DOTS AND LINES: GEOMETRIC SEMIGROUP THEORY AND FINITE PRESENTABILITY

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Abstract

Geometric semigroup theory means different things to different people, but it is agreed that it involves associating a geometric structure to a semigroup and deducing properties of the semigroup based on that structure.

One such property is finite presentability. In geometric group theory, the geometric structure of choice is the Cayley graph of the group. It is known that in group theory finite presentability is an invariant under quasi-isometry of Cayley graphs.

We choose to associate a metric space to a semigroup based on a Cayley graph of that semigroup. This metric space is constructed by removing directions, multiple edges and loops from the Cayley graph. We call this a skeleton of the semigroup.

We show that finite presentability of certain types of direct products, completely (0-)simple, and Clifford semigroups is preserved under isomorphism of skeletons. A major tool employed in this is the Švarc-Milnor Lemma.

We present an example that shows that in general, finite presentability is not an invariant property under isomorphism of skeletons of semigroups, and in fact is not an invariant property under quasi-isometry of Cayley graphs for semigroups.

We give several skeletons and describe fully the semigroups that can be associated to these.

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Contents

Chapter 1

Introduction and Background

> When you are a Bear of Very Little Brain, and you Think of Things, you find sometimes that a Thing which seemed very Thingish inside you is quite different when it gets out into the open and has other people looking at it.
>
> Winnie the Pooh

Geometric group theory is well explored and described terrain, much like reading an Ordnance Survey map, in comparison to the strange and murky ocean floor charts of geometric semigroup theory. Much of this is due to the underlying geometric structures of groups already being developed and understood in their own right. Semigroups, however, do not immediately lend themselves to a well-known geometric structure. The idea of viewing semigroups from a geometrical standpoint has become increasingly common in recent years, and various different approaches can be found in [8, 10, 15]. A popular choice for a geometric structure is the Cayley graph of a semigroup, which is an inherently directed graph. The machinery for understanding directed spaces is less well studied than that of undirected spaces, namely semi-metrics and metrics respectively. This thesis is an attempt to mesh semigroup theory with a well-understood notion of geometry and discover if properties can be sensibly transferred from such a space to the semigroup.

1.1 Groups and Finite Presentability

It has been observed (see for example [5, 4]) that the property of being finitely presented is preserved under quasi-isometries (and therefore also isometries) of groups. We give a geometric proof for the quasi-isometry invariance of finite presentability, and a more combinatorial proof for isometry invariance of finite presentability.

Theorem 1.1 ([4, Proposition 8.24])
Let G and H be groups such that $\mathrm{Cay}(G)$ *is quasi-isometric to* $\mathrm{Cay}(H)$ *(G is quasi-isometric to H). Then G is finitely presented if and only if H is.*

This proof relies on a lot of fairly involved topological methods which are outside the scope of this thesis. Since our focus will lean more towards graphs that are isometric, we give an alternative proof for groups that have isometric Cayley graphs which reduces the amount of topological concepts needed. We introduce the concept of a *skeleton* graph which is the fundamental object considered in this thesis.

Definition 1.2
A graph *is a tuple (V, E, ι, τ). V is a set of* vertices *of the graph and E is a set of* edges*. $\iota : E \to V$ is a map denoting the initial or start vertex, and $\tau : E \to V$ is a map giving the terminal or end vertex. A graph may be* labelled*, where each edge is assigned a label by a labelling function $\lambda : E \to A$, from some set of labels A.*

The Cayley graph *of a semigroup $S = \mathrm{sgp}\langle A\rangle$ with respect to the generating set A is the labelled directed graph $(V, E, \iota, \tau, \lambda)$, where $V = S$ and there is one edge $e \in E$ for each $x \in S$ and $a \in A$, namely the edge with start vertex $(e)\iota = x$, end vertex $(e)\tau = y$ and label $(e)\lambda = a$.*

Let S be a semigroup generated by A and let $\mathrm{Cay}(S, A) = (S, E, \iota, \tau, \lambda)$ *be the Cayley graph of S with respect to A. We define a new set of edges*

$$F = \{(\iota(e), \tau(e)), (\tau(e), \iota(e)) \mid e \in E, \iota(e) \neq \tau(e)\}.$$

We then define two functions on this set $\overline{\iota} : F \to S$ and $\overline{\tau} : F \to S$. Let $f = (x, y)$ be *an edge in F then*

$$(f)\iota = x$$

$$(f)\tau = y$$

Then the skeleton *of S with respect to A is the (undirected) graph* $\dagger(S, A) = (S, F, \bar{\iota}, \bar{\tau})$.

We can intuitively think of the skeleton graph as the graph which is obtained by taking the undirected version of the Cayley graph and removing any multiple edges or loops that occur.

Theorem 1.3
Let $G = \mathrm{gp}\langle A\rangle$ and $H = \mathrm{gp}\langle B\rangle$ be groups such that $\dagger(G, A)$ is isometric to $\dagger(H, B)$. G is finitely presented if and only if H is.

Proof: Suppose G is finitely presented. This means that G is finitely generated, say $|A| = k$, so each vertex of $\dagger(G, A)$ has degree less than or equal to $2k$. Let H have generating set B. Since $\dagger(G, A) \cong \dagger(H, B)$, each vertex of $\dagger(H, B)$ also has degree of less than or equal to $2k$. If B is finite there is nothing to show. For a contradiction, assume B is infinite. Each vertex in $\dagger(H, B)$ must have a finite number of neighbours since $\dagger(H, B)$ is locally finite. This implies that there are an infinite number of edges between (at least) two vertices, say u and v. Without loss of generality, we consider only two edges from u to v; edge a and edge b. From this we see $ua = ub$ and since H is a group, $a = b$. Hence there are only finitely many edges at each vertex in $\mathrm{Cay}(H, B)$ and thus B is finite.

G is finitely presented so there exists a finite set of relators for G, say $R = \{r_1, \ldots, r_k\}$. Consider a vertex v in $\mathrm{Cay}(G, A)$. Each r_i forms a simple cycle starting and ending at v - if there is some $r = s_1 \ldots s_m$ say, which does not form a simple cycle, consider the vertex f which represents the longest subword $l = s_1 \ldots s_l$ of r_i such that $f s_l s_{l+1} \ldots s_{l+h} = f$ for some h and $f s_l s_{l+1} \ldots s_{l+h}$ is a product of t simple cycles. Then $s_l s_{l+1} \ldots s_{l+h}$ represents at most t relators, $\{t_1, \ldots t_t\}$ say, and will form at most t simple cycles at v so we can add these to our set R. Then the relation r can be written as

$$(s_1 \dots s_{l-1})t_1(s_1 \dots s_{l-1})^{-1} \dots (s_1 \dots s_{l-1})t_t(s_1 \dots s_{l-1})^{-1}(s_1 \dots s_{l-1})(s_m s_{m-1} \dots s_{l+h+1})^{-1}.$$

We iterate this process over the last two terms until we are left with a product of conjugates of relations in $R \setminus \{r\}$, so we can remove r from R.

Now we claim that these are all the simple cycles based at v. Suppose there exists another simple cycle labelled $c \neq r_i$ beginning and ending at v. Since G is a group then $c = 1$, and c is a product of conjugates of relators in R, say

$$c = g_1 r_1 {g_1}^{-1} g_2 r_n {g_2}^{-1} \dots g_n r_n {g_n}^{-1}.$$

Then c cannot be a simple cycle, but a series of cycles already in R and "lollipops" (a simple path followed by a simple cycle from R followed by the inverse of the simple path). Hence R describes all simple cycles at v.

Now consider † (H, B). Since it is isomorphic to † (G, A), for any vertex v there are a finite number of simple cycles starting and ending there, in particular for the vertex corresponding to $1 \in H$. For these cycles based at 1, there are only $|B|$ ways to label each edge in a cycle, and hence at most $n|B|$ different relators described by each cycle. We then claim that this (finite) set of relators S, is a sufficient set of relations for H. Suppose there exists some relator q that is not a consequence of any relators in S. Then q must form a cycle based at 1, but this cannot be a simple cycle, as these are all contained in S. We now use the method from above to rewrite q as a product of conjugates of relations in S. Hence S is sufficient, and H is finitely related, and thus finitely presented. The proof of the converse is analogous. □

1.2 Semigroups and semimetric spaces

In order to generalise these notions from group theory in to semigroup theory, connections between semigroups and what we here call *semimetric spaces* have been made in

[8, 7, 9]. There is some debate about how to correctly name these spaces equipped with asymmetric distance functions. Many authors refer to these as quasimetric spaces [18], yet others use the term quasimetric to refer to a metric satisfying a generalised form of the triangle inequality [6]. We choose to follow [8] and use semimetric here (a pleasing choice, give that we associate them to *semi*groups).

Definition 1.4

A semimetric space *is a pair* (X, d) *where* X *is a set and* $d : X \to [0, \infty]$ *is a distance function that satisfies:*

i $d(x, y) = 0$ *if and only if* $x = y$

ii $d(x, z) \leq d(x, y) + d(y, z)$

for all $x, y, z \in X$.

A natural semimetric space associated to a semigroup $S = \text{sgp}\langle A \rangle$ is given by (S, d_A), where

$$d_A(a, b) = \inf\{|w| \mid w \in A^*, aw = b\},$$

that is, the smallest length of word w, such that $aw = b$.

A map $\varphi : X \to \overline{X}$ between two semimetric spaces (X, d_X) and $(\overline{X}, d_{\overline{X}})$ is a (λ, ε)-quasi-isometric embedding if for all $x, y \in X$

$$\frac{1}{\lambda} d_X(x, y) - \varepsilon \leq d_{\overline{X}}(\varphi(x), \varphi(y)) \leq \lambda d_X(x, y) + \varepsilon.$$

A subset $Z \subseteq \overline{X}$ is called μ-quasi-dense if for every $\overline{x} \in \overline{X}$ there exists a $z \in Z$ with $d_{\overline{X}}(\overline{x}, z) \leq \mu$ and $d_{\overline{X}}(z, \overline{x}) \leq \mu$. If $\varphi : X \to \overline{X}$ is a (λ, ε)-quasi-isometric embedding and its image is μ-quasi-dense, then φ is called a $(\lambda, \varepsilon, \mu)$-quasi-isometry and X and $\overline{X}$ are said to be quasi-isometric.

Two semigroups $S = \text{sgp}\langle A \rangle$and $T = \text{sgp}\langle B \rangle$ are said to be quasi-isometric if the spaces (S, d_A) and (T, d_B) are quasi-isometric. Gray and Kambites show in [8, 9] some results for finite presentability as a quasi-isometric invariant for certain types of semigroups.

Theorem 1.5 ([9, Theorem A])

Let M and N be left cancellative, finitely generated monoids which are quasi-isometric. Then M is finitely presentable if and only if N is finitely presentable.

Theorem 1.6 ([8, Theorem 4])

For finitely generated monoids with finitely many left and right ideals, finite presentability is a quasi-isometry invariant.

In particular, Theorem 1.6 includes Clifford monoids and completely (0-)simple semigroups [8, Corollary 2].

We take a different approach in this thesis, and instead of associating a semimetric space to a semigroup, we will associate a metric space to the semigroup. There are both advantages and disadvantages to our approach when compared to the semimetric space approach.

The main advantage is that we are able to apply techniques associated with metric spaces, such as the Švarc-Milnor lemma. Another advantage is that the rigidity of the skeletons will also allow us to more easily find semigroups which possess given skeletons, such as in Chapters 6 and 7. The fact that we consider isomorphic skeletons allows us to approach the search in a combinatorial way by considering all possible edge directions and labellings. Were we to look at quasi-isometries of skeletons here, we would have to account for stretching and squashing of the graph, and our approach would become much more difficult to implement.

A disadvantage to our approach is that we must always consider the generating set when discussing skeletons. We are looking for isomorphic skeletons, and changing the generating set of a semigroup will not necessarily result in an isomorphic skeleton: for example the integers generated by $\{-1, 0, 1\}$ has a different skeleton to the integers generated by $\{-1, 0, 2\}$. This is not an issue for the approach taken by Gray and Kambites, as Proposition 4 of [8] tells us that for two generating sets A and B of a semigroup S the semimetric spaces given by the word metrics are quasi-isometric.

We will consider preservation of finite presentability under isomorphism of skeleton graphs in Chapters 3, 4 and 5, in which we will look at both Clifford semigroups and completely simple semigroups amongst others. When a property is preserved under isomorphism of skeletons, we will say that this property is *skeleton invariant*.

Chapter 6 answers an open question posed by Gray and Kambites

Question 1.7 ([8, Question 1])
Is finite presentability a quasi-isometry invariant of finitely generated semigroups in general?

In Chapter 7, we will present some examples of skeletons, and prove that these skeletons represent only a finite number of semigroups, which are described therein.

Chapter 2

Definitions

It is a capital mistake to theorize before one has data.

Sherlock Holmes

In this chapter we establish basic definitions, notation and concepts that will be used throughout the thesis.

2.1 Semigroups

Definition 2.1

A semigroup *is a set* S *together with an operation* $\cdot : S \times S \to S$ *such that for all* $a, b, c \in S$

$$(a \cdot b) \cdot c = a \cdot (b \cdot c)$$

that is, $\cdot$ *is* associative.

Unless required for emphasis, we will often omit the $\cdot$ notation for the binary operation, often referred to as *multiplication*, and simply juxtapose elements to denote multiplication. There are two types of semigroup with special properties that we would like to be able to refer to explicitly.

Definition 2.2

A monoid *is a semigroup M which contains an element e such that*

$$m \cdot e = e \cdot m = m$$

for all $m \in M$.

The element e is known as an *identity*, or more properly, *the identity*, since it is unique. We will often write 1 to represent the identity element in a monoid. The family of monoids forms a subfamily of semigroups. We now define a subfamily of monoids, the family of groups.

Definition 2.3

A group *is a monoid G in which for all $g \in G$, there exists a $g' \in G$ such that*

$$gg' = g'g = 1.$$

The element g' is the *inverse* of g, and is unique. The inverse of g will often be denoted by g^{-1}

Given two semigroups, we might like to compare their structure via mappings.

Definition 2.4

Let $(S, \cdot)$ and $(T, \bar{\cdot})$ be two semigroups. A semigroup homomorphism *is a map $\varphi : S \to T$ such that*

$$(s)\varphi \bar{\cdot} (s')\varphi = (s \cdot s')\varphi$$

for all $s, s' \in S$.

An injective homomorphism, that is, one such that

$$(x)\varphi = (y)\varphi \implies x = y$$

for all x, y in S is known as a *monomorphism*. A surjective homomorphism, where for all $t \in T$ there exists an $s \in S$ such that

$$(s)\varphi = t$$

is an *epimorphism*.

If S and T are both monoids and

$$(1_S)\varphi = 1_T$$

then a semigroup homomorphism, epimorphism or monomorphism φ is called a monoid homomorphism, epimorphism or monomorphism. If S and T are both groups, then φ is automatically a group homomorphism, epimorphism or monomorphism and requires no extra conditions.

Finally, the map φ is a *semigroup isomorphism* between S and T if φ is a bijective semigroup homomorphism. If there exists a semigroup isomorphism between S and T, we say S and T are isomorphic, and write $S \cong T$.

2.2 Presentations

We will wish to describe various semigroups without listing all their elements and multiplication. One such way of doing this is via a presentation, which first require the concept of generation.

Definition 2.5

Let S be a semigroup and let X be a non-empty subset of S. Let $\{T_i\}_{i\in I}$ be the collection of all subsemigroups of S which contain X. Then $T = \bigcap_{i\in I} T_i$ is the subsemigroup of S generated by X. *We denote this by $T = \text{sgp}\langle X\rangle$.*

If $S = \text{sgp}\langle X\rangle$ then we say that X is a generating set *for S.*

A presentation requires two elements: a set of generators and a set of relations.

Definition 2.6

A semigroup presentation *is a pair $\text{sgp}\langle A \mid R\rangle$ where A is an alphabet and a set of generators, and $R \subseteq A^+ \times A^+$ is a set of* relations.

For a relation $(u, v) \in R$ we will normally write $u = v$.

Definition 2.7

The semigroup S defined by a presentation $\text{sgp}\langle A \mid R\rangle$ is any semigroup isomorphic to A^+/ρ, where ρ is the least congruence containing R.

If a semigroup S is isomorphic to A^+/ρ for a given presentation $\text{sgp}\langle A \mid R\rangle$, we will write $S = \text{sgp}\langle A \mid R\rangle$.

Definition 2.8

A semigroup S is finitely presented if there exists a presentation $\text{sgp}\langle S \mid R\rangle$ *such that A and R are finite, and $S \cong A^+/\rho$.*

We will write $w_1 \equiv w_2$ if two words are equal in A^+, and $w_1 = w_2$ if two words are equal in S. For two words $w_1, w_2 \in A^+$, we say that w_2 is obtained from w_1 by applying a relation in R if we can write $w_1 \equiv \alpha u \beta$ and $w_2 \equiv \alpha v \beta$ where either $u = v$ or $v = u$ is a relation in R, and $\alpha, \beta \in A^*$.

Definition 2.9

An elementary sequence *from w_1 to w_2 is a sequence*

$$w_1 \equiv s_1, s_2, \ldots, s_n \equiv w_2$$

where $s_i \in A^+$ and for each $1 \le i \le k-1$ we have that either $s_i \equiv s_{i+1}$, or s_{i+1} is obtained from s_i by applying a relation from R. If such a sequence exists, we say that the relation $w_1 = w_2$ is a consequence of relations in R.

A relation $w_1 = w_2$ holds in $S = \text{sgp}\langle A \mid R\rangle$ if and only if it is a consequence of relations in R.

Stable semigroups (as found in [16]), which are those avoiding critical pairs, will be useful to us in this thesis when deciding whether direct products of semigroups have finite presentations. A critical pair may be thought of as a relation in S, where all elementary sequences go via shorter words.

Definition 2.10

Let $S = \text{sgp}\langle A \mid R\rangle$, and let $w_1, w_2 \in A^+$ be arbitrary words. The pair (w_1, w_2) is called a critical pair *if:*

(i) the relation $w_1 = w_2$ holds in S;

(ii) for every elementary sequence $w_1 \equiv s_1, s_2, \ldots, s_n \equiv w_2$ from w_1 to w_2, there exists an $1 \le i \le k$ such that $|s_i| < \min(|w_1|, |w_2|)$.

Definition 2.11
Let S be a semigroup with finite generating set A. We say that S is stable *with respect to A if there exists a finite presentation* $\mathrm{sgp}\langle A \mid R\rangle$ *for S, with respect to which S has no critical pairs.*

Stability is in fact invariant under change of generating set [Proposition 3.4,[16]), so we may refer to a semigroup being stable without reference to a specific generating set.

2.3 Graphs

Graphs describe a collection of objects, or vertices, and the connections, or edges, between them. There are many ways to define a graph: the definition chosen here is to allow us to work with graphs that have multiple edges, loops and labels.

Definition 2.12
A graph *is a tuple (V, E, ι, τ). V is a set of* vertices *of the graph and E is a set of* edges. *$\iota : E \to V$ is a map denoting the initial or start vertex, and $\tau : E \to V$ is a map giving the terminal or end vertex.*

Graphs can be *directed* or *undirected*. A graph is undirected if and only if for every edge $e \in E$ with $(e)\iota = x$ and $(e)\tau = y$, there exists an edge $f \in E$ such that $(f)\iota = y$ and $(f)\tau = x$.

We note that in this definition of graph we allow for multiple edges between vertices and loops on edges. We call a graph that contains no multiple edges or loops a *simple graph*.

Graphs may also be *labelled*, where each edge is assigned a label by a labelling function $\lambda : E \to A$, from some set of labels A.

The *indegree* of a vertex is the number of edges terminating at that vertex. The *outdegree* is the number of edges originating at that vertex. For a directed graph, the *degree* of a vertex is the sum of the indegree and outdegree. For an undirected graph, we define the degree of a vertex to be equal to the number of vertices adjacent to that vertex. Note that a vertex can be adjacent to itself by way of a loop.

A graph is finite if it has finitely many vertices and edges. A graph is infinite if it has infinitely many vertices or edges. A graph is *locally finite* if each vertex has finite degree.

Definition 2.13

The (vertex)-induced subgraph *of* $\Gamma = (V, E, \iota, \tau)$ *induced by* $W \subseteq V$ *is the graph with vertex set* W *and edge set* $F \subseteq E$*, where* $F = \{e \in E \mid \iota(e), \tau(e) \in W\}$*, and the maps* $\iota \restriction_F$ *and* $\tau \restriction_F$*.*

An important definition for us is that of the Cayley graph, which allows us to associate a graph to a semigroup. It gives a representation of the multiplicative structure of the semigroup with respect to a particular generating set.

Definition 2.14

A Cayley graph *of a semigroup* $S = \mathrm{sgp}\langle A\rangle$ *is a labelled directed graph* $(V, E, \iota, \tau, \lambda)$*, where* $V = S$*. For all* $x, y \in S$ *and* $a \in A$ *such that* $xa = y$ *there exists an edge* e*, with the start vertex* $(e)\iota = x$*, the end vertex* $(e)\tau = y$ *and the label* $(e)\lambda = a$*.*

Cayley graphs can also be constructed for monoids and groups. In the case of groups with symmetric generating sets, that is, a generating set that contains an inverse for every element, the Cayley graph of a group is an undirected graph.

We will wish to compare the structure of graphs so we introduce the notion of isomorphism for graphs.

Definition 2.15

Let $\Gamma_1 = (V_1, E_1, \iota_1, \tau_1)$ *and* $\Gamma_2 = (V_2, E_2, \iota_2, \tau_2)$ *be graphs. A* graph isomorphism $\varphi : \Gamma_1 \to \Gamma_2$ *is a pair of bijective maps* $f : V_1 \to V_2$ *and* $g : E_1 \to E_2$ *such that for any* $e \in E_1$ *we have*

$$((e)g)\tau_2 = ((e)\tau_1)f$$

$$((e)g)\iota_2 = ((e)\iota_1)f$$

If there exists a graph isomorphism between two graphs Γ_1 and Γ_2, we say that they are isomorphic, and write $\Gamma_1 \cong \Gamma_2$.

2.4 Spaces

A *metric space* is a set with a function describing distance between elements defined on it.

Definition 2.16

Let X be a set and let $d : X \times X \to \mathbb{R}_{\geq 0}$ be a function such that the following hold:

- $d(x, x) = 0$ *for all* $x \in X$
- $d(x, y) = 0$ *if and only if* $x = y$
- $d(x, y) = d(y, x)$ *for all* $x, y \in X$
- $d(x, z) \leq d(x, y) + d(y, z)$ *for all* $x, y, z \in X$

Then d *is a* metric *and* (X, d) *is a* metric space

If we are given two metric spaces, we will want to be able to compare them and have some notion of similarity between them. There are two such notions that we will use.

Definition 2.17

Let (X, d) *and* $(\overline{X}, \overline{d})$ *be two metric spaces. An* isometric embedding *is a map* $\varphi : X \to \overline{X}$ *such that* $\overline{d}(\varphi(x), \varphi(y)) = d(x, y)$ *for all* $x, y, \in X$*. If* φ *is onto, then it is an* isometry *and* $X, \overline{X}$ *are* isometric.

Definition 2.18

Let (X, d) *and* $(\overline{X}, \overline{d})$ *be two metric spaces. For constants* $1 \leq \lambda < \infty$, $0 < \varepsilon < \infty$ *and* $0 < \mu < \infty$*, a* $(\lambda, \varepsilon, \mu)$*-quasi-isometry is a map* $\varphi : X \to \overline{X}$ *such that*

(i) for all $x, y \in X$
$\frac{1}{\lambda} d(x, y) - \varepsilon \leq \overline{d}((x)\varphi, (y)\varphi) \leq \lambda d(x, y) + \varepsilon$*, and*

(ii) for every $y \in \overline{X}$ *there exists an* $x \in X$ *with* $\overline{d}(y, (x)\varphi) \leq \mu$*.*

If there exists a quasi-isometry between two spaces they are said to be quasi-isometric.

Definition 2.19

A geodesic *between two points* x, y *in a metric space* (X, d) *is the image of an isometric embedding* φ *of the interval* $[0, l]$ *into* X *such that* $(0)\varphi = x$*,* $(l)\varphi = y$ *and* $d(x, y) = l$*.*

A metric space is called geodesic *if any two points can be joined by at least one geodesic.*

Definition 2.20
Let (X, d) be a metric space. An open ball *of radius n centred at a point x for $x \in X$, $n \in \mathbb{R}_{\geq 0}$ is the set of all points $y \in X$ such that $d(x, y) < n$. A* closed ball *of radius $n \in \mathbb{R}_{\geq 0}$ centred at a point $x \in X$ is the set of all points $y \in X$ such that $d(x, y) \leq n$.*

Where metric spaces consider the distance between elements of the set, *topological spaces* are concerned with describing the closeness of subsets of the set.

Definition 2.21
Let X be a set together with a non-empty collection τ of subsets of X (known as open sets*) such that the following are satisfied:*

- *Any union of open sets is itself open*
- *Any finite intersection of open sets is open*
- *The empty set and X are both open*

The collection of open sets τ is a topology *and (X, τ) is called a* topological space.

We will often refer to a topological space (X, τ) simply by X, when it is clear what the topology is. One such topology that we will make frequent use of is the metric topology.

Definition 2.22
Let (X, d) be a metric space. We define an open set in X to be a subset of X that can be written as the union of open balls with respect to d. The metric topology *τ is the collection of such open sets, and we call (X, τ) the topological space induced by the metric d.*

For the following definitions we let X be a topological space throughout.

Definition 2.23
A family $\mathcal{F}$ of open subsets of X is called an open cover *of X if $\bigcup \mathcal{F} = X$. A subfamily $\overline{\mathcal{F}}$ of $\mathcal{F}$ is called a* subcover *of $\mathcal{F}$ if $\bigcup \overline{\mathcal{F}} = X$.*

A topological space X is called compact *if every open cover of X has a finite subcover.*

Definition 2.24

A topological (or metric) space is proper *if its closed balls of finite radius are compact.*

For semigroups we had the notion of isomorphism to show when semigroups look the same. In topological spaces, this concept of sameness is described by homeomorphisms.

Definition 2.25

Let (X_1, τ_1) and (X_2, τ_2) be two topological spaces. Then a map $\sigma : X_1 \to X_2$ is continuous *if for every open set $V \subset X_2$, the preimage σ^{-1} is an open subset of X_1. The map σ is a homeomorphism if*

(i) σ is a bijection;

(ii) σ is continuous, and

(iii) σ has a continuous inverse.

If there exists such a homeomorphism we say that (X_1, τ_1) and (X_2, τ_2) are homeomorphic.

2.5 Actions

A central concept in group theory, and one that will be used in this thesis, is that of a group action. A group action provides a way of thinking about a group as a set of bijective maps of a given object which allows us to understand properties of the group more easily. We will want to act on spaces which have some sort of structure on them, and our group actions should preserve this structure. In this thesis will use the following definitions and notation for group actions.

Definition 2.26

Let G be a group and let X be a set. An action *of G on X is a map $G \times X \to X$ denoted by $(g, x) \mapsto {}^g x$ such that:*

(i) ${}^1g = g$

(ii) ${}^g({}^hx) = {}^{gh}x$,

for all $g, h \in G$ *and* $x \in X$. *If we have such an action, we will say that* G acts on X.

We will mostly be interested in acting on topological spaces, and so we wish for our actions to be by homeomorphisms, that is for each $g \in G$ the map defined by $x \mapsto {}^gx$ is a homeomorphism for all $g \in G$. The following properties of actions are important.

Definition 2.27

An action of a group G *on a topological space* X *is* proper *if for every compact subset* $K \subset X$ *the set* $\{g \in G \mid {}^gK \cap K \neq \varnothing\}$ *is finite. If such an action is proper, we will say that* G acts properly *on* X.

Definition 2.28

Let G *be a group acting on a set* X. *The* quotient space *of the action is the set of all orbits of* X *under the action of* G. *We denote the quotient space by* ${}^X/_G$. *Let* $\pi : X \to {}^X/_G$ *be the projection from* X *onto* ${}^X/_G$. *We define a set* $U \subseteq {}^X/_G$ *to be open if and only if its preimage* $\pi^{-1}(U)$ *is open in* X. *This collection of open sets is known as the* quotient topology.

Definition 2.29

An action of a group G *on a topological space* X *is* cocompact *if the quotient space* ${}^X/_G$ *is compact.*

Chapter 3

Direct Products with Groups and How to Draw a Graph

> Pooh looked at his two paws. He knew that one of them was the right, and he knew that when you had decided which one of them was the right, then the other one was the left, but he never could remember how to begin.
>
> The House At Pooh Corner
> A.A. Milne

For our first foray into the world of semigroups, we shall tackle direct products of groups and semigroups. We recall that a *direct product* of two semigroups $(G, \cdot)$, $(S, *)$ is the set of ordered pairs

$$G \times S = \{(g, s) \mid g \in G, s \in S\}$$

along with the operation defined by applying the operations from G and S component-wise

$$(g, s)(h, t) = (g \cdot h, s * t).$$

Throughout the chapter (indeed, throughout the whole thesis) we shall (very sensibly) choose G to represent a group, and S to represent a semigroup.

We begin first by looking at family of semigroups known as *left zero semigroups*.

Definition 3.1

A left zero *is an element $l \in S$ such that $ls = l$ for all $s \in S$. A left zero semigroup is one in which all elements are left zeros. We denote a left zero semigroup of size n by*

$$L_n = \{l_1, l_2, \dots, l_n \mid l_i l_j = l_i \textit{ for all } 1 \leq i, j \leq n\}$$

Since we claimed to be interested in direct products in this chapter, we will consider those of groups and left zero semigroups now. The following small lemma will be useful.

Lemma 3.2

Let $S = G \times L_n$ be a semigroup with G a group. If S if finitely presented, then G is also.

Proof: Let $A = \{(a_i, l_{j_i}) \mid i \in I\}$ be a finite generating set for S and let

$$\mathcal{P} = \text{sgp}\langle A \mid R \rangle$$

be a presentation for S. Let $\pi_G : S \to G$ be the projection of S onto G. Since this map is onto, $\pi_G(A)$ is a generating set for G. We will show that

$$\text{gp}\langle \pi_G(A) \mid \pi_G(R) \rangle$$

is a presentation for G. Let

$$(a_{i_1}, l_{j_1})(a_{i_2}, l_{j_2}) \dots (a_{i_m}, l_{j_m}) = (a_{k_1}, l_{o_1})(a_{k_1}, l_{o_1}) \dots (a_{k_q}, l_{o_q})$$

be a relation in R. We may simplify this relation by carrying out the multiplication to give

$$(u, l_{j_1}) = (v, l_{o_1})$$

where $u = a_{i_1} a_{i_2} \dots a_{i_m}$ and $v = a_{k_1} a_{k_2} \dots a_{k_q}$.

Now if $w = x$ is a relation in G we have $(w, l_i) = (x, l_i)$ for any $l_i \in L_n$. This means $(w, l_i) = (x, l_i)$ is a consequence of relations in R, that is

$$(w, l_i) \equiv s_1, s_2, \ldots s_k \equiv (x, l_i)$$

where for each $1 \leq y \leq k$ the element s_{y+1} is obtained from s_y by the application of a relation in R. That is for each $1 \leq y \leq k$ we have $s_y = \alpha(u, l_{j_1})\beta$ and $s_{y+1} = \alpha(v, l_{o_1})\beta$ where $(u, l_{j_1}) = (v, l_{o_1})$ is a relation in R and $\alpha, \beta \in S^1$. Now considering the projection of this elementary sequence on to G we see that $\pi_G(s_y) = \pi_G(s_{y+1})$ is a consequence of $u = v$ for each y. This shows that $w = x$ is a consequence of the various relations $u = v$ obtained by projecting the relations in R onto G.

We have shown that

$$\text{gp}\langle \pi_G(A) \mid \pi_G(R) \rangle$$

is a presentation for G, and since both A and R are finite, this is a finite presentation for G. $\square$

We note that Lemma 3.2 can also be proved using Theorem 3.5 from [16].

Now we may show that finite presentability is preserved under isomorphism of skeletons for direct products of groups with left zero semigroups.

Theorem 3.3

Let G, H be groups and let $S = G \times L_n$ and $T = H \times L_m$ for some $m, n \in \mathbb{N}$, with $\dagger(S, A) \cong \dagger(T, B)$. Then $m = n$ and S is finitely presented if and only if T is.

Proof: We will assume that S is finitely presented and so first look at $\dagger(S, A)$. For $1 \leq i \leq n$, let $G_i = \{(g, l_i) \mid g \in G\}$. Since for $(g, l_i) \in G_i, (h, l_j) \in S$ we have

$$(g, l_i)\,(h, l_j) = (gh, l_i)$$

we may find a path between any two vertices (g, l_i) and (gh, l_i), that is the path labelled by (h, l_j) written as a product of generators for any $l_j \in L_n$. This product also shows us that we can never find a path between two vertices with different L_n components, and so each G_i forms a strongly connected component of Cay(S, A), and hence a connected component of $\dagger(S, A)$. This tells us that $\dagger(S, A)$ has n connected components.

Similarly for T, if we let $H_i = \{(h, l_i) \mid h \in H\}$, each H_i is a connected component in $\dagger\,(T, B)$. Since $\dagger\,(S, A) \cong \dagger\,(T, B)$, both graphs must have the same number of connected components, that is, $n = m$.

We show that each component induced by the set of vertices G_i is isomorphic to $\dagger\,(G, \pi_G\,(A))$, where $\pi_G : S \to G$ is the projection onto the group G. When restricted to G_i, the projection $\pi_G \restriction : G_i \to G$ is a bijection. Let e be an edge with initial vertex (g, l_i) and terminal vertex (ga, l_i), with label (a, l_j) in Cay(S, A). Then the images of these vertices are g and ga respectively, which are the initial and terminal vertices of an edge labelled $a \in \pi_G\,(A)$ in the graph Cay$(G, \pi_G\,(A))$. Similarly if g and ga are initial and terminal vertices of an edge labelled a in Cay$(G, \pi_G\,(A))$, then the preimages are (g, l_i) and (ga, l_i) respectively, which form an edge with label (a, l_j) in Cay(G_i, A). Hence since $\pi_G \restriction$ is a bijection which maps edges to edges, it is a graph isomorphism.

Similarly each component H_i is isomorphic to $\dagger\,(H, \pi_H\,(B))$. Therefore $\dagger\,(G, \pi_G\,(A)) \cong \dagger\,(H, \pi_H\,(B))$. We have S finitely presented, and so by Lemma 3.2, G is finitely presented also. We can then apply Theorem 1.3 to show that H is finitely presented, and thus T. $\square$

This works very neatly, so it seems like it might be a good idea to investigate a closely related class, *right zero semigroups*.

Definition 3.4

A right zero *is an element $r \in S$ such that $sr = r$ for all $s \in S$. A right zero semigroup is one in which all elements are right zeros. We denote a right zero semigroup of size n by*

$$R_n = \{r_1, r_2, \ldots, r_n \mid r_i r_j = r_j \textit{ for all } 1 \leq i, j \leq n\}$$

It would be nice if we could take the same approach here, and show that given two semigroups $S = G \times R_n$, $T = H \times R_m$, we can find copies of $\dagger\,(G, A)$ and $\dagger\,(H, B)$ nestled inside, and apply Theorem 1.3. Sadly, this is not the case. If we consider the subgraph of $\dagger\,(S, A)$ induced by the set of vertices $G_i = \{(g, r_i) \mid g \in G\}$, we do not necessarily find a copy of $\dagger\,(G, \pi_G\,(A))$ as the following example illustrates.

Example 3.5

Let $S = V_4 \times R_2$, the direct product of the Klein four group (generated by $\{a, b\}$) and the right zero semigroup R_2 which has size 2. This can be generated by $A = \{(a, r_1), (b, r_2)\}$. The Cayley graph of S with generating set A is given in figure 3.1.

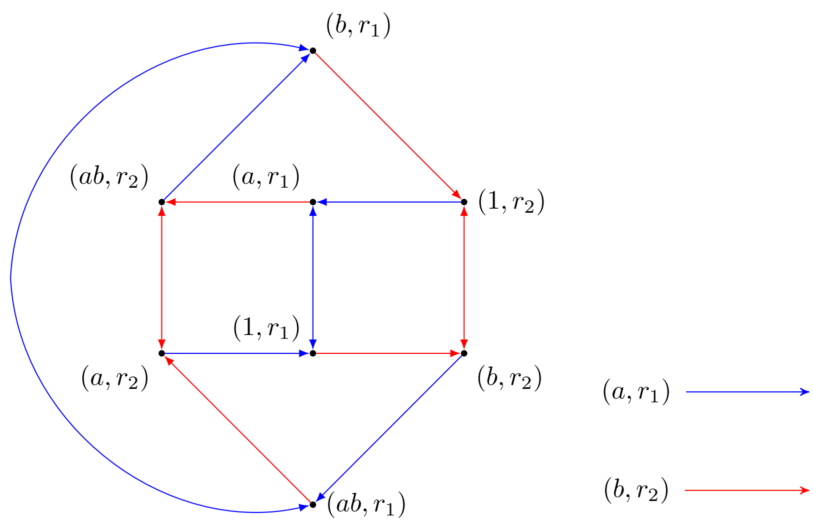


Figure 3.1: Cayley graph of S with generating set A

We then consider the subgraph of Cay(S, A) induced by the set of vertices $(V_4)_1 = \{(g, r_1) \mid g \in V_4\}$, which is given in Figure 3.2. Since this graph has two components, then it is not a copy of Cay$(V_4, \{a, b\})$.

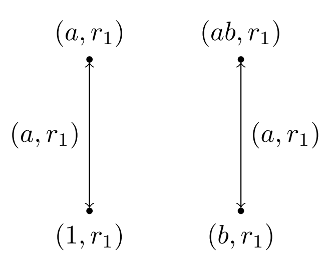


Figure 3.2: Subgraph of Cay(S, A) restricted to F

Thus it seems it will be tricky to follow the same procedure as with left zero semigroups. We turn instead to geometric group theory, which provides us with a handy tool to apply to the problem. Sometimes referred to as the "Fundamental Observation of Geometric Group Theory" (see [5]), the Švarc-Milnor Lemma, which was originally proven independently by both Švarc [19] and Milnor [13] tells us that if we can act nicely with a finitely generated group on a nice metric space, then the group, equipped with the word metric, is quasi-isometric to that space. We note that a group along with the word metric is a metric space, and we can refer to a group as being equipped with the word metric without specifying a generating set, as we are concerned only with quasi-isometries, and we have the following theorem, for which a proof can be found in [3, Proposition 4.3].

Theorem 3.6

Let G be a group and let A,B be two finite generating sets. Let d_A and d_B be the word metrics for the respective generating sets. Then (G, d_A) is quasi-isometric to (G, d_B).

We give the formulation of the lemma found in [4].

Theorem 3.7 (Švarc-Milnor Lemma)

Let X be a proper, geodesic metric space. Let G be a group acting co-compactly and properly by isometries on X. Then G is finitely generated and for any $x_0 \in X$ the map $G \to X$ given by $g \mapsto {}^g x_0$ is a quasi-isometry.

In order to use the Švarc-Milnor lemma, we will need to turn our graphs into appropriate metric spaces, and understand the quotient spaces. This turns out to require a small amount of work, which we see in the following section.

3.1 Turning Graphs into Metric Spaces

Any connected graph Γ can be equipped with the graph metric m where the distance between any two vertices is the length of a shortest path between them.

We can define a topological space from any graph Γ by adding a copy of the interval $[0, 1]$ to each edge. We take the set $Z = V \cup (E \times [0, 1])$ and define an equivalence relation $\rho \subseteq Z \times Z$. For any point $z \in Z$ we define $z\rho z$. Additionally, given an edge e with $\iota(e) = u$ and $\tau(e) = v$ we will let $u\rho(e, 0)$ and $v\rho(e, 1)$, and take the symmetric and transitive closure of these. This is reflexive, symmetric and transitive by construction.

We may then define a topological space by taking the set $X = Z/\rho$. We have a map $z : Z \to Z/\rho$, and we will write $z(x) = x'$ or $z((e, \mu)) = (e, \mu)'$ depending on the level of detail required. The open sets are then given by defining a metric $d : X \to [0, \infty)$. We will denote the topological space created from Γ by Γ'.

Let $x' = (e, \mu)'$ and $y' = (f, \nu)'$ be non-equal elements of X and define a path q from x' to y' as a sequence

$$(e, \mu)', v_1, v_2, \ldots, v_n, (f, \nu)'$$

where v_i, v_{i+1} are adjacent for all $1 \leq i \leq n-1$ and v_1 (resp. v_n) is an endpoint of the edge e (resp. f). The shortest length of a path between two adjacent vertices is 1. The length of a path $(e, \mu)', \iota(e)$ is μ and the length of a path $(e, \mu)', \tau(e)$ is $1 - \mu$. Denote the length of a path q by $l(q)$. Let $Q_{x',y'}$ be the set of all paths from x' to y'. Then the metric d is defined by

$$d(x', y') = \begin{cases} \min_{q \in Q} l(q). & \text{for } x' \neq y' \\ 0 & \text{for } x' = y' \end{cases}$$

Note that if two points $(e, \mu)'$ and $(e, \nu)'$ are on the same edge, then the shortest path between them is simply along the edge and so $d((e, \mu)', (e, \nu)') = |\mu - \nu|$.

Claim 3.8

The map d is a metric.

Proof: Let x', y' be defined as above and let $z' \in X$ also.

By definition $d(x', x') = 0$ and $d(x', y') \geq 0$.

Suppose that $d(x', y') = k$. Then there exists some path

$$q = x', v_1, v_2, \ldots, v_n, y'$$

such that $l(q) = k$. Traversing this path backwards, that is

$$b = y', v_n, v_{n-1}, \ldots, v_1, x'$$

is a path of length k from y' to x' so $d(y', x') = k$.

Finally, we wish to show that the triangle inequality holds. Suppose q_1 is the shortest path from x' to y' and q_2 is the shortest path between y' and z'. Then q_1q_2 is a path from x' to z' and has length $d(x', y') + d(y', z')$ and so $d(x', z') \leq d(x', y') + d(y', z')$.

□

The topology on X is then the topology induced by the metric d.

We let Γ be a connected, locally finite graph, and let G be a group that acts on Γ, such that the action of each element $g \in G$ results in an automorphism of the underlying undirected graph. We will do this by first defining an action on the vertices and edges. We will then extend this to an action on points in the topological space. The following example illustrates why we must be careful about how the action is defined.

Example 3.9

Let Γ be the 3-cycle graph as given in Figure 3.3. Let the starts and ends of edges be as follows:

$$\iota(e_1) = v_1, \tau(e_1) = v_2$$
$$\iota(e_2) = v_2, \tau(e_2) = v_3$$
$$\iota(e_3) = v_1, \tau(e_3) = v_3.$$

We act on Γ with $C_3 = \text{gp}\langle x \rangle$ by rotation, that is

$$^{x}v_1 = v_2$$
$$^{x}v_2 = v_3$$
$$^{x}v_3 = v_1$$

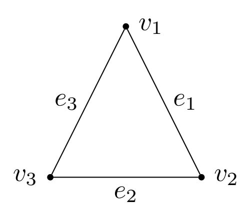


Figure 3.3: 3-cycle graph

and

$$
\begin{aligned}
{}^{x}e_1 &= e_2 \\
{}^{x}e_2 &= e_3 \\
{}^{x}e_3 &= e_1.
\end{aligned}
$$

Notice that $\iota({}^{x}e_3) = \iota(e_1) = v_1$, but ${}^{x}\iota(e_3) = {}^{g}v_1 = v_2 = \tau(e_1)$, and so we have an automorphism of the underlying undirected graph, but not of Γ itself, as the action has swapped the start and end of an edge.

Suppose then that we had a point $(e_3, \mu)'$. If we were to naïvely define an action ${}^{x}(e_3, \mu)' = ({}^{x}e_3, \mu)' = (e_1, \mu)'$, this is the point that is distance μ from v_1. This action then is not an isometry, as an isometry should move $(e_3, \mu)'$ to the point that is distance μ from ${}^{x}\iota(e_3) = v_2$.

Thus we must be more careful about defining the action on points. Let $(e_i, \mu)'$ be a point in Γ' and define

$$
{}^{x}(e_i, \mu)' = \begin{cases} ({}^{x}e_i, \mu)' & \text{if } {}^{x}\iota(e_i) = \iota({}^{x}e_i) \\ ({}^{x}e_i, 1-\mu)' & \text{if } {}^{x}\iota(e_i) = \tau({}^{x}e_i) \end{cases}
$$

Then our previous troublesome point under the action becomes

$$
\begin{aligned}
{}^{x}(e_3, \mu)' &= ({}^{x}e_3, 1-\mu)' \\
&= (e_1, 1-\mu)'
\end{aligned}
$$

which is indeed the point that is distance μ from v_2.

Hence given an action of G on V and E such that $\{\iota({}^{g}e), \tau({}^{g}e)\} = \{{}^{g}\iota(e), {}^{g}\tau(e)\}$, that is ends of an edge are mapped to ends of an edge, we define a map on X by

$$
{}^{x}(e, \mu)' = \begin{cases} ({}^{x}e, \mu)' & \text{if } {}^{x}\iota(e) = \iota({}^{x}e), \\ ({}^{x}e, 1-\mu)' & \text{if } {}^{x}\iota(e) = \tau({}^{x}e). \end{cases} \tag{+}
$$

Claim 3.10

The map (+) *is well defined and is an action.*

Proof: Let $g, h \in G$ and $x' \in X$. The only type of point in X that can have multiple representatives is one which was a vertex in the original graph. Let $(e,0)' \in X$ and $\iota(e)'$ be two representations of x', so $(e,0)\rho\iota(e)$ in Z. If ${}^g\iota(e) = \iota({}^g e)$ then

$${}^g\iota(e)' = \iota({}^g e)'$$

and

$${}^g(e,0)' = ({}^g e, 0)'.$$

Then since $({}^g e, 0)\rho\iota({}^g e)$ we have

$$\begin{aligned} {}^g\iota(e)' &= \iota({}^g e)' \\ &= ({}^g e, 0)' \\ &= {}^g(e,0)' \end{aligned}$$

Alternatively, if ${}^g\iota(e) = \tau({}^g e)$ then

$${}^g\iota(e)' = \tau({}^g e)'$$

and

$${}^g(e,0)' = ({}^g e, 1)'.$$

Then since $({}^g e, 1)\rho\tau({}^g e)$ we have

$$\begin{aligned} {}^g\iota(e)' &= \tau({}^g e)' \\ &= ({}^g e, 1)' \\ &= {}^g(e,0)' \end{aligned}$$

This follows analogously if instead we have $(e,1)'$ and $\tau(e)'$ being two representatives for $x' \in X$.

The final situation we may have is that we have two representatives $(e,\mu)', (f,\nu)' \in X$ of x', where $\mu, \nu \in \{0,1\}$. This means $(e,\mu)\rho(f,\nu)$ in Z. Suppose that $\mu = 0$ and $\nu = 1$. If ${}^g\iota(e) = \iota({}^g e)$ and ${}^g\iota(f) = \iota({}^g f)$ then

$${}^g(e,\mu)' = ({}^g e, \mu)'$$

and

$$^{g}(f,\nu)' = (^{g}f,\nu)'.$$

Then since $(^{g}e,\mu)\rho(^{g}f,\nu)$ we have $(^{g}e,\mu)' = (^{g}f,\nu)'$.

If $^{g}\iota(e) = \tau(^{g}e)$ and $^{g}\iota(f) = \iota(^{g}f)$ then

$$^{g}(e,\mu)' = (^{g}e,1-\mu)'$$

and

$$^{g}(f,\nu)' = (^{g}f,\nu)'.$$

Then since $(^{g}e,1-\mu)\rho(^{g}f,\nu)$ we have $(^{g}e,1-\mu)' = (^{g}f,\nu)'$.

If $^{g}\iota(e) = \iota(^{g}e)$ and $^{g}\iota(f) = \tau(^{g}f)$ then

$$^{g}(e,\mu)' = (^{g}e,\mu)'$$

and

$$^{g}(f,\nu)' = (^{g}f,1-\nu)'.$$

Then since $(^{g}e,\mu)\rho(^{g}f,1-\nu)$ we have $(^{g}e,\mu)' = (^{g}f,1-\nu)'$.

Finally, if $^{g}\iota(e) = \tau(^{g}e)$ and $^{g}\iota(f) = \tau(^{g}f)$ then

$$^{g}(e,\mu)' = (^{g}e,1-\mu)'$$

and

$$^{g}(f,\nu)' = (^{g}f,1-\nu)'.$$

Then since $(^{g}e,1-\mu)\rho(^{g}f,1-\nu)$ we have $(^{g}e,1-\mu)' = (^{g}f,1-\nu)'$. Hence the map is well defined.

Now we can show that the map is an action. We observe that if only one of g or h swaps the endpoints of the edge under the action, for example $^{h}\iota(e) = \tau(^{h}e)$, then we will have

$$\begin{aligned}
^{gh}\iota(e) &= {}^{g}(^{h}\iota(e)) \\
&= {}^{g}(\tau(^{h}e)) \\
&= \tau(^{gh}e).
\end{aligned}$$

If both g and h swap the endpoints of the edge under the action, so ${}^{h}\iota(e) = \tau({}^{h}e)$ and ${}^{g}\iota({}^{h}e) = \tau({}^{g}({}^{h}e))$ then

$$\begin{aligned} {}^{gh}\iota(e) &= {}^{g}({}^{h}\iota(e)) \\ &= {}^{g}\tau({}^{h}e) \\ &= \iota({}^{g}he). \end{aligned}$$

This means we have two cases to examine when showing (+) is an action, namely ${}^{gh}\iota(e) = \iota({}^{g}he)$ and ${}^{gh}\iota(e) = \tau({}^{gh}e)$.

$${}^{g}({}^{h}(e,\mu)') = \begin{cases} ({}^{gh}(e,\mu)') & \text{if } {}^{gh}\iota(e) = \iota({}^{gh}e) \\ ({}^{gh}e, 1-\mu)' & \text{if } {}^{gh}\iota(e) = \tau({}^{gh}e) \end{cases}$$

which is equal to ${}^{gh}(e,\mu)$. If $1 \in G$ is the identity then

$$\begin{aligned} {}^{1}(e,\mu)' &= ({}^{1}e,\mu)' \\ &= (e,\mu)'. \end{aligned}$$

Hence (+) is an action. $\square$

We will write $[v]$ and $[e]$ for the orbits of v and e under the action of G.

