  $q \in \mathbb{R}$ . Now in addition assume that  $C$  is a closed set. Then the following holds: There is a unique real number  $\mathcal{F}(C)$  such that

$$\lim_{r \rightarrow 0} \sigma_{\text{rad}}(\zeta_{B(C,r)}^{\text{dyn-con}}(\mathcal{F}(C), \cdot)) = 1.$$

Then,

$$\mathcal{F}(C) = \sup_{\alpha \in C} \beta^*(\alpha).$$

Specifically if  $C = \{\alpha\}$  there is a unique real number  $\mathcal{F}(\alpha)$  such that

$$\lim_{r \rightarrow 0} \sigma_{\text{rad}}(\zeta_{B(\alpha,r)}^{\text{dyn-con}}(\mathcal{F}(\alpha), \cdot)) = 1.$$

Then

$$\mathcal{F}(\alpha) = \beta^*(\alpha).$$

Additionally if the OSC is satisfied then for each  $v \in V$  we have that

$$\begin{aligned} \mathcal{F}(C) &= \sup_{\alpha \in C} \dim_{\text{H}} \left\{ x \in K_v \mid \lim_{r \rightarrow 0} \frac{\log \mu_v(B(x, r))}{\log r} = \alpha \right\} \\ &= \dim_{\text{H}} \left\{ x \in K_v \mid \text{acc}_{r \rightarrow 0} \frac{\log \mu_v(B(x, r))}{\log r} \subset C \right\}. \end{aligned}$$

Specifically if  $C = \{\alpha\}$  then for each  $v \in V$  we have

$$\mathcal{F}(\alpha) = \dim_{\text{H}} \left\{ x \in K_v \mid \lim_{r \rightarrow 0} \frac{\log \mu_v(B(x, r))}{\log r} = \alpha \right\}.$$

Proof is analogous to proof of 2.3.11 and hence omitted.

### Multifractal zeta functions for mixed multifractal spectra of self-conformal measures

**Theorem 2.3.7** (Geometric multifractal zeta function for mixed multifractal spectra of a self-conformal measure). *Let for  $m = 1, \dots, M$ , measure  $\mu_m$  be the self-conformal measure associated with the list*

$$(U, X, S_1, S_2, \dots, S_N, p_{m,1}, p_{m,2}, \dots, p_{m,N}).$$

*I.e.  $\mu_m$  is a unique probability measure such that*

$$\mu_m = \sum_{i=1}^N p_{m,i} \mu_m \circ S_i^{-1}.$$

For  $\mathbf{i} \in \Sigma^*$  define

$$s_{\mathbf{i}} = \sup_{\mathbf{j} \in \Sigma^{\mathbb{N}}} |D(S_{\mathbf{i}} \pi(\mathbf{j}))|.$$

For set  $C \subset \mathbb{R}$  let us define zeta function

$$\zeta_C^{\text{con}}(s) = \sum_{\substack{\mathbf{i} \in \Sigma_A^* \\ \left( \frac{\log p_{1,\mathbf{i}}}{\log |K_{\mathbf{i}}|}, \dots, \frac{\log p_{M,\mathbf{i}}}{\log |K_{\mathbf{i}}|} \right) \in C}} s_{\mathbf{i}}^s.$$

For  $\mathbf{i} \in \Sigma^{\mathbb{N}}$ , define  $\Lambda(\mathbf{i}) = \log |D(S_{i_1} \pi(S\mathbf{i}))|$ . For  $\mathbf{i} \in \Sigma^{\mathbb{N}}$  and  $1 \leq m \leq M$  define  $\Phi_m(\mathbf{i}) = \log p_{m,i_1}$ . Let  $\Phi = (\Phi_1, \dots, \Phi_M)$ . Let  $\beta : \mathbb{R}^M \rightarrow \mathbb{R}$  be defined by

$$P(\beta(q)\Lambda + \langle q, \Phi \rangle) = 0.$$

for  $q \in \mathbb{R}^M$ . Now in addition assume that  $C$  is a closed set. Then the following holds:

1. For  $r > 0$ ,

$$\lim_{r \rightarrow 0} \sigma_{ab}(\zeta_{B(C,r)}^{\text{con}}(\cdot)) = \sup_{\alpha \in C} \beta^*(\alpha).$$

Specifically if  $C = \{\alpha\}$

$$\lim_{r \rightarrow 0} \sigma_{ab}(\zeta_{B(\alpha,r)}^{\text{con}}(\cdot)) = \beta^*(\alpha).$$

Additionally if the OSC is satisfied then

$$\begin{aligned} \lim_{r \rightarrow 0} \sigma_{ab}(\zeta_{B(C,r)}^{\text{con}}(\cdot)) &= \sup_{\alpha \in C} \dim_{\text{H}} \left\{ x \in K \mid \lim_{r \rightarrow 0} \left( \frac{\log \mu_1(B(x,r))}{\log r}, \dots, \frac{\log \mu_M(B(x,r))}{\log r} \right) = \alpha \right\} \\ &= \dim_{\text{H}} \left\{ x \in K \mid \text{acc}_{r \rightarrow 0} \left( \frac{\log \mu_1(B(x,r))}{\log r}, \dots, \frac{\log \mu_M(B(x,r))}{\log r} \right) \subset C \right\}. \end{aligned}$$

Specifically if  $C = \{\alpha\}$

$$\lim_{r \rightarrow 0} \sigma_{ab}(\zeta_{B(\alpha,r)}^{\text{con}}(\cdot)) = \dim_{\text{H}} \left\{ x \in K \mid \lim_{r \rightarrow 0} \left( \frac{\log \mu_1(B(x,r))}{\log r}, \dots, \frac{\log \mu_M(B(x,r))}{\log r} \right) = \alpha \right\}.$$

2. If  $C$  is convex and  $\text{int} C \cap -\nabla \beta(\mathbb{R}^M) \neq \emptyset$  then we have

$$\sigma_{ab}(\zeta_C^{\text{con}}(\cdot)) = \sup_{\alpha \in C} \beta^*(\alpha).$$

In addition if OSC is satisfied we have

$$\begin{aligned} \sigma_{ab}(\zeta_C^{\text{con}}(\cdot)) &= \sup_{\alpha \in C} \dim_{\text{H}} \left\{ x \in K \mid \lim_{r \rightarrow 0} \left( \frac{\log \mu_1(B(x,r))}{\log r}, \dots, \frac{\log \mu_M(B(x,r))}{\log r} \right) = \alpha \right\} \\ &= \dim_{\text{H}} \left\{ x \in K \mid \text{acc}_{r \rightarrow 0} \left( \frac{\log \mu_1(B(x,r))}{\log r}, \dots, \frac{\log \mu_M(B(x,r))}{\log r} \right) \subset C \right\}. \end{aligned}$$

Proof.

First let us note that  $s_{\mathbf{i}}$  defined above is exactly  $s_{\mathbf{i}}(\Lambda)$ . Following lemma allow us to use results for  $\zeta_C^{U,\Lambda}(s)$  to investigate  $\zeta_C^{\text{con}}$ .

**Lemma 2.3.8.** Let  $U : \mathcal{P}(\Sigma_A^{\mathbb{N}}) \rightarrow \mathbb{R}$  be defined as  $U(\mu) = \frac{\int_{\Sigma_A^{\mathbb{N}}} \Phi d\mu}{\int_{\Sigma_A^{\mathbb{N}}} \Lambda d\mu}$ . Then we have

$$\lim_{r \rightarrow 0} \sigma_{ab}(\zeta_{B(C,r)}^{\text{con}}(\cdot)) = \lim_{r \rightarrow 0} \sigma_{ab}(\zeta_{B(C,r)}^{U,\Lambda}(\cdot)).$$

Proof.

First let us note that for  $\mathbf{i} \in \Sigma_A^{\mathbb{N}}$  we have  $U(L_n \mathbf{i}) = \frac{\frac{1}{n} S_n \Phi(\mathbf{i})}{\frac{1}{n} S_n \Lambda(\mathbf{i})} = \frac{S_n \Phi(\mathbf{i})}{S_n \Lambda(\mathbf{i})}$ . It is easy to see that  $S_n \Phi(\mathbf{i}) = (p_{1,\mathbf{i}|n}, p_{2,\mathbf{i}|n}, \dots, p_{M,\mathbf{i}|n})$ . There is a constant  $C$  such that for each  $\mathbf{i} \in \Sigma_A^{\mathbb{N}}$  we have

$$\frac{1}{C} |K_{\mathbf{i}|n}| \leq \exp S_n \Lambda(\mathbf{i}) \leq C |K_{\mathbf{i}|n}|,$$

and therefore

$$\begin{aligned} \log |K_{\mathbf{i}|n}| - \log C &\leq S_n \Lambda(\mathbf{i}) \leq \log C + \log |K_{\mathbf{i}|n}| \\ 1 - \frac{\log C}{\log |K_{\mathbf{i}|n}|} &\leq \frac{S_n \Lambda(\mathbf{i})}{\log |K_{\mathbf{i}|n}|} \leq 1 + \frac{\log C}{\log |K_{\mathbf{i}|n}|}. \end{aligned}$$

Hence for each  $k$

$$\left| \frac{\log p_{k,\mathbf{i}|n}}{\log |K_{\mathbf{i}|n}|} - \frac{\log p_{k,\mathbf{i}|n}}{S_n \Lambda(\mathbf{i})} \right| = \left| \frac{\log p_{k,\mathbf{i}|n}}{S_n \Lambda(\mathbf{i})} \left( \frac{S_n \Lambda(\mathbf{i})}{\log |K_{\mathbf{i}|n}|} - 1 \right) \right| \leq \frac{\log p_{k,\mathbf{i}|n}}{S_n \Lambda(\mathbf{i})} \frac{\log C}{\log |K_{\mathbf{i}|n}|}.$$

Now  $\frac{\log p_{k,\mathbf{i}|n}}{S_n \Lambda(\mathbf{i})}$  is bounded and for any  $\epsilon_1$  we can find  $m$  such that for any  $n > m$  we have  $\log |K_{\mathbf{i}|n}| < \epsilon_1$ . So using that and 3.0.2 for any  $\epsilon$  we can chose  $n_0$  such that for each  $n > n_0$  we have that for each  $\mathbf{i} \in \Sigma_A^{\mathbb{N}}$

$$\max_{\mathbf{i} \in \Sigma_A^{\mathbb{N}}} \left\{ \text{diam} U(L_n[\mathbf{i}|n]), d \left( \left( \frac{p_{1,\mathbf{i}|n}}{\log |K_{\mathbf{i}|n}|}, \frac{p_{2,\mathbf{i}|n}}{\log |K_{\mathbf{i}|n}|}, \dots, \frac{p_{M,\mathbf{i}|n}}{\log |K_{\mathbf{i}|n}|} \right), U L_n(\mathbf{i}) \right) \right\} < \frac{\epsilon}{2}.$$

Hence we have

$$\begin{aligned} \left( \frac{p_{1,\mathbf{i}|n}}{\log |K_{\mathbf{i}|n}|}, \frac{p_{2,\mathbf{i}|n}}{\log |K_{\mathbf{i}|n}|}, \dots, \frac{p_{M,\mathbf{i}|n}}{\log |K_{\mathbf{i}|n}|} \right) \in B(C, r) &\Rightarrow U(L_n[\mathbf{i}|n]) \subset B(C, r + \epsilon), \\ U(L_n[\mathbf{i}|n]) \subset B(C, r) &\Rightarrow \left( \frac{p_{1,\mathbf{i}|n}}{\log |K_{\mathbf{i}|n}|}, \frac{p_{2,\mathbf{i}|n}}{\log |K_{\mathbf{i}|n}|}, \dots, \frac{p_{M,\mathbf{i}|n}}{\log |K_{\mathbf{i}|n}|} \right) \in B(C, r + \epsilon). \end{aligned}$$

Therefore  $\sigma_{ab}(\zeta_{B(C, r+\epsilon)}^{U, \Lambda}(\cdot)) \geq \sigma_{ab}(\zeta_{B(C, r)}^{\text{con}}(\cdot))$ , and  $\sigma_{ab}(\zeta_{B(C, r+\epsilon)}^{U, \Lambda}(\cdot)) \geq \sigma_{ab}(\zeta_{B(C, r)}^{\text{con}}(\cdot))$ . Hence,

$$\sigma_{ab}(\zeta_{B(C, 3r)}^{U, \Lambda}(\cdot)) \geq \sigma_{ab}(\zeta_{B(C, 2r)}^{\text{con}}(\cdot)) \geq \sigma_{ab}(\zeta_{B(C, r)}^{U, \Lambda}(\cdot))$$

Letting  $r$  to tend to 0 we get our lemma. ■

Using Lemma above and theorem 2.2.4 we have

$$\lim_{r \rightarrow 0} \sigma_{ab}(\zeta_{B(C, r)}^{\text{con}}(\cdot)) = \lim_{r \rightarrow 0} \sigma_{ab}(\zeta_{B(C, r)}^{U, \Lambda}(\cdot)) = \sup_{\substack{\mu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}}) \\ U\mu \in C}} - \frac{h(\mu)}{\int_{\Sigma_A^{\mathbb{N}}} \Lambda d\mu}. \quad (2.14)$$

The following claim is well-known in case  $M = 1$ , and for  $M > 1$  the proof is very similar. We will include the proof for the sake of completeness.

**Lemma 2.3.9.** *Let  $U : \mathcal{P}(\Sigma_A^{\mathbb{N}}) \rightarrow \mathbb{R}$  be defined as  $U(\mu) = \frac{\int_{\Sigma_A^{\mathbb{N}}} \Phi d\mu}{\int_{\Sigma_A^{\mathbb{N}}} \Lambda d\mu}$ . Then for  $\alpha \in \mathbb{R}^M$ , we have*

$$\sup_{\substack{\mu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}}) \\ U\mu = \alpha}} - \frac{h(\mu)}{\int_{\Sigma_A^{\mathbb{N}}} \Lambda d\mu} = \beta^*(\alpha).$$



Proof.

Let us define  $F : \mathbb{R} \times \mathbb{R}^M \rightarrow \mathbb{R}$  as  $F(x, y) = P(x\Lambda + \langle y, \Phi \rangle)$ . Now recalling definition of  $\beta$  for every  $q = (q_1, q_2, \dots, q_M)$  we have

$$F(\beta(q), q) = P(\beta(q)\Lambda + \langle q, \Phi \rangle) = 0. \quad (2.15)$$

Since  $\Phi_1, \Phi_2, \dots, \Phi_M, \Lambda \in H(\Sigma_A^{\mathbb{N}})$  from 1.2.8 we have that  $F$  is differentiable. Differentiating with respect to  $q_1, q_2, \dots, q_M$  and using 1.2.8 and chain rule we get

$$\sum_{i=1}^m \frac{d}{dq_i} P(\beta(q)\Lambda + \langle q, \Phi \rangle) = \int_{\Sigma_A^{\mathbb{N}}} \nabla \beta(q)\Lambda + \langle q, \Phi \rangle d\mu_q = 0$$

where  $\mu_q$  is the unique ergodic Gibbs measure associated to potential  $\beta(q)\Lambda + \langle q, \Phi \rangle$ . From there we have

$$\nabla \beta(q) = - \frac{\int_{\Sigma_A^{\mathbb{N}}} \Phi d\mu_q}{\int_{\Sigma_A^{\mathbb{N}}} \Lambda d\mu_q} = -U(\mu_q). \quad (2.16)$$

Let us choose  $q$  such that  $U(\mu_q) = \alpha$ . Now we have

$$\begin{aligned} \sup_{\substack{\mu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}}) \\ U(\mu) = \alpha}} - \frac{h(\mu)}{\int_{\Sigma_A^{\mathbb{N}}} \Lambda d\mu} &\geq - \frac{h(\mu_q)}{\int_{\Sigma_A^{\mathbb{N}}} \Lambda d\mu_q} \\ &= \frac{\int_{\Sigma_A^{\mathbb{N}}} \beta(q)\Lambda + \langle q, \Phi \rangle d\mu_q}{\int_{\Sigma_A^{\mathbb{N}}} \Lambda d\mu_q} && \text{using 2.15 and 1.2.7} \\ &= \beta(q) + \langle q, \alpha \rangle && \text{using } U(\mu_q) = \alpha \\ &\geq \inf_{q \in \mathbb{R}} \{ \beta(q) + \langle q, \alpha \rangle \} \\ &= \beta^*(\alpha) \end{aligned}$$

Let us prove the other side of equality. Let us again start from definition of  $\beta$ . Then we have

$$P(\beta(q)\Lambda + \langle q, \Phi \rangle) = \sup_{\nu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}})} \left\{ h(\nu) + \int_{\Sigma_A^{\mathbb{N}}} \beta(q)\Lambda + \langle q, \Phi \rangle d\nu \right\} = 0.$$

In this case there is a unique invariant equilibrium state  $\mu_q$

$$\begin{aligned} 0 = P(\beta(q)\Lambda + \langle q, \Phi \rangle) &\geq h(\nu) + \int_{\Sigma_A^{\mathbb{N}}} \beta(q)\Lambda + \langle q, \Phi \rangle d\nu && \text{using 1.2.7} \\ \beta(q) + \left\langle q, \frac{\int_{\Sigma_A^{\mathbb{N}}} \Phi d\nu}{\int_{\Sigma_A^{\mathbb{N}}} \Lambda d\nu} \right\rangle &\geq - \frac{h(\nu)}{\int_{\Sigma_A^{\mathbb{N}}} \Lambda d\nu}. \end{aligned}$$

It is true for every  $q$  and if we restrict ourselves to measures  $\nu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}})$  such that  $U(\nu) = \alpha$  we got

$$\inf_q \{ \beta(q) + \langle q, \alpha \rangle \} \geq \sup_{\substack{\nu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}}) \\ U\nu = \alpha}} - \frac{h(\nu)}{\int_{\Sigma_A^{\mathbb{N}}} \Lambda d\nu}.$$

Hence we have our result. ■

Note that from the lemma above we have

$$\sup_{\alpha \in C} \beta^*(\alpha) = \sup_{\alpha \in C} \sup_{\substack{\mu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}}) \\ U\mu = \alpha}} - \frac{h(\mu)}{\int_{\Sigma_A^{\mathbb{N}}} \Lambda d\mu} = \sup_{\substack{\mu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}}) \\ U\mu \in C}} - \frac{h(\mu)}{\int_{\Sigma_A^{\mathbb{N}}} \Lambda d\mu}. \quad (2.17)$$

Now using 2.2.4 and the result above for closed set  $C$  we have

$$\lim_{r \rightarrow 0} \sigma_{ab}(\zeta_{B(C,r)}^{\text{con}}(\cdot)) = \lim_{r \rightarrow 0} \sigma_{ab}(\zeta_{B(C,r)}^{U,\Lambda}(\cdot)) = \sup_{\substack{\mu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}}) \\ U\mu \in C}} - \frac{h(\mu)}{\int_{\Sigma_A^{\mathbb{N}}} \Lambda d\mu} = \sup_{\alpha \in C} \beta^*(\alpha). \quad (2.18)$$

In addition if  $OSC$  is satisfied from 1.3.6 we have that for each  $v \in V$  the following holds

$$\lim_{r \rightarrow 0} \sigma_{ab}(\zeta_{B(C,r)}^{\text{con}}(\cdot)) = \sup_{\alpha \in C} \beta^*(\alpha) = \dim_{\mathbb{H}} \left\{ x \in K \mid \text{acc}_{r \rightarrow 0} \left( \frac{\log \mu_1 B(x, r)}{\log r}, \dots, \frac{\log \mu_M B(x, r)}{\log r} \right) \subset C \right\}.$$

**Lemma 2.3.10.** *If  $C$  is convex and  $\text{int}C \cap (-\nabla\beta(\mathbb{R}^M)) \neq \emptyset$ , then  $f^{U,\Lambda}$  is inner continuous at  $C$ .*

Proof.

For  $\beta^*$  we have that  $\{\alpha \in \mathbb{R}^M \mid \beta^*(\alpha) > -\infty\} = \nabla\beta(\mathbb{R}^M)$  (see [Ro, Corollary 26.4.1]).

$$\begin{aligned} f^{U,\Lambda}(I[C, r]) &= \sup_{\substack{\mu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}}) \\ U\mu \in I[C, r]}} - \frac{h(\mu)}{\int \Lambda d\mu} && \text{by 2.1.2} \\ &= \sup_{\alpha \in I[C, r]} \beta^*(\alpha) && \text{by 2.17.} \end{aligned}$$

Hence

$$\lim_{r \rightarrow 0} f^{U,\Lambda}(I(C, r)) = \lim_{r \rightarrow 0} \sup_{\alpha \in I[C, r]} \beta^*(\alpha) \quad (2.19)$$

$$= \sup_{\alpha \in C} \beta^*(\alpha) = f^{U,\Lambda}(C) \quad (2.20)$$

■

Using 2.2.6, 2.3 we have that

$$\sigma_{ab}(\zeta_C^{\text{con}}(\cdot)) \leq \lim_{r \rightarrow 0} \sigma_{ab}(\zeta_{B(C,r)}^{\text{con}}(\cdot)) = f^{U,\Lambda}(C) \quad \text{by 2.18} \quad (2.21)$$

Next by 2.17 and fact that  $\sigma_{ab}(\zeta_C^{\text{con}}) \geq \sigma_{ab}(\zeta_{I(C,r)}^{\text{con}})$

$$\sigma_{ab}(\zeta_C^{\text{con}}(\cdot)) \geq f^{U,\Lambda}(I[C, r])$$

Hence

$$\sigma_{ab}(\zeta_C^{\text{con}}(\cdot)) \geq \lim_{r \rightarrow 0} f^{U,\Lambda}(I[C, r]) = f^{U,\Lambda}(C) \quad (2.22)$$

Hence

$$\begin{aligned} \sigma_{ab}(\zeta_C^{\text{con}}(\cdot)) &= f^{U,\Lambda}(C) && \text{by 2.22, 2.21} \\ &= \sup_{\alpha \in C} \beta^*(\alpha) && \text{by 2.18} \end{aligned}$$

In addition, if OSC is satisfied we have

$$\sigma_{ab}(\zeta_C^{\text{con}}(\cdot)) = \sup_{\alpha \in C} \beta^*(\alpha) = \dim_{\text{H}} \left\{ x \in K \mid \text{acc}_{r \rightarrow 0} \left( \frac{\log \mu_1 B(x, r)}{\log r}, \dots, \frac{\log \mu_M B(x, r)}{\log r} \right) \subset C \right\}. \quad \text{by 1.3.6}$$

Which ends the proof of the theorem. ■

**Theorem 2.3.11** (Dynamical zeta function for mixed multifractal spectra of self-conformal measures). *Let for  $m = 1, \dots, M$ , measure  $\mu_m$  be the self-conformal measure associated with the list*

$$(U, X, S_1, S_2, \dots, S_N, p_{m,1}, p_{m,2}, \dots, p_{m,N}).$$

*I.e.  $\mu_m$  is a unique probability measure such that*

$$\mu_m = \sum_{i=1}^N p_{m,i} \mu_m \circ S_i^{-1}.$$

*For  $\mathbf{i} \in \Sigma^*$  define*

$$s_{\mathbf{i}} = \sup_{\mathbf{j} \in \Sigma^{\mathbb{N}}} |D(S_{\mathbf{i}} \pi(\mathbf{j}))|.$$

*For set  $C \subset \mathbb{R}$  let us define zeta function*

$$\zeta_C^{\text{dyn-con}}(s, z) = \sum_{n=1}^{+\infty} \frac{z^n}{n} \sum_{\substack{\mathbf{i} \in \Sigma_A^n \\ \left( \frac{\log p_{1,\mathbf{i}}}{\log |K_{\mathbf{i}}|}, \dots, \frac{\log p_{M,\mathbf{i}}}{\log |K_{\mathbf{i}}|} \right) \in C}} s_{\mathbf{i}}^s.$$

*For  $\mathbf{i} \in \Sigma^{\mathbb{N}}$ , define  $\Lambda(\mathbf{i}) = \log |D(S_{i_1} \pi(S\mathbf{i}))|$ . For  $\mathbf{i} \in \Sigma^{\mathbb{N}}$  and  $1 \leq m \leq M$  define  $\Phi_m(\mathbf{i}) = \log p_{m,i_1}$ . Let  $\Phi = (\Phi_1, \dots, \Phi_M)$ . Let  $\beta : \mathbb{R}^M \rightarrow \mathbb{R}$  be defined by*

$$P(\beta(q)\Lambda + \langle q, \Phi \rangle) = 0.$$

*for  $q \in \mathbb{R}^M$ . Now in addition assume that  $C$  is a closed set. Then the following holds: There is a unique real number  $\mathcal{F}(C)$  such that*

$$\lim_{r \rightarrow 0} \sigma_{\text{rad}}(\zeta_{B(C,r)}^{\text{dyn-con}}(\mathcal{F}(C), \cdot)) = 1.$$

*and we have*

$$\mathcal{F}(C) = \sup_{\alpha \in C} \beta^*(\alpha).$$

*Additionally if the OSC is satisfied then we have*

$$\begin{aligned} \mathcal{F}(C) &= \sup_{\alpha \in C} \dim_{\text{H}} \left\{ x \in K \mid \lim_{r \rightarrow 0} \left( \frac{\log \mu_1(B(x, r))}{\log r}, \dots, \frac{\log \mu_M(B(x, r))}{\log r} \right) = \alpha \right\} \\ &= \dim_{\text{H}} \left\{ x \in K \mid \text{acc}_{r \rightarrow 0} \left( \frac{\log \mu_1(B(x, r))}{\log r}, \dots, \frac{\log \mu_M(B(x, r))}{\log r} \right) \subset C \right\}. \end{aligned}$$

*Specifically if  $C = \{\alpha\}$*

$$\mathcal{F}(\alpha) = \dim_{\text{H}} \left\{ x \in K \mid \lim_{r \rightarrow 0} \left( \frac{\log \mu_1(B(x, r))}{\log r}, \dots, \frac{\log \mu_M(B(x, r))}{\log r} \right) = \alpha \right\}.$$

Proof.

