DECISION PROBLEMS IN GROUPS OF HOMEOMORPHISMS OF CANTOR SPACE

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Abstract

The Thompson groups F, T and V are important groups in geometric group theory: T and V being the first discovered examples of finitely presented infinite simple groups. There are many generalisations of these groups including, for n and r natural numbers and $1 < r < n$, the groups F_n, $T_{n,r}$ and $G_{n,r}$ ($T \cong T_{2,1}$ and $V \cong G_{2,1}$). Automorphisms of F and T were characterised in the seminal paper of Brin ([16]) and, later on, Brin and Guzman ([17]) investigate automorphisms of $T_{n,n-1}$ and F_n for $n > 2$. However, their techniques give no information about automorphisms of $G_{n,r}$.

The second chapter of this thesis is dedicated to characterising the automorphisms of $G_{n,r}$. Presenting results of the author's article [10], we show that automorphisms of $G_{n,r}$ are homeomorphisms of Cantor space induced by transducers (finite state machines) which satisfy a strong synchronizing condition.

In the rest of Chapter 2 and early sections of Chapter 3 we investigate the group $\mathrm{Out}(G_{n,r})$ of outer automorphisms of $G_{n,r}$. Presenting results of the forthcoming article [6] of the author's, we show that there is a subgroup $\mathcal{H}_n$ of $\mathrm{Out}(G_{n,r})$, independent of r, which is isomorphic to the group of automorphisms of the one-sided shift dynamical system. Most of Chapter 3 is devoted to the order problem in $\mathcal{H}_n$ and is based on [44]. We give necessary and sufficient conditions for an element of $\mathcal{H}_n$ to have finite order, although these do not yield a decision procedure.

Given an automorphism ϕ of a group G, two elements $f, g \in G$ are said to be ϕ-twisted conjugate to one another if for some $h \in G$, $g = h^{-1}f(h)\phi$. This defines an equivalence relation on G and G is said to have the R_∞ property if it has infinitely many ϕ-twisted conjugacy classes for all automorphisms $\phi \in \mathrm{Aut}(G)$. In the final chapter we show, using the description of $\mathrm{Aut}(G_{n,r})$, that for certain automorphisms, $G_{n,r}$ has infinitely many twisted conjugacy classes. We also show that for certain $\phi \in \mathrm{Aut}(G_{2,1})$ the problem of deciding when two elements of $G_{2,1}$ are ϕ-twisted conjugate to one another is soluble.

Declaration

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I, Feyisayo Aderonle Olukoya, do hereby certify that this thesis, submitted for the degree of PhD, which is approximately 61,319 words in length, has been written by me, and that it is the record of work carried out by me, or principally by myself in collaboration with others as acknowledged, and that it has not been submitted in any previous application for any degree.
I was admitted as a research student at the University of St Andrews in September 2014.
I, Feyisayo Aderonle Olukoya, received assistance in the writing of this thesis in respect of language, grammar, spelling and syntax, which was provided by Dr Martyn Quick.
I received funding from an organisation or institution and have acknowledged the funder(s) in the full text of my thesis.

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"I try all things, I achieve what I can."— Herman Melville, Moby-Dick.

"Say what you know, do what you must, come what may." — Sofia Kovalevskaya.

Contents

Preface

This thesis investigates certain subgroups of the *rational group* $\mathcal{R}_n$ of [30] and its generalisations $\mathcal{R}_{n,r}$ for natural numbers n and r satisfying $1 \leqslant r < n$. It is a fact, which we do not make use of in this work, that $\mathcal{R}_{n,r}$ and $\mathcal{R}_n$ are isomorphic to the group $\mathcal{R}_2$. Hence, in this preface we shall refer to both by the name 'the rational group' and phrase our discussion in terms of the group $\mathcal{R}_{n,r}$. In subsequent paragraphs we shall use the notation $\mathfrak{D}_{n,r}$, for some symbol $\mathfrak{D}$, to represent certain subgroups of $\mathcal{R}_{n,r}$, however, to have uniformity in our notation and to make correspondences clear, when $r = 1$ we set $\mathfrak{D}_{n,r} = \mathfrak{D}_n$; for instance $\mathcal{R}_{n,1} = \mathcal{R}_n$.

Since the rational group $\mathcal{R}_{n,r}$ was first introduced in [30] it has generated a lot of research activity. It has been shown to have insoluble order problem ([4]), and demonstrated to be simple ([5]) and not finitely-generated ([5],[30]). Moreover, all of its finitely generated subgroups have soluble word problem ([30]). The group $\mathcal{R}_{n,r}$ also contains many important classes of groups. For instance it contains the Thompson groups F, T and V and the Higman-Thompson groups $G_{n,r}$ generalising V ([30]); the Brin-Thompson group nV ([4]); the groups of automorphisms of the shift dynamical system ([30]); and groups generated by automata.

The groups T and V were the first discovered examples of finitely presented infinite simple groups, whilst Thompson's group F, also finitely presented, has a simple derived subgroup. Groups generated by automata are a vast source of groups with interesting properties. For instance the Grigorchuk group, which is an infinite torsion group of intermediate growth ([29], [31]), belongs to the class of groups generated by automata. The automorphisms of the shift dynamical system are an important class of groups in symbolic dynamics. These are defined as those homeomorphisms of the Cantor space of one-sided or two-sided infinite sequences over a finite alphabet, which commute with the shift map. The group $\mathrm{Aut}(\{0,1,\ldots,n-1\}^{\omega}, \sigma_n)$ of shift commuting homeomorphisms of the one-sided infinite sequence space is called the *group of automorphisms of the one-sided shift (dynamical system)*. In a similar manner, the group $\mathrm{Aut}(\{0,1,\ldots,n-1\}^{\mathbb{Z}}, \sigma_n)$ of shift commuting homeomorphisms of the two-sided infinite sequence space is called the *group of automorphisms of the two-sided shift dynamical system*.

This thesis will mainly be concerned with the subgroup $G_{n,r}$, its automorphism group, and a subgroup of its outer automorphisms group which is isomorphic to the group of automorphisms of the one-sided shift dynamical system. We also consider the monoid consisting of continuous functions from the Cantor space of one or two-sided infinite sequences to itself which commute with the shift map. These are the so called *endomorphisms of the one or two-sided shift dynamical system*

We begin in Chapter 1 by defining, for $1 \leqslant r < n$, the groups $\mathcal{R}_{n,r}$. These are groups of homeomorphisms of Cantor space $\mathfrak{C}_{n,r}$, the disjoint union of r copies of $\mathfrak{C}_n$, which may be represented by finite state machines called transducers. Hence, in Chapter 1, we also introduce automata and transducers and present various algorithms for multiplying, inverting and minimising transducers. We distinguish *synchronous transducers*, which always write a single output letter on consuming an input letter, as a subclass of *asynchronous transducers* which are allowed to write strings (including the empty string) on reading a single input. Additionally, we show how to construct from an element of $\mathcal{R}_{n,r}$, a transducer representing this homeomorphism. This is a technical chapter carrying out the fundamental constructions from [30] in full detail in the context of the groups $\mathcal{R}_{n,r}$. These constructions are particularly relevant to later work in Chapter 2.

We begin our investigation of subgroups of the rational group in Chapter 2, by considering the Higman-Thompson groups $G_{n,r}$. As mentioned above, these are a family of groups, which are either simple or have an index two subgroup which is simple ([34]), generalising Thompson group V. More specifically, in this chapter we characterise the automorphisms of the group $G_{n,r}$, presenting the results of the author's article [10]. This completes a line of research first begun by Brin in his seminal paper [16] characterising the automorphisms of the Thompson groups F and T. Later on, Brin together with Guzman, in the article [17], also investigate automorphisms of groups generalising Thompson's groups F and T. However their techniques do not apply to the groups $G_{n,r}$.

The article [10], making use of the transitivity of the action of $G_{n,r}$ on $\mathfrak{C}_{n,r}$ and a deep result of Rubin ([47]), shows that automorphisms of $G_{n,r}$ are elements of $\mathcal{R}_{n,r}$ for which the transducers representing these homeomorphisms have a particular property. This property is called the *synchronizing property*. The set of all rational homeomorphisms of $\mathfrak{C}_{n,r}$ whose transducers have the synchronizing property forms a submonoid $\widetilde{\mathcal{B}}_{n,r}$. The largest inverse closed subset of $\widetilde{\mathcal{B}}_{n,r}$ is the group $\mathcal{B}_{n,r}$. The results of [10] demonstrate that $\mathrm{Aut}(G_{n,r})$ is isomorphic to $\mathcal{B}_{n,r}$.