Definition 3.11

The quotient graph *of* Γ *by* G *is* ${}^{\Gamma}/_{G} = ({}^{V}/_{G}, {}^{E}/_{G})$ *where* ${}^{V}/_{G} = \{[v] \mid v \in V\}$ *and* ${}^{E}/_{G} = \{[e] \mid e \in E\}$.

We are correct here to call this a graph, as we have specified that our action takes endpoints of an edge to endpoint of an(other) edge. Hence an edge $[e]$ has endpoints $[\iota(e)] = \iota([e])$ and $[\tau(e)] = \tau([e])$.

Since ${}^{\Gamma}/_{G}$ is a graph, we can create a topological space Y in an analogous way to X, with equivalence relation $\hat{\rho}$ and metric $\hat{d}$. Points in Y are denoted $[x]' = [(e,\mu)]'$.

We recall that the Švarc-Milnor Lemma requires our action to be an isometry, so we should now show that we have defined a useful action here.

Claim 3.12

The map $x' \mapsto {}^{g}x'$ *is an isometry.*

PROOF: Let $x', y' \in X$ such that $x' = (e, \mu)'$, $y' = (f, \nu)'$ and $d(x', y') = l$. Suppose a shortest path p from x' to y' is $x', v_0, v_1, \ldots, v_{n-1}, v_n, y'$ where v_i are vertices such that v_i is adjacent to v_{i-1} and v_{i+1}, and v_0, v_n are endpoints of e and f respectively.

Consider the image of two adjacent vertices v_i, v_{i+1} under the action, which is ${}^g v_i$ and ${}^g v_{i+1}$ respectively. These vertices are endpoints of some edge e_i. The image of this edge under the action has endpoints

$$\begin{aligned}\{\iota({}^g e_i), \tau({}^g e_i)\} &= \{{}^g \iota(e_i), {}^g \tau(e_i)\} \\ &= \{{}^g v_i, {}^g v_{i+1}\}.\end{aligned}$$

Hence the vertices ${}^g v_i$ and ${}^g v_{i+1}$ are adjacent. We now check that the point ${}^g x'$ (and analogously ${}^g y'$) remains the same distance from ${}^g v_0$ (respectively ${}^g v_n$) as x' (respectively y') from v_0 (respectively v_n) . Suppose that $v_0 = \tau(e)$, then x' is a distance of $1 - \mu$ from v_0. Under the action, we may find either ${}^g \tau(e) = \tau({}^g e)$ or ${}^g \tau(e) = \iota({}^g e)$. For the former, the point x' is mapped to $({}^g e, \mu)'$, which is indeed distance $1 - \mu$ from ${}^g v_0$. For the latter, our point is mapped to $({}^g e, 1 - \mu)'$. Since $\iota({}^g e) = {}^g v_0$, our point is still a distance of $1 - \mu$ from ${}^g v_0$. This works analogously for $v_0 = \iota(e)$.

From this we deduce that ${}^g x', {}^g v_0, {}^g v_1, \ldots, {}^g v_{n-1}, {}^g v_n, {}^g y'$ is a path from ${}^g x'$ to ${}^g y'$, with length l. Suppose now that there exists some path

$${}^g x', u_0, u_1, \ldots, u_{n-1}, u_m, {}^g y'$$

from ${}^g x'$ to ${}^g y'$, with length $k < l$. Then by an analogous argument,

$${}^{g^{-1}} {}^g x', {}^{g^{-1}} u_0, {}^{g^{-1}} u_1, \ldots {}^{g^{-1}}, {}^{g^{-1}} u_{n-1}, u_m, {}^{g^{-1}} {}^g y'$$

is a path from ${}^{g^{-1}} {}^g x' = x'$ to ${}^{g^{-1}} {}^g y' = y'$ of length $k < l$, which contradicts our original path being of shortest length. Hence $d({}^g x', {}^g y') = l$.

□

Definition 3.13

Consider the equivalence relation on X with equivalence classes

$$[x'] = \{x'_i \mid x'_i = {}^g x' \text{ for some } g \in G\},$$

that is orbits of points in X under the action of G. The quotient space *of X by this equivalence relation is the set of these equivalence classes and is denoted $X/_G$. Let π*

be the projection of X onto ${}^{X}/_{G}$, then a set $U \subseteq {}^{X}/_{G}$ is open if and only if $\pi^{-1}(U)$ is open in X.

Claim 3.14

The set of points in the topological space Y created from the quotient graph and the set of points ${}^{X}/_{G}$ are in bijection.

Proof: We define a map $\sigma : {}^{X}/_{G} \to Y$ by $\sigma([(e,\mu)']) = [(e,\mu)]'$ for $(e,\mu)' \in X$. This map is surjective as for every $[(e,\mu)]' \in Y$, $\sigma([(e,\mu)']) = [(e,\mu)]'$. It is injective since if $\sigma([(e,\mu)']) = \sigma([(f,\nu)'])$, then $[(e,\mu)]' = [(f,\nu)]'$. Hence $(e,\mu) = {}^{g}(f,\nu)$ for some $g \in G$ and so $(e,\mu)' = {}^{g}(f,\nu)'$ in X. Therefore $[(e,\mu)'] = [(f,\nu)']$ in ${}^{X}/_{G}$. $\square$

Finally we wish to show that Y and ${}^{X}/_{G}$ are homeomorphic as topological spaces. To show that these spaces are homeomorphic, we must show that the map $\sigma : {}^{X}/_{G} \to Y$ given by $\sigma([x']) = [(e,\mu)]'$ is continuous, and its inverse is continuous. That is, we must show that sets are open in Y if and only if their preimage is open in ${}^{X}/_{G}$.

Claim 3.15

U is open in Y if and only if $\sigma^{-1}(U)$ is open in ${}^{X}/_{G}$.

Proof: Y is equipped with the metric topology, so U is a union of open balls. In particular, we may write U as a union of open balls that contain a single vertex and open balls that contain points from a single edge. We therefore consider two cases. We first assume that $U \subset Y$ is an open ball containing no vertices, that is $U = B_n([(e,\mu)]')$ where $[(e,\mu)]$ is a distance greater than n from $\iota(e)$ and $\tau(e)$. We wish to show that the preimage of U in ${}^{X}/_{G}$ is open. We will denote $\sigma^{-1}(U)$ by $\overline{U}$. Since ${}^{X}/_{G}$ has the quotient topology, then to show that $\overline{U}$ is open, we must show that $\pi^{-1}(\overline{U})$ is open in X.

We will show that $\pi^{-1}(\overline{U})$ is a union of open balls. Let $U = B_n([(e,\mu)]') = \{[(e,\mu_i)]' \mid i \in I\}$ for some index set I. Then

$$\pi^{-1}(\overline{U}) = \{(f,\mu_i)' \mid f = {}^{g}e \text{ for some } g \in G, i \in I\}.$$

We then show that for any $f \in E$ such that $f = {}^{g}e$ for some $g \in G$, the set $W =$

$\{(f, \mu_i)' \mid i \in I\}$ is an open ball, in particular, $W = B_n((f, \mu)')$.

Let $(f, \mu_j)'$ be an arbitrary point in $B_n((f, \mu)')$. Then

$$d((f, \mu_j)', (f, \mu)') = |\mu_j - \mu| < n.$$

We then have $\hat{d}([(f, \mu_j)]', [(f, \mu)]') = |\mu_j - \mu| < n$, and so $[(f, \mu_j)]' \in U$ and hence $(f, \mu_j)' \in W$.

Similarly, if $(f, \mu_j)'$ is an arbitrary point in W then

$$\hat{d}([(f, \mu_j)]', [(f, \mu)]') = |\mu_j - \mu| < n,$$

and since G acts by isometries, $d((f, \mu_j)', (f, \mu)') = |\mu_j - \mu| < n$ and so $(f, \mu_j)' \in B_n((f, \mu)')$.

Hence $\pi^{-1}(\overline{U})$ is a union of open balls $B_n((f, \mu)')$ where $f \in \{{}^g e \mid g \in G\}$. Hence $\overline{U}$ is open in ${}^{X}\!/_{G}$.

Now suppose that $U \subset Y$ is an open set containing one vertex $[v]'$. Without loss of generality, let $U = B_n([v]')$, that is U is the ball of radius n centred at $[v]'$. Let I, J be index sets, then $U = \{[(e_i, \mu_j)]' \mid i \in I, j \in J, \hat{d}([v]', [(e_i, \mu_j)]') < n\}$. Then the preimage of this set in ${}^{X}\!/_{G}$ is

$$\sigma^{-1}(U) = \{[(e_i, \mu_j)'] \mid [(e_i, \mu_j)]' \in U, i \in I, j \in J\}$$

We will write $\sigma^{-1}(U) = \overline{U}$. The set $\overline{U}$ is open in ${}^{X}\!/_{G}$ if and only if $\pi^{-1}(\overline{U})$ is open in X. This preimage is

$$\pi^{-1}(\overline{U}) = \{(f, \mu_j)' \mid f = {}^g e_i, \hat{d}([v]', [(e_i, \mu_j)]') < n \\ \text{for some } g \in G, \text{for all } i \in I, j \in J\}.$$

Fix $k \in I$ and let $W = \{(e_k, \mu_j)' \mid j \in J, \hat{d}([v]', [(e_k, \mu_j)]') < n\}$. We wish to show that W is an open ball of radius n centred at ${}^g v'$. Suppose $[(e_i, \mu_j)]'$ is a point in U. Then the distance of this point from $[v]'$ is

$$\hat{d}([(e_i, \mu_j)]', [v]') = \begin{cases} \mu & \text{if } \iota([e_i]) = [v] \\ 1 - \mu & \text{if } \tau([e_i]) = [v] \end{cases}$$

which must be less than n.

Let $(e_k, \mu_j)'$ be an arbitrary point in W. We want to show that $(e_k, \mu_j)'$ is distance less than n from v'.

$$d((e_k, \mu_j)', v') = \begin{cases} \mu_j & \text{if } \iota(e_k) = v \\ 1 - \mu_j & \text{if } \tau(e_k) = v \end{cases}$$

Now recall that $\iota([e_k]) = [\iota(e_k)]$ so if $\iota(e_k) = v$ then $\iota([e_k]) = [\iota(e_k)] = [v]$ and so $d((e_k, \mu_j)', v') = \mu_j < n$. On the other hand if $\tau(e_k) = v$ then $\tau([e_k]) = [\tau(e_k)] = [v]$ and $d((e_k, \mu_j)', v') = 1 - \mu_j < n$. Hence all points in W on the edge e_k are a distance less than n from v'. Therefore $W \subset B_n({}^g v')$.

Suppose now that $(e_k, \mu_j)'$ is an arbitrary point in $B_n({}^g v')$. Then by a symmetric argument $B_n({}^g v') \subset W$.

Now since G acts by isometries, then all points in ${}^g W$ are a distance less than n from from ${}^g v'$ for all $g \in G$, and so ${}^g W$ is also an open ball. Since $\pi^{-1}(\overline{U}) = \bigcup_{g \in G} {}^g W$, then $\pi^{-1}(\overline{U})$ is a union of open balls and so $\overline{U}$ is open in ${}^{X}/_{G}$.

Conversely, if $\pi^{-1}(U)$ is open in ${}^{X}/_{G}$, then an analogous argument shows that U is open in Y.

$\square$

Let $t_X : \Gamma \to X$ be the map from the graph to its associated topological space, and let $t_Y : {}^{\Gamma}/_{G} \to Y$ be the map from the quotient graph to its associated topological space. Additionally, let $p : \Gamma \to {}^{\Gamma}/_{G}$ be the map from the graph to the quotient graph. Then we have the following commutative diagram.

$$\begin{tikzcd} \Gamma \arrow[r, "t_X"] \arrow[d, "p"'] & X \arrow[d, "\sigma\pi"] \\ {}^{\Gamma}/_{G} \arrow[r, "t_Y"'] & Y \end{tikzcd}$$

This means that when we speak of the quotient space of the topological space created from Γ, it does not matter if we first take the quotient by the action on the graph and then create the space Y, or if we first create the topological space X and then take the quotient by the action on X.

When using the Švarc-Milnor lemma, we will want to know that our quotient space is compact, so we had best decide when a set in our topological space is compact. This will involve us looking at unions of parts of edges in our space, so we define a *formal path* to be a continuous map $p : [0, 1] \to X$.

Lemma 3.16

A set $K \subseteq X$ is compact if and only if K is a finite union of images of formal paths.

PROOF: First, suppose K is a set such that $K = \bigcup_{i=1}^{n} p_i([0, 1])$ for formal paths p_i, and let $\mathcal{F}$ be an open cover of K. Let $\mathcal{P}_i \subseteq \mathcal{F}$ denote the subset that covers each $p_i([0, 1])$. A formal path is compact, since it is the continuous image of the compact set $[0, 1]$, and so there exists some finite $\overline{\mathcal{P}_i} \subseteq \mathcal{P}_i$ that covers $p_i([0, 1])$. $\overline{\mathcal{F}} = \bigcup_{i=1}^{n} \overline{\mathcal{P}_i}$ is then a finite subcover for K, and so K is compact.

Now let K be a compact set. Let Δ be the set of all edges δ for which $(\delta, \mu) \in K$ for some $\mu \in [0, 1]$. Then we construct an open cover $\mathcal{F}$ by taking the union of the set of open balls of radius $2/3$ centred at $(\gamma, 1/2)$ for each $\gamma \in \Gamma$. Since K is compact, there exists a finite subcover, $\mathcal{F}' \subseteq \mathcal{F}$. For each open ball $\mathcal{B}$ in $\mathcal{F}'$ we can find a formal path p_i such that $p_i([0, 1]) = \mathcal{B} \cap K$, and hence K is the finite union of images of formal paths. $\square$

For the above to hold, it is important that our graph is locally finite, as it is possible to construct a compact set in the space constructed from a non-finitely generated semigroup that is the image of infinitely many formal paths.

Example 3.17

Let S be a non-finitely generated semigroup with generators e_i for $i \in \mathbb{N}$. Take the set containing all points between $((a, ae_i), 0)$ and $((a, ae_i), 1/2^i)$ for all i. This corresponds to an infinite union of images of paths,for example, the paths which map $[0, 1]$ to the interval $[0, 1/2^i]$ on each edge (a, ae_i) but is in fact compact. Any open cover must contain a set covering the centre point $((a, ae_i), 0)$. This open set must contain an open ball of some radius r containing the centre point. This open ball covers infinitely many of the intervals, those for which $1/2^i < r$, on the edges (a, ae_i), leaving only a finite number of closed intervals $[r, 1/2^i]$, which are compact, and so any open cover of these has a finite subcover.

Finally, we show that the metric space we have created from a graph is proper.

Lemma 3.18

Let X be the metric space created from a connected, locally finite graph Γ as outlined above. Then X is proper.

Proof: Let $B_r(x') = \{y' \in X \mid d(x', y') \leq r\}$ be the closed ball of radius r centred at x'. Then since $B_r(x')$ has finite radius it contains finitely many vertices, and since Γ is locally finite $B_r(x')$ contains finitely many edges. Hence $B_r(x')$ is a finite union of paths by Lemma 3.16 is compact. Thus X is proper. □

Now we are ready to dive in to the wonderful world of the Švarc-Milnor lemma!

3.2 Applying the Švarc-Milnor Lemma

Fundamentally, the Švarc-Milnor lemma is a theorem about groups and how they act, so it seems like a good idea for us to pick a group and decide how and where it should act. We recall that we wished to investigate direct products of groups and right-zero semigroups, and so it may be useful to know something about when direct products of semigroups are finitely presented.

Theorem 3.19 ([16, Theorem 3.5])

Let C and D be two infinite semigroups. The direct product $C \times D$ is finitely presented if and only if the following conditions are satisfied:

(i) $C^2 = C$ *and* $D^2 = D$*;*

(ii) C *and* D *are finitely presented and stable.*

We note that it is possible for a finite semigroup to fail to be stable, as demonstrated in [16] by the following example.

Example 3.20 ([16, Example 8.4])

Let S be the semigroup defined by the presentation

$$\mathrm{sgp}\langle a, x, y \mid xa = a, ya = a, xy = x, a^3 = a^2, x^2 = x, y^2 = y\rangle.$$

Then S has 11 elements but is not stable.

We will prove the following theorem.

Theorem 3.21

Let $S = G \times U$ and $T = H \times V$ be semigroups such that G, H are groups and $U^2 = U$, $V^2 = V$, and U, V are both finite and stable. Then if $\dagger\,(S, A) \cong \dagger\,(T, B)$ for finite generating sets A and B, S is finitely presented if and only if T is.

We let S and T be as in Theorem 3.21, and suppose that S is finitely presented, which by Theorem 3.19 implies that G is finitely presented. We choose to act with G on the metric space $\dagger\,(S, A)'$ obtained from $\dagger\,(S, A)$ as outlined in § 3.1. For a vertex $(x, u) \in S$, we define an action of G on the vertices by

$$^{g}(x, u) = (gx, u).$$

We can then extend this to an action on edges. Let e be the edge originating at $\iota(e) = (x, u)$, labelled $\lambda(e) = (a, b)$ and terminating at $\tau(e) = (xa, ub)$. Then we define $^{g}e = f$ where $\iota(f) = {}^{g}\iota(e) = (x, u)$, $\tau(f) = {}^{g}\tau(e) = (gxa, ub)$ and the label is $\lambda(f) = \lambda(e) = (a, b)$. This can then be extended to an action on the space $\dagger\,(S, A)'$.

Claim 3.22

The action of G on $\dagger\,(S, A)'$ is by isometries.

Proof: To show this is action is by isometries we first check that adjacency of vertices in the graph $\dagger\,(S, A)$ is preserved under the action. Let (x_1, u_1) and (x_1a_1, u_1v_1) be adjacent vertices, which were connected by an edge labelled (a_1, v_1) in $\mathrm{Cay}(S, A)$. Let $g \in G$, then $^{g}(x_1, u_1) = (gx_1, u_1)$ and $^{g}(x_1a_1, u_1v_1) = (gx_1a_1, u_1v_1)$. These two vertices are adjacent in $\dagger\,(S, A)$ since $(gx_1, u_1)(a_1, v_1) = (gx_1a_1, u_1v_1)$.

We then show that non-adjacency is also preserved under the action. Let (x_2, u_2) and (x_3, u_3) be two non-adjacent vertices such that for some $g \in G$, $^{g}(x_2, u_2)$ and

${}^{g}(x_3, u_3)$ are adjacent, say ${}^{g}(x_2, u_2)(a_2, v_2) = {}^{g}(x_3, u_3)$ for $(a_2, v_2) \in A$. Then $(gx_3, u_3) = (gx_2a_2, u_2v_2)$. If we multiply both sides by $(g^{-1}, 1)$ then we see $(x_3, u_3) = (x_2a_2, u_2v_2) = (x_2, u_2)(a_2, v_2)$ and so (x_2, u_2) and (x_3, u_3) are adjacent, connected by an edge labelled (a_2, v_2). This is a contradiction and so non-adjacent vertices cannot be mapped to adjacent vertices.

Since both adjacency and non-adjacency are preserved, then paths of shortest length are preserved and so the action is by isometries. □

Claim 3.23

The action of G on $\dagger\,(S, A)'$ is cocompact.

Proof: By Lemma 3.16 we must show that the quotient space is a finite union of paths. The orbit of a vertex is $[(x, u)] = \{(g, u) \mid g \in G\}$, and so our quotient spaces has finitely many vertices. We show that the size of the set ${}^{E}\!/_{G}$ is $|U||A|$. Given an edge e with start vertex $\iota(e) = (g, u)$ with label $\lambda(e) = (a, v)$, the orbit of this edge is

$$\begin{aligned}[e] = \{f \in E \mid \iota(f) = {}^{x}(g, u) = (xg, u),\\ \tau(f) = {}^{x}(ga, uv) = (xga, uv)\\ \lambda(f) = (a, v)\\ \}.\end{aligned}$$

If we consider the orbit of the vertex $\iota(e)$, we note that every vertex (xg, u) in it has an edge leaving it labelled (a, v), ending at (xga, uv). Hence the orbit of our edge e is all edges labelled (a, v), beginning at any vertex (x, u).

$$[e] = \{f \in E \mid \iota(f) = (x, u) \text{ for } x \in G, \lambda(f) = (a, v)\}.$$

Thus ${}^{E}\!/_{G}$ contains one edge per generator for each copy of G, of which there are $|U|$, and so ${}^{E}\!/_{G}$ contains $|U||A|$ edges, so it is certainly finite. We now have that the quotient graph has finitely many vertices and edges, and hence the quotient space is a finite union of paths, that is, compact. □

This means the action on $\dagger\,(S, A)'$ by G meets the first criterion for the application of the Švarc-Milnor lemma. We would also like to show this action is proper.

Claim 3.24

The action of G on $\dagger\,(S, A)'$ is proper.

Proof: Let $K \subseteq M$ be a compact set, that is, a finite union of images of paths, say $K = \bigcup_{i=1}^{n} p_i([0,1])$. Let Q be the set of group elements

$$Q = \{g \in G \mid (g, u) \text{ is an endpoint of an edge that contains a point in } K \text{ for some } u \in U\}.$$

Then the set $P = \{g \in G | K^g \cap K \neq \varnothing\}$ is precisely

$$\begin{aligned} P &= \{g \in G \mid {}^g(e, \mu)' \in K \text{ for some } (e, \mu)' \in K\} \\ &= \{g \in G \mid gx \in Q \text{ for } x \in Q\} \\ &= \{g \in G \mid gx = q \text{ for } x, q \in Q\} \\ &= \{g \in G \mid g(xq^{-1}) = 1 \text{ for } x, q \in Q\} \end{aligned}$$

Since Q is a finite subset of a group, there are finitely many products xq^{-1}, and hence finitely many g that are inverses for these products. Therefore P is finite. Hence the action is proper. □

Now we can follow the exact same procedure for T and act on its metric space $\dagger\,(T, B)'$ with H. Applying the Švarc-Milnor lemma with G acting on $\dagger\,(S, A)'$, we see that G is quasi-isometric to $\dagger\,(S, A)'$, and similarly we have H is quasi-isometric to $\dagger\,(T, B)'$. Now since $\dagger\,(S, A) \cong \dagger\,(T, B)$ with graph isomorphism α, the metric spaces $\dagger\,(S, A)'$ and $\dagger\,(T, B)'$ are also isometric. The isometry φ is given by mapping the endpoints of an edge to their images under α.

This allows us to prove a useful lemma, one showing that since G is quasi-isometric to $\dagger\,(S, A)'$, and H is quasi-isometric to $\dagger\,(T, B)'$, then via the isometry of these topological spaces φ, we can show that G is quasi-isometric to H.

Lemma 3.25

The group G is quasi-isometric to H.

Proof: Let G act on $\dagger\,(S, A)'$ as outlined above. This is a cocompact and proper action on a proper geodesic metric space, so by Theorem 3.7, there exists a quasi-isometry

$\psi : G \to M$. Using the isomorphism φ above, the map $\psi' : G \to N$ given by $\psi'(g) = \varphi(\psi(g))$ is also a quasi-isometry. By acting in an analogous way with H on $\dagger(T, B)'$, we can also establish a quasi-isometry $\theta : H \to N$, and hence we have a quasi-isometry $\beta : G \to H$ given by $\beta(g) = \theta^{-1}(\psi'(g))$. $\square$

Now we recall that Theorem 1.1 tells us that finite presentability of groups is preserved under quasi-isometry, giving the following corollary to Lemma 3.25.

Corollary 3.26

The group H is finitely presented.

Hence both H and V meet the conditions of Theorem 3.19, and so T is finitely presented. This completes the proof of Theorem 3.21.

We recall that we began this chapter by looking at direct products with left zero semigroups, and lamenting the fact that we we not able to apply a similar method to direct products with right zero semigroups. Happily, right zero semigroups are now just a special case of what we have just proved.

Corollary 3.27

Let G and H be groups, let m, n be positive integers and let $S = G \times R_n$ and $T = H \times R_m$ where $S = \mathrm{sgp}\langle A\rangle$ and $T = \mathrm{sgp}\langle B\rangle$, with A and B finite. If $\dagger(S, A) \cong \dagger(T, B)$, then S is finitely presented if and only if T is.

Proof: Consider R_n and R_m given by the presentations $R_n = \mathrm{sgp}\langle r_1, \ldots, r_n \mid r_i r_j = r_j \text{ for all } 1 \leq i, j \leq n\rangle$ and $R_m = \mathrm{sgp}\langle r_1, \ldots, r_m \mid r_i r_j = r_j \text{ for all } 1 \leq i, j \leq m\rangle$. Both R_n and R_m are clearly finite, and since for any right zero r_i we have $r_i r_i = r_i$, then $R_n^2 = R_n$ and $R_m^2 = R_m$.

We show that R_n (similarly R_m) is stable. Let $r_{i_1} \ldots r_{i_k}$ and $r_{j_1} \ldots r_{j_l}$ be two words over $r_1, \ldots, r_n{}^+$ such that the relation

$$r_{i_1} \ldots r_{i_k} = r_{j_1} \ldots r_{j_l}$$

holds in R_n. We note that two words are equal in R_n if and only if the last letters of the word are equal, so $r_{i_n} = r_{j_m}$. Suppose that $k > l$. Then an elementary sequence is

$$\begin{aligned} r_{i_1} \dots r_{i_k} \equiv & \, r_{i_1} \dots r_{i_k}, r_{i_1} \dots r_{i_{k-2}} r_{i_k}, \\ & r_{i_1} \dots r_{i_{k-3}} r_{i_k}, \dots, \\ & r_{i_1} \dots r_{i_l} r_{i_k}, r_{i_1} \dots r_{i_{l-1}} r_{i_k}, \\ & r_{i_1} \dots r_{j_{l-1}} r_{i_{l-1}} r_{i_k}, r_{i_1} \dots r_{j_{l-1}} r_{i_k}, \\ & \dots r_{i_1} r_{j_2} \dots r_{j_{l-1}} r_{i_k}, \\ & r_{i_1} r_{j_1} r_{j_2} \dots r_{j_{l-1}} r_{i_k}, r_{j_1} r_{j_2} \dots r_{j_{l-1}} r_{i_k} \\ & \equiv r_{j_1} r_{j_2} \dots r_{j_l}. \end{aligned}$$

The shortest word in this sequence is $r_{i_1} \dots r_{i_{l-1}} r_{i_k}$ which has length l, and so there is no word shorter than the shortest of our two original words. Now if $k < l$, there exists an elementary sequence

$$\begin{aligned} r_{i_1} \dots r_{i_k} \equiv & \, r_{i_1} \dots r_{i_k}, r_{j_{l-1}} r_{i_1} \dots r_{i_k}, \\ & r_{j_{l-2}} r_{j_{l-1}} r_{i_1} \dots r_{i_k}, \dots, r_{j_1} \dots r_{j_{l-1}} r_{i_1} \dots r_{i_k}, \\ & r_{j_1} \dots r_{j_{l-1}} r_{i_1} \dots r_{i_{k-2}} r_{i_k}, \dots, \\ & r_{j_1} \dots r_{j_{l-1}} r_{i_1} r_{i_k}, r_{j_1} \dots r_{j_{l-1}} r_{i_k} \\ & \equiv r_{j_1} \dots r_{j_{l-1}} r_{i_l}. \end{aligned}$$

The shortest word in this sequence is $r_{j_{l-1}} r_{i_1} \dots r_{i_k}$ which has length $k+1$, so this sequence never goes via a word shorter than the original words. Hence R_n and R_m have no critical pairs, and so are stable. Thus we may apply Theorem 3.19, and S is finitely presented if and only if T is. $\square$

Chapter 4

Completely Simple Semigroups

> Of course, of course! Absurdly simple,
> like most riddles when you see the answer.
>
> Gandalf The Grey

A sensible path to follow is to investigate semigroups which feature groups somewhere in their construction, as this allows us to use the Švarc-Milnor Lemma (provided we can construct a sensible action of course). Here we will look into completely simple and completely 0-simple semigroups which, due to Rees [14], have a construction which may be thought of as the direct product of a left zero semigroup, a group and a right zero semigroup, with a twisted form of group multiplication. We will extend our ideas on left and right zero semigroups from Chapter 3 to take into account the difficulties introduced here. We begin with the definition of the objects we will work with.

Definition 4.1

A semigroup S is simple *if it has no proper two-sided ideals. A semigroup is* completely simple *if it is simple and has minimal left and right ideals.*

A famous theorem of Rees allows us to give a construction of such semigroups.

Definition 4.2

Let T be a semigroup, let I, Λ be arbitrary sets and let $P = (p_{\lambda i})$ be a $|\Lambda| \times |I|$ matrix *(known as the* sandwich matrix*) with entries in T. The set $I \times T \times \Lambda$ is a semigroup with the multiplication*

$$(i, g, \lambda)(j, h, \mu) = (i, gp_{\lambda j}h, \mu)$$

called a Rees matrix semigroup*. We write $\mathcal{M}[T; I, \Lambda; P]$ to denote this semigroup.*

Theorem 4.3 (Rees, [14])

A semigroup S is completely simple if and only if $S \cong \mathcal{M}[G; I, \Lambda; P]$, for some group G, sets I, Λ and sandwich matrix P with entries in G.

When considering a completely simple semigroup, we will choose to work with the Rees matrix semigroup to which it is isomorphic. We let $S = \mathrm{sgp}\langle A \rangle$ and $T = \mathrm{sgp}\langle B \rangle$ be two completely simple semigroups, with $S \cong \mathcal{M}[G; I_S, \Lambda_S; P_S]$ and $T \cong \mathcal{M}[H; I_T, \Lambda_T; P_T]$. We of course wish to show that if we have $\dagger(S, A) \cong \dagger(T, B)$, then S is finitely presented if and only if T is.

One approach we may consider is to see that if given an $S \cong \mathcal{M}[G; I_S, \Lambda_S; P_S]$, we can detect clusters of vertices that represent the group G. That is, given a fixed $i \in I_S$ and $\lambda \in \Lambda_S$, can we identify the set of vertices $V_{i,\lambda} = \{(i, g, \lambda) \mid g \in G\}$ in $\dagger(S, A)$. We will see in Lemmas 4.6 and 4.7 that I_S can be viewed as a left zero semigroup and so $\dagger(S, A)$ has at least $|I_S|$ components. We can therefore identify the larger set of vertices $V_i = \{(i, g, \mu) \mid g \in G, \mu \in \Lambda\}$ that contains $V_{i,\lambda}$. Now considering the subgraph induced by this set of vertices V_i we would like to be able to find all vertices that represent elements with λ as their third component. However, similarly to Example 3.5, restricting to $V_{i,\lambda}$ does not necessarily give us a copy of $\dagger(G, \pi_G(A))$. Hence we cannot deduce which vertices belong to $V_{i,\lambda}$, and therefore cannot even decide the size of Λ_S.

We show by means of an example that two completely simple semigroups with isomorphic skeletons can in fact have differently sized Λ sets.

Example 4.4

Let $S = \mathcal{M}[C_2; \{l_1\}, \{r_1, r_2\}; I]$ and let $T = \mathcal{M}[V_4; \{j_1\}, \{s_1\}; I]$. Let

$$S = \mathrm{sgp}\langle \{(l_1, x, r_1), (l_1, x, r_2)\} \rangle$$

and let

$$T = \text{sgp}\langle\{(j_1, a, s_1), (j_1, b, s_1)\}\rangle.$$

We then have the following Cayley graphs.

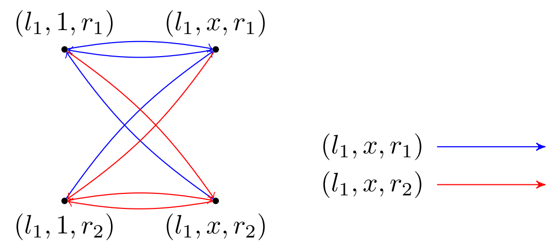


Figure 4.1: Graph Cay$(S, \{(l_1, x, r_1), (l_1, x, r_2)\})$

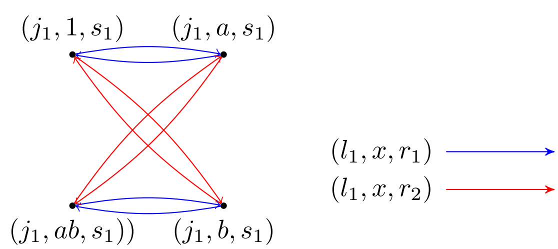


Figure 4.2: Graph Cay$(T, \{(j_1, a, s_1), (j_1, b, s_1)\})$

The skeletons of both S and T with respect to these generating sets are then as follows.

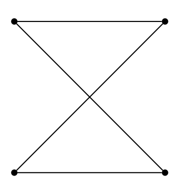


Figure 4.3: Graph $\dagger\,(S, \{(l_1, x, r_1), (l_1, x, r_2)\})$ and $\dagger\,(T, \{(j_1, a, s_1), (j_1, b, s_1)\})$

Lemma 4.5

Given an $S \cong \mathcal{M}[G; I_S, \Lambda_S; P_S]$ we cannot necessarily identify the set of vertices $V_{i,\lambda} = \{(i, g, \lambda) \mid g \in G\}$ in $\dagger\,(S, A)$ for a fixed $i \in I_S$ and $\lambda \in \Lambda_S$. Given two semi-

groups $S \cong \mathcal{M}[G; I_S, \Lambda_S; P_S]$ *and* $T \cong \mathcal{M}[H; I_T, \Lambda_T; P_T]$ *with* $\dagger(S, A) \cong \dagger(T, B)$ *we do not necessarily have* $|\Lambda_S| = |\Lambda_T|$.

PROOF: In Example 4.4 we see that

$$\dagger(S, \{(l_1, x, r_1), (l_1, x, r_2)\}) \cong \dagger(T, \{(j_1, a, s_1), (j_1, b, s_1)\}).$$

If we attempt to identify the set V_{l_1,r_1}, which consists of the upper two vertices in Figure 4.1, in the skeleton $\dagger(S, \{(l_1, x, r_1), (l_1, x, r_2)\})$ we are unable to see which set of two connected vertices this should be. We also have that $|\Lambda_S| = 2$ and $|\Lambda_T| = 1$. $\square$

Hence we see that we can determine neither the location of $V_{i,\lambda}$ nor the size of Λ by inspection of the skeleton, and as such there is no clear way to describe when two completely simple semigroups have isomorphic skeletons.

We will assume that S is finitely presented, with finite generating set

$$A = \{(i_k, g_k, \lambda_k) \mid 1 \leq k \leq n \text{ for some } n \in \mathbb{N}\}.$$

Lemma 4.6

The sets I_S *and* Λ_S *can be viewed as left and right zero semigroups respectively.*

PROOF: If we consider the projection $\pi_{I_S} : S \to I_S$, then the set $\pi_{I_S}(A)$ forms a generating set for the set I_S under the multiplication $ij = i$. This tells us that I_S is in fact a finitely generated left zero semigroup, and hence is finite. Similarly, the projection $\pi_{\Lambda_S} : S \to \Lambda_S$ gives a finite generating set $\pi_{\Lambda_S}(A)$ for Λ under the multiplication $\lambda\mu = \mu$, and so Λ_S is a finite right zero semigroup. $\square$

Lemma 4.7

The graph $\dagger(S, A)$ *has at least* $|I_S|$ *components.*

PROOF: Consider now an element (i, g, λ). When we multiply this on the right with an element (j, h, μ), say, our product is $(i, gp_{\lambda j}h, \mu)$. We notice that since I_S is a left zero semigroup, the I_S component remains the same as in the original element. This means that in the Cayley graph of S, there are no edges between vertices with different I_S components, and so $\dagger(S, A)$ has at least $|I_S|$ components. $\square$

We now restrict ourself to looking at the set $R = \{(i, g, \lambda) \mid g \in G, \lambda \in \Lambda_S\}$ for a fixed $i \in I_S$. This set is a subsemigroup of S, but it is not necessarily generated by the set $A \cap R$, and even if it were, the graph $\dagger\,(R, A \cap R)$ would not neccesarily be isomorphic to the subgraph of $\dagger\,(S, A)$ induced by the set R. Hence we will work with the graph $\dagger\,(R, A)$. Recall that $\dagger\,(R, A)'$ denotes the metric space created from the graph $\dagger\,(R, A)$ as described in § 3.1. This space is geodesic, since R is a right ideal of S, meaning we can find a path between any two points in $\dagger\,(R, A)$ that realises the shortest distance. We will define an action of G on R by

$$^{g}(i, x, \lambda) = (i, gx, \lambda) \qquad (\star)$$

Claim 4.8

$(\star)$ *is an action.*

Proof: Let $g, h \in G$ and $(i, x, \lambda) \in S$. Then

$$\begin{aligned} ^{gh}(i, x, \lambda) &= (i, (gh)x, \lambda) \\ &= (i, g(hx), \lambda) \\ &= {}^{g}(i, hx, \lambda) \\ &= {}^{g}({}^{h}(i, x, \lambda)) \end{aligned}$$

and

$$^{1}(i, x, \lambda) = (i, x, \lambda).$$

□

We define an action on edges, by defining $^{g}e = f$ where $\iota(f) = {}^{g}\iota(e), \tau(f) = {}^{g}\tau(e)$ and $\lambda(f) = \lambda(e)$. We can extend our action to an action of G on $\dagger\,(R, A)'$, which we desire to be by isometries, cocompact and proper.

Claim 4.9

The action $(\star)$ *is by isometries.*

Proof: Let (i, x, λ) and (i, xa, μ) be two vertices in $\dagger\,(R, A)'$ joined by an edge labelled (j, a, μ). Under the action $\star$ we have $^{g}(i, x, \lambda) = (i, gx, \lambda)$ and $^{g}(i, xa, \mu) =$

(i, gxa, μ). Since $(i, x, \lambda)(j, a, \mu) = (i, gxa, \mu)$ then the action preserves adjacency of vertices. Now let (i, x, λ) and (i, xa, μ) be two vertices such that there exists no edge between them, but the exists an edge labelled (j, a, μ) between ${}^{g}(i, x, \lambda) = (i, gx, \lambda)$ and ${}^{g}(i, xa, \mu) = (i, gxa, \mu)$. Then since $(i, gx, \lambda)(j, a, \mu) = (i, gxa, \mu)$ we have

$$\begin{aligned} {}^{g^{-1}}(i, gx, \lambda)(j, a, \mu) &= (i, x, \lambda)(j, a, \mu) \\ &= {}^{g^{-1}}(i, gxa, \mu) \\ &= (i, xa, \mu). \end{aligned}$$

Hence there does exist an edge between (i, x, λ) and (i, xa, μ) labelled (j, a, μ). Thus the action preserves non-adjacency also. Since both adjacency and non-adjacency are preserved, then paths of shortest length are preserved and so our action $\star$ is by isometries. □

Claim 4.10

The action $(\star)$ *is cocompact.*

Proof: To check our action is cocompact, we simply need to establish that the quotient graph of $\dagger(R, A)$ is finite. The orbit of a vertex (i, x, λ) is

$$\begin{aligned} [(i, x, \lambda)] &= {}^{G}(i, x, \lambda) \\ &= \{(i, gx, \lambda) \mid g \in G\} \\ &= \{(i, g, \lambda) \mid g \in G\}. \end{aligned}$$

Thus we have $|\Lambda|$ distinct orbits, and so $\dagger(R, A)/G$ has finitely many vertices.

An upper bound for the size of the set E/G is $|\Lambda||A|$, as given an edge e with start vertex $\iota(e) = (i, x, \lambda)$ with label $\lambda(e) = (i, a, \mu)$, the orbit of this edge is

$$\begin{aligned} [e] = \{f \in E \mid & \iota(f) = {}^{g}(i, x, \lambda) = (i, gx, \lambda), \\ & \tau(f) = {}^{g}(i, xp_{\lambda j}a, \mu) = (i, gxp_{\lambda j}a, \mu), \\ & \lambda(f) = (i, a, \mu)\}. \end{aligned}$$

Consider the orbit of the vertex $\iota(e)$. For every vertex (i, gx, λ) in this orbit, there is an edge leaving it labelled (i, a, μ), terminating at $(i, gxp_{\lambda j}a, \mu)$. Hence the orbit of our edge e is all edges labelled $(i, gxp_{\lambda j}a, \mu)$, beginning at any vertex (i, g, λ).

$$[e] = \{f \in E \mid \iota(f) = (i, g, \lambda) \text{ for } g \in G, \lambda(f) = (i, a, \mu)\}$$

Thus E/G contains one edge per generator for each copy of G, of which there are $|\Lambda|$, and so E/G contains $|\Lambda||A|$ edges, so it is certainly finite. Hence the quotient graph has finitely many vertices and edges, and so the quotient space is a finite union of paths. By Lemma 3.16, the quotient space is compact, and hence the action is cocompact. □

Claim 4.11

The action $(\star)$ is proper.

PROOF: We now let $K \subseteq \dagger(R, A)'$ be a compact set, so by Lemma 3.16, K is a finite union of paths. Let Q be the set of group elements

$$Q = \{g \in G \mid (i, g, \lambda) \text{ is an endpoint of an edge that contains a point in } K \text{ for some } \lambda\}.$$

Then the set $P = \{g \in G | {}^gK \cap K \neq \varnothing\}$ is precisely

$$\begin{aligned} P &= \{g \in G \mid {}^g(e, \mu)' \in K \text{ for some } (e, \mu)' \in K\} \\ &= \{g \in G \mid gx \in Q \text{ for } x \in Q\} \\ &= \{g \in G \mid gx = q \text{ for } x, q \in Q\} \\ &= \{g \in G \mid g(xq^{-1}) = 1 \text{ for } x, q \in Q\}. \end{aligned}$$

Since Q is a finite subset of a group, there are finitely many such g that are inverses for xq^{-1}, and so P is finite, and so the action is proper. □

We notice that in order to construct a nice action here, we required Λ_S to be finite. We will want to be able to act in a similar way with H on $\dagger(T, B)'$ so we must show that Λ_T is finite.

Lemma 4.12

The sets Λ_T and I_T are finite.

PROOF: Given the multiplication of T, we know that each set of vertices $\{(i, h, \lambda) \mid h \in H, \lambda \in \Lambda_T\}$ forms a component of $\dagger(T, B)$, meaning $\dagger(T, B)$ has $|I_T|$ components.

Since $\dagger(S, A) \cong \dagger(T, B)$, and $\dagger(S, A)$ has $|I_S|$ components, then $\dagger(T, B)$ has $|I_S|$ components also, and so $|I_T| = |I_S|$ meaning I_T is finite.

Now suppose that Λ_T is infinite. This implies that there exists a generator (i, a, λ) for all $\lambda \in \Lambda_T$; that is, infinitely many generators. Let $v = (j, h, \mu)$. For each generator (i, a, λ) we have

$$(j, h, \mu)(i, a, \lambda) = (j, hp_{\mu,i}a, \lambda).$$

Since there are infinitely many $\lambda \in \Lambda_T$, the vertex v is therefore connected to infinitely many other vertices, but this is a contradiction, as $\dagger(T, B)$ is locally finite. Therefore Λ_T is finite. $\square$

Lemma 4.13

The group G is quasi-isometric to H.

PROOF: We apply the Švarc-Milnor Lemma, which tells us G is quasi-isometric to $\dagger(R, A)'$, via a map ψ. Since we have $\varphi : \dagger(S, A)' \to \dagger(T, B)'$, an isomorphism, there exists a subset U of T such that $\dagger(R, A)'$ is isomorphic to $\dagger(U, B)'$. We can then construct a quasi-isometry $\psi' : G \to \dagger(U, B)'$, where for an element $g \in G$ we have $\psi'(g) = \varphi(\psi(g))$. Acting analogously on $\dagger(U, B)'$ with H, we establish a quasi-isometry $\sigma : H \to \dagger(U, B)'$, which allows us to build a quasi-isometry $\beta : G \to H$ given by $\beta(g) = \sigma^{-1}(\psi'(g))$. Hence G is quasi-isometric to H. $\square$

Thus far, we have not mentioned finite presentability in the context of Rees matrix semigroups. The following result will be useful.

Theorem 4.14 (Ayik and Ruškuc, [2])

Let S be a Rees matrix semigroup $\mathcal{M}[T; I, \Lambda; P]$, and let W be the ideal of T generated by the set $\{p_{\lambda i} \mid \lambda \in \Lambda, i \in I\}$ of all entries of P. Then S is finitely generated (respectively, finitely presented) if and only if the following three conditions are satisfied:

(i) both I and Λ are finite,

(ii) T is finitely generated (respectively, finitely presented)

(iii) the set $T \setminus W$ is finite.

Our semigroup S is assumed to be finitely presented, and so by Theorem 4.14 the group G is also. Since quasi-isometries preserve finite presentability of groups, we establish a lemma.

Lemma 4.15

If S is finitely presented H is finitely presented.

PROOF: By Theorem 4.14, S being finitely presented implies that G is finitely presented. Lemma 4.13 tells us that G is quasi-isometric to H, and since quasi-isometry of groups preserves finite presentability, then H is finitely presented.

□

Consequently we have the following theorem.

Theorem 4.16

Let $S \cong \mathcal{M}[G; I_S, \Lambda_S; P_S]$ with finite generating set A and $T \cong \mathcal{M}[H; I_T, \Lambda_T; P_T]$ with finite generating set B. If $\dagger(S, A) \cong \dagger(T, B)$, then the semigroup S is finitely presented if and only if T is.

PROOF: Let S be finitely presented. Then by Lemma 4.15, H is finitely presented and by Lemma 4.12 I_T and Λ_T are finite. Since H is a group, the ideal $W = T^1\{p_{\lambda i} \mid \lambda \in \Lambda_T, i \in I_T\}T^1$ is the whole group, as groups have no proper ideals. Hence $H \setminus W = \varnothing$ which is clearly finite. By Theorem 4.14 we then have that T is finitely presented. □

We can now turn our attention to completely simple semigroups sibling, the *completely* 0*-simple semigroup*. First we define a 0*-simple* semigroup S as a semigroup with 0, for which the only proper two-sided ideal is $\{0\}$. We then say that an ideal I of S is 0*-minimal* if I and $\{0\}$ are the only ideals contained in I. Then S is completely 0-simple if it is 0-simple and its left and right ideals are 0-minimal. For completely simple semigroups, we were able to find a characterisation in the form of Rees matrix semigroups. Completely 0-simple semigroups are similarly nice, and so we introduce the *Rees matrix semigroup with* 0. We let T be a semigroup not containing a 0, let I, Λ be index sets, and let P be a $|\Lambda| \times |I|$ matrix with entries in $T \cup \{0\}$. Then $(I \times T \times \Lambda) \cup \{0\}$

is a semigroup with multiplication

$$(i, g, \lambda)(j, h, \mu) = \begin{cases} (i, gp_{\lambda j}h, \mu) & \text{if } p_{\lambda j} \neq 0 \\ 0 & \text{if } p_{\lambda j} = 0 \end{cases}$$

and

$$0(i, g, \lambda) = (i, g, \lambda)0 = 0 = 00.$$

This semigroup is a Rees matrix semigroup with 0 and is denoted $\mathcal{M}^0[T; I, \Lambda; P]$. Our characterisation in then given by the following theorem.

Theorem 4.17 (Rees, [14])

A semigroup S is completely 0-simple if and only it is isomorphic to a Rees matrix semigroup with 0, $\mathcal{M}^0[G; I, \Lambda; P]$ where G is a group, and P is regular.

Note that a matrix is *regular* if every row and every column contains at least one non-zero element.

We let $S = \mathcal{M}^0[G; I_S, \Lambda_S; P_S]$ with P_S regular, and $T = \mathcal{M}^0[H; I_T, \Lambda_T; P_T]$ with P_T regular. We prove the following.

Theorem 4.18

Let $S = \mathcal{M}^0[G; I_S, \Lambda_S; P_S]$ where P_S is regular, and $T = \mathcal{M}^0[H; I_T, \Lambda_T; P_T]$ where P_T is regular. Let $S = \mathrm{sgp}\langle A\rangle$ and $T = \mathrm{sgp}\langle B\rangle$ with A and B finite, and $\dagger\,(S, A) \cong \dagger\,(T, B)$, then S is finitely presented if and only if T is.

Proof: Suppose that S is finitely presented, then by Lemma 4.23 H is finitely presented. Then by [2] and Lemmas 4.21 and 4.24 T is finitely presented. $\square$

We now proceed to prove the lemmas used in the proof of Theorem 4.18. In order to do this we will need to construct a slightly different space to the standard skeletons to act on.

Suppose that $\dagger\,(S, A) \cong \dagger\,(T, B)$, and assume S is finitely presented. As with completely simple semigroups, we can deduce that both I_S and Λ_S are finite (cf. Lemma 4.6). We assume that G is infinite, as if G is finite, then S is finite since I_S and Λ_S are finite and so it is trivial to show T is finitely presented. Therefore, given an infinite G, we claim that in $\dagger\,(S, A)$, that 0 is the unique vertex of infinite degree.