**Lemma 2.3.12.** *Let  $U(\mu) = \frac{\int_{\Sigma_A^{\mathbb{N}}} \Phi d\mu}{\int_{\Sigma_A^{\mathbb{N}}} \Lambda d\mu}$  we have*

$$\lim_{r \rightarrow 0} \sigma_{rad}(\zeta_{B(C,r)}^{\text{dyn-con}}(s, \cdot)) = \lim_{r \rightarrow 0} \sigma_{rad}(\zeta_{B(C,r)}^{\text{dyn}, U}(s\Lambda, \cdot)).$$

Proof is similar to 2.3.8 hence omitted.

Now using 2.3.12 and 2.2.14 we got that there is a unique real number  $\mathcal{F}(C)$  such that

$$\lim_{r \rightarrow 0} \sigma_{rad}(\zeta_{B(C,r)}^{\text{dyn-con}}(\mathcal{F}(C), \cdot)) = \lim_{r \rightarrow 0} \sigma_{rad}(\zeta_{B(C,r)}^{U, \text{dyn}}(\mathcal{F}(C)\Lambda, \cdot)) = 1.$$

Then we have

$$\mathcal{F}(C) = \sup_{\substack{\mu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}}) \\ U\mu \in C}} - \frac{h(\mu)}{\int_{\Sigma_A^{\mathbb{N}}} \Lambda d\mu}.$$

Using 2.3.9 we get

$$\mathcal{F}(C) = \sup_{\substack{\mu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}}) \\ U\mu \in C}} - \frac{h(\mu)}{\int_{\Sigma_A^{\mathbb{N}}} \Lambda d\mu} = \sup_{\alpha \in C} \beta^*(\alpha).$$

■

### Multifractal zeta functions for Gibbs measures

**Theorem 2.3.13** (Geometric Multifractal zeta function for Gibbs measures). *Let  $\Sigma_A^{\mathbb{N}}$  be given. Let  $\Lambda \in H(\Sigma_A^{\mathbb{N}})$ ,  $\Lambda < 0$ . Fix metric  $d_\Lambda$ . Let Hausdorff dimension on  $\Sigma_A^{\mathbb{N}}$  be defined with respect to  $d_\Lambda$ . Let  $\Phi \in H(\Sigma_A^{\mathbb{N}})$ . Let  $\mu_\Phi$  be Gibbs measure with potential  $\Phi$ . Denote  $s_{\mathbf{i}}(\Lambda)$  as  $s_{\mathbf{i}}$ . For set  $C \subset \mathbb{R}$  let us define zeta function*

$$\zeta_C^{\text{gib}}(s) = \sum_{\substack{\mathbf{i} \in \Sigma_A^* \\ \frac{\log \mu_\Phi([\mathbf{i}])}{\log s_{\mathbf{i}}} \in C}} s_{\mathbf{i}}^s.$$

Define  $\beta : \mathbb{R} \rightarrow \mathbb{R}$  by

$$P(\beta(q)\Lambda + q\Phi) = 0,$$

for  $q \in \mathbb{R}$ . Now in addition assume that  $C$  is a closed set. Then the following holds:

1. For  $r > 0$ ,

$$\lim_{r \rightarrow 0} \sigma_{ab}(\zeta_{B(C,r)}^{\text{gib}}(\cdot)) = \sup_{\alpha \in C} \beta^*(\alpha) = \dim_H \left\{ \mathbf{i} \in \Sigma_A^{\mathbb{N}} \mid \text{acc}_{n \rightarrow +\infty} \frac{\log \mu_\Phi([\mathbf{i}|n])}{\log s_{\mathbf{i}|n}} \subset C \right\}.$$

Specifically if  $C = \{\alpha\}$

$$\lim_{r \rightarrow 0} \sigma_{ab}(\zeta_{B(\alpha,r)}^{\text{gib}}(\cdot)) = \beta^*(\alpha) = \dim_H \left\{ \mathbf{i} \in \Sigma_A^{\mathbb{N}} \mid \lim_{n \rightarrow +\infty} \frac{\log \mu_\Phi([\mathbf{i}|n])}{\log s_{\mathbf{i}|n}} = \alpha \right\}.$$

2. If  $C$  is a closed interval and  $\text{int}C \cap -\beta'(\mathbb{R}) \neq \emptyset$  then we have

$$\sigma_{ab}(\zeta_C^{\text{gib}}(\cdot)) = \sup_{\alpha \in C} \beta^*(\alpha) = \dim_H \left\{ \mathbf{i} \in \Sigma_A^{\mathbb{N}} \mid \text{acc}_{n \rightarrow +\infty} \frac{\log \mu_\Phi([\mathbf{i}|n])}{\log s_{\mathbf{i}|n}} \subset C \right\}.$$

Proof is similar to proof of 2.3.7 and hence omitted.

**Theorem 2.3.14** (Dynamical Multifractal zeta function for Gibbs measures). *Let  $\Sigma_A^{\mathbb{N}}$  be given. Let  $\Lambda \in H(\Sigma_A^{\mathbb{N}})$ ,  $\Lambda < 0$ . Fix metric  $d_\Lambda$ . Let Hausdorff dimension on  $\Sigma_A^{\mathbb{N}}$  be defined with respect to  $d_\Lambda$ . Let  $\Phi \in H(\Sigma_A^{\mathbb{N}})$ . Let  $\mu_\Phi$  be Gibbs measure with potential  $\Phi$ . Denote  $s_{\mathbf{i}}(\Lambda)$  as  $s_{\mathbf{i}}$ . Let  $C \subset \mathbb{R}$  be closed set. Then we have*

*For set  $C \subset \mathbb{R}$  let us define zeta function*

$$\zeta_C^{\text{dyn-gib}}(s, z) = \sum_{n=1}^{+\infty} \frac{z^n}{n} \sum_{\substack{\mathbf{i} \in \Sigma_A^n \\ \frac{\log \mu_\Phi([\mathbf{i}])}{\log s_{\mathbf{i}}} \in C}} s_{\mathbf{i}}^s$$

Define  $\beta : \mathbb{R} \rightarrow \mathbb{R}$  by

$$P(\beta(q)\Lambda + q\Phi) = 0,$$

for  $q \in \mathbb{R}$ . Now in addition assume that  $C$  is a closed set. Then there is a unique real number  $\mathcal{F}(C)$  such that

$$\lim_{r \rightarrow 0} \sigma_{\text{rad}}(\zeta_{B(C,r)}^{\text{dyn-gib}}(\mathcal{F}(C), \cdot)) = 1.$$

and we have

$$\mathcal{F}(C) = \dim_{\text{H}} \left\{ \mathbf{i} \in \Sigma_A^{\mathbb{N}} \mid \lim_{n \rightarrow +\infty} \frac{\log \mu_\Phi([\mathbf{i}|n])}{\log s_{\mathbf{i}|n}} \in C \right\}.$$

Proof is similar to proof of 2.3.11 and hence omitted.

### Multifractal zeta functions for ergodic averages of continuous function

**Theorem 2.3.15** (Geometric zeta function for ergodic averages of continuous function). *Let self-conformal graph-directed IFS be given by*

$$(V, E, (U_v)_{v \in V}, (X_v)_{v \in V}, (S_e)_{e \in E}, (p_e)_{e \in E}).$$

Then let us fix continuous function  $f : \Sigma_A^{\mathbb{N}} \rightarrow \mathbb{R}$ . For  $\mathbf{i} \in \Sigma_A^*$  define

$$s_{\mathbf{i}} = \sup_{\mathbf{j} \in \Sigma_A^{\mathbb{N}}} |D(S_{\mathbf{i}}\pi(\mathbf{j}))|.$$

For set  $C \subset \mathbb{R}$  let us define zeta function

$$\zeta_C^{\text{erg}}(s) = \sum_{\substack{\mathbf{i} \in \Sigma_A^* \\ \forall \mathbf{j} \in [\mathbf{i}] : \frac{1}{|\mathbf{i}|} S_{|\mathbf{i}|} f(\mathbf{j}) \in C}} s_{\mathbf{i}}^s.$$

For  $\mathbf{i} \in \Sigma_A^{\mathbb{N}}$ , define  $\Lambda(\mathbf{i}) = \log |D(S_{i_1}\pi(\mathbf{Si}))|$ .

Now in addition assume that  $C$  is a closed set. Then the following holds:

For  $r > 0$ ,

$$\lim_{r \rightarrow 0} \sigma_{ab}(\zeta_{B(C,r)}^{\text{erg}}(\cdot)) = \sup_{\substack{\mu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}}) \\ \int_{\Sigma_A^{\mathbb{N}}} f d\mu \in C}} \frac{h(\mu)}{\int_{\Sigma_A^{\mathbb{N}}} \Lambda d\mu}.$$

Specifically if  $C = \{\alpha\}$

$$\lim_{r \rightarrow 0} \sigma_{ab}(\zeta_{B(\alpha,r)}^{\text{erg}}(\cdot)) = \sup_{\substack{\mu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}}) \\ \int_{\Sigma_A^{\mathbb{N}}} f d\mu = \alpha}} \frac{h(\mu)}{\int_{\Sigma_A^{\mathbb{N}}} \Lambda d\mu}.$$

Additionally if the OSC is satisfied then

$$\begin{aligned} \lim_{r \rightarrow 0} \sigma_{ab}(\zeta_{B(C,r)}^{\text{erg}}(\cdot)) &= \sup_{\alpha \in C} \dim_{\text{H}} \pi \left\{ \mathbf{i} \in \Sigma_A^{\mathbb{N}} \mid \lim_{n \rightarrow +\infty} \frac{1}{n} S_n f(\mathbf{i}) = \alpha \right\} \\ &= \dim_{\text{H}} \pi \left\{ \mathbf{i} \in \Sigma_A^{\mathbb{N}} \mid \text{acc}_{n \rightarrow +\infty} \frac{1}{n} S_n f(\mathbf{i}) \subset C \right\}. \end{aligned}$$

Specifically if  $C = \{\alpha\}$

$$\lim_{r \rightarrow 0} \sigma_{ab}(\zeta_{B(\alpha,r)}^{\text{erg}}(\cdot)) = \dim_{\text{H}} \pi \left\{ \mathbf{i} \in \Sigma_A^{\mathbb{N}} \mid \lim_{n \rightarrow +\infty} \frac{1}{n} S_n f(\mathbf{i}) = \alpha \right\}$$

Proof.

Note that  $\zeta^{\text{dyn-erg}}(s, z)$  is exactly  $\zeta^{\text{dyn}, U}(s\Lambda, z)$ , where for  $U : \mathcal{P}(\Sigma_A^{\mathbb{N}}) \rightarrow \mathbb{R}$  we choose  $U(\mu) = \int_{\Sigma_A^{\mathbb{N}}} f d\mu$ . Hence we can use theorem 2.2.4. When OSC is satisfied we apply theorem 1.3.11, which ends our proof. ■

**Theorem 2.3.16** (Dynamical zeta function for ergodic averages of continuous function). *Let self-conformal graph-directed IFS be given by*

$$(V, E, (U_v)_{v \in V}, (X_v)_{v \in V}, (S_e)_{e \in E}, (p_e)_{e \in E}).$$

Then let us fix continuous function  $f : \Sigma_A^{\mathbb{N}} \rightarrow \mathbb{R}$ . For  $\mathbf{i} \in \Sigma_A^*$  define

$$s_{\mathbf{i}} = \sup_{\mathbf{j} \in \Sigma_A^{\mathbb{N}}} |D(S_{\mathbf{i}}\pi(\mathbf{j}))|.$$

For set  $C \subset \mathbb{R}$  let us define zeta function

$$\zeta_C^{\text{dyn-erg}}(s, z) = \sum_{n=1}^{+\infty} \frac{z^n}{n} \sum_{\substack{\mathbf{i} \in \Sigma_A^n \\ \forall \mathbf{j} \in [\mathbf{i}] : \frac{1}{n} S_n f(\mathbf{j}) \in C}} s_{\mathbf{i}}^s.$$

For  $\mathbf{i} \in \Sigma_A^{\mathbb{N}}$ , define  $\Lambda(\mathbf{i}) = \log |D(S_{i_1}\pi(\mathbf{Si}))|$ . Now in addition assume that  $C$  is a closed set. Then the following holds:

1. There is a unique real number  $\mathcal{F}(C)$  such that

$$\lim_{r \rightarrow 0} \sigma_{\text{rad}}(\zeta_{B(C,r)}^{\text{dyn-erg}}(\mathcal{F}(C), \cdot)) = 1.$$

and we have

$$\mathcal{F}(C) = \sup_{\substack{\mu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}}) \\ \int_{\Sigma_A^{\mathbb{N}}} f d\mu \in C}} - \frac{h(\mu)}{\int_{\Sigma_A^{\mathbb{N}}} \Lambda d\mu}.$$

Additionally if the OSC is satisfied then we have

$$\begin{aligned} \mathcal{F}(C) &= \sup_{\alpha \in C} \dim_{\text{H}} \pi \left\{ \mathbf{i} \in \Sigma_A^{\mathbb{N}} \mid \lim_{n \rightarrow +\infty} \frac{1}{n} S_n f(\mathbf{i}) = \alpha \right\} \\ &= \dim_{\text{H}} \pi \left\{ \mathbf{i} \in \Sigma_A^{\mathbb{N}} \mid \text{acc}_{n \rightarrow +\infty} \frac{1}{n} S_n f(\mathbf{i}) \subset C \right\}. \end{aligned}$$

2. If  $C$  is convex, closed and  $\text{int}C \cap \left\{ \int_{\Sigma_A^{\mathbb{N}}} f d\mu \mid \mu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}}) \right\} \neq \emptyset$  then we have

$$\sigma_{\text{rad}}(\zeta_C^{\text{dyn-erg}}(\mathcal{F}(C), \cdot)) = 1.$$

and we have

$$\mathcal{F}(C) = \sup_{\substack{\mu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}}) \\ \int_{\Sigma_A^{\mathbb{N}}} f d\mu \in C}} - \frac{h(\mu)}{\int_{\Sigma_A^{\mathbb{N}}} \Lambda d\mu}.$$

Additionally if the OSC is satisfied then we have

$$\begin{aligned} \mathcal{F}(C) &= \sup_{\alpha \in C} \dim_{\text{H}} \pi \left\{ \mathbf{i} \in \Sigma_A^{\mathbb{N}} \mid \lim_{n \rightarrow +\infty} \frac{1}{n} S_n f(\mathbf{i}) = \alpha \right\} \\ &= \dim_{\text{H}} \pi \left\{ \mathbf{i} \in \Sigma_A^{\mathbb{N}} \mid \text{acc}_{n \rightarrow +\infty} \frac{1}{n} S_n f(\mathbf{i}) \subset C \right\}. \end{aligned}$$

Proof.

Note that  $\zeta^{\text{dyn-erg}}(s, z)$  is exactly  $\zeta^{\text{dyn}, U}(s\Lambda, z)$ , where for  $U : \mathcal{P}(\Sigma_A^{\mathbb{N}}) \rightarrow \mathbb{R}$  we choose  $U(\mu) = \int_{\Sigma_A^{\mathbb{N}}} f d\mu$ .

Hence we can use theorems 2.2.14 and 2.2.16. When OSC is satisfied we apply theorem 1.3.11, which ends our proof. ■

### Multifractal zeta functions for mixed ergodic averages continuous functions

**Theorem 2.3.17** (Geometric zeta function for mixed ergodic averages of continuous functions). *Let self-conformal graph-directed IFS be given*

$$(V, E, (U_v)_{v \in V}, (X_v)_{v \in V}, (S_e)_{e \in E}, (p_e)_{e \in E}).$$

Let us fix  $M \in \mathbb{N}$ . Then for each  $m$ ,  $1 \leq m \leq M$  let us fix continuous function  $f_m : \Sigma_A^{\mathbb{N}} \rightarrow \mathbb{R}$ . For  $\mathbf{i} \in \Sigma_A^*$  define

$$s_{\mathbf{i}} = \sup_{\mathbf{j} \in \Sigma_A^{\mathbb{N}}} |D(S_{\mathbf{i}}\pi(\mathbf{j}))|.$$

For set  $C \subset \mathbb{R}^M$  let us define zeta function

$$\zeta_C^{\text{erg}}(s) = \sum_{\mathbf{i} \in \Sigma_A^*} s_{\mathbf{i}}^s \cdot \left( \frac{1}{|\mathbf{i}|} S_{|\mathbf{i}|} f_1(\bar{\mathbf{i}}), \dots, \frac{1}{|\mathbf{i}|} S_{|\mathbf{i}|} f_M(\bar{\mathbf{i}}) \right) \in C$$

For  $\mathbf{i} \in \Sigma_A^{\mathbb{N}}$ , define  $\Lambda(\mathbf{i}) = \log |D(S_{i_1}\pi(S\mathbf{i}))|$ .

Now in addition assume that  $C$  is a closed set. Then the following holds:

For  $r > 0$ ,

$$\lim_{r \rightarrow 0} \sigma_{ab}(\zeta_{B(C, r)}^{\text{erg}}(\cdot)) = \sup_{\mu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}})} - \frac{h(\mu)}{\int_{\Sigma_A^{\mathbb{N}}} \Lambda d\mu} \cdot \left( \int_{\Sigma_A^{\mathbb{N}}} f_1 d\mu, \dots, \int_{\Sigma_A^{\mathbb{N}}} f_M d\mu \right) \in C$$

Specifically if  $C = \{\alpha\}$

$$\lim_{r \rightarrow 0} \sigma_{ab}(\zeta_{B(\alpha, r)}^{\text{erg}}(\cdot)) = \sup_{\mu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}})} - \frac{h(\mu)}{\int_{\Sigma_A^{\mathbb{N}}} \Lambda d\mu} \cdot \left( \int_{\Sigma_A^{\mathbb{N}}} f_1 d\mu, \dots, \int_{\Sigma_A^{\mathbb{N}}} f_M d\mu \right) = \alpha$$

Additionally if the OSC is satisfied then

$$\begin{aligned} \lim_{r \rightarrow 0} \sigma_{ab}(\zeta_{B(C,r)}^{\text{erg}}(\cdot)) &= \sup_{\alpha \in C} \dim_H \pi \left\{ \mathbf{i} \in \Sigma_A^{\mathbb{N}} \mid \lim_{n \rightarrow +\infty} \left( \frac{1}{n} S_n f_1(\mathbf{i}), \dots, \frac{1}{n} S_n f_M(\mathbf{i}) \right) = \alpha \right\} \\ &= \dim_H \pi \left\{ \mathbf{i} \in \Sigma_A^{\mathbb{N}} \mid \lim_{n \rightarrow +\infty} \left( \frac{1}{n} S_n f_1(\mathbf{i}), \dots, \frac{1}{n} S_n f_M(\mathbf{i}) \right) \subset C \right\}. \end{aligned}$$

Proof.