We do not present all the results of the article [10] in Chapter 2, we instead show that the normalizer of $G_{n,r}$ in the rational group is the group $\mathcal{B}_{n,r}$. By appealing to Rubin's Theorem and finiteness results on local actions of rational homeomorphisms in Yonah Maissel's thesis ([39]), which also appear in [10]), we conclude that the map which takes an element $\tau \in \mathcal{B}_{n,r}$ to the automorphism of $G_{n,r}$ induced by conjugation by τ is an isomorphism. Additionally, in this chapter we explore various consequences and characterisations of the synchronizing property. We close the chapter by examining the outer automorphisms of the group $G_{n,r}$.

It is immediate from the characterisation of $\mathrm{Aut}(G_{n,r})$ as those homeomorphisms in $\mathcal{R}_{n,r}$ that can be represented by transducers with the synchronizing property, that $\mathrm{Out}(G_{n,r})$ is best thought of as a group consisting of non-initial transducers with the synchronizing property. (Observe that since groups are inverse closed, then inverses of elements of $\mathrm{Out}(G_{n,r})$ also have the synchronizing property.) Perhaps surprisingly, it turns out that for $r = n-1$, $\mathrm{Out}(G_{n,r})$ contains a subgroup which is isomorphic to the quotient of the group of automorphisms of the two-sided shift dynamical system, by the group generated by the shift map. We observe that, by Ryan's Theorem ([48]), the group generated by the shift map is the centre of the group of automorphisms of the two-sided shift dynamical system.

The author's forthcoming paper [6] explores this connection between $\mathrm{Out}(G_{n,r})$ and the group of automorphisms of the two-sided shift dynamical system further. One of the results contained in this article, is that shift commuting homeomorphisms of the two-sided infinite sequence space can be represented by elements of $\mathrm{Out}(G_{n,r})$, together with natural combinatorial data arising from the structure of the non-initial transducers. A consequence of the results of [6], is that there is a subgroup $\mathcal{H}_n$ independent of r, that is $\mathcal{H}_n \leqslant \cap_{1 \leqslant r \leqslant n-1} \mathrm{Out}(G_{n,r})$, which is isomorphic to the group of automorphisms of the one-sided shift dynamical system. This subgroup is the focus of Chapter 3.

In Chapter 3, we begin our investigation of $\mathcal{H}_n$ by first demonstrating the isomorphism between $\mathcal{H}_n$ and the group of automorphisms of the one-sided shift dynamical system. As a somewhat natural starting point, we show that a submonoid of endomorphisms of the shift dynamical system (those requiring no 'future information'), is isomorphic to a monoid of non-initial transducers $\mathcal{P}_n$ which contains the group $\mathcal{H}_n$. We then deduce, as a corollary of this fact, that the group $\mathcal{H}_n$ is isomorphic to the automorphisms of the one-sided shift dynamical system.

The rest of the chapter, is based on the paper [44] which explores the order problem in $\mathcal{H}_n$. Now, elements of $\mathcal{H}_n$ are precisely those elements of $\mathrm{Out}(G_{n,r})$ which can be represented by *synchronous transducers* with the synchronizing property. We observe that, as a synchronous transducer on reading a single input letter, will always write a single output letter, inverses of elements of $\mathcal{H}_n$ are also synchronous; they are synchronizing since $\mathrm{Out}(G_{n,r})$ is a group. It is a standard result in the literature that each state of an invertible synchronous transducer induces an element of $\mathcal{R}_n$. Thus, invertible synchronous transducers naturally generate subgroups of $\mathcal{R}_n$ by taking the group generated by the homeomorphisms of $\mathfrak{C}_n$ induced by the states of such transducers. Groups generated in this way are called *automata groups*, and, as we mentioned in an earlier paragraph, this class of groups contain many groups with interesting properties.

In the latter half of Chapter 3, we also investigate the finiteness and order problems in the automata groups generated by elements of $\mathcal{H}_n$. These are problems of interest highlighted by [30] for elements of the group $\mathcal{R}_{n,r}$. The finiteness problem asks if there is an algorithm which, given an invertible synchronous transducer, will decide in finite time if the automata group generated by the transducer is finite. The order problem asks if there is an algorithm, which, given an invertible synchronous transducer and an element of the group as a product of the generators, decides in finite time if the given element has finite order.

Bleak and Belk in [4] investigate the finiteness and order problems in their full generality. For groups generated by finitely many elements of the rational group, they show that the finiteness and order problem are undecidable in general. However, the decidability of these problems for groups generated by invertible, synchronous transducers, remained open.

Recently, the order problem for groups generated by invertible, synchronous, transducers was shown to be insoluble in general. This was done independently by Gillibert ([27]), and Bartholdi and Mitrofanov ([3]). The finiteness problem remains open for synchronous transducers. As it turns out, the finiteness problem for transducers in $\mathcal{H}_n$ is equivalent to the order problem for automata groups generated by elements of $\mathcal{H}_n$, and to the order problem in $\mathcal{H}_n$. The equivalence of the finiteness problem for transducers in $\mathcal{H}_n$ to the order problem in $\mathcal{H}_n$ was shown in the paper [23]. However, we provide an independent proof all three equivalences in Chapter 3.

We also provide in Chapter 3 some new and sufficient conditions for when an element of $\mathcal{H}_n$ has finite or infinite order. These conditions focus on studying properties of the *dual transducer* for elements of $\mathcal{H}_n$. This is a technique which has been used effectively in investigating the order problem, as in the paper [1]. We prove some new results about the dual transducer of finite order elements of $\mathcal{H}_n$ which go some way towards resolving a conjecture of Picantin in [40]. These properties of the dual automata for elements of $\mathcal{H}_n$ lead us to study a new combinatorial object we call the *graph of bad pairs*. For each element of $\mathcal{H}_n$, we associate infinitely many such graphs. Properties of this graph, for instance if it contains a circuit, help in determining whether an element of $\mathcal{H}_n$ has infinite or finite order. In fact we conjecture that whenever an element of $\mathcal{H}_n$ has infinite order, then eventually one of its graph of bad pairs contains a circuit. It is a consequence of our structure results for the dual transducer of a finite order element of $\mathcal{H}_n$ that for a given finite element of $\mathcal{H}_n$, eventually all of its graphs of bad pairs are precisely the empty graph.

To close Chapter 3, we investigate the growth rate of automata groups generated by transducers in $\mathcal{H}_n$. As indicated above, this is a standard question to ask about groups generated by invertible, synchronous transducers. We show that all such groups, whenever they are infinite, have exponential growth rate. This result, as it happens, is already implied by work of Chou in [21] and results of Silva and Steinberg in [51] showing, amongst other things, that groups generated by transducers in $\mathcal{H}_n$ are elementary amenable groups. (We should perhaps also remark that Chou's proof showing that finitely generated elementary amenable groups either have polynomial or exponential growth rate contains a gap. The gap, however, is fixed in a paper by Rosset ([46]) from which one also deduces that, in the case where a group generated by an element of $\mathcal{H}_n$ has polynomial growth, it is finite. The author is grateful to Bartholdi for drawing his attention to the work of Chou and for pointing out both the error in Chou's proof and the fix by Rosset.)

Chapter 3 also contains results which do not fit under the umbrella of order problem and growth, but which naturally arise in our consideration of these problems: for instance we present certain embedding results for the groups $\mathcal{H}_n$, and some conjugacy invariants in $\mathcal{H}_n$.