Claim 4.19

In $\dagger(S, A)$, 0_S *is the unique vertex of infinite degree.*

PROOF: We first show that 0_S has infinite degree. We know there are two possible ways to get to 0 via multiplication. The first is that 0_S appears already in our generating set, and $s0_S = 0_S$ for all $s \in S$, which would mean 0 trivially has infinite degree. We therefore assume that 0_S is not in our generating set, and so in order to reach 0_S we must find two elements $(i, g, \lambda), (j, h, \mu)$ such that $p_{\lambda j} = 0_S$, and then their product will be zero. Let $p_{\lambda j} = 0_S$ be an entry of the matrix with value 0_S. Now since I can be viewed as a left zero semigroup, for each $i \in I$ there must be a generator of S that contains i as its left most component. Therefore, there exists some generator (j, a, μ) with j as the left component. Let

$$D = \{(i, g, \lambda) \mid g \in G, i \in I\}.$$

This is an infinite set, and for all $(i, g, \lambda) \in D$ we have

$$\begin{aligned}(i, g, \lambda)(j, a, \mu) &= (i, gp_{\lambda,j}a, \mu)\\ &= 0_S.\end{aligned}$$

Hence 0 has infinite degree.

Now suppose $v = (i, g, \lambda) \neq 0_S$ is a vertex in $\dagger(S, A)$ with infinite degree. Since S is finitely generated, v must have finite outdegree in Cay(S, A) and so has infinite indegree. In particular, there are infinitely many vertices w such that $wa_1 = v$ for some generator $a_1 = (j, a, \lambda)$. Since I_S can be viewed as a left zero semigroup, we know that each w has the same I_S component i as v. As Λ_S is finite, there must be infinitely many such w that share the same Λ_S component, say μ. We thus consider the subset of elements $W = \{w \in T \mid w = (i, h, \mu), h \in H\}$, which is an infinite set. We have that for all $w = (i, h, \mu) \in W$

$$\begin{aligned}(i, h, \mu)(j, a, \mu) &= (i, hp_{\mu j}a, \mu)\\ &= (i, g, \lambda).\end{aligned}$$

Since W is infinite, but i, μ are fixed, there are an infinite number of elements $h \in G$ such that $hp_{\mu j}a = g$. Since $v \neq 0_S$, we know that $p_{\mu j} \neq 0_S$, so $p_{\mu j} \in G$. This implies there are infinitely many different h in G such that $h(p_{\mu j}ag^{-1}) = 1$, that is the fixed element $p_w ag^{-1}$ has infinitely many inverses. This is a contradiction, and so v must have finite degree. Hence we have shown that 0_S is the unique vertex of infinite degree in $\dagger\,(S, A)$. $\square$

We now know that there exists exactly one vertex of infinite degree in $\dagger\,(S, A)$, and hence also in $\dagger\,(T, B)$. Since T also contains a zero element 0_T, we may show in an identical way as in S that the vertex representing 0_T has infinite degree, and hence in both $\dagger\,(S, A)$ and $\dagger\,(T, B)$ the unique vertex of infinite degree represent the zero element.

Since both graphs $\dagger\,(S, A)$ and $\dagger\,(T, B)$ have a single vertex of infinite degree, the subgraphs generated by removing these vertices and their associated edges will remain isomorphic. We will call these graphs $\dagger\,(S_z, A_z)$ and $\dagger\,(T_z, B_z)$ respectively. We wish to work with a geodesic metric space, so we restrict our attention to the subgraph induced by the set of vertices $R_z = \{(i, g, \lambda) \mid g \in G, \lambda \in \Lambda_S\}$. We call this graph $\dagger\,(R_z, A_z)$. We then create a metric space from $\dagger\,(R_z, A_z)$ using the method outlined in § 3.1. We call this metric space $\dagger\,(R_z, A_z)'$.

Lemma 4.20

The group G is quasi-isometric to the metric space $\dagger\,(R_z, A_z)'$.

Proof: We define an action of G on $R_z = \{(i, g, \lambda) \mid g \in G, \lambda \in \Lambda_S\}$ by

$$^{g}(i, x, \lambda) = (i, gx, \lambda)$$

This is a map from $G \times R_z \to R_z$, as it is never the case that the multiplication gx returns zero, and the action axioms are fulfilled in the same way as for Rees matrix semigroups. This extends to give us an action on $\dagger\,(R_z, A_z)'$, which is cocompact and proper, following a similar line of reasoning to Rees matrix semigroups. We apply the Švarc-Milnor lemma to obtain a quasi-isometry between G and $\dagger\,(R_z, A_z)'$ $\square$

In order for such an action to be cocompact, we required Λ_S to be finite, and so if we hope to act on the component $\dagger\,(U_z, B_z)'$ of $\dagger\,(T_z, B_z)'$ which is isomorphic to $\dagger\,(R_z, A_z)'$, we will want to show that Λ_T is finite.

Lemma 4.21

The sets Λ_T and I_T are finite.

Proof: To see that I_T is finite, consider the subgraph of $\dagger\,(T, B)$ induced by the set of vertices $T \setminus \{0_T\}$. Since I_T can be viewed as a left zero semigroup, then this graph has $|I_T|$ components. Since this is isomorphic to the subgraph of $\dagger\,(S, A)$ induced by the set of vertices $S \setminus \{0_S\}$, which has $|I_S|$ components, then $|I_T| = |I_S|$ and so I_T is finite.

Assume that Λ_T is infinite. We will show that this means there exists a non-zero vertex with infinite indegree. The matrix P_T then has finitely many columns and infinitely many rows. Since it is regular, this means there exists some column, say the one indexed by $j \in I_T$, which contains infinitely many non-zero entries. There exists at least one generator, (j, g, λ) say, with this j as its first component. Now consider the vertex $(i, 1_H, \lambda)$ for some $i \in I_T$. For every element of the form $(i, g^{-1}{p_{\mu j}}^{-1}, \mu)$, provided that $p_{\mu j} \neq 0$, we have

$$\begin{aligned}(i, g^{-1}{p_{\mu j}}^{-1}, \mu)(j, g, \lambda) &= (i, g^{-1}{p_{\mu j}}^{-1}p_{\mu j}g, \lambda) \\ &= (i, 1_H, \lambda).\end{aligned}$$

Now since we picked j such that the column indexed by it in P_T has infinitely many non-zero entries, there are infinitely many values μ such that the above product is realised, and hence $(i, 1_H, \lambda)$ has infinite indegree. This is a contradiction to the fact that 0_T is the only vertex of infinite degree in $\dagger\,(T, B)$, and so Λ_T is finite. $\square$

Lemma 4.22

The group H is quasi-isometric to G.

Proof: We can then show similarly to Lemma 4.20 that H is quasi-isometric to the subset $\dagger\,(U_z, B_z)' \cong \dagger\,(R_z, A_z)'$ of $\dagger\,(T_z, B_z)'$. Hence G is quasi-isometric to H via the isomorphism between $\dagger\,(U_z, B_z)'$ and $\dagger\,(R_z, A_z)'$. $\square$

There exists an analogous theorem to 4.14 for Rees matrix semigroups with zero in [2], which states identical conditions for a semigroup $\mathcal{M}^0[T; I, \Lambda; P]$ to be finitely presented.

Lemma 4.23

If S is finitely presented then H is finitely presented.

Proof: Since S is finitely presented, G is also. Quasi-isometry preserves finite presentability of groups, and so by Lemma 4.22 H has a finite presentation. □

Lemma 4.24

Let U is the ideal of H generated by the set $\{p_{\lambda i} \mid \lambda \in \Lambda_T, i \in I_T\}$ of all entries in P_T, then $H \setminus U$ is finite.

Proof: Since H is a group, $H \setminus U = \varnothing$, and hence finite. □

This completes the proof of Theorem 4.18.

Chapter 5

Clifford Semigroups

> "I invoke the All Nations Agreement article number 39436175880932/B."
> "39436175880932/B? 'All nations attending the conference are only allocated one car parking space?' Is that entirely relevant, sir?"
>
> Red Dwarf

In the previous chapter, we discovered that completely simple semigroups were a fruitful area of study for our question. This is because they are constructed from groups, and retain some of the group structure for us to exploit. It is perhaps then a sensible idea to look for more families of semigroups which are based on groups. One such family is Clifford semigroups. We may construct these by taking a semilattice, placing a group at each element of the semilattice, and defining a sensible multiplication. We expand on this construction below.

Definition 5.1

A partially ordered set $(X, \leq)$ *is a set* X *together with a binary relation* $\leq$ *such that for all* $a, b, c \in X$

(i) $a \leq a$ *(reflexivity);*

(ii) if $a \leq b$ *and* $b \leq a$ *then* $a = b$ *(anti-symmetry);*

(iii) if $a \leq b$ and $b \leq c$ then $a \leq c$ (transitivity).

If a pair of elements $(a, b) \in X \times X$ are not in the relation $\leq$, we say they are incomparable and we write $a \| b$.

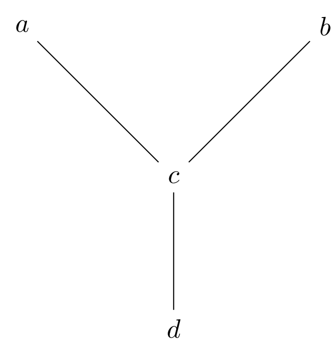
Definition 5.2

Let $(X, \leq)$ be a partially ordered set and let $a, b \in X$. An element c is called the meet *of a and b if $c \leq a$ and $c \leq b$, and for any $d \in X$ such that $d \leq a, b$, then $d \leq c$. If the meet of two elements exists, then it is unique, and we write $a \wedge b = c$.*

Definition 5.3

*Let $(X, \leq)$ be a partially ordered set such that for all $a, b \in X$, $a \wedge b$ exists. Then $(X, \leq)$ is a (*lower*)* semilattice.

We may draw a Hasse diagram to represent a semilattice, for example let $X = \{a, b, c, d\}$ with $c \leq a, c \leq b$ and $d \leq c$, and a, b incomparable.



We can observe from the diagram here that every pair of elements does have a meet:

$$a \wedge b = c$$

$$a \wedge x = x$$

$$b \wedge a = c$$

$$b \wedge x = x$$

$$c \wedge d = d$$

$$c \wedge x = c$$

$$d \wedge x = d$$

where x is any element of X that has not already been specified. Hence this is a semilattice.

We can now construct a semigroup as follows. Suppose we have Y, a semilattice, and a set of semigroups S_λ indexed by Y. For every $\lambda, \mu \in Y$ where $\lambda \geq \mu$ let $\varphi_{\lambda,\mu} : S_\lambda \to S_\mu$ be a homomorphism such that $\varphi_{\lambda,\lambda}$ is the identity map on S_λ and for all $\lambda, \mu, \nu \in Y$ such that $\lambda \geq \mu \geq \nu$

$$\varphi_{\lambda,\mu}\varphi_{\mu,\nu} = \varphi_{\lambda,\nu}.$$

Our semigroup is the set $S = \bigcup_{\lambda \in Y} S_\lambda$, where multiplication of two elements $x \in S_\lambda$ and $y \in S_\mu$ is given by

$$xy = (x\varphi_{\lambda,\lambda\wedge\mu})(y\varphi_{\mu,\lambda\wedge\mu}).$$

We call S a *strong semilattice of semigroups*, and denote it by $S = \mathcal{S}[Y; S_\lambda; \varphi_{\lambda,\mu}]$.

To define Clifford semigroups, we first define completely regular semigroups. A semigroup S is *completely regular* if there exists a unary operation $a \mapsto a^{-1}$ on S such that

$$(a^{-1})^{-1} = a \ , \ aa^{-1}a = a \ , \ aa^{-1} = a^{-1}a$$

for every $a \in S$. An enlightening theorem which suggests to us why this may be a sensible area to look is the following:

Theorem 5.4 (Proposition 4.1.1, [12])

Let S be a semigroup. Then S is completely regular if and only if every $\mathcal{H}$-class of S is a group.

A *Clifford semigroup* is a completely regular semigroup such that for all $x, y \in s$

$$(xx^{-1})(yy^{-1}) = (yy^{-1})(xx^{-1}).$$

We would like a characterisation of Clifford semigroups that allows us to easily work with the group structure found within. The following theorem gives us such a characterisation:

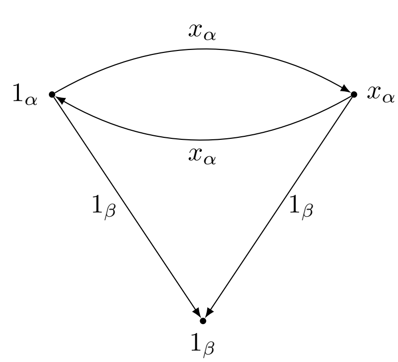
Theorem 5.5 (Theorem 4.2.1, [12])

Let S be a semigroup. Then S is Clifford if and only if it is a strong semilattice of groups.

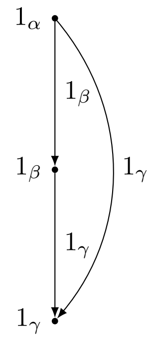
It is tricky to work with arbitrary Clifford semigroups, as given an arbitrary skeleton of a Clifford semigroup, we may not even be able to distinguish the semilattice underlying it.

Example 5.6

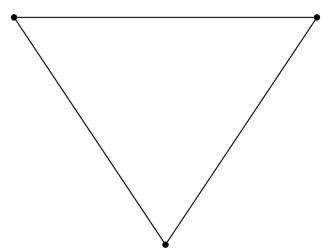
One very small example of this is to let $S = \mathcal{S}[Y; S_\lambda; \varphi_{\lambda,\mu}]$, where $Y = \{\alpha, \beta\}$ is the two-element semilattice with $\alpha \geq \beta$, and $S_\alpha = C_2 = \text{gp}\langle x_\alpha \rangle$ and $S_\beta = \{1_\beta\}$, with the obvious homomorphism $\varphi_{\alpha,\beta}$. Then if we let $S = \text{sgp}\langle x_\alpha, 1_\beta \rangle$, we have $\text{Cay}(S, \{x_\alpha, 1_\beta\})$ as the following graph.



Now let $T = \mathcal{T}[Z; T_\lambda; \psi_{\lambda,\mu}]$ with $Z = \{\alpha, \beta, \gamma\}$ being a 3-element linear order and for all $\lambda \in Z$, we have T_λ trivial groups. We let the homomorphisms be the obvious ones. We have $T = \mathrm{sgp}\langle 1_\alpha, 1_\beta, 1_\gamma \rangle$, and $\mathrm{Cay}(T, \{1_\alpha, 1_\beta, 1_\gamma\})$ is given below.



Now if we are to draw $\dagger\,(S, \{x_\alpha, 1_\beta\})$ and $\dagger\,(T, \{1_\alpha, 1_\beta, 1_\gamma\})$, both result in the following graph.



These are indistinguishable from their skeletons, but clearly have different semilattice structures.

We might wonder if this occurs only if the component groups are finite. To dispel such notions, we provide a further example.

Example 5.7

Let $Y = \{\alpha, \beta\}$ be the two element semilattice, and let $S_\alpha = \mathbb{Z} \times C_2 = S_\beta$. Note that we write $C_2 = \{1, x\}$. Let $\varphi_{\alpha,\beta}$ be the identity map; then $S = \mathcal{S}[Y; S_\lambda; \varphi_{\lambda,\mu}]$ is a semigroup generated by $A = \{(-1_\alpha, 1_\alpha), (1_\alpha, 1_\alpha), (0_\alpha, x_\alpha)(0_\beta, 1_\beta), (0_\beta, x_\beta)\}$. A section of $\mathrm{Cay}(S, A)$ is given in Figure 5.1.

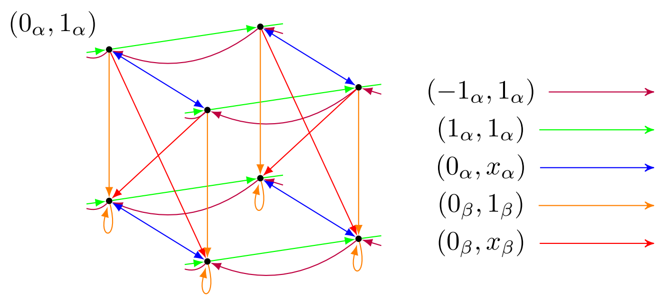


Figure 5.1: Cayley graph of S

Let $Z = \{\alpha, \beta, \gamma, \delta\}$ be the four element linear order, and $T_\lambda = \mathbb{Z}$ for all $\lambda \in Z$. For each $\lambda, \mu \in Z$, define $\varphi_{\lambda,\mu}$ by $x \mapsto 0_\mu$. Then $T = \mathcal{T}[Z; T_\lambda; \varphi_{\lambda,\mu}]$ is a Clifford semigroup, which can be generated by $B = \{-1_\alpha, 1_\alpha, 1_\beta, 1_\gamma, 1_\delta\}$. A section of Cay$(T, B)$ is given in Figure 5.2.

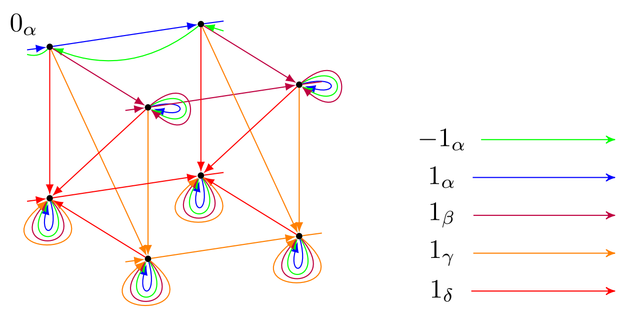


Figure 5.2: Cayley graph of T

Now for S and T we have $\dagger\,(S, A)$ and $\dagger\,(T, B)$ as shown in Figure 5.3.

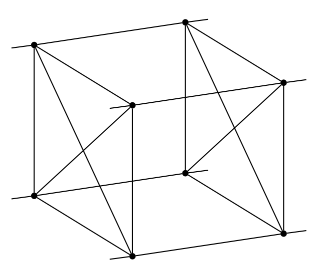


Figure 5.3: Skeleton of S and T

These infinite Clifford semigroups have different semilattice structures, but $\dagger(S, A)$ and $\dagger(T, B)$ are isomorphic.

Sadly, then, it is not possible to deduce the structure of the semilattice simply by inspection of the skeleton. We can, however, cheer ourselves slightly by proving a theorem linking isomorphism of skeletons of Clifford semigroups and finite presentability. In order to do this, it will be useful to know exactly when Clifford semigroups are finitely presented.

When proving this, it will be useful to know when Clifford semigroups are finitely presented. The following theorem gives us conditions for when this occurs.

Theorem 5.8 ([1, Theorem 6.1])
A strong semilattice of semigroups $S = \mathcal{S}[Y; S_\lambda; \varphi_{\lambda,\mu}]$ is finitely presented if and only if Y is finite and every semigroup S_λ is finitely presented.

We will also find the following lemma to be useful.

Lemma 5.9
Let G be a finitely generated group with a finite normal subgroup N. Then G is quasi-isometric to ${}^{G}/_{N}$.

Proof: Define an action of G on ${}^{G}/_{N}$ by ${}^{g}aN = (ga)N$. Then the orbit of a point $aN \in {}^{G}/_{N}$ is the entire group ${}^{G}/_{N}$, and so the action is cocompact. If we let $K \subseteq {}^{G}/_{N}$ be a compact set, then K is finite. Let $P = \{g \in G \mid {}^{g}K \cap K \neq \varnothing\}$, then

$$\begin{aligned} P &= \{g \in G \mid {}^{g}K \cap K \neq \varnothing\} \\ &= \{g \in G \mid gaN \in K \text{ for any } aN \in K\}. \end{aligned}$$

Since N is finite, there are only finitely many $g \in G$ that move elements of K back to itself, and so P is finite. Hence this action is proper, and so by the Švarc-Milnor lemma the map $g \mapsto (ga)N$ for any $aN \in {}^{G}/_{N}$ is a quasi-isometry between G and ${}^{G}/_{N}$ equipped with the word metric. □

We will now prove the following theorem linking isomorphism of skeletons of Clifford semigroups and finite presentability.

Theorem 5.10

Let $S = \mathcal{S}[Y; G_\lambda; \varphi_{\lambda,\mu}]$ and $T = \mathcal{S}[Z; H_\lambda; \theta_{\lambda,\mu}]$ where Y, Z are finite and homomorphisms $\varphi_{\lambda,\mu}, \theta_{\lambda,\mu}$ are such that

(i) $\varphi_{\lambda,\mu}, \theta_{\lambda,\mu}$ *are surjective,*

(ii) $\ker \varphi_{\lambda,\mu}, \ker \theta_{\lambda,\mu}$ *are finite.*

Let $S = \mathrm{sgp}\langle A\rangle$ and $T = \mathrm{sgp}\langle B\rangle$ with A and B finite. If $\dagger(S, A) \cong \dagger(T, B)$ then S is finitely presented if and only if T is.

PROOF:

Let $S = \mathcal{S}[Y; G_\lambda; \varphi_{\lambda,\mu}]$ and $T = \mathcal{S}[Z; H_\lambda; \theta_{\lambda,\mu}]$ be as in Theorem 5.10, and suppose that S is finitely presented. Let ω be the least element of Y.

We will prove this theorem by constructing a new semigroup Ω with skeleton $\dagger\left(\Omega, \overline{A}\right)$ for some generating set $\overline{A}$, on which we can define an action of the group G_ω. We will show that G_ω is quasi-isometric to $\dagger\left(\Omega, \overline{A}\right)$ and that $\dagger\left(\Omega, \overline{A}\right)$ is quasi-isometric to $\dagger(S, A)$. Similarly we let ψ be the least element of Z and define a new semigroup Ψ with skeleton $\dagger\left(\Psi, \overline{B}\right)$ for some generating set $\overline{B}$. We show that H_ψ is quasi-isometric to $\dagger\left(\Psi, \overline{B}\right)$ and that $\dagger\left(\Psi, \overline{B}\right)$ is quasi-isometric to $\dagger(T, B)$. Then since $\dagger(S, A) \cong \dagger(T, B)$ we show that G_ω is quasi-isometric to H_ψ, and hence H_ψ is finitely presented and thus T is finitely presented.

We begin by describing how to construct the semigroup Ω. Recall that ω is the least element of Y and let $K_{\lambda,\omega}$ denote $\ker \varphi_{\lambda,\omega}$. We create a semigroup Ω by taking the union of ${G_\lambda}/{K_{\lambda,\omega}}$ for all $\lambda \in Y$ as elements. Elements of ${G_\lambda}/{K_{\lambda,\omega}}$ are denoted by $sK_{\lambda,\omega}$ where $s \in G_\lambda$. Let $s_1K_{\lambda_1,\omega}, s_2K_{\lambda_2,\omega} \in \Omega$ then we define the following multiplication:

$$s_1K_{\lambda,\omega}s_2K_{\alpha,\omega} = (s_1\varphi_{\lambda,\lambda\wedge\alpha}\, s_2\varphi_{\alpha,\lambda\wedge\alpha})K_{\lambda\wedge\alpha,\omega}.$$

Let $s_1K_{\lambda_1,\omega} = s_2K_{\lambda_1,\omega}$ and $t_1K_{\lambda_2,\omega} = t_2K_{\lambda_2,\omega}$. Then

$$\begin{aligned} s_1K_{\lambda_1,\omega}t_1K_{\lambda_2,\omega} &= (s_1\varphi_{\lambda_1,\lambda_1\wedge\lambda_2}t_1\varphi_{\lambda_2,\lambda_1\wedge\lambda_2})K_{\lambda_1\wedge\lambda_2,\omega} \\ &= (s_1\varphi_{\lambda_1,\lambda_1\wedge\lambda_2})K_{\lambda_1\wedge\lambda_2,\omega}(t_1\varphi_{\lambda_2,\lambda_1\wedge\lambda_2})K_{\lambda_1\wedge\lambda_2,\omega} \\ &= (s_2\varphi_{\lambda_1,\lambda_1\wedge\lambda_2})K_{\lambda_1\wedge\lambda_2,\omega}(t_2\varphi_{\lambda_2,\lambda_1\wedge\lambda_2})K_{\lambda_1\wedge\lambda_2,\omega} \\ &= (s_2\varphi_{\lambda_1,\lambda_1\wedge\lambda_2}t_2\varphi_{\lambda_2,\lambda_1\wedge\lambda_2})K_{\lambda_1\wedge\lambda_2,\omega} \\ &= s_2K_{\lambda_1,\omega}t_2K_{\lambda_2,\omega} \end{aligned}$$

Hence this multiplication is well defined. Let

$$\overline{A} = \{aK_{\alpha,\omega} \mid a \in A \text{ where } a \in G_\alpha\}.$$

We then have the following claim.

Claim 5.11

$\overline{A}$ is a generating set for Ω.

Proof: Let $sK_{\lambda,\omega} \in \Omega$ and let $s = a_1 \ldots a_n$ with $a_i \in G_{\alpha_i} \cap A$ for each $1 \leq i \leq n$. We note that since $s \in G_\lambda$, then $\alpha_1 \wedge \ldots \wedge \alpha_n = \lambda$. Now we have

$$\begin{aligned} a_1K_{\alpha_1,\omega} \ldots a_nK_{\alpha_n,\omega} &= sK_{\alpha_1\wedge\ldots\wedge\alpha_n,\omega} \\ &= sK_{\lambda,\omega} \end{aligned}$$

and all $a_iK_{\alpha_i,\omega} \in \overline{A}$. Hence $\overline{A}$ is a generating set for Ω. □

We will now show that our constructed space is quasi-isometric to the original metric space.

Claim 5.12

The metric spaces $\dagger\,(S,A)'$ and $\dagger\left(\Omega,\overline{A}\right)'$ are quasi-isometric.

Proof: For each group G_λ, let $f_\lambda : G_\lambda \to {}^{G_\lambda}/_{K_{\lambda,\omega}}$ be the quasi-isometry as given in Lemma 5.9, and denote the image of an element $g \in G_\lambda$ by $gf_\lambda = gK_{\lambda,\omega}$. We then define a map $f : \dagger\,(S,A)' \to \dagger\left(\Omega,\overline{A}\right)'$ by defining the map for vertices, which induces a map for the space. We define $sf = sf_\lambda$ where $s \in G_\lambda$. Let $\alpha \in Y$ be such that $|K_{\alpha,\omega}| \geq |K_{\lambda,\omega}|$ for any $\lambda \in Y$. Then we claim that f is a $(|K_{\alpha,\omega}|, 1, 1)$-quasi-isometry.

Suppose that s and t are two elements in S, with $sf = sK_{\beta,\omega}$ and $tf = tK_{\gamma,\omega}$. Let $v = v_1 \dots v_n$ be a word labelling a shortest path between s and t where $v_i \in G_{\lambda_i}$. Then $vK_{\gamma,\omega} = v_1 K_{\lambda_1,\omega} \dots v_n K_{\lambda_n,\omega}$ is a path from $sK_{\beta,\omega}$ to $tK_{\gamma,\omega}$ which has length n, and so

$$\begin{aligned} d_{\dagger\left(\Omega,\overline{A}\right)'}(sK_{\beta,\omega}, tK_{\gamma,\omega}) &\leq d_{\dagger(S,A)'}(s,t) \\ &\leq |K_{\alpha,\omega}| d_{\dagger(S,A)'}(s,t) + 1. \end{aligned}$$

We now consider how much shorter than $d_\dagger \left(S,A\right)'(s,t)$ a path between $sK_{\beta,\omega}$ and $tK_{\gamma,\omega}$ can become. If a sequence of vertices $v_i \dots v_j$ in the path v are found in the same coset, then in $\dagger\left(\Omega,\overline{A}\right)'$ these vertices are mapped to a single vertex and so our path length is reduced by $j - i$. The size of $j - i$ can be at most the size of the coset these vertices were found in, which we recall is finite, as all kernels $K_{\lambda,\omega}$ are finite. Now suppose the path v travels through a number of cosets, the largest possible size of which we know is of size $|K_{\alpha,\omega}|$. Then for each coset that the path travels through, we know that we are mapping at most $|K_{\alpha,\omega}|$ vertices to a single vertex in $\dagger\left(\Omega,\overline{A}\right)'$. Thus we reduce the length of a path in $\dagger\left(S,A\right)'$ by a factor of at most $|K_{\alpha,\omega}|$ when moving into $\dagger\left(\Omega,\overline{A}\right)'$. We note that we must then also subtract a constant of 1 from $\frac{1}{|K_{\alpha,\omega}|} d_{\dagger(S,A)'}(s,t)$ to find the true lower bound as we will always have at least one edge which becomes a loop when passing to $\dagger\left(\Omega,\overline{A}\right)'$.

Therefore we have that

$$\frac{1}{|K_{\alpha,\omega}|} d_{\dagger(S,A)'}(s,t) - 1 \leq d_{\dagger\left(\Omega,\overline{A}\right)'}(sK_{\beta,\omega}, tK_{\gamma,\omega}) < |K_{\alpha,\omega}| d_{\dagger(S,A)'}(s,t) + 1.$$

Now since $\dagger\left(\Omega,\overline{A}\right)'$ is connected for any vertex $y \in \dagger\left(\Omega,\overline{A}\right)$ there exists an $x \in \dagger\left(\Omega,\overline{A}\right)$ such that $y = (x)faK_{\lambda,\omega}$ for some $a \in \overline{A}$, that is $d_{\dagger\left(\Omega,\overline{A}\right)}(y,(x)f = 1$.

Hence f is a $(|K_{\alpha,\omega}|, 1, 1)$-quasi-isometry. □

Let $\hat{g}$ be a preimage of $g \in G_\omega$ under $\varphi_{\lambda,\omega}$. We now define an action of G_ω first on the vertices of $\dagger\left(\Omega,\overline{A}\right)$ by

$$^{g}sK_{\lambda,\omega} = \hat{g}sK_{\lambda,\omega}. \tag{$\star\star$}$$

Claim 5.13

($\star\star$) *is an action of G_ω on the vertices of* $\dagger\left(\Omega, \overline{A}\right)$.

Proof: Let $g, h \in G_\omega$ and $sK_{\lambda,\omega} \in \Omega$. We first note that since $(s^{-1}\hat{gh}^{-1}\hat{g}\hat{h}s)\varphi_{\lambda,\omega} = 1$ then $\hat{g}\hat{h}sK_{\lambda,\omega} = \hat{gh}sK_{\lambda,\omega}$. Therefore

$$\begin{aligned}
{}^{g}({}^{h}sK_{\lambda,\omega}) &= {}^{g}\hat{h}sK_{\lambda,\omega} \\
&= \hat{g}\hat{h}sK_{\lambda,\omega} \\
&= \hat{gh}sK_{\lambda,\omega} \\
&= {}^{gh}(sK_{\lambda,\omega}).
\end{aligned}$$

We also have

$$\begin{aligned}
{}^{1_\omega}sK_{\lambda,\omega} &= \hat{1_\omega}sK_{\lambda,\omega} \\
&= 1_\lambda sK_{\lambda,\omega} \\
&= sK_{\lambda,\omega}.
\end{aligned}$$

Hence ($\star\star$) is an action. $\square$

This is then extended to an action on edges, and then to an action on the metric space $\dagger\left(\Omega, \overline{A}\right)'$. We will show that this action is by isometries.

Claim 5.14

The action ($\star\star$) is by isometries.

Proof: Let $g \in G_\omega$. We show that the action of g preserves adjaceny of vertices. Let $sK_{\lambda_1,\omega}$ and $saK_{\lambda_2,\omega}$ be two vertices in $\dagger\left(\Omega, \overline{A}\right)'$ that are connected by an edge labelled $aK_{\lambda_2,\omega}$. The images of these vertices under the action of g are $\hat{g}sK_{\lambda_1,\omega}$ and $\hat{g}tK_{\lambda_2,\omega}$ respectively. Then

$$\begin{aligned}
\hat{g}sK_{\lambda_1,\omega}aK_{\lambda_2,\omega} &= \hat{g}saK_{\lambda_2,\omega} \\
&= \hat{g}tK_{\lambda_2,\omega}.
\end{aligned}$$

Hence edges are preserved under the action. Suppose now that $sK_{\lambda_1,\omega}$ and $saK_{\lambda_2,\omega}$ be two vertices in $\dagger\left(\Omega, \overline{A}\right)'$ that are not connected by any edge, but $\hat{g}sK_{\lambda_1,\omega}aK_{\lambda_2,\omega} =$

$\hat{g}tK_{\lambda_2,\omega}$ for some $aK_{\lambda_2,\omega} \in \overline{A}$. We multiply both sides of this equation by $\hat{g}^{-1}K_{\lambda_1,\omega}$ and see that

$$\hat{g}^{-1}K_{\lambda_1,\omega}\hat{g}sK_{\lambda_1,\omega}aK_{\lambda_2,\omega} = \hat{g}^{-1}K_{\lambda_1,\omega}\hat{g}tK_{\lambda_2,\omega}$$
$$sK_{\lambda_1,\omega}aK_{\lambda_2,\omega} = \hat{g}tK_{\lambda_2,\omega}.$$

This tells us there exists an edge between $sK_{\lambda_1,\omega}$ and $\hat{g}tK_{\lambda_2,\omega}$ labelled $aK_{\lambda_2,\omega}$. Hence the action ($\star\star$) preserves non-edges also. Since both adjacency and non-adjacency are preserved, then paths of shortest length are preserved and so the action is by isometries. $\square$

We now show that the action ($\star\star$) is cocompact.

Claim 5.15

The action ($\star\star$) *is cocompact.*

Proof: Let $sK_{\lambda,\omega} \in \Omega$. We consider the orbit of $sK_{\lambda,\omega}$ under G_ω.

$$\begin{aligned} {}^{G_\omega}sK_{\lambda,\omega} &= \{\hat{g}sK_{\lambda,\omega} \mid g \in G_\omega\} \\ &= {}^{G_\lambda}/_{K_{\lambda,\omega}} \end{aligned}$$

Hence in the quotient of $\dagger\left(\Omega, \overline{A}\right)'$ by the action there exists a single vertex for each $\lambda \in Y$. Since $\overline{A}$ is finite, then the quotient space is a finite union of paths and so by Lemma 3.16 the action ($\star\star$) is cocompact. $\square$

The final thing we will show for the action ($\star\star$) is that it is proper.

Claim 5.16

The action ($\star\star$) *is proper.*

Proof: Let K be a compact set and let W be the set of vertices that are endpoints of edges containing points in K. We will assume without loss of generality that W only contains points from a single group ${}^{G_\lambda}/_{K_{\lambda,\omega}}$. If this is not the case we simply split W into the union of sets containing vertices in only single groups and consider the sets individually. We will denote $W \cap {}^{G_\lambda}/_{K_{\lambda,\omega}}$ by C.

$$\begin{aligned}
P &= \{g \in G_\omega \mid {}^{g}K \cap K = \varnothing\} \\
&= \{g \in G_\omega \mid {}^{g}W \cap W = \varnothing\} \\
&= \{g \in G_\omega \mid {}^{g}cK_{\lambda,\omega} \in C \text{ for some } cK_{\lambda,\omega} \in C\} \\
&= \{g \in G_\omega \mid \hat{g}cK_{\lambda,\omega} = dK_{\lambda,\omega} \text{ for some } cK_{\lambda,\omega}, dK_{\lambda,\omega} \in C\}.
\end{aligned}$$

Since C is a finite subset of a group then P is a finite set and so the action ($\star\star$) is proper. $\square$

Hence we apply the Švarc-Milnor lemma which tells us that G_ω equipped with the word metric is quasi-isometric to $\dagger\left(\Omega, \overline{A}\right)'$. We will let $\delta : G_\omega \to \dagger\left(\Omega, \overline{A}\right)'$ denote this quasi-isometry. We can perform the same construction for T, creating a metric space $\dagger\left(\Psi, \overline{B}\right)'$ which is quasi-isometric to the group H_ψ equipped with the word metric where $\psi \in Z$ is the least element of Z. We will let $\zeta : H_\psi \to \dagger\left(\Psi, \overline{B}\right)'$ denote this quasi isometry.

Now recall from Claim 5.12 that $f : \dagger(S, A)' \to \dagger\left(\Omega, \overline{A}\right)'$ is a quasi-isometry. Similarly we may define a quasi-isometry $g : \dagger(T, B)' \to \dagger\left(\Psi, \overline{B}\right)'$. Additionally, since $\dagger(S, A) \cong \dagger(T, B)$ we have an isometry $\xi : \dagger(S, A)' \to \dagger(T, B)'$. We therefore have the following diagram.

$$\begin{tikzcd}
\dagger(S, A)' \arrow[r, "\xi"] \arrow[d, "f"'] & \dagger(T, B)' \arrow[d, "g"'] \\
\dagger\left(\Omega, \overline{A}\right)' & \dagger\left(\Psi, \overline{B}\right)' \\
G_\omega \arrow[u, "\delta"'] & H_\psi \arrow[u, "\zeta"']
\end{tikzcd}$$

This means that G_ω is quasi-isometric to H_ψ, and since these are groups then G_ω being finitely presented means that H_ψ is also finitely presented. Now since H_ψ is finitely presented, then so is ${H_\lambda}/{K_{\lambda,\psi}}$ for every $\lambda \in Z$. Then by Lemma 5.9 H_λ is quasi-isometric to ${}^{H_\lambda}/_{K_{\lambda,\psi}}$ and so H_λ is finitely presented by Theorem 1.1 for all $\lambda \in Z$. Hence by Theorem 5.8 T is finitely presented. $\square$

We have managed to show that finite presentability is invariant under our skeleton operation for Clifford semigroups in which both semigroups have finite semilattices where the homomorphisms are epimorphisms with finite kernel. Further work may look at whether it is possible to relax the conditions on the homomorphisms.

Conjecture 5.17

Let $S = \mathcal{S}[Y; G_\lambda; \varphi_{\lambda,\mu}]$ *and* $T = \mathcal{S}[Z; H_\lambda; \theta_{\lambda,\mu}]$ *where* Y, Z *are finite and homomorphisms* $\varphi_{\lambda,\mu}, \theta_{\lambda,\mu}$ *are such that*

(i) $\operatorname{im}\varphi_{\alpha,\beta}, \operatorname{im}\theta_{\alpha,\beta}$ *have finite index.*

(ii) $\ker\varphi_{\alpha,\beta}, \ker\theta_{\alpha,\beta}$ *are finite.*

If $\dagger(S) \cong \dagger(T)$ *then* S *is finitely presented if and only if* T *is.*

In order to follow a similar proof method here, we would need to find a sensible action for the Švarc-Milnor Lemma, and it is not immediately clear how we would define such an action.

Chapter 6

A Counterexample

Der Graf ist nicht das, was er mal war
Ja, der Graf wirkt heut seltsam und bizarr

Der Graf
Die Ärzte

Having developed the theory of finite presentability as a skeleton-invariant for certain families of semigroups, we now change tack, and demonstrate that it is not, in general, a skeleton-invariant property. This chapter therefore presents a counterexample to the conjecture that finite presentability is a skeleton and quasi-isometry invariant of semigroups by proving the following lemma.

Lemma 6.1

Let

$$S = \mathrm{mon}\langle a, b \mid ab^n a = aba \textit{ for } n \in \mathbb{N}\rangle$$

and

$$T = \mathrm{mon}\langle c, d \mid cdc = cd^2c = cd^4 = cd^3c^2 = cd^3cdc\rangle.$$

Then $\dagger\,(S, \{a, b\}) = \dagger\,(T, \{c, d\})$.

We can see from the given presentation that T is finitely presented. We will shortly prove that S does not have a finite presentation.

Lemma 6.2 ([17, Proposition 1.3.1])

Let S be a semigroup and A and B be two finite generating sets for S. If there exists a finite presentation for S in terms of generators A, then there exists a finite presentation for S in terms of generators B.

PROOF: Since B is a generating set for S, there exists an onto mapping $\hat{\varphi} : A^+ \to B^+$ such that a and $a\hat{\varphi}$ represent the same element for all $a \in A$. This can be extended to a homomorphism $\varphi : A^+ \to B^+$ such that for all $w \in A^+$, $w\varphi$ represents the same element as w in S. Similarly there exists a homomorphism $\sigma : B^+ \to A^+$ such that w and $w\sigma$ represent the same element in S for all $w \in B^+$.

Let sgp$\langle A \mid R\rangle$ be a finite presentation for S, and let $R\varphi = \{u\varphi = v\varphi \mid (u = v) \in R\}$. We show that

$$P = \text{sgp}\langle B \mid R\varphi, b = b\sigma\varphi \text{ for } b \in B\rangle$$

is a finite presentation for S.

The relations of P are satisfied by S, so it remains to show that if $w_1 = w_2$ is a relation in S, then it is a consequence of P. Suppose w_1, w_2 are words in B^+ representing the same element in S, then $w_1\sigma$ and $w_2\sigma$ are in A^+ and represent the same element of S. Since sgp$\langle A \mid R\rangle$ is a presentation, $w_2\sigma$ can be obtained from $w_1\sigma$ by applying relations from R. Hence $w_2\sigma\varphi$ can be obtained from $w_1\sigma\varphi$ by application of relations from $R\varphi$. Now if $w_1 \equiv b_1b_2\dots b_k$ for $b_i \in B$, then $w_1\sigma\varphi \equiv (b_1\sigma\varphi)(b_2\sigma\varphi)\dots(b_k\sigma\varphi)$ since σ and φ are both homomorphisms. Hence $w_1 = w_1\sigma\varphi$ is a consequence of the relations $b = b\sigma\varphi$ for $b \in B$. The relation $w_2 = w_2\sigma\varphi$ is obtained similarly, and hence $w_1 = w_2$ is a consequence of the presentation P. $\square$

We note that a monoid has a finite semigroup presentation if and only if it has a finite monoid presentation, and hence the above lemma may be applied to our monoid S.

Lemma 6.3 ([17, Example 1.3.2])

The monoid S does not admit a finite presentation.

PROOF: Suppose that S has a finite presentation. Then by Lemma 6.2, there exists a finite presentation of the form mon$\langle a, b \mid R\rangle$. Since $ab^na = aba$ holds in S for all

$n \in \mathbb{N}$, then aba can be obtained by applying relations in R to $ab^n a$. However, since ab^n and $b^n a$ do not satisfy any non-trivial relations, then for each $n > 1$, R must contain a relation whose left or right-hand side is $ab^n a$, and hence R must be infinite, a contradiction. $\square$

6.1 Normal Forms

In order to see that we have correctly established the Cayley graphs for S and T, we will need to have some way of representing the elements of both. We do this by finding normal forms for S and T, via the use of complete rewriting systems. Throughout, let $\underline{k} = \{1, \ldots, k\}$ for $k \in \mathbb{N}_0$.

Recall that the semigroup defined by a presentation is the quotient of the free semigroup over the generators by the smallest congruence containing the relations, and hence is a set of congruence classes. Given two words in A^+, we may be able to see if they are in the same congruence class using a *rewriting system* to determine *normal forms* for words.

Definition 6.4

A rewriting system *is a set of ordered pairs* LHS $\to$ RHS*, a left-hand side and a right-hand side, with* $\mathrm{LHS}, \mathrm{RHS} \in A^+$*. Words in* A^+ *are* rewritten *by replacing subwords that are left hand sides of rewriting rules with the corresponding right hand side. We denote a word* v *being rewritten to* w *using a single rule by* $v \to w$*. If we can rewrite a word* v *to a word* w *by applying* $n \geq 1$ *rewrite rules, we will write* $v \to^* w$*. A word* w *that cannot be rewritten, that is, there exists no word* v *such that* $w \to v$*, is called* irreducible.

Definition 6.5

We denote by $\leftrightarrow^*$ *the symmetric closure of* $\to$ *and so* $u \leftrightarrow^* v$ *if and only if for some* $n \geq 0$ *there exist* $u = u_0, u_1, \ldots, u_n = v$ *with*

$$\begin{tikzcd}
 & u_1 \arrow[dl, "*"'] \arrow[dr, "*"] & & u_3 \arrow[dl, "*"'] \arrow[dr, "*"] & & u_{n-1} \arrow[dl, "*"'] \arrow[dr, "*"] & \\
u_0 & & u_2 & & \cdots & & u_n.
\end{tikzcd}$$

Then $\leftrightarrow *$ forms a congruence.

We wish to use certain rewriting systems that allow us to find unique representatives for congruence classes, which we will call *normal forms*. To do this, we will need to order words in A^+.

Definition 6.6

The shortlex order *on a set of words W, denoted here by $\prec$, is a well order, in which for $u, v \in W$, we have $u \prec v$ if u has shorter length than v, or if u and v have the same length but u precedes v lexicographically.*

We will use the shortlex order when defining the following properties. Given a semigroup presentation $S = \text{sgp}\langle A \mid R\rangle$ let ρ be the least congruence containing R. We will assume without loss of generality that given a relation $(u, v) \in R$ that $u \prec v$ and we create a rewriting system from R by defining $u \to v$ for all $(u, v) \in R$. Then the congruences $\leftrightarrow *$ and ρ coincide.

Definition 6.7

A rewriting system is locally confluent *if for $a, b_1, b_2 \in A^+$ with $a \to b_1$ and $a \to b_2$, there exists some $c \in A^+$ such that b_1 and b_2 can be rewritten to c in any number of steps. A system is* Noetherian *if there is no infinite chain of strings a_i where $a_i \to a_{i+1}$ for all $i > 0$.*

Definition 6.8

A rewriting system is complete *if it is both Noetherian and confluent.*

Lemma 6.9 ([11, Lemma 12.15])

A rewriting system is complete if it is both Noetherian and locally confluent.

We can now establish a condition for when we can find normal forms for elements of our semigroups.

Lemma 6.10 ([11, Lemma 12.16])

If a complete rewriting system exists for A^+/ρ, then each congruence class contains a unique irreducible element which is the normal form for that congruence class.

Lemma 6.11 ([11, Lemma 12.17])

For an alphabet A and rewriting system $\mathcal{R}$, the rewriting system is locally confluent if and only if the following conditions hold for any two rules $v_1 \to u_1$ and $v_2 \to u_2$ in $\mathcal{R}$.

(i) If $v_1 = xy$ and $v_2 = yz$ for $x, y, z \in A^*$ and $y \neq \varepsilon$, then $u_1 z \to^* t$ and $xu_2 \to^* t$ for some $t \in A^*$.

(ii) If $v_1 = xu_2y$ for $x, y \in A^*$, then $u_1 \to^* w$ and $xu_2y \to^* w$ for some $w \in A^*$.

We can now apply this theory to our semigroups S and T.

Claim 6.12

Elements of S have normal forms given by $\prod_{i \in \underline{k}} b^{\alpha_i} a^{\beta_i}$ where $\alpha_i = 1$ for $i \neq 1, k$, $\beta_i \neq 0$ for $i < k$, and if $\alpha_k > 1$ then $\beta_k = 0$. The empty string, ε_S is also a normal form, representing the identity.

Proof: Using the shortlex order with $a \prec b$, the following is an infinite complete rewriting system made from the relations of the presentation of S:

$$ab^n a \to aba \qquad\qquad \text{for all } n > 1, n \in \mathbb{N}$$

This system is locally confluent[11, Lemma 12.17], as there exists only one form of overlap when rewriting. We provide a diagram to demonstrate this below. The underlined subwords are rewritten following the upper and lower arrow respectively. In this case, the only overlap is the central a in the diagram.

$$\begin{tikzcd}
 & abab^{n_2}a \arrow[r] & ababa \\
\underline{ab^{n_1}a}b^{n_2}a \arrow[ur] \arrow[dr] & & \\
 & ab^{n_1}aba \arrow[r] & ababa
\end{tikzcd}$$

Since both paths rewrite to the same word, we have local confluence. Now for all rules we have RHS $\prec$ LHS, that is if a rewrite rule is applied to a word, it will always be rewritten to a shorter word. Thus the rewriting system is Noetherian, and hence is complete.

We must now check that our proposed normal forms are indeed the correct representatives for their congruence classes, so we must show they are irreducible. Since all rules have a left hand side with ab^na for $n > 1$, elements of the form $\prod_{i \in \underline{k}} b^{\alpha_i} a^{\beta_i}$, with the restrictions as above, cannot be reduced any further, since they do not contain any subwords of the correct form, and so are contained in the set of normal forms. The empty word ε_S cannot be rewritten and hence is also irreducible.

We are sure now that our proposed normal forms are indeed that, but we must also show that we have listed all normal forms. Suppose that $s = \prod_{i \in \underline{k}} b^{\alpha_i} a^{\beta_i} \in S$ where $\alpha_i, \beta_i \in \mathbb{N}$ is an element in normal form not given in the above set. For each $i \in \underline{k}$ with $i \neq k$ such that $\alpha_i \neq 1$, we apply the following rewrite to s:

$$b^{\alpha_1} a^{\beta_1} \ldots b^{\alpha_{i-1}} a^{\beta_{i-1}} b^{\alpha_i} a^{\beta_i} \ldots b^{\alpha_k} a^{\beta_k} \to b^{\alpha_1} a^{\beta_1} \ldots b^{\alpha_{i-1}} a^{\beta_{i-1}} b a^{\beta_i} \ldots b^{\alpha_k} a^{\beta_k}$$

Hence s was not irreducible.

□

The normal forms for S are perhaps better understood by the following description: words beginning with any number of b, which may be followed by any non-zero number of a, then a single b, again any non-zero number of a and a single b, etc., then finally ending in any number of occurrences of the letter a or any number of occurrences of the letter b.

The normal forms for T are somewhat more complicated than for S.

Claim 6.13

Elements of T have one of the following as a canonical forms:

(i) $\prod_{i \in \underline{k}} d^{\alpha_i} c^{\beta_i}$ *where* $\alpha_i = 1$ *for* $i \neq 1$, $\beta_i \neq 0$ *for* $i \leq k$. *These words follow a similar pattern to the normal forms of S, but must end in at least one c.*

(ii) Elements of the form $v \left(d^3 c\right)^l d^r$, *where v is an element of the form given in (i) and* $r \in \{0, 1, 2, 3\}$

(iii) The empty string ε_T is also a normal form, representing the identity.