Note that  $\zeta^{\text{erg}}$  is exactly  $\zeta^{\text{dyn}, U}$ , where for  $U : \mathcal{P}(\Sigma^{\mathbb{N}}) \rightarrow \mathbb{R}^M$  we choose

$$U(\mu) = \left( \int_{\Sigma_A^{\mathbb{N}}} f_1 d\mu, \int_{\Sigma_A^{\mathbb{N}}} f_2 d\mu, \dots, \int_{\Sigma_A^{\mathbb{N}}} f_M d\mu \right).$$

Hence we can use theorem 2.2.4. When OSC is satisfied we apply theorem 1.3.12, which ends our proof. ■

**Theorem 2.3.18** (Dynamical zeta function for mixed ergodic averages of continuous functions). *Let self-conformal graph-directed IFS be given*

$$(V, E, (U_v)_{v \in V}, (X_v)_{v \in V}, (S_e)_{e \in E}, (p_e)_{e \in E}).$$

Let us fix  $M \in \mathbb{N}$ . Then for each  $m$ ,  $1 \leq m \leq M$  let us fix continuous function  $f_m : \Sigma_A^{\mathbb{N}} \rightarrow \mathbb{R}$ . For  $\mathbf{i} \in \Sigma_A^*$  define

$$s_{\mathbf{i}} = \sup_{\mathbf{j} \in \Sigma_A^{\mathbb{N}}} |D(S_{\mathbf{i}} \pi(\mathbf{j}))|.$$

For set  $C \subset \mathbb{R}^M$  let us define zeta function

$$\zeta_C^{\text{dyn-erg}}(s, z) = \sum_{n=1}^{+\infty} \frac{z^n}{n} \sum_{\substack{\mathbf{i} \in \Sigma_A^n \\ \mathbf{j} \in [\mathbf{i}] : \left( \frac{1}{n} S_n f_1(\mathbf{j}), \dots, \frac{1}{n} S_n f_M(\mathbf{j}) \right) \in C}} s_{\mathbf{i}}.$$

For  $\mathbf{i} \in \Sigma_A^{\mathbb{N}}$ , define  $\Lambda(\mathbf{i}) = \log |D(S_{\mathbf{i}} \pi(\mathbf{S}\mathbf{i}))|$ . Now in addition assume that  $C$  is a closed set. Then the following holds:

1. There is a unique real number  $\mathcal{F}(C)$  such that

$$\lim_{r \rightarrow 0} \sigma_{\text{rad}}(\zeta_{B(C,r)}^{\text{dyn-erg}}(\mathcal{F}(C), \cdot)) = 1.$$

and we have

$$\mathcal{F}(C) = \sup_{\substack{\mu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}}) \\ \left( \int_{\Sigma_A^{\mathbb{N}}} f_1 d\mu, \dots, \int_{\Sigma_A^{\mathbb{N}}} f_M d\mu \right) \in C}} - \frac{h(\mu)}{\int_{\Sigma_A^{\mathbb{N}}} \Lambda d\mu}.$$

Additionally if the OSC is satisfied then we have

$$\begin{aligned} \mathcal{F}(C) &= \sup_{\alpha \in C} \dim_H \pi \left\{ \mathbf{i} \in \Sigma_A^{\mathbb{N}} \mid \lim_{n \rightarrow +\infty} \left( \frac{1}{n} S_n f_1(\mathbf{i}), \dots, \frac{1}{n} S_n f_M(\mathbf{i}) \right) = \alpha \right\} \\ &= \dim_H \pi \left\{ \mathbf{i} \in \Sigma_A^{\mathbb{N}} \mid \text{acc}_{n \rightarrow +\infty} \left( \frac{1}{n} S_n f_1(\mathbf{i}), \dots, \frac{1}{n} S_n f_M(\mathbf{i}) \right) \subset C \right\}. \end{aligned}$$



2. If  $C$  is convex, closed and  $\text{int}C \cap \left\{ \left( \int_{\Sigma_A^{\mathbb{N}}} f_1 d\mu, \dots, \int_{\Sigma_A^{\mathbb{N}}} f_M d\mu \right) \mid \mu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}}) \right\} \neq \emptyset$  then we have

$$\sigma_{\text{rad}}(\zeta_C^{\text{dyn-erg}}(\mathcal{F}(C), \cdot)) = 1.$$

and we have

$$\mathcal{F}(C) = \sup_{\substack{\mu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}}) \\ \left( \int_{\Sigma_A^{\mathbb{N}}} f_1 d\mu, \dots, \int_{\Sigma_A^{\mathbb{N}}} f_M d\mu \right) \in C}} - \frac{h(\mu)}{\int_{\Sigma_A^{\mathbb{N}}} \Lambda d\mu}.$$

Additionally if the OSC is satisfied then we have

$$\begin{aligned} \mathcal{F}(C) &= \sup_{\alpha \in C} \dim_{\text{H}} \pi \left\{ \mathbf{i} \in \Sigma_A^{\mathbb{N}} \mid \lim_{n \rightarrow +\infty} \left( \frac{1}{n} S_n f_1(\mathbf{i}), \dots, \frac{1}{n} S_n f_M(\mathbf{i}) \right) = \alpha \right\} \\ &= \dim_{\text{H}} \pi \left\{ \mathbf{i} \in \Sigma_A^{\mathbb{N}} \mid \text{acc}_{n \rightarrow +\infty} \left( \frac{1}{n} S_n f_1(\mathbf{i}), \dots, \frac{1}{n} S_n f_M(\mathbf{i}) \right) \subset C \right\}. \end{aligned}$$

Note that  $\zeta^{\text{dyn-erg}}(s, z)$  is exactly  $\zeta^{\text{dyn}, U}(s\Lambda, z)$ , where for  $U : \mathcal{P}(\Sigma^{\mathbb{N}}) \rightarrow \mathbb{R}^M$  we choose

$$U(\mu) = \left( \int_{\Sigma_A^{\mathbb{N}}} f_1 d\mu, \int_{\Sigma_A^{\mathbb{N}}} f_2 d\mu, \dots, \int_{\Sigma_A^{\mathbb{N}}} f_M d\mu \right).$$

Hence we can use theorems 2.2.14 and 2.2.16. When OSC is satisfied we apply theorem 1.3.12 which ends our proof. ■

### Multifractal zeta functions for relative ergodic averages of continuous functions

**Theorem 2.3.19** (Geometric zeta function for mixed relative ergodic averages with exponent of continuous functions). *Let self-conformal graph-directed IFS be given*

$$(V, E, (U_v)_{v \in V}, (X_v)_{v \in V}, (S_e)_{e \in E}, (p_e)_{e \in E}).$$

Let us fix  $M \in \mathbb{N}$ . Then let us fix continuous function  $f_1, f_2, \dots, f_M : \Sigma_A^{\mathbb{N}} \rightarrow \mathbb{R}$  and  $g_1, g_2, \dots, g_M : \Sigma_A^{\mathbb{N}} \rightarrow \mathbb{R}^+$ . For  $\mathbf{i} \in \Sigma_A^*$  define

$$s_{\mathbf{i}} = \sup_{\mathbf{j} \in \Sigma_A^{\mathbb{N}}} |D(S_{\mathbf{i}}\pi(\mathbf{j}))|.$$

For set  $C \subset \mathbb{R}^M$  let us define zeta function

$$\zeta_C^{\text{erg}}(s) = \sum_{\mathbf{i} \in \Sigma_A^*} s_{\mathbf{i}}^s.$$

$$\forall \mathbf{j} \in [\mathbf{i}] : \left( \frac{S_{[\mathbf{i}]} f_1(\mathbf{j})}{S_{[\mathbf{i}]} g_1(\mathbf{j})}, \dots, \frac{S_{[\mathbf{i}]} f_M(\mathbf{j})}{S_{[\mathbf{i}]} g_M(\mathbf{j})} \right) \in C$$

For  $\mathbf{i} \in \Sigma_A^{\mathbb{N}}$ , define  $\Lambda(\mathbf{i}) = \log |D(S_{i_1}\pi(\mathbf{S}\mathbf{i}))|$ .

Now in addition assume that  $C$  is a closed set. Then the following holds:

1. For  $r > 0$ ,

$$\lim_{r \rightarrow 0} \sigma_{ab}(\zeta_{B(C, r)}^{\text{erg}}(\cdot)) = \sup_{\substack{\mu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}}) \\ \left( \frac{\int_{\Sigma_A^{\mathbb{N}}} f_1 d\mu}{\int_{\Sigma_A^{\mathbb{N}}} g_1 d\mu}, \dots, \frac{\int_{\Sigma_A^{\mathbb{N}}} f_M d\mu}{\int_{\Sigma_A^{\mathbb{N}}} g_M d\mu} \right) \in C}} - \frac{h(\mu)}{\int_{\Sigma_A^{\mathbb{N}}} \Lambda d\mu}.$$

Specifically if  $C = \{\alpha\}$

$$\lim_{r \rightarrow 0} \sigma_{ab}(\zeta_{B(\alpha, r)}^{\text{erg}}(\cdot)) = \sup_{\mu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}})} - \frac{h(\mu)}{\int_{\Sigma_A^{\mathbb{N}}} \Lambda d\mu} \cdot \left( \frac{\int_{\Sigma_A^{\mathbb{N}}} f_1 d\mu}{\int_{\Sigma_A^{\mathbb{N}}} g_1 d\mu}, \dots, \frac{\int_{\Sigma_A^{\mathbb{N}}} f_M d\mu}{\int_{\Sigma_A^{\mathbb{N}}} g_M d\mu} \right) = \alpha$$

Additionally if the OSC is satisfied then

$$\begin{aligned} \lim_{r \rightarrow 0} \sigma_{ab}(\zeta_{B(C, r)}^{\text{erg}}(\cdot)) &= \sup_{\alpha \in C} \dim_{\mathbb{H}} \pi \left\{ \mathbf{i} \in \Sigma_A^{\mathbb{N}} \mid \lim_{n \rightarrow +\infty} \left( \frac{S_n f_1(\mathbf{i})}{S_n g_1(\mathbf{i})}, \dots, \frac{S_n f_M(\mathbf{i})}{S_n g_M(\mathbf{i})} \right) = \alpha \right\} \\ &= \dim_{\mathbb{H}} \pi \left\{ \mathbf{i} \in \Sigma_A^{\mathbb{N}} \mid \text{acc}_{n \rightarrow +\infty} \left( \frac{S_n f_1(\mathbf{i})}{S_n g_1(\mathbf{i})}, \dots, \frac{S_n f_M(\mathbf{i})}{S_n g_M(\mathbf{i})} \right) \subset C \right\}. \end{aligned}$$

2. If  $C$  is convex, closed and  $\text{int} C \cap \left\{ \left( \frac{\int_{\Sigma_A^{\mathbb{N}}} f_1 d\mu}{\int_{\Sigma_A^{\mathbb{N}}} g_1 d\mu}, \dots, \frac{\int_{\Sigma_A^{\mathbb{N}}} f_M d\mu}{\int_{\Sigma_A^{\mathbb{N}}} g_M d\mu} \right) \mid \mu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}}) \right\} \neq \emptyset$  then we have

$$\sigma_{ab}(\zeta_C^{\text{erg}}(\cdot)) = \sup_{\mu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}})} - \frac{h(\mu)}{\int_{\Sigma_A^{\mathbb{N}}} \Lambda d\mu} \cdot \left( \frac{\int_{\Sigma_A^{\mathbb{N}}} f_1 d\mu}{\int_{\Sigma_A^{\mathbb{N}}} g_1 d\mu}, \dots, \frac{\int_{\Sigma_A^{\mathbb{N}}} f_M d\mu}{\int_{\Sigma_A^{\mathbb{N}}} g_M d\mu} \right) \in C$$

In addition if OSC is satisfied we have

$$\begin{aligned} \sigma_{ab}(\zeta_C^{\text{erg}}(\cdot)) &= \sup_{\alpha \in C} \dim_{\mathbb{H}} \pi \left\{ \mathbf{i} \in \Sigma_A^{\mathbb{N}} \mid \lim_{n \rightarrow +\infty} \left( \frac{S_n f_1(\mathbf{i})}{S_n g_1(\mathbf{i})}, \dots, \frac{S_n f_M(\mathbf{i})}{S_n g_M(\mathbf{i})} \right) = \alpha \right\} \\ &= \dim_{\mathbb{H}} \pi \left\{ \mathbf{i} \in \Sigma_A^{\mathbb{N}} \mid \text{acc}_{n \rightarrow +\infty} \left( \frac{S_n f_1(\mathbf{i})}{S_n g_1(\mathbf{i})}, \dots, \frac{S_n f_M(\mathbf{i})}{S_n g_M(\mathbf{i})} \right) \subset C \right\}. \end{aligned}$$

Note that  $\zeta^{\text{erg}}$  is exactly  $\zeta^{U, \Lambda}$ , where for  $U : \mathcal{P}(\Sigma^{\mathbb{N}}) \rightarrow \mathbb{R}^M$  we choose

$$U = \left( \frac{\int_{\Sigma_A^{\mathbb{N}}} f_1 d\mu}{\int_{\Sigma_A^{\mathbb{N}}} g_1 d\mu}, \dots, \frac{\int_{\Sigma_A^{\mathbb{N}}} f_M d\mu}{\int_{\Sigma_A^{\mathbb{N}}} g_M d\mu} \right).$$

Hence we can use theorem 2.2.4. When OSC is satisfied the rest of the proof follows from [43]. ■

**Theorem 2.3.20** (Dynamical zeta function for mixed relative ergodic averages with exponent of continuous functions). *Let self-conformal graph-directed IFS be given*

$$(V, E, (U_v)_{v \in V}, (X_v)_{v \in V}, (S_e)_{e \in E}, (p_e)_{e \in E}).$$

Let us fix  $M \in \mathbb{N}$ . Then for each  $m$ ,  $1 \leq m \leq M$  let us fix continuous function  $f_m : \Sigma_A^{\mathbb{N}} \rightarrow \mathbb{R}$ . For  $\mathbf{i} \in \Sigma_A^*$  define

$$s_{\mathbf{i}} = \sup_{\mathbf{j} \in \Sigma_A^{\mathbb{N}}} |D(S_{\mathbf{i}} \pi(\mathbf{j}))|.$$

For set  $C \subset \mathbb{R}^M$  let us define zeta function

$$\zeta_C^{\text{dyn-erg}}(s, z) = \sum_{n=1}^{+\infty} \frac{z^n}{n} \sum_{\mathbf{i} \in \Sigma_A^n} s_{\mathbf{i}}^s \cdot \prod_{\mathbf{j} \in [\mathbf{i}] : \left( \frac{S_n f_1(\mathbf{j})}{S_n g_1(\mathbf{j})}, \dots, \frac{S_n f_M(\mathbf{j})}{S_n g_M(\mathbf{j})} \right) \in C}$$

For  $\mathbf{i} \in \Sigma_A^{\mathbb{N}}$ , define  $\Lambda(\mathbf{i}) = \log |D(S_{\mathbf{i}} \pi(\mathbf{S}\mathbf{i}))|$ . Now in addition assume that  $C$  is a closed set. Then the following holds:

1. There is a unique real number  $\mathcal{F}(C)$  such that

$$\lim_{r \rightarrow 0} \sigma_{rad}(\zeta_{B(C,r)}^{\text{dyn-erg}}(\mathcal{F}(C), \cdot)) = 1.$$

and we have

$$\mathcal{F}(C) = \sup_{\mu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}})} - \frac{h(\mu)}{\int_{\Sigma_A^{\mathbb{N}}} \Lambda d\mu} \cdot \left( \frac{\int_{\Sigma_A^{\mathbb{N}}} f_1 d\mu}{\int_{\Sigma_A^{\mathbb{N}}} g_1 d\mu}, \dots, \frac{\int_{\Sigma_A^{\mathbb{N}}} f_M d\mu}{\int_{\Sigma_A^{\mathbb{N}}} g_M d\mu} \right) \in C$$

Additionally if the OSC is satisfied then we have

$$\begin{aligned} \mathcal{F}(C) &= \sup_{\alpha \in C} \dim_{\mathbb{H}} \pi \left\{ \mathbf{i} \in \Sigma_A^{\mathbb{N}} \mid \lim_{n \rightarrow +\infty} \left( \frac{S_n f_1(\mathbf{i})}{S_n g_1(\mathbf{i})}, \dots, \frac{S_n f_M(\mathbf{i})}{S_n g_M(\mathbf{i})} \right) = \alpha \right\} \\ &= \dim_{\mathbb{H}} \pi \left\{ \mathbf{i} \in \Sigma_A^{\mathbb{N}} \mid \text{acc}_{n \rightarrow +\infty} \left( \frac{S_n f_1(\mathbf{i})}{S_n g_1(\mathbf{i})}, \dots, \frac{S_n f_M(\mathbf{i})}{S_n g_M(\mathbf{i})} \right) \subset C \right\}. \end{aligned}$$

2. If  $C$  is convex, closed and  $\text{int}C \cap \left\{ \left( \frac{\int_{\Sigma_A^{\mathbb{N}}} f_1 d\mu}{\int_{\Sigma_A^{\mathbb{N}}} g_1 d\mu}, \dots, \frac{\int_{\Sigma_A^{\mathbb{N}}} f_M d\mu}{\int_{\Sigma_A^{\mathbb{N}}} g_M d\mu} \right) \mid \mu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}}) \right\} \neq \emptyset$  then we have

$$\sigma_{rad}(\zeta_C^{\text{dyn-erg}}(\mathcal{F}(C), \cdot)) = 1.$$

and we have

$$\mathcal{F}(C) = \sup_{\mu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}})} - \frac{h(\mu)}{\int_{\Sigma_A^{\mathbb{N}}} \Lambda d\mu} \cdot \left( \frac{\int_{\Sigma_A^{\mathbb{N}}} f_1 d\mu}{\int_{\Sigma_A^{\mathbb{N}}} g_1 d\mu}, \dots, \frac{\int_{\Sigma_A^{\mathbb{N}}} f_M d\mu}{\int_{\Sigma_A^{\mathbb{N}}} g_M d\mu} \right) \in C$$

Additionally if the OSC is satisfied then we have

$$\begin{aligned} \mathcal{F}(C) &= \sup_{\alpha \in C} \dim_{\mathbb{H}} \pi \left\{ \mathbf{i} \in \Sigma_A^{\mathbb{N}} \mid \lim_{n \rightarrow +\infty} \left( \frac{S_n f_1(\mathbf{i})}{S_n g_1(\mathbf{i})}, \dots, \frac{S_n f_M(\mathbf{i})}{S_n g_M(\mathbf{i})} \right) = \alpha \right\} \\ &= \dim_{\mathbb{H}} \pi \left\{ \mathbf{i} \in \Sigma_A^{\mathbb{N}} \mid \text{acc}_{n \rightarrow +\infty} \left( \frac{S_n f_1(\mathbf{i})}{S_n g_1(\mathbf{i})}, \dots, \frac{S_n f_M(\mathbf{i})}{S_n g_M(\mathbf{i})} \right) \subset C \right\}. \end{aligned}$$

Proof.

Note that  $\zeta^{\text{dyn-erg}}(s, z)$  is exactly  $\zeta^{\text{dyn}, U}(s\Lambda, z)$ , where for  $U : \mathcal{P}(\Sigma^{\mathbb{N}}) \rightarrow \mathbb{R}^M$  we choose

$$U = \left( \frac{\int_{\Sigma_A^{\mathbb{N}}} f_1 d\mu}{\int_{\Sigma_A^{\mathbb{N}}} g_1 d\mu}, \dots, \frac{\int_{\Sigma_A^{\mathbb{N}}} f_M d\mu}{\int_{\Sigma_A^{\mathbb{N}}} g_M d\mu} \right).$$

Hence we can use theorems 2.2.14 and 2.2.16. When OSC is satisfied the rest of the proof follows from [43]. ■.

### Multifractal zeta functions for relative ergodic averages with exponent of continuous functions

**Theorem 2.3.21** (Geometric zeta function for mixed relative ergodic averages with exponents of continuous functions). *Let  $\Sigma_A^{\mathbb{N}}$  be given. Let  $\Lambda \in H(\Sigma_A^{\mathbb{N}})$ ,  $\Lambda < 0$ . Fix metric  $d_{\Lambda}$ . Let Hausdorff dimension on  $\Sigma_A^{\mathbb{N}}$  be defined with respect to  $d_{\Lambda}$ . Let us fix  $M \in \mathbb{N}$ . Then let us fix continuous function*

$f_1, f_2, \dots, f_M, g_1, g_2, \dots, g_M : \Sigma_A^{\mathbb{N}} \rightarrow \mathbb{R}^+$ . And fix numbers  $t_1, t_2, \dots, t_M, a_1, a_2, \dots, a_M > 0$ . Let us  $s_{\mathbf{i}}(\Lambda)$  denote as  $s_{\mathbf{i}}$ . For set  $C \subset \mathbb{R}^M$  let us define zeta function

$$\zeta_C^{\text{erg}}(s) = \sum_{\mathbf{i} \in \Sigma_A^*} s_{\mathbf{i}}^s \cdot \forall \mathbf{j} \in [\mathbf{i}] : \left( \frac{(s_{[\mathbf{i}]} f_1(\mathbf{j}))^{t_1}}{(s_{[\mathbf{i}]} g_1(\mathbf{j}))^{a_1}}, \dots, \frac{(s_{[\mathbf{i}]} f_M(\mathbf{j}))^{t_M}}{(s_{[\mathbf{i}]} g_M(\mathbf{j}))^{a_M}} \right) \in C$$

For  $\mathbf{i} \in \Sigma_A^{\mathbb{N}}$ , define  $\Lambda(\mathbf{i}) = \log |D(S_{i_1} \pi(S\mathbf{i}))|$ .

Now in addition assume that  $C$  is a closed set. Then the following holds:

For  $r > 0$ ,

$$\lim_{r \rightarrow 0} \sigma_{ab}(\zeta_{B(C,r)}^{\text{erg}}(\cdot)) = \sup_{\mu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}})} - \frac{h(\mu)}{\int_{\Sigma_A^{\mathbb{N}}} \Lambda d\mu} \cdot \left( \frac{(\int_{\Sigma_A^{\mathbb{N}}} f_1 d\mu)^{t_1}}{(\int_{\Sigma_A^{\mathbb{N}}} g_1 d\mu)^{a_1}}, \dots, \frac{(\int_{\Sigma_A^{\mathbb{N}}} f_M d\mu)^{t_M}}{(\int_{\Sigma_A^{\mathbb{N}}} g_M d\mu)^{a_M}} \right) \in C$$

Specifically if  $C = \{\alpha\}$

$$\lim_{r \rightarrow 0} \sigma_{ab}(\zeta_{B(\alpha,r)}^{\text{erg}}(\cdot)) = \sup_{\mu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}})} - \frac{h(\mu)}{\int_{\Sigma_A^{\mathbb{N}}} \Lambda d\mu} \cdot \left( \frac{(\int_{\Sigma_A^{\mathbb{N}}} f_1 d\mu)^{t_1}}{(\int_{\Sigma_A^{\mathbb{N}}} g_1 d\mu)^{a_1}}, \dots, \frac{(\int_{\Sigma_A^{\mathbb{N}}} f_M d\mu)^{t_M}}{(\int_{\Sigma_A^{\mathbb{N}}} g_M d\mu)^{a_M}} \right) = \alpha$$

Additionally,

$$\lim_{r \rightarrow 0} \sigma_{ab}(\zeta_{B(C,r)}^{\text{erg}}(\cdot)) = \dim_{\mathbb{H}} \left\{ \mathbf{i} \in \Sigma_A^{\mathbb{N}} \mid \text{acc}_{n \rightarrow +\infty} \left( \frac{(S_n f_1(\mathbf{i}))^{t_1}}{(S_n g_1(\mathbf{i}))^{a_1}}, \dots, \frac{(S_n f_M(\mathbf{i}))^{t_M}}{(S_n g_M(\mathbf{i}))^{a_M}} \right) \subset C \right\}.$$

Proof.