In Chapter 4, we again consider the group $G_{n,r}$. Having now an understanding of the automorphisms of the group $G_{n,r}$, we consider the *twisted conjugacy problem* in $G_{n,r}$. The twisted conjugacy problem is a generalisation of the conjugacy problem and asks for a finitely presented group G, if there is an algorithm which, given elements $f, g \in G$ and an automorphism ϕ of G, decides in finite time if there is an element $h \in G$ such that $f = h^{-1}g(h)\phi$. If such an h exists then f and g are said to be ϕ-*twisted conjugate*. Moreover if ϕ is the identity automorphism of G, then f and g are conjugate in G. For a given ϕ in $\mathrm{Aut}(G)$, the relation on G defined by, $f \sim_{\phi} g$ if and only if f and g are ϕ-twisted conjugate to one another, is an equivalence relation. A natural question to ask then, is if there are finitely many or infinitely many ϕ-twisted conjugacy classes for ϕ an automorphism of a finitely presented group. We note that a group which has infinitely many twisted conjugacy classes for all automorphisms is said to have the *R_∞ property*.

The twisted conjugacy problem and corresponding R_∞ question have been investigated for Thompson's group F and T, whose automorphisms were classified by Brin in the article [16]. The paper [7] shows that F has the R_∞ property, whilst the paper [19] shows that the twisted conjugacy problem in F is soluble, and demonstrates that both F and T have the R_∞-property.

In the final chapter we begin an investigation of the twisted conjugacy problem and R_∞

property for the Higman-Thompson groups $G_{n,r}$. We demonstrate that for automorphisms ϕ of $G_{n,r}$ such that the image of ϕ in the quotient $\mathrm{Out}(G_{n,r})$ is in $\mathcal{H}_n$, there are infinitely many ϕ-twisted conjugacy classes. The remainder of the chapter focuses on the group $G_{2,1}$, and, for automorphisms ϕ whose image in $\mathrm{Out}(G_{n,r})$ is in $\mathcal{H}_2$, we solve the ϕ-twisted conjugacy problem.

The majority of the computations with transducers appearing in this work were vastly helped by the GAP software ([25]) together with the GAP packages "AutomGrp" ([41]) and "aaa". The former package deals only with synchronous transducers and was helpful in the computations appearing in Chapter 3; the latter package handles asynchronous transducers and is currently still being developed by Collin Bleak (the author's supervisor), Fernando Flores-Brito, Plamena Minerva, the author, and Angela Richardson. Already implemented in this package are many of the algorithms in the paper [30], moreover, this package allows for graphical visualization of transducers which was a great help to the author as a starting point for many of the figures of transducers appearing in this document.

Throughout the document we state several open questions and conjectures indicating the current state of research and future work.

Chapter 1

The Rational Group $\mathcal{R}_n$ and Related groups $\mathcal{R}_{n,r}$

This chapter shall be concerned primarily with defining the groups $\mathcal{R}_n$ and $\mathcal{R}_{n,r}$. Along the way we shall introduce certain algorithms that will play a key role in future discussion. We shall keep the exposition as self-contained and detailed as possible, illustrating key ideas with examples so the reader can become as familiar as possible with these groups and some of the ways we will be working with them.

We begin by introducing the notion of words, then graphs and trees, then automata and finally transducers. Afterwards, we define the groups $\mathcal{R}_n$ and $\mathcal{R}_{n,r}$ with which this thesis will be primarily concerned. In the process we establish some notation that we will be using throughout this work. Subsequent notation and terminology shall be introduced as we require them. The contents of the next few sections should therefore be thought of as containing the 'essentials'. Our exposition throughout this chapter is based on those given in the author's article [10] and the paper [30].

For a given j an element of $\mathbb{Z}$ or $\mathbb{N}$, we often need to refer to the subsets of $\mathbb{Z}$ or $\mathbb{N}$ consisting of those elements which are greater than or equal to j. We establish the following general notation to refer to these subsets.

Notation 1.0.1. Let $\mathbf{I}$ be one of $\mathbb{Z}$, $\mathbb{N}$ or $\mathbb{R}$, and let $i \in \mathbf{I}$, then we denote by $\mathbf{I}_i$ the subset of $\mathbf{I}$ consisting of those elements greater than or equal to i.

1.1 Words

Definition 1.1.1. A finite set of symbols $X := \{x_1, x_2, \ldots, x_n\}$ will be called an *alphabet*. An element $x \in X$ will be called a *letter*. Let X be an alphabet, then the *size of* X, denoted $|X|$ is the number of symbols in X.

Definition 1.1.2. Let $X := \{x_1, x_2, \ldots, x_n\}$ be an alphabet, a *word or string* (over the alphabet X) is a finite or infinite sequence $w_1 w_2 w_3 \ldots$ of elements of X. The *length of a word* Γ over X, is the length of the sequence representing the word, and we denote it by $|\Gamma|$. The *empty word*, denoted by the symbol ϵ, is the unique word of length zero. Let Γ be a word over X, then we call Γ an *infinite word* if $|\Gamma| = \infty$ otherwise we call Γ a *finite word*.

Notation 1.1.3. Let X be a finite alphabet. Set X^* to be the set of all finite words over X, and set $X^+ := X^* \backslash \{\epsilon\}$. For $k \in \mathbb{N}$, set $X^k := \{\Gamma \in X_n^* \mid |\Gamma| = k\}$.

Remark 1.1.4. For $n \in \mathbb{N}_1$ let $\overline{\mathbf{n}}$ be the ordered set $\{0, \ldots, n-1\}$. Let $N \in \{\overline{\mathbf{n}} \mid n \in \mathbb{N}_1\} \sqcup \{\mathbb{N}\}$ and $X := \{x_1, \ldots, x_n\}$ be an alphabet. A function $f : N \to X$ defines a non-empty word $f(0)f(1)\ldots$ over X. In particular, every non-empty word over X can be identified with a function $f : N \to X$ for some $N \in \{\overline{\mathbf{n}} \mid n \in \mathbb{N}_1\} \sqcup \{\mathbb{N}\}$ with cardinality equal to the length of the word.

Definition 1.1.5. Let X be a finite alphabet and let $\Gamma \in X^*$ be a word. Suppose that Γ is the sequence $\gamma_1\gamma_2\ldots\gamma_k$, $\gamma_i \in X$ for all $1 \leqslant i \leqslant k$. Let $1 \leqslant j \leqslant |\Gamma|$, then we call γ_j the j^{th} *letter* of Γ.

Definition 1.1.6. Given two words $\Gamma_1, \Gamma_2 \in X^*$ the *concatenation of* Γ_1 with Γ_2 is the word $\Gamma_1\Gamma_2$.

Remark 1.1.7. Concatenation with the empty word returns the original word. Thus X^* forms a monoid under concatenation.

Notation 1.1.8. Let X be a finite alphabet and let $\Gamma \in X^*$ be a word. For $j \in \mathbb{N}$, $j \geqslant 1$, we shall denote by Γ^j the word $\Gamma_1\Gamma_2 \ldots \Gamma_j$ where $\Gamma_i = \Gamma$ for all $1 \leqslant i \leqslant j$. Set $\Gamma^0 := \epsilon$.

Definition 1.1.9. Let X be a finite alphabet. By Definition 1.1.4 we may identify a map $f : \mathbb{N} \to X$ with the infinite word $f(0)f(1)f(2)\ldots$. Analogously, a map $g : \mathbb{Z} \to X$ defines a *bi-infinite word* consisting of the sequence $\ldots g(-2)g(-1)g(0)g(1)g(2)\ldots$. We shall identify the map g with the bi-infinite sequence $\ldots g(-2)g(-1)g(0)g(1)g(2)\ldots$.

Remark 1.1.10. We shall sometimes omit the prefixes infinite or bi-infinite when it is clear from the context that the word in question is infinite or bi-infinite.

Notation 1.1.11. Let X be a finite alphabet. Set X^{ω} to be the set of all infinite words over X and set $X^{\mathbb{Z}}$ to be the set of all bi-infinite words over X. Given a bi-infinite word $g \in X^{\mathbb{Z}}$ and $i \in \mathbb{Z}$ we shall sometimes denote by g_i the letter $g(i)$ and we shall call this letter the i^{th} letter of g.

Definition 1.1.12. Given an alphabet X and two words u, v over X such that $u \in X^*$ and $v \in X^* \sqcup X^{\omega}$, we say that u is a *prefix* of v if $v = uv_1$ for some word $v_1 \in X^* \sqcup X^{\omega}$; we say that v_1 is a *suffix* of v.

Remark 1.1.13. The empty word is a prefix and suffix of every word.