Proof: Using the shortlex order with $c \prec d$, the following is a finite complete rewriting system made from the relations of the presentation of T:

$$cd^2c \to cdc$$
$$cd^4 \to cdc$$
$$cd^3c^2 \to cdc$$
$$cd^3cdc \to cdc$$

To show local confluence for this system we must consider the following overlaps:

$$\begin{tikzcd} & cdcd^2c \arrow[r] & cdcdc \arrow[dd, "\equiv"'] \\ \underline{cd^{\mathbf{2}}}\underline{cd^2c} \arrow[ur] \arrow[dr] & & \\ & cd^2cdc \arrow[r] & cdcdc \end{tikzcd}$$

$$\begin{tikzcd} & cdcd^4 \arrow[r] & cdcdc \arrow[dd, "\equiv"'] \\ \underline{cd^{\mathbf{2}}}\underline{cd^4} \arrow[ur] \arrow[dr] & & \\ & cd^2cdc \arrow[r] & cdcdc \end{tikzcd}$$

$$\begin{tikzcd} & cdcd^3c^2 \arrow[r] & cdcdc \arrow[dd, "\equiv"'] \\ \underline{cd^{\mathbf{2}}}\underline{cd^3c^2} \arrow[ur] \arrow[dr] & & \\ & cd^2cdc \arrow[r] & cdcdc \end{tikzcd}$$

$$\begin{tikzcd} & cdcd^3cdc \arrow[r] & cdcdc \arrow[dd, "\equiv"'] \\ \underline{cd^{\mathbf{2}}}\underline{cd^3cdc} \arrow[ur] \arrow[dr] & & \\ & cd^2cdc \arrow[r] & cdcdc \end{tikzcd}$$

$$\begin{tikzcd} & cdcd^2c \arrow[r] & cdcdc \arrow[dd, "\equiv"'] \\ \underline{cd^{\mathbf{3}}c}\underline{cd^2c} \arrow[ur] \arrow[dr] & & \\ & cd^3ccdc \arrow[r] & cdcdc \end{tikzcd}$$

$$\begin{tikzcd} & cdcd^4 \arrow[r] & cdcdc \arrow[dd, "\equiv"'] \\ \underline{cd^{\mathbf{3}}c}\underline{cd^4} \arrow[ur] \arrow[dr] & & \\ & cd^3ccdc \arrow[r] & cdcdc \end{tikzcd}$$

$$\begin{tikzcd} & cdcd^3cdc \arrow[r] & cdcdc \arrow[dd, "\equiv"'] \\ \underline{cd^{\mathbf{3}}c}\underline{cd^3cdc} \arrow[ur] \arrow[dr] & & \\ & cd^3ccdc \arrow[r] & cdcdc \end{tikzcd}$$

$$\begin{tikzcd} & cdcd^3cdc \arrow[r] & cdcdc \arrow[dd, "\equiv"'] \\ \underline{cd^{\mathbf{3}}c}\underline{cd^3cdc} \arrow[ur] \arrow[dr] & & \\ & cd^3ccdc \arrow[r] & cdcdc \end{tikzcd}$$

$$\begin{tikzcd} & cdcd^2c \arrow[r] & cdcdc \arrow[dd, "\equiv"'] \\ \underline{cd^{\mathbf{3}}cd}\underline{cd^ic} \arrow[ur] \arrow[dr] & & \\ & cd^3cdcdc \arrow[r] & cdcdc \end{tikzcd}$$

$$\begin{tikzcd} & cdcd^4 \arrow[r] & cdcdc \arrow[dd, "\equiv"'] \\ \underline{cd^{\mathbf{3}}cd}\underline{cd^4} \arrow[ur] \arrow[dr] & & \\ & cd^3cdcdc \arrow[r] & cdcdc \end{tikzcd}$$

$$\begin{tikzcd} & cdcd^3c^2 \arrow[r] & cdcdc \arrow[dd, "\equiv"'] \\ \underline{cd^{\mathbf{3}}cd}\underline{cd^3c^2} \arrow[ur] \arrow[dr] & & \\ & cd^3cdcdc \arrow[r] & cdcdc \end{tikzcd}$$

$$\begin{tikzcd} & cdcd^3cdc \arrow[r] & cdcdc \arrow[dd, "\equiv"'] \\ \underline{cd^{\mathbf{3}}cd}\underline{cd^3cdc} \arrow[ur] \arrow[dr] & & \\ & cd^3cdcdc \arrow[r] & cdcdc \end{tikzcd}$$

These describe all the possible overlaps, and hence the rewriting system is locally confluent. All rewrite rules have LHS $\prec$ RHS, so a shorter word is never rewritten to a longer word, and so the system is Noetherian, and thus complete.

Since we have a complete rewriting system, we know that normal forms for elements of T exist. We wish to check that we have found the correct representative for each congruence class, that is, the unique irreducible element in each class.

Suppose we have an element s of the form (i). All left hand sides of rules can only be applied to words containing a power of d greater than 1. Since s does not contain any such powers, s is irreducible, and thus a normal form.

Now suppose s is in form (ii), that is, $s = v\left(d^3c\right)^l d^r$. As before, we cannot reduce v, so we are concerned with reducing the substring $c\left(d^3c\right)^l d^r$. Given the restrictions on l and r we can see by inspection that this contains no left hand sides of rules, and thus s is irreducible and so a normal form.

The empty string cannot be rewritten in any way, and is trivially a normal form.

Now conversely we must check that we have not forgotten any normal forms. We let s be an element of T in normal form that does not have any of the above forms. We can write $s = \prod\limits_{i\in\underline{k}} d^{\alpha_i}c^{\beta_i}$ for $\alpha_i, \beta_i \in \mathbb{N}$. We will assume that there is exactly one $\alpha_i \neq 1$, as otherwise we will apply the process described below to each $\alpha_i \neq 1$ until we are left with only one.

Now if $\alpha_i = 2$ then

$$d^{\alpha_1}c^{\beta_1}\dots d^{\alpha_{i-1}}c^{\beta_{i-1}}d^2c^{\beta_i}\dots d^{\alpha_k}c^{\beta_k} \to d^{\alpha_1}c^{\beta_1}\dots d^{\alpha_{i-1}}c^{\beta_{i-1}}dc^{\beta_i}\dots d^{\alpha_k}c^{\beta_k}.$$

For $\alpha_i = 3$, if we have $\beta_i = 1$ then

$$\begin{aligned} &d^{\alpha_1}c^{\beta_1}\dots d^{\alpha_{i-1}}c^{\beta_{i-1}}d^3cd^{\alpha_{i+1}}c^{\beta_{i+1}}\dots d^{\alpha_k}c^{\beta_k} \\ &\qquad\qquad \to d^{\alpha_1}c^{\beta_1}\dots d^{\alpha_{i-1}}c^{\beta_{i-1}}dcd^{\alpha_{i+2}}c^{\beta_{i+2}}\dots d^{\alpha_k}c^{\beta_k}. \end{aligned}$$

If $\beta_i \geq 2$ then

$$d^{\alpha_1}c^{\beta_1}\dots d^{\alpha_{i-1}}c^{\beta_{i-1}}d^3c^{\beta_i}\dots d^{\alpha_k}c^{\beta_k} \to d^{\alpha_1}c^{\beta_1}\dots d^{\alpha_{i-1}}c^{\beta_{i-1}}dc^{\beta_i-1}\dots d^{\alpha_k}c^{\beta_k}.$$

Finally, if $\alpha_i \geq 4$ we can rewrite the subword $c^{\beta_{i-1}}d^{\alpha_i}$ as

$$c^{\beta_{i-1}}d^{\alpha_i} \to c^{\beta_{i-1}-1}(cd)^m d^{n-4m}$$

where $m = \lfloor n/4 \rfloor$ and then apply one of the above rewrites if $n - 4m > 1$. Hence s is not irreducible and not in normal form. $\square$

6.2 Graphs

Before providing an isomorphism between $\dagger(S) = \dagger(T)$ we will first establish an intuitive description of these graphs. Since the Cayley graphs have many repeated subgraphs, we build them from smaller graphs to aid understanding. We will zoom in on a section of the graph and describe fully the edges and vertices there. We will then zoom out and gloss over occurrences of the section we examined previously.

Cay(S) is built from four sections, the first of which, our deepest zoom level, is C_1. This consists of the set of vertices $V_1 = \{b^i \mid i \in \mathbb{N}_0\} \cup \{ba\}$, and the edges $E_1 = \{\left(b^i, b^{i+1}\right), \mid i \in \mathbb{N}_0\} \cup \{\left(b^i, ba\right) \mid i \in \mathbb{N}_0\}$.

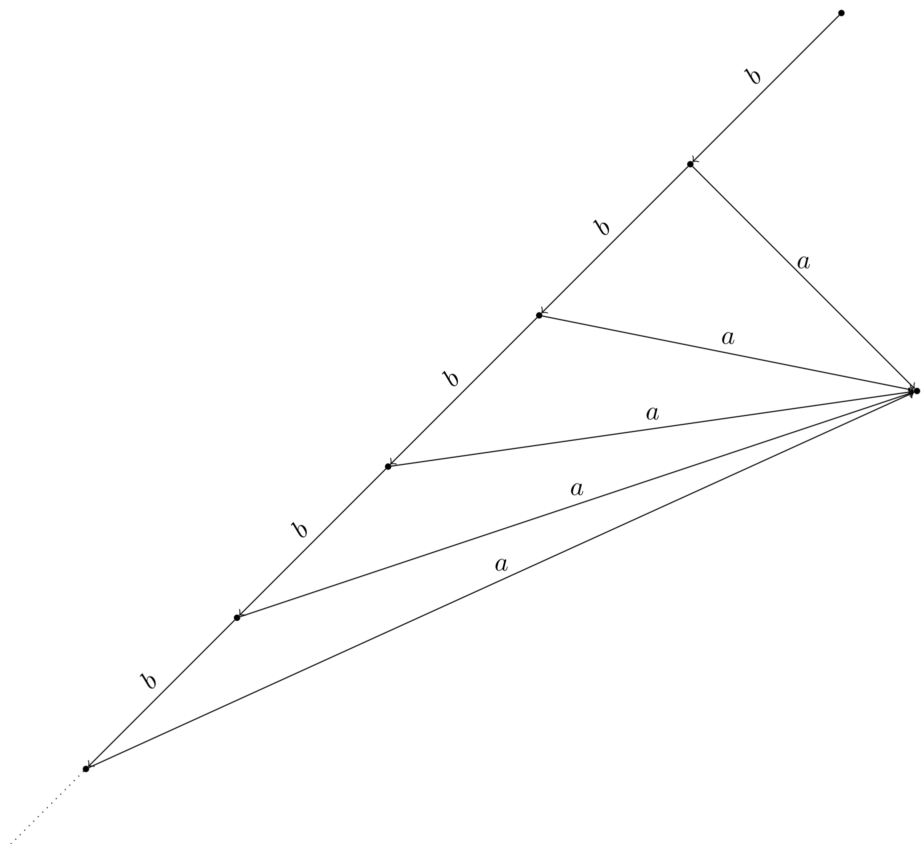


Figure 6.1: C_1

Zooming out, the second section, C_2 consists of vertices $V_2 = \{(\prod_{i \in \underline{k}} ba^{\alpha_i})v_1 \mid k \in \mathbb{N}_0, v_1 \in V_1, \alpha_1 \geq 1\}$. Let $w \in V_2$ be a word ending in a, or the empty word in V_2 ε_{V_2}. Then edges in C_2 are given by $E_2 = \{(w, wa), \left(wb^i, wb^{i+1}\right), \left(wb^j, wba\right) \mid i \in \mathbb{N}_0, j \in \mathbb{N}\} \setminus \{(\varepsilon_{V_2}, a)\}$. Subgraphs isomorphic to C_1 are represented in Figure 6.2 of C_2 by dashed lines.

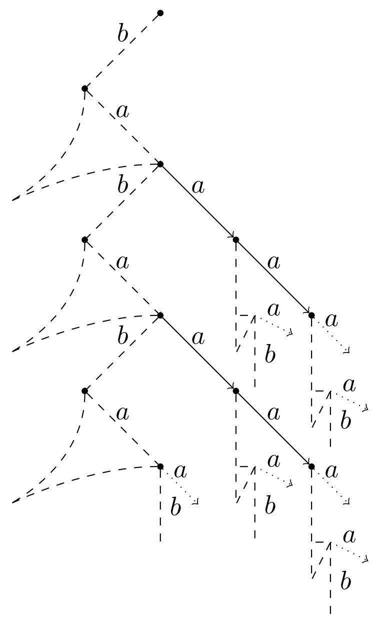


Figure 6.2: C_2

The third section contains C_3 has vertices $V_3 = \{a^i v_2 \mid i \in \mathbb{N}, v_2 \in V_2\} \cup \{\varepsilon_S\}$. Let $w \in V_3$ where w ends in a, then edges are given by

$$E_3 = \{(w, wa), \left(wb^i, wb^{i+1}\right), \left(wb^j, wba\right) \mid i \in \mathbb{N}_0, j \in \mathbb{N}\} \cup \{(\varepsilon_S, a)\}.$$

In Figure 6.3 representing C_3, dashed lines represent subgraphs isomorphic to C_2.

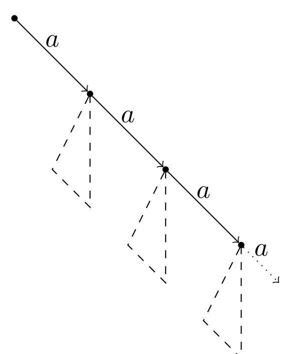


Figure 6.3: C_3

Finally, we now zoom out as far as possible to the fourth section C_4. This has vertices $V_4 = \{b^i v_3 \mid i \in \mathbb{N}_0, v_3 \in V_3\}$ and if $w \in V_3$ with w being ε_S or a word ending in a, the edge set is $E_4 = \{(w, wa), \left(wb^i, wb^{i+1}\right), \left(wb^j, wba\right) \mid w \in V_4, i \in \mathbb{N}_0, j \in \mathbb{N}\}$. Dashed lines in Figure 6.4 represent subgraphs isomorphic to C_3 in the graph C_4. The Cayley graph of S is given by C_4.

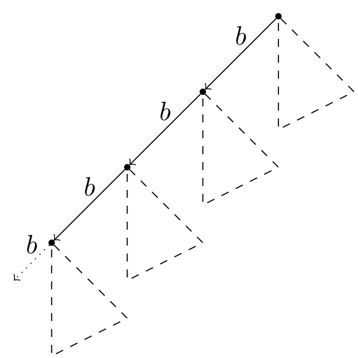


Figure 6.4: C_4

Now the vertex set of C_4 can be expanded upon so that we better understand what it is. We must expand the term v_3 for $v_3 \in V_3$, which gives us the following set

$$
\begin{aligned}
V_4 =& \{b^i a^j (\prod_{l \in \underline{k}} ba^{\alpha_l}) b^m \mid i, j, k, m \in \mathbb{N}_0, \alpha_l \in \mathbb{N}\} \\
& \cup \{b^i a^j (\prod_{l \in \underline{k}} ba^{\alpha_l}) ba \mid i, j, k \in \mathbb{N}_0, \alpha_l \in \mathbb{N}\} \\
& \cup \{b^i a^j b^k \mid i, j, k \in \mathbb{N}_0\} \\
& \cup \{b^i a^j ba \mid i, j \in \mathbb{N}_0\}
\end{aligned}
$$

If we were to describe this set, we might say something like it consists of words that can begin with any number of b (including zero), then any number of a (including zero), which may be followed by a single b, then any non-zero number of a, which we may repeat some number of times, before finally ending in either some number of a or some number of b. That is, alternating products of any number of a and single a b, where the first and last part of the word may be any number of b. This is exactly a description of the normal forms of S, and so the set of vertices V_4 is equal to the set of elements of S.

From the set of relations, we see that there exists an edge between wab^i and $waba$ labelled by a for $w \in S$. This corresponds to the edges $\left(wb^j, wba\right)$ in E_4. Since there are no other relations, the rest of the edges labelled by a are $\left(wa^i, wa^{i+1}\right)$ and those labelled by b are (w, wb) for $w \in S$, which completes the set E_4.

Having established the structure for S, we can now use a very similar method to describe T. Cay(T) is built from four sections, the first of which is D_1. This consists of the set of vertices $W_1 = \{(d^3c)^l d^r \mid 0 \leq r \leq 3, l \in \mathbb{N}_0\} \cup \{dc\}$. Let ε_{W_1} be the empty word in V_1, then the set of edges for C_1 is

$$
\begin{aligned}
F_1 = &\{\left(\left(d^3c\right)^l d^i, \left(d^3c\right)^l d^{i+1}\right) \mid i \in \{0,1,2\}, l \in \mathbb{N}_0\} \\
&\cup \{\left(\left(d^3c\right)^l d^3, \left(d^3c\right)^{l+1}\right) \mid l \in \mathbb{N}_0\} \\
&\cup \{\left(\left(d^3c\right)^l d^i, dc\right) \mid i \in \{0,1,2,3\}, l \in \mathbb{N}_0\} \setminus \{(\varepsilon_{W_1}, dc)\}.
\end{aligned}
$$

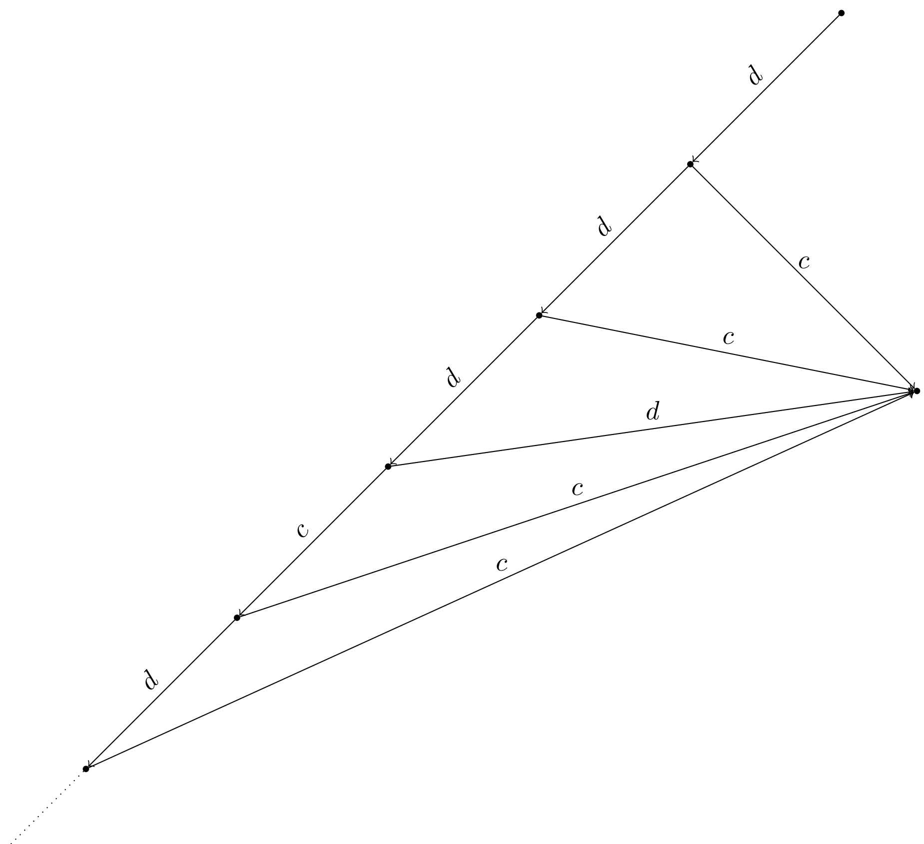


Figure 6.5: D_1

The second section D_2 has vertices $W_2 = W_1 \cup \{(\prod_{i\in\underline{k}}(dc^{\alpha_i}))w_1 \mid$ where $k \in \mathbb{N}_0, w_1 \in W_1 \alpha_1 \geq 1\}$. Let $w \in V_2$ be a word ending in c and ε_{W_2} be the empty word in W_2, then the edges of D_2 are

$$
\begin{aligned}
F_2 = &\{\left(wc^i, wc^{i+1}\right) \mid i \in \mathbb{N}_0\} \\
&\cup \{\left(w\left(d^3c\right)^l d^i, w\left(d^3c\right)^l d^{i+1}\right) \mid i \in \{0,1,2\}, l \in \mathbb{N}_0\} \\
&\cup \{\left(w\left(d^3c\right)^l d^3, w\left(d^3c\right)^{l+1}\right) \mid l \in \mathbb{N}_0\} \\
&\cup \{\left(w\left(d^3c\right)^l d^i, wdc\right) \mid i \in \{0,1,2,3\}, l \in \mathbb{N}_0\} \setminus \{(\varepsilon_{W_2}, dc)\}.
\end{aligned}
$$

Subgraphs isomorphic to D_1 are represented in the figure D_2 by dashed lines.

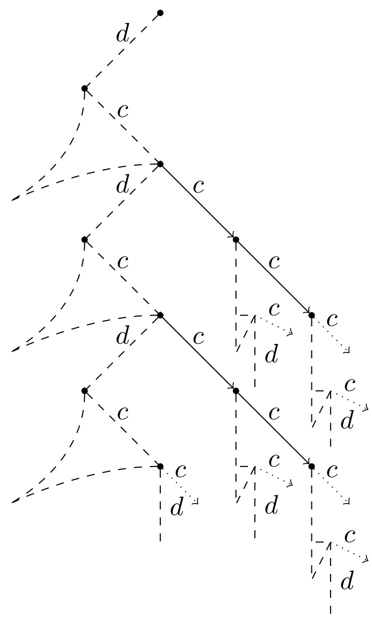


Figure 6.6: D_2

The third section D_3 has vertices $W_3 = \{c^i w_2 \mid i \in \mathbb{N}, w_2 \in W_2\} \cup \{\varepsilon_T\}$. We let $w \in W_3$ be a word ending in c, then the edges of D_3 are

$$
\begin{aligned}
F_3 = &\{(\varepsilon_T, c)\} \\
&\{\left(wc^i, wc^{i+1}\right) \mid i \in \mathbb{N}_0\} \\
&\cup \{\left(w\left(d^3c\right)^l d^i, w\left(d^3c\right)^l d^{i+1}\right) \mid i \in \{0,1,2\}, l \in \mathbb{N}_0\} \\
&\cup \{\left(w\left(d^3c\right)^l d^3, w\left(d^3c\right)^{l+1}\right) \mid l \in \mathbb{N}_0\} \\
&\cup \{\left(w\left(d^3c\right)^l d^i, wdc\right) \mid i \in \{0,1,2,3\}, l \in \mathbb{N}_0\} \setminus \{(\varepsilon_T, dc)\}.
\end{aligned}
$$

In the figure D_3 dashed lines represent subgraphs isomorphic to D_2.

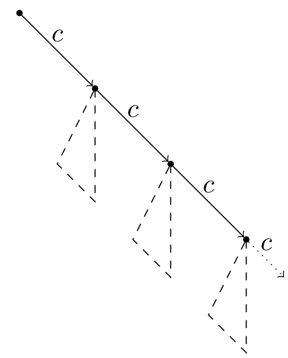


Figure 6.7: D_3

Finally, the fourth section D_4 has vertex set $W_4 = \{d^i w_3 \mid i \in \mathbb{N}_0, w_3 \in W_3\} \cup \{\varepsilon_T\}$. If we let $w \in T$ be a word ending in c, then D_4 has edge set

$$
\begin{aligned}
F_4 = & \{(\varepsilon_T, c), (\varepsilon_T, d)\} \\
& \cup \{\left(d^i, d^{i+1}\right) \mid i \in \mathbb{N}_0\} \\
& \cup \{\left(wc^i, wc^{i+1}\right) \mid i \in \mathbb{N}_0\} \\
& \cup \{\left(w\left(d^3 c\right)^l d^i, w\left(d^3 c\right)^l d^{i+1}\right) \mid i \in \{0,1,2\}, l \in \mathbb{N}_0\} \\
& \cup \{\left(w\left(d^3 c\right)^l d^3, w\left(d^3 c\right)^{l+1}\right) \mid l \in \mathbb{N}_0\} \\
& \cup \{\left(w\left(d^3 c\right)^l d^i, wdc\right) \mid i \in \{0,1,2,3\}, l \in \mathbb{N}_0\} \setminus \{(\varepsilon_T, dc)\}.
\end{aligned}
$$

Dashed lines represent subgraphs isomorphic to D_3 in the figure D_4. The Cayley graph of T is given by D_4.

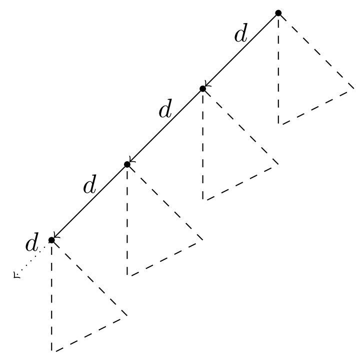


Figure 6.8: D_4

Now the vertex set of D_4 can be expanded upon so that we better understand what it is. We must expand the term w_3 for $w_3 \in W_3$, which gives us the following set

$$\begin{aligned} W_4 = & \{d^i c^j (\prod_{l \in \underline{k}} dc^{\alpha_l})(d^3c)^m d^r \mid i,j,k,m \in \mathbb{N}_0, 0 \leq r \leq 3, \alpha_l \in \mathbb{N}\} \\ & \cup \{c^i c^j (\prod_{l \in \underline{k}} dc^{\alpha_l}) dc \mid i,j,k \in \mathbb{N}_0, \alpha_l \in \mathbb{N}\} \\ & \cup \{d^i c^j (d^3c)^l d^r \mid i,j,l \in \mathbb{N}_0, 0 \leq r \leq 3\} \\ & \cup \{d^i c^j dc \mid i,j \in \mathbb{N}_0\} \end{aligned}$$

The second and fourth sets cover all normal forms of type (ii) and (iii), whilst the first and third sets cover all normal forms of type (i). Thus the set of vertices $V_4 = T$.

From the relations, we can infer the following types of edge:

- $\{\big(vcd^2, vcdc\big)\}$ for $v \in T$, labelled c.
- $\{\big(vcd^3, vcdc\big)\}$ for $v \in T$, labelled d.
- $\{\big(vcd^3c, vcdc\big)\}$ for $v \in T$, labelled c.
- $\{\big(vcd^3cd, vcdc\big)\}$ for $v \in T$, labelled c.

These are all the edges found in the subset $\{\Big(w\left(d^3c\right)^l d^i, wdc\Big) \mid i \in \{0,1,2,3\}, l \in \mathbb{N}_0\}$ of W_4. We can describe the edges labelled by the remaining generator for the first word in each of these pairs:

- $\{(vcd^2, vcd^3)\}$ for $v \in T$, labelled d.
- $\{(vcd^3, vcd^3c)\}$ for $v \in T$, labelled c.
- $\{(vcd^3c, vcd^3cd)\}$ for $v \in T$, labelled d.
- $\{(vcd^3cd, vcd^3cd^2)\}$ for $v \in T$, labelled d.

Edges of the first, second and fourth type are found in the subset $\{\left(w\left(d^3c\right)^l d^i, w\left(d^3c\right)^l d^{i+1}\right) \mid i \in \{0,1,2\}, l \in \mathbb{N}_0\}$ of W_4 and describe almost all edges in this subset, save for the edge (c, cd). Edges of the third type are all the edges found in the subset $\{\left(w\left(d^3c\right)^l d^3, w\left(d^3c\right)^{l+1}\right) \mid l \in \mathbb{N}_0\}$.

Suppose then that $u \in T$ but $u \notin \{vcd^2, vcd^3, vcd^3c, vcd^3cd\}$. Then we have edges of the type (u, uc) and (u, ud). These cover edges in the subsets $\{(\varepsilon_T, c), (\varepsilon_T, d)\}$, $\{\left(d^i, d^{i+1}\right) \mid i \in \mathbb{N}_0\}$ $\{\left(wc^i, wc^{i+1}\right) \mid i \in \mathbb{N}_0\}$ and the missing edge (c, cd). Hence the edges W_4 are indeed the edges we have in Cay(T).

Given these diagrams, we can intuitively see that these graphs are isomorphic. In the next section we provide an explicit graph isomorphism.

6.3 Graph Isomorphism

Having established normal forms for elements, it is now possible to define a map between the vertex sets S and T. We have previously looked at normal forms as products, and from that perspective we would define a map $f : S \to T$ as follows. Let $w = \prod_{i \in \underline{k}} b^{\alpha_i} a^{\beta_i}$ be a word that ends in either a single b, or any number of a. That is, let α_i and β_i be as above, but $\alpha_k = 1$ also. Then

$$(w) f = \prod_{i \in \underline{k}} d^{\alpha_i} c^{\beta_i} = v$$

Now suppose w is such that $\beta_k \neq 0$, that is, w ends in an a. We define the following:

$$\left(wb^j\right) f = v\left(d^3c\right)^l d^r \qquad \text{where } j = 4l + r$$

Finally we observe that

$$(\varepsilon_S) f = \varepsilon_T.$$

However, it will be more instructive to categorise our normal forms into different types, from which we can describe the types of edges arising. We then provide two mutually inverse maps between the types, which preserve edges and so form an isomorphism of the graphs Cay(S) and Cay(T).

For S, let $U = \{a^{i_0}ba^{i_1}b \dots ba^{i_k} \mid k \geq 0, i_m > 0\}$. Then our normal forms in S are words of the form b^j, $b^j u$ and $b^j ub^q$. It will be useful later on to view information in the form of tables, so Table 6.1 describes our classification of normal forms of S.

Label	Normal Form	Parameters
NFS1	b^j	$j \geq 0$
NFS2	$b^j u$	$j \geq 0, u \in U$
NFS3	$b^j ub^q$	$j \geq 0, u \in U, q > 0$

Table 6.1:

We will assume these parameters apply throughout. For each vertex in Cay(S), there are two edges leaving, one labelled a and one labelled b. Consider a vertex $b^j ub^q$ of type NFS3. The edge labelled a initiating at this vertex will terminate at the vertex $b^j uba$, due to the relation $ab^n a = aba$, giving us an edge $(b^j ub^q, b^j uba)$. The edge labelled b that begins at $b^j ub^q$ will terminate at $b^j ub^{q+1}$, as there are no relations to apply in this instance, giving us an edge $(b^j ub^q, b^j ub^{q+1})$. All types of edges in Cay(S) are listed in Table 6.2, categorised by their initial vertex type followed by the edge label.

Label	Edge	Vertex Types
NFS1a	$(b^j, b^j a)$	(NFS1,NFS2)
NFS1b	(b^j, b^{j+1})	(NFS1,NFS1)
NFS2a	$(b^j u, b^j ua)$	(NFS2,NFS2)
NFS2b	$(b^j u, b^j ub)$	(NFS2,NFS3)
NFS3a	$(b^j ub^q, b^j uba)$	(NFS3,NFS2)
NFS3b	$(b^j ub^q, b^j ub^{q+1})$	(NFS3, NFS3)

Table 6.2:

For T, let $V = \{c^{i_0}dc^{i_1}d \dots dc^{i_k} \mid k \geq 0, i_m > 0\}$. We then have three categories of normal forms in T, which are d^j, d^jv and $d^jv(d^3c)^ld^r$, which are tabulated in Table 6.3.

Label	Normal Form	Parameters
NFT1	d^j	$j \geq 0$
NFT2	d^jv	$j \geq 0, v \in V$
NFT3	$d^jv(d^3c)^ld^r$	$j, l \geq 0, 0 \leq r \leq 3, v \in V, l + r > 0$

Table 6.3:

As with S, we will assume these parameters apply to normal forms of T throughout. For vertices in Cay(T), each has an edge labelled c and one labelled d. The most complex case here is for a vertex $d^jv(d^3c)^ld^r$ of type NFT3. If $0 \leq r \leq 2$, then the edge labelled c will terminate at the vertex d^jvdc thanks to the relations $cd^2c = cd^3c^2 = cd^3cdc = cdc$. If $r = 3$, then simple multiplication by c gives the terminating vertex as $d^jv(d^3c)^{l+1}$. Now for the edge labelled d starting at this vertex, if $0 \leq r \leq 2$, then the edge terminates at $d^jv(d^3c)^ld^{r+1}$. However if $r = 3$, then by the relation $cd^4 = cdc$, the terminating vertex is d^jvdc. Other edges in Cay(T) are expanded upon in Table 6.4.

Label	Edge	Vertex Type
NFT1c	(d^j, d^jc)	(NFT1,NFT2)
NFT1d	(d^j, d^{j+1})	(NFT1,NFT1)
NFT2c	(d^jv, d^jvc)	(NFT2,NFT2)
NFT2d	(d^jv, d^jvd)	(NFT2, NFT3)
NFT3c	$(d^jv(d^3c)^ld^r, d^jvdc)$ for $0 \leq r \leq 2$	(NFT3, NFT2)
	$(d^jv(d^3c)^ld^3, d^jv(d^3c)^{l+1})$	(NFT3, NFT3)
NFT3d	$(d^jv(d^3c)^ld^r, d^jv(d^3c)^ld^{r+1})$ for $0 \leq r \leq 2$	(NFT3, NFT3)
	$(d^jv(d^3c)^ld^3, d^jvdc)$	(NFT3, NFT2)

Table 6.4:

We now define a bijection between normal forms of S and those of T. We first define a natural bijection between the sets U and V. Let $u = a^{i_0}ba^{i_1}b\dots ba^{i_k} \in U$ and $\bar{u} = c^{i_0}dc^{i_1}d\dots dc^{i_k} \in V$. We then define the bijection by $u \mapsto \bar{u}$. The general bijection $S \to T$ is then given in Table 6.5. It is easy to find the image of most normal forms by exchanging alphabets, however for those of type NFS3 the situation is slightly more complicated. Here we must take the power q and divide it by 4 to give an appropriate l and r for a normal form of type NFT3.

Type of w	w	$(w)f$	Type of $(w)f$
NFS1	b^j	d^j	NFT1
NFS2	$b^j u$	$d^j \bar{u}$	NFT2
NFS3	$b^j u b^q$ where $q = 4l + r$	$d^j \bar{u}(d^3c)^l d^r$	NFT3

Table 6.5:

The inverse of f is given in Table 6.6, and is found by exchanging the columns of Table 6.5.

Type of w	w	$(w)f^{-1}$	Type of $(w)f^{-1}$
NFT1	d^j	b^j	NFS1
NFT2	$d^j \bar{u}$	$b^j u$	NFS2
NFT3	$d^j \bar{u}(d^3c)^l d^r$	$b^j u b^q$ where $q = 4l + r$	NFS3

Table 6.6:

We check finally that the map f is a graph isomorphism by checking that it maps edges in Cay(S) to edges in Cay(T), and vice-versa. Consider, for example an edge of type NFS3b, which begins at $b^j u b^q$. This initial vertex is mapped to $d^j \bar{u}(d^3c)^l d^r$, however the image of the terminal vertex $b^j u b^{q+1}$ is dependent on the value of q. Given a q such that $0 \leq r \leq 2$, then the remainder of $q+1$ on division by 4 is $r+1$, and so the image of the vertex $b^j ub^{q+1}$ is simply $d^j \bar{u}(d^3c)^l d^{r+1}$. However if q is such that $r = 3$, then upon division by 4, $q + 1$ has quotient $l + 1$ and remainder 0, hence the image of the terminal vertex is $d^j \bar{u}(d^3c)^{l+1}$.

Edge Type	(w, wx)	$((w, wx))f$	Edge Type
NFS1a	$(b^j, b^j a)$	$(d^j, d^j c)$	NFT1c
NFS1b	(b^j, b^{j+1})	(d^j, d^{j+1})	NFT1d
NFS2a	$(b^j u, b^j ua)$	$(d^j \bar{u}, d^j \bar{u}c)$	NFT2c
NFS2b	$(b^j u, b^j ub)$	$(d^j \bar{u}, d^j \bar{u}d)$	NFT2d
NFS3a	$(b^j ub^q, b^j uba)$	$(d^j \bar{u}(d^3c)^l d^r, d^j \bar{u}dc)$	NFT3c
NFS3b	$(b^j ub^q, b^j ub^{q+1})$	$(d^j \bar{u}(d^3c)^l d^r, d^j \bar{u}(d^3c)^l d^{r+1})$ for $0 \leq r \leq 2$ $(d^j \bar{u}(d^3c)^l d^r, d^j \bar{u}(d^3c)^{l+1})$ for $r = 3$	NFT3d

Table 6.7:

In the opposite direction, we note that edge types NFT3c will map to edges of type NFS3a or NFS3b depending on the value of r (similarly for NFT3d), but this mapping is otherwise straightforward, and can be found in Table 6.8.

Edge Type	(w, wx)	$((w, wx))f^{-1}$	Edge Type
NFT1c	$(d^j, d^j c)$	$(b^j, b^j a)$	NFS1a
NFT1d	(d^j, d^{j+1})	(b^j, b^{j+1})	NFS1b
NFT2c	$(d^j \bar{u}, d^j \bar{u} c)$	$(b^j u, b^j ua)$	NFS2a
NFT2d	$(d^j \bar{u}, d^j \bar{u} d)$	$(b^j u, b^j ub)$	NFS2b
NFT3c	$(d^j \bar{u}(d^3c)^l d^r, d^j \bar{u} dc)$ for $0 \leq r \leq 2$	$(b^j ub^q, b^j uba)$	NFS3a
	$(d^j \bar{u}(d^3c)^l d^3, d^j \bar{u}(d^3c)^{l+1})$ for $r = 3$	$(b^j ub^q, b^j ub^{q+1})$	NFS3b
NFT3d	$(d^j \bar{u}(d^3c)^l d^r, d^j \bar{u}(d^3c)^l d^{r+1})$ for $0 \leq r \leq 2$	$(b^j ub^q, b^j ub^{q+1})$	NFS3b
	$(d^j \bar{u}(d^3c)^l d^3, d^j \bar{u} dc)$ for $r = 3$	$(b^j ub^q, b^j uba)$	NFS3a

Table 6.8:

Hence S is Cayley graph isomorphic to T via the map f.

6.4 Conclusions

We have demonstrated here two semigroups that have not only isomorphic skeleton graphs, but isomorphic Cayley graphs, where one is finitely presented, and the other infinitely presented. This provides an answer to the question asked in [8, Question 1] and shows that in general, finite presentation of semigroups is not a quasi-isometry, or even isometry invariant property.

Remark 6.14
There is in fact an infinite family of finitely presented semigroups $\{J_n\}$ *where* $\dagger(S)$ *and* $\dagger(J_n)$ *are isomorphic.*

$$J_n = \text{mon}\langle c, d \mid cdc = cd^2c = \ldots = cd^{n-2}c = cd^n = cd^{n-1}c^2 = cd^{n-1}cdc\rangle$$

where $n \geq 4$.

Each monoid J_n has normal forms:

(i) $\prod_{i \in \underline{k}} d^{\alpha_i} c^{\beta_i}$ where $\alpha_i = 1$ for $i \neq 1$, $\beta_i \neq 0$ for $i \leq k$.

(ii) Let v be an element of the form given above (where the last letter is a c). Then we have also elements of the form $v \left(d^{n-1}c\right)^l d^r$, where $r \in \{0, 1, 2, 3, \ldots, n-1\}$.

We can generalise the isomorphism f to $f_n : S \to J_n$ where

$$(w)\, f = \prod_{i \in \underline{k}} d^{\alpha_i} c^{\beta_i} = v$$

for $w = \prod_{i \in \underline{k}} b^{\alpha_i} a^{\beta_i}$ which ends in either a single b, or any number of as. Now suppose w is such that $\beta_k \neq 0$, that is, w ends in an a,then:

$$\left(wb^j\right) f = v \left(d^{n-1}c\right)^l d^r \qquad \text{where } j = nl + r$$

We can also note that throughout, we have been working with monoids. We may also regard these as semigroup presentations

$$\hat{S} = \mathrm{sgp}\langle a, b \mid ab^n a = aba \text{ for } n \in \mathbb{N}\rangle$$

and

$$\hat{T} = \mathrm{sgp}\langle c, d \mid cdc = cd^2c = cd^4 = cd^3c^2 = cd^3cdc\rangle.$$

Theorem 6.15

The semigroup $\hat{S}$ is infinitely presented and has $\dagger\left(\hat{S}\right) \cong \dagger\left(\hat{T}\right)$.

Proof: From Lemma 6.3, we have that $\hat{S}$ is not finitely presentable. We observe that $\mathrm{Cay}\left(\hat{S}\right)$ is isomorphic to the subgraph of $\mathrm{Cay}(S)$ induced by $S \setminus \{\varepsilon_S\}$. Similarly, $\mathrm{Cay}\left(\hat{T}\right)$ is the subgraph of $\mathrm{Cay}(T)$ induced by $T \setminus \{\varepsilon_T\}$. Then since the map f from above is an isomorphism, and maps $\varepsilon_S \mapsto \varepsilon_T$, then $f \upharpoonright_{\hat{S}}$ is a graph isomorphism between $\dagger\left(\hat{S}\right)$ and $\dagger\left(\hat{T}\right)$. $\square$

Similarly all the monoids J_n can be considered as semigroups.

Chapter 7

Cayley Spectra of Semigroups

Your leafy screens throw down,
And show like those you are.

Macbeth
Shakespeare

In this chapter we leave behind notions of preserving finite presentability (or not preserving as we have seen in Chapter 6), and look at whether we can deduce any information about semigroups by simply looking at a skeleton. One semigroup and skeleton that we will consider in detail is that of the infinite monogenic semigroup, or natural numbers under addition. This semigroup gives us the following proposition.

Proposition 7.1

It is not always possible to see if a semigroup has an identity by inspecting its skeleton.

Proof: Let $\mathbb{N} = \mathrm{sgp}\langle 1 \rangle$ be the natural numbers under addition, and let $\mathbb{N}_0 = \mathrm{sgp}\langle 0, 1 \rangle$ be the natural numbers under addition with an identity. Then $\dagger(\mathbb{N}, 1) \cong \dagger(\mathbb{N}_0, \{0, 1\})$, but the latter has an identity whilst the former does not. $\square$

There are however some special cases in which we can detect the presence of an identity, which we will see in 7.2.

We introduce the notion of a Cayley spectrum of a semigroup.

Definition 7.2

Let S be a semigroup with skeleton $\dagger\,(S, A)$. *Then the* Cayley spectrum of S with respect to A *is the maximal set of pairwise non-isomorphic semigroups such that if T_i is in $\mathcal{C}(S, A)$ then there exists some generating set B_i for T_i with $\dagger\,(T_i, B_i) \cong \dagger\,(S, A)$.*

The Cayley spectra of a semigroup with respect to a generating set tells us which other semigroups share this skeleton. We will investigate the Cayley spectra of four types of semigroup, and find that in some cases the skeleton is unique, and in others it is shared by many other semigroups. Where the skeleton is not unique, we will describe exactly the semigroups that have that skeleton.

We will explore the Cayley spectrum of the natural numbers, which we will prove contains only the natural numbers themselves, either with or without an identity element. We then expand to considering free monoids with generating sets of size at least two, and find that the Cayley spectrum here is as small as possible, containing only itself. It is interesting then, that when we move to consider free semigroups with generating sets of size at least 2, the number of semigroups we find in the Cayley spectrum increases significantly. We give an exact number in Theorem 7.56, and exact descriptions of these semigroups in 7.69. Finally we consider the Cayley spectrum of the integers. Here we find that there are seven semigroups in the spectrum, details of which are found in Theorem 7.74, which we note is a curious number.

The proofs in this section will involve a significant amount of cases, but throughout the cases we will rely on one common technique. We will attempt to find unique features of the skeletons, such as vertices with unique degrees, and study the possible relations that can arise from that feature. We will then take those relations and attempt to translate them to another point in the graph and see whether this induces a contradiction.

7.1 The Natural Numbers

The first semigroup we look at is the natural numbers (without 0) under addition. We will look at this with respect to the generating set $\{1\}$, and show that there are only two semigroups in the Cayley spectrum of $\mathbb{N}$ with respect to this generating set. We first prove the following theorem.

Theorem 7.3

Let $S = \mathrm{sgp}\langle a, b\rangle$ with $\dagger\,(S, \{a, b\}) \cong \dagger\,(\mathbb{N}, 1)$. Then either $S \cong \mathbb{N}$ or $S \cong \mathbb{N}_0$.

Remark 7.4

Let $S = \mathrm{sgp}\langle a, b\rangle$, where $S \ncong \mathbb{N}$ and $S \ncong \mathbb{N}_0$, be such that $\dagger\,(S) \cong \dagger\,(\mathbb{N}, 1)$. When visualising $\dagger\,(S, \{a, b\})$, we think of it as a one-ended infinite line. We will place the unique vertex u of degree one to the left, and all subsequent vertices to the right of this. When traversing this line starting at u we will let the first generator we encounter be a, and the second generator we encounter be b.

An example diagram is given in Figure 7.1, and we will refer to edges point left and right, or vertices being left and right of each other in accordance with this diagram.

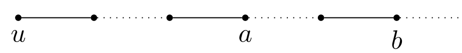


Figure 7.1: Visualisation of $\dagger\,(S, \{a, b\})$

The convention established in this remark will be followed throughout the rest of this section.

In this proof we will first consider the orders of our generators a and b, where order is defined as follows.

Definition 7.5

Let S be a semigroup and let $s \in S$. Then the order of s is the size of the semigroup generated by s, that is $\mathrm{sgp}\langle s\rangle = \{s^i \mid i \in \mathbb{N}\}$. If $|\mathrm{sgp}\langle s\rangle|$ is finite, then s has finite order. If $\mathrm{sgp}\langle s\rangle$ is infinite, then the order of s is infinite.

A relatively short argument will eliminate either having infinite order, however when both have finite order the argument is not so short. We will require a breakdown into a large amount of cases. In some of the cases we will have that the unique vertex u is equal to some word w and will pick a vertex s that is distance at least $|w| + 1$ right from b and inspect the path labelled w from s to draw conclusions from. The breakdown of cases will be as follows.

1 a and b both have infinite order.

2 Exactly one of a and b has infinite order.

3 Both a and b have finite order.

3.1 u is not a generator.

3.1.1 All edges from s labelled by w are right arrows.

3.1.2 There exists at least one right edge and one non-right edge in the path labelled by w from s.

3.1.3 All edges in paths labelled w from s onwards are left edges.

3.2 u is a generator, that is $u = a$.

3.2.1 There exists an edge going left from b.

3.2.2 There exist only loops or right edges from b.

3.2.2.1 $ba = bb = b$

3.2.2.2 $ba = bb \neq b$

3.2.2.2.1 $abb = ab = a$

3.2.2.2.2 $abb = ab \neq a$

3.2.2.2.3 $abb = a \neq ab$

3.2.2.2.4 $abb \neq a \neq ab$

3.2.2.3 $ba = b \neq bb$

3.2.2.3.1 $ab = a$

3.2.2.3.2 $ab \neq a$ label

3.2.2.3.2.1 $aa = a$

3.2.2.3.2.2 $aa \neq a$

3.2.2.4 $bb = b \neq ba$

3.2.2.4.1 $abb = ab = a$

3.2.2.4.2 $abb = ab \neq a$ label

3.2.2.4.2.1 $aa \neq ab$

3.2.2.4.2.2 $aa = a$

In subcases of 3, our unique feature that we will exploit here is the vertex u of degree one in $\dagger(\mathbb{N}, 1)$. For 3.1 we will find some word w that represents this vertex, and then attempt to follow the path labelled by this word from any vertex s sufficiently far from the generators. The assumptions we make in these cases will lead us to finding that the skeleton graph does not have the correct shape, and so we obtain contradictions.

For subcases of 3.2, we will look at the products of length 2 starting at b, and then translate these equalities to the unique vertex of degree one, that is $u = a$. From this we will either deduce that we do not obtain the correct shape of graph, or that the graph and semigroup that we do obtain is in fact either $\mathbb{N}$ or $\mathbb{N}_0$.

We will approach the proof of Theorem 7.3 by examining the cases above in a series of claims. The relation of claims to cases is given in the following table.

Case	**Claim**
1	7.6
2	7.7
3	see subcases
3.1	see subcases
3.1.1	7.10
3.1.2	7.11
3.1.3	7.12
3.2	see subcases
3.2.1	7.13, 7.14
3.2.2	see subcases
3.2.2.1	7.15
3.2.2.2	see subcases
3.2.2.2.1	7.16
3.2.2.2.2	7.17

Case	**Claim**
3.2.2.2.3	7.18
3.2.2.2.4	7.19
3.2.2.3	see subcases
3.2.2.3.1	7.20
3.2.2.3.2	see subcases
3.2.2.3.2.1	7.21
3.2.2.3.2.1	7.22
3.2.2.4	see subcases
3.2.2.4.1	7.23
3.2.2.4.2	see subcases
3.2.2.4.2.1	7.24
3.2.2.4.2.2	7.25

We now prove Theorem 7.3.

PROOF:

First suppose both a and b are of infinite order (Case 1) and consider the graph Cay$(S, \{a, b\})$.

Claim 7.6

In Case 1, that is, a and b both have infinite order, every edge in Cay$(S, \{a, b\})$ *is an edge pointing right, and $S \cong \mathbb{N}$.*

PROOF: We first consider all vertices including and to the right of b. If any edge here does not point right, we have either $b^i = b^{i+x}$ or $a^i = a^{i+x}$ for some $i \in \mathbb{N}$ and $x \in \{1, 2\}$; that is, either a or b has finite order. Thus all edges to the right of b are right arrows.

Consider next all vertices between a and b, including a. All edges labelled a must be right arrows, as otherwise we would have $a^i = a^{i+x}$ for some $i \in \mathbb{N}$ and $x \in \{1, 2\}$; that is, a has finite order. Assume then that there exists an edge that is either a loop or a left arrow labelled b. This means we have $a^i b = a^{i+x}$ for some $i \in \mathbb{N}$ and $x \in \{0, 1\}$. We know that since all edges from the vertices right of b are right arrows, we have $bb^i = ba^i$ and so

$$\begin{aligned} bb^i b &= ba^i b \\ &= ba^{i+x} \\ &= bb^{i+x}. \end{aligned}$$

This gives us that b has finite order, which is a contradiction. Hence all edges between a and b must be right arrows. Since we have only two generators, a and b, both of which point right from a, there can be no more vertices to the left of a.