Result follows directly from 2.2.4 by choosing  $U : \mathcal{P}(\Sigma_A^{\mathbb{N}}) \rightarrow \mathbb{R}^M$  to be

$$U(\mu) = \left( \frac{(\int_{\Sigma_A^{\mathbb{N}}} f_1 d\mu)^{t_1}}{(\int_{\Sigma_A^{\mathbb{N}}} g_1 d\mu)^{a_1}}, \dots, \frac{(\int_{\Sigma_A^{\mathbb{N}}} f_M d\mu)^{t_M}}{(\int_{\Sigma_A^{\mathbb{N}}} g_M d\mu)^{a_M}} \right).$$

And then using 2.1.2 we get our result. ■

**Theorem 2.3.22** (Dynamical zeta function for mixed relative ergodic averages with exponents of continuous functions). *Let  $\Sigma_A^{\mathbb{N}}$  be given. Let  $\Lambda \in H(\Sigma_A^{\mathbb{N}})$ ,  $\Lambda < 0$ . Fix metric  $d_{\Lambda}$ . Let Hausdorff dimension on  $\Sigma_A^{\mathbb{N}}$  be defined with respect to  $d_{\Lambda}$ . Let us fix  $M \in \mathbb{N}$ . Then let us fix continuous function  $f_1, f_2, \dots, f_M, g_1, g_2, \dots, g_M : \Sigma_A^{\mathbb{N}} \rightarrow \mathbb{R}^+$ . And fix numbers  $t_1, t_2, \dots, t_M, a_1, a_2, \dots, a_M > 0$ . Let us  $s_{\mathbf{i}}(\Lambda)$  denote as  $s_{\mathbf{i}}$ . For set  $C \subset \mathbb{R}^M$  let us define zeta function*

$$\zeta_C^{\text{dyn-erg}}(s, z) = \sum_{n=1}^{+\infty} \frac{z^n}{n} \sum_{\mathbf{i} \in \Sigma_A^n} s_{\mathbf{i}}^s \cdot \forall \mathbf{j} \in [\mathbf{i}] : \left( \frac{(s_n f_1(\mathbf{j}))^{t_1}}{(s_n g_1(\mathbf{j}))^{a_1}}, \dots, \frac{(s_n f_M(\mathbf{j}))^{t_M}}{(s_n g_M(\mathbf{j}))^{a_M}} \right) \in C$$

For  $\mathbf{i} \in \Sigma_A^{\mathbb{N}}$ , define  $\Lambda(\mathbf{i}) = \log |D(S_{i_1} \pi(S\mathbf{i}))|$ . Now in addition assume that  $C$  is a closed set. Then there is a unique real number  $\mathcal{F}(C)$  such that

$$\lim_{r \rightarrow 0} \sigma_{rad}(\zeta_{B(C,r)}^{\text{dyn-erg}}(\mathcal{F}(C), \cdot)) = 1.$$

and we have

$$\mathcal{F}(C) = \sup_{\mu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}})} - \frac{h(\mu)}{\int_{\Sigma_A^{\mathbb{N}}} \Lambda d\mu} \cdot \left( \frac{(\int_{\Sigma_A^{\mathbb{N}}} f_1 d\mu)^{t_1}}{(\int_{\Sigma_A^{\mathbb{N}}} g_1 d\mu)^{a_1}}, \dots, \frac{(\int_{\Sigma_A^{\mathbb{N}}} f_M d\mu)^{t_M}}{(\int_{\Sigma_A^{\mathbb{N}}} g_M d\mu)^{a_M}} \right) \in C$$

Additionally,

$$\mathcal{F}(C) = \dim_{\mathbb{H}} \left\{ \mathbf{i} \in \Sigma_A^{\mathbb{N}} \mid \text{acc}_{n \rightarrow +\infty} \left( \frac{(S_n f_1(\mathbf{i}))^{t_1}}{(S_n g_1(\mathbf{i}))^{a_1}}, \dots, \frac{(S_n f_M(\mathbf{i}))^{t_M}}{(S_n g_M(\mathbf{i}))^{a_M}} \right) \subset C \right\}.$$

Proof.

The result follows directly from 2.2.14 by choosing  $U : \mathcal{P}(\Sigma_A^{\mathbb{N}}) \rightarrow \mathbb{R}^M$  to be

$$U(\mu) = \left( \frac{(\int_{\Sigma_A^{\mathbb{N}}} f_1 d\mu)^{t_1}}{(\int_{\Sigma_A^{\mathbb{N}}} g_1 d\mu)^{a_1}}, \dots, \frac{(\int_{\Sigma_A^{\mathbb{N}}} f_M d\mu)^{t_M}}{(\int_{\Sigma_A^{\mathbb{N}}} g_M d\mu)^{a_M}} \right).$$

Then using 2.1.2 we get our result. ■.



# Chapter 3

## Proofs

### 3.0.1 Hölder function on Subshift of finite type

In following two proposition we will state properties of Hölder functions and of  $s_i$  that will be often used through the chapter.

**Proposition 3.0.1.** *Let  $\phi \in H(\Sigma_A^{\mathbb{N}})$ ,  $\phi < 0$  then the following holds*

1.  $(\exists c_{min}, c_{max} \in \mathbb{R}), -\infty < c_{min} \leq \phi \leq c_{max} < 0$ .

2. *There is  $M > 0$  such that for each  $\mathbf{i}, \mathbf{j} \in \Sigma_A^{\mathbb{N}}$  we have*

$$|\mathbf{i}|_n = |\mathbf{j}|_n \Rightarrow M^{-1} \leq \frac{\exp(\sum_{k=0}^{n-1} \phi(S^k \mathbf{i}))}{\exp(\sum_{k=0}^{n-1} \phi(S^k \mathbf{j}))} \leq M$$

1 follows from  $\phi$  being continuous and  $\Sigma_A^{\mathbb{N}}$  being compact hence  $\phi$  reaches its maximum and minimum on  $\Sigma_A^{\mathbb{N}}$ .

2 is written in the form which is appropriate due to fact that functions that we work with are often logarithms of derivatives. But it could be rewritten as

$$-\log M \leq \sum_{k=0}^{n-1} (\phi(S^k \mathbf{i}) - \phi(S^k \mathbf{j})) \leq \log M$$

Then from  $|\mathbf{i}|_n = |\mathbf{j}|_n$  and  $\phi \in H(\Sigma_A^{\mathbb{N}})$  we have  $|\phi(S^k \mathbf{i}) - \phi(S^k \mathbf{j})| \leq \gamma^{n\alpha}$ . Hence

$$\begin{aligned} \left| \sum_{k=0}^{n-1} (\phi(S^k \mathbf{i}) - \phi(S^k \mathbf{j})) \right| &\leq \sum_{k=0}^{n-1} |\phi(S^k \mathbf{i}) - \phi(S^k \mathbf{j})| \\ &\leq \sum_{k=0}^{+\infty} \gamma^{k\alpha} < +\infty \end{aligned} \quad \text{due to } \gamma^\alpha < 1.$$

So if we choose  $M = \log \sum_{k=0}^{+\infty} \gamma^{k\alpha}$  we got desired formula. Now will investigate some properties of  $s_i(\phi)$ .

**Proposition 3.0.2.** *Let  $\phi \in H(\Sigma_A^{\mathbb{N}})$ ,  $\phi < 0$ . Let  $c_{min}, c_{max}, M$  be from 3.0.1. Let  $s_{min} = e^{c_{min}}, s_{max} = e^{c_{max}}$ . Then we have:*

1. For each  $\mathbf{i} \in \Sigma_A^*$  we have  $0 < s_{\min}^{|\mathbf{i}|} \leq s_{\mathbf{i}} \leq s_{\max}^{|\mathbf{i}|} < 1$
2. For each  $\mathbf{i}, \mathbf{j} \in \Sigma_A^*$ ,  $\mathbf{ij} \in \Sigma_A^*$  we have  $s_{\mathbf{ij}} \leq s_{\mathbf{i}} s_{\mathbf{j}} \leq M s_{\mathbf{ij}}$ .
3. For each  $\mathbf{i} \in \Sigma_A^*$  we have  $s_{\mathbf{i}} \leq s_{\hat{\mathbf{i}}}$ . Where  $\hat{\mathbf{i}} = \mathbf{i}(|\mathbf{i}| - 1)$ .
4. For  $\mathbf{k} \in \Sigma_A^{\mathbb{N}}$ ,  $n \in \mathbb{N}$ , we have  $\exp(S_n \phi(\mathbf{k})) \leq s_{\mathbf{k}|n} \leq M \exp(S_n \phi(\mathbf{k}))$ .
5. For  $\mathbf{k} \in \Sigma_A^{\mathbb{N}}$  the following two are equivalent

- $\lim_{n \rightarrow +\infty} \frac{1}{n} S_n \phi(\mathbf{k}) = \alpha$ ,
- $\lim_{n \rightarrow +\infty} \frac{1}{n} \log s_{\mathbf{k}|n} = \alpha$ .

Proof.

1 follows directly from the definitions and 3.0.1.1.

To prove 2 let us note following

$$\begin{aligned} s_{\mathbf{ij}} &= \sup_{\mathbf{k} \in [\mathbf{ij}]} \exp(S_{|\mathbf{ij}|} \phi(\mathbf{k})) \\ &\leq \sup_{\mathbf{k} \in [\mathbf{i}]} \exp(S_{|\mathbf{i}|} \phi(\mathbf{k})) \sup_{\mathbf{w} \in [\mathbf{j}]} \exp(S_{|\mathbf{j}|} \phi(\mathbf{w})) = s_{\mathbf{i}} s_{\mathbf{j}}. \end{aligned}$$

Next let  $\mathbf{v} \in [\mathbf{i}]$ ,  $\mathbf{w} \in [\mathbf{j}]$  and let  $\mathbf{u} = (\mathbf{iw})$  then

$$\begin{aligned} \exp(S_{|\mathbf{i}|} \phi(\mathbf{v})) \exp(S_{|\mathbf{j}|} \phi(\mathbf{w})) &= \frac{\exp(S_{|\mathbf{i}|} \phi(\mathbf{v})) \exp(S_{|\mathbf{j}|} \phi(\mathbf{w}))}{\exp(S_{|\mathbf{ij}|} \phi(\mathbf{u}))} \exp(S_{|\mathbf{ij}|} \phi(\mathbf{u})) \\ &= \frac{\exp(S_{|\mathbf{i}|} \phi(\mathbf{v}))}{\exp(S_{|\mathbf{i}|} \phi(\mathbf{u}))} \exp(S_{|\mathbf{ij}|} \phi(\mathbf{u})) \\ &\leq M \exp(S_{|\mathbf{ij}|} \phi(\mathbf{u})) && \text{by 3.0.2 part 2.} \\ &\leq M s_{\mathbf{ij}} \end{aligned}$$

Since this is true for each  $\mathbf{v} \in [\mathbf{i}]$ ,  $\mathbf{w} \in [\mathbf{j}]$  we have that 2 easily follows from there. 3 follows directly from definition of  $s_{\mathbf{i}}$  and  $\phi < 0$ . 4 follows from definitions and 3.0.2.2, proof is somewhat similar to proof of 2 so we will skip it. 5 follows directly from 4. ■

### 3.1 Multifractal Pressure and Dynamical multifractal zeta function

Here we will show the connection between the radius of convergence of dynamical zeta function and pressure, and then provide the rest of proofs needed for the main results regarding dynamical zeta function/multifractal pressure.

#### Relationship between Multifractal Pressure and the radius of convergence of Dynamical multifractal zeta function

Let us first recall the definition of dynamical zeta function.

Let  $X$  be metric space. Let  $C \subset X$ , let  $U : \mathcal{P}_S(\Sigma_A^{\mathbb{N}}) \rightarrow X$  be continuous with respect to weak topology, and let  $\phi : \Sigma_A^{\mathbb{N}} \rightarrow \mathbb{R}$  be a continuous function. Then, recall, dynamical multifractal zeta function is defined as

$$\zeta_C^{\text{dyn}, U}(\phi, z) = \sum_{n=1}^{+\infty} \frac{z^n}{n} \sum_{\substack{\mathbf{i} \in \Sigma_A^{\mathbb{N}} \\ UL_n[\mathbf{i}] \subset C}} s_{\mathbf{i}}(\phi).$$



Recall that multifractal pressure is defined as

$$P_C^U(\phi) = \limsup_{n \rightarrow +\infty} \frac{1}{n} \log \sum_{\substack{\mathbf{i} \in \Sigma_A^n \\ UL_n[\mathbf{i}] \subset C}} s_{\mathbf{i}}(\phi).$$

$\zeta_C^{\text{dyn}, U}(\phi, z)$  is power series as function of  $z$  so radius of convergence could be calculated as follows:

$$\begin{aligned} \sigma_{\text{rad}}(\zeta_C^{\text{dyn}, U}(\phi, \cdot)) &= \limsup_{n \rightarrow +\infty} \left( \frac{1}{n} \sum_{\substack{\mathbf{i} \in \Sigma_A^n \\ UL_n[\mathbf{i}] \subset C}} s_{\mathbf{i}}(\phi) \right)^{-1/n} \\ &= \limsup_{n \rightarrow +\infty} \exp \left( -\frac{1}{n} \log \sum_{\substack{\mathbf{i} \in \Sigma_A^n \\ UL_n[\mathbf{i}] \subset C}} s_{\mathbf{i}}(\phi) + \frac{\log n}{n} \right) \\ &= \exp(-P_C^U(\phi)). \end{aligned}$$

Hence we conclude

$$-\log \sigma_{\text{rad}}(\zeta_C^{\text{dyn}, U}(\phi, \cdot)) = P_C^U(\phi). \quad (3.1)$$

### 3.1.1 Variational Principles for Multifractal Pressure

Here we will provide required proofs regarding main results for multifractal pressure/dynamical zeta function, first in the case of shrinking targets and then in the case of fixed target. To do so we will introduce modified multifractal pressure for which it will be easier to prove variational results. Then we will show that modified multifractal pressure is "close" to multifractal pressure and use that to prove our results in the shrinking targets case. Then we will prove results in the fixed target settings. In order to introduce modified multifractal pressure we will alter a bit the sequence of functions  $L_n$ . Recall in definition 2.1.3 we defined  $\mathbf{i}', \bar{\mathbf{i}}$ . And from the comment below definition we have that there is  $M \in \mathbb{N}$  such that for each  $\mathbf{i} \in \Sigma_A^*$  we have  $|\mathbf{i}'| < M$ . We will now introduce the sequence of functions  $M_n : \Sigma_A^{\mathbb{N}} \rightarrow \mathcal{P}_S(\Sigma_A^{\mathbb{N}})$ .

**Definition 3.1.1.** For  $n \in \mathbb{N}$ , function  $M_n : \Sigma_A^{\mathbb{N}} \rightarrow \mathcal{P}_S(\Sigma_A^{\mathbb{N}})$  is defined as

$$M_n(\mathbf{i}) = \frac{1}{|(\mathbf{i}|n)(\mathbf{i}|n)'|} \sum_{k=0}^{|(\mathbf{i}|n)(\mathbf{i}|n)'|-1} \delta_{S^k \mathbf{i}|n}.$$

Note that for each  $n \in \mathbb{N}$  and  $\mathbf{i} \in \Sigma_A^{\mathbb{N}}$  we have

$$n \leq |(\mathbf{i}|n)(\mathbf{i}|n)'| \leq n + M. \quad (3.2)$$

Note as well that  $M_n$  maps, unlike  $L_n$ , whole cylinder to same measure. Function  $M_n$  "approximates well" function  $L_n$ , i.e. the following holds

**Lemma 3.1.2.** There is a constant  $D > 0$  such that for each  $n \in \mathbb{N}$  and each  $\mathbf{i} \in \Sigma_A^{\mathbb{N}}$  we have

$$d_L(L_n \mathbf{i}, M_n \mathbf{i}) < \frac{D}{n}.$$

Proof.

We will do simple calculations using definition of  $d_L$  between measures. Let  $f \in Lip(\Sigma_A^{\mathbb{N}})$ . Since  $f$  is continuous on the compact set  $\Sigma_A^{\mathbb{N}}$  it is bounded. Therefore we can choose constant  $M'$  such that for each  $\mathbf{i} \in \Sigma_A^{\mathbb{N}}$  we have  $|f(\mathbf{i})| < M'$ . Then

$$\begin{aligned}
\left| \int_{\Sigma_A^{\mathbb{N}}} f(\mathbf{i}) dL_n \mathbf{i} - \int_{\Sigma_A^{\mathbb{N}}} f(\mathbf{i}) dM_n \mathbf{i} \right| &= \left| \frac{1}{n} \sum_{k=0}^{n-1} f(S^k \mathbf{i}) - \frac{1}{|(\mathbf{i}|n)(\mathbf{i}|n)'|} \sum_{k=0}^{|(\mathbf{i}|n)(\mathbf{i}|n)'|-1} f(S^k \overline{\mathbf{i}|n}) \right| \\
&= \left| \frac{1}{n} \sum_{k=0}^{n-1} (f(S^k \mathbf{i}) - f(S^k \overline{\mathbf{i}|n})) - \frac{1}{|(\mathbf{i}|n)(\mathbf{i}|n)'|} \sum_{k=n}^{|(\mathbf{i}|n)(\mathbf{i}|n)'|-1} f(S^k \overline{\mathbf{i}|n}) + \left( \frac{1}{n} - \frac{1}{|(\mathbf{i}|n)(\mathbf{i}|n)'|} \right) \sum_{k=0}^{n-1} f(S^k \overline{\mathbf{i}|n}) \right| \\
&= \left| \frac{1}{n} \sum_{k=0}^{n-1} (f(S^k \mathbf{i}) - f(S^k \overline{\mathbf{i}|n})) - \frac{1}{|(\mathbf{i}|n)(\mathbf{i}|n)'|} \sum_{k=n}^{|(\mathbf{i}|n)(\mathbf{i}|n)'|-1} \left( f(S^k \overline{\mathbf{i}|n}) - \frac{1}{n} \sum_{k=0}^{n-1} f(S^k \overline{\mathbf{i}|n}) \right) \right| \\
&\leq \frac{1}{n} \sum_{k=0}^{n-1} \gamma^n + \frac{2M'M}{n} \quad \text{using } Lip(f) < 1, M \text{ is constant from 3.2}
\end{aligned}$$

Which finishes the proof. ■

Now let us introduce modified multifractal pressure changing  $L_n$  for  $M_n$  in definition of multifractal pressure.

**Definition 3.1.3** (Modified Multifractal Pressure). *For set  $F \subset X$  define Modified Multifractal Pressure  $\overline{P}_F : C(\Sigma_A^{\mathbb{N}}) \rightarrow \mathbb{R}$  by*

$$\overline{P}_F(\phi) = \limsup_{n \rightarrow +\infty} \frac{1}{n} \log \sum_{\substack{\mathbf{i} \in \Sigma_A^n \\ UM_n[\mathbf{i}] \subset F}} s_i(\phi).$$

As we have seen pressure could be introduced as well via variational principle so we introduce the following notation. For set  $F \subset X$  and function  $\phi \in C(\Sigma_A^{\mathbb{N}})$  we define Variational Multifractal Pressure as

$$\widehat{P}_F^U(\phi) = \sup_{\substack{\mu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}}) \\ U\mu \in F}} \left\{ h(\mu) + \int \phi d\mu \right\}.$$

The main property of modified multifractal pressure is the following theorem:

**Theorem 3.1.4** (Variational Principle for Modified Multifractal Pressure). *For set  $F \subset X$  and function  $\phi \in C(\Sigma_A^{\mathbb{N}})$  we have*

$$\sup_{\substack{\mu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}}) \\ U\mu \in \text{int} F}} \left\{ \int_{\Sigma_A^{\mathbb{N}}} \phi d\mu + h(\mu) \right\} = \widehat{P}_{\text{int} F}^U(\phi) \leq \overline{P}_{\text{int} F}^U(\phi) \quad (3.3)$$

$$\leq \overline{P}_F^U(\phi) \leq \widehat{P}_F^U(\phi) = \sup_{\substack{\mu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}}) \\ U\mu \in \overline{F}}} \left\{ \int_{\Sigma_A^{\mathbb{N}}} \phi d\mu + h(\mu) \right\}. \quad (3.4)$$

Proof is due to length postponed to the next subsection. ■

Continuity result that we need of  $\widehat{P}^U$  is described in the following proposition.

**Proposition 3.1.5.** *Let  $C \subset X$  be closed set,  $\phi \in C(\Sigma_A^{\mathbb{N}})$  then*

$$\lim_{r \rightarrow 0} \widehat{P}_{B(C,r)}^U(\phi) = \lim_{r \rightarrow 0} \widehat{P}_{B[C,r]}^U(\phi) = \widehat{P}_C^U(\phi).$$

Proof.

Let us first prove the following lemma:

**Lemma 3.1.6.** *Let  $F : \mathcal{P}(\Sigma_A^{\mathbb{N}}) \rightarrow \mathbb{R}$  be upper semi-continuous function, and let  $C \subset \mathcal{P}(\Sigma_A^{\mathbb{N}})$  be a closed set. Then*

$$\lim_{r \rightarrow 0} \sup_{x \in B[C, r]} F(x) = \inf_{r > 0} \left\{ \sup_{x \in B[C, r]} F(x) \right\} = \sup_{x \in C} F(x)$$

Proof.

$F$  is upper semi-continuous, which means that if we choose any  $\epsilon > 0$  then

$$(\forall x \in P(\Sigma_A^{\mathbb{N}}))(\exists B[x, r_x])(\forall y \in B(x, r_x))F(y) < F(x) + \epsilon.$$

Now let us fix some  $\epsilon > 0$ . And let us for every  $x \in C$ , find  $r_x$  like above. Next  $C \subset \cup_{x \in C} B(x, r_x)$  since  $C$  compact we have  $C \subset \cup_{i=0}^n B(x_n, r_n)$ . Let us denote  $U = \cup_{i=0}^n B(x_n, r_n)$ . Since  $U$  is open  $U^C$  is compact. Next continuous function  $f(x) = d(x, U^C)$  restricted to  $C$ , since  $C$  is compact, reaches its minimum at point  $x_0 \in C$ . Then  $d(x_0, U^C) > 0$  because  $U^C$  is closed and  $x_0 \notin U^C$ . So  $d(U^c, C) = d_1 > 0$  hence we conclude that

$$(\forall y \in B[C, d_1/2]) \sup_{x \in C} F(x) + \epsilon > F(y).$$

So we have that

$$(\forall \epsilon > 0)(\exists r > 0) \sup_{x \in B[C, r]} F(x) \leq \sup_{x \in C} F(x) + \epsilon.$$

So we have

$$\begin{aligned} (\forall \epsilon > 0) \inf_{r > 0} \left\{ \sup_{x \in B[C, r]} F(x) \right\} &\leq \sup_{x \in C} F(x) + \epsilon, \\ \sup_{x \in B[C, r]} F(x) &\geq \sup_{x \in C} F(x). \end{aligned}$$

Combining the last two inequalities we get

$$\sup_{x \in C} F(x) = \inf_{r > 0} \left\{ \sup_{x \in B[C, r]} F(x) \right\}. \blacksquare$$

Let us define  $F(\mu) : \mathcal{P}(\Sigma_A^{\mathbb{N}}) \rightarrow \mathbb{R}$  as

$$F(\mu) = h(\mu) + \int_{\Sigma_A^{\mathbb{N}}} \phi d\mu.$$

Hence  $F(\mu)$  is upper semi-continuous and for any set  $B \subset X$  we have

$$\widehat{P}_B^U(\phi) = \sup_{\substack{\mu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}}) \\ U\mu \in B}} F(\mu)$$

Note that  $\cap_{r>0} U^{-1}B(C, r) = U^{-1}(\cap_{r>0} B(C, r)) = U^{-1}C$ . Using that, the fact that  $A \subset B \Rightarrow \widehat{P}_A^U(\phi) \leq \widehat{P}_B^U(\phi)$ , and lemma 3.1.6 we get

$$\lim_{r \rightarrow 0} \widehat{P}_{B(C, r)}^U(\phi) = \lim_{r \rightarrow 0} \widehat{P}_{B[C, r]}^U(\phi) = \widehat{P}_C^U(\phi).$$

■

The following result specifies similarity between Multifractal Pressure and Modified Multifractal Pressure.

**Proposition 3.1.7.** *For  $F \subset X, \phi \in C(\Sigma_A^{\mathbb{N}})$  we have*

$$\begin{aligned} \limsup_{r \rightarrow 0} \bar{P}_{B(F,r)}^U(\phi) &= \limsup_{r \rightarrow 0} P_{B(F,r)}^U(\phi), \\ \liminf_{r \rightarrow 0} \bar{P}_{B(F,r)}^U(\phi) &= \liminf_{r \rightarrow 0} P_{B(F,r)}^U(\phi). \end{aligned}$$

Proof.

It is easy to see from definitions of multifractal pressure and modified multifractal pressure and lemma 3.1.2 that for each  $r > 0$  we have

$$P_{B(F,r)}^U(\phi) \leq \bar{P}_{B(F,2r)}^U(\phi) \leq P_{B(F,3r)}^U(\phi).$$

Now taking  $\limsup$  and  $\liminf$  as appropriate we prove our proposition. ■

Now using similarity between multifractal pressure and modified multifractal pressure we prove variational principle for multifractal pressure.