Definition 1.1.14. Define a relation '$\leqslant$' on $X^* \sqcup X^{\omega}$ by $\nu \leqslant \eta$, for $\nu \in X^*$ and $\eta \in X^* \sqcup X^{\omega}$, if and only if ν is a prefix of η. If $\nu \nleqslant \eta$ and $\eta \nleqslant \nu$, then we say that ν and η are *incomparable* and we denote this by $\nu \perp \eta$.

Remark 1.1.15. It is easy to see that the relation '$\leqslant$' on $X^* \sqcup X^{\omega}$ of Definition 1.1.14 is a partial order.

Notation 1.1.16. Let $\nu, \mu \in X^*$ and suppose $\nu \leqslant \mu$. Let $\tau \in X^*$ be such that $\mu = \nu\tau$, then we set $\mu - \nu := \tau$. Let $U \subset X^* \sqcup X^{\omega}$. If $\nu \in X^*$ is such that ν is a prefix of every element of U then we shall indicate this by writing $\nu \leqslant U$. Furthermore, for $U \subset X^* \sqcup X^{\omega}$ and $\nu \in X^*$ such that $\nu \leqslant U$, set $U - \nu := \{\delta \in X^* \sqcup X^{\omega} \mid \nu\delta \in U\}$.

Definition 1.1.17. Let X be an alphabet, and let $\delta \in X^{\omega} \sqcup X^*$. Let $i \in \mathbb{N}$ be such that $1 \leqslant i \leqslant |\delta|$, and let $\delta_i \in X^i$ be the unique word of length i such that $\delta_i \leqslant \delta$, then we call δ_i the *length* i *prefix of* δ.

Definition 1.1.18. Given two alphabets X and Y, and subsets $V \subset X^*$, and $W \subset Y^* \sqcup Y^{\omega}$ we shall denote by VW the set $\{vw \mid v \in V, w \in W\}$.

We now introduce the notion of antichains.

Definition 1.1.19. A finite, ordered, subset $\overline{\mathbf{u}} \subset X^*$ is called an *antichain* if for any distinct pair $\nu, \eta \in \overline{\mathbf{u}}$ we have $\nu \perp \eta$. An antichain $\overline{\mathbf{u}}$ is called *complete* if for any $\tau \in X^*$ there is some $\nu \in \overline{\mathbf{u}}$ such that $\tau \leqslant \nu$ or $\nu \leqslant \tau$.

Remark 1.1.20. Observe that the definition of an antichain only depends on the partial order on the set X^*. Thus, for an arbitrary set Y with a partial order $\prec$ on Y we may define antichains on Y as in Definition 1.1.19.

Definition 1.1.21. Let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite alphabet and let $\overline{\mathbf{u}} = \{u_1, u_2, \ldots, u_l\} \subset X^+$ be an antichain of length l. For $1 \leqslant i \leqslant l$, a *single expansion of* $\overline{\mathbf{u}}$ *(over* u_i*)* is the ordered set $\{u_1, \ldots, u_{i-1}, u_i x_1, u_i x_2, \ldots, u_i x_n, u_{i+1}, \ldots, u_n\}$. A k*-fold expansion of* $\overline{\mathbf{u}}$ for $k \in \mathbb{N}_1$ is a set $\overline{\mathbf{u}}'$ such that there is a finite sequence $\overline{\mathbf{u}} := \overline{\mathbf{u}}_0, \overline{\mathbf{u}}_1, \ldots, \overline{\mathbf{u}}_k := \overline{\mathbf{u}}'$ where $\overline{\mathbf{u}}_i$, $1 \leqslant i \leqslant k$, is a single expansion of $\overline{\mathbf{u}}_{i-1}$. An antichain $\overline{\mathbf{v}}$ is called an *expansion* of $\overline{\mathbf{u}}$ if it is equal to $\overline{\mathbf{u}}$ or it is a k-fold expansion of $\overline{\mathbf{u}}$ for some $k \in \mathbb{N}_1$.

Remark 1.1.22. Let $\overline{\mathbf{u}}$ be a (complete) antichain over some finite alphabet X. If $\overline{\mathbf{u}}'$ is a single expansion of $\overline{\mathbf{u}}$ over some element $\nu \in \overline{\mathbf{u}}$, then $\overline{\mathbf{u}}'$ is still a (complete) antichain. Thus any expansion of $\overline{\mathbf{u}}$ is a (complete) antichain. Moreover, if $\overline{\mathbf{w}}$ is an expansion of $\overline{\mathbf{u}}$ then $|\overline{\mathbf{u}}| \equiv |\overline{\mathbf{w}}| \mod |X| - 1$; if $\overline{\mathbf{w}}$ is a k-fold expansion of $\overline{\mathbf{u}}$ then $|\overline{\mathbf{w}}| = |\overline{\mathbf{u}}| + k(|X| - 1)$.

Now that we have the necessary preliminaries on words and related notions, in the next section we establish the essential definitions and facts concerning graphs and trees that we will require in this work.

1.2 Graphs and trees

This section shall be concerned with introducing some of the main geometric objects we will be working with.

Definition 1.2.1. A *directed graph* is a tuple $G = (V, E, \iota, \tau)$ where V is a set of symbols and is called the *vertex set* of G, E is a *set of edges* of G and $\iota, \tau : E \to V$ are maps. For an edge $e \in E$, $(e)\iota$ is called the *start of the edge* and $(e)\tau$ is called the *end or terminus of the edge*. Elements of V are called *vertices or nodes* and elements of E are called *edges*. The *size of* G, denoted $|G|$, is the size of the vertex set of G. If $|G|$ is finite then we say that G is *finite*, otherwise we say that G is *infinite*.

Definition 1.2.2. Let $G = (V, E, \iota, \tau)$ be a directed graph and let $V' \subset V$. The *subgraph* of G *induced* by V' is the graph $G' = (V', E', \iota', \tau')$ where $E' \subset E$ consists precisely of those edges e such that $(e)\iota, (e)\tau \in V'$; the maps ι' and τ' are the restrictions of ι and τ to E'.

Definition 1.2.3. Let $G = (V, E, \iota, \tau)$ be a directed graph. A *path (in* G*)* is a sequence of edges (alternatively a finite or infinite word over the set of edges) $p := e_1, e_2, e_3 \ldots$ such that, for any $1 \leqslant i \leqslant |p|$, $(e_{i+1})\iota = (e_i)\tau$. If the sequence is finite then we say that the path is *finite* otherwise we say that the path is *infinite*. The *length of the path*, denoted $|p|$, is the length of the sequence. We call $(e_1)\iota$ the *initial vertex of* p and, if $|p| < \infty$, we call $(e_{|p|})\tau$ the *terminal vertex of* p.

Definition 1.2.4. Let $G = (V, E, \iota, \tau)$ be a directed graph. Two vertices u and ν of G are called *connected* if there is a path p in G with initial vertex u and final vertex ν. The directed graph G is called *connected*, if for any pair u, ν of distinct vertices of G, there is a path p with initial and final vertices μ and ν, respectively, such that $\{\mu, \nu\} = \{u, \nu\}$. If for any pair of vertices u, ν of G there is a path with initial vertex u and final vertex ν then we say that G is *strongly connected.*

Definition 1.2.5. An *(undirected) graph* is a directed $G = (V, E, \iota, \tau)$ such that for every edge $e \in E$, there is an edge $e^{-1} \in E$, called the *inverse of* e, such that $(e^{-1})\iota = (e)\tau$ and $(e^{-1})\tau = (e)\iota$.

Remark 1.2.6. For a graph $G = (V, E, \iota, \tau)$ we shall identify every edge $e \in E$ with its inverse e^{-1} and denote both by e. Thus a graph can be described by a triple $G = (V, E, \xi)$, where V is the vertex set of G, E is the set of edges, and $\xi : E \to V(2)$ where $V(2) := \{\{u, \nu\} \mid u, \nu \in V\}$. For an edge $e \in E$, $(e)\xi = \{u, \nu\}$ is called the *ends of* e and we say e *is an edge between* u *and* ν. For an undirected graph $G = (V, E, \xi)$, a path (in G) is a sequence $p := e_1, e_2, e_3, \ldots$ of edges with ends $\{\nu_1, \nu_2\}, \{\nu_2, \nu_3\}, \{\nu_3, \nu_4\} \ldots$. Notice that an undirected graph is connected if and only if it is strongly connected.