Hence all edges in Cay$(S, \{a, b\})$ are right arrows. Therefore we deduce that $b = a^i$ for some $i \in \mathbb{N}$, and hence S is monogenic, meaning that $S \cong \mathbb{N}$. $\square$

Claim 7.7

In Case 2, that is exactly one of a and b has finite order, then either $S \cong \mathbb{N}$ or $S \cong \mathbb{N}_0$.

PROOF: First suppose that a has infinite order and b has finite order. Then since a is found to the left of b we have $b = a^i$ for some $i \in \mathbb{N}$, so S is monogenic and infinite and therefore isomorphic to $\mathbb{N}$. Hence we must have a with finite order and b with infinite order, that is either $a^m = a^{m+1}$ or $a^m = a^{m+2}$ for some $m \in \mathbb{N}$. Suppose first that we have the former. Then for all $t \in \mathbb{N}$ there are three possibilities for $b^t a$. First, we may have $b^t a = b^{t-1}$ for some t, but then

$$\begin{aligned} b^{t+m+1} &= b^{m+2} b^{t-1} \\ &= b^{m+m} b^t a^{m+1} \\ &= b^{m+m} b^t a^m \\ &= b^{t+m} \end{aligned}$$

which is a contradiction. Second, we may have $b^t a = b^{t+1}$ for some t, which means

$$\begin{aligned} b^{t+m+1} &= b^m b^{t+1} \\ &= b^t a^{m+1} \\ &= b^t a^m \\ &= b^{t+m} \end{aligned}$$

again a contradiction to the infinite order of b. Finally, if $b^t a = b^t$ for all $t \in \mathbb{N}$ then all arrows labelled b are directed right, and so $u = a$ and a is an identity. Hence $S \cong \mathbb{N}_0$.

We now consider the case where $a^m = a^{m+2}$, and again consider the possibilities for $b^t a$. If we have $b^t a = b^{t-1}$ for some $t \in \mathbb{N}$ we derive a contradiction as follows:

$$\begin{aligned} b^{t+m-2} &= b^{m-1} b^{t-1} \\ &= b^{m+m} b^t a^{m+2} \\ &= b^{m+m} b^t a^m \\ &= b^{t+m}. \end{aligned}$$

If $b^t a = b^{t+1}$ for some $t \in \mathbb{N}$ we have

$$\begin{aligned} b^{t+m+2} &= b^m b^{t+2} \\ &= b^t a^{m+2} \\ &= b^t a^m \\ &= b^{t+m}. \end{aligned}$$

Finally if $b^t a = b^t$ for all t we have $S \cong \mathbb{N}_0$ as above.

□

We therefore are left to consider the case where both a and b have finite order (Case 3). We first make two observations that will be used throughout the proofs of subcases of 3.1.

Lemma 7.8

In Case 3.1, where u does not represent a generator, we have that u is equal to some word $w \in \{a,b\}^*$ where $w = w_1 w_2 \ldots w_k$ for some k and $w_i \in \{a,b\}$ and for any generator $x \in \{a,b\}$ we have either $wx = w$ or $wx = w_1 \ldots w_{n-1}$.

PROOF: Since u is not a generator we must be able to write it as a product of at least two generators, that is, as a word over $\{a,b\}^*$. Then since u has degree one and edges from it may only end at u or the vertex immediately right of u we have either $wx = w$ or $wx = w_1 \ldots w_{n-1}$ as demonstrated in this figure.

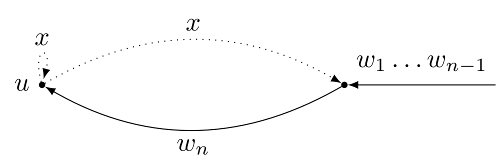


Figure 7.2: Observation 1

□

Claim 7.9

Let $w_1 w_2 \ldots w_k$ be a word in S for some k and $w_i \in \{a,b\}$. Then we have that for any $1 \leq i \leq k-1$ and generator $x \in \{a,b\}$ we have either $w_1 \ldots w_i x = w_1 \ldots w_{i-1}$, $w_1 \ldots w_i x = w_1 \ldots w_i$ or $w_1 \ldots w_i x = w_1 \ldots w_i w_{i+1}$.

PROOF:

This follows since edges may only end at the either their starting vertex, or the ones immediately left or right of the starting vertex. It may be visualised as follows.

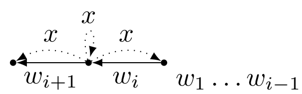


Figure 7.3: Observation 2

□

We will now choose a vertex s that is a distance of at least $|w| + 1$ to the right of generator b. This is to ensure that when we examine what happens when we follow the path labelled by w from this vertex, we know that we will never end up either at or to the left of any generators in the graph. We look at Cases 3.1.1,3.1.2 and 3.1.3.

Claim 7.10

In Case 3.1.1, that is a and b have finite order, u is not a generator, and all edges from s labelled by w are right arrows we have that $\dagger(S, \{a, b\})$ is finite.

PROOF:

Suppose that every edge in the path from s labelled by w points right. Then we have the following section of graph, where, using Observation 7.2 x represents the possible edges for generators leaving sw.

Now the vertex sw is found to the right of any generators in the graph, and so there must be an edge going right from this vertex. However, Observation 7.2 means that all edges leaving sw are either loops or point towards the left, which is a contradiction to $\dagger(S, \{a, b\})$ being an infinite graph. □

Note that this eliminates any case in which when following the path w from s we find w_n as a rightwards edge. Thus for cases 3.1.2 and 3.1.3 we can assume that w_n is a leftwards edge.

Claim 7.11

In Case 3.1.2, that is a and b have finite order, u is not a generator, and there exists at least one non-right edge in the path labelled by w from s we have that $\dagger\,(S, \{a, b\})$ *is finite.*

Proof: Suppose that when following the path labelled w from s, at some point the path folds back on itself, so $sw_1 \ldots w_{i-1} = sw_1 \ldots w_{i+1}$ for some $1 \leq i \leq k$. Since we have established that the edge w_n must be leftwards, then we may assume that the edge on this path labelled w_i is a rightwards edge. Let x represent the generators of S, then by Observation 7.3 we see the following in the Cayley graph of S.

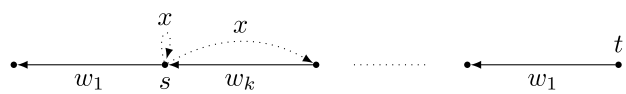
We now note that similarly to the first case, since $sw_1 \ldots w_i$ is found at a vertex to the right of any generator, in order to have the correct graph structure for $\dagger\,(S, \{a, b\})$ there must be an edge from $sw_1 \ldots w_i$ that goes right. Observation 7.3 tells us that for any generator x of S the edge labelled by x at $sw_1 \ldots w_i$ is either a loop or a leftwards edge, which means $\dagger\,(S, \{a, b\})$ is finite. □

We now come to Case 3.1.3. Claims 7.10 and 7.11 eliminate the possibility that, for any vertex t that is either s or found to the right of s, we find $tw_1 \ldots w_i$ for any $1 \leq i \leq k$ to the right of t. The second case eliminates any situation in which the path labelled w folds back on itself. Hence for any such vertex t, if we follow the path labelled by w, then each edge in it points left.

Claim 7.12

In Case 3.1.3, that is a and b have finite order, u is not a generator, and all edges in paths labelled w from s onwards are left edges we have that $\mathrm{Cay}(S, \{a, b\}) \ncong \mathrm{Cay}(\mathbb{N}, \{1\})$.

Proof: Let t be a vertex that is a distance exactly $|w|$ right of the vertex s. Since we are in Case 3.1.3, we have that $tw = s$. Let x represent any generator of S, then by Observation 7.2, our graph has the following structure.



Due to Observation 7.2, we have that for any generator x either $twx = s$ or $twx = tw_1 \dots w_{n-1}$. In particular, there is no generator left to label the edge labelled w_1 from s in our diagram above and so $\text{Cay}(S, \{a, b\}) \ncong \text{Cay}(\mathbb{N}, \{1\})$. $\square$

Therefore the unique vertex u of degree one in $\dagger(S, \{a, b\})$ is not a generator. We therefore now study Case 3.2 where u represents a generator.

Claim 7.13

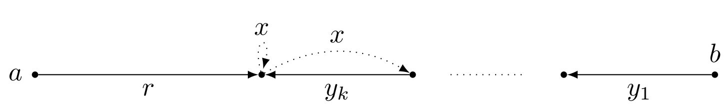
In Case 3.2.1, that is a and b have finite order and u is a generator, then $u = a$ and we have $a \neq bv$ for any word $v \in \{a, b\}^*$.

Proof: Suppose that $a = bv$ for some $v \in \{a, b\}^*$. Then we may apply an analogous argument to Case 3.1 when u is not a generator. Hence $a \neq bv$ for any $v \in \{a, b\}^*$. $\square$

Claim 7.14

In Case 3.2.1, that is a and b have finite order, u is the generator a and there exists an edge going left from b we have that $\dagger(S, \{a, b\}) \ncong \dagger(\mathbb{N}, \{1\})$.

Proof: Suppose first that there is an edge that goes left from b and let $y = y_1 \dots y_k$ for some k be the longest word that does not traverse any loops or cycles such that $by = ar$ for some $r \in \{a, b\}^*$. This means that y labels the longest leftwards path that does not include loops or cycles from b. We can now make an identical argument to when the vertex u did not represent a generator using the word by in place of w. This works because we have either $byx = by$ or $byx = by_1 \dots y_n$ for any generator x of S. This is visualised below.



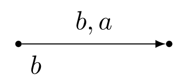
We can see that the vertex by behaves in much the same way as u with respect to its out edges, and so the arguments from Case 3.1 and its subcases are easily applied with by instead of w.

$\square$

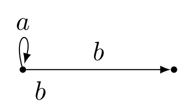
Last but not least, we are left to consider when there is no path from b to a, and in fact no edge left from b. This is Case 3.2.2. Below we demonstrate sections of graph occurring in Cases 3.2.2.1,3.2.2.2,3.2.2.3 and 3.2.2.4.

(i) $ba = bb = b$ (see 3.2.2.1),

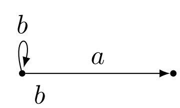
(ii) $ba = bb \neq b$ (see 3.2.2.2),



(iii) $ba = b \neq bb$ (see 3.2.2.3),



(iv) or $bb = b \neq ba$(see 3.2.2.4).



Claim 7.15

In Case 3.2.2.1, that is a and b have finite order, u is the generator a, there are only loops and edges right from b and $ba = bb = b$ we have that $\mathrm{Cay}(S, \{a, b\})$ *is finite.*

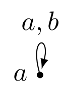
PROOF: There are no generators available to label an edge going right from b and so $\mathrm{Cay}(S, \{a, b\})$ is finite. $\square$

We now examine subcases of 3.2.2.2, by examining what happens at a, in particular by looking at $a \cdot bb$.

Claim 7.16

In Case 3.2.2.2.1, that is a and b have finite order, u is the generator a, there are only loops and edges right from b, $ba = bb \neq b$ and $abb = ab = a$ we have that $\dagger(S, \{a, b\}) \ncong \dagger(\mathbb{N}, \{1\})$.

Proof: Since we have $abb = a$ and $ba = bb$ then $aba = a$ as shown below.

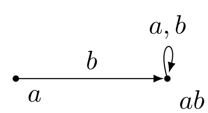


There are no more generators left to label an edge right from a and so $\dagger(S, \{a, b\}) \ncong \dagger(\mathbb{N}, \{1\})$. □

Claim 7.17

In Case 3.2.2.2.2, that is a and b have finite order, u is the generator a, there are only loops and edges right from b, $ba = bb \neq b$ and $abb = ab \neq a$ we have that $\dagger(S, \{a, b\}) \ncong \dagger(\mathbb{N}, \{1\})$.

Proof: Since $abb = ab$ and $bb = ba$ then $aba = ab$ as shown.

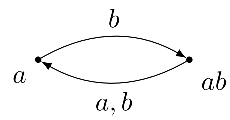


There are no more generators left to label an edge right from ab and so $\dagger(S, \{a, b\}) \ncong \dagger(\mathbb{N}, \{1\})$ □

Claim 7.18

In Case 3.2.2.2.3, that is a and b have finite order, u is the generator a, there are only loops and edges right from b, $ba = bb \neq b$ and $abb = a \neq ab$ we have that $\dagger(S, \{a, b\}) \ncong \dagger(\mathbb{N}, \{1\})$.

Proof: Since $abb = a$ and $bb = ba$ then $aba = a$ as shown below.

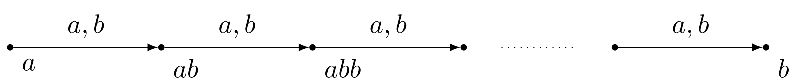


There are no more generators left to label an edge right from ab and so $\dagger(S, \{a, b\}) \ncong \dagger(\mathbb{N}, \{1\})$ $\square$

Claim 7.19

In Case 3.2.2.2.4, that is a and b have finite order, u is the generator a, there are only loops and edges right from b, $ba = bb \neq b$ and $abb \neq a \neq ab$ we have that $S \cong \mathbb{N}$.

Proof: If we consider the edge b leaving the vertex abb, this must be a right edge, as otherwise we have both $abbb$ and $abba$ equal to either ab or abb, which results in no more generators left to label an edge right from abb. This is true for all vertices ab^i between a and b, and since $bb = ba$ we have $ab^i = aa^i$ for all such vertices. This is visualised in the following diagram.



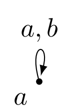
This gives us $b = a^i$ for some i. This means $a \cdot a^i = a \cdot b$ and so $b = a$. Hence we have $S \cong \mathbb{N}$. $\square$

We next consider subcases of 3.2.2.3. Here we will be concerned with looking at $a \cdot ba$.

Claim 7.20

In Case 3.2.2.3.1, that is a and b have finite order, u is the generator a, there are only loops and edges right from b, $ba = b \neq bb$ and $ab = a$ we have that $\dagger(S, \{a, b\}) \ncong \dagger(\mathbb{N}, \{1\})$.

Proof: Since $ab = a$ and $ba = b$ then $aa = a$ shown.

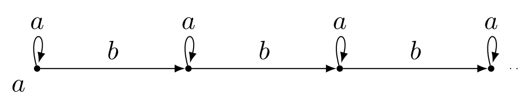


Then there are no generators left to label an edge right from a and so $\dagger(S, \{a, b\}) \ncong \dagger(\mathbb{N}, \{1\})$. $\square$

Claim 7.21

In Case 3.2.2.3.2.1, that is a and b have finite order, u is the generator a, there are only loops and edges right from b, $ba = b \neq bb$, $ab \neq a$ and $aa = a$ we have that $S \cong \mathbb{N}_0$.

Proof: Since $ba = b$, then $aba = ab$ then the edge going right from ab must be labelled b. Using $ba = b$, we can see that in the rest of the graph we must have b as a right edge and a as a loop on every vertex.

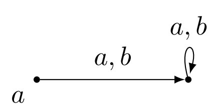


From this we can say that $b = ab^i$ for some $i \geq 1$. This means that $ab = aab^i$, which in fact means $ab = ab^i$ and so $i = 1$. Hence $b = ab$. The element a is an identity for S, since $ab = b$, $aa = a$ and $ba = b$. Removing the identity from S leaves us with an infinite monogenic semigroup, and so here we have that $S \cong \mathbb{N}_0$. $\square$

Claim 7.22

In Case 3.2.2.3.2.2, that is a and b have finite order, u is the generator a, there are only loops and edges right from b, $ba = b \neq bb$, $ab \neq a$ and $aa \neq a$ we have that $\dagger(S, \{a, b\}) \ncong \dagger(\mathbb{N}, \{1\})$.

Proof: Since $ab \neq a$ and $aa \neq a$ then $ab = aa$, and since $ba = b$ then $aaa = aba = ab$ as shown here.



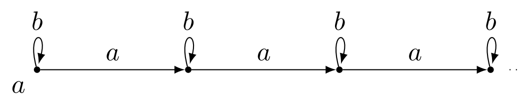
Then there are no generators left to label an edge right from ab and so $\dagger(S, \{a, b\}) \ncong \dagger(\mathbb{N}, \{1\})$. $\square$

Finally we look into subcases of 3.2.2.4.

Claim 7.23

In Case 3.2.2.4.1, that is a and b have finite order, u is the generator a, there are only loops and edges right from b, $bb = b \neq ba$, and $ab = a$ we have that $\dagger(S, \{a, b\}) \ncong \dagger(\mathbb{N}, \{1\})$.

Proof: We know that there must be an edge leaving a that goes right, and since $ab = a$ this edge must be labelled by a. Since $ab = a$, then $aab = aa$, and so b forms a loop on aa. This can be extended for any a^i between a and b, so $a^i a = a^{i+1}$ and $a^i b = a^i$ (see 3.2.2.4.1).

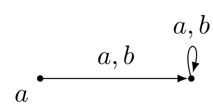


This shows us that $b = a^i$ for some $i \geq 1$, but since $a \cdot b = a \cdot a^i = a$, then $i = 0$, and so $\dagger(S, \{a, b\}) \ncong \dagger(\mathbb{N}, \{1\})$. □

Claim 7.24

In Case 3.2.2.4.2.1, that is a and b have finite order, u is the generator a, there are only loops and edges right from b, $bb = b \neq ba$, $ab \neq a$ and $aa \neq a$ we have that $\dagger(S, \{a, b\}) \ncong \dagger(\mathbb{N}, \{1\})$.

Proof: Since $ab \neq a$ and $aa \neq a$ then $ab = aa$. Since $bb = b$ then $aaa = aab = abb = ab$ as shown below.

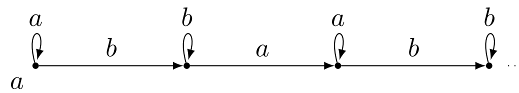


We can see here that we have no generator left to label an edge which goes right from ab, and so $\dagger(S, \{a, b\}) \ncong \dagger(\mathbb{N}, \{1\})$. □

Claim 7.25

In Case 3.2.2.4.2.2, that is a and b have finite order, u is the generator a, there are only loops and edges right from b, $bb = b \neq ba$, $ab \neq a$ and $aa = a$ we have that S is not a semigroup.

PROOF: Since $abb = ab$, the edge leaving ab in the right direction must be labelled a, and due to $aa = a$ we have $abaa = aba$. We may continue this line of reasoning to show that between a and b, the non loop edges always alternate between a and b, and a non loop edge is always followed by a loop. This is visualised here.



In this case, we see that b is equal to some word beginning with a, say $b = aw$. Then $ab \cdot b = ab$ but $ab \cdot aw \neq ab$, which is a contradiction to S being a semigroup. □

This completes the proof that there is no 2-generated semigroup S with $\dagger(S, \{a, b\}) \cong \dagger(\mathbb{N}, 1)$ such that $S \ncong \mathbb{N}$ and $S \ncong \mathbb{N}_0$.

□

Having shown that there are no 2-generated semigroups that are skeleton isomorphic to $\mathbb{N} = \mathrm{sgp}\langle 1 \rangle$ other than by adjoining an identity to $\mathbb{N}$, it would be nice if we could show that there are no n-generated semigroups with this property for any $n \in \mathbb{N}$. Fortunately, we can extended the work we have done for the 2-generated case to fit any n. We will now suppose that $S = \mathrm{sgp}\langle a_1, a_2, \ldots, a_n \rangle$ with $S \ncong \mathbb{N}$ and $S \ncong \mathbb{N}_0$, where $\dagger(S, \{a_1, a_2, \ldots, a_n\}) \cong \dagger(\mathbb{N}, 1)$. As before, we view the graph as an infinite line from left to right, with the unique vertex of degree 1 placed at the left, and the generator a_i being found to the left of generator a_{i+1} for all $1 \leq i < n$. We prove the following theorem.

Theorem 7.26

Let S be a finitely generated semigroup with generating set A such that $\dagger(S, A) \cong \dagger(\mathbb{N}, 1)$. Then either $S \cong \mathbb{N}$ or $S \cong \mathbb{N}_0$.

Now using the 2-generated case as a template, we have the following breakdown of cases.

1 u is not a generator.

2 u is a generator, say a.

2.1 There exists a path left from a_n.

2.2 There exists no path left from a_n.

2.2.1 $a_na_1 = a_n$ and $a_na_n \neq a_n$

2.2.1.1 $a_1a_1 = a_1a_n \neq a_1$

2.2.1.2 $a_1a_1 = a_1a_n = a_1$

2.2.1.3 $a_1a_1 \neq a_1a_n = a_1$

2.2.1.4 $a_1a_1 = a_1 \neq a_1a_n$

2.2.2 $a_na_1 \neq a_n$ and $a_na_n = a_n$

2.2.3 $a_na_1 = a_n$ and $a_na_n = a_n$ with $a_na_j \neq a_n$ for some j

2.2.4 $a_na_1 \neq a_n$ and $a_na_n \neq a_n$

Our proof techniques will also follow similarly to those in Theorem 7.3. For case 1 we will find some word w such that $u = w$. We will then pick a special vertex s such that s is sufficiently far right from any generator, and follow the path labelled by w from s. This will lead us to contradictions in the form of incorrect graph shapes, allowing us to rule out this case. This technique will also be use in case 2.1.

For subcases of 2.2, we will consider the location of the elements a_na_1 and a_na_n. We will then take these equalities to the unique vertex of degree one, that is $u = a_1$ and see what shape of graph these equalities force upon us. This will either lead to contradictions, or to the semigroup produced being $\mathbb{N}$ or $\mathbb{N}_0$.

We will approach the proof of Theorem 7.26 by examining the cases above in a series of claims. The relation of claims to cases is given in the following table.

Case	**Claim**
1	7.27
2	7.28
2.1	7.29
2.2	see subcases
2.2.1	see subcases
2.2.1.1	7.31
2.2.1.2	7.32
2.2.1.3	7.33
2.2.1.4	7.34
2.2.2	7.35
2.2.3	7.36, 7.36
2.2.4	7.38

We now begin the proof of Theorem 7.26.

PROOF:

We will first investigate what happens when we assume $u \neq a_i$ for all i.

Claim 7.27

In Case 1, that is u does not represent a generator we have that $\dagger\,(S, A) \ncong \dagger\,(\mathbb{N}, \{1\})$.

PROOF: We let $u = w$ where w is a word over $\{a_1, a_2, \ldots, a_n\}^*$, x represent any generator a_i and pick a vertex s to be distance $|w| + 1$ right of a_n. The proof then follows analogously to Claims 7.8, 7.10, 7.11 and 7.12. $\square$

Therefore we will now consider the case u represents a generator, that is $u = a_1$. This is Case 2.

Claim 7.28

In Case 2, that is $u = a_1$ is a generator we have that $a_1 \neq a_i v$ for any generator a_i and word v over the generators.

PROOF: This follow analogously to Claim 7.13 $\square$

Claim 7.29

In Case 2.1, that is $u = a_1$ and there exists an edge going left from any a_i we have that $\dagger(S, A) \ncong \dagger(\mathbb{N}, \{1\})$.

Proof: This follows analogously to Claim 7.14. □

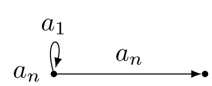
Claim 7.30

In Case 2.2, that is $u = a_1$ and there are only loops or right edges from all a_i we have that there exists a path from a_1 to a_n.

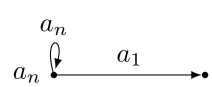
Proof: Since there are no leftwards paths from any generators and the graph is connected, there must exist a path from a_1 to a_n. □

We will now consider all subcases of 2.2. We will be concerned for the most part with the behaviour of a_1 and a_n at their respective vertices. Considering the vertex a_n, we have four possible cases for the location of edges a_1 and a_n, these are 2.2.1, 2.2.2, 2.2.3 and 2.2.4. We illustrate these below.

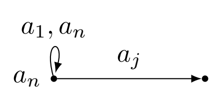
(i) $a_n a_1 = a_n$ and $a_n a_n \neq a_n$ (Case 2.2.1).



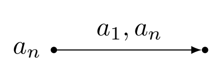
(ii) $a_n a_1 \neq a_n$ and $a_n a_n = a_n$ (Case 2.2.2).



(iii) $a_n a_1 = a_n$ and $a_n a_n = a_n$ with $a_n a_j \neq a_n$ for some j (Case 2.2.3).



(iv) $a_n a_1 \neq a_n$ and $a_n a_n \neq a_n$ (Case 2.2.4).

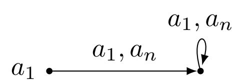


We will look first at Case 2.2.1 and its subcases.

Claim 7.31

The case 2.2.1.1, that is $u = a_1$, there are only loops or right edges from all a_i, $a_na_1 = a_n$, $a_na_n \neq a_n$ and $a_1a_1 = a_1a_n \neq a_1$ is not possible as the relations are not compatible.

Proof: Since $a_1a_1 = a_1a_n$ and $a_na_1 = a_n$ we have $a_1a_1a_1 = a_1a_na_n = a_1a_1$ as shown.

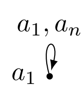


This implies that $a_n \cdot a_1a_1 = a_n \cdot a_1a_1a_n$, which is a contradiction as these are not equal in this case. □

Claim 7.32

In Case 2.2.1.2, that is $u = a_1$, there are only loops or right edges from all a_i, $a_na_1 = a_n$, $a_na_n \neq a_n$ and $a_1a_1 = a_1a_n = a_1$ is not possible as the relations are not compatible.

Proof:

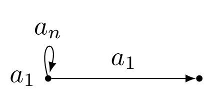


Here we have that $a_n \cdot a_1 = a_n \cdot a_1a_n$, which is not compatible. □

Claim 7.33

In Case 2.2.1.3, that is $u = a_1$, there are only loops or right edges from all a_i, $a_na_1 = a_n$, $a_na_n \neq a_n$ and $a_1a_1 \neq a_1a_n = a_1$ is not possible as the relations are not compatible.

Proof:

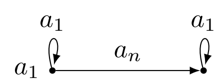


This situation gives us $a_n \cdot a_1 a_n = a_n \cdot a_1$ which is not compatible. $\square$

Claim 7.34

In Case 2.2.1.4, that is $u = a_1$, there are only loops or right edges from all a_i, $a_n a_1 = a_n$, $a_n a_n \neq a_n$ and $a_1 a_1 = a_1 \neq a_1 a_n$ we have that S is 2-generated.

Proof:



By Claim 7.30 we have $a_n = a_1 a_n w$ for some word w, where w labels a shortest path. Now $a_1 \cdot a_n = a_1 \cdot a_1 a_n w$, that is $a_1 a_n = a_1 a_n w$. The word w must then be empty, and so $a_n = a_1 a_n$. Hence we are in the 2-generated case and we may refer to the proof there. $\square$

In the next case, Case 2.2.2 we let a_n label a loop on a_n and a_1 label a right edge.

Claim 7.35

In Case 2.2.2, that is $u = a_1$, there are only loops or right edges from all a_i, $a_n a_1 \neq a_n$ and $a_n a_n = a_n$ we have that $\dagger(S, A) \ncong \dagger(\mathbb{N}, \{1\})$.

Proof: By Claim 7.30 we know that there exists a path from a_1 to a_n. Let w be a shortest word labelling this path, so $a_1 w = a_n$. Now we have $a_n \cdot a_n = a_n \cdot a_1 w$, that is $a_n = a_n a_1 w$. Since $a_n a_1 \neq a_n$, the word w must have non-zero length, and when following the path labelled w from $a_n a_1$, this path must fold back on itself. We may then follow an argument analogous to that in Case 7.1 to show that there are no generators left to label a required right edge. Thus $\dagger(S, A) \ncong \dagger(\mathbb{N}, \{1\})$. $\square$

Claim 7.36

In Case 2.2.3, that is $u = a_1$, there are only loops or right edges from all a_i, $a_n a_1 = a_n$ and $a_n a_n = a_n$ we have that $a_n a_j \neq a_n$ for some $1 < j < n$.

Proof: Suppose that $a_n a_j = a_n$ for all j. Then there are no generators left to label an edge right from a_n and so $\dagger(S, A) \ncong \dagger(\mathbb{N}, \{1\})$. Hence $a_n a_j \neq a_n$ for some $1 < j < n$. $\square$

Claim 7.37

In Case 2.2.3, that is $u = a_1$, there are only loops or right edges from all a_i, $a_n a_1 = a_n$ and $a_n a_n = a_n$ we have that $\dagger(S, A) \ncong \dagger(\mathbb{N}, \{1\})$.

Proof: By considering the behaviour of a_1 and a_j at a_1, we may apply the exact same arguments as in Claims 7.31, 7.32, 7.33, and 7.34 substituting a_j for a_n. □

Claim 7.38

In Case 2.2.4, that is $u = a_1$, there are only loops or right edges from all a_i, $a_n a_1 = a_n$ and $a_n a_n = a_n$ we have that $S \cong \mathbb{N}$.

Proof: From Claim 7.30 we know that $a_n = a_1 w$ for some word w that labels a shortest path. Therefore, $a_n a_n = a_n a_1 w$, and by applying an argument analogous to Case 7.1, we see that w must be empty. Hence we have $a_n = a_1$, and so S is an infinite monogenic semigroup. □

This concludes our look at possible configuration for an n-generated semigroup S where $\dagger(S, A) \cong \dagger(\mathbb{N}, 1)$. Thus we have proved Theorem 7.26.

□

As a corollary we now know the Cayley spectra of $\mathbb{N}$.

Corollary 7.39

The Cayley spectrum of $\mathbb{N}$ is $\mathcal{C}(\mathbb{N}, 1) = \{\mathbb{N}, \mathbb{N}_0\}$.

7.2 Free Monoids

Since the free monogenic semigroup has been a source of so much joy, it seems a sensible idea to cast an eye over the free semigroups on n generators for $n > 1$. It turns out that it makes more sense to first consider the free monoid, then return to the free semigroup. We will take the natural generating set A of size n for each of these, and observe that the skeleton of the free monoid A^* is an n-ary rooted tree. We suppose that we have some finitely generated semigroup S with generating set B such that $\dagger(S, B) \cong \dagger(A^*, A)$. We will let u be the unique vertex of valency n in $\text{Cay}(S)$. For ease of understanding we will refer to the root vertex u as being found at the top of the graph, and child vertices are found below. In this way we may also refer to edges having direction up or down, if they are oriented towards or away from u respectively.

We will prove the following theorem.

Theorem 7.40

Let S be a semigroup generated by B such that $\dagger(S, B) \cong \dagger(A^*, A)$ where $|A| > 1$. Then $S \cong A^*$.

Our key technique in this section will be to determine the exact structure at the root vertex u, and then translate this structure to a vertex s that is sufficiently far below any generators. Using Lemma 7.42 we will then conclude that we do not have enough edges to create the correct tree structure, and so eliminate all semigroups save for the free monoid. The outline of cases will be as follows.

1 u is not an identity or generator and $u = wb_i$ for some word w.

  1.1 $wb_ib_j = w$ for some b_j.

  1.2 $wb_ib_j \neq w$ for any b_j.

2 u is an identity.

3 u is a generator, say b_1.

  3.1 $|B| < n$.

  3.2 $|B| = n$.

  3.3 $|B| > n$ and B includes some element b_2.

    3.3.1 $b_1b_1 \neq b_1b_2 \neq b_1$

    3.3.2 $b_1b_1 = b_1b_2 \neq b_1$

    3.3.3 $b_1b_1 = b_1b_2 = b_1$

    3.3.4 $b_1b_1 \neq b_1b_2 = b_1$

    3.3.5 $b_1b_1 = b_1 \neq b_1b_2$

In case 1 we will use the word wb_i as the structure we translate to s, and on following this path we will find we have insufficient edges to create the correct tree. In cases 2 and 3 we will make an important observation that there can be no edges in the upwards direction (see Claim 7.46). From this we will use the fact that any generator can be written as the product of b_1 and some word w. In particular we will use the equality $b_2 = b_1 w$. In cases 2 and 3.2 we see that S is simply the free semigroup on n generators. In case 3.3 we will again make use of $b_2 = b_1 w$ and apply this to the possible scenarios for edges labelled b_1 and b_2 from the root of the tree. In each of these we will see that we either do not have enough edges to form our tree, or the semigroup that we have formed is in fact A^*.

We will approach the proof of Theorem 7.26 by examining the cases above in a series of claims. The relation of claims to cases is given in the following table.

Case	**Claim**
1	see subcases
1.1	7.43
1.2	7.44
2	7.45
3	see subcases
3.1	7.41
3.2	7.47
3.3	see subcases
3.3.1	7.49
3.3.2	7.49
3.3.3	7.50
3.3.4	7.51
3.3.5	7.52

We now begin the proof of Theorem 7.40.

PROOF:

First we observe that S must be at least n-generated, otherwise there exists an unreachable subtree in Cay(S).

Claim 7.41

S is at least n-generated.

Proof: Suppose that S has a generating set $B = \{b_1, b_2, \ldots, b_{n-m}\}$ of size $n - m$ and let s be a vertex where no generators are found below it in Cay(S). Then s has valency of $n + 1$, and an outdegree of at most $n - m$. In-edges for s can only come from the vertices sb_i, and the vertex found above s in the tree. This is because for the remaining m neighbouring vertices, there is no way of reaching them as we have assigned all generators to edges already. Hence we have disconnected our graph in an unacceptable way and so B must have size at least n. □

Since u is unique in the graph, we will use it as a focus point to establish the possible forms of the semigroup S. The following lemma will be integral to the proofs made in this section.

Lemma 7.42

Let s be a vertex in Cay(S, A) *such that in the subtree rooted at s, there are no generators found below s. Then for every vertex v_i on the level below s that is connected to s, there must be an edge that starts at s and ends at v_i.*

Proof: Suppose that v_i is a vertex one level below s that is connected to s such that there is no edge starting at s and terminating at v_i. Since Cay(S, A) is a tree, the only other vertices that we can reach v_i from are those in the subtree rooted at v_i. However, since these are in the subtree rooted at s, we know that none of these vertices are generators and so v_i cannot be reached from a generator. This is a contradiction, and so we must have at least one edge from s to every vertex on the level below that is connected to s.

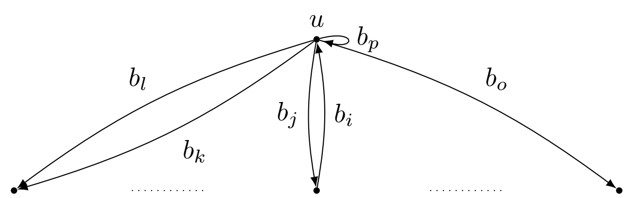
□

Suppose then that S has a generating set of size $n + m$ for $m \in \mathbb{N}_0$, say $B = \{b_1, \ldots, b_{n+m}\}$ and that $u \neq 1$ and $u \neq b_i$ for any $1 \leq i \leq n + m$. Then $u = wb_i$ for some $w \in B^*$ and some b_i. We examine two cases, the first where there exists an edge back to w from u, and the second where no such edge exists.

Claim 7.43

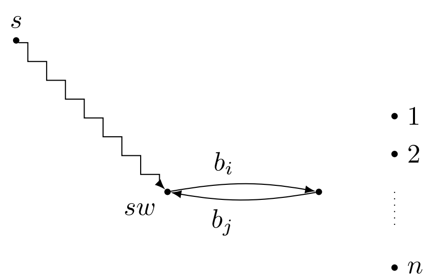
In Case 1.1, that is u is not a generator or identity and $wb_ib_j = w$ for some $b_j \in B$ we have that $\dagger\,(S, B) \ncong \dagger\,(A^*, A)$.

Proof: We note that it may be the case that $wb_ib_k = wb_ib_l$, for some pairs $b_k, b_l \in B$, or $wb_k = w$ for $b_k \in B$, or indeed both may occur. An example is shown here.



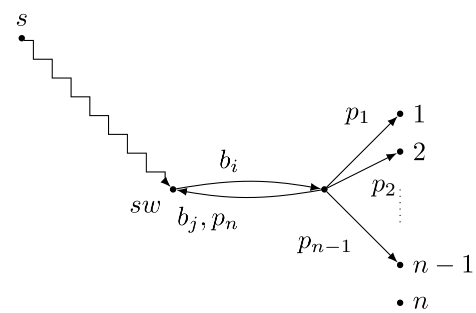
This means there are at most n different vertices that can be reached using a single edge from n.

Assuming that the tree is oriented in the usual fashion (with u at the top), we chose a vertex s that is a distance at least $|wb_i| + 1$ below a generator. We then examine the vertex swb_i, which has precisely $n + 1$ neighbours as it cannot be u. These are the vertices 1 to n and sw as shown below.



By Lemma 7.42 we know that there must be an edge from swb_i to every vertex 1 to n in the above diagram.

Let $P = \{p_k\}$ be a maximal set of generators such that no two generators in P visit the same vertex from u. That is,$up_k \neq up_l$ for any p_k , $p_l \in P$. The set P has size at most n, which means we can reach at most n different vertices from swb_i using the edges from P. We know that there exists an edge from swb_i to sw, say p_n, leaving at most $n - 1$ sets of edges with which we need to reach vertices 1 to n.



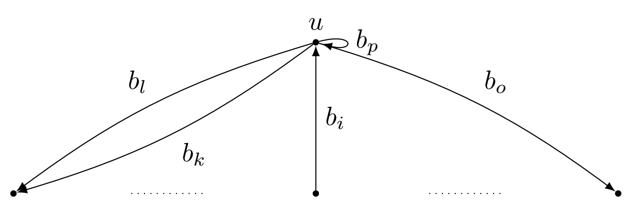
All generators must label one of the edges in the above diagram, leaving at least one vertex unreachable. This means that $\dagger\,(S,B) \ncong \dagger\,(A^*,A)$. □

Since we assumed there existed a generator b_j such that $wb_ib_j = w$, we must now then assume that there is no such generator

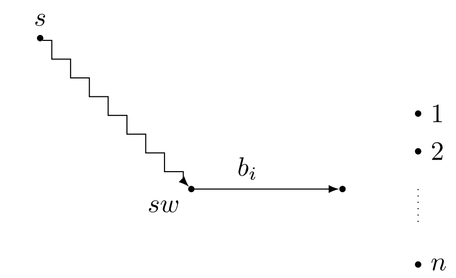
Claim 7.44

In Case 1.2, that is u is not a generator or identity and $wb_ib_j \neq w$ for any $b_j \in B$ we have that $\dagger\,(S,B) \ncong \dagger\,(A^*,A)$.

Proof: In this case, u now has outdegree at most $n-1$, as opposed to n previously. Since S is $n+m$ generated, there exist pairs (b_j, b_k) such that $ub_j = ub_k$, or generators b_k such that $ub_k = u$, or both.

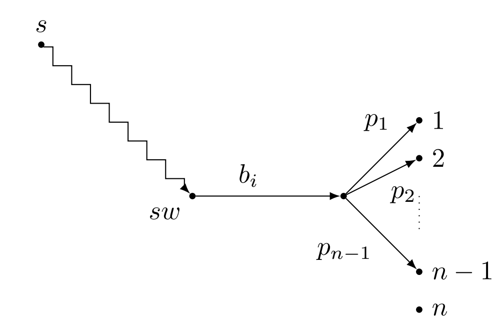


We locate our favourite vertex s, found distance at least $|wb_i|+1$ below any generators, and study the graph at swb_i.



By Lemma 7.42 we know that there must be an edge from swb_i to every vertex 1 to n in the above diagram.

Let $P = \{p_k\}$ be a maximal set of generators such that no two generators in P visit the same vertex from u. That is,$up_k \neq up_l$ for any p_k , $p_l \in P$. The set P has size at most $n-1$, which means we can reach at most $n-1$ different vertices from swb_i using these edges in P.



Any remaining generators b_k must lie on edges parallel to those we have drawn already, and so we are left with nothing to label an edge to vertex n. Hence $\dagger\,(S, B) \ncong \dagger\,(A^*, A)$. □

Therefore u is either a generator or an identity element.

Claim 7.45

In Case 2, that is u is an identity element, then S is n-generated and is free.

PROOF: If u is an identity element, then S must be precisely n generated as u has valency n. Since every other vertex has degree $n+1$, S must be the free monoid on n generators. □

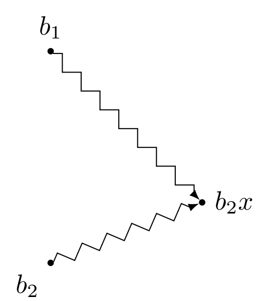
Now having u as an identity element allows us to show that S is isomorphic to A^+. It would be nice to see this the other way around, and show that if S is precisely n-generated we also have S isomorphic to A^+. In order to do this more neatly, we will first say something about the direction of edges in $\text{Cay}(S)$.

Claim 7.46

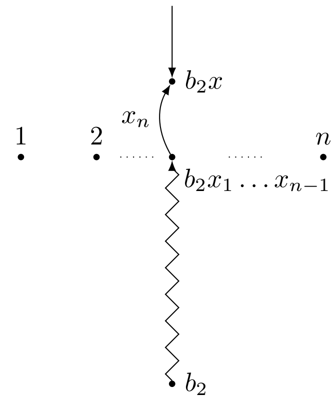
In Case 3, that is u is a generator, we have that no edges in $\text{Cay}(S)$ *have upwards direction.*

Proof: Assume u is some generator, say $u = b_1$. We select some other generator b_2 and look at the existence of paths between b_1 and b_2. Suppose there exists a path from b_2 to b_1. We may then apply the arguments from the proofs of Claims 7.43 and 7.44, using b_1 in place of u.

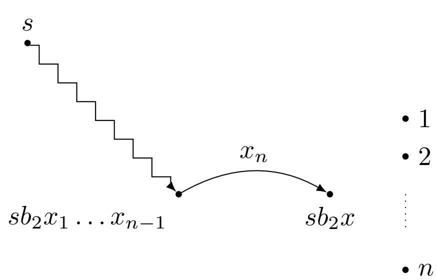
Suppose then that there exists some path labelled x from b_2 that goes towards b_1 but does not reach it.



We zoom in on the vertex b_2x and see what happens at this vertex. The vertex b_2x must be connected to $n+1$ vertices, which are shown here in the diagram as the vertices 1 to n, and the vertex above b_2x.



The edges leaving vertex b_2x must all terminate at the vertices labelled 1 to n, as an edge going up from b_2x would mean that x was not the longest path towards b_1 from b_2. We let $P = \{p_i\}$ be a maximal set of edges such that no two p_i label an edge starting at b_2x and terminating at the same vertex. The set P has size at most n, but we note that if P has size exactly n, then we have $b_2xp_j = b_2x_1 \cdots x_{n-1}$ for some $p_j \in P$. We now find our favourite vertex s which sits a distance at least $|b_2x| + 1$ below b_2, which means that no generators are found below sb_2x and so we can apply Lemma 7.42. We then have the following.



We now consider the edges labelled by all $p_i \in P$. These can reach at most n different vertices from sb_2x, as P has size at most n. If P has size precisely n then the n vertices we can reach from sb_2x must include $sb_2x_1 \dots x_{n-1}$. We can then reach an additional $n-1$ vertices, say those labelled 1 to $n-1$. All remaining generators not in P must label edges parallel to those in P and so we are left with an unreachable vertex n. If P has size less than n, then we clearly cannot reach all of the vertices 1 to n.

Hence our scenario must be that between b_1 and all other generators there are no edges which point in an upwards direction. $\square$

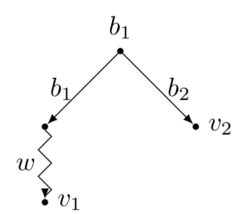
Hence we may deduce from this claim that there exists a path from b_1 to any generator we choose, say b_2, with $b_2 = b_1 w$. Note that as always w is a shortest word, so no loops or cycles are traversed when walking the path.

We are now able to assert what happens when S is precisely n-generated.

Claim 7.47

In Case 3.2 that is u is a generator and S is n-generated, then u is an identity element and S is the free monoid on n generators.

PROOF: Suppose S is n-generated but u is not an identity element. We have that u is a generator, b_1 say. Now by Claim 7.46, no edges may be in the upwards direction, and so all products $b_1 b_i$ are found on level 1 of the tree and $b_1 b_i \neq b_1 b_j$ for any $i \neq j$, since all vertices on level 1 must be reached via an edge from the root. Now since no edges are upwards, we have that for any generator, b_2 say, $b_2 = b_1 w$ for some word $w \in B^+$. Therefore, we must have $b_1 \cdot b_1 w = b_1 \cdot b_2$.



Hence in this picture, the vertices v_1 and v_2 must in fact be the same vertex, and as our graph is a tree, cycles are disallowed. Hence w must in fact be the empty word and $b_1 b_1 = b_1 b_2$, a contradiction to our earlier statement. Hence u cannot be equal to some generator and is therefore an identity, and by Claim 7.45 is free. $\square$

We can now assume that B has size $n+m$ where $m > 0$. To elicit our contradictions, we will look at what happens when we multiply b_1 by both b_2 and $b_1 w$, which are words that should represent the same element of S. This is split into four cases depending on what b_1 and b_2 do at vertex b_1. We first establish a small claim which will be useful throughout these cases.

Claim 7.48

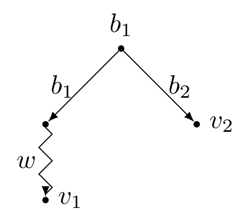
In Case 3.3, that is S is generated by more than n elements with $u = b_1$ we have for any generator $b_i \in B$ we have $b_i = b_1 w$ for some $w \in B^*$.

Proof: By Claim 7.46 there are no edges in the upwards direction. The graph $\mathrm{Cay}(S, B)$ must be an n-ary rooted tree and so wherever an edge occurs from a parent to child vertex, the edge must be directed down. Hence we can find a path from b_1 to any vertex v in $\mathrm{Cay}(S, B)$, and in particular to all b_i. This path is labelled by some word $w \in B^*$, and hence $b_i = b_1 w$. □

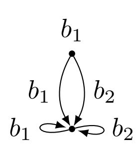
Claim 7.49

In Cases 3.3.1 and 3.3.2, that is S is generated by more than n elements with $u = b_1$ and either $b_1 b_1 \neq b_1 b_2 \neq b_1$ or $b_1 b_1 = b_1 b_2 \neq b_1$ we have that $\dagger\,(S, B) \ncong \dagger\,(A^*, A)$.

Proof:



Now since $b_1 \cdot b_1 w = b_1 \cdot b_2$ by Claim 7.48 , the vertices labelled v_1 and v_2 must in fact be the same vertex, and since there are no cycles in the graph or in w then $b_1 b_1 = b_1 b_2$. Additionally, from this we also have that $b_2 = b_1 b_1$, which allows to us deduce that since $b_1 \cdot b_1 b_1 = b_1 \cdot b_2$, then $b_2 b_1 = b_2$. Finally, we have $b_2 \cdot b_1 b_1 = b_2 \cdot b_2$, which gives us $b_2 b_2 = b_2$.

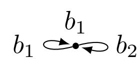


Now there are n sets of edges that have distinct destinations on leaving b_1, and so at most n distinct locations may be visited from b_2. However, we see above that we have already assigned one of these locations, namely b_2, and so this leaves us with at most $n-1$ sets of edges to travel to n different vertices, and so $\dagger(S,B) \ncong \dagger(A^*,A)$. $\square$

Claim 7.50

In Case 3.3.3, that is S is generated by more than n elements with $u = b_1$ and $b_1 b_1 = b_1 = b_1 b_2$, then $b_1 = b_2$.

Proof:

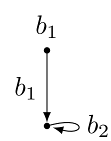


This is easily dismissed. By Claim 7.48 we have $b_1 \cdot b_2 = b_1 \cdot b_1 w$, and so we can conclude that $w = \varepsilon$ and $b_1 = b_2$. $\square$

Claim 7.51

In Case 3.3.4, that is S is generated by more than n elements with $u = b_1$ and $b_1 b_2 = b_1 \neq b_1 b_1$ we have that there must exist an upwards edge in $\mathrm{Cay}(S, B)$*, a contradiction to Claim 7.46.*

Proof:



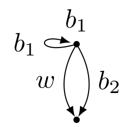
This case is also easy to eliminate. Since $b_1 \cdot b_2 = b_1$, then $b_1 \cdot b_1 w = b_1$ using Claim 7.48. However, we are not allowed to travel up towards b_1 by Claim 7.46. $\square$

We then come to the final case, 3.3.5.

Claim 7.52

In Case 3.3.5, that is S is generated by more than n elements with $u = b_1$ and $b_1 b_1 = b_1 \neq b_1 b_2$ then $S \cong A^*$.

Proof: Using our favourite equality from Claim 7.48, $b_1 \cdot b_2 = b_1 \cdot b_1 w$, we see that $b_1 w = b_1 b_2$ and so $b_2 = b_1 b_2$.



We have assumed here that $w \neq b_2$. If that is the case, then since w is some generator, then $w = b_1 v$ for some word v. Then

$$\begin{aligned} b_1 \cdot w = b_1 \cdot b_1 v &\implies b_1 v = b_1 w \\ &\implies b_1 v = b_2 \\ &\implies w = b_2 \end{aligned}$$

So in fact $w = b_2$, and we can generalise this argument to show that there are no multiple edges leaving b_1. Similarly, we can show there is no $b_i \neq b_1$ such that $b_1 b_i = b_1$, as if there is, we have $b_i = b_1 v$ for some word v. Then

$$\begin{aligned} b_1 \cdot b_i = b_1 \cdot b_1 v &\implies b_1 = b_1 v \\ &\implies v = \varepsilon \\ &\implies b_i = b_1. \end{aligned}$$

Hence there are exactly $n + 1$ generators in B. We will now show that b_1 is an identity. Suppose first that $sb_1 \neq s$ for some element $s \in S$. Then $sb_1 b_1 = sb_1$ and $sb_2 = sb_1 b_2$ which tells us that $sb_1 = sb_2$ and additionally $sb_1 b_2 = sb_2$. Then from the vertex sb_1, we need to reach n vertices, but now have only $n - 1$ edges left to do this with. Hence b_1 is a right identity. We have already established that $b_1 b_i = b_i$ for all generators b_i, and so b_1 is also a left identity. Thus our semigroup S is actually A^* generated as a semigroup. □

This completes the proof of Theorem 7.40.