**Proposition 3.1.8** (Variational Principle for Multifractal Pressure - Shrinking Targets). *Let  $C \subset X$  be closed set,  $\phi \in C(\Sigma_A^{\mathbb{N}})$  then we have*

$$\lim_{r \rightarrow 0} P_{B(C,r)}^U(\phi) = \lim_{r \rightarrow 0} \bar{P}_{B(C,r)}^U(\phi) = \lim_{r \rightarrow 0} \bar{P}_{B[C,r]}^U(\phi) = \sup_{\substack{\mu \in \mathcal{P}_s(\Sigma_A^{\mathbb{N}}) \\ U\mu \in C}} \left\{ \int_{\Sigma_A^{\mathbb{N}}} \phi d\mu + h(\mu) \right\}.$$

Proof.

We first have

$$\begin{aligned} \liminf_{r \rightarrow 0} \bar{P}_{B[C,r]}^U(\phi) &\geq \liminf_{r \rightarrow 0} \bar{P}_{B(C,r)}^U(\phi) \\ &\geq \liminf_{r \rightarrow 0} \hat{P}_{B(C,r)}^U(\phi) && \text{using theorem 3.1.4} \\ &= \hat{P}_C^U(\phi) && \text{from 3.1.5.} \end{aligned}$$

Next

$$\begin{aligned} \hat{P}_C^U(\phi) &= \limsup_{r \rightarrow 0} \hat{P}_{B[C,r]}^U(\phi) && \text{from 3.1.5} \\ &\geq \limsup_{r \rightarrow 0} \bar{P}_{B[C,r]}^U(\phi) && \text{using theorem 3.1.4} \\ &\geq \limsup_{r \rightarrow 0} \bar{P}_{B(C,r)}^U(\phi). \end{aligned}$$

Hence using 3.1.7 follows our result. ■

Let us state the result needed for multifractal Bowen's formula.

**Proposition 3.1.9.** *Let  $C \subset X$  be closed set,  $C \cap U(\mathcal{P}_s(\Sigma_A^{\mathbb{N}})) \neq \emptyset$ , and  $\phi \in C(\Sigma_A^{\mathbb{N}}), \phi < 0$ . Then there is a unique number  $t_0 \in \mathbb{R}$  such that*

$$\hat{P}_C^U(t_0\phi) = 0.$$

Then

$$t_0 = \sup_{\substack{\mu \in \mathcal{P}_s(\Sigma_A^{\mathbb{N}}) \\ U\mu \in C}} - \frac{h(\mu)}{\int_{\Sigma_A^{\mathbb{N}}} \phi d\mu}.$$

Additionally,

$$\begin{aligned}\widehat{P}^U(t\phi) &> 0 \text{ for } t < t_0, \\ \widehat{P}^U(t\phi) &< 0 \text{ for } t > t_0.\end{aligned}$$

Proof.

First recall that

$$\widehat{P}_C^U(t\phi) = \sup_{\substack{\mu \in \mathcal{P}_s(\Sigma_A^{\mathbb{N}}) \\ U\mu \in C}} \left\{ \int_{\Sigma_A^{\mathbb{N}}} t\phi d\mu + h(\mu) \right\}.$$

$h(\mu)$  is upper semi-continuous function and  $\int_{\Sigma_A^{\mathbb{N}}} \phi d\mu$  is continuous and less than zero. So we have that function  $H$  defined as

$$H(\mu) = -\frac{h(\mu)}{\int_{\Sigma_A^{\mathbb{N}}} \phi d\mu}$$

is upper semi-continuous.

$U^{-1}C \cap \mathcal{P}_s(\Sigma_A^{\mathbb{N}})$  is a compact non-empty set. Upper semi-continuous function reaches its maximum on compact set. Hence there is  $\mu_0$  such that

$$t_0 = H(\mu_0) = \sup_{\substack{\mu \in \mathcal{P}_s(\Sigma^{\mathbb{N}}) \\ U\mu \in C}} H(\mu) = \sup_{\substack{\mu \in \mathcal{P}_s(\Sigma^{\mathbb{N}}) \\ U\mu \in C}} -\frac{h(\mu)}{\int_{\Sigma_A^{\mathbb{N}}} \phi d\mu}.$$

If  $H(\mu) \neq H(\mu_0)$  then

$$\begin{aligned}t_0 &> \frac{-h(\mu)}{\int_{\Sigma_A^{\mathbb{N}}} \phi d\mu}, \\ t_0 \int_{\Sigma_A^{\mathbb{N}}} \phi d\mu &< -h(\mu) \text{ due to } \phi < 0, \\ \int_{\Sigma_A^{\mathbb{N}}} t_0 \phi d\mu + h(\mu) &< 0.\end{aligned}$$

Hence we conclude

$$\sup_{\substack{\mu \in \mathcal{P}_s(\Sigma_A^{\mathbb{N}}) \\ U\mu \in C}} \int_{\Sigma_A^{\mathbb{N}}} t_0 \phi d\mu + h(\mu) \leq 0.$$

Next

$$\int_{\Sigma_A^{\mathbb{N}}} t_0 \phi d\mu_0 + h(\mu_0) = \int_{\Sigma_A^{\mathbb{N}}} \sup_{\substack{\mu \in \mathcal{P}_s(\Sigma_A^{\mathbb{N}}) \\ U\mu \in C}} -\frac{h(\mu_0)}{\int_{\Sigma_A^{\mathbb{N}}} \phi d\mu_0} \phi d\mu_0 + h(\mu_0) = 0.$$

So we conclude

$$\sup_{\substack{\mu \in \mathcal{P}_s(\Sigma_A^{\mathbb{N}}) \\ U\mu \in C}} \left\{ \int_{\Sigma_A^{\mathbb{N}}} t_0 \phi d\mu + h(\mu) \right\} = 0.$$

It is left to prove that  $t_0$  is unique.

Let assume that there is  $t_1 \neq t_0$  such that

$$\sup_{\substack{\mu \in \mathcal{P}_s(\Sigma_A^{\mathbb{N}}) \\ U\mu \in C}} \left\{ \int_{\Sigma_A^{\mathbb{N}}} t_1 \phi d\mu + h(\mu) \right\} = 0. \quad (3.5)$$

Since  $h(\mu)$  is upper semi-continuous function and  $\int_{\Sigma_A^{\mathbb{N}}} \phi d\mu$  is continuous as function of  $\mu$  we have that  $G(\mu) = \int_{\Sigma_A^{\mathbb{N}}} t_0 \phi d\mu + h(\mu)$  is upper semi-continuous, and hence reaches maximum at compact set  $\mathcal{P}_s(\Sigma_A^{\mathbb{N}}) \cap U^{-1}(C)$ . So there is  $\mu_1$  such that

$$t_1 \int_{\Sigma_A^{\mathbb{N}}} \phi d\mu_1 + h(\mu_1) = 0.$$

Next because of  $t_1 \neq t_0$  we have

$$t_1 = H(\mu_1) < H(\mu_0) = t_0.$$

Hence we have

$$t_1 < -\frac{h(\mu_0)}{\int_{\Sigma_A^{\mathbb{N}}} \phi d\mu_0},$$

$$t_1 \int_{\Sigma_A^{\mathbb{N}}} \phi d\mu_0 + h(\mu_0) > 0.$$

Hence

$$\sup_{\substack{\mu \in \mathcal{P}_s(\Sigma_A^{\mathbb{N}}) \\ U\mu \in C}} \left\{ t_1 \int_{\Sigma_A^{\mathbb{N}}} \phi d\mu + h(\mu) \right\} > 0.$$

Which is contradictory to assumption 3.5. So uniqueness has been proved. The rest follows directly from  $\phi < 0$ . ■

Now we will need the following continuity property of variational pressure.

**Proposition 3.1.10.** *Let  $V \subset X$  be open set. Then we have*

$$\lim_{r \rightarrow 0} \hat{P}_{I[V,r]}^U = \hat{P}_V^U$$

Proof.

Let us first prove the following lemma:

**Lemma 3.1.11.** *Let  $F : \mathcal{P}(\Sigma_A^{\mathbb{N}}) \rightarrow \mathbb{R}$  be upper semi-continuous function, let  $U \subset \mathcal{P}(\Sigma_A^{\mathbb{N}})$  be open set then*

$$\lim_{r \rightarrow 0} \sup_{x \in I(U,r)} F(x) = \sup_{r > 0} \left\{ \sup_{x \in I(U,r)} F(x) \right\} = \sup_{x \in U} F(x)$$

Proof. From the definition of supremum we have

$$(\forall \epsilon > 0)(\exists y \in U) \sup_{x \in U} F(x) \leq F(y) + \epsilon.$$

Since  $U$  is open there is such  $r$  that  $y \in I(U, r), r > 0$ . So we have

$$\sup_{x \in U} F(x) \leq F(y) + \epsilon \leq \sup_{I(U,r)} F(x) + \epsilon \leq \sup_{r > 0} \sup_{I(U,r)} F(x) + \epsilon \leq \sup_{x \in U} F(x) + \epsilon.$$

Hence we have

$$\sup_{r > 0} \left\{ \sup_{x \in I(U,r)} F(x) \right\} = \sup_{x \in U} F(x). \blacksquare$$

Let us define  $F(\mu) : \mathcal{P}(\Sigma_A^{\mathbb{N}}) \rightarrow \mathbb{R}$  as

$$F(\mu) = h(\mu) + \int_{\Sigma_A^{\mathbb{N}}} \phi d\mu.$$

Hence  $F(\mu)$  is upper semi-continuous and for any set  $B \subset X$  we have

$$\widehat{P}_B^U(\phi) = \sup_{\substack{\mu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}}) \\ U\mu \in B}} F(\mu)$$

Note that  $\cap_{r>0} U^{-1}I[V, r] = U^{-1}(\cap_{r>0} I[V, r]) = U^{-1}V$ . Using that, the fact that  $A \subset B \Rightarrow \widehat{P}_A^U(\phi) \leq \widehat{P}_B^U(\phi)$ , and lemma 3.1.11 we get

$$\lim_{r \rightarrow 0} \widehat{P}_{I(V, r)}^U(\phi) = \lim_{r \rightarrow 0} \widehat{P}_{I[V, r]}^U(\phi) = \widehat{P}_V^U(\phi).$$

■

Now we will prove one small technical lemma and then we will prove variational principle for multifractal pressure in the case of fixed targets.

**Lemma 3.1.12.** *Let  $\Delta : \mathcal{P}(\Sigma_A^{\mathbb{N}}) \rightarrow \mathbb{R}$  be continuous with  $\Delta(\mu) \neq 0$ . Then we have  $\Delta(\mu) > 0$  for all  $\mu \in \mathcal{P}(\Sigma_A^{\mathbb{N}})$  or  $\Delta < 0$  for all  $\mu \in \mathcal{P}(\Sigma_A^{\mathbb{N}})$ .*

This is clear since  $\mathcal{P}(\Sigma_A^{\mathbb{N}})$  is convex and therefore, in particular, connected.

**Proposition 3.1.13.** *Let  $X$  be a normed vector space. Let  $\Gamma : \mathcal{P}(\Sigma_A^{\mathbb{N}}) \rightarrow X$  be continuous and affine and let  $\Delta : \mathcal{P}(\Sigma_A^{\mathbb{N}}) \rightarrow \mathbb{R}$  be continuous and affine with  $\Delta(\mu) \neq 0$  for all  $\mu \in \mathcal{P}(\Sigma_A^{\mathbb{N}})$ . Define  $U : \mathcal{P}(\Sigma_A^{\mathbb{N}}) \rightarrow X$  by  $U = \frac{\Gamma}{\Delta}$ . Let  $C$  be a closed and convex subset of  $X$  and assume that*

$$\text{int}C \cap U(\mathcal{P}(\Sigma_A^{\mathbb{N}})) \neq \emptyset.$$

Then

$$\widehat{P}_C^U(\phi) = \widehat{P}_{\text{int}C}^U(\phi).$$

I.e.

$$\sup_{\substack{\mu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}}) \\ U\mu \in C}} \left\{ h(\mu) + \int_{\Sigma_A^{\mathbb{N}}} \phi d\mu \right\} = \sup_{\substack{\mu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}}) \\ U\mu \in \text{int}C}} \left\{ h(\mu) + \int_{\Sigma_A^{\mathbb{N}}} \phi d\mu \right\}.$$

For brevity let us denote  $F(\mu) = h(\mu) + \int_{\Sigma_A^{\mathbb{N}}} \phi d\mu$  for  $\mu \in \mathcal{P}(\Sigma_A^{\mathbb{N}})$ . It clearly suffices to show that

$$\sup_{\substack{\mu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}}) \\ U\mu \in C}} F(\mu) \leq \sup_{\substack{\mu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}}) \\ U\mu \in \text{int}C}} F(\mu)$$

Write  $s = \sup_{\substack{\mu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}}) \\ U\mu \in C}} F(\mu)$ . Fix  $\epsilon > 0$ . It follows from the definition of  $s$  that we can choose  $\lambda \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}})$

with  $U\lambda \in C$  and  $F(\lambda) > s - \epsilon$ . Also, since  $\text{int}C \cap U(\mathcal{P}(\Sigma_A^{\mathbb{N}})) \neq \emptyset$  we can find  $\nu \in \mathcal{P}(\Sigma_A^{\mathbb{N}})$  with  $U\nu \in \text{int}C$ . For  $t \in (0, 1)$  we now define  $\gamma_t \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}})$  by  $\gamma_t = t\nu + (1-t)\lambda$ . Next, we prove the following two lemmas.

**Lemma 3.1.14.** *For every  $t \in (0, 1)$  we have  $U\gamma_t \in \text{int}C$*

Proof.

Fix  $t \in (0, 1)$ . write  $a = \frac{t\Delta(\nu)}{t\Delta(\nu) + (1-t)\Delta(\lambda)}$  and  $b = \frac{(1-t)\Delta(\lambda)}{t\Delta(\nu) + (1-t)\Delta(\lambda)}$  we have

$$\begin{aligned} U\gamma_t &= \frac{\Gamma(\gamma_t)}{\Delta(\gamma_t)} = \frac{t\Gamma(\nu) + (1-t)\Gamma(\lambda)}{t\Delta(\nu) + (1-t)\Delta(\lambda)} \\ &= aU(\nu) + bU(\lambda) \end{aligned}$$

since  $a + b = 1$  and  $U(\lambda) \in \text{int}C$  and  $U(\nu) \in C$  from [[13], p. 102, Proposition 1.11] we have our lemma. ■

**Lemma 3.1.15.** *There is  $\mu_0 \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}})$  such that  $U\mu_0 \in \text{int}C$  such that  $F(\mu_0) > s - \epsilon$ .*

Since the entropy function  $h : \mathcal{P}_S(\Sigma_A^{\mathbb{N}}) \rightarrow \mathbb{R}$  is affine (see [Wa]), we conclude that  $F$  is affine, and so  $F(\gamma_t) = F(t\nu + (1-t)\lambda) = tF(\nu) + (1-t)F(\lambda) \rightarrow F(\lambda) > s - \epsilon$ . This implies that there is  $t_0 \in (0, 1)$  with  $F(\gamma_{t_0}) > s - \epsilon$ . Now put  $\mu_0 = \gamma_{t_0}$ . Then  $F(\mu_0) = F(\gamma_{t_0}) > s - \epsilon$  and lemma 3.1.14 implies that  $U\mu_0 = U\gamma_{t_0} \in \text{int}C$ . This completes the proof of lemma. ■

We can now prove inequality. Indeed, it follows from the previous lemma that there is  $\mu_0 \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}})$  such that  $U\mu_0 \in \text{int}C$  such that  $F(\mu_0) > s - \epsilon$ , whence

$$s - \epsilon < F(\mu_0) \leq \sup_{\substack{\mu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}}) \\ U\mu \in \text{int}C}} F(\mu)$$

Finally, letting  $\epsilon \rightarrow 0$  gives  $s \leq \sup_{\substack{\mu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}}) \\ U\mu \in \text{int}C}} F(\mu)$ , which concludes proof.

**Proposition 3.1.16.** *Let  $X$  be a normed vector space. Let  $\Gamma : \mathcal{P}(\Sigma_A^{\mathbb{N}}) \rightarrow X$  be continuous and affine and let  $\Delta : \mathcal{P}(\Sigma_A^{\mathbb{N}}) \rightarrow \mathbb{R}$  be continuous and affine with  $\Delta(\mu) \neq 0$  for all  $\mu \in \mathcal{P}(\Sigma_A^{\mathbb{N}})$ . Define  $U : \mathcal{P}(\Sigma_A^{\mathbb{N}}) \rightarrow X$  by  $U = \frac{\Gamma}{\Delta}$ . Let  $C$  be a closed and convex subset of  $X$  and assume that*

$$\text{int}C \cap U(\mathcal{P}(\Sigma_A^{\mathbb{N}})) \neq \emptyset.$$

*Then*

$$P_C^U = \sup_{\substack{\mu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}}) \\ U\mu \in C}} \left\{ h(\mu) + \int_{\Sigma_A^{\mathbb{N}}} \phi d\mu \right\}.$$

Proof.

Note that using 3.1.4 for  $r > 0$  we have

$$\widehat{P}_{I(C,r)}^U(\phi) \leq \overline{P}_{I(C,r)}^U(\phi) \leq P_{I(C,r/2)}^U(\phi) \leq P_{B(C,r)}^U$$

So letting  $r \rightarrow 0$ , and using propositions 3.1.5, 3.1.10, 3.1.13 and 3.1.8 we get our result. ■

### 3.1.2 Proof of Variational Principle for Modified Multifractal Pressure

Here we will prove variational principle for modified multifractal spectra i.e. theorem 3.1.4. We will use Varadhan's [69] large deviation proposition 3.1.21 below, and a non-trivial application of this i.e. 3.1.21 (2). In order to do so, we will tweak a bit the results by Orey & Pelikan which we state below. First, let us define what Large deviation property means.



**Definition 3.1.17** (Large Deviation Property - LDP). *Let  $X$  be complete separable metric space and let  $(\mu_n)_{n \in \mathbb{N}}$  be a sequence of probability measures on  $X$ , and let  $(a_n)_{n \in \mathbb{N}}$  be a sequence of positive numbers such that  $\lim_{n \rightarrow +\infty} a_n = +\infty$ . Then if there is a lower semi-continuous function  $I : X \rightarrow \mathbb{R}$  with compact level sets, such that for any  $A \subset X$  we have:*

$$-\inf_{x \in \text{int} A} I(x) \leq \liminf_{n \rightarrow +\infty} \frac{1}{a_n} \log \mu_n(\text{int} A) \leq \limsup_{n \rightarrow +\infty} \frac{1}{a_n} \log \mu_n(\overline{A}) \leq -\inf_{x \in \overline{A}} I(x).$$

*Then it is said that sequence  $(\mu_n)_{n \in \mathbb{N}}$  has large deviation property with respect to sequence  $(a_n)_n$  and rate function  $I$ .*

We will abbreviate Large Deviation Property as LDP. The following theorem will provide us with class of measures that satisfy LDP which we will use later for our main proof.

**Theorem 3.1.18** ([52],[53]). *Let  $\phi : \Sigma_A^{\mathbb{N}} \rightarrow \mathbb{R}$  be Hölder continuous function and let  $\mu_\phi$  be invariant Gibbs measure with potential  $\phi$ . Let us denote  $\mu_{\phi,n} = \mu_\phi \circ L_n^{-1}$ . Then define  $I_\phi : \mathcal{P}(\Sigma_A^{\mathbb{N}}) \rightarrow \mathbb{R}$  by*

$$I_\phi(\mu) = \begin{cases} P(\phi) - \int_{\Sigma_A^{\mathbb{N}}} \phi d\mu - h(\mu), & \mu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}}) \\ +\infty, & \text{otherwise} \end{cases}$$

*Then sequence  $\mu_{\phi,n}$  poses Large deviation property with respect to sequence  $(n)_n$ , and rate function  $I_\phi$ .*

Let us now introduce Parry measure  $P$  on  $\Sigma_A^{\mathbb{N}}$ . Let  $u$  and  $v$  be positive eigenvectors of  $A$  and  $A^T$  such that  $\sum_{i=1}^N u_i v_i = 1$ . And let  $\lambda_A$  be maximal (which is unique and positive by Perron-Frobenius since matrix  $A$  is prime). Then Parry measure is defined as

$$P([i_1 i_2 \cdots i_n]) = u_{i_1} \lambda_A^{-n} v_{i_n}.$$

Note that in the theorem above, if we choose  $\phi = 0$  we get Parry measure. So from the theorem above sequence  $(P \circ L_n^{-1})_n$  has LDP with sequence  $(n)_n$  and rate function

$$I_\phi(\mu) = \begin{cases} P(0) - h(\mu), & \mu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}}) \\ +\infty, & \text{otherwise} \end{cases}$$

In our proof of variational principle we will make use of  $M_n$  instead of  $L_n$ . So first we will prove the following

**Proposition 3.1.19.** *Let  $P$  be Parry measure on  $\Sigma_A^{\mathbb{N}}$ . Then the sequence of measures  $(R_n)_n$ , where  $R_n = P \circ M_n^{-1}$ , poses LDP with the same sequence and with the same rate function as the sequence of measures  $(P_n)_n$ , where  $P_n = P \circ L_n^{-1}$ .*

*Proof.* The idea of the proof is that we will use  $P_n$  to approximate  $R_n$  and utilize upper semi-continuity of  $-I$ . First let  $C$  be closed subset of  $\mathcal{P}(\Sigma_A^{\mathbb{N}})$ . Then let  $r > 0$ . Then there from proposition 3.1.2 there is  $n_0 \in \mathbb{N}$  such that for each  $n > n_0$  and each  $\mathbf{i} \in \Sigma_A^{\mathbb{N}}$  we have  $d_L(M_n(\mathbf{i}), L_n(\mathbf{i})) \leq r$ . So for  $n$  big enough we have

$$M_n^{-1}(C) \subset L_n^{-1}(B[C, r]) \Rightarrow R_n(C) \leq P_n(B[C, r]).$$

So we can conclude that

$$\limsup_{n \rightarrow +\infty} \frac{1}{n} \log R_n(C) \leq \limsup_{n \rightarrow +\infty} \frac{1}{n} \log P_n(B[C, r]) \leq \sup_{x \in B[C, r]} -I(x)$$

Since it is true for every  $r > 0$ , using lemma 3.1.6 and the fact that  $-I$  is upper semi-continuous we have

$$\limsup_{n \rightarrow +\infty} \frac{1}{n} \log R_n(C) \leq \inf_{r > 0} \left\{ \sup_{x \in B[C, r]} -I(x) \right\} = \sup_{x \in C} -I(x).$$

Let us prove the other side of LDP. Let  $V$  be an open subset of  $\mathcal{P}(\Sigma_A^{\mathbb{N}})$ . Then let  $r > 0$ . Then there from 3.1.2 there is  $n_0 \in \mathbb{N}$  such that for each  $n > n_0$  and each  $\mathbf{i} \in \Sigma_A^{\mathbb{N}}$  we have  $d_L(M_n(\mathbf{i}), L_n(\mathbf{i})) \leq r$ . So for  $n$  big enough we have:

$$L_n^{-1}(I(V, r)) \subset M_n^{-1}(V) \Rightarrow P_n(I(V, r)) \leq R_n(V).$$

Hence we have that

$$\sup_{x \in I(V, r)} -I(x) \leq \liminf_{n \rightarrow +\infty} \frac{1}{n} \log P_n(B(V, -r)) \leq \liminf_{n \rightarrow +\infty} \frac{1}{n} \log R_n(V).$$

Since it is true for every  $r > 0$  using lemma above and the fact that  $-I$  is upper semi-continuous we have

$$\sup_{x \in V} -I(x) = \sup_{r > 0} \left\{ \sup_{x \in B(V, -r)} -I(x) \right\} \leq \liminf_{n \rightarrow +\infty} \frac{1}{n} \log R_n(V).$$

Hence the claim is proved. ■

The above result combined with 3.1.18 gives us the following result.