Definition 1.2.7. Let $G = (V, E, \xi)$ be a graph, and let $\nu_1, \nu_2 \in V$. Then we say that ν_1 is *incident* to ν_2 if there is an edge $e \in E$ with $(e)\xi = \{\nu_1, \nu_2\}$. A vertex $\nu \in V$ is said to *belong to* an edge $e \in E$ if $\nu \in (e)\xi$. If a vertex ν_1 is incident to a vertex ν_2 then we call ν_2 a *neighbour* of ν_1.

Definition 1.2.8. Let $G = (V, E, \xi)$ be a graph. An edge e is called a *loop* if $(e)\xi$ is a singleton.

Definition 1.2.9. Let $G = (V, E, \xi)$ be a graph. A path $p = e_1, e_2 \ldots$ defines a sequence $\nu_1, \nu_2 \ldots$ of vertices of length $|p| + 1$ ($|p|$ for an infinite path), where, for $1 \leqslant i \leqslant p$, $(e_i)\xi = \{\nu_i, \nu_{i+1}\}$. We call this sequence the *vertex sequence of* p. For a path p of G and a vertex $\nu \in V$, we say that p *visits* ν *or* ν *is visited by* p if ν belongs to an edge in p (alternatively, ν occurs in the vertex sequence of p). For a finite path $p = e_1, \ldots, e_k$, with vertex sequence $\nu_1, \nu_2, \ldots, \nu_{k+1}$, we call ν_1 the *initial vertex (of the path)* and ν_{k+1} the *final vertex (of the path)*. A finite path is called a *circuit* if its initial vertex is equal to its final vertex. A circuit, $e_1, e_2, \ldots, e_k$ is called *basic* if $k \geqslant 3$ and the only repeated vertex in its vertex sequence is the start vertex. A *geodesic* is a shortest path connecting any two vertices.

Definition 1.2.10. Let $G = (V, E, \xi)$ be a graph. Then two vertices v_1 and v_2 are said to be *connected* if there is a path in G with start vertex v_1 and final vertex v_2. A graph $G = (V, E, \xi)$ is said to be *connected* if any pair of distinct vertices are connected.

Definition 1.2.11. Let $G_1 = (V_1, E_1, \xi_1)$ and $G_2 = (V_2, E_2, \xi_2)$ be graphs. A pair of maps $\gamma_E : E_1 \to E_2, \gamma_V : V_1 \to V_2$ is called a *graph homomorphism* if whenever $e \in E_1$ with $(e)\xi_1 = \{u, v\}$, then $((e)\gamma_E)\xi_2 = \{(u)\gamma_V, (v)\gamma_V\}$. If the maps γ_E and γ_V are also injective/ surjective/ bijective then we say that γ is a *graph monomorphism/ epimorphism/ isomorphism*.

Definition 1.2.12. Let $G = (V, E, \iota, \tau)$ be a directed or undirected graph and let S be an alphabet. A map $\mathscr{L} : E \to S$ is called an *edge labelling of* G. The pair $(G, \mathscr{L})$ is called a *labelled graph*.

Definition 1.2.13. A graph $G = (V, E, \xi)$ is called *simple* if it has no loops and for any pair v_1, v_2 of vertices of V, there is at most one edge $e \in E$ for which $(e)\xi = \{v_1, v_2\}$.

Remark 1.2.14. For a simple graph $G = (V, E, \xi)$, an edge $e \in E$ is characterised precisely by $(e)\xi$. In particular, a simple graph can be represented by a pair $G = (V, E)$ where $E \subset V(2) = \{\{u, v\} \mid u, v \in V\}$. From henceforth, we denote all simple graphs by pairs $G = (V, E)$ of vertices and edges, where $E \subset V(2)$. For a simple graph $G = (V, E)$ we represent a path p as a sequence $v_1, v_2, v_3, \ldots$ of vertices where for all $1 \leqslant i \leqslant |p|$, $\{v_i, v_{i+1}\} \in E$.

Definition 1.2.15. A simple graph G is called a *tree* if it is connected and contains no basic circuits. A tree G is said to be *rooted* if there is a distinguished vertex; we call this distinguished vertex the *root (of the tree)*. Let G be a rooted tree, then we call a vertex of G with only one neighbour a *leaf* of the tree G. A vertex which is neither a leaf nor the root will be called an *internal vertex*.

Remark 1.2.16. Observe that in a tree T there is a unique geodesic connecting any two vertices.

Definition 1.2.17. A *rooted* n*-ary tree* $\mathcal{T}_n$, is a rooted tree with an infinite set of vertices such that every vertex apart from the root has precisely $n + 1$ neighbours, and the root has n distinct neighbours.

Remark 1.2.18. There is only one rooted n-ary tree up to isomorphism, thus, we fix a representative tree $\mathcal{T}_n$. We phrase all subsequent discussion with regards to this tree, and refer to it as *the rooted* n*-ary tree*.

Definition 1.2.19. Let $\mathcal{T}_n$ be the rooted n-ary tree. For $i \in \mathbb{N}_1$, let v be a vertex such that there is a geodesic from the root ϵ of $\mathcal{T}_n$ to v of length i, then we say that v *is at level* i *or the level of* v *is* i. We set the level of the root to be 0. For a vertex v of $\mathcal{T}_n$, we denote by $l(v)$ the level of the vertex v.

Remark 1.2.20. Let $\mathcal{T}_n$ be the rooted n-ary tree, as there is a unique geodesic connecting any two vertices it follows that every vertex of $\mathcal{T}_n$ has a unique level. Moreover, for $i \in \mathbb{N}_1$, every vertex at level i is incident to precisely one vertex at level $i - 1$. Thus, since every vertex apart from the root is incident to $n + 1$ distinct vertices, the number of vertices at level i of $\mathcal{T}_n$, for $i \in \mathbb{N}$, is precisely n^i.

Definition 1.2.21. Let $\mathcal{T}_n$ be the rooted n-ary tree with root ϵ. Let v be a vertex of $\mathcal{T}_n$ not equal to the root, then we call the unique neighbour of v at level $l(v) - 1$, the *parent of* v. Let v be any vertex of $\mathcal{T}_n$, the *children of* v are the n distinct neighbours of v at level $l(v) + 1$. A *child* of v is therefore a neighbour of v at level $l(v) + 1$.

Definition 1.2.22. Let $\mathcal{T}_n$ be the rooted n-ary tree and let v be an internal vertex of $\mathcal{T}_n$. The *subtree* $\mathcal{T}_n(v)$ *of* $\mathcal{T}_n$ *rooted at* v, is the rooted subtree of $\mathcal{T}_n$, with root v, induced by v and all vertices of $\mathcal{T}_n$ which are connected to v by a path which does not visit the parent of v.

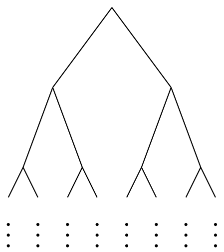
Remark 1.2.23. Let $\mathcal{T}_n$ be the rooted n-ary tree and let v be a vertex of $\mathcal{T}_n$, then $\mathcal{T}_n(v)$ is a rooted n-ary tree. Moreover, any internal vertex of $\mathcal{T}_n(v)$ is at a level of $\mathcal{T}_n$ strictly greater than $l(v)$.

Definition 1.2.24. We define a partial ordering of the nodes of the n-ary tree $\mathcal{T}_n$ as follows: for two vertices v_1, v_2 of $\mathcal{T}_n$, we say that $v_1 \leqslant v_2$, if v_2 is a vertex of $\mathcal{T}_n(v)$. Two vertices v_1, v_2 are incomparable if $v_1 \nleqslant v_2$ and $v_2 \nleqslant v_1$. An *antichain* is, as in Section 1.1, an ordered set $\{v_1, v_2, \ldots, v_k\}$ of pairwise incomparable vertices. An antichain $\overline{\mathbf{v}}$ is called *complete*, if for every vertex u of $\mathcal{T}_n$ there is a vertex $v \in \overline{\mathbf{v}}$ such that $u \leqslant v$ or $v \leqslant u$.