□

Having established this, we can then write down the Cayley spectrum of A^*.

Corollary 7.53

Let A be an alphabet of size strictly greater than 1. Then the Cayley spectrum of A^* is $\mathcal{C}(A^*, A) = \{A^*\}$.

7.3 Free Semigroups

It seems now that it would be a simple task to prove an analogous theorem about the free semigroup on generated by A, that is A^+. Recall that Cay(A^+, A) has the form of n n-ary rooted trees, compared to Cay(A^*, A) which is a single n-ary rooted tree. We might wonder then, that since they share many similar features as graphs, whether we might apply similar methods of proof to free semigroups as we had for free monoids. Indeed, many of the steps in the proof for A^* are applicable to A^+. However, the last step, in which we find we have drawn A^* generated as a semigroup leads us to find many more semigroups sharing the graph $\dagger\,(A^+, A)$.

In fact, we will enumerate all semigroups that have a skeleton isomorphic to $\dagger\,(A^+, A)$ and provide a presentation for each of them. To state the exact number of such semigroups, we first need to establish the concept of *partitions*.

Definition 7.54

A partition *of a number $n \in \mathbb{N}$ is an expression for n as unordered sum of natural numbers. We write a partition P of n into i parts as $(p_1, p_2, \ldots, p_i)$ where $n = p_1 + p_2 + \ldots + p_i$. The number of partitions of n is given by the* partition function*, $p(n)$.*

Definition 7.55

A restricted partition *is a partition in which the largest part has size $\leq N$ and the number of parts is $\leq M$ for some $N, M \in \mathbb{N}$. We denote a restricted partition of n into M parts with largest part size N as $P(N, M, n)$. The* restricted partition function *$p(N, M, n)$ gives the number of restricted partitions of n with largest part $\leq N$ and number of parts $\leq M$.*

The following theorem is the main result of this section. A fully worked example of this theorem for $n = 4$ can be found in subsection 7.3.1.

Theorem 7.56

Let $n \in \mathbb{N}$, then for $i \leq m \leq n$ and let $P_{m,f}$ for $1 \leq f \leq p(m)$ be partitions of m. For each partition P_f, if i is its size then define

$$\alpha_{f,m,i} = p(n-i, n+1-m, n-i) - p(n-i, n-m, n-i)$$

and

$$\beta_{f,m,i} = p(n-i, n-m, n-i) - p(n-i, n-m-1, n-i).$$

Let $Q_{f,x}$ for $1 \leq x \leq \alpha_{m,i}$ be a partition of $n-i$ into $n+1-m$ parts, and let $r_{Q_{f,x}}$ denote the number of distinct parts in $Q_{f,x}$. For a given i, define

$$U_{P_f} = \begin{cases} \sum_{x=1}^{\alpha_{f,m,i}} r_{Q_{f,x}} + \sum_{x=1}^{\beta_{m,i}} (r_{Q_{f,x}} + 1) & \textit{if } m \neq i \\ 1 & \textit{if } m = i \end{cases}$$

Then there are exactly

$$\#S = \sum_{m=1}^{n} \sum_{f=1}^{p(m)} U_{P_{m,f}}$$

semigroups $S = \mathrm{sgp}\langle B \rangle$ such that $\dagger\,(S, B) \cong \dagger\,(A^+, A)$.

In this section we prove this theorem, as well as provide an explicit presentation and description for each semigroup S. We will show that S must be generated by more than n elements using Lemma 7.42, as otherwise it will not have the correct skeleton, or will be isomorphic to A^+. We will then focus on a single tree in $\mathrm{Cay}(S)$ which contains at least two generators, b_1 and b_{n+1} say. We will call the unique vertex of degree n in this tree u. The outline of cases will be as follows.

1 u is not a generator.

2 u is a generator, say b_1 and $b_{n+1} = b_1 w$ for some w.

2.1 $b_{n+1} \neq b_1 w$ for some w.

2.2 $b_{n+1} = b_1 w$ for some w.

2.2.1 $b_1 b_1 \neq b_1 b_{n+1} \neq b_1$

2.2.2 $b_1 b_1 = b_1 b_{n+1} \neq b_1$

2.2.3 $b_1b_1 = b_1b_{n+1} = b_1$

2.2.4 $b_1b_1 \neq b_1b_{n+1} = b_1$

2.2.5 $b_1b_1 = b_1 \neq b_1b_{n+1}$

These cases will be eliminated similarly to their analogues in Section 7.2, save for case 2.2.5. In this case, we will find that w is in fact situated on level 1, and that we can construct many different semigroups with the correct skeleton, which we will describe in full.

Case	**Claim**
1	7.58
2	see subcases
2.1	7.60
2.2	see subcases
2.2.1	7.61
2.2.2	7.61
2.2.3	7.62
2.2.4	7.63
2.2.5	7.64, 7.65, 7.66, 7.67, 7.68, 7.69

We now begin the proof of Theorem 7.56.

Proof:

Let $S = \text{sgp}\langle B \rangle$ be a semigroup such that $\dagger\,(S, B) \cong \dagger\,(A^+, A)$. Then S is at least n generated as $\dagger\,(S)$ is n disjoint n-ary rooted trees.

Claim 7.57

If S is exactly n-generated, then the generators are found at the n vertices of degree n and $S \cong A^+$.

Proof: Suppose S is n-generated; then there must be precisely one generator found in each tree. Consider the tree containing b_1, and suppose that b_1 is found at some vertex v of degree $n + 1$. Since we must be able to reach all vertices in the tree containing b_1

from v, there exists a path from v to the unique vertex of degree n which we will call u. This means we have $u = b_1 b_i w$ for some $b_i \in B$, $w \in B^+$, that is, the edge labelled b_i is in the upwards direction. Now consider the vertices below v. By Lemma 7.42 these must also be reached using arrows from b_1. However, since S is n-generated there are only $n-1$ edges available and n vertices to reach, a contradiction. Hence b_1 must be found at u, and all other generators at the vertices of degree n.

Now we must reach all other vertices in each tree from the root vertex, and since we are n-generated this means every vertex save the root vertex has indegree of 1 and outdegree of n and so S is free. $\square$

Hence we have that S is at least $n+1$ generated. Suppose that B has size $n+m$ for some $m \geq 1$, and consider the subtree $\mathrm{Cay}(S)_1$ of $\mathrm{Cay}(S)$ which contains at least 2 generators, say b_1 and b_{n+1}.

Let u be the vertex of degree n in $\mathrm{Cay}(S)_1$.

Claim 7.58

In Case 1, that is the vertex u is not a generator we have that $\dagger\,(S, B) \ncong \dagger\,(A^+, A)$.

Proof: Assume that u is not a generator, then $u = wb_i$ for some $b_i \in B$ and $w \in B^+$. As when considering A^*, we have either $wb_ib_j = w$ for some $b_j \in B$, or $wb_ib_j \neq w$ for any $b_j \in B$. In both cases(cf. Claims 7.43 and 7.44) , we find a vertex s that is a sufficient distance below any generators and consider swb_i. We find in both cases that we are trying to reach n vertices with only $n-1$ sets of edges available to us. Therefore $\dagger\,(S, B) \ncong \dagger\,(A^+, A)$. $\square$

Suppose that $u = b_1$. We have the following observation on the direction of edges in $\mathrm{Cay}(S)$.

Claim 7.59

In Case 2, that is $u = b_1$ we have that there are no upwards edges in $\mathrm{Cay}(S)$.

Proof: The proof follows identically to Claim 7.46. $\square$

This claim will help us rule out Case 2.1.

Claim 7.60

In Case 2.1, that is $u = b_1$ *and* $b_{n+1} \neq b_1 w$ *for any* $w \in B^*$ *we have that* $\dagger\,(S, B) \ncong \dagger\,(A^+, A)$.

Proof: Recall that b_1 is the root vertex of the tree that b_{n+1} is found in. If $b_{n+1} \neq b_1 w$ this means there exists no path from b_1 to b_{n+1}, and by Claim 7.59 we have that there are no edges in the upwards direction, so there cannot be a path from b_{n+1} to b_1 either. Hence this graph is disconnected and so $\dagger\,(S, B) \ncong \dagger\,(A^+, A)$. □

We can then break down Case 2.2 into five cases as we did for A^* by considering the terminating vertices of the edges labelled by b_1 and b_{n+1} that come from b_1. The first four cases have a similar result to their counterparts in the free monoid case.

Claim 7.61

In Cases 2.2.1 and 2.2.2, that is $u = b_1$, $b_{n+1} = b_1 w$ *and either* $b_1 b_1 \neq b_1 b_{n+1} \neq b_1$ *or* $b_1 b_1 = b_1 b_{n+1} \neq b_1$ *we have that* $\dagger\,(S, B) \ncong \dagger\,(A^+, A)$

Proof: The proof is analogous to Claim 7.49. □

Claim 7.62

In Case 2.2.3, that is $u = b_1$, $b_{n+1} = b_1 w$ *and* $b_1 b_1 = b_1 b_{n+1} = b_1$ *we have that* $\dagger\,(S, B) \ncong \dagger\,(A^+, A)$

Proof: The proof is analogous to Claim 7.50. □

Claim 7.63

In Case 2.2.4, that is $u = b_1$, $b_{n+1} = b_1 w$ *and* $b_1 b_1 \neq b_1 b_{n+1} = b_1$ *we have that* $\dagger\,(S, B) \ncong \dagger\,(A^+, A)$

Proof: The proof is analogous to Claim 7.51. □

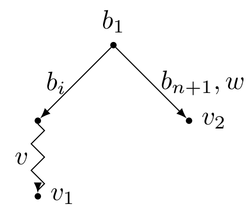
However the argument of the last case for A^*, 7.52, does not transfer to A^+. This means we must study what happens when $b_1 b_1 = b_1$ but $b_1 b_{n+1} \neq b_1$, that is case 2.2.5.

From Claim 7.59 we know that there must be a path from b_1 to b_{n+1}, say $b_{n+1} = b_1 w$. It is not immediately obvious to us where the vertex labelled by w is found in Cay(S), but by establishing the following claims we will be able to locate it.

Claim 7.64

In Case 2.2.5, that is $u = b_1$, $b_{n+1} = b_1 w$ and $b_1 b_1 = b_1 \neq b_1 b_{n+1}$ we have that w is either a level 0 or level 1 vertex.

Proof: Suppose otherwise. By Claim 7.59 all edges are in the downwards direction, so all roots must be generators, and so $w = b_i v$ for some $b_i \in B$ and $v \in AA^*$. Then $b_{n+1} = b_1 b_i v$.



Now these two vertices v_1, v_2 are in fact the same vertex, but v has non-zero length and so we have introduced a cycle into the graph, which is a contradiction. Hence w must indeed be found on level 0 or 1. $\square$

This helps us narrow down the location of w to a selection of $n + n^2$ vertices. The following claim will allow us to specify exactly where w is located in Cay(S).

Claim 7.65

In Case 2.2.5, that is $u = b_1$, $b_{n+1} = b_1 w$ and $b_1 b_1 = b_1 \neq b_1 b_{n+1}$, we have that if b_j is a generator found on level 1 of the tree with root b_i then $b_i b_i = b_i$ and $b_i b_j = b_j$. Additionally, we never have $b_i b_k = b_j$ for some $k \neq j$.

Proof:

If $b_i b_i = b_i$ and $b_i b_j = b_j$ is not the case, we may refer to Case 7.49, 7.50 and 7.51.

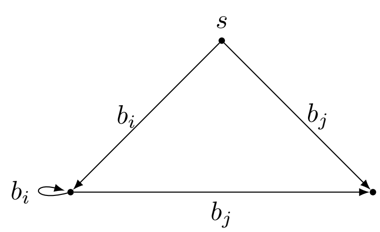
Suppose that $b_i b_k = b_j$ for some $k \neq j$. Then b_j is written as the product of two other generators, and can be removed without affecting the shape of $\dagger\,(S)$ and leaves a semigroup isomorphic to S. Since we want our generating set as small as possible, then b_j should be removed. $\square$

This shows us that since $b_1 b_{n+1} = b_{n+1}$ we must have $w = b_{n+1}$ and so w was hiding in front of us all along. We can now deduce even more information about S.

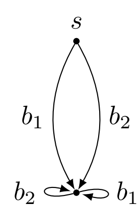
Claim 7.66

In Case 2.2.5, that is $u = b_1$, $b_{n+1} = b_1 w$ and $b_1 b_1 = b_1 \neq b_1 b_{n+1}$, we have that if b_i, b_j are a generators of S such that $b_i b_j = b_j$, then $sb_i = s$ for all $s \in S$.

Proof: Suppose there exists some $s \in S$ such that $sb_i \neq s$. By Claim 7.65, $b_i b_i = b_i$, and so we must have $sb_i b_i = sb_i$. Since $b_i b_j = b_j$, then $sb_i b_j = sb_j$ which has two possible realisations in the graph. If $sb_i \neq sb_j$ our graph has the following section.



This has created a cycle, which is forbidden in our graph, so we must have $sb_i = sb_j$.



Now in this case, we must remember that we have to reach n new vertices from sb_1, but since $sb_1 b_2 = sb_1$, we are left with only $n - 1$ sets of edges to achieve this with, a contradiction.

Hence $sb_i = s$.

$\square$

Now it would seem that we can have a generating set of size up to $n + n^2$ as this is the number of level 0 and level 1 vertices. However, as we are interested in counting the semigroups up to isomorphism, we in fact have the following.

Claim 7.67

In Case 2.2.5, that is $u = b_1$, $b_{n+1} = b_1 w$ and $b_1 b_1 = b_1 \neq b_1 b_{n+1}$ we have that S has a generating set with size at most $n + n$.

PROOF: Suppose that S has a generating set $B = \{b_1, \ldots, b_{n+n+1}\}$ of size $n+n+1$. Of the generators, n are found at level 0, say $b_1, b_2, \ldots, b_n$, and the rest are found at level 1. Suppose that b_{n+i} for $1 \leq i \leq n+1$ are distributed across level 1 vertices in trees rooted at b_i where $1 \leq i \leq k$ for some k. At least one of these trees contains at least two level 1 generators. Without loss of generality, say the tree rooted at b_1 contains generators b_{n+1} and b_{n+2}. Now by Claim 7.65 we know that $b_1 b_i = b_1$ for $1 \leq i \leq k$ and $b_1 b_{n+1} = b_{n+1}$ and $b_1 b_{n+2} = b_{n+2}$. We now have $n-2$ vertices on level 1 for which we have not determined the edge that meets it from level 0.

Let b_j be a generator that is found on level 1 in the tree rooted at b_i say, and assume $b_1 b_j = b_1$. Then $b_i b_1 b_j = b_i b_j = b_j$, but $b_i b_1 b_j = b_i b_1 = b_1$. Hence we have that $b_1 b_j$ is always found on level 1 for any generator b_j where $j > n$. Now there are $n+1$ such generators, and only n level 1 vertices, so it must be that $b_1 b_x = b_1 b_y$ for $x, y > n$. Let b_y be found in the tree rooted at b_i. Then $b_i b_1 b_y = b_i b_y = b_y$ and $b_i b_1 b_y = b_i b_1 b_x = b_i b_x$ so $b_i b_x = b_y$. This means that b_y can be written in terms of other generators, and its removal does not affect the shape of $\dagger(S)$. Hence we can remove b_y from the generating set and have a semigroup isomorphic to S with the same skeleton. $\square$

Claim 7.68

In Case 2.2.5, that is $u = b_1$, $b_{n+1} = b_1 w$ and $b_1 b_1 = b_1 \neq b_1 b_{n+1}$ we have that for $n+1 \leq j \leq n+m$, then $b_1 b_j \neq b_1$.

PROOF:

Suppose that $b_1 b_j = b_1$ for such a j. Then for $n+i < k \leq n+m$ where $k \neq j$ we have

$$\begin{aligned} b_k b_1 b_j &= b_k b_j \\ &= b_j \\ &= b_k \\ &= b_k b_1 \\ &= b_k b_1 b_j. \end{aligned}$$

That is, $b_j = b_k$, which is a contradiction. $\square$

Given these restrictions on the structure of Cay(S, B), we may now construct precisely #S different semigroups such that $\dagger\,(S, B) \cong \dagger\,(A^+, A)$ but where S is not isomorphic to A^+ itself.

Let the generating set of S have size $n + m$ for some $1 \leq m \leq n$. We will let the root generators be $b_1, b_2, \ldots, b_n$, leaving us with m generators, which by Claim 7.64 are found on level 1.

The distribution of the m generators across vertices on level 1 gives rise to a partition of m, so we create at least $p(m)$ semigroups in this way. Suppose then that $P = (p_1, p_2, \ldots, p_i)$ is a partition of m. Without any loss of generality, we will let generators $\{b_{n+1}, \ldots, b_{n+p_1}\}$ be found in the tree rooted at b_1 and in general

$$\{b_{n+p_1+\ldots+p_{j-1}+1}, \ldots, b_{n+p_1+\ldots+p_{j-1}+p_j}\}$$

are in the tree with root b_j. By Claim 7.65, we have $b_j b_j = b_j$ for $1 \leq j \leq i$, that is, for each root vertex b_j where $1 \leq j \leq i$ we have a loop labelled by b_j on that vertex. Claim 7.65 also determines precisely edges labelled $b_{n+p_1+\ldots+p_{j-1}+k}$ where $1 \leq k \leq p_j$, since $b_j b_{n+p_1+\ldots+p_{j-1}+k} = b_{n+p_1+\ldots+p_{j-1}+k}$ for $1 \leq j \leq i$. That is, for each $b_{n+p_1+\ldots+p_{j-1}+k}$ with $1 \leq k \leq p_j$ and $1 \leq j \leq i$, there is an edge from b_j to $b_{n+p_1+\ldots+p_{j-1}+k}$ labelled by $b_{n+p_1+\ldots+p_{j-1}+k}$. We now consider what edges labelled b_k for $n + i < k \leq n + m$ can do.

By Claim 7.66 $sb_1 = s$ for all $s \in S$, then by determining $b_1 b_k$, we determine sb_k for all $s \in S$. We therefore need only consider the end vertex of the remaining edges leaving b_1 in order to find all edges in Cay(S, B).

We have already determined the destination of p_1 of the edges leaving b_1, specifically those labelled b_z for $1 \leq z \leq i$ and $n \leq z \leq n + p_1$. This leaves $n + m - i - p_1$ edges for which we need to find the end point, and $n - p_1$ level 1 vertices which have not yet featured as the end point of an edge.

By Claims 7.65 and 7.68 we know that for any generator found on level 1, that is, b_j for $n+1 \leq j \leq n+m$, such a generator labels neither a loop on b_1, nor a multiple edge from b_1. We therefore have that generators b_j for $n + 1 \leq j \leq n + m$ label m edges terminating at m different level 1 vertices from b_1, and generators b_j for $1 \leq j \leq i$ label loops starting and terminating at b_1. This gives us $m + i$ edges for which we know the start and end, and therefore $n-i$ edges yet to be discovered, with $n-m$ level 1 vertices as yet unreachable.

There are now no further restrictions on edge locations, and so the remaining $n-i$ edges can be found either as loops on b_1 or as edges to the $n-m$ level 1 vertices that have not yet featured as a terminating vertex. There are a number of different ways this can happen, and we shall now count these.

We observe that we must have an edge from b_1 to the $n-m$ vertices that we have not visited yet, and so we need to partition the $n-i$ generators into partitions with at least $n-m$ parts. If we create a partition $Q = (q_1, \ldots, q_{n-m})$ of precisely $n-m$ parts then for each q_j we have the generators $\{b_{i+q_1+\ldots+q_{j-1}+1}, \ldots, b_{i+q_1+\ldots+q_{j-1}+q_j}\}$ labelling an edge from b_1 to one of the $n-m$ vertices we have not seen yet. To count the number of partitions of $n-i$ into $n-m$ parts we calculate

$$\beta = p(n-i, n-m, n-i) - p(n-i, n-m-1, n-i)$$

Now since there are only $n-m+1$ permitted terminal vertices for our edges, that is the $n-m$ unvisited level 1 vertices and b_1, our partition of $n-i$ must be no bigger than $n-m+1$. Suppose we have such a partition $Q = (q_1, \ldots, q_{n-m+1})$ of $n-i$. We notice that from this partition of edges we must select one set of edges to be loops on b_1, as opposed to edges to level 1. If two parts of Q are equal, say $q_x = q_y$, then it makes no difference to our semigroup whether we find the set of edges relating to q_x as loops, or the set relating to q_y as loops. Therefore, we let r represent the number of distinct parts of Q, and so a partition Q gives rise to r different semigroups. We count this by first calculating the number of partitions of the correct size:

$$\alpha = p(n-i, n-m+1, n-i) - p(n-i, n-m, n-i).$$

Then given a partition Q_x of the correct size, let r_x represent the number of distinct parts. We then have

$$\sum_{x=1}^{\alpha} r_x$$

non-isomorphic semigroups found from these configurations.

$\square$

Hence given an m, a partition $P = (p_1, p_2, \ldots, p_i)$ of m, a partition $Q = (q_1, \ldots, q_j)$ of $n - i$ and a choice $i + q_k$ of how many generators should label the loop on b_1, we have the semigroup

$$\begin{aligned} S[m, P, Q, q_k] = \mathrm{sgp}\langle b_1, \ldots, b_{n+m} \; | & b_x b_l = b_x \text{ for } 1 \leq l \leq i, 1 \leq x \leq n+m \\ & b_1 b_{n+l} = b_{n+l} \text{ for } 1 \leq l \leq p_1 \\ & b_2 b_{n+p_1+l} = b_{n+p_1+l} \text{ for } 1 \leq l \leq p_2 \\ & \ldots \\ & b_i b_{n+p_1+\ldots+p_{i-1}+l} = b_{n+p_1+\ldots+p_{i-1}+l} \text{ for } 1 \leq l \leq p_i \\ & b_1 b_{i+n-m+l} = b_1 b_{i+1} \text{ for } 1 \leq l \leq q_1 \\ & b_1 b_{i+n-m+q_1+l} = b_1 b_{i+2} \text{ for } 1 \leq l \leq q_2 \\ & \ldots \\ & b_1 b_{i+n-m+q_1+\ldots+q_{k-2}+l} = b_1 b_{i+k-1} \text{ for } 1 \leq l \leq q_{k-1} \\ & b_1 b_{i+n-m+q_1+\ldots+q_{k-1}+l} = b_1 \text{ for } 1 \leq l \leq q_k \\ & b_1 b_{i+n-m+q_1+\ldots+q_k+l} = b_1 b_{i+k} \text{ for } 1 \leq l \leq q_{k+1} \\ & b_1 b_{i+n-m+q_1+\ldots+q_j+l} = b_1 b_{i+j} \text{ for } 1 \leq l \leq q_j \rangle. \end{aligned}$$

We wish to prove the following theorem.

Theorem 7.69

Let A be an alphabet with n elements and let A^+ be the free semigroup on A. Let $m \leq n$ and let $P = (p_1, p_2, \ldots, p_i)$ be a partition of m. Let $Q = (q_1, \ldots, q_j)$ be a partition of $n - i$ and a choose a q_k. Then $\dagger\,(S[m, P, Q, q_k], \{b_1, \ldots b_{n+m}\}) \cong \dagger\,(A, A^+)$ and given the choices of m, P, Q, q_k there are precisely #S different semigroups $S[m, P, Q, q_k]$.

Proof: This is a consequence of Claims 7.70, 7.72 and 7.73 □

We first show that there are the number of semigroups $S[m, P, Q, q_k]$ that we want to have. We then use a rewriting system based on this presentation to find normal forms for $S[m, P, Q, q_k]$ and then show that the skeleton graph of this semigroup is isomorphic to that of $A^+ = \mathrm{sgp}\langle A \rangle$.

Claim 7.70

Let $S_1 = S[m_1, P_1, Q_1, q_{k_1}], S_2 = S[m_2, P_2, Q_2, q_{k_2}]$ be two semigroups as defined above. Then $S_1 \cong S_2$ if and only if $m_1 = m_2$, $P_1 = P_2$, $Q_1 = Q_2$ and $q_{k_1} = q_{k_2}$.

PROOF: If $m_1 = m_2$, $P_1 = P_2$, $Q_1 = Q_2$ and $q_{k_1} = q_{k_2}$, then clearly $S_1 \cong S_2$.

For the converse, suppose that $S_1 \cong S_2$, and let $\varphi : S_1 \to S_2$ be an isomorphism. Let $S_1 = \text{sgp}\langle b_1, \ldots, b_{n+m_1}\rangle$ and $\cong S_2 = \text{sgp}\langle c_1, \ldots, c_{n+m_2}\rangle$. Since φ must map generators to generators and no generator in S_1 or S_2 can be written as a product of other generators both semigroups must have generating sets of the same size, and so $m_1 = m_2$.

Let the number of parts of P_1 and P_2 be denoted by i_1 and i_2 respectively. Then $b_z b_z = b_z$ for $1 \leq z \leq i_1$ and $\varphi(b_z)\varphi(b_z) = \varphi(b_z)$. The only such elements in S_2 with this property are c_z for $1 \leq z \leq i_2$. Since we must also have $\varphi^{-1}(c_z)\varphi^{-1}(c_z) = \varphi^{-1}(c_z)$ then we conclude that $i_1 = i_2$.

Let $P_1 = (p_1, \ldots, p_{i_1})$ and $P_2 = (\pi_1, \ldots, \pi_{i_1})$. Given some $1 \leq z \leq i_1$, the tree rooted at b_z we have p_z generators b_y such that $b_z b_y = b_y$. Under φ the element b_z must be mapped to some element c_v such that there are p_z generators b_y such that $c_v c_y = c_y$. Hence $P_1 = P_2$.

Now Q_1 and Q_2 can only have $n - m_1$ or $n - m_1 + 1$ parts. Suppose without loss of generality that Q_1 has $n - m_1 + 1$ parts and Q_2 has $n - m_1$ parts. Then we have $b_1 b_z = b_1$ for $i_1 + q_{k_1}$ element b_z, then we must be able to find $i_1 + q_{k_1}$ elements c_z such that $c_1 c_z = c_1$. However, since Q_2 has only $n - m_1$ parts this means there are only i_1 such elements. Hence Q_2 must in fact have $n - m_1 + 1$ and furthermore, $q_{k_1} = q_{k_2}$. A similar argument applies if Q_1 has $n - m_1$ parts.

Finally, let $Q_1 = (q_1, \ldots, q_j)$. If $j = n - m + 1$ then we know that $q_{k_1} = q_{k_2}$. For each $1 \leq z \leq j$ where $z \neq k_1$ such that , there are exactly q_z generators b_v, b_w such that $b_1 b_v = b_1 b_w$. Using φ, there must also be exactly q_z generators c_v, c_w such that $c_1 c_v = c_1 c_w$, and hence $Q_1 = Q_2$. □

Hence there are #S non-isomorphic semigroups that can be found for a given n using the above presentation. We will shortly show that semigroups given by these presentations are indeed skeleton-isomorphic to † (A^+, A), however we will first describe an example of such a semigroup to aid understanding.

Example 7.71

As an illustrative example, suppose that $A = \{a_1, a_2, a_3, a_4\}$, and create the following semigroup T. We choose the number of extra generators to be $m = 3$. The partition of m is $3 = 2 + 1$, and so $i = 2$. We then have $n - i = 2$. We choose the one possible partition into $n - m = 1$ parts, that is the partition $2 = 2$. Finally this means we must have exactly i generators labelling a loop on b_1. Using the general presentation for S, this gives us the specific presentation

$$\begin{aligned} T = \text{sgp}\langle b_1, b_2, b_3, b_4, b_5, b_6, b_7 \mid\, & b_i b_1 = b_i, b_i b_2 = b_i \text{ for } 1 \leq i \leq 7, \\ & b_1 b_5 = b_5, b_1 b_6 = b_6, b_2 b_7 = b_7, \\ & b_1 b_3 = b_1 b_4 \rangle. \end{aligned}$$

We will show that T has the graph that is partially shown in Figure 7.4 by establishing normal forms via a complete rewriting system, and considering the edges that arise when using these as the vertices of Cay(T).

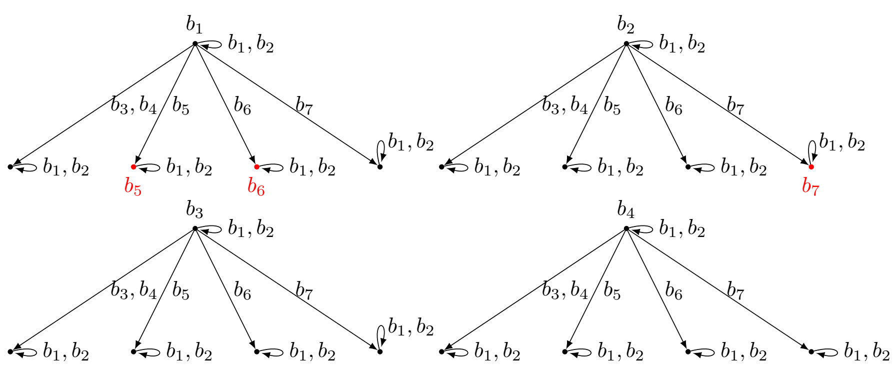


Figure 7.4: Cayley graph of T

We show that T has the normal forms

I b_1 and b_2;

II $b_1 b_3\{b_3, b_5, b_6, b_7\}^\star$, $b_1 b_7\{b_3, b_5, b_6, b_7\}^\star$

$b_2b_3\{b_3, b_5, b_6, b_7\}^\star$,$b_2b_5\{b_3, b_5, b_6, b_7\}^\star$,$b_2b_6\{b_3, b_5, b_6, b_7\}^\star$

III $b_3\{b_3, b_5, b_6, b_7\}^\star$, $b_4\{b_3, b_5, b_6, b_7\}^\star$, $b_5\{b_3, b_5, b_6, b_7\}^\star$,
$b_6\{b_3, b_5, b_6, b_7\}^\star$, $b_7\{b_3, b_5, b_6, b_7\}^\star$.

Using the shortlex order with $b_x < b_y$ for $x < y$, we have a rewriting system

RW1 $b_xb_l \to b_x$ for $1 \leq l \leq 2, 1 \leq x \leq 7$,

RW2 $b_1b_l \to b_l$ for $5 \leq l \leq 6$,

RW3 $b_2b_7 \to b_7$,

RW4 $b_xb_4 \to b_xb_3$ for $1 \leq x \leq 7$.

We will show that this system is locally confluent by considering possible overlaps of rules when rewriting words. If we encounter the rule RW1 on the left, this can overlap with RW1, RW2, RW3 and RW4 on the right.

RW1 RW1

$$\begin{tikzcd}
& b_lb_{x_2} \arrow[r] & b_{x_2} \arrow[dd] \\
\underline{b_{x_1}b_l}b_{x_2} \arrow[ur] \arrow[dr] & & \equiv \\
& b_{x_1}b_{x_2} \arrow[r] & b_{x_2}
\end{tikzcd}$$

RW1 RW2

$$\begin{tikzcd}
& b_xb_l \arrow[dd] \\
\underline{b_xb_1}b_l \arrow[ur] \arrow[dr] & \equiv \\
& b_xb_l
\end{tikzcd}$$

RW1 RW3

$$\begin{tikzcd}
& b_xb_7 \arrow[r, dotted] & b_7 \arrow[dd] \\
\underline{b_xb_2}b_7 \arrow[ur] \arrow[dr] & & \equiv \\
& b_xb_7 \arrow[r, dotted] & b_7
\end{tikzcd}$$

RW1 RW4

$$\begin{tikzcd}
& b_xb_4 \arrow[r] & b_xb_3 \arrow[dd] \\
\underline{b_xb_l}b_4 \arrow[ur] \arrow[dr] & & \equiv \\
& b_xb_lb_3 \arrow[r] & b_xb_3
\end{tikzcd}$$

Rule RW2 overlaps with only RW1,RW2 and RW4.

RW2 RW1

$$\begin{tikzcd}
& b_{l_1}b_{l_2} \arrow[r] & b_{l_1} \arrow[dd, "\equiv"'] \\
b_1b_{l_1}b_{l_2} \arrow[ur] \arrow[dr] & & \\
& b_1b_{l_1} \arrow[r] & b_{l_1}
\end{tikzcd}$$

RW2 RW2

$$\begin{tikzcd}
& b_1b_l \arrow[r] & b_l \arrow[dd, "\equiv"'] \\
b_1b_1b_l \arrow[ur] \arrow[dr] & & \\
& b_1b_l \arrow[r] & b_l
\end{tikzcd}$$

RW2 RW4

$$\begin{tikzcd}
& b_lb_4 \arrow[r] & b_lb_3 \arrow[dd, "\equiv"'] \\
b_1b_lb_4 \arrow[ur] \arrow[dr] & & \\
& b_1b_lb_3 \arrow[r] & b_lb_3
\end{tikzcd}$$

The rule RW3 can only overlap with RW1 and RW4.

RW3 RW1

$$\begin{tikzcd}
& b_7b_l \arrow[r] & b_7 \arrow[dd, "\equiv"'] \\
b_2b_7b_l \arrow[ur] \arrow[dr] & & \\
& b_2b_7 \arrow[r] & b_7
\end{tikzcd}$$

RW3 RW4

$$\begin{tikzcd}
& b_7b_4 \arrow[r] & b_7b_3 \arrow[dd, "\equiv"'] \\
b_2b_7b_4 \arrow[ur] \arrow[dr] & & \\
& b_2b_7b_3 \arrow[r] & b_7b_3
\end{tikzcd}$$

Similarly RW4 also only overlaps with RW1 and RW4.

RW4 RW1

$$\begin{tikzcd}
& b_xb_3b_l \arrow[r] & b_xb_3 \arrow[dd, "\equiv"'] \\
b_xb_4b_l \arrow[ur] \arrow[dr] & & \\
& b_xb_4 \arrow[r] & b_xb_3
\end{tikzcd}$$

RW4 RW4

$$\begin{tikzcd}
& b_xb_3b_4 \arrow[r] & b_xb_3b_3 \arrow[dd, "\equiv"'] \\
b_xb_4b_4 \arrow[ur] \arrow[dr] & & \\
& b_xb_4b_3 \arrow[r] & b_xb_3b_3
\end{tikzcd}$$

This shows that our system is locally confluent. Under the ordering that we have used, all right-hand sides of rules are shorter than left-hand sides, and so when a word is rewritten, it becomes shorter. Thus this system is Noetherian and so by Lemma 6.9 we have a complete rewriting system. Then by Lemma 6.10 there exist unique normal forms for T.

We must now show that the elements (I)-(III) are normal forms by showing they are irreducible. An element of type I is a single letter and so a rewrite rule can never be applied to it, hence it is irreducible. For a word of type II or III, we note that rewrite rule RW1 requires either b_1 or b_2 to appear as a non-leading letter which is never the case for such a word. Rule RW2 requires b_5 or b_6 to appear after b_1, and this will never occur in a word of type II or III. The third rewrite rule RW3 only operates on the subword b_2b_7,

Generator	b_1	b_2	b_3	b_4	b_5	b_6	b_7
b_1	(b_1, b_1)	(b_1, b_1)	(b_1, b_1b_3)	(b_1, b_1b_3)	(b_1, b_5)	(b_1, b_6)	(b_1, b_1b_7)
b_2	(b_2, b_2)	(b_2, b_2)	(b_2, b_2b_3)	(b_2, b_2b_3)	(b_2, b_2b_5)	(b_2, b_2b_6)	(b_2, b_7)
b_3	(b_3, b_3)	(b_3, b_3)	(b_3, b_3b_3)	(b_3, b_3b_3)	(b_3, b_3b_5)	(b_3, b_3b_6)	(b_3, b_7)
b_4	(b_4, b_4)	(b_4, b_4)	(b_4, b_4b_3)	(b_4, b_4b_3)	(b_4, b_4b_5)	(b_4, b_4b_6)	(b_4, b_4b_7)
b_5	(b_5, b_5)	(b_5, b_5)	(b_5, b_5b_3)	(b_5, b_5b_3)	(b_5, b_5b_5)	(b_5, b_5b_6)	(b_5, b_5b_7)
b_6	(b_6, b_6)	(b_6, b_6)	(b_6, b_6b_3)	(b_6, b_6b_3)	(b_6, b_6b_5)	(b_6, b_6b_6)	(b_6, b_6b_7)
b_7	(b_7, b_7)	(b_7, b_7)	(b_7, b_7b_3)	(b_7, b_7b_3)	(b_7, b_7b_5)	(b_7, b_7b_6)	(b_7, b_7b_7)

Table 7.1:

which cannot be made from any of the expressions listed in II and III. Finally, rule RW4 requires the letter b_4 to occur as a non-leading letter, and this does not happen in any words of type II and II. Hence all the elements of types I, II, and III are irreducible and thus are normal forms.

We would now like to show that we have listed all the normal forms, so we will suppose that $w \in \{b_1, \ldots, b_7\}^*$ is a normal form for T but is not of type I, II, or III. This means that w must have length at least 2, as all normal forms of length 1 are found as types I or III. First suppose that w begins with b_1. Then w either has b_y for $y \in \{1, 2, 4, 5, 6\}$ as a second letter, or b_z for $z \in \{1, 2, 4\}$ as a third or later letter. If we have b_y as a second letter, then we can apply RW1, RW2 or RW4 to w, and so w was not irreducible. If we have b_z as a third or later letter, then we can apply RW1 or RW4 to w and so w was not irreducible.

If w begins with b_2, then w must have either b_y where $y \in \{1, 2, 4, 7\}$ as a second letter, or b_z where $z \in \{1, 2, 4\}$ as a third or later letter. For b_y as a second letter, we can apply either RW1, RW3 or RW4 to w, so w is not irreducible. If we have b_z as a third or later letter, then we can rewrite w using RW1 or RW4, and so w was not irreducible.

We can now consider the Cayley graph of T. The set of vertices V is the set of all normal forms as described above. We will describe the edges of Cay(T) by first considering the edges that arise from the vertices representing the generators of T.

From Table 7.1 we see that b_1, b_2, b_3 and b_4 have at least 4 neighbouring vertices, and b_5, b_6 and b_7 have at least 5. Now for any normal form of length 2 or greater,

Generator	b_1	b_2	b_3	b_4	b_5	b_6	b_7
wb_3	(wb_3, wb_3)	(wb_3, wb_3)	(wb_3, wb_3b_3)	(wb_3, wb_3b_3)	(wb_3, wb_3b_5)	(wb_3, wb_3b_6)	(wb_3, wb_3b_7)
wb_5	(wb_5, wb_5)	(wb_5, wb_5)	(wb_5, wb_5b_3)	(wb_5, wb_5b_3)	(wb_5, wb_5b_5)	(wb_5, wb_5b_6)	(wb_5, wb_5b_7)
wb_6	(wb_6, wb_6)	(wb_6, wb_6)	(wb_6, wb_6b_3)	(wb_6, wb_6b_3)	(wb_6, wb_6b_5)	(wb_6, wb_6b_6)	(wb_6, wb_6b_7)
wb_7	(wb_7, wb_7)	(wb_7, wb_7)	(wb_7, wb_7b_3)	(wb_7, wb_7b_3)	(wb_7, wb_7b_5)	(wb_7, wb_7b_6)	(wb_7, wb_7b_7)

Table 7.2:

we know that the last letter is either b_3, b_5, b_6 or b_7. Therefore we let w be a word in normal form of length at least one, and consider the edges originating at vertices wb_i for $i \in \{3, 5, 6, 7\}$.

Together, Tables 7.1 and 7.2 describe all edges in Cay(T). We can see here that normal forms beginning with b_1, b_5 and b_6 are contained in a single connected component of Cay(T) as there exist edges from b_1 to b_5 and b_6, and all other edges from normal forms that begin b_1, b_5 or b_6 are to other normal forms that begin b_1, b_5 or b_6 respectively. Having listed all edges, we now know that b_1 has exactly four neighbours (of which two are b_5 and b_6), and b_5 and b_6 have exactly five neighbours (of which one is b_1). If we consider an arbitrary vertex wb_i for $i \in \{3, 5, 6, 7\}$ in this component, we see that this vertex is the child of precisely one vertex, that is, the only edges for which wb_i is a terminal vertex are those beginning at w. The vertex wb_i has precisely four child vertices, specifically wb_ib_3, wb_ib_5, wb_ib_6, and wb_ib_7. Hence if we consider this component in $\dagger(T)$ by removing all directions, loops and multiple edges from the component in Cay(T), we have a 4-ary tree rooted at the vertex b_1.

By similar arguments we can show that Cay(T) contains three more components, one containing normal forms beginning with b_2 and b_7, one containing normal forms beginning with b_3, and one containing normal forms beginning with b_4. All of these components in $\dagger(T)$ are 4-ary trees rooted at b_2, b_3 and b_4 respectively.

Hence $\dagger(T)$ is isomorphic to $\dagger(A^*)$, concluding the example.

We will now let A be of size n for some $n > 1$ and pick an m such that $1 \leq m \leq n$, a partition P of m into i parts, a partition Q of $n - i$ into j parts, and a number k such that $1 \leq k \leq j$. We will show that the semigroup S given by the presentation defined above has $\dagger(S) \cong \dagger(A^*)$. We follow a similar method to the example T, and establish normal forms for S by creating a complete rewriting system, and then examining the edges that arise in $\text{Cay}(S)$ using these normal forms as the set of vertices.

Claim 7.72

The semigroup S has normal forms

- b_g *for* $1 \leq g \leq i$
- $b_g b_h \{b_f \mid i+1 \leq f \leq i+n-m, n+1 \leq f \leq n+m\}^*$ *for*
  - $1 \leq g \leq i$,
  - $i+1 \leq h \leq i+n-m$,
  - $n+1 \leq h \leq n+p_1+\ldots+p_{g-1}$,
  - $n+p_1+\ldots+p_g+1 \leq h \leq n+m$;
- $b_g\{b_h \mid i+1 \leq h \leq i+n-m, n+1 \leq h \leq n+m\}^*$ *for* $i < g \leq n+m$.

PROOF: We first show that S does in fact have normal forms by showing it has a complete rewriting system. Using the shortlex order with $b_x < b_y$ for $x < y$ we have the following rewriting system.

$b_x b_l \to b_x$ for $1 \leq l \leq i, 1 \leq x \leq n+m$

$b_1 b_{n+l} \to b_{n+l}$ for $1 \leq l \leq p_1$

$b_2 b_{n+p_1+l} \to b_{n+p_1+l}$ for $1 \leq l \leq p_2$

...

$b_i b_{n+p_1+\ldots+p_{i-1}+l} \to b_{n+p_1+\ldots+p_{i-1}+l}$ for $1 \leq l \leq p_i$

$b_x b_{i+n-m+l} \to b_x b_{i+1}$ for $1 \leq l \leq q_1, 1 \leq x \leq n+m$

$b_x b_{i+n-m+q_1+l} \to b_x b_{i+2}$ for $1 \leq l \leq q_2, 1 \leq x \leq n+m$

...

$b_x b_{i+n-m+q_1+\ldots+q_{k-2}+l} \to b_x b_{i+k-1}$ for $1 \leq l \leq q_{k-1}, 1 \leq x \leq n+m$

$b_x b_{i+n-m+q_1+\ldots+q_{k-1}+l} \to b_x$ for $1 \leq l \leq q_k, 1 \leq x \leq n+m$

$b_x b_{i+n-m+q_1+\ldots+q_k+l} \to b_x b_{i+k}$ for $1 \leq l \leq q_{k+1}, 1 \leq x \leq n+m$

$b_x b_{i+n-m+q_1+\ldots+q_{j-1}+l} \to b_x b_{i+j}$ for $1 \leq l \leq q_j, 1 \leq x \leq n+m$

We will show that this system is locally confluent by considering possible overlaps of words when rewriting. We will break the rules down into five categories:

(i) $b_x b_l \to b_l$ for $1 \leq l \leq i, 1 \leq x \leq n+m$

(ii) $b_y b_g \to b_g$ for $1 \leq y \leq i, g = n + p_1 + \ldots + p_f + e$ where $1 \leq f \leq i-1$ and $1 \leq e \leq p_{f+1}$

(iii) $b_x b_r \to b_x b_{i+u+1}$ for $1 \leq x \leq n+m, r = i+n-m+q_1+\ldots+q_u+v$ where $1 \leq v \leq q_{u+1}$ and $1 \leq u \leq k-2$

(iv) $b_x b_w \to b_x$ for $1 \leq x \leq n+m, r = i+n-m+q_1+\ldots+q_{k-1}+v$ where $1 \leq v \leq q_{k+1}$

(v) $b_x b_t \to b_x b_{i+u}$ for $1 \leq x \leq n+m, t = i+n-m+q_1+\ldots+q_u+v$ where $1 \leq v \leq q_{u+1}$ and $k \leq u \leq j-1$

Save for those of type (ii), all rewrite rules have any generator as the first letter of the left-hand side, we can overlap all those rules. Rules of type (ii) may overlap its second letter with the first letter of any rule except those of type two. They may also overlap their first letter with the second letter of rules of type (i).

The overlaps with rules of type (i) at the front are:

(i)
$$\begin{tikzcd}
& b_lb_{x_2} \arrow[r] & b_{x_2} \arrow[dd, "\equiv"'] \\
b_{x_1}b_lb_{x_2} \arrow[ur] \arrow[dr] & & \\
& b_{x_1}b_{x_2} \arrow[r] & b_{x_2}
\end{tikzcd}$$

(ii)
$$\begin{tikzcd}
& b_lb_g \arrow[dd, "\equiv"'] \\
b_xb_lb_g \arrow[ur] \arrow[dr] & \\
& b_xb_g
\end{tikzcd}$$

(iii)
$$\begin{tikzcd}
& b_lb_r \arrow[r] & b_lb_{i+u+1} \arrow[dd, "\equiv"'] \\
b_xb_lb_r \arrow[ur] \arrow[dr] & & \\
& b_xb_lb_{i+u+1} \arrow[r] & b_lb_{i+u+1}
\end{tikzcd}$$

(iv)
$$\begin{tikzcd}
& b_1b_w \arrow[r] & b_l \arrow[dd, "\equiv"'] \\
b_xb_lb_w \arrow[ur] \arrow[dr] & & \\
& b_xb_l \arrow[r] & b_l
\end{tikzcd}$$

(v)
$$\begin{tikzcd}
& b_lb_t \arrow[r] & b_lb_{i+u} \arrow[dd, "\equiv"'] \\
b_xb_lb_t \arrow[ur] \arrow[dr] & & \\
& b_xb_lb_{i+u} \arrow[r] & b_lb_{i+u}
\end{tikzcd}$$

Overlaps with rules of type (ii) at the front:

(i)
$$\begin{tikzcd}
& b_gb_l \arrow[r] & b_l \arrow[dd, "\equiv"'] \\
b_xb_gb_l \arrow[ur] \arrow[dr] & & \\
& b_xb_l \arrow[r] & b_l
\end{tikzcd}$$

(iii)
$$\begin{tikzcd}
& b_gb_r \arrow[r] & b_gb_{i+u+1} \arrow[dd, "\equiv"'] \\
b_xb_gb_r \arrow[ur] \arrow[dr] & & \\
& b_xb_gb_{i+u+1} \arrow[r] & b_gb_{i+u+1}
\end{tikzcd}$$

(iv)
$$\begin{tikzcd}
& b_gb_w \arrow[r] & b_g \arrow[dd, "\equiv"'] \\
b_xb_gb_w \arrow[ur] \arrow[dr] & & \\
& b_xb_g \arrow[r] & b_g
\end{tikzcd}$$

(v)
$$\begin{tikzcd}
& b_gb_t \arrow[r] & b_gb_{i+u} \arrow[dd, "\equiv"'] \\
b_xb_gb_t \arrow[ur] \arrow[dr] & & \\
& b_xb_gb_{i+u} \arrow[r] & b_gb_{i+u}
\end{tikzcd}$$

Overlaps with rules of type (iii) at the front:

(i)

$$\begin{tikzcd}
 & b_xb_{i+u+1}b_l \arrow[r] & b_xb_l \arrow[r] & b_l \arrow[ddl, "\equiv"'] \\
b_xb_rb_l \arrow[ur] \arrow[dr] & & & \\
 & b_xb_l \arrow[r] & b_l &
\end{tikzcd}$$

(iii)

$$\begin{tikzcd}
 & b_xb_{i+u_1+1}b_{r_2} \arrow[r] & b_xb_{i+u_1+1}b_{i+u_2+1} \arrow[dd, "\equiv"'] \\
b_xb_{r_1}b_{r_2} \arrow[ur] \arrow[dr] & & \\
 & b_xb_{r_1}b_{i+u_2+1} \arrow[r] & b_xb_{i+u_1+1}b_{i+u_2+1}
\end{tikzcd}$$

(iv)

$$\begin{tikzcd}
 & b_xb_{i+u+1}b_w \arrow[r] & b_xb_{i+u+1} \arrow[dd, "\equiv"'] \\
b_xb_rb_w \arrow[ur] \arrow[dr] & & \\
 & b_xb_r \arrow[r] & b_xb_{i+u+1}
\end{tikzcd}$$

(v)

$$\begin{tikzcd}
 & b_xb_{i+u_1+1}b_t \arrow[r] & b_xb_{i+u_1+1}b_{i+u_2} \arrow[dd, "\equiv"'] \\
b_xb_rb_t \arrow[ur] \arrow[dr] & & \\
 & b_xb_rb_{i+u_2} \arrow[r] & b_xb_{i+u_1+1}b_{i+u_2}
\end{tikzcd}$$

Overlaps with rules of type (iv) at the front:

(i)

$$\begin{tikzcd}
 & b_xb_l \arrow[r] & b_l \arrow[dd, "\equiv"'] \\
b_xb_wb_l \arrow[ur] \arrow[dr] & & \\
 & b_xb_l \arrow[r] & b_l
\end{tikzcd}$$

(iii)

$$\begin{tikzcd}
 & b_xb_r \arrow[r] & b_xb_{i+u+1} \arrow[dd, "\equiv"'] \\
b_xb_wb_r \arrow[ur] \arrow[dr] & & \\
 & b_xb_wb_{i+u+1} \arrow[r] & b_xb_{i+u+1}
\end{tikzcd}$$

(iv)

$$\begin{tikzcd}
 & b_xb_{w_2} \arrow[r] & b_x \arrow[dd, "\equiv"'] \\
b_xb_{w_1}b_{w_2} \arrow[ur] \arrow[dr] & & \\
 & b_xb_{w_1} \arrow[r] & b_x
\end{tikzcd}$$

(v)

$$\begin{tikzcd}
 & b_xb_t \arrow[r] & b_xb_{i+u} \arrow[dd, "\equiv"'] \\
b_xb_wb_t \arrow[ur] \arrow[dr] & & \\
 & b_xb_wb_{i+u} \arrow[r] & b_xb_{i+u}
\end{tikzcd}$$

Overlaps with rules of type (v) at the front:

(i)

$$\begin{tikzcd}
 & b_xb_{i+u}b_l \arrow[r] & b_xb_l \arrow[r] & b_l \arrow[ddl, "\equiv"'] \\
b_xb_tb_l \arrow[ur] \arrow[dr] & & & \\
 & b_xb_l \arrow[r] & b_l &
\end{tikzcd}$$

(iii)

$$\begin{tikzcd}
 & b_xb_{i+u_1}b_r \arrow[r] & b_xb_{i+u_1}b_{i+u_2+1} \arrow[dd, "\equiv"'] \\
b_xb_tb_r \arrow[ur] \arrow[dr] & & \\
 & b_xb_tb_{i+u_2+1} \arrow[r] & b_xb_{i+u_1}b_{i+u_2+1}
\end{tikzcd}$$

(iv)

$$\begin{tikzcd}
 & b_xb_{i+u}b_w \arrow[r] & b_xb_{i+u} \arrow[dd, "\equiv"'] \\
b_xb_tb_w \arrow[ur] \arrow[dr] & & \\
 & b_xb_t \arrow[r] & b_xb_{i+u}
\end{tikzcd}$$

(v)

$$\begin{tikzcd}
 & b_xb_{i+u_1}b_{t_2} \arrow[r] & b_xb_{i+u_1}b_{i+u_2} \arrow[dd, "\equiv"'] \\
b_xb_{t_1}b_{t_2} \arrow[ur] \arrow[dr] & & \\
 & b_xb_{t_1}b_{i+u_2} \arrow[r] & b_xb_{i+u_1}b_{i+u_2}
\end{tikzcd}$$

These are all our overlaps, and in each case we always arrive at the same word, so we have local confluence. Now if we consider left and right-hand sides of rules, we can see that under the order we have imposed, the right-hand side is always shorter under the shortlex ordering, and so this system is Noetherian. Hence this is a complete rewriting system.