**Proposition 3.1.20.** *Let  $P$  be Parry measure on  $\Sigma_A^{\mathbb{N}}$ , which we will denote by  $P$ . So from the theorem above, sequence  $(P \circ L_n^{-1})_n$  has LDP with sequence  $(n)_n$  and rate function*

$$I_\phi(\mu) = \begin{cases} P(0) - h(\mu), & \mu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}}) \\ +\infty, & \text{otherwise} \end{cases}$$

Now we will state Varadhan's [69] large deviation theorem below (3.1.21, (1)), with application stated as 3.1.21, (2).

**Theorem 3.1.21.** *Let  $X$  be a complete separable metric space and let  $(P_n)_n$  be a sequence of probability measures on  $X$ . Assume that the sequence  $(P_n)_n$  has the large deviation property constants  $(a_n)_n$  and the rate function  $I$ . Let  $F : X \rightarrow \mathbb{R}$  be a continuous function satisfying the following two conditions:*

(i)

$$(\forall n) \int_X \exp(a_n F) dP_n < \infty.$$

(ii) *We have*

$$\lim_{M \rightarrow \infty} \limsup_{n \rightarrow \infty} \frac{1}{a_n} \log \int_{\{M \leq F\}} \exp(a_n F) dP_n = -\infty.$$

*Then we have*

(1)

$$\lim_{n \rightarrow +\infty} \frac{1}{a_n} \log \int_X \exp(a_n F) dP_n = - \inf_{x \in X} (I(x) - F(x)).$$

(2) Define

$$R_n(E) = \frac{\int_E \exp(a_n F) dP_n}{\int_X \exp(a_n F) dP_n}.$$

Then sequence  $(R_n)_{n \in \mathbb{N}}$  has large deviation property with constants  $(a_n)_{n \in \mathbb{N}}$  and rate function

$$(I - F) - \inf_{x \in X} (I(x) - F(x)).$$

Statement (1) follows from [[18], Theorem II.7.1] or [[14], Theorem 4.3.1], and statement (2) follows from [[18], Theorem II.7.2].

Note that it is enough for function  $F$  to be bounded in order to satisfy the conditions of the theorem. Before we finally start proving theorem 3.1.4 let us state just one more technical proposition.

**Proposition 3.1.22.** *Let  $\phi : \Sigma_A^{\mathbb{N}} \rightarrow \mathbb{R}$  be a continuous function. Then let us define a sequence of positive numbers  $(C_n)_n$ , where  $C_n$  is defined infimum of all constants  $C$  such that for each  $\mathbf{i}, \mathbf{j} \in \Sigma_A^{\mathbb{N}}$  if  $\mathbf{i}|_n = \mathbf{j}|_n$  we have*

$$|S_n \phi(\mathbf{i}) - S_n \phi(\mathbf{j})| \leq C. \quad (3.6)$$

Then we have

$$C_n = o(n). \quad (3.7)$$

Proof.

Note that  $\phi$  is continuous on compact space and hence bounded. So it is clear that for each  $n$ , number  $C_n$  is well defined (i.e. no infimum of empty set in definition of  $C_n$ ) finite number. Let us prove 3.7. Note that

$$C_{n+1} \leq C_n + \sup_{\mathbf{i}|_{n+1} = \mathbf{j}|_{n+1}} |\phi(\mathbf{i}) - \phi(\mathbf{j})|.$$

Function  $\phi$  is uniformly continuous, so for each  $\epsilon > 0$  there is  $n_\epsilon$ , such that for each  $n > n_\epsilon$  we have  $\mathbf{i}|_n = \mathbf{j}|_n$  that  $|\phi(\mathbf{i}) - \phi(\mathbf{j})| < \epsilon$ . Hence  $\lim_{n \rightarrow +\infty} C_{n+1} - C_n = 0$ . Now it is easy to see that  $C_n = o(n)$ . ■

Finally we can continue to the proof of 3.1.4.

**Proposition 3.1.23.** *(Variational Principle part 1) Let  $\phi : \Sigma_A^{\mathbb{N}} \rightarrow \mathbb{R}$  be a continuous function. If  $C \subset X$  is a closed set then*

$$\overline{P}_C^U(\phi) \leq \sup_{\substack{\mu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}}) \\ U\mu \in C}} \left\{ \int_{\Sigma_A^{\mathbb{N}}} \phi d\mu + h(\mu) \right\}.$$

Proof.

Let  $P$  be Parry measure on  $\Sigma_A^{\mathbb{N}}$ . Let  $(Q_n)_n$  be a sequence defined by  $Q_n = P \circ M_n^{-1}$ . Denote  $s_{\mathbf{i}}(\phi)$  as  $s_{\mathbf{i}}$ .

$$\begin{aligned} \sum_{\substack{\mathbf{i} \in \Sigma_A^{\mathbb{N}} \\ UM_n[\mathbf{i}] \in C}} s_{\mathbf{i}} &= \lambda_A^n \sum_{\substack{\mathbf{i} \in \Sigma_A^{\mathbb{N}} \\ UM_n[\mathbf{i}] \in C}} s_{\mathbf{i}} \frac{1}{u_{i_1} v_{i_n}} u_{i_1} \lambda_A^{-n} v_{i_n} && \text{denote } A_n = \{\mathbf{i} \mid \mathbf{i} \in \Sigma_A^{\mathbb{N}}, M_n[\mathbf{i}]_n \subset C\} \\ &\leq \frac{1}{\min_i u_i \cdot \min_i v_i} \lambda_A^n \int_{A_n} s_{\mathbf{i}|_n} dP(\mathbf{i}) \\ &\leq e^{C_n} \lambda_A^n \int_{A_n} \exp \left( n \cdot \frac{1}{n} \sum_{k=0}^{n-1} \phi(S^k \mathbf{i}) \right) dP(\mathbf{i}) && \text{where } C_n \text{ is from 3.1.22} \end{aligned}$$

Since  $\phi$  is continuous on the compact set  $\Sigma_A^{\mathbb{N}}$  it is bounded. Therefore we can choose constant  $M'$  such that for each  $\mathbf{i} \in \Sigma_A^{\mathbb{N}}$  we have  $|\phi(\mathbf{i})| < M'$ . Now note

$$\begin{aligned}
& \left| \frac{1}{n} \sum_{k=0}^{n-1} \phi(S^k \mathbf{i}) - \frac{1}{|(\mathbf{i}|n)(\mathbf{i}|n)'|} \sum_{k=0}^{|(\mathbf{i}|n)(\mathbf{i}|n)'|-1} \phi(S^k \mathbf{i}|n) \right| \\
&= \left| \frac{1}{n} \sum_{k=0}^{n-1} \left( \phi(S^k \mathbf{i}) - \phi(S^k \mathbf{i}|n) \right) - \frac{1}{|(\mathbf{i}|n)(\mathbf{i}|n)'|} \sum_{k=n}^{|(\mathbf{i}|n)(\mathbf{i}|n)'|-1} \phi(S^k \mathbf{i}|n) + \left( \frac{1}{n} - \frac{1}{|(\mathbf{i}|n)(\mathbf{i}|n)'|} \right) \sum_{k=0}^{n-1} \phi(S^k \mathbf{i}|n) \right| \\
&= \left| \frac{1}{n} \sum_{k=0}^{n-1} \left( \phi(S^k \mathbf{i}) - \phi(S^k \mathbf{i}|n) \right) - \frac{1}{|(\mathbf{i}|n)(\mathbf{i}|n)'|} \sum_{k=n}^{|(\mathbf{i}|n)(\mathbf{i}|n)'|-1} \left( \phi(S^k \mathbf{i}|n) - \frac{1}{n} \sum_{k=0}^{n-1} \phi(S^k \mathbf{i}|n) \right) \right| \\
&\leq \frac{1}{n} C_n + \frac{2M'M}{n} \quad M \text{ is constant from note after 2.1.3, } C_n \text{ is from 3.1.22}
\end{aligned}$$

We have that  $\lim_{n \rightarrow +\infty} (\frac{1}{n} C_n + \frac{2M'M}{n}) = 0$  (by 3.1.22) hence there is a constant  $D$  such that for each  $n \in \mathbb{N}$  we have

$$\left| \frac{1}{n} \sum_{k=0}^{n-1} \phi(S^k \mathbf{i}) - \frac{1}{|(\mathbf{i}|n)(\mathbf{i}|n)'|} \sum_{k=0}^{|(\mathbf{i}|n)(\mathbf{i}|n)'|-1} \phi(S^k \mathbf{i}|n) \right| < D. \quad (3.8)$$

Hence

$$\begin{aligned}
&\leq e^{C_n} \lambda_A^n \int_{A_n} \exp \left( n \cdot \frac{1}{n} \sum_{k=0}^{n-1} \phi(S^k \mathbf{i}) \right) dP(\mathbf{i}) \\
&\leq e^{C_n+D} \lambda_A^n \int_{A_n} \exp \left( n \cdot \frac{1}{A} \sum_{k=0}^{A-1} \phi(S^k \mathbf{i}|n) \right) dP(\mathbf{i}) \quad \text{by 3.8, where } A = |(\mathbf{i}|n)(\mathbf{i}|n)'| \\
&= e^{C_n+D} \lambda_A^n \int_{A_n} \exp \left( n \int_{\Sigma_A^{\mathbb{N}}} \phi(\mathbf{j}) dM_n(\mathbf{i}) \right) dP(\mathbf{i}) \\
&= e^{C_n+D} \lambda_A^n \int_{M_n(A_n)} \exp \left( n \int_{\Sigma_A^{\mathbb{N}}} \phi(\mathbf{j}) d\mu \right) dP \circ M_n^{-1}(\mu) \\
&= e^{C_n+D} \int_{U^{-1}(C)} \exp \left( n(\log \lambda_A + \int_{\Sigma_A^{\mathbb{N}}} \phi(\mathbf{j}) dM_n(\mu)) \right) dP \circ M_n^{-1}(\mu) \quad \text{due to } M_n^{-1}(U^{-1}C \setminus M_n(A_n)) = \emptyset
\end{aligned}$$

Let us put  $F(\mu) = \int_{\Sigma_A^{\mathbb{N}}} \phi d\mu$ . Then define a sequence of probability measures  $(W_n)_n$  by

$$W_n(A) = \frac{\int_A \exp(n(\log \lambda_A + F(\mu))) dP \circ M_n^{-1}}{\int_{\mathcal{P}(\Sigma_A^{\mathbb{N}})} \exp(n(\log \lambda_A + F(\mu))) dP \circ M_n^{-1}}$$

Due to 3.1.20, we can use 3.1.21(2), and hence conclude that  $(W_n)_n$  possess LDP with sequence  $(n)_n$  and rate function

$$I(\mu) = \begin{cases} P(0) - h(\mu) - \log \lambda_A - \int_{\Sigma_A^{\mathbb{N}}} \phi d\mu + \sup_{\mu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}})} \{h(\mu) + \int_{\Sigma_A^{\mathbb{N}}} \phi d\mu\}, & \mu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}}) \\ +\infty, & \text{otherwise} \end{cases}$$

Due to  $\lambda_A = \exp(P(0))$  we have

$$I(\mu) = \begin{cases} -h(\mu) - \int_{\Sigma_A^{\mathbb{N}}} \phi d\mu + \sup_{\mu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}})} \{h(\mu) + \int_{\Sigma_A^{\mathbb{N}}} \phi d\mu\}, & \mu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}}) \\ +\infty, & \text{otherwise} \end{cases}$$

Now we have

$$\limsup_{n \rightarrow +\infty} \frac{1}{n} \log \sum_{\substack{\mathbf{i} \in \Sigma_A^{\mathbb{N}} \\ M_n[\mathbf{i}] \subset C}} s_{\mathbf{i}} \leq \limsup_{n \rightarrow +\infty} \left( \frac{1}{n} (C_n + D) + \frac{1}{n} \log W_n(U^{-1}(C)) + \frac{1}{n} \log \left( \int_{\mathcal{P}(\Sigma_A^{\mathbb{N}})} \exp(n(\log \lambda_A + F(\mu))) dP \circ M_n^{-1} \right) \right)$$

Using continuity of  $U$ , i.e. the fact that  $U^{-1}(C)$  is a closed set, fact that  $\lim_{n \rightarrow +\infty} \frac{1}{n} C_n = 0$  by 3.1.22 and using LDP of  $(W_n)_n$  and 3.1.21(1) we continue by

$$\begin{aligned} &\leq \sup_{\mu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}})} \left\{ h(\mu) + \int_{\Sigma_A^{\mathbb{N}}} \phi d\mu \right\} - \sup_{\mu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}})} \left\{ h(\mu) + \int_{\Sigma_A^{\mathbb{N}}} \phi d\mu \right\} - \inf_{\substack{\mu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}}) \\ U\mu \in C}} \left\{ - \int_{\Sigma_A^{\mathbb{N}}} \phi d\mu - h(\mu) \right\} \\ &= \sup_{\substack{\mu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}}) \\ U\mu \in C}} \left\{ \int_{\Sigma_A^{\mathbb{N}}} \phi d\mu + h(\mu) \right\}. \end{aligned}$$

So we conclude

$$\overline{P}_C^U(\phi) = \limsup_{n \rightarrow +\infty} \frac{1}{n} \log \sum_{\substack{\mathbf{i} \in \Sigma_A^{\mathbb{N}} \\ M_n[\mathbf{i}] \subset C}} s_{\mathbf{i}} \leq \sup_{\substack{\mu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}}) \\ U\mu \in C}} \left\{ \int_{\Sigma_A^{\mathbb{N}}} \phi d\mu + h(\mu) \right\}. \blacksquare$$

**Proposition 3.1.24.** (Variational Principle part 2) Let  $\phi : \Sigma_A^{\mathbb{N}} \rightarrow \mathbb{R}$  be a continuous function. If  $V$  is an open set then

$$\overline{P}_V^U(\phi) \geq \sup_{\substack{\mu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}}) \\ U\mu \in V}} \left\{ \int_{\Sigma_A^{\mathbb{N}}} \phi d\mu + h(\mu) \right\}.$$

Proof.

Proof is similar as in the case above and hence omitted.  $\blacksquare$

Therefore our proof of 3.1.4 is concluded.  $\blacksquare$

## 3.2 Geometric multifractal zeta function

Here we will assume that  $X$  is metric space.  $\Lambda \in H(\Sigma_A^{\mathbb{N}})$ ,  $\Lambda < 0$ ,  $U : \mathcal{P}(\Sigma_A^{\mathbb{N}}) \rightarrow X$  is continuous with respect to weak topology. And we will denote  $s_{\mathbf{i}}(\Lambda)$  as  $s_{\mathbf{i}}$ . And let  $C \subset X$  be closed. We will prove the following inequalities

$$\overline{f}_c^{U,\Lambda}(C) \leq \limsup_{\epsilon \rightarrow 0} \sigma_{ab}(\zeta_{B(C,\epsilon)}^{U,\Lambda}(\cdot)) \leq \sup_{\substack{\mu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}}) \\ U\mu \in C}} - \frac{h(\mu)}{\int \Lambda d\mu} \leq \underline{f}_c^{U,\Lambda}(C) \leq \liminf_{\epsilon \rightarrow 0} \sigma_{ab}(\zeta_{B(C,\epsilon)}^{U,\Lambda}(\cdot)) \quad (3.9)$$

Due to the lack of a better name,  $\sup_{\substack{\mu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}}) \\ U\mu \in C}} \left\{ - \frac{h(\mu)}{\int \Lambda d\mu} \right\}$  we will call zero of variational multifractal pressure. We will also show that  $\underline{f}_c^{U,\Lambda}(C) \leq \overline{f}_c^{U,\Lambda}(C)$  as proposition 3.2.6. This completes proof of 2.2.4. Combining 3.2.6 with 3.9 important immediate corollary is that

**Proposition 3.2.1.** *Let  $C \subset X$  be a closed set and let  $\Lambda \in H(\Sigma_A^{\mathbb{N}}), \Lambda < 0$ . Then*

$$\underline{f}_c^{U,\Lambda}(C) = \overline{f}_c^{U,\Lambda}(C) = f_c^{U,\Lambda}(C) = f^{U,\Lambda}(C) = \sup_{\substack{\mu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}}) \\ U\mu \in C}} \left\{ -\frac{h(\mu)}{\int_{\Sigma_A^{\mathbb{N}}} \Lambda d\mu} \right\}.$$

In the end of the section we will prove 2.2.6.

### Abscissa of convergence is bigger or equal to upper coarse multifractal spectra

This result is proven directly.

**Proposition 3.2.2.** *Let  $C \subset X$  be a closed set and let  $\Lambda \in H(\Sigma_A^{\mathbb{N}}), \Lambda < 0$ . Then*

$$\overline{f}_c^{U,\Lambda}(C) \leq \limsup_{\epsilon \rightarrow 0} \sigma_{ab}(\zeta_{B(C,\epsilon)}^{U,\Lambda}(\cdot)).$$

Proof.

Recall the following definition. For any set  $D \subset X$ ,

$$\overline{f}_c^{U,\Lambda}(D) = \lim_{\delta \rightarrow 0} \overline{f}_c^{U,\Lambda}(D, \epsilon) = \lim_{\delta \rightarrow 0} \limsup_{r \rightarrow 0} \frac{\log N_r^{U,\Lambda}(D, \epsilon)}{-\log r}.$$

It is enough to prove that  $\overline{f}_c^{U,\Lambda}(C, \epsilon) \leq \sigma_{ab}(\zeta_{B(C,\epsilon)}^{U,\Lambda}(\cdot))$ .

There is a descending sequence  $(r_n)_n, r_n \rightarrow 0$ , such that

$$\overline{f}_c^{U,\Lambda}(C, \epsilon) = \lim_{n \rightarrow +\infty} \frac{\log N_{r_n}^{U,\Lambda}(C, \epsilon)}{-\log r_n}.$$

We will now construct sequence  $(r'_n)_n$ , subsequence of  $(r_n)_n$ , such that for each  $\mathbf{i}$  there is at most one  $n$  such that  $s_{\mathbf{i}} \sim r'_n$ .

Let  $r < s_{\min}^2$ . We will construct  $(r'_n)_{n \in \mathbb{N}}$  by removing elements of  $r_n$  until we have not more than one of the elements in each of intervals  $[r^{k+1}, r^k], k \in \mathbb{N}$ .

Note that now  $s_{\mathbf{i}} \sim r'_n$ , we have

$$r'_{n+1} < s_{\min} r'_n \leq s_{\mathbf{i}} \leq r'_n.$$

Hence  $r'_n$  has the desired property. Let us assume that  $t$  is such that  $\zeta_{B(C,\epsilon)}^{U,\Lambda}(t)$  converge. Now let us write

$$\begin{aligned} \zeta_{B(C,\epsilon)}^{U,\Lambda}(t) &= \sum_{\substack{\mathbf{i} \in \Sigma_A^* \\ UL_{|\mathbf{i}|}[\mathbf{i}] \in B(C,\epsilon)}} s_{\mathbf{i}}^t \\ &\geq \sum_n \sum_{\substack{\mathbf{i} \in \Sigma_A^*, s_{\mathbf{i}} \sim r'_n \\ UL_{|\mathbf{i}|}[\mathbf{i}] \in B(C,\epsilon)}} s_{\mathbf{i}}^t \end{aligned}$$

Note that the last inequality holds since due to the property of  $(r'_n)_n$  we have counted every  $\mathbf{i}$  at most once. Note that since  $s_{\mathbf{i}} \sim r'_n \Rightarrow s_{\mathbf{i}} \geq s_{\min} r'_n$ . The number of elements in set  $\{\mathbf{i} \mid \mathbf{i} \in \Sigma_A^*, s_{\mathbf{i}} \sim r'_n, UL_{|\mathbf{i}|}[\mathbf{i}] \in B(C,\epsilon)\}$  is exactly  $N_{r'_n}^{U,\Lambda}(C, \epsilon)$ . Hence we got

$$\sum_n \sum_{\substack{\mathbf{i} \in \Sigma_A^*, s_{\mathbf{i}} \sim r'_n \\ UL_{|\mathbf{i}|}[\mathbf{i}] \in B(C,\epsilon)}} s_{\mathbf{i}}^t \geq C s_{\min}^t \sum_n N_{r'_n}^{U,\Lambda}(C, \epsilon) (r'_n)^t.$$

The sum above converges so we have that

$$\lim_{n \rightarrow +\infty} N_{r'_n}^{U,\Lambda}(C, \epsilon)(r'_n)^t = 0$$

Therefore for all  $n$  bigger than some  $n_0$  we have

$$N_{r'_n}^{U,\Lambda}(C, \epsilon)(r'_n)^t < 1.$$

Which implies

$$N_{r'_n}^{U,\Lambda}(C, \epsilon) < (r'_n)^{-t}.$$

Hence finally we have

$$\begin{aligned} \bar{f}_c^{U,\Lambda}(C, \epsilon) &= \lim_{n \rightarrow +\infty} \frac{\log N_{r'_n}^{U,\Lambda}(C, \epsilon)}{-\log r'_n} \\ &\leq \frac{\log((r'_n)^{-t})}{-\log r'_n} \\ &= t. \end{aligned}$$

Since the above is true whenever  $\zeta_{B(C,\epsilon)}^{U,\Lambda}(t)$  converges, we have  $\bar{f}_c^{U,\Lambda}(C, \epsilon) \leq \sigma_{ab}(\zeta_{B(C,\epsilon)}^{U,\Lambda}(\cdot))$ . ■

### Zero of variational multifractal pressure is less or equal to lower multifractal spectra

In this section we will prove that for a closed set  $C \subset \mathcal{P}(\Sigma_A^{\mathbb{N}})$ , and function  $\Lambda \in H(\Sigma_A^{\mathbb{N}})$ ,  $\Lambda < 0$  we have

$$\sup_{\substack{U\mu \in C \\ \mu \in \mathcal{P}_s(\Sigma_A^{\mathbb{N}})}} -\frac{h(\mu)}{\int_{\Sigma_A^{\mathbb{N}}} \Lambda d\mu} \leq \bar{f}_c^{U,\Lambda}(C). \quad (3.10)$$

Let us start with the left side. By upper semi continuity we have that there is  $\mu_0 \in \mathcal{P}_s(\Sigma_A^{\mathbb{N}})$  such that

$$\sup_{\substack{U\mu \in C \\ \mu \in \mathcal{P}_s(\Sigma_A^{\mathbb{N}})}} -\frac{h(\mu)}{\int_{\Sigma_A^{\mathbb{N}}} \Lambda d\mu} = -\frac{h(\mu_0)}{\int_{\Sigma_A^{\mathbb{N}}} \Lambda d\mu_0}. \quad (3.11)$$

Hence let us investigate the expression  $-\frac{h(\mu_0)}{\int_{\Sigma_A^{\mathbb{N}}} \Lambda d\mu_0}$ . Let us note the following. It is folklore but we will include proof for the sake of completeness.

**Proposition 3.2.3.** *Let  $\mu \in \mathcal{P}_s(\Sigma_A^{\mathbb{N}})$ . And let  $\mu$  be ergodic. Then  $\dim_H \mu = -\frac{h(\mu)}{\int_{\Sigma_A^{\mathbb{N}}} \Lambda d\mu}$ .*

Proof.