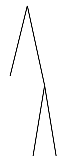
Remark 1.2.25. Let $\mathcal{T}_n$ be the rooted n-ary tree, and let $\{v_1, v_2, \ldots v_k\}$ be an antichain. Observe that the subtrees $\mathcal{T}_n(v_1), \ldots, \mathcal{T}_n(v_k)$ have no vertices in common.

Definition 1.2.26. Let $\mathcal{T}_n = (V, E)$ be the rooted n-ary tree with root ϵ, and let $\overline{\mathbf{v}} := \{v_1, v_2, \ldots v_k\}$ be an antichain. The subtree $\mathcal{T}_n^{\overline{\mathbf{v}}}$ is the rooted subtree of $\mathcal{T}_n$, with root ϵ, induced by the all the vertices of $\mathcal{T}_n$ except those which are internal vertices of some $\mathcal{T}_n(v_i)$ for some $1 \leqslant i \leqslant k$. We may also refer to $\mathcal{T}_n^{\overline{\mathbf{v}}}$ as *the complement of the subtrees* $\mathcal{T}_n(v_1), \ldots, \mathcal{T}_n(v_k)$.

Remark 1.2.27. Let $\mathcal{T}_n$ be the rooted n-ary tree, and let $\overline{\mathbf{v}}$ be an antichain of $\mathcal{T}_n$. All elements of $\overline{\mathbf{v}}$ are vertices of $\mathcal{T}_n^{\overline{\mathbf{v}}}$, moreover, if v is not the antichain consisting only of the root, any $v \in \overline{\mathbf{v}}$ is a leaf. All other vertices of the tree $\mathcal{T}_n^{\overline{\mathbf{v}}}$, apart from the root which has n neighbours, have $n+1$ neighbours. Thus the leaves of $\mathcal{T}_n^{\overline{\mathbf{v}}}$ are precisely the vertices in $\overline{\mathbf{v}}$. If $\overline{\mathbf{v}}$ is the antichain consisting only of the root, then $\overline{\mathbf{v}}$ is the rooted tree with precisely one vertex. If $\overline{\mathbf{v}}$ is a complete antichain, then $\mathcal{T}_n^{\overline{\mathbf{v}}}$ is a finite rooted tree.



(a) The binary tree $\mathcal{T}_2$.



(b) A finite rooted tree.

Figure 1.1: The binary tree and a finite subtree

Remark 1.2.28. All rooted trees will be drawn with the root at the top, and with the neighbours of a vertex v at level $l(v)+1$, immediately below the vertex v. Thus, as in the examples in Figure 1.1, we mainly leave the roots of such trees unlabelled.

Definition 1.2.29. Let $\mathcal{T}_n$ be the rooted n-ary tree, then a *finite (rooted) subtree* of $\mathcal{T}_n$ is a (rooted) subtree $\mathcal{T}_n^{\overline{\mathbf{v}}}$ for some complete antichain $\overline{\mathbf{v}}$ of $\mathcal{T}_n$.

Remark 1.2.30. Note that the definition of finite subtrees that we have above is stronger than definitions that occur elsewhere in the literature, since in our definition, all internal vertices of a subtree have n children.

Definition 1.2.31. A simple graph $G = (V, E)$ is called a *forest* if it is a disjoint union of trees.

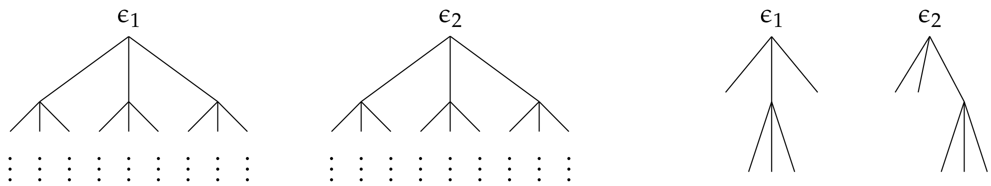
Definition 1.2.32. An r*-rooted* n*-ary forest* $\mathcal{T}_{n,r}$, is a forest which is the disjoint union of r rooted n-ary trees.

Remark 1.2.33. There is a unique r-rooted, n-ary forest up to isomorphism. We shall thus fix a representative $\mathcal{T}_{n,r}$ for the r-rooted n-ary forest. We refer to this representative forest as the r*-rooted* n*-ary forest* in subsequent discussions.

Definition 1.2.34. Let $\mathcal{T}_{n,r}$ be the r-rooted n-ary forest. A *finite* r*-rooted subforest* of $\mathcal{T}_{n,r}$ is the disjoint union of r finite subtrees with roots corresponding to the roots of the r n-ary trees making up the forest.

Remark 1.2.35. We partially order the nodes of $\mathcal{T}_{n,r}$ as follows. Let ϵ_i, $1 \leqslant i \leqslant r$ be the roots of the rooted n-ary trees making up the forest $\mathcal{T}_{n,r}$. For each n-ary tree in the forest, we order the nodes of the tree as in Definition 1.2.24. For $i \neq j$, $i, j \in \{1, \ldots, r\}$, set any node connected to the root ϵ_i to be incomparable to any node connected to the root ϵ_j. This gives a partial ordering of $\mathcal{T}_{n,r}$. We may thus refer to antichains and complete antichains of $\mathcal{T}_{n,r}$. By definition a complete antichain $\overline{\mathbf{u}}$ of $\mathcal{T}_{n,r}$ can be written as a disjoint union $\overline{\mathbf{u}}_1 \sqcup \overline{\mathbf{u}}_2 \sqcup \ldots \sqcup \overline{\mathbf{u}}_r$ where $\overline{\mathbf{u}}_i$, $1 \leqslant i \leqslant r$, is a complete antichain for the n-ary tree with root ϵ_i. Thus $\mathcal{T}_n^{\overline{\mathbf{u}}_i}$ is a finite subtree of the n-ary tree with root

ϵ_i. Hence $\mathcal{T}_n^{\overline{\mathbf{u}}_i} \sqcup \ldots \sqcup \mathcal{T}_n^{\overline{\mathbf{u}}_r}$ is finite subforest of $\mathcal{T}_{n,r}$. Observe that, as in Remark 1.2.27, the leaves of $\sqcup_{1 \leqslant i \leqslant r} \mathcal{T}_n^{\overline{\mathbf{u}}_i}$ are precisely the vertices of $\overline{\mathbf{u}}$ not equal to a root. Moreover, by Definition 1.2.34, it follows that all finite subforests of $\mathcal{T}_{n,r}$ are obtained in this way. Thus, for a complete antichain $\overline{\mathbf{u}}$ of $\mathcal{T}_{n,r}$, we write $\mathcal{T}_{n,r}^{\overline{\mathbf{u}}}$ for the finite subforest $\sqcup_{1 \leqslant i \leqslant r} \mathcal{T}_n^{\overline{\mathbf{u}}_i}$, where the $\overline{\mathbf{u}}_i$ $(1 \leqslant i \leqslant r)$ are disjoint and each $\overline{\mathbf{u}}_i$ $(1 \leqslant i \leqslant r)$ is a complete antichain for the n-ary tree with root ϵ_i.



(a) The forest $\mathcal{T}_{3,2}$

(b) A finite rooted subforest of $\mathcal{T}_{3,2}$

Figure 1.2: The r-rooted n-ary forest and a subforest

In the next section we introduce special types of labelled graphs called *transducers* and *automata*. These may also be viewed as machines with a set of states which process inputs according to certain rules.

1.3 Automata and Transducers

In this section we describe some of the key machinery that we use to understand the various groups of homeomorphisms that this work considers. The 'machinery' referred to are called automata and transducers. We begin by first introducing automata, then we define transducers. We introduce in some sense the basics of these objects, further concepts will be revealed as needed in relevant sections.

1.3.1 Automata

As a transducer is a special type of automaton we shall begin by first defining automata.

Definition 1.3.1. An *automaton* is a triple $A := \langle X, Q_A, \pi_A \rangle$ where:

(1) X is a finite alphabet,

(2) Q_A is a finite or infinite set of states of the automaton,

(3) $\pi_A : X \times Q_A \to Q_A$ is the *transition function*.

Remark 1.3.2. Note that finite automata as defined in Definition 1.3.1 are elsewhere in the literature called deterministic finite automata, see for instance [35].