In order to show that what we have given are actually normal forms, we check that they are irreducible. First suppose we have a word of the form b_g for $1 \leq g \leq i$. This has length one and so cannot be rewritten using any rules.

Suppose we have a word w in the form $b_g b_h \{b_f \mid i+1 \leq f \leq i+n-m, n+1 \leq f \leq n+m\}^*$ for $1 \leq g \leq i, i+1 \leq h \leq i+n-m, n+1 \leq h \leq n+p_1+\ldots+p_{g-1}, n+p_1+\ldots+p_g+1 \leq h \leq n+m$. We cannot apply rewrite rules of type (i) as these rule require a letter b_y for $1 \leq y \leq i$ to appear after some other letter which is not possible in words of this form. For rules of type (ii), we observe that the only place we could apply these is to the first two letters of w because this is the only place we find find a letter b_y for $1 \leq y \leq i$. Now for a given b_y with $1 \leq y \leq i$ we can only apply a rule of type (ii) if b_y is followed by a letter b_z where $n+p_1+\ldots+p_{y-1}+1 \leq z \leq n+p_1+\ldots+p_y$, however these letters are not found in any normal form of this type beginning with our chosen b_y. Finally to apply rules of type (iii), (iv) and (v) we require a letter b_y for $i+n-m+1 \leq y \leq n$, and none of these appear in our word w. Thus w is irreducible.

Suppose then we have a word in the form $b_l\{b_h \mid i+1 \leq h \leq i+n-m$ or $n+1 \leq h \leq n+m\}^*$ for $i < l \leq n+m$. We can never apply rewrite rules of types (i) or (ii) to this word as it does not feature any letters b_y for $1 \leq y \leq i$. The remaining rule types are also not applicable, as they require a letter b_y for $i+n-m+1 \leq y \leq i+n-m+m-i$ to follow some letter which is not possible given the restrictions on h. Therefore, all our suggested normal forms are irreducible.

Finally, suppose we have some element $w \in B^*$ which is in normal form but not listed above. The element w must have at least length two since all elements of length one are covered by our normal forms. Suppose then that w begins with b_g for $1 \leq g \leq i$, then either w has b_y where $1 \leq y \leq i, i+n-m+1 \leq y \leq n$, or $n+p_1+\ldots+p_{g-1}+1 \leq y \leq n+p_1+\ldots+p_g$ as a second letter or b_y where $1 \leq y \leq i$ or $i+n-m+1 \leq y \leq n$ as some letter that is in the third or later place.

Suppose that w has b_y where $1 \leq y \leq i$, $i+n-m+1 \leq y \leq n$, or $n+p_1+\ldots+p_{g-1}+1 \leq y \leq n+p_1+\ldots+p_g$ as a second letter then if y is the first,

$1 \leq y \leq i$, then we may apply a rewrite rule of type (i). If y in the second interval, $i+n-m+1 \leq y \leq n$, then we may apply a rule of type (iii), (iv) or (v). Finally, if y is in the interval $n+p_1+\ldots+p_{g-1}+1 \leq y \leq n+p_1+\ldots+p_g$ then we can apply a rule of type (ii).

Therefore we suppose w has b_y where $1 \leq y \leq i$ or $i+n-m+1 \leq y \leq n$ as some letter that is in the third or later place. If y is in the first interval then we can apply a rule of type (i). For y in the second interval then we can apply rules of type (iii), (iv) or (v).

Hence w must begin with b_g for $i+1 \leq g \leq n+m$ and contain some letter b_y where $1 \leq y \leq i$ or $i+n-m < y \leq n$. For the former case, we apply rules of type (i). For y in the latter case, we apply rules of type (iii), (iv) or (v). Hence w can be rewritten and is reducible, so is not a normal form.

□

Claim 7.73

For a given m, P, Q and q_k we have $\dagger\,(S[m,P,Q,q_k],B) \cong \dagger\,(A^+,A)$.

Proof: We may label all the vertices in Cay$(S[m,P,Q,q_k],B)$ by the normal forms given in Claim 7.72. Consider a vertex b_x for $i < x \leq n+m$. We will use the relations to determine what edges occur within the graph. The relations can be classified into the following classes:

(i) $b_x b_l = b_l$ for $1 \leq l \leq i, 1 \leq x \leq n+m$

(ii) $b_y b_g = b_g$ for $1 \leq y \leq i, g = n+p_1+\ldots+p_f+e$ where $1 \leq f \leq i-1$ and $1 \leq e \leq p_{f+1}$

(iii) $b_1 b_r = b_x b_{i+u+1}$ for $r = i+n-m+q_1+\ldots+q_u+v$ where $1 \leq v \leq q_{u+1}$ and $1 \leq u \leq k-2$

(iv) $b_1 b_w = b_x$ for $r = i+n-m+q_1+\ldots+q_{k-1}+v$ where $1 \leq v \leq q_{k+1}$

(v) $b_1 b_t = b_x b_{i+u}$ for $t = i+n-m+q_1+\ldots+q_u+v$ where $1 \leq v \leq q_{u+1}$ and $k \leq u \leq j-1$

Generator Type	Parameter
b_α	$1 \leq \alpha \leq i$
b_β	$i+1 \leq \beta \leq i+n-m$
b_γ	$i+n-m+1 \leq \gamma \leq i+n-m+q_1+\ldots+q_{k-1}$
b_δ	$i+n-m+q_1+\ldots+q_{k-1}+1 \leq \delta \leq i+n-m+q_1+\ldots+q_k$
b_ε	$i+n-m+q_1+\ldots+q_k+1 \leq \varepsilon \leq i+n-m+q_1+\ldots+q_j = n$
b_ζ	$n+1 \leq \zeta \leq n+m$

Table 7.3:

Generator	b_α	b_β	b_γ	b_δ	b_ε	b_ζ
b_α	(b_α, b_α)	$(b_\alpha, b_\alpha b_\beta)$	$(b_\alpha, b_\alpha b_{i+u+1})$	(b_α, b_α)	$(b_\alpha, b_\alpha b_{i+u})$	(b_α, b_ζ) $(b_\alpha, b_\alpha b_\zeta)$
b_β	(b_β, b_β)	$(b_\beta, b_\beta b_\beta)$	$(b_\beta, b_\beta b_{i+u+1})$	(b_β, b_β)	$(b_\beta, b_\beta b_{i+u})$	$(b_\beta, b_\beta b_\zeta)$
b_γ	(b_γ, b_γ)	$(b_\gamma, b_\gamma b_\beta)$	$(b_\gamma, b_\gamma b_{i+u+1})$	(b_γ, b_γ)	$(b_\gamma, b_\gamma b_{i+u})$	$(b_\gamma, b_\gamma b_\zeta)$
b_δ	(b_δ, b_δ)	$(b_\delta, b_\delta b_\beta)$	$(b_\delta, b_\delta b_{i+u+1})$	(b_δ, b_δ)	$(b_\delta, b_\delta b_{i+u})$	$(b_\delta, b_\delta b_\zeta)$
b_ε	$(b_\varepsilon, b_\varepsilon)$	$(b_\varepsilon, b_\varepsilon b_\beta)$	$(b_\varepsilon, b_\varepsilon b_{i+u+1})$	$(b_\varepsilon, b_\varepsilon)$	$(b_\varepsilon, b_\varepsilon b_{i+u})$	$(b_\varepsilon, b_\varepsilon b_\zeta)$
b_ζ	(b_ζ, b_ζ)	$(b_\zeta, b_\zeta b_\beta)$	$(b_\zeta, b_\zeta b_{i+u+1})$	(b_ζ, b_ζ)	$(b_\zeta, b_\zeta b_{i+u})$	$(b_\zeta, b_\zeta b_\zeta)$

Table 7.4:

We will first consider the graph Cay$(S[m,P,Q,q_k],B)$. We will categorise the generators of $S[m,P,Q,q_k]$ in to useful types, and examine the edges that arise from these. From this we can then deduce the edges arising from all normal forms, and hence we can see where multiple edges and loops occur, and thus understand the edges in † $(S[m,P,Q,q_k],B)$.

Edges leaving each type of generator are found by multiplying each type by all other types and applying appropriate relations. For example, a vertex of type b_α has an edge labelled b_α forming a loop on it, due to the relations of type (i). If we multiply a vertex of type b_α by a generator of type b_ζ, then depending on whether these particular generators appear in a relation of type (ii) or not, we have either an edge (b_α, b_ζ) or an edge $(b_\alpha, b_\alpha b_\zeta)$.

Now Table 7.4 allows us to see the type of edges leaving each generator vertex in

Generator	b_α	b_β	b_γ	b_δ	b_ε	b_ζ
wb_β	(wb_β, wb_β)	$(wb_\beta, wb_\beta b_\beta)$	$(wb_\beta, wb_\beta b_{i+u+1})$	(wb_β, wb_β)	$(wb_\beta, wb_\beta b_{i+u})$	$(wb_\beta, wb_\beta b_\zeta)$
wb_ζ	(wb_ζ, wb_ζ)	$(wb_\zeta, wb_\zeta b_\beta)$	$(wb_\zeta, wb_\zeta b_{i+u+1})$	(wb_ζ, wb_ζ)	$(wb_\zeta, wb_\zeta b_{i+u})$	$(wb_\zeta, wb_\zeta b_\zeta)$

Table 7.5:

Cay$(S[m, P, Q, q_k], B)$. In particular, we see that columns b_α and b_δ always give rise to loop type edges, and columns b_γ and b_ε give multiple edges - more specifically, they have the same initial and terminal vertices as edges in column b_β.

We would now like to see which edges arise from non-generator vertices. Given our normal forms from Claim 7.72, we see that a normal form of length greater than or equal to two always ends in a generator of type b_β or b_ζ. Since all relations have left hand side length of at most two, we may view our normal forms of length at least two in two different ways. We let w be a word in normal form (including those of length one), then all normal forms of length at least two may be written as either wb_β or wb_ζ. The edges that these vertices give rise to are elaborated on in Table 7.5, which follows much the same reasoning as Table 7.4.

We see that the generators of type b_α and b_δ again give rise to loops, and generators of b_γ and b_ε give multiple edges with the same initial and terminal vertices as edges labelled b_β. This allows us to count the outdegree of all vertices in $\dagger\,(S)$, by ignoring the generators that result in loops or multiple edges in Cay(S). This is given by counting the number of edges with labels of type b_β and b_ζ, that is

$$n - m + m = n.$$

Hence each vertex in $\dagger\,(S[m, P, Q, q_k], B)$ has outdegree n.

We then consider the indegree of each vertex in $\dagger\,(S)$. We can see that for those vertices in Cay(S) which correspond to normal forms of length at least two, if we ignore multiple edges and loops, they appear only once as a terminal vertex, and so have indegree one. For normal forms of length precisely one, we see that those of type b_α, b_β, b_γ , b_δ and b_ε never appear as a terminal vertex, provided we ignore loops, and so these vertices have indegree zero. Those of type b_ζ do appear as terminal vertices, but appear precisely once, and so have indegree one.

Suppose then that we have a vertex corresponding to a normal form b_1, which is of type b_α. This has exactly n child vertices, which are all those of the form b_1b_β, b_ζ for $n+1 \leq \zeta \leq n+p_1$ and b_1b_ζ for $n+p_1+1 \leq \zeta \leq n+m$. For those of type b_1b_β and b_1b_ζ, each has n distinct child vertices, and there are no edges between vertices of the same generation, as we see from Table 7.5 that the non-loop edges are never between words of the same length. Similarly, for the vertices of type b_ζ, each has n distinct child vertices and no non-loop edges within the same generation via Table 7.4. Every vertex in this third level now has n child vertices, which from Table 7.5 we see are all unique, and are not connected to each other by edges. Via a recursive process, we find that we have in fact an n-ary rooted tree, rooted at b_1.

This argument applies analogously to any vertex of type b_α, and a simplified argument applies to all vertices of type b_β, b_γ, b_δ and b_ε as on each level the length of normal form increases by one and we need not worry about vertices of type b_ζ here. Thus for each vertex of these types, we have an n-ary rooted tree forming $\dagger\,(S)$, and all vertices of type b_ζ are found within the trees rooted at b_α vertices.

Now we may count the number of vertices of type b_α, b_β, b_γ, b_δ and b_ε, which is simply $n+m-m=n$. Thus $\dagger\,(S[m,P,Q,q_k],B)$ has the form of n n-ary rooted trees, and is hence isomorphic to $\dagger\,(A^+,A)$.

$\square$

7.3.1 An Example

Let $A=\{a,b,c,d\}$ and let A^+ be the free semigroup generated by A. We will now look at how many different semigroups B_i we can construct such that $\dagger\,(A^+) \cong \dagger\,(B_i)$.

First we try adding a single generator e. There is only one possible configuration for this, which is shown in Figure 7.5.

Adding two new generators, e and f, gives rise to three possible semigroups. Since we have added two new generators, we have two possible partitions for these; into 2 and $1+1$. The partition 2 results in semigroups B_2 (Figure 7.6) and B_3 (Figure 7.7).

The partition of $1+1$ gives us one single semigroup, displayed in Figure 7.8

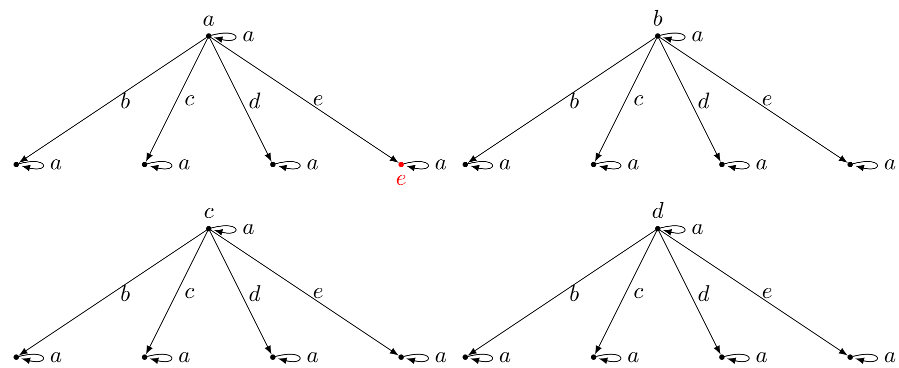


Figure 7.5: B_1

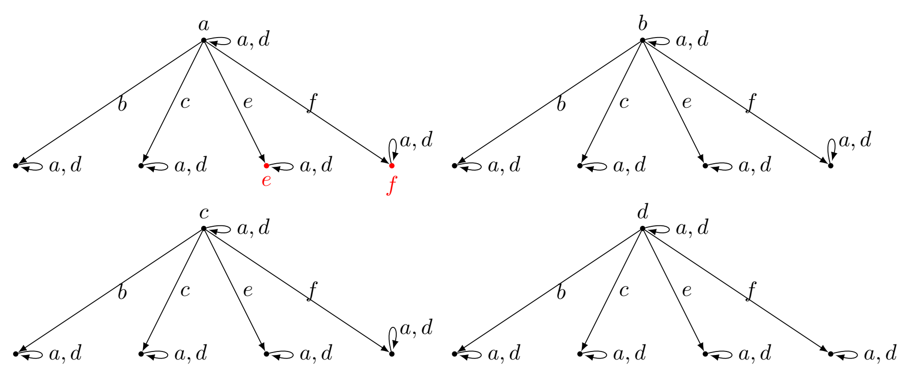


Figure 7.6: B_2

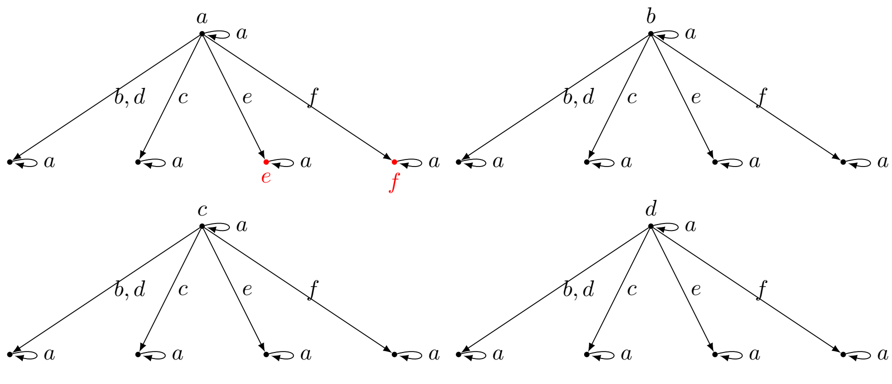


Figure 7.7: B_3

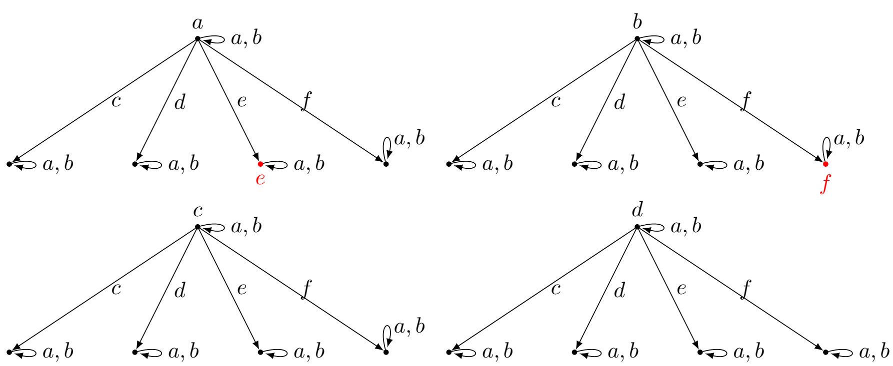


Figure 7.8: B_4

These are all possible semigroups for a generating set of size $4 + 2$. We then add another generator, g, giving us $m = 3$, and $p(3) = 3$. We can calculate U for each of these partitions of 3. Let $P_1 = 3$, $P_2 = 2 + 1$ and $P_3 = 1 + 1 + 1$. For P_1, we have $i = 1$, so $n - i = 3$. There is exactly one partition of of 3 into $n - m + 1 = 2$ parts, giving us an α value of 1. This partition is $3 = 2+1$, which has two unique parts giving an r_x value of 2. Similarly there is exactly one partition of 2 in to $n - m = 1$ part, and so $\beta = 1$. Hence

$$\begin{aligned} U_{P_1} &= \sum_{x=1}^{1} r_x + 1 \\ &= 2 + 1 \\ &= 3. \end{aligned}$$

For partition P_2, we have $i = 2$ and so $n - i = 2$. We can partition 2 into $n - m + 1 = 2$ in one way, that is $2 = 1 + 1$, which has only one unique part. Similarly there is only one partition of 2 into $n - m = 1$ part, so $\beta = 1$.

$$\begin{aligned} U_{P_2} &= \sum_{x=1}^{1} r_x + 1 \\ &= 1 + 1 \\ &= 2. \end{aligned}$$

Finally for P_3, we have $n - i = 1$ and so there are no partitions of 1 into $n - m + 1 = 2$ parts, giving $\alpha = 0$. There is precisely one partition of 1 into $n - m = 1$ parts, so $\beta = 1$.

$$\begin{aligned} U_{P_3} &= \sum_{x=1}^{0} r_x + 1 \\ &= 0 + 1 \\ &= 1. \end{aligned}$$

Thus with the addition of three generators, we find $U_{P_1} + U_{P_2} + U_{P_3} = 3 + 2 + 1 = 6$ new semigroups.

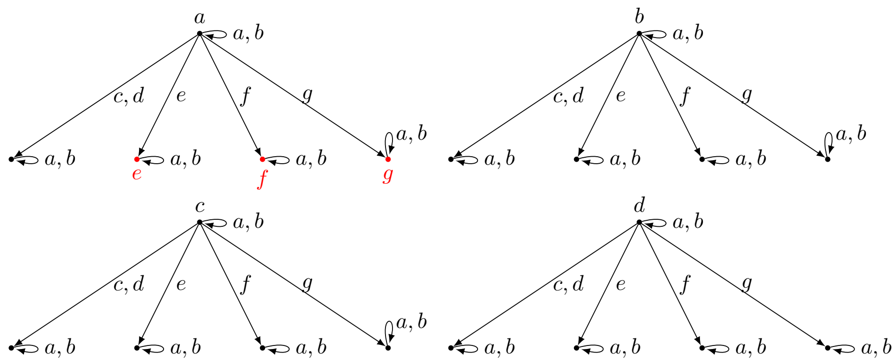


Figure 7.9: B_5

A semigroup of type P_1 is shown in Figure 7.9 and one of type P_2 is shown in Figure 7.10. The rest may be found in the appendix A.1.

Finally, we add a fourth generator h, giving $m = 4$. For each partition of 4, the U value is 1, and so since $p(4) = 5$, there are five new semigroups. One of these is displayed in Figure 7.11, the rest can be found in appendix A.2.

These are all the semigroups B_i such that $\dagger(B_i) \cong \dagger(A^*)$ for an alphabet of size four. According to our formula, we get a sum of

$$\begin{aligned}
\#S &= \sum_{m=1}^{4} \sum_{y=1}^{p(m)} U_y \\
&= \sum_{y=1}^{p(1)} U_{P_y} + \sum_{y=1}^{p(2)} U_{P_y} + \sum_{y=1}^{p(3)} U_{P_y} + \sum_{y=1}^{p(4)} U_{P_y} \\
&= (1) + (2+1) + (3+2+1) + (1+1+1+1+1) \\
&= 15.
\end{aligned}$$

This matches the number of semigroups which we have demonstrated.

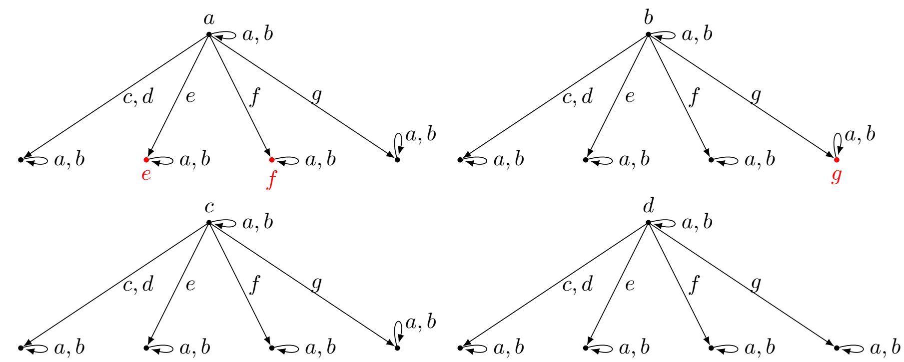


Figure 7.10: B_6

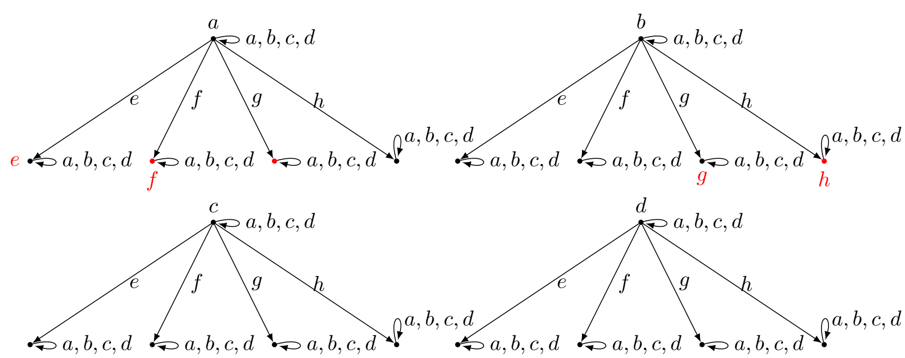


Figure 7.11: B_7

7.4 The Integers

When we considered the natural numbers, we looked at a one-ended infinite line. Why not, then, look at a two-ended infinite line, which, at first glance represents the integers $\mathbb{Z}$. We will prove the following theorem.

Theorem 7.74

The spectrum of the integers $\mathbb{Z} = \text{sgp}\langle 1, -1\rangle$ *is* $\mathcal{C}(\mathbb{Z}) = \{\mathbb{Z}, C_2 \star C_2, T_1, T_2, T_3, T_4, T_5\}$, *where*

$$T_1 = \text{sgp}\langle a, b, c \mid aa = ac, ab = a, ba = a, bb = b, bc = c, cc = ca, cb = c\rangle$$
$$T_2 = \text{sgp}\langle a, b, c \mid aa = a, ab = a, ba = a, bb = b, bc = c, cb = c, cc = c\rangle$$
$$T_3 = \text{sgp}\langle a, b, c \mid ab = a, ac = a, ba = a, bb = c, bc = b, cb = c, cc = c\rangle$$
$$T_4 = \text{sgp}\langle a, b, c \mid ab = a, ac = a, ba = a, bb = b, bc = c, cb = c, cc = c\rangle$$
$$T_5 = \text{sgp}\langle a, b \mid ab = a, ba = a, bbb = bb\rangle,$$

and $C_2 \star C_2$ *is the monoid free product of two copies of the cyclic group of order* 2.

Suppose we have a semigroup S generated by $B = \{b_1, b_2, \ldots, b_n\}$ such that $\dagger(S, B) \cong \dagger(\mathbb{Z}, \{1, -1\})$. As with $\mathbb{N}$, we shall speak about the graph as being oriented horizontally, so vertices may be described as being left or right of each other. Let b_1 and b_n be the left and right-most generators respectively. Figure 7.12 gives a visualisation of this.

Figure 7.12: Visualisation of $\dagger(S)$

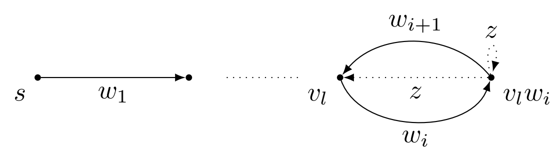
We will first establish a claim that will be useful throughout.

Claim 7.75

Let $w = w_1 \ldots w_k$ *be any word over the generators of* S *and let* s *be any element of* S. *Let the sequence of vertices* $v_1 \ldots v_j$ *be the path labelled by* w *starting from* s. *Suppose* *that we encounter some vertex twice. Let v_l be the first vertex that is encountered twice. Then the edges leaving the vertex $v_{l+1} = v_l w_i$ are either loops or terminate at v_l. In particular edges that are not loops leaving $v_l w_i$ go either left or right, but not both.*

Proof: Since $\dagger(S)$ is a straight line graph, we know that for any $1 \leq i \leq k$ we have either $w_1 \dots w_i z = w_1 \dots w_{i-1}$, $w_1 \dots w_i z = w_1 \dots w_i$ or $w_1 \dots w_i z = w_1 \dots w_{i+1}$.

Now suppose that we have a vertex s such that the path labelled w encounters a repeat vertex, so $sw_1 \dots w_{i-1} = sw_1 \dots w_{i+1}$ for some $1 \leq i \leq k$. We will orient our diagram with s to the left and $sw_1 \dots w_i$ to the right, but this applies equally to any orientation of these vertices.



Now any generator z must label one of the dotted edges in the diagram, and so there are no generators left to label any edges going right from $sw_1 \dots w_i$. □

We will prove that only $\mathbb{Z}, C_2 \star C_2$ and T_i for $1 \leq i \leq 5$ have this skeleton by examining three cases for S. These cases are based on the existence of paths between the two extremal generators b_1 and b_n, and contain several subcases. The case structure is outlined as follows.

1 There exists a path from b_1 to b_n and from b_n to b_1.

- 1.1 $b_1 b_n$ is equal to b_1
  - 1.1.1 $b_1 b_1$ is equal to b_1
  - 1.1.2 $b_1 b_1$ is right of b_1
  - 1.1.3 $b_1 b_1$ is left of b_1
- 1.2 $b_1 b_n$ is right of b_1
  - 1.2.1 $b_1 b_1$ is equal to b_1
  - 1.2.2 $b_1 b_1$ is right of b_1

1.2.3 b_1b_1 is left of b_1

1.3 b_1b_n is left of b_1

1.3.1 b_1b_1 is equal to b_1

1.3.2 b_1b_1 is right of b_1

1.3.3 b_1b_1 is left of b_1

2 There exists a path from b_n to b_1 but no path from b_1 to b_n

2.1 There exists an edge going right from b_1

2.2 The exists no edge going right from b_1

2.2.1 b_1b_1 equals b_1 and b_1b_n is left of b_1

2.2.2 b_1b_1 is left of b_1 and b_1b_n equals b_1

2.2.3 b_1b_1 and b_1b_n are both left of b_1

3 There exists a path from b_1 to b_n but no path from b_n to b_1

4 There exists no path from b_1 to b_n and no path from b_n to b_1

4.1 There exists an edge going left from b_n and an edge right from b_1.

4.2 There does not exist an edge going left from b_n but there exists an edge right from b_1.

4.3 There exists an edge going left from b_n but there does not exist an edge right from b_1.

4.4 There exists neither an edge going left from b_n nor an edge right from b_1, and we consider b_m for some $1 < m < n$ (see Claim 7.76).

4.4.1 b_1b_1 is left of b_1 and b_1b_m equals b_1

4.4.1.1 b_nb_n is right of b_n and b_nb_m equals b_n

4.4.1.2 b_nb_n equals b_n and b_nb_m is right of b_n

4.4.1.3 b_nb_n equals b_n and b_nb_m equals b_n

4.4.1.4 b_nb_n is right of b_n and b_nb_m is right of b_n

4.4.2 b_1b_1 equals b_1 and b_1b_m is left of b_1

4.4.2.1 $b_n b_n$ is right of b_n and $b_n b_m$ equals b_n

4.4.2.2 $b_n b_n$ equals b_n and $b_n b_m$ is right of b_n

4.4.2.3 $b_n b_n$ equals b_n and $b_n b_m$ equals b_n

4.4.2.4 $b_n b_n$ is right of b_n and $b_n b_m$ is right of b_n

4.4.3 $b_1 b_1$ equals b_1 and $b_1 b_m$ equals b_1

4.4.3.1 $b_n b_n$ is right of b_n and $b_n b_m$ equals b_n

4.4.3.2 $b_n b_n$ equals b_n and $b_n b_m$ is right of b_n

4.4.3.3 $b_n b_n$ equals b_n and $b_n b_m$ equals b_n

4.4.3.4 $b_n b_n$ is right of b_n and $b_n b_m$ is right of b_n

4.4.4 $b_1 b_1$ if left of b_1 and $b_1 b_m$ is left of b_1

4.4.4.1 $b_n b_n$ is right of b_n and $b_n b_m$ equals b_n

4.4.4.2 $b_n b_n$ equals b_n and $b_n b_m$ is right of b_n

4.4.4.3 $b_n b_n$ equals b_n and $b_n b_m$ equals b_n

4.4.4.4 $b_n b_n$ is right of b_n and $b_n b_m$ is right of b_n

Claim 7.76

In case 4 there exists at least one generator b_m where $1 < m < n$.

PROOF: Since there are no edges right (respectively left) from b_1 (respectively b_n), we know there must exist some generator b_m for $1 < m < n$ such that $b_m w = b_1$ for some word w and $b_m v = b_n$ for some v otherwise the Cayley graph of S would be disconnected. $\square$

We will show that the only possible constructions for a Cayley graph with this skeleton are the aforementioned semigroups. The general method we will use is to establish certain equalities in a given case, for example $b_1 = b_n w$, and then starting at specific vertex we will follow the paths labelled by both sides of this equality. This will either allow us to deduce the location of more edges in the graph, or lead us to a contradiction. If we write for example $b_1 \cdot b_n w$, this indicates that we are starting at vertex b_1 and following the path labelled by $b_n w$.

The outcomes for each case can be found in the claims outlined in the following table.

Case	Claim
1	see subcases
1.1	see subcases
1.1.1	7.78
1.1.2	7.79
1.1.3	7.80
1.2	see subcases
1.2.1	1.2.1
1.2.2	7.82
1.2.3	7.83
1.3	see subcases
1.3.1	7.84
1.3.2	7.86
1.3.3	7.85
2	see subcases
2.1	7.87
2.2	see subcases
2.2.1	7.88
2.2.2	7.89
2.2.3	7.90
3	7.91
4.1	7.92
4.2	7.92
4.3	7.92

Case	Claim
4.4	see subcases
4	see subcases
4.4.1	see subcases
4.4.1.1	7.94
4.4.1.2	7.93
4.4.1.3	7.95
4.4.1.4	7.96
4.4.2	see subcases
4.4.2.1	7.93
4.4.2.2	7.97
4.4.2.3	7.98
4.4.2.4	7.99
4.4.3	see subcases
4.4.3.1	7.95
4.4.3.2	7.98
4.4.3.3	7.100
4.4.3.4	7.101
4.4.4	see subcases
4.4.4.1	7.96
4.4.4.2	7.99
4.4.4.3	7.101
4.4.4.4	7.102

Case (1)

There exists a path from b_1 to b_n and a path from b_n to b_1.

We first prove a claim that will be useful in the subcases of Case 1.

Claim 7.77

In Case 1, that is there exists a path from b_1 to b_n and from b_n to b_1 then $b_n = b_1w$ and $b_1 = b_nv$ for some $w, v \in B^*$.

PROOF: Let the path from b_1 to b_n be labelled by a word $w = w_1 \dots w_m$, where w is as short as possible. In the other direction let the path from b_n to b_1 be labelled by $v = v_1 \dots v_m$ similarly. Hence we have $b_n = b_1w$ and $b_1 = b_nv$. □

We will make use of the following equality to establish which semigroups arise from this construction.

$$b_1 \cdot b_n = b_1 \cdot b_1w_1 \dots w_n \tag{\#}$$

By Claim 7.75 we know that if we follow the path w from vertex b_1b_1 we do not encounter any folds or loops, so the vertex b_1b_1w is distance exactly $|w|$ from b_1b_1.

We now further subdivide this case by looking at the placement of b_1b_n. There are three possibilities:

- b_1b_n is equal to b_1;
- b_1b_n is the vertex to the right of b_1;
- b_1b_n is the vertex to the left of b_n.

Case ($b_1b_n = b_1$, 1.1)

The element b_1b_n is equal to b_1.

We now consider the placement of b_1b_1. This gives us three subcases, that is Cases 1.1.1,1.1.2 and 1.1.3.

Claim 7.78

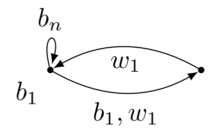
In Case 1.1.1, that is if there exists a path from b_1 to b_n and vice versa, $b_1b_n = b_1$ and $b_1b_1 = b_1$ we have that $\dagger(S, B) \ncong \dagger(\mathbb{Z}, \{1, -1\})$.

PROOF: First, if the edge leaving b_1 labelled b_1 is a loop (case 1.1.1) then $b_1b_1 = b_1$ and using the equality (#) we deduce that w must be the empty word and so $b_1 = b_n$. This would mean S is monogenic, and it is not possible to construct the graph $\dagger(S)$ from a monogenic semigroup. □

Claim 7.79

In Case 1.1.2, that is if there exists a path from b_1 to b_n and vice versa, $b_1 b_n = b_1$ and b_1 labels an edge going right from b_1 we have that $\dagger(S, B) \not\cong \dagger(\mathbb{Z}, \{1, -1\})$.

Proof: We use our equality (#) and Claim 7.75 to deduce that w must have length 1, giving us the following part of the graph.

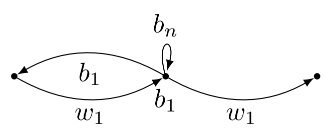


Now this means $b_n = b_1 b_1$ and so S has only two generators. If we examine the graph, we see that we have already used both of these generators to label edges from b_1, but we have not labelled an edge going left from b_1. This means $\dagger(S, B) \not\cong \dagger(\mathbb{Z}, \{1, -1\})$. □

Claim 7.80

In Case 1.1.3, that is if there exists a path from b_1 to b_n and vice versa, $b_1 b_n = b_1$ and b_1 labels an edge going left from b_1 we have that $\dagger(S, B) \not\cong \dagger(\mathbb{Z}, \{1, -1\})$.

Proof: Using equality (#) we deduce that our graph has the following shape.



We see that w only has length 1, and so S has only two generators. This is not enough generators to label all the edges that leave b_1 and so $\dagger(S, B) \not\cong \dagger(\mathbb{Z}, \{1, -1\})$.

□

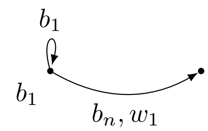
Case ($b_1 b_n$ right of b_1, 1.2)

The element $b_1 b_n$ is found to the right of b_1. We again consider which vertex represents the element $b_1 b_1$.

Claim 7.81

In Case 1.2.1, that is if there exists a path from b_1 to b_n and vice versa, b_1b_n is found to the right of b_1 and $b_1b_1 = b_1$ we have that $\dagger\,(S, B) \ncong \dagger\,(\mathbb{Z}, \{1, -1\})$.

Proof: By equality (#) and Claim 7.75 we draw a section of the Cayley graph of S.

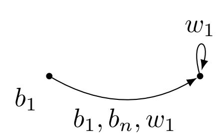


This tells us that w has length 1 and so S is only 2-generated. This does not leave any generators to label an edge left from b_1 and so $\dagger\,(S, B) \ncong \dagger\,(\mathbb{Z}, \{1, -1\})$. □

Claim 7.82

In Case 1.2.2, that is if there exists a path from b_1 to b_n and vice versa, b_1b_n is found to the right of b_1 and b_1 labels an edge going right from b_1 we have that $\dagger\,(S, B) \ncong \dagger\,(\mathbb{Z}, \{1, -1\})$.

Proof: We can deduce that $b_1b_1 = b_1b_n$. By Claim 7.75 and (#) w has length 1, and so we have the following graph.



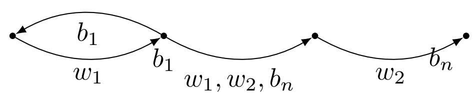
This again has S having only two generators and no label for the left edge from b_1, and hence $\dagger\,(S, B) \ncong \dagger\,(\mathbb{Z}, \{1, -1\})$. □

Finally, the most complicated option here is that b_1b_1 is found to the left of b_1.

Claim 7.83

In Case 1.2.3, that is if there exists a path from b_1 to b_n and vice versa, b_1b_n is found to the right of b_1 and b_1 labels an edge going right from b_1 we have that $S \cong \mathbb{Z}$.

Proof: From the given information we know that our graph must have the following structure.



First we show that $w_2 = b_n$. If this is not the case, then w_2 must be equal to a third generator, b_2 say, which is found at the vertex between b_1 and b_n. Then we would have $b_2 = b_1 b_n$, and $b_1 b_1 b_n = b_1 b_2 = b_2$, which would require an edge labelled b_n to leap from $b_1 b_1$ to b_2, which is not allowed. Hence $w_2 = b_n$. We then have the equality $b_n = b_1 b_n b_n$. Multiplying b_1 by this tells us that

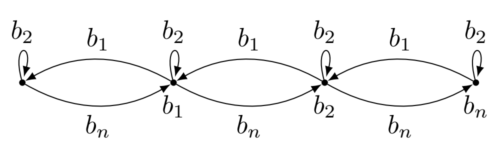
$$b_1 \cdot b_1 b_n b_n = b_1 b_n$$

and so we must have that $b_1 b_1 b_n = b_1$.

If b_2 does exist, then since $b_2 = b_1 b_n$, we can deduce $b_1 b_2 = b_1$, which gives us also that $b_1 b_1 b_2 = b_1 b_1$. We note that this means $w_1 = b_n$ as there are no more possible generators that it could be. We then see that $b_2 b_2$ does not equal b_1 or b_n as $b_1 \cdot b_2 b_2 \neq b_1 \cdot b_1$ and $b_1 \cdot b_2 b_2 \neq b_1 \cdot b_n$. Hence $b_2 b_2 = b_2$.

Now we wish to establish what happens at the vertex b_n. By multiplying b_1 by $b_1 b_n b_1$ we can see that $b_1 b_n b_1$ is not equal to either b_n or b_2. Hence $b_1 b_n b_1 = b_1$. Now since $b_n \cdot b_1 = b_n \cdot b_1 b_n b_1$, the only viable way for this to hold in the graph structure is if $b_n b_1 = b_1 b_n$. Next, we know that we cannot have $b_n b_n$ being equal to b_2 as this is contradicted by the fact that $b_1 \cdot b_2 \neq b_1 \cdot b_n b_n$. We also do not have $b_n b_n$ equal to b_n, as this would be contradicted by the fact that $b_1 \cdot b_n \neq b_1 \cdot b_n b_n$. Thus $b_n b_n$ is found to the right of b_n. Finally it can be seen that $b_n b_2 = b_n$, as any other possibility results in a contradiction.

We now have enough information to fully describe this semigroup.



We can write down a presentation for S.

$$S = \text{sgp}\langle b_1, b_2, b_n \mid b_1 b_n = b_2, b_1 b_n b_n = b_n, b_n b_1 = b_2,$$
$$b_n b_1 b_1 = b_1, b_1 b_2 = b_2, b_2 b_2 = b_2, b_n b_2 = b_2\rangle$$

This is in fact a presentation for the integers as a semigroup. If we consider $\mathbb{Z} = \text{sgp}\langle 1, -1, 0\rangle$ to be the standard generating set for the integers as a semigroup, then the map sending $b_1 \mapsto -1$, $b_2 \mapsto 0$ and $b_n \mapsto 1$ is an isomorphism. Notice that we assumed here that there were three generators. If we removed the need for this third generator, we would have a semigroup that was isomorphic to the integers presented as a monoid, using the same isomorphism. □

This completes all the cases where $b_1 b_n$ is found right of b_1.

Case ($b_1 b_n$ left of b_1,1.3)

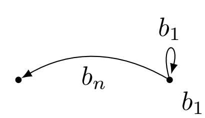
The element $b_1 b_n$ is found left of b_1.

We first have two easy cases.

Claim 7.84

In Case 1.3.1, that is if there exists a path from b_1 to b_n and vice versa, $b_1 b_n$ is found left of b_1 and $b_1 b_1 = b_1$ we have that $\dagger\,(S, B) \ncong \dagger\,(\mathbb{Z}, \{1, -1\})$.

Proof: From the information we have we know that our graph contains the following section.

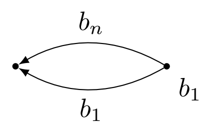


Since w is not allowed to fold, the only way to have equality (#) $b_1 b_n = b_1 b_1 w$ hold would be for w_1 to label an edge going left from b_1, but by definition w_1 goes right. Hence $\dagger\,(S, B) \ncong \dagger\,(\mathbb{Z}, \{1, -1\})$. □

Claim 7.85

In Case 1.3.3, that is if there exists a path from b_1 to b_n and vice versa, $b_1 b_n$ is found left of b_1 and $b_1 b_1 = b_1 b_n$ we have that $\dagger\,(S, B) \ncong \dagger\,(\mathbb{Z}, \{1, -1\})$.

Proof: From the relations defined so far we have the following graph.

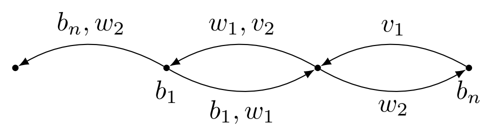


There is no way to achieve $b_1 b_n = b_1 b_1 w$ without folding in w here, so $\dagger\,(S, B) \ncong \dagger\,(\mathbb{Z}, \{1, -1\})$. $\square$

Claim 7.86

In Case 1.3.2, that is if there exists a path from b_1 to b_n and vice versa, $b_1 b_n$ is found left of b_1 and $b_1 b_1$ right of b_1 we have that $S \cong C_2 \star C_2$.

Proof: Due to equality (#), we know that w has length 2, and thus picture the graph as follows.



We will assume that there are three generators here: if there are only two, the same argument holds, ignoring any statements about the third generator b_2. We show that $w_1 = b_1$. Suppose not, then we must have $w_1 = b_2$. Now since $b_2 = b_1 b_1$ then $b_1 \cdot b_1 b_1 = b_1 \cdot b_2 = b_2$. This implies that $b_2 b_1 = b_2$. However, $b_2 \cdot b_2 = b_1$ and $b_2 \cdot b_2 b_1 = b_2$, which is a contradiction, and so $w_1 = b_1$.

We then show $w_2 = b_n$. Suppose otherwise, then $w_2 = b_2 = b_1 b_1$. However $b_1 \cdot b_1 b_1 = b_1$ and $b_1 \cdot b_2 = b_1 b_n$, so we must have made an incorrect assumption. Therefore $w_2 = b_2$.

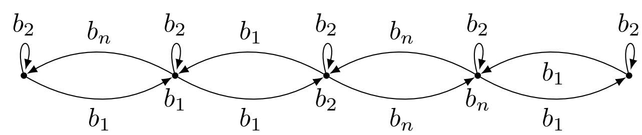
We can also also see that $b_2 b_2 \neq b_1$ and $b_2 b_2 \neq b_n$, as if the former were an equality, we would have $b_1 b_2 = b_1$, but then $b_1 b_2 b_2 \neq b_1 b_1$. The latter follows similarly and we are left with $b_2 b_2 = b_2$.

Considering the product $b_1 b_2$, we can see that if $b_1 b_2 = b_2$, then $b_1 b_2 = b_1 b_1$. However, $b_1 \cdot b_1 b_1 = b_1 \neq b_2 = b_1 \cdot b_1 b_2$. If, then, $b_1 b_2 = b_1 b_n$, we can look at $b_1 \cdot b_1 b_2 = b_2$, but $b_1 \cdot b_1 b_n = b_n$, and thus we can deduce that $b_1 b_2 = b_2$.

We now wish to understand the edge v_1. Suppose that $v_1 = b_1$, then $b_n \cdot b_1 b_1 = b_1$, which should be equal to $b_n \cdot b_2$, but this would require an edge leaping from b_n to b_1 which is not allowed. In a similar manner, $v_1 = b_2$ is also disallowed as $b_n \cdot b_2 b_1 = b_1$, but we cannot have $b_n b_1 = b_1$ as it should in this scenario. Hence, $v_1 = b_n$.

Next, we establish the edge b_1 leaving b_n. We know already that $b_n b_1 \neq b_2$. If $b_n b_1 = b_n$, then we should have $b_n b_n b_1 = b_n b_n$, but we do not, so we are left with only one option, that $b_n b_1$ is found to the right of b_n.

Finally, to find where $b_n b_2$ lies, we rule out $b_n b_2 = b_n b_1$ as this would require $b_n b_n b_2 = b_n b_n b_1$, which is not the case. We already established that $b_n b_2 \neq b_2$, and so we must have $b_n b_2 = b_2$. This set of relations defines the whole graph, as shown here.



This semigroup has the following presentation.

$$S = \text{sgp}\langle b_1, b_2, b_n \mid b_1 b_1 = b_n b_n = b_2, b_i b_2 = b_i \text{ for } i = 1, 2, n\rangle$$

This is the free product of two copies of the cyclic group of order 2, $C_2 \star C_2$, presented as a semigroup. A standard semigroup presentation for $C_2 \star C_2$ is $C_2 \star C_2 = \text{sgp}\langle a, b, 1 \mid a^2 = 1 = b^2, a1 = 1a = a, b1 = 1b = b, 1^2 = 1\rangle$. Clearly we can then construct an isomorphism by mapping $b_1 \mapsto a$, $b_n \mapsto b$ and $b_2 \mapsto 1$. Notice that if we had removed all reference to the generator (or indeed, identity) b_2, we would have our semigroup being $C_2 \star C_2$ presented as a monoid. $\square$

This completes the case in which there exists a path from b_1 to b_n and vice versa.