From  $\mu$  being ergodic using Birkhoff ergodic theorem we have

$$\lim_{n \rightarrow +\infty} \frac{\sum_{k=0}^{n-1} \Lambda(S^k \mathbf{i})}{n} = \int_{\Sigma_A^{\mathbb{N}}} \Lambda d\mu, \mu\text{-a.s.}$$

Now remember that from 3.0.2 we have that

$$\lim_{n \rightarrow +\infty} \frac{\log s_{\mathbf{i}|n}(\Lambda)}{n} = \int_{\Sigma_A^{\mathbb{N}}} \Lambda d\mu \quad \mu\text{-a.s.}$$

By Shannon-MacMillan-Breiman theorem we have that

$$\lim_{n \rightarrow +\infty} \frac{\log \mu([i]_n)}{n} = -h(\mu) \text{ } \mu\text{-a.s..}$$

Hence combining the equalities above we have that

$$\lim_{n \rightarrow +\infty} \frac{\log \mu([i]_n)}{\log s_{i|n}} = \lim_{n \rightarrow +\infty} \frac{\log \mu([i]_n)}{n} \frac{n}{\log s_{i|n}} = -\frac{h(\mu)}{\int \Lambda d\mu} \text{ for } \mu\text{-a.s..} \quad (3.12)$$

Now let us fix  $i \in \Sigma_A^{\mathbb{N}}$ . Let  $n(r) : \mathbb{R} \rightarrow \mathbb{N}$  be such that  $s_{i|n(r)} \sim r$ . Note that  $s_{i|n(r)} \sim r$  implies that  $s_{i|n(r)} \leq r \leq s_{\max} s_{i|n(r)}$ . Now we have

$$\begin{aligned} \dim_{loc} \mu(i) &= \lim_{r \rightarrow 0} \frac{\log \mu(B(i, r))}{\log r} \\ &= \lim_{r \rightarrow 0} \frac{\log \mu([i]_{n(r)})}{\log r} \\ &= \lim_{r \rightarrow 0} \frac{\log \mu([i]_{n(r)})}{\log s_{i|n(r)}} && \text{using } s_{i|n(r)} \leq r \leq s_{\max} s_{i|n(r)} \\ &= \lim_{n \rightarrow +\infty} \frac{\log \mu([i]_n)}{\log s_{i|n}} \\ &= -\frac{h(\mu)}{\int_{\Sigma_A^{\mathbb{N}}} \Lambda d\mu} \text{ for } \mu\text{-a.s.} && \text{using 3.12} \end{aligned}$$

Therefore from the definitions it follows that

$$\dim_H \mu = -\frac{h(\mu)}{\int_{\Sigma_A^{\mathbb{N}}} \Lambda d\mu}. \blacksquare \quad (3.13)$$

Measure  $\mu_0$  is invariant but not ergodic, hence in order to use the result above, we will need some approximation argument. The following lemma tells about approximation of invariant measures, supported on whole set, with ergodic measures.

**Lemma 3.2.4.** *Let  $\mu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}})$  with  $\text{supp } \mu = \Sigma_A^{\mathbb{N}}$ . Then there is a sequence  $(\mu_n)_n \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}})$  satisfying the following three conditions:*

1.  $\mu_n \rightarrow \mu$  weakly.
2.  $\mu_n$  is ergodic.
3. We have  $h(\mu_n) \rightarrow h(\mu)$ .

*Proof.*

First let us define  $\mu_n([i]_n) = \mu([i]_n)$ . Hence we will get that  $\mu_n$  is equal to  $\mu$  on first  $n$  cylinders. Then let us define  $\mu_n$  on cylinders of length greater than  $n$  as follows.

For  $m > n$  we put

$$\mu_n([i]_m) = \prod_{k=1}^{m-n} \frac{\mu_n([S^{(k-1)}i]_n)}{\mu_n([S^k i]_{n-1})} \mu_n([S^{m-n} i]_n).$$

This being measure could be seen at (cf. [Wa, p.5]).

Note that  $\mu_n$  being equal to  $\mu$  on first  $n$  cylinders implies that  $\mu_n \rightarrow \mu$  weakly. Hence (1) is satisfied. We will prove ergodicity by proving that  $\mu_n$  is invariant Gibbs measure, and therefore ergodic, with Hölder potential

$$\phi_n(i) = \log \left( \frac{\mu([i]_n)}{\mu([S i]_{n-1})} \right).$$



Let us first prove that  $\mu_n$  is Gibbs measure associated with potential  $\phi_n$  with  $P(\phi_n) = 0$ . We will do that directly.

$$\begin{aligned} \exp(S_m \phi_n(\mathbf{i})) &= \prod_{k=1}^m \frac{\mu_n[S^{k-1}\mathbf{i}]_n}{\mu_n[S^k\mathbf{i}]_{n-1}} \\ &= \prod_{k=1}^{m-n} \frac{\mu_n[S^{k-1}\mathbf{i}]_n}{\mu_n[S^k\mathbf{i}]_{n-1}} \prod_{k=m-n+1}^m \frac{\mu_n[S^{k-1}\mathbf{i}]_n}{\mu_n[S^k\mathbf{i}]_{n-1}} \end{aligned}$$

Now we have

$$\frac{\mu_n[\mathbf{i}]_m}{\exp(S_m \phi_n(x))} = \frac{\mu_n([S^{m-n}\mathbf{i}]_n)}{\prod_{k=m-n+1}^m \frac{\mu_n[S^{k-1}\mathbf{i}]_n}{\mu_n[S^k\mathbf{i}]_{n-1}}}.$$

Which ends the proof, since the left term is bounded between two constants, as can be seen below:

$$\frac{\min_{\mathbf{i} \in \Sigma_A^n} \{\mu([\mathbf{i}]_n)\}}{\max_{\mathbf{i} \in \Sigma_A^n} \left\{ \frac{\mu([\mathbf{i}]_n)}{\mu([S\mathbf{i}]_{n-1})} \right\}^n} \leq \frac{\mu_n([S^{m-n}\mathbf{i}]_n)}{\prod_{k=m-n+1}^m \frac{\mu_n[S^{k-1}\mathbf{i}]_n}{\mu_n[S^k\mathbf{i}]_{n-1}}} \leq \frac{\min_{\mathbf{i} \in \Sigma_A^n} \{\mu([\mathbf{i}]_n)\}}{\max_{\mathbf{i} \in \Sigma_A^n} \left\{ \frac{\mu([\mathbf{i}]_n)}{\mu([S\mathbf{i}]_{n-1})} \right\}^n}.$$

Since  $\mu_n$  is equilibrium measure for  $\phi_n$ , and  $\phi_n$  is Hölder continuous we only need to show that it is shift invariant in order to prove its ergodicity. We will do that straight from the definitions. It is enough to prove that on cylinders.

$$\begin{aligned} \sum_{i \in \Sigma, \mathbf{i} \in S^{-1}([\mathbf{i}])} \mu_n([i\mathbf{i}]_m) &= \sum_{i \in \Sigma, \mathbf{i} \in S^{-1}([\mathbf{i}])} \prod_{k=1}^{m-n} \frac{\mu_n([S^{k-1}i\mathbf{i}]_n)}{\mu_n([S^k\mathbf{i}]_{n-1})} \mu_n([S^{m-n}i\mathbf{i}]_n) \\ &= \prod_{k=1}^{m-n-1} \frac{\mu_n([S^{k-1}\mathbf{i}]_n)}{\mu_n([S^k\mathbf{i}]_{n-1})} \mu_n([S^{m-n-1}\mathbf{i}]_n) \sum_{i \in \Sigma, \mathbf{i} \in S^{-1}([\mathbf{i}])} \frac{\mu_n([i\mathbf{i}]_n)}{\mu_n([\mathbf{i}]_{n-1})} \\ &= \mu_n([\mathbf{i}]_{m-1}). \end{aligned}$$

Which ends the proof of (2).

From  $P(\phi_n) = 0$ , and the fact that  $\mu_n$  is Gibbs measure of  $\phi_n$ , using variational principle we have.

$$h(\mu_n) = - \int \phi_n d\mu_n.$$

Now

$$\begin{aligned} - \int \phi_n d\mu_n &= - \sum_{\mathbf{i} \in \Sigma^n} \mu_n[\mathbf{i}]_n \log \frac{\mu_n[\mathbf{i}]_n}{\mu_n[S\mathbf{i}]_{n-1}} \\ &= - \sum_{\mathbf{i} \in \Sigma^n} \mu_n[\mathbf{i}]_n \log \frac{\mu_n[\mathbf{i}]}{\sum_{j \in \Sigma} \mu_n[jS\mathbf{i}]_n} \\ &= H(C_n | S^{-1}C_n) \end{aligned}$$

Now we have

$$\lim_{n \rightarrow +\infty} H(C_n | S^{-1}C_n) = h(\mu).$$

Hence the lemma is proven. ■

So at this moment, starting from the left side of inequality we are roughly able to derive, at least for measures that are supported at all  $\Sigma_A^{\mathbb{N}}$ , something like

$$-\frac{h(\mu)}{\int_{\Sigma_A^{\mathbb{N}}} \Lambda d\mu} = \dim_H \mu.$$

Recall in 2.1 we have defined

$$E_C = \left\{ \mathbf{i} \in \Sigma_A^{\mathbb{N}} \mid \lim_{n \rightarrow +\infty} \text{dist}(UL_n \mathbf{i}, C) = 0 \right\}.$$

And  $f^{U, \Lambda}(C) = \dim_H E_C$ . Let us first prove one simple lemma

**Lemma 3.2.5.** *There is a constant  $C$  such that for each  $\mathbf{i} \in \Sigma_A^{\mathbb{N}}$  we have  $\text{diam} L_n[\mathbf{i}]_n \leq \frac{C}{n}$ .*

Proof.

Let  $\mathbf{i}, \mathbf{j} \in [\mathbf{i}]_n$ . Then

$$\begin{aligned} d_L(L_n \mathbf{i}, L_n \mathbf{j}) &= \sup_{\text{Lip}(f) < 1} \left| \int_{\Sigma_A^{\mathbb{N}}} f dL_n \mathbf{i} - \int_{\Sigma_A^{\mathbb{N}}} f dL_n \mathbf{j} \right| \\ &= \sup_{\text{Lip}(f) < 1} \left| \sum_{k=0}^{n-1} (f(S^k \mathbf{i}) - f(S^k \mathbf{j})) \right| \\ &\leq \frac{1}{n} \sum_{k=0}^{+\infty} \gamma^n \end{aligned} \quad \text{recall } \gamma < 1, \gamma \text{ generates } d_\gamma.$$

Now we have

**Proposition 3.2.6.** *Let  $C \subset X$  then*

$$f^{U, \Lambda}(C) \leq \underline{f}_C^{U, \Lambda}(C).$$

Proof.

First fix  $\epsilon > 0$ . Let  $\mathbf{i} \in E_C$ . Then due to  $\lim_{n \rightarrow +\infty} d(U(L_n \mathbf{i}), C) = 0$ , we have that there is  $n_{\mathbf{i}}$  such that for  $n \geq n_{\mathbf{i}}$  we have  $U(L_n \mathbf{i}) \in B(C, \epsilon)$ . Let us now define

$$\Gamma_n(C, \epsilon) = \{ \mathbf{i} \mid m \geq n \Rightarrow U(L_m \mathbf{i}) \in B(C, \epsilon) \}.$$

From above it is clear that for every  $\mathbf{i} \in E_C$  and  $n \geq n_{\mathbf{i}}$  we have  $\mathbf{i} \in \Gamma_n(C, \epsilon)$ . Combining that with monotonicity of Hausdorff dimension we conclude that

$$\begin{aligned} \dim_H E_C &\leq \dim_H \bigcup_{n \in \mathbb{N}} \Gamma_n(C, \epsilon) \\ &= \sup_{n \in \mathbb{N}} \{ \dim_H \Gamma_n(C, \epsilon) \} \end{aligned} \quad \text{by 1.1.1}$$

Now let us fix  $\epsilon_1 > 0$ . Then,

$$\begin{aligned} \sup_{n \in \mathbb{N}} \{ \dim_H \Gamma_n(C, \epsilon) \} &\leq \dim_H \Gamma_{n_0}(C, \epsilon) + \epsilon_1 \quad \text{for some } n_0 \in \mathbb{N} \\ &\leq \underline{\dim}_B \Gamma_{n_0}(C, \epsilon) + \epsilon_1 \quad \text{by 1.1.6} \end{aligned}$$

For point  $\mathbf{i}$ ,  $B(\mathbf{i}, r)$  is cylinder  $[\mathbf{i}]_n$  such that  $s_{\mathbf{i}|n} \sim r$ . As we have seen, the only way to cover  $\Gamma_{n_0}(C, \epsilon)$  with balls of diameter  $r$  is simply by taking balls of diameter  $r$  at every point  $\mathbf{i} \in \Gamma_{n_0}(C, \frac{\epsilon}{2})$ . Hence the last line could be written as

$$\underline{\dim}_B \Gamma_{n_0}(C, \epsilon) = \liminf_{r \rightarrow 0} \frac{\log | \{ \mathbf{j} \in \Sigma_A^* \mid s_{\mathbf{j}} \sim r, [\mathbf{j}] \cap \Gamma_{n_0}(C, \frac{\epsilon}{2}) \neq \emptyset \} |}{-\log r}$$

By 3.2.5 there is a constant  $D$  such that for each  $\mathbf{i} \in \Sigma_A^{\mathbb{N}}$  we have  $\text{diam} L_n[\mathbf{i}]_n \leq \frac{D}{n}$ . Due to uniform continuity of  $U$  there exists  $n_1$  such that for  $n \geq n_1$  we have  $U(L_n[\mathbf{i}]_n) < \frac{\epsilon}{2}$ . Now note that for  $r$  small enough we have that from  $s_{\mathbf{i}} \sim r$  we have that  $|\mathbf{i}| \geq \max\{n_1, n_0\}$ . Using that we got

$$\begin{aligned} \liminf_{r \rightarrow 0} \frac{\log |\{\mathbf{j} \in \Sigma_A^* | s_{\mathbf{j}} \sim r, [\mathbf{j}] \cap \Gamma_{n_0}(C, \frac{\epsilon}{2}) \neq \emptyset\}|}{-\log r} &\leq \liminf_{r \rightarrow 0} \frac{\log |\{\mathbf{j} \in \Sigma_A^* | s_{\mathbf{j}} \sim r, U(L_{|\mathbf{j}|}[\mathbf{j}]) \in B(C, \epsilon)\}|}{-\log r} \\ &= \underline{f}_c^{U, \Lambda}(C, \epsilon). \end{aligned}$$

This being true for any  $\epsilon$  we have

$$f^{U, \Lambda}(C) = \dim_{\text{H}} E_C \leq \liminf_{\epsilon \rightarrow 0} \underline{f}_c^{U, \Lambda}(C, \epsilon) + \epsilon_1 = \underline{f}_c^{U, \Lambda}(C) + \epsilon_1.$$

Since  $\epsilon_1$  could be arbitrarily small we have our lemma proved. ■

We will need the following continuity result for  $\underline{f}_c^{U, \Lambda}(C)$ . So we will first define the appropriate topology and then state result.

**Definition 3.2.7** (Topology of closed sets). *Denote*

$$\mathcal{F}(\Sigma_A^{\mathbb{N}}) := \{F \subset \Sigma_A^{\mathbb{N}} | F \text{ closed and non-empty}\}. \quad (3.14)$$

**Definition 3.2.8** (Hausdorff Metric). *Metric  $D$  on  $\mathcal{F}(\Sigma_A^{\mathbb{N}})$  is defined by*

$$D(A, B) = \min \left\{ 1, \max \left( \inf_{x \in A} \{ \text{dist}(x, B) \}, \inf_{x \in B} \{ \text{dist}(x, A) \} \right) \right\}.$$

**Proposition 3.2.9.**  $\underline{f}_c^{U, \Lambda} : \mathcal{F}(\Sigma_A^{\mathbb{N}}) \rightarrow \mathbb{R}$  is upper semicontinuous, i.e.

$$(\forall F \in \mathcal{F}(\Sigma_A^{\mathbb{N}}))(\forall \epsilon > 0)(\exists \delta > 0) A \in \mathcal{F}(X), D(A, F) < \delta \Rightarrow \underline{f}_c^{U, \Lambda}(A) \leq \underline{f}_c^{U, \Lambda}(F) + \epsilon.$$

Proof.

Let us recall

$$\begin{aligned} \underline{f}_c^{U, \Lambda}(F, r) &= \liminf_{\delta \rightarrow 0} \frac{\log |\{\mathbf{j} \in \Sigma_A^* | s_{\mathbf{j}} \sim \delta, U(L_{|\mathbf{j}|}[\mathbf{j}]) \in B(F, r)\}|}{\log \delta}. \\ \underline{f}_c^{U, \Lambda}(F) &= \lim_{r \rightarrow 0} \underline{f}_c^{U, \Lambda}(F, r) \end{aligned}$$

So there  $r_0$  such that if

$$r \leq r_0 \Rightarrow \underline{f}_c^{U, \Lambda}(F, r) \leq \underline{f}_c^{U, \Lambda}(F) + \epsilon.$$

Now if we let  $\delta = \frac{r_0}{2}$  then from  $D(A, F) < \delta \Rightarrow A \subset B(F, r_0)$ , and

$$\underline{f}_c^{U, \Lambda}(A) \leq \underline{f}_c^{U, \Lambda}(F, r) \leq \underline{f}_c^{U, \Lambda}(F) + \epsilon.$$

Hence the claim. ■

Finally let us prove

**Theorem 3.2.10.** *Let  $C$  be closed set and let  $\Lambda \in H(\Sigma_A^{\mathbb{N}})$ ,  $\Lambda < 0$ . Then*

$$\sup_{\substack{\mu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}}) \\ U\mu \in C}} - \frac{h(\mu)}{\int_{\Sigma_A^{\mathbb{N}}} \Lambda d\mu} \leq \underline{f}_c^{U, \Lambda}(C).$$

Proof.

Let  $\mu_0$  be from 3.11. Let  $\nu$  be invariant probability measure such that  $\text{supp}\nu = \Sigma_A^{\mathbb{N}}$  (for example it could be Parry measure). Then define  $\mu_t = (1-t)\mu_0 + t\nu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}})$ . Note that for  $1 > t > 0$ ,  $\text{supp}\mu_t = \Sigma_A^{\mathbb{N}}$ , and  $\mu_t$  is invariant measure. Then by 3.2.4 there is a sequence of ergodic measures such that  $\mu_{t,n} \rightarrow \mu_t$  and  $h(\mu_{t,n}) \rightarrow h(\mu_t)$ . Note as well that since entropy is affine map we have  $h((1-t)\mu + t\nu) \geq (1-t)h(\mu) + th(\nu)$ . Hence now we have

$$\begin{aligned}
-\frac{h(\mu_0)}{\int_{\Sigma_A^{\mathbb{N}}} \Lambda d\mu_0} &= \lim_{t \rightarrow 0} -\frac{(1-t)h(\mu_0) + th(\nu)}{\int_{\Sigma_A^{\mathbb{N}}} \Lambda d\mu_t} \\
&\leq \lim_{t \rightarrow 0} -\frac{h(\mu_t)}{\int_{\Sigma_A^{\mathbb{N}}} \Lambda d\mu_t} && \text{by the fact that } h \text{ is affine} \\
&= \lim_{t \rightarrow 0} \lim_{n \rightarrow +\infty} \frac{h(\mu_{t,n})}{\int_{\Sigma_A^{\mathbb{N}}} \Lambda d\mu_{t,n}} && \text{using 3.2.4} \\
&= \lim_{t \rightarrow 0} \lim_{n \rightarrow +\infty} \dim_{\mathbb{H}} \mu_{t,n} && \text{by 3.2.3} \\
&\leq \lim_{t \rightarrow 0} \lim_{n \rightarrow +\infty} \dim_{\mathbb{H}} \left\{ \mathbf{i} \mid \lim_{k \rightarrow +\infty} L_k \mathbf{i} = \mu_{t,n} \right\} \\
&= \lim_{t \rightarrow 0} \lim_{n \rightarrow +\infty} \dim_{\mathbb{H}} \left\{ \mathbf{i} \mid \lim_{k \rightarrow +\infty} UL_k \mathbf{i} = U\mu_{t,n} \right\} && \text{due to the continuity of } U
\end{aligned}$$

Let us now fix  $r > 0$  and  $t > 0$  and  $n$  big enough  $\mu_{t,n}$  will be close to  $\mu_t$ , i.e.  $U(\mu_{t,n}) \in B(U\mu_t, \frac{r}{2})$ . Hence

$$\lim_{n \rightarrow +\infty} \dim_{\mathbb{H}} \left\{ \mathbf{i} \mid \lim_{k \rightarrow +\infty} UL_k \mathbf{i} = U\mu_{t,n} \right\} \leq \lim_{n \rightarrow +\infty} \dim_{\mathbb{H}} \left\{ \mathbf{i} \mid \lim_{k \rightarrow +\infty} UL_k \mathbf{i} \in B(U\mu_t, \frac{r}{2}) \right\}$$

Now let us fix  $r > 0$ . Because of  $\mu_t \rightarrow \mu_0$ , and due to the continuity of  $U$ , we have that for  $t$  small enough  $U\mu_t \in B(\mu_0, \frac{r}{2})$ . Combining with the above we got.

$$\lim_{t \rightarrow 0} \lim_{n \rightarrow +\infty} \dim_{\mathbb{H}} \left\{ \mathbf{i} \mid \lim_{k \rightarrow +\infty} UL_k \mathbf{i} \in B(U\mu_t, \frac{r}{2}) \right\} \leq \lim_{t \rightarrow 0} \lim_{n \rightarrow +\infty} \dim_{\mathbb{H}} \left\{ \mathbf{i} \mid \lim_{k \rightarrow +\infty} UL_k \mathbf{i} \in B(U\mu_0, r) \right\}$$

Now combining the expressions above we have

$$\begin{aligned}
\lim_{t \rightarrow 0} \lim_{n \rightarrow +\infty} \dim_{\mathbb{H}} \left\{ \mathbf{i} \mid \lim_{k \rightarrow +\infty} UL_k \mathbf{i} = U\mu_{t,n} \right\} &\leq \lim_{t \rightarrow 0} \lim_{n \rightarrow +\infty} \dim_{\mathbb{H}} \left\{ \mathbf{i} \mid \lim_{k \rightarrow +\infty} UL_k \mathbf{i} \in B(\mu_0, r) \right\} \\
&\leq f_c^{U,\Lambda}(B(C, r)) \\
&\leq \underline{f}_c^{U,\Lambda}(B(C, r)) && \text{by 3.2.6}
\end{aligned}$$

$\underline{f}_c^{U,\Lambda}$  is upper semi-continuous, and the expression above is true for any  $r$ , so we could choose  $r$  such that for any  $\epsilon > 0$

$$\underline{f}_c^{U,\Lambda}(B(C, r)) \leq \underline{f}_c^{U,\Lambda}(C) + \epsilon. \tag{3.15}$$

Connecting inequalities got that for any  $\epsilon > 0$

$$\sup_{\substack{\mu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}}) \\ U\mu \in C}} -\frac{h(\mu)}{\int_{\Sigma_A^{\mathbb{N}}} \Lambda d\mu} = -\frac{h(\mu_0)}{\int_{\Sigma_A^{\mathbb{N}}} \Lambda d\mu_0} \leq \underline{f}_c^{U,\Lambda}(C) + \epsilon.$$

Which concludes the proof of our theorem. ■

**Abscissa of convergence is less or equal to zero of multifractal variational pressure**

$$\begin{aligned}\zeta_C^{U,\Lambda}(s) &= \sum_{\substack{\mathbf{i} \in \Sigma_A^* \\ UL_{|\mathbf{i}|}[\mathbf{i}] \in B(C,r)}} s_{\mathbf{i}}^s \\ &= \sum_{n=1}^{+\infty} \sum_{\substack{\mathbf{i} \in \Sigma_A^n \\ UL_n[\mathbf{i}] \in B(C,r)}} s_{\mathbf{i}}^s\end{aligned}$$

It obviously converges when

$$\limsup_{n \rightarrow +\infty} \left( \sum_{\substack{\mathbf{i} \in \Sigma_A^n \\ UL_n[\mathbf{i}] \in B(C,r)}} s_{\mathbf{i}}^s \right)^{1/n} < 1,$$

or equivalently when

$$\limsup_{n \rightarrow +\infty} \frac{1}{n} \log \sum_{\substack{\mathbf{i} \in \Sigma_A^n \\ UL_n[\mathbf{i}] \in B(C,r)}} s_{\mathbf{i}}^s < 0. \quad (3.16)$$

Note that

$$\limsup_{n \rightarrow +\infty} \frac{1}{n} \log \sum_{\substack{\mathbf{i} \in \Sigma_A^n \\ UL_n[\mathbf{i}] \in B(C,r)}} s_{\mathbf{i}}^s = P_{B(C,r)}^U(s\Lambda) \leq \widehat{P}_{B[C,r]}^U(s\Lambda) \quad \text{by 3.1.4 .} \quad (3.17)$$

Let now assume that  $s > \sup_{\substack{\mu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}}) \\ U\mu \in C}} - \frac{h(\mu)}{\int_{\Sigma_A^{\mathbb{N}}} \Lambda d\mu}$ . From 3.1.9 we have that  $\widehat{P}_C^U(s\Lambda) < 0$ . Next recall 3.1.8

says that

$$\lim_{r \rightarrow 0} P_{B(C,r)}^U(s\Lambda) = \widehat{P}_C^U(s\Lambda).$$

Hence there is  $r_0 > 0$  such that for each  $r$  such that  $r < r_0$  we have  $P_{B(C,r)}(s\Lambda) < 0$ . Hence from 3.16 and 3.17 we have that  $\zeta_{B(C,r)}^{U,\Lambda}(s)$  converges so  $\sigma_{ab}(\zeta_{B(C,r)}^{U,\Lambda}(\cdot)) < s$ . So we finally conclude

$$\limsup_{r \rightarrow 0} \sigma_{ab}(\zeta_{B(C,r)}^{U,\Lambda}(\cdot)) \leq \sup_{\substack{\mu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}}) \\ U\mu \in C}} - \frac{h(\mu)}{\int_{\Sigma_A^{\mathbb{N}}} \Lambda d\mu}.$$

### 3.2.1 Geometric zeta function - the fixed target settings

Now we will prove the main result for Geometric zeta function - in the fixed target settings.