Inductively we may extend the domain of π_A to X^* according to the following rules:

$$\pi_A(\epsilon, q) = q \text{ for all } q \in Q_A, \tag{1.1}$$
$$\text{for } \Gamma \in X^* \text{ and } x \in X \text{ we have } \pi_A(\Gamma x, q) = \pi_A(x, \pi(\Gamma, q)) \text{ for all } q \in Q_A. \tag{1.2}$$

Hence an automaton can be thought of as a machine with a finite set of states which reads letters from an input tape and changes states according to some rules. Notice that by (1.1) and (1.2) above, all states of an automata process words from left to right.

Definition 1.3.3. Let $A = \langle X, Q_A, \pi_A \rangle$, then the *size of* A, denoted $|A|$, is the number of states of A. If $|Q_A| < \infty$ then we say that A is *finite*, otherwise we say that A is *infinite*.

Usually we represent a finite automaton A by a finite, labelled directed graph as in Figure 1.3. The vertices of the graph correspond to the states of the automaton. There is an edge *from* a state q_1 *to* a state q_2 labelled with an x, whenever, in the automata A we have $\pi_A(x, q_1) = q_2$. We shall identify the automata A with the labelled graph representing it. Thus when we refer to an automaton, A, we shall mean both the labelled graph representing it, and the tuple $\langle X, Q_A, \pi_A \rangle$.

Definition 1.3.4. Let $A = \langle X, Q_A, \pi_A \rangle$ and $B = \langle X, Q_B, \pi_b \rangle$ be automata over the same alphabet X. We say that A and B are *isomorphic* if there is a bijection $\phi : Q_A \to Q_B$ satisfying, for any $i \in X_n$, $\pi_A(i, q) = p$ if and only if $\pi_B(i, (q)\phi) = (p)\phi$.

Given an automaton $A = \langle X, Q_A, \pi_A \rangle$ we may fix a state q of A from which to begin reading inputs, in this case we say that A is *initialised* at the state q and we denote this by A_q. In the graph of the automaton, we indicate that A is initialised at q by double circling the state q.

Below we have an example of an initial automaton:

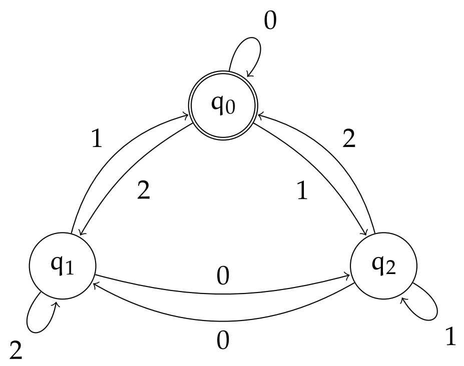


Figure 1.3: An example of an initial automaton.

1.3.2 Transducers

Definition 1.3.5. A *transducer* is a quintuple $T = \langle X_I, X_O, Q_T, \pi_T, \lambda_T \rangle$ such that:

(1) X_I and X_O are finite alphabets called the *input* and *output* alphabets respectively.

(2) Q_T is a set consisting of states of the transducer.

(3) $\pi_T : X_I \times Q_T \to Q_T$ is the *transition function*.

(4) $\lambda_T : X_I \times Q_T \to X_O^*$ is the *re-write* function.

A transducer can be thought of as an automaton which, as well as reading inputs from an input tape, may also write strings from the output alphabet onto the output tape.

As in (1.1) and (1.2) we may extend the domain of π_T to $X_I^* \times Q_T$. We may also analogously extend the domain of λ_T to $X_I^* \times Q_T$. We do this as follows. First set $\lambda_T(\epsilon, q) = \epsilon$ and $\pi_T(\epsilon, q) = q$ for all states $q \in Q_T$. Now for $\Gamma \in X_I^*$ and $x \in X_I$ we have:

$$\lambda_T(\Gamma x, q) = \lambda_T(\Gamma, q)\lambda_T(x, \pi_T(\Gamma, q)). \tag{1.3}$$

Notice that we process inputs from left to right.

Now let $\delta \in X_I^\omega$, and, for $i \in \mathbb{N}$, let $\delta_i \in X_I^i$ be the length i prefix of δ. Observe that, for a given state $q \in Q_T$, there is a unique element ρ of $X_I^\omega \sqcup X_I^*$ satisfying the following conditions:

(i) $\lambda_T(\delta_i, q) \leqslant \rho$ for all $i \in \mathbb{N}$,

(ii) For any prefix ν of ρ there is an $i \in N$ such that $\nu \leqslant \lambda_T(\delta_i, q)$.

Notation 1.3.6. Let $T = \langle X_I, X_O, Q_T, \pi_T, \lambda_T \rangle$ be a transducer. For $\delta \in X_I^{\omega}$ we denote by $\lambda_T(\delta, q)$ the unique word $\rho \in X_I^{\omega} \sqcup X_I^*$ satisfying conditions (i) and (ii).

Given a transducer $T = \langle X_I, X_O, Q_T, \pi_T, \lambda_T \rangle$ and an element $\gamma \in X_I^*$ we shall use the language *read* γ *through a state* q_1 *(to a state* q_2*)* or *read* γ *from a state* q_1 *(to a state* q_2*)* to indicate the transition $\pi_T(\gamma, q_1) = q_2$. We may also append the phrase *with output* ξ, for some $\xi \in X_O^*$, if in addition $\lambda_T(\gamma, q_1) = \xi$.

Notation 1.3.7. Given a transducer $T = \langle X_I, X_O, Q_T, \pi_T, \lambda_T \rangle$, denote by $\mathcal{A}(T)$ the automaton $\langle X_I, Q_T, \pi_T \rangle$. We call $\mathcal{A}(T)$ the *underlying automaton* of T. Given a word $\gamma \in X_I^*$ and $q \in Q_T$, it will sometimes be convenient to use the notation $(\gamma)T_q$ for the word $\lambda_T(\gamma, q)$. On rare occasions we also extend the notation $(\gamma)T_q$ to words $\gamma \in X_I^{\omega}$.

If we fix a state $q \in Q_T$ from which we begin processing inputs, then we say that T is initialised at state q and we denote this by T_q. We call T_q an *initial transducer*. We say the transducer T is *finite* if the underlying automaton $\mathcal{A}(T)$ is finite, otherwise we say that T is *infinite*. The *size of a transducer* T is the size of the underlying automaton $\mathcal{A}(T)$.

Definition 1.3.8. Let T_{q_0} be an initial transducer and let q be any state of T not equal to q_0. We call q a *non-initial state (of* T_{q_0}*)*.

Notation 1.3.9. If T_{q_0} is an initial transducer, then we shall write T for the transducer T_{q_0} with no states initialised. We shall sometimes call the transducer T the *underlying transducer of* T_{q_0}. This is not to be confused with the underlying automaton $\mathcal{A}(T)$ of T.

Notation 1.3.10. For a transducer T with input and output alphabet both equal to an alphabet X, we write $T = \langle X, Q_T, \pi_T, \lambda_T \rangle$. We call such a transducer a *transducer over the (alphabet)* X.

Usually we represent a finite transducer $T = \langle X_I, X_O, Q_T, \pi_T, \lambda_T \rangle$ by a finite labelled graph. The vertices of the graph will correspond to the states of T. For every state, $q \in Q$, and for every $x \in X_I$, there is a directed edge from q to $\pi(x, q)$ labelled by '$x|\lambda(x, q)$'. If we fix an initial state $q \in Q_T$, then we represent this in the graph of the transducer T, by doubly circling the state q.

We have the following definitions:

Definition 1.3.11. A transducer, T is said to be *synchronous* if for every letter $x \in X_I$, and all states $q \in Q_T$, we have $|\lambda_T(x, q)| = 1$. Otherwise we say the transducer is *asynchronous*.

Definition 1.3.12. Let T be a transducer, and let q_0 be a state of T so that T_{q_0} is an initial transducer. A state $q \in Q_T$ is said to be *accessible* if there is a word $w \in X_I^*$ such that $\pi_T(w, q_0) = q$, and we say that q is *accessible (in* T_{q_0}*) by* w. If $w \in X_I^+$ then we say that q *is strictly accessible (in* T_{q_0} *by* w*)*. If all the states of T_{q_0} are (strictly) accessible, we say that T_{q_0} *is (strictly) accessible*.