Case (2)

There does not exist a path from b_1 to b_n, but there exists some path from b_n to b_1.

Let $v = v_1 \dots v_m$ be a word labelling the shortest path from b_n to b_1. There are two subcases here depending on what occurs at b_1. We may have that either there exists an edge right from b_1, or that there exists no such edge.

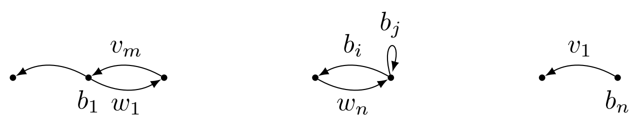
Case (There exists a path right from b_1,2.1)

This case has no subcases, so we need only show the following claim.

Claim 7.87

In Case 2.1, that is there exists a path from b_n to b_1 but no path from b_1 to b_n and there exists an edge going right from b_1 we have that $\dagger(S, B) \ncong \dagger(\mathbb{Z}, \{1, -1\})$.

Proof: Let $w = w_1 \dots w_k$ be a word labelling the longest path right from b_1 that does not visit any vertex more than once, and does not reach b_n. This may be visualised as follows.



In this diagram b_k and b_j represent all generators in B. Now by an analogous proof to Claim 7.8 using the word w, and finding an appropriate vertex right of b_n we see that this case $\dagger(S, B) \ncong \dagger(\mathbb{Z}, \{1, -1\})$. □

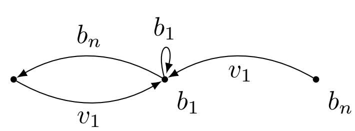
Case (There are no edges going right from b_1, 2.2)

Recall that $b_1 = b_n v$, and so $b_1 \cdot b_1 = b_1 \cdot b_n v$. Due to the nature of the graph, this means v has length 1 and S is 2-generated. We now ask what the edges labelled b_1 and b_n do at b_1.

Claim 7.88

In Case 2.2.1, that is there exists a path from b_n to b_1 but no path from b_1 to b_n,there are no edges going right from b_1 and $b_1 b_1 = b_1$ we have that $\dagger(S, B) \ncong \dagger(\mathbb{Z}, \{1, -1\})$.

Proof: From the assumptions of the claim we have the following section of graph.

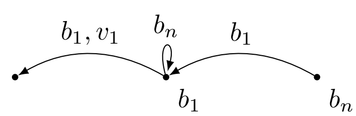


Now $v_1 = b_i$ for either $i = 1$ or $i = n$. There exists an edge left from the vertex $b_1 b_n$, which must be labelled b_j where $j \neq i$. Consider now $b_1 b_n \cdot b_n b_i$. This must be equal to $b_1b_n \cdot b_1$. If $i = 1$ then an edge skipping a vertex would be required to make this hold. If $i = n$ then $b_1b_n \cdot b_nb_n = b_1b_n$ and so for this to hold, we would need $b_1b_n \cdot b_1 = b_1$ also, and hence $\dagger(S, B) \ncong \dagger(\mathbb{Z}, \{1, -1\})$. $\square$

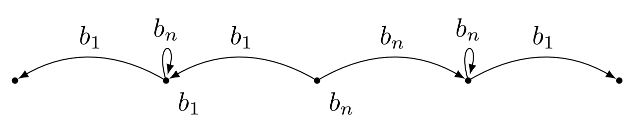
Claim 7.89

In Case 2.2.2, that is there exists a path from b_n to b_1 but no path from b_1 to b_n,there are no edges going right from b_1 and $b_1b_n = b_1$ we have that $S \cong T_5$.

Proof: The following diagram shows a section of our graph.



Since there are no other generators left to label edges from b_1, we must have $v_1 = b_1$. We can then immediately deduce that b_nb_n is found right of b_n. We also deduce that $b_nb_nb_1$ is not equal to b_nb_n or b_n, since $b_1 \cdot b_n = b_1 \cdot b_nb_n = b_1$ which is not equal to $b_1 \cdot b_nb_nb_1$. This also shows us $b_nb_nb_n = b_nb_n$, because otherwise we would have $b_nb_nb_n = b_nb_nb_1$ which would give us the same contradiction.



These relations now determine the entire graph, in that each vertex left of b_1 has an edge labelled b_1 going left and a loop labelled b_n, and each vertex right of b_nb_n has an edge labelled b_1 going right and a loop labelled b_n. Hence this is the Cayley graph of a semigroup with presentation

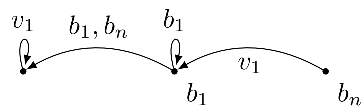
$$\text{sgp}\langle b_1, b_n \mid b_1b_n = b_1, b_nb_1 = b_1, b_nb_nb_n = b_nb_n\rangle.$$

This is the semigroup T_5. $\square$

Claim 7.90

In Case 2.2.3, that is there exists a path from b_n to b_1 but no path from b_1 to b_n,there are no edges going right from b_1 and $b_1b_1 = b_1b_n \neq b_1$ we have that $\dagger(S, B) \ncong \dagger(\mathbb{Z}, \{1, -1\})$.

Proof: This means that v_1 has to label a loop on the vertex $b_1 b_1$



Since $b_1 b_1 = b_1 b_n$, and v_1 is equal to one of b_1 or b_n, we have $b_1 \cdot b_1 b_1 = b_1 \cdot b_1 b_n = b_1 b_1$. This means we have run out of generators to label the edge that must go left from $b_1 b_1$ and so $\dagger\,(S, B) \ncong \dagger\,(\mathbb{Z}, \{1, -1\})$. $\square$

Case (3)

There exists a path from b_1 to b_n but no path from b_n to b_1

Claim 7.91

This case is symmetric to Case 2.

There is now one final case to consider.

Case (4)

There exists no path from b_1 to b_n and no path from b_n to b_1

For the first three subcases we have the following.

Claim 7.92

In cases 4.1, 4.2 and 4.3, that is there exists some edge going towards b_1 or b_n originating at the other generator, we have that $\dagger\,(S, B) \ncong \dagger\,(\mathbb{Z}, \{1, -1\})$.

Proof: This follows analogously to the proof of Claim 7.87. $\square$

The remaining subcase has several subcases of its own.

Case (Case 4.4)

There exists neither an edge going left from b_n nor an edge right from b_1.

Now in this case, there exists some generator b_m such that $b_m w = b_1$ and $b_m v = b_n$ for some words $w = w_1 w_2 \dots w_k$ and $v = v_1 v_2 \dots v_l$. Consider the product $b_1 b_1$, which is either found left of b_1, or is equal to b_1. In either case, this must be equal to the product $b_1 b_m w$. Now by Claim 7.75 we know that when following the path w from

$b_1 b_m$ we must not encounter any folding or loops. Hence w must have length 1. This means $w = w_1$, and there are only three configurations which this works in. Since there are no edges going left from b_1 then both b_1 and b_m must label either loops on b_1 or right edges from b_1.

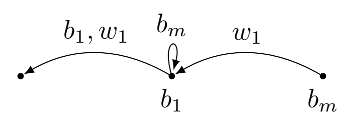


Figure 7.13: Section A

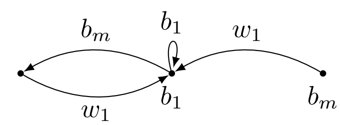


Figure 7.14: Section B

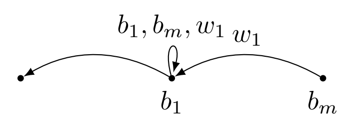


Figure 7.15: Section C

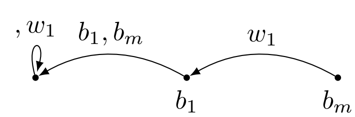


Figure 7.16: Section D

These arguments are all equally applicable to b_n and $b_m v$, so we end up with three possible layouts for the b_n side of the graph, as shown here.

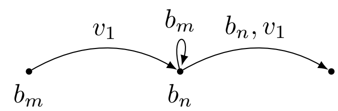


Figure 7.17: Section 1

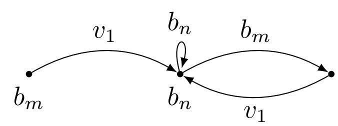


Figure 7.18: Section 2

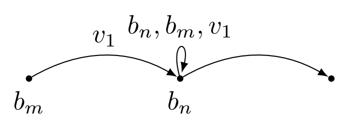


Figure 7.19: Section 3

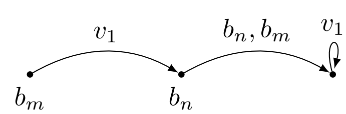


Figure 7.20: Section 4

We can combine these sections in an attempt to construct Cayley graphs of semigroups. We note that since these sections are symmetrical, then we need only study ten of the possible sixteen combinations. We shall look at the following:

- Section A and 1(7.13 and 7.17, case 4.4.1.1)
- Section A and 2 (7.13 and 7.18, case 4.4.1.2) (isomorphic to B and 1, case 4.4.2.1)
- Section A and 3 (7.13 and7.19, case 4.4.1.3) (isomorphic to C and 1, case 4.4.3.1)
- Section A and 4 (7.13 and 7.20, case 4.4.1.4) (isomorphic to D and 1, case 4.4.4.1)

- Section B and 2 (7.14 and 7.18, case 4.4.2.2)
- Section B and 3 (7.14 and 7.19, case 4.4.2.3) (isomorphic to C and 2, case 4.4.3.2)
- Section B and 4 (7.14 and 7.20, case 4.4.2.4) (isomorphic to D and 2, case 4.4.4.2)
- Section C and 3 (7.15 and 7.19, case 4.4.3.3)
- Section C and 4 (7.15 and 7.20, case 4.4.3.4) (isomorphic to D and 3, case 4.4.4.3)
- Section D and 4 (7.16 and 7.20, case 4.4.4.4).

When examining the combinations, we shall try to establish the destinations of all edges leaving the generators, and decide whether this forms a Cayley graph of a semigroup or not.

Case (Section A and Section 2)

The combination of these sections are visualised as follows.

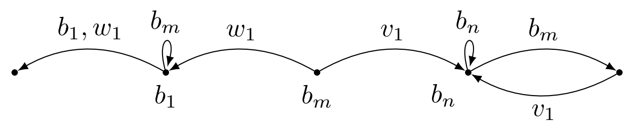


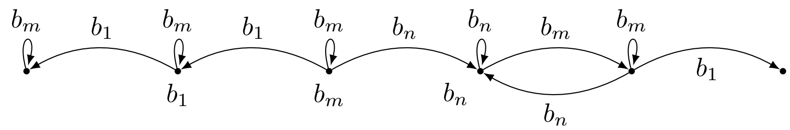
Figure 7.21: Sections A and 2

Claim 7.93

In Cases 4.4.1.2 and 4.4.2.1 we have that the graph is not the Cayley graph of a semigroup.

Proof: We first show that $b_m b_m = b_m$. Suppose otherwise, then either $b_m b_m = b_1$ or $b_m b_m = b_n$. In the former case, we see an immediate contradiction because $b_1 \cdot b_m b_m = b_1 \neq b_1 \cdot b_1$. In the latter, then we should have $b_m \cdot b_m b_m = b_n b_m = b_m \cdot b_n$, but this would create an edge that jumped over a vertex which is forbidden. Hence $b_m b_m = b_m$, and as a consequence $b_n b_m b_m = b_n b_m$. Since $b_1 b_m = b_1$, then $b_1 b_1 b_m = b_1 b_1$.

This leaves us with only two generators to assign to v_1 and w_1. If $w_1 = b_n$ and $v_1 = b_1$, then we have $b_m b_1 = b_n$. However, $b_m \cdot b_m b_1 = b_n$ but $b_m \cdot b_n = b_1$, which is a contradiction. Hence $w_1 = b_1$ and $v_1 = b_n$.



Now we will examine the placement of the product b_1b_n. Recalling that there are no edges going right from b_1, there are only two options of where the edge labelled b_n from vertex b_1 goes, either a loop on b_1 or a left edge. Suppose b_n labels a loop on b_1, so $b_1b_n = b_1$. This means $b_n \cdot b_1b_n = b_n \cdot b_1$ and hence $b_nb_1 = b_n$. Returning to b_1, we now have $b_1 \cdot b_nb_1 = b_1b_1$, but $b_1 \cdot b_n = b_1$. Hence our assumption that $b_1b_n = b_1$ must be incorrect.

Thus b_n must label an edge going left from b_1, and so $b_1b_n = b_1b_1$. This implies that $b_nb_1 = b_n$. Now looking at $b_1 \cdot b_nb_1 = b_1 \cdot b_1$, we see that b_1 labels a loop on b_1b_1. Similarly, since $b_1 \cdot b_nb_n = b_1 \cdot b_n$, there is also a loop labelled b_n on b_1b_1. Now we have used up all our generators, but have not yet labelled an edge going left from b_1b_1. This is a contradiction, and so we must not have $w_1 = b_1$ and $v_1 = b_n$.

Thus sections A and 2 do not create a graph that is the Cayley graph of a semigroup.

□

We will look now at using sections A and 1.

Case (Section A and Section 1)

These sections combine to make the following section of graph.

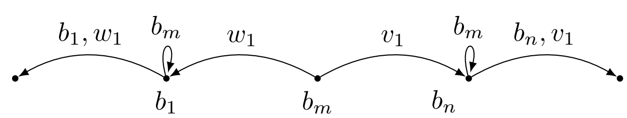


Figure 7.22: Sections A and 1

Claim 7.94

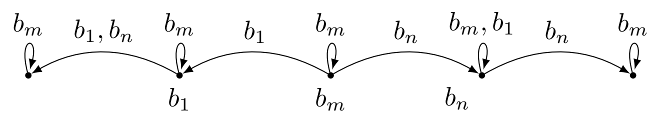
In Case 4.4.1.1 we have that $S \cong T_1$.

PROOF: Similarly to Claim 7.93, we can show that $b_mb_m = b_m$.

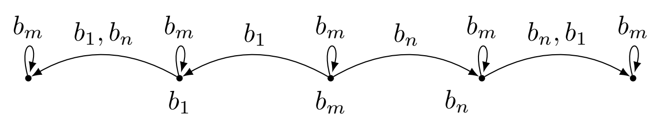
Suppose $w_1 = b_n$ and $v_1 = b_1$, then we have $b_mb_n = b_1$, but $b_m \cdot b_mb_n = b_1$ and

$b_m \cdot b_1 = b_n$, and so $w_1 = b_1$ and $v_1 = b_n$.

We look at where the product $b_n b_1$ lies. If $b_n b_1 = b_n$, then since we require $b_1 \cdot b_n b_1 = b_1 \cdot b_n$, we must have $b_1 b_n = b_1 b_1$.



However, $b_n \cdot b_1 b_n = b_n b_n$ and $b_n \cdot b_1 b_1 = b_n$, a contradiction. Hence $b_n b_1 = b_n b_n$. This implies that $b_1 b_n = b_1 b_1$, as otherwise $b_1 b_n = b_n$ and this means that $b_n b_1 b_n = b_n b_1$ and $b_n b_n b_n = b_n b_1 b_n$. This would mean every generator labels a loop on the vertex $b_n b_n$ leaving no generators to label the edge right from here. Therefore $b_1 b_n = b_1 b_1$.



Now we must have $b_1 \cdot b_n b_1 = b_1 \cdot b_n b_n$, and so $b_1 b_1 b_1 = b_1 b_1 b_n \neq b_1 b_1$ as otherwise there would be no generators going left from $b_1 b_1$. Similarly $b_n b_n b_n = b_n b_n b_1 \neq b_n b_n$. The relations defined here create a graph which for every vertex left of b_1 has left edges labelled b_1, b_n and loops labelled b_m. Symmetrically for every vertex right of b_n there are right edges labelled b_1, b_n and loops labelled b_m.

Hence this is the Cayley graph of a semigroup with the presentation

$$\begin{aligned} \text{sgp}\langle b_1, b_m, b_n \mid\ & b_1 b_1 = b_1 b_n, b_n b_n = b_n b_1, b_1 b_m = b_1, \\ & b_m b_m = b_m, b_n b_m = b_m, b_m b_1 = b_1, b_m b_n = b_n \rangle \end{aligned}$$

This is the semigroup T_1. $\square$

Case (Section A and Section 3)

Sections A and 3 have the following graph section.

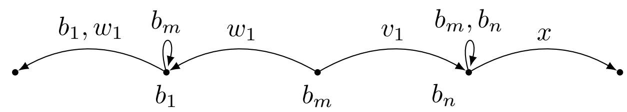
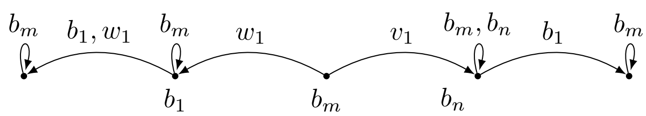


Figure 7.23: Sections A & 3

Claim 7.95

In Cases 4.4.1.3 and 4.4.3.1 we have that either $S \cong T_4$, $S \cong T_3$ or $S \cong T_5$.

Proof: By elimination we can determine that $x = b_1$. Since $b_1 b_m = b_1$, then $b_n b_1 b_m = b_n b_1$ and $b_1 b_1 b_m = b_1 b_1$.



We show that $w_1 = b_1$. Suppose not, then $w_1 = b_i$ for $i = n$ or $i = m$, and $b_m b_i = b_1$. However, $b_n \cdot b_m b_i = b_n \neq b_n \cdot b_1$. Hence $w_1 = b_1$. Now we can consider the possibilities for the edges labelled b_m and b_n leaving vertex b_m. We have determined that these cannot go left to meet the vertex b_1, and so they may either label a loop on b_m or an edge right to meet b_n. At least one of b_n, b_m must label an edge right to meet b_n.

Suppose first that we have $b_m b_m = b_m$ and $b_m b_n = b_n$. From this we can show that $b_1 b_n = b_1$, as otherwise if $b_1 b_n = b_1 b_1$, then we must have $b_1 \cdot b_n b_n = b_1 \cdot b_n$, so b_n labels a loop on the vertex $b_1 b_n$. Then since $b_1 \cdot b_1 b_1 = b_1 \cdot b_1 b_n$, we also have b_1 labelling a loop on the vertex $b_1 b_n$. This means all generator label a loop on the vertex $b_1 b_n$, which leaves no generators to label an edge left of here. Thus, $b_1 b_n = b_1$.

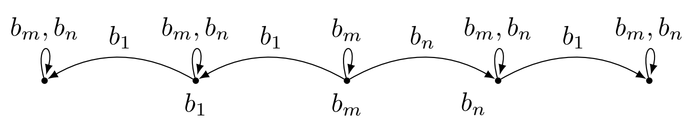


Figure 7.24: $b_m b_m = b_m$ and $b_m b_n = b_n$

Now using only the equalities defined so far, we see that since $b_1 b_n = b_1 b_m = b_1$

then we have a left edge labelled b_1 from every vertex left of b_1 and all such vertices have a loops labelled b_n and b_m. Similarly every vertex right of b_n has a right edge labelled b_1 and loops labelled b_n and b_m. Hence this is the Cayley graph of semigroup with the presentation

$$\begin{aligned}\mathrm{sgp}\langle b_1, b_m, b_n \mid\ & b_1 b_m = b_1, b_1 b_n = b_1, b_m b_1 = b_1,\\ & b_m b_m = b_m, b_m b_n = b_n, b_n b_m = b_n, b_n b_n = b_n\rangle.\end{aligned}$$

This is the semigroup T_4.

Secondly, we can have that b_n labels a loop on b_m and b_m labels a right edge from b_m. This gives us $b_m b_m = b_n$ and $b_m b_n = b_m$. This immediately implies that $b_1 b_n = b_1$ since $b_1 \cdot b_m b_m = b_1 \cdot b_n$. These relations now establish the entire graph, as $b_1 b_n = b_1 b_m = b_1$ tells us that every vertex left of b_1 has an edge labelled b_1 going left from it and loops labelled b_n and b_m. It also means every vertex right of b_n has an edge labelled b_1 going left from it and loops labelled b_n and b_m.

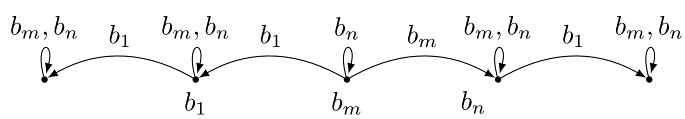


Figure 7.25: $b_m b_m = b_n$ and $b_m b_n = b_m$

This is therefore the Cayley graph of a semigroup with presentation

$$\begin{aligned}\mathrm{sgp}\langle b_1, b_m, b_n \mid\ & b_1 b_m = b_1, b_1 b_n = b_1, b_m b_1 = b_1,\\ & b_m b_m = b_n, b_m b_n = b_m, b_n b_m = b_n, b_n b_n = b_n\rangle.\end{aligned}$$

This is the semigroup T_3.

Finally we have both b_n and b_m label an edge going right from the vertex b_m and so $b_m b_m = b_m b_n = b_n$. From this we deduce that $b_1 b_n = b_m$, since we have $b_1 \cdot b_m b_n = b_1 \cdot b_m b_m = b_1$. Then for every vertex left of b_1 there is an edge going left labelled b_1 and loops labelled b_n and b_m. For every vertex right of b_n there is an edge going right labelled b_1 and loops labelled b_n and b_m.

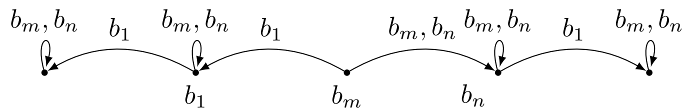


Figure 7.26: $b_m b_m = b_m b_n = b_n$

Now we notice that $b_n = b_m b_m$ and so b_n is a superfluous generator. Hence this Cayley graph defines a semigroup presented by

$$\mathrm{sgp}\langle b_1, b_m \mid b_1 b_m = b_1, b_m b_1 = b_1, b_m b_m b_m = b_m b_m \rangle.$$

This is the semigroup T_5. $\square$

Case (Section A and Section 4)

Sections A and 4 have the following graph section.

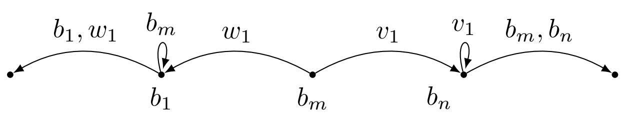
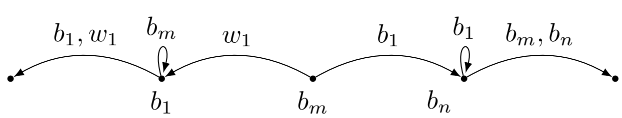


Figure 7.27: Sections A & 4

Claim 7.96

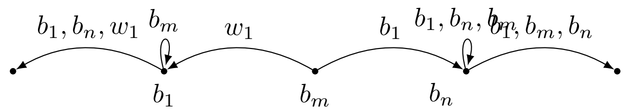
In Cases 4.4.1.4 and 4.4.4.1 we have that $\dagger\,(S, B) \not\cong \dagger\,(\mathbb{Z}, \{-1, 1\})$.

Proof: Consider v_1. If $v_1 = b_n$ or $v_1 = b_m$, then since $b_n b_n = b_n b_m$, we get $b_n b_n b_n = b_n b_n b_m = b_n b_n$. Since $b_n b_1$ is equal to either b_n or $b_n b_n$, then $b_n b_n b_1 = b_n b_n$ also, and we have no generator left to label an edge going right from $b_n b_n$. Therefore, $v_1 = b_1$.



Now we have $b_m b_1 = b_n$. If we consider $b_1 \cdot b_m b_1 = b_1 b_1$ we see that we must have $b_1 \cdot b_n = b_1 b_1$. Additionally, if we look at $b_n \cdot b_1$, we know that this is either equal to b_n or $b_n b_n$. If it is the former, then we have a contradiction as $b_n \cdot b_1 b_m \neq b_n \cdot b_1$. Therefore $b_n \cdot b_1 = b_n b_n$.

We may now use the equality $b_1b_n = b_1b_1$ to see that $b_n \cdot b_1b_n = b_n \cdot b_1b_1 = b_1b_n$ and the equality $b_1b_m = b_1$ to see $b_n \cdot b_1b_m = b_n \cdot b_1 = b_nb_n$.



Examining the vertex b_nb_n we see that we have used all generators to label a loop, and so there are none left to label the edge right from here. This means that$\dagger\,(S, B) \ncong \dagger\,(\mathbb{Z}, \{-1, 1\})$. $\square$

Case (Section B and 2)

On combining these sections we create the following graph.

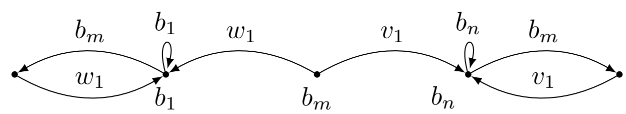


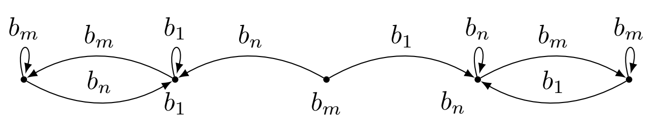
Figure 7.28: Sections B & 3

Claim 7.97

In Case 4.4.2.2 we have that $\dagger\,(S, B) \ncong \dagger\,(\mathbb{Z}, \{-1, 1\})$.

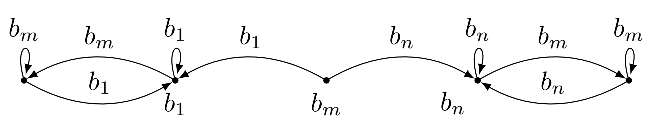
Proof: We can show similarly to Claim 7.93 that $b_mb_m = b_m$.

Suppose that $w_1 = b_n$ and $v_1 = b_1$, then we have $b_mb_n = b_1$ and $b_mb_1 = b_n$.



Now $b_1b_m \cdot b_mb_1$ must equal $b_1b_m \cdot b_n = b_1$, and so all generators leaving b_1b_m have been assigned, leaving none to go left.

If $w_1 = b_1$ and $v_1 = b_n$ then, we have $b_mb_n = b_n$ and $b_mb_1 = b_1$.



Then $b_1b_m \cdot b_mb_n$ must equal $b_1b_m \cdot b_1 = b_1$ and there are no generators left to label an edge going left from b_1b_m so $\dagger\,(S, B) \ncong \dagger\,(\mathbb{Z}, \{-1, 1\})$. $\square$

Case (Section B and 3)

These sections combine to give the following graph section.

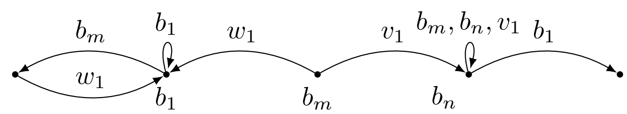


Figure 7.29: Sections B & 3

Claim 7.98

In Cases 4.4.2.3 and 4.4.3.2 we have that $\dagger\,(S, B) \ncong \dagger\,(\mathbb{Z}, \{-1, 1\})$.

Proof:

Now we can show that $w_1 = b_1$, since otherwise $b_mb_i = b_1$ for either $i = m$ or $i = m$, and $b_n \cdot b_mb_i \neq b_n \cdot b_1$.

We now ask where the edges labelled b_m and b_n go from the vertex b_m.

First we may have b_m labelling a loop on b_m and b_n as an edge right meeting b_n, so $b_mb_m = b_m$ and $b_mb_n = b_n$. Now this means that the edge going left from b_1b_m must be labelled b_n.

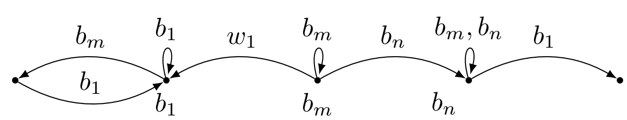


Figure 7.30: Sections B & 3

Now we look at $b_1 \cdot b_mb_n$. This product must be equal to $b_1 \cdot b_n$, but this requires an edge to jump over a vertex which is not allowed in this graph. Hence we do not have this configuration of edges.

Second, we may have b_n labelling a loop on b_m and b_m labelling an edge going right from b_m, so $b_mb_m = b_n$ and $b_mb_n = b_m$. We can then see that $b_1 \cdot b_mb_n = b_1 \cdot b_m$, and so there is a loop labelled by b_n on the vertex b_1b_m.

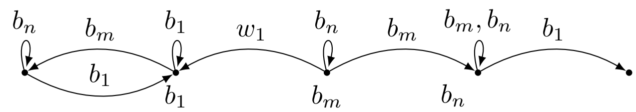


Figure 7.31: Sections B & 3

Now b_m must label an edge right from b_1b_m as it is the only generator left to do so. Now $b_1b_m \cdot b_n = b_1b_m \neq b_1 \cdot b_mb_m$ which is a contradiction.

Hence we are left with the final option, which is to have both b_n and b_m label an edge right from b_m and so $b_mb_m = b_mb_n = b_n$. Now since $b_1 \cdot b_mb_m = b_1 \cdot b_n$, we must have b_m labelling a loop on b_1b_m as else b_n would have to jump over a generator. As a consequence, b_n also labels a loop on b_1b_m.

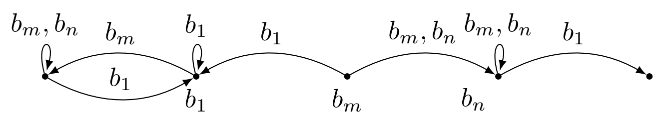


Figure 7.32: Sections B & 3

Now we can see from this that there are no edges left to label an edge left from b_1b_m and so $\dagger\,(S, B) \ncong \dagger\,(\mathbb{Z}, \{-1, 1\})$. $\square$

Thus the combination of sections B and 3 does not create a Cayley graph of a semigroup.

Case (Section B and Section 4)

Sections B and 3 have the following graph section.

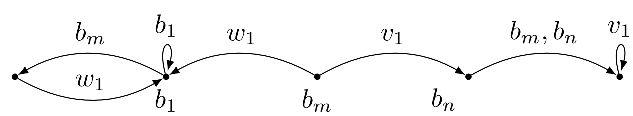


Figure 7.33: Sections B & 4

Claim 7.99

In Cases 4.4.2.4 and 4.4.4.2 we have that $\dagger\,(S, B) \ncong \dagger\,(\mathbb{Z}, \{-1, 1\})$.

PROOF:

Similarly to Claim 7.96, we have that $v_1 = b_1$.

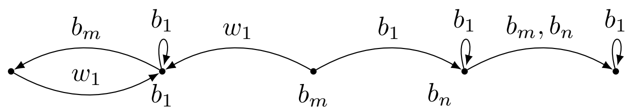


Figure 7.34: Sections B & 4

Now consider $b_m \cdot b_m$. If $b_m \cdot b_m = b_n$, then $b_n \cdot b_m b_m = b_n b_n$ and since $b_n b_m = b_n b_n$ we also have $b_n \cdot b_n b_n = b_n \cdot b_n b_m = b_n b_n$, which means there are no generators left to label an edge going right from $b_n b_n$. Suppose then that $b_m \cdot b_m = b_m$, then $b_n \cdot b_m b_m = b_n b_m$. Since $b_n b_n = b_n b_m$ then $b_n \cdot b_n b_n = b_n \cdot b_n b_m = b_n b_m$, and so there are no generators left to label an edge right from $b_n b_n$. Hence $b_m b_m = b_1$.

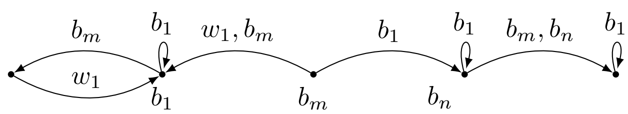


Figure 7.35: Sections B & 4

Now $b_m \cdot b_m b_m = b_1 b_m$, but $b_m \cdot b_1 = b_n \neq b_1 b_m$. This means the sections B and 4 do not form a graph with $\dagger\,(S, B) \cong \dagger\,(\mathbb{Z}, \{-1, 1\})$. $\square$

Case (Sections C and 3)

In combining these sections we create the following piece of graph.

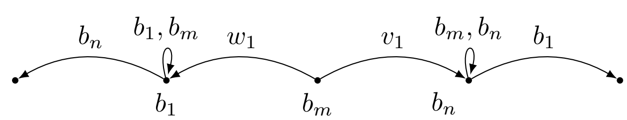


Figure 7.36: Sections C & 3

Claim 7.100

In Case 4.4.3.3 we have that $S \cong T_3$.

PROOF:

We can show that $w_1 = b_1$ since otherwise we would have either $b_m b_n = b_1$ or $b_m b_m = b_1$. In the first case a contradiction is found by inspecting $b_1 \cdot b_m b_n$ and $b_1 \cdot b_1$. In the second we find a contradiction at $b_n \cdot b_m b_m$ and $b_n \cdot b_1$. Similarly, we find that $v_1 = b_n$. As a consequence of these arguments we also deduce that $b_m b_m = b_m$.

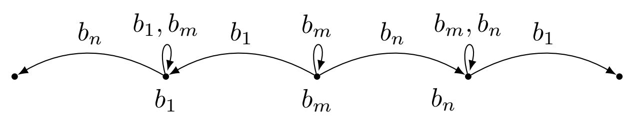


Figure 7.37: Sections C & 3

Now using these relations defined at the generators, we can draw a graph which has the correct structure. Every vertex has a loop labelled b_m. Going left from b_1 the left edges alternate between b_n and b_1, with the remaining generator forming a loop on the vertex. Going right from b_n we alternate in the opposite order between b_1 and b_n, with the remaining generator again forming a loop. This is the Cayley graph of a semigroup with the presentation

$$\begin{aligned}\mathrm{sgp}\langle b_1, b_m, b_n \mid {} & b_1 b_1 = b_1, b_1 b_m = b_1, b_m b_1 = b_1, \\ & b_m b_m = b_m, b_m b_n = b_n, b_n b_m = b_n, b_n b_n = b_n\rangle.\end{aligned}$$

This is the semigroup T_2. $\square$

Case (Section C and Section 4)

Sections C and 4 have the following graph section.

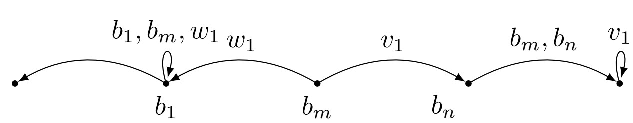


Figure 7.38: Sections C & 4

Claim 7.101

In Cases 4.4.3.4 and 4.4.4.3 we have that $\dagger\,(S, B) \not\cong \dagger\,(\mathbb{Z}, \{-1, 1\})$.

Proof: Similarly to Claim 7.96, we have that $v_1 = b_1$.

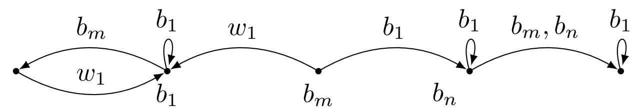


Figure 7.39: Sections C & 4

Now $b_1 \cdot b_m b_1 = b_1$ and so $b_1 \cdot b_n = b_1$. There are then no generators left to label the edge going left from b_1, and so $\dagger\,(S, B) \ncong \dagger\,(\mathbb{Z}, \{-1, 1\})$. $\square$

Case (Section D and Section 4)

Sections D and 4 have the following graph section.

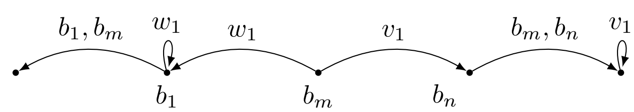


Figure 7.40: Sections D & 4

Claim 7.102

In Case 4.4.4.4 we have that $\dagger\,(S, B) \ncong \dagger\,(\mathbb{Z}, \{-1, 1\})$.

Proof: Similarly to Claim 4.4.1.4, we have that $v_1 = b_1$.

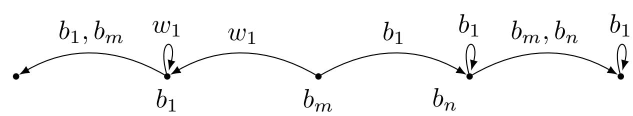


Figure 7.41: Sections D & 4

Now consider $b_m \cdot b_m$. If $b_m \cdot b_m = b_n$, then $b_n \cdot b_m b_m = b_n \cdot b_n = b_n b_n$. Now $b_n b_m = b_n b_n$ and so $b_n \cdot b_n b_n = b_n \cdot b_n b_m = b_n b_n$. There are then no generators left to label an edge right from $b_n b_n$. Suppose then that $b_m \cdot b_m = b_m$, and so $b_n \cdot b_m b_m = b_n \cdot b_m = b_n b_m$. Now since $b_n b_n = b_n b_m$, then $b_n \cdot b_n b_n = b_n \cdot b_n b_m = b_n b_m$. Then there are no generators left to label an edge right from $b_n b_m$. Therefore $b_m b_m = b_1$, and so $b_m \cdot b_m b_m = b_1 b_m$, but $b_m \cdot b_1 = b_n \neq b_1 b_m$. Hence $\dagger\,(S, B) \ncong \dagger\,(\mathbb{Z}, \{-1, 1\})$. $\square$

This completes the case analysis and the proof of Theorem 7.74.

Chapter 8

Conclusions and Further Work

> When we asked Pooh what the opposite of an Introduction was, he said ”The what of a what?” which didn’t help us as much as we had hoped, but luckily Owl kept his head and told us that the Opposite of an Introduction, my dear Pooh, was a Contradiction; and, as he is very good at long words, I am sure that that’s what it is.
>
> The House At Pooh Corner
> A.A. Milne

In this thesis we have presented a novel way of connecting semigroups with geometric structures. The results that we have acheived show us that this is a useful perspective to take on geometric semigroup theory.

In chapters 3, 4, and 5 we saw that certain semigroups with group-like properties preserved finite presentability under isomorphism of skeletons. Futher work in this area should include relaxing some of the conditions found in these chapters, and extending the results to other categories of semigroups.

Conjecture 8.1
Let $S = \mathcal{S}[Y; G_\lambda; \varphi_{\lambda,\mu}]$ and $T = \mathcal{S}[Z; H_\lambda; \theta_{\lambda,\mu}]$ where Y, Z are finite and homomorphisms $\varphi_{\lambda,\mu}, \theta_{\lambda,\mu}$ are such that

(i) $\operatorname{im}\varphi_{\alpha,\beta}, \operatorname{im}\theta_{\alpha,\beta}$ *have finite index.*

(ii) $\ker\varphi_{\alpha,\beta}, \ker\theta_{\alpha,\beta}$ *are finite.*

If $\dagger(S) \cong \dagger(T)$ *then* S *is finitely presented if and only if* T *is.*

Work by Gray and Kambites [8, Theorem 4] shows that when working with semimetric spaces as the geometric structure, finite presentability is a quasi-isometry invariant for finitely generated monoids with finitely many left and right ideals.

Question 8.2
Is finite presentability a skeleton-invariant for semigroups with finitely many left and right ideals?

We note that the proof of Gray and Kambites theorem relies on identifying $\mathcal{R}$-classes, which is not possible using the skeleton structure.

A further line of inquiry inspired by [9, Theorem A] is to consider cancellative semigroups.

Question 8.3
Is finite presentability a skeleton-invariant for left cancellative semigroups?

Question 8.4
Are there futher classes of semigroups which have finite presentability as a skeleton-invariant property?

In chapter 6, we presented an example of two semigroups, one finitely presented and one not, which are skeleton isomorphic, and in fact have isometric Cayley graphs. This answers [8, Question 1]. A possible extension of work in this area would be to find further examples, and to establish the Cayley spectra of the skeleton. We also note that for both semigroups in this example, we found a regular language of unique representatives. This leads us to ask the following question.

Question 8.5
If S *and* T *are such that* $\dagger(S) \cong \dagger(T)$, *is it true that* S *has a regular language of unique representative if and only if* T *does.*

Finally, the work of chapter 7 can be extended to incorporate many more skeletons and the related semigroups. We suggest that it is perhaps not a sensible idea to attempt to stretch the definition of Cayley spectra to disregard the generating set; that is, to try find all semigroups that are skeleton isomorphic to a given semigroup S for any given generating set of S. One issue with this sort of definition would be that for infinite semigroups we can find infinitely many generating sets and may end up with infinitely many skeletons which we would want to investigate. A final observation is that it may be that the techniques we employ in this chapter are difficult to extend to more general semigroups, as the semigroups we worked with here had very special structures as skeletons which we were able to use in our arguments. We may not be so lucky with an arbitrary semigroup.

Appendix A

Lots of trees that look like A^+

This appendix contains diagrams of semigroups constructed in subsection 7.3.1, that is semigroups which are skeleton-isomorphic to $A^* = \text{sgp}\langle a, b, c, d \mid \rangle$.

A.1 Adding 3 generators

Figure A.1 gives a semigroup with partition P_1 and partition of $n - i = 3$ into one part, where a, b, c are all idempotents.

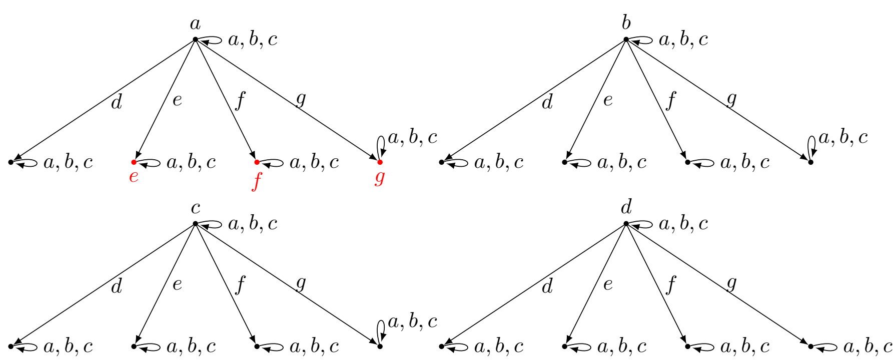


Figure A.1: B_8

Figure A.2 gives a semigroup with partition P_1 and partition of $n - i = 3$ into one part, where only a is idempotent.

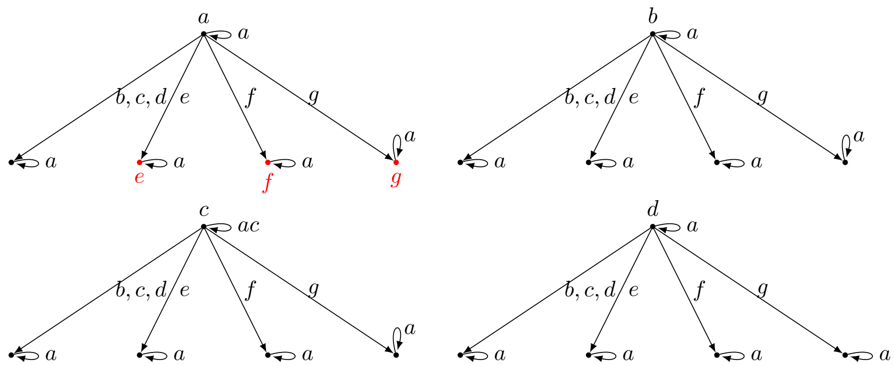


Figure A.2: B_9

Figure A.3 gives a semigroup with partition P_2 where a, b and c are idempotents.

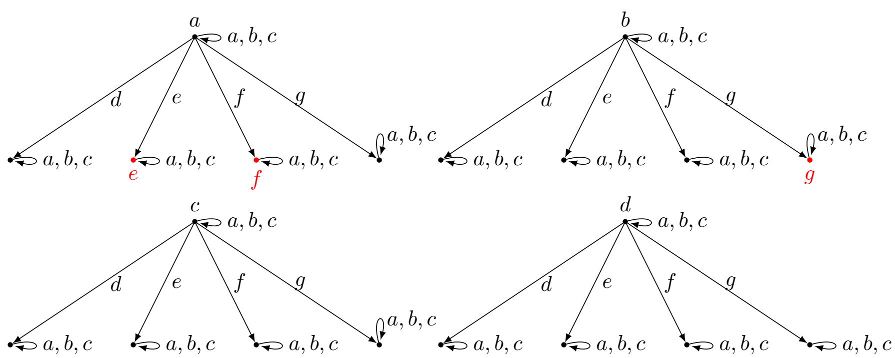


Figure A.3: B_{10}

Figure A.4 gives a semigroup with partition P_3.

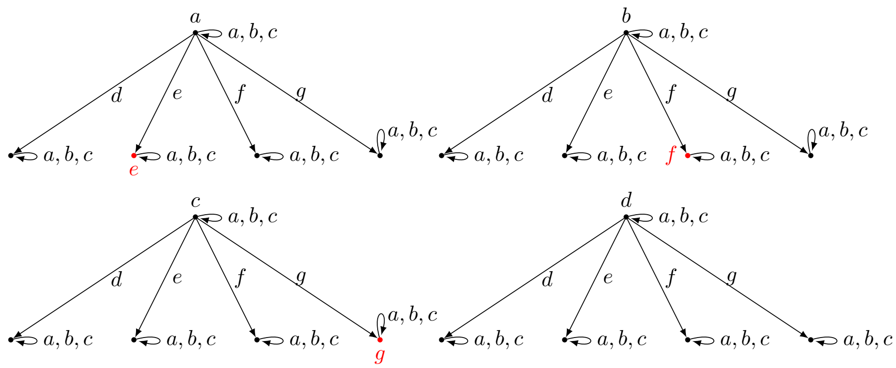


Figure A.4: B_{11}

A.2 Adding 4 generators

Figure A.5 gives a semigroup with the partition $4 = 4$.

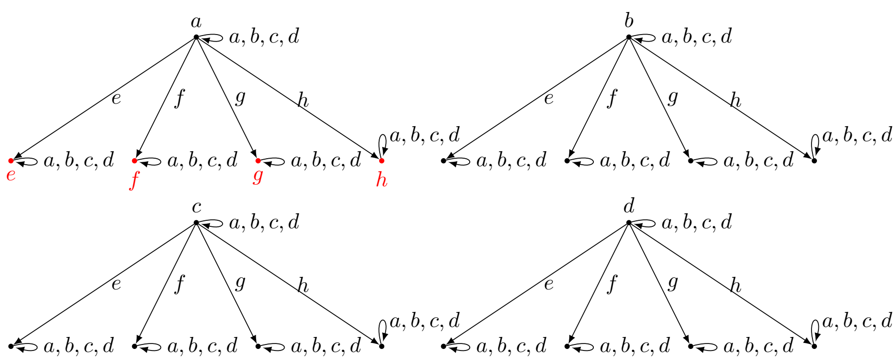


Figure A.5: B_{12}

Figure A.6 gives a semigroup with the partition $4 = 3 + 1$.

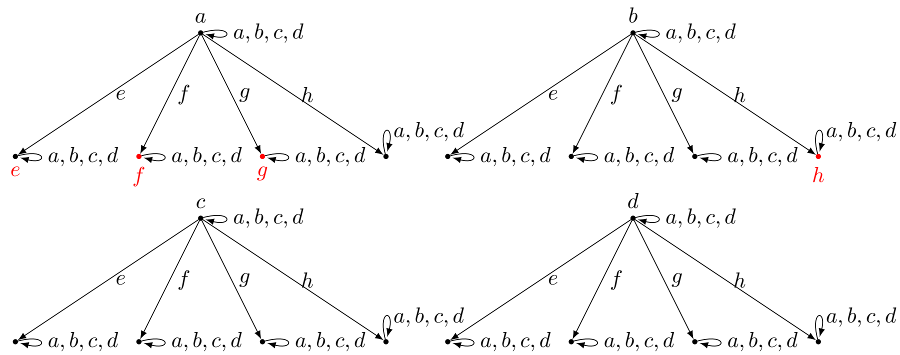


Figure A.6: B_{13}

Figure A.7 gives a semigroup with the partition $4 = 2 + 1 + 1$.

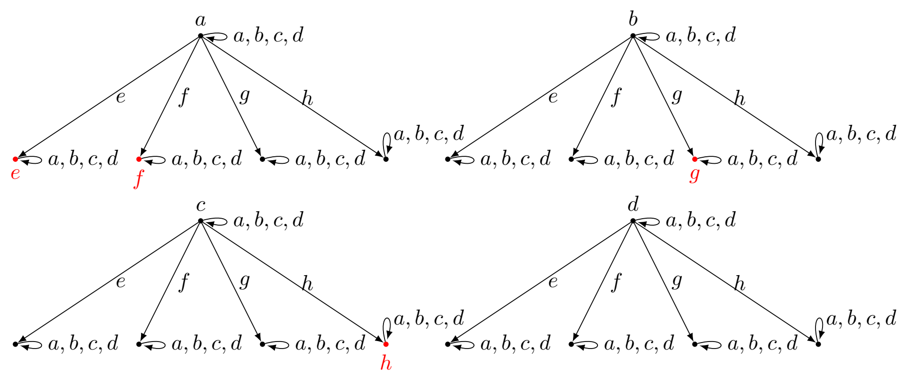


Figure A.7: B_{14}

Figure A.7 gives a semigroup with the partition $4 = 1 + 1 + 1 + 1$.

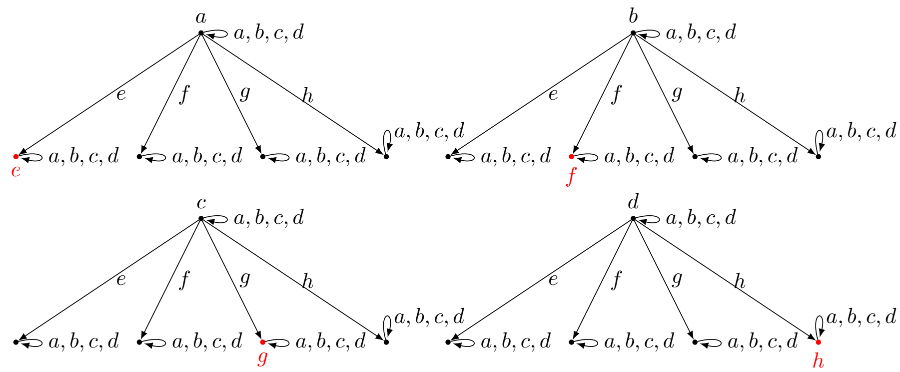


Figure A.8: B_{15}

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