Fix  $M \in \mathbb{N}$ . For  $x, y \in \mathbb{R}^M$ , write

$$\llbracket x, y \rrbracket = \{(1-t)x + ty | t \in [0, 1]\}$$

**Lemma 3.2.11.** *Let  $E \subset \mathbb{R}^M$  and let  $x \in E$  and  $y \in \mathbb{R}^M \setminus E$ . Then  $\llbracket x, y \rrbracket \cap \partial E \neq \emptyset$*

*Proof.*

If we let  $t_0 = \sup\{t \in [0, 1] | (1-t)x + ty \in E\}$ . Then it is easy to see that  $t_0x + (1-t_0)y \in \partial E$ . ■

**Lemma 3.2.12.** *Let  $C \subset \mathbb{R}^M$  and let  $x \in E$  be a closed subset of  $\mathbb{R}^M$  and let  $r, \epsilon > 0$  with  $r < \epsilon$ . Then  $B(I[C, \epsilon], r) \subset C$ .*

Proof.

Let  $y \in B(I[C, \epsilon], r)$ . We must prove that  $y \in C$ . Assume, in order to reach a contradiction, that  $y \notin C$ . Since  $I[C, \epsilon]$  is closed there is  $x \in I[C, \epsilon]$  such that  $|x - y| = \text{dist}(y, I[C, \epsilon])$ . Also since  $x \in I[C, \epsilon]$  and  $y \notin C$  from lemma above, there is  $v \in \llbracket x, y \rrbracket \cap \partial C$ . We now conclude that

$$\begin{aligned} r &\geq \text{dist}(y, I[C, \epsilon]) && \text{since } y \in B(I[C, \epsilon], r) \\ &= |y - x| \\ &\geq |v - x| && \text{since } v \in \llbracket x, y \rrbracket \\ &\geq \text{dist}(x, \partial C) && \text{since } v \in \partial C \\ &\geq \epsilon && \text{since } x \in I[C, \epsilon] \end{aligned}$$

Which is a contradiction. Hence we conclude that  $y \in C$ . ■

Now let us prove theorem 2.2.6. Note that  $\sigma_{ab}(\zeta_{B(C, r)}^{U, \Lambda}(\cdot))$  is non increasing as  $r$  decreases. Hence  $\lim_{r \rightarrow 0} \sigma_{ab}(\zeta_{B(C, r)}^{U, \Lambda}(\cdot)) \geq \sigma_{ab}(\zeta_C^{U, \Lambda}(\cdot))$ . Hence using theorem 2.2.4  $\sigma_{ab}(\zeta_C^{U, \Lambda}(\cdot)) \leq f^{U, \Lambda}(C)$ . Next for  $\epsilon < r$ , using lemma above we have that  $\sigma_{ab}(\zeta_C^{U, \Lambda}(\cdot)) \geq \sigma_{ab}(\zeta_{B(I[C, r], \epsilon)}^{U, \Lambda}(\cdot)) \geq f^{U, \Lambda}(I[C, r])$ . So now taking  $r \rightarrow 0$  and using theorem 2.2.4 we prove our result. ■

### 3.3 Renyi dimension

Now we will prove the existence of Renyi dimension in general case i.e. let us state again and prove theorem 2.1.4.

**Theorem 2.1.4.** *Let  $X$  be scalar space with scalar product  $\langle \cdot, \cdot \rangle$ . Let  $\Lambda \in H(\Sigma_A^{\mathbb{N}})$ ,  $\Lambda < 0$  then we can define  $\tau^{U, \Lambda} : X \rightarrow \mathbb{R}$*

$$\tau^{U, \Lambda}(q) = \lim_{r \rightarrow 0} \frac{\log \sum_{s_i \sim r} s_i^{\langle q, U(L_{|i|} \bar{i}) \rangle}}{-\log r}.$$

Additionally

$$\tau^{U, \Lambda}(q) = \sup_{\mu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}})} \left\{ -\frac{h(\mu)}{\int \Lambda d\mu} - \langle q, U(\mu) \rangle \right\}.$$

Proof.

Idea of the proof is a rather simple one. Heuristic is as follows

$$\begin{aligned} \sum_{s_i \sim r} s_i^{qU(L_{|i|} \bar{i})} &\approx \sum_i \sum_{s_i \sim r, U(L_{|i|} \bar{i}) \approx c_i} s_i^{qU(L_{|i|} \bar{i})} \\ &\approx \max_i \sum_{s_i \sim r, U(L_{|i|} \bar{i}) \approx c_i} s_i^{qU(L_{|i|} \bar{i})} && \text{only the biggest element of sum affects asymptotics} \\ &\approx \max_i N^{U, \Lambda}(c_i) r^{qc_i} \\ &\approx \max_i r^{-f^{U, \Lambda}(c_i)} (c_i) r^{qc_i} \\ &\approx \max_i r^{-\sup_{\mu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}}), U(\mu) = c_i} \frac{-h(\mu)}{\int \Lambda d\mu}} r^{qc_i} \\ &\approx \max_i r^{-\sup_{\mu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}}), U(\mu) = c_i} \left\{ \frac{-h(\mu)}{\int \Lambda d\mu} - qc_i \right\}} \end{aligned}$$

Hence

$$\lim_{r \rightarrow 0} \frac{\log \sum_{s_i \sim r} s_i^{qU(L_{|i|}\bar{i})}}{-\log r} = \sup_{\mu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}})} \left\{ -\frac{h(\mu)}{\int_{\Sigma_A^{\mathbb{N}}} \Lambda d\mu} - qU(\mu) \right\}.$$

Let us be more detailed now.

Function  $\left\{ -\frac{h(\mu)}{\int_{\Sigma_A^{\mathbb{N}}} \Lambda d\mu} - \langle q, U(\mu) \rangle \right\}$  is upper semi-continuous, hence it reaches maximum on  $\mathcal{P}(\Sigma_A^{\mathbb{N}})$ .

So we can chose  $\mu_q$  such that

$$\sup_{\mu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}})} \left\{ -\frac{h(\mu)}{\int_{\Sigma_A^{\mathbb{N}}} \Lambda d\mu} - \langle q, U(\mu) \rangle \right\} = -\frac{h(\mu_q)}{\int_{\Sigma_A^{\mathbb{N}}} \Lambda d\mu_q} - \langle q, U(\mu_q) \rangle.$$

And let us denote  $U(\mu_q) = c_q$ . Note that we have

$$\sup_{\mu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}})} \left\{ -\frac{h(\mu)}{\int_{\Sigma_A^{\mathbb{N}}} \Lambda d\mu} - \langle q, U(\mu) \rangle \right\} = -\frac{h(\mu_q)}{\int_{\Sigma_A^{\mathbb{N}}} \Lambda d\mu_q} - \langle q, U(\mu_q) \rangle = \sup_{\substack{\mu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}}) \\ U\mu = c_q}} \left\{ -\frac{h(\mu)}{\int_{\Sigma_A^{\mathbb{N}}} \Lambda d\mu} - \langle q, U(\mu) \rangle \right\} \quad (3.18)$$

We will first prove

$$\liminf_{r \rightarrow 0} \frac{\log \sum_{s_i \sim r} s_i^{\langle q, U(L_{|i|}\bar{i}) \rangle}}{-\log r} \geq \sup_{\mu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}})} \left\{ -\frac{h(\mu)}{\int_{\Sigma_A^{\mathbb{N}}} \Lambda d\mu} - \langle q, U(\mu) \rangle \right\}.$$

Let us fix  $\epsilon > 0$ . Then let us choose  $\frac{\epsilon}{3} > \epsilon_1 > 0$  such that

$$\underline{f}_c^{U, \Lambda}(c_q, \epsilon_1) - f^{U, \Lambda}(c_q) \leq \frac{\epsilon}{3}. \quad (3.19)$$

Recall

$$\liminf_{r \rightarrow 0} \frac{\log N_r^{U, \Lambda}(c_q, \epsilon_1)}{-\log r} = \underline{f}_c^{U, \Lambda}(c_q, \epsilon_1).$$

Next for  $r$  smaller than some  $r_0$  we have

$$\frac{\log N_r^{U, \Lambda}(c_q, \epsilon_1)}{-\log r} \geq \underline{f}_c^{U, \Lambda}(c_q, \epsilon_1) - \frac{\epsilon}{3}$$

Therefore

$$N_r^{U, \Lambda}(C, \epsilon_1) \geq r^{-\underline{f}_c^{U, \Lambda}(c_q, \epsilon_1) + \frac{\epsilon}{3}}. \quad (3.20)$$

Recall that from  $s_i \sim r$  implies  $s_i \geq s_{\min} r$ . And recall that the number of elements in set  $\{i | i \in \Sigma_A^*, s_i \sim r, UL_{|i|}[i] \in B(c_q, \epsilon_1)\}$  is exactly  $N_r^{U, \Lambda}(c_q, \epsilon_1)$ . Hence now for  $r < r_0$  we have

$$\begin{aligned} \sum_{s_i \sim r} s_i^{\langle q, U(L_{|i|}\bar{i}) \rangle} &\geq \sum_{\substack{i \in \Sigma_A^*, s_i \sim r \\ UL_{|i|}[i] \in B(c_q, \epsilon_1)}} s_i^{\langle q, U(L_{|i|}\bar{i}) \rangle} \\ &\geq N_r^{U, \Lambda}(C, \epsilon_1) (s_{\min} r)^{\langle q, c_q \rangle + \frac{\epsilon}{3}} \\ &\geq s_{\min}^{\langle q, c_q \rangle + \frac{\epsilon}{3}} r^{-\underline{f}_c^{U, \Lambda}(c_q, \epsilon_1) + \langle q, c_q \rangle + 2\frac{\epsilon}{3}} \quad \text{by 3.20} \end{aligned}$$

Now we have

$$\begin{aligned}
-\underline{f}_c^{U,\Lambda}(c_q, \epsilon_1) + \langle q, c_q \rangle + 2\frac{\epsilon}{3} &\leq -f^{U,\Lambda}(c_q) - \langle q, c_q \rangle + \epsilon && \text{by 3.19} \\
&= -\sup_{\substack{\mu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}}) \\ U\mu = c_q}} \left\{ -\frac{h(\mu)}{\int_{\Sigma_A^{\mathbb{N}}} \Lambda d\mu} \right\} - \langle q, c_q \rangle + \epsilon && \text{by 3.2.1} \\
&= -\sup_{\substack{\mu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}}) \\ U\mu = c_q}} \left\{ -\frac{h(\mu)}{\int_{\Sigma_A^{\mathbb{N}}} \Lambda d\mu} - \langle q, U(\mu) \rangle \right\} + \epsilon \\
&= -\sup_{\mu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}})} \left\{ -\frac{h(\mu)}{\int_{\Sigma_A^{\mathbb{N}}} \Lambda d\mu} - \langle q, U(\mu) \rangle \right\} + \epsilon && \text{by 3.18.}
\end{aligned}$$

So

$$s_{\min}^{\langle q, c_q \rangle + \frac{\epsilon}{3}} r^{-\underline{f}_c^{U,\Lambda}(c_q, \epsilon_1) + \langle q, c_q \rangle + 2\frac{\epsilon}{3}} \geq s_{\min}^{\langle q, c_q \rangle + \frac{\epsilon}{3}} r^{-\sup_{\mu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}})} \left\{ -\frac{h(\mu)}{\int_{\Sigma_A^{\mathbb{N}}} \Lambda d\mu} - \langle q, U(\mu) \rangle \right\} + \epsilon}.$$

Therefore for every  $\epsilon > 0$  we have that

$$\liminf_{r \rightarrow 0} \frac{\log \sum_{s_i \sim r} s_i^{\langle q, U(L_{|\mathbf{i}|}) \bar{\mathbf{i}} \rangle}}{-\log r} \geq \sup_{\mu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}})} \left\{ -\frac{h(\mu)}{\int_{\Sigma_A^{\mathbb{N}}} \Lambda d\mu} - \langle q, U(\mu) \rangle \right\} - \epsilon,$$

which concludes the proof of this part.

$$\text{Now let us prove that } \limsup_{r \rightarrow 0} \frac{\log \sum_{s_i \sim r} s_i^{\langle q, U(L_{|\mathbf{i}|}) \bar{\mathbf{i}} \rangle}}{-\log r} \leq \sup_{\mu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}})} \left\{ -\frac{h(\mu)}{\int_{\Sigma_A^{\mathbb{N}}} \Lambda d\mu} - \langle q, U(\mu) \rangle \right\}.$$

Let us note that since  $\Sigma_A^{\mathbb{N}}$  is compact there is  $[a, b]$  such that  $U(\mathcal{P}_S(\Sigma_A^{\mathbb{N}})) \subset [a, b]$ . Let us choose  $\mathcal{B} := \{B_i\}_i$  an finite disjoint  $\frac{\epsilon}{4}$  - cover of  $[a, b]$ . Then let us use cover  $\mathcal{B}$  to define cover with small overlaps  $\mathcal{C}$  defined by  $\mathcal{C} = \{C_i | C_i := B[B_i, \frac{\epsilon}{24}]\}$ . Next let us for every  $C_i \in \mathcal{C}$  choose  $\epsilon_i, \frac{\epsilon}{24} > \epsilon_i > 0$ , such that

$$\bar{f}_c^{U,\Lambda}(C_i, \epsilon_i) - f^{U,\Lambda}(C_i) \leq \frac{\epsilon}{4}. \quad (3.21)$$

Since set  $\mathcal{C}$  is finite, there is  $r_0$  such that for  $r > 0$ ,  $r_0 > r$  and for each  $C_i \in \mathcal{C}$  we have that

$$\frac{\log N_r^{U,\Lambda}(C_i, \epsilon_i)}{-\log r} \leq \bar{f}_c^{U,\Lambda}(C_i, \epsilon) + \frac{\epsilon}{4},$$

and so we have

$$N_r^{U,\Lambda}(C_i, \epsilon_i) \leq r^{-\bar{f}_c^{U,\Lambda}(C_i, \epsilon) - \epsilon/4}. \quad (3.22)$$

Note that interval  $[a, b]$  is covered by sets in  $\mathcal{C}$  in such a way that if there is interval of length less than  $\frac{\epsilon}{42}$  then that interval is entirely at least in one of sets in  $\mathcal{C}$ . Next from 3.2.5 there is  $n_0 \in \mathbb{N}$  such that for  $n > n_0$  we have that  $UL_n[\mathbf{i}]_n$  is contained in interval of length less than  $\frac{\epsilon}{420}$ . There is  $r_1 > 0$ , such that for  $r < r_1$  from  $s_i \sim r$  we have  $|\mathbf{i}| > n_0$ . This condition is here to make sure that no  $\mathbf{i}$  is counted less than once i.e. to ensure first inequality below.

$$\sum_{\mathbf{i} \in \Sigma_A^*, s_i \sim r} s_i^{\langle q, U(L_{|\mathbf{i}|}) \bar{\mathbf{i}} \rangle} \leq \sum_{C_i \in \mathcal{C}} \sum_{\substack{\mathbf{i} \in \Sigma_A^*, s_i \sim r \\ UL_{|\mathbf{i}|}([\mathbf{i}]) \subset C_i}} s_i^{\langle q, U(L_{|\mathbf{i}|}) \bar{\mathbf{i}} \rangle}$$



Note that the number of elements in set  $\{\mathbf{i} | \mathbf{i} \in \Sigma_A^*, \mathbf{i} \sim r, UL_{|\mathbf{i}|}[\mathbf{i}] \subset C_i\}$  is exactly  $N_r^{U,\Lambda}(C_i)$ .

$$\begin{aligned} \sum_{C_i \in \mathcal{C}} \sum_{\substack{\mathbf{i} \in \Sigma_A^*, s_{\mathbf{i}} \sim r \\ UL_{|\mathbf{i}|}([\mathbf{i}]) \in C_i}} s_{\mathbf{i}}^{\langle q, U(L_{|\mathbf{i}|} \bar{\mathbf{i}}) \rangle} &\leq \sum_{C_i \in \mathcal{C}} N_r^{U,\Lambda}(C_i, \epsilon_i) (s_{\max} r)^{\inf_{c \in C_i} \langle q, c \rangle} \\ &\leq \sum_{C_i \in \mathcal{C}} s_{\max}^{\inf_{c \in C_i} \langle q, c \rangle} r^{-\bar{f}_c^{U,\Lambda}(C_i, \epsilon_i) - \epsilon/4} r^{\inf_{c \in C_i} \langle q, c \rangle} \quad \text{by 3.22} \end{aligned}$$

Let  $p = \inf_{\mu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}})} \langle q, U(\mu) \rangle$ , then

$$\begin{aligned} \sum_{C_i \in \mathcal{C}} s_{\max}^{\inf_{c \in C_i} \langle q, c \rangle} r^{-\bar{f}_c^{U,\Lambda}(C_i, \epsilon_i) - \epsilon/4} r^{\inf_{c \in C_i} \langle q, c \rangle} &\leq s_{\max}^{\langle q, p \rangle} \sum_{C_i \in \mathcal{C}} r^{-f^{U,\Lambda}(C_i) - \epsilon/2} r^{-\sup_{c \in C_i} -\langle q, c \rangle} \quad \text{by 3.21} \\ &\leq s_{\max}^p \sum_{C_i \in \mathcal{C}} r^{-\sup_{\substack{\mu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}}) \\ U(\mu) \in C_i}} \left\{ -\frac{h(\mu)}{\int \Lambda d\mu} \right\} - \sup_{c \in C_i} \{-\langle q, c \rangle\} - \epsilon/2} \quad \text{by 3.2.1} \end{aligned}$$

Note that for  $C_i \in \mathcal{C}$  diameter is smaller than  $\frac{\epsilon}{2}$ . Keeping that in mind we get:

$$\begin{aligned} -\sup_{\substack{\mu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}}) \\ U(\mu) \in C_i}} \left\{ -\frac{h(\mu)}{\int \Lambda d\mu} \right\} + \inf_{c \in C_i} \{\langle q, c \rangle\} - \epsilon/2 &= -\sup_{\substack{\mu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}}) \\ U(\mu) \in C_i}} \left\{ -\frac{h(\mu)}{\int \Lambda d\mu} + (\langle q, U(\mu) \rangle - \inf_{c \in C_i} \{\langle q, c \rangle\} - \langle q, U(\mu) \rangle) \right\} - \epsilon/2 \\ &\geq -\sup_{\substack{\mu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}}) \\ U(\mu) \in C_i}} \left\{ -\frac{h(\mu)}{\int \Lambda d\mu} - \langle q, U(\mu) \rangle \right\} - \epsilon/2 - \langle q, q \rangle \epsilon/2. \end{aligned}$$

Therefore

$$\begin{aligned} s_{\max}^p \sum_{C_i \in \mathcal{C}} r^{-\sup_{\substack{\mu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}}) \\ U(\mu) \in C_i}} \left\{ -\frac{h(\mu)}{\int \Lambda d\mu} \right\} - \sup_{c \in C_i} \{-\langle q, c \rangle\} - \epsilon/2} &\leq s_{\max}^p \sum_{C_i \in \mathcal{C}} r^{-\sup_{\substack{\mu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}}) \\ U(\mu) \in C_i}} \left\{ -\frac{h(\mu)}{\int \Lambda d\mu} - qU(\mu) \right\} - \epsilon/2 - \langle q, q \rangle \epsilon/2} \\ &\leq s_{\max}^p |\mathcal{C}| r^{-\sup_{\mu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}})} \left\{ -\frac{h(\mu)}{\int \Lambda d\mu} - \langle q, U(\mu) \rangle \right\} - \epsilon/2 - \langle q, q \rangle \epsilon/2}, \end{aligned}$$

where  $|\mathcal{C}|$  is number of sets in cover  $\mathcal{C}$ .

Hence for every  $\epsilon > 0$  we have that

$$\limsup_{r \rightarrow 0} \frac{\log \sum_{s_{\mathbf{i}} \sim r} s_{\mathbf{i}}^{\langle q, U(L_{|\mathbf{i}|} \bar{\mathbf{i}}) \rangle}}{-\log r} \leq \sup_{\mu \in \mathcal{P}_S(\Sigma_A^{\mathbb{N}})} \left\{ -\frac{h(\mu)}{\int \Lambda d\mu} - \langle q, U(\mu) \rangle \right\} + \frac{\epsilon}{2} + \langle q, q \rangle \frac{\epsilon}{2},$$

we fixed  $q$  at the begin, but this is true for every  $q$  which concludes the proof. ■



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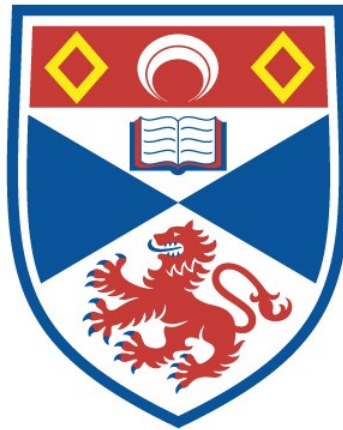
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# MULTIFRACTAL ZETA FUNCTIONS

**Vuksan Mijović**

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