Definition 1.3.13. Let T_{q_0} be an initial transducer. A state $q \in Q_T$ is called *non-trivially accessible* if there is a word $w \in X_I^*$ such that q is accessible in T_{q_0} by w and $\lambda_T(w, q_0) \neq \epsilon$.

Below are examples of synchronous and asynchronous transducers.

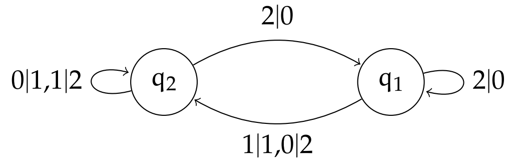


Figure 1.4: Example of a synchronous transducer

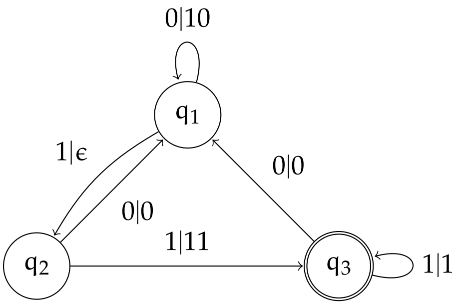


Figure 1.5: Example of an asynchronous transducer with initial state q_3

Eventually we shall see that transducers which satisfy certain non-degeneracy conditions induce continuous functions on Cantor space. To this end we begin by introducing Cantor space: we present several different ways we will be viewing Cantor space in this work.

1.4 Cantor Space from different points of view

In this section we introduce different ways of thinking of Cantor space. Subsequently we shall move between these different points of view as necessary.

From henceforth we shall fix $n \in \mathbb{N}_2$ and $r \in \mathbb{N}_1$ a number strictly less than n. We shall also fix $X_n := \{0, 1, \ldots, n-1\}$ a finite alphabet, and fix the ordering $0 < 1 < 2 < \ldots < n-1$ of elements of X_n. Fix also $\dot{\mathbf{r}} := \{\dot{1}, \dot{2}, \ldots, \dot{r}\}$ and let the elements of $\dot{\mathbf{r}}$ be ordered $\dot{1} < \dot{2} < \ldots < \dot{r}$. We begin by establishing some further notation.

Notation 1.4.1. Set $X_{n,r}^{+} := \dot{\mathbf{r}}X_n^{*}$, $X_{n,r}^{*} := X_{n,r}^{+} \sqcup \{\epsilon\}$ and $X_{n,r}^{\omega} := \dot{\mathbf{r}}X_n^{\omega}$. For $k \in \mathbb{N}$ let $X_{n,r}^{k}$ be the subset of $X_{n,r}^{*}$ consisting of all elements of length k. Observe that $X_{n,r}^{1} = \dot{\mathbf{r}}$. We identify, as in Definition 1.1.9, $X_{n,r}^{\omega}$ with the set of maps $f : \mathbb{N} \to \dot{\mathbf{r}} \sqcup X_n$ such that $f(i) \in \dot{\mathbf{r}}$ if and only if $i = 0$. Therefore given an infinite word w in $X_{n,r}^{\omega}$, we denote by w_i the letter $w(i)$, for $i \in \mathbb{N}$ and we call this letter the i^{th} letter of w.

We extend the partial order $\leqslant$ of Definition 1.1.14 to a partial order on the set $X_{n,r}^{*}$ in the natural way and again denote this partial order by $\leqslant$. We retain the symbol '$\perp$' for two incomparable words in $X_{n,r}^{*}$. More specifically, for two words $\nu, \eta \in X_{n,r}^{*}$, we say that $\nu \leqslant \eta$ if ν is a prefix of η; if $\nu \nleqslant \eta$ and $\eta \nleqslant \nu$ then ν is incomparable to η i.e $\nu \perp \eta$. We also extend the notion of antichains and complete antichains (Definition 1.1.19) to subsets of $X_{n,r}^{*}$.

We now define a metric on X_n^{ω}, $X_n^{\mathbb{Z}}$ and $X_{n,r}^{\omega}$ which makes each of these sets homeomorphic to Cantor space.

First we define a metric d_n on X_n^{ω}.

Definition 1.4.2. Let $d_n : X_n^{\omega} \times X_n^{\omega} \to \mathbb{R}_0$ be given by:

$$d_n(w_1, w_2) = \begin{cases} \dfrac{1}{k+1}, \text{ k is minimal such that } w_1(k) \neq w_2(k) \\ 0, \text{ otherwise.} \end{cases} \tag{1.4}$$

The metric d_∞ on $X_n^{\mathbb{Z}}$, which we define below, is a natural extension of d_n to bi-infinite sequences.

Definition 1.4.3. Let $d_\infty : X_n^{\mathbb{Z}} \times X_n^{\mathbb{Z}} \to \mathbb{R}_0$ be given by:

$$d_\infty(w_1, w_2) = \begin{cases} \dfrac{1}{k+1}, k \in \mathbb{N} \text{ is minimal such that } w_1(k) \neq w_2(k) \text{ or } w_1(-k) \neq w_2(-k) \\ 0, \text{ otherwise.} \end{cases} \tag{1.5}$$

Finally the metric on $X_{n,r}^{\omega}$ is the natural extension of d_n to $X_{n,r}^{\omega}$ and we again denote this new metric by d_n appealing to the context to clarify any ambiguities that may arise.

Note that each of the spaces $X_{n,r}^{\omega}$, X_n^{ω} and $X_n^{\mathbb{Z}}$ are homeomorphic to each other. This follows from the well known result that any compact, totally disconnected, perfect, metric space is homeomorphic to Cantor space. Moreover, one can also define homeomorphisms from each of these spaces into X_2^{ω}.

Notation 1.4.4. We also use the symbol $\mathfrak{C}_{n,r}$ for the Cantor space $X_{n,r}^{\omega}$, and the symbol $\mathfrak{C}_n$ for the Cantor space X_n^{ω}

We will be considering groups of self-homeomorphisms of each of the spaces $X_{n,r}^{\omega}, X_n^{\omega}, X_n^{\mathbb{Z}}$. We begin with the spaces $X_{n,r}^{\omega}$ and X_n^{ω}. Later on we consider groups of self-homeomorphisms of $X_n^{\mathbb{Z}}$.

Notation 1.4.5. Let $H(\mathfrak{C}_{n,r})$ and $H(\mathfrak{C}_n)$ denote the group of self-homeomorphisms of $\mathfrak{C}_{n,r}$ and $\mathfrak{C}_n$ respectively.

We now describe a basis for each of the topologies induced by the metrics d_n on X_n^{ω} and $X_{n,r}^{\omega}$. However, we first establish some further notation.

Notation 1.4.6. Let $\nu \in X_n^*$, set $U_\nu := \{\nu\delta \mid \delta \in \mathfrak{C}_n\}$. For $\nu \in X_{n,r}^+$, set $U_\nu := \{\nu\delta \mid \delta \in \mathfrak{C}_n\}$. For $\nu = \epsilon$ take $U_\nu := \mathfrak{C}_{n,r}$. For a subset $I \subset X_n^* \sqcup X_{n,r}^*$ we shall set $U(I) = \{U_\nu \mid \nu \in I\}$.

Remark 1.4.7. For $\nu \in X_n^*$ or $\nu \in X_{n,r}^*$, the subset of $\mathfrak{C}_n$ or $\mathfrak{C}_{n,r}$ given by U_ν is a clopen set. That U_ν is open follows since for any point $\delta \in U_\nu$, and letting $\mathcal{X}$ be one of $\mathfrak{C}_n$ or $\mathfrak{C}_{n,r}$ as appropriate, the open ball $\mathbf{B}(\delta, \dfrac{1}{|\nu|+1}) := \{\rho \in \mathcal{X} \mid d_n(\delta,\rho) \leqslant 1/|\nu|+1\}$ is a subset of U_ν. That U_ν is closed follows since the complement of U_ν is either empty if $\nu = \epsilon$ otherwise it is equal to the union $\cup_{\mu \in X_n^{|\nu|}, \mu \neq \nu} U_\mu$.

Notation 1.4.8. Let $\mathbf{B}_n := U(X_n^*) = \{U_\nu \mid \nu \in X_n^*\}$ and let $\mathbf{B}_{n,r} := U(X_{n,r}^*) = \{U_\nu \mid \nu \in X_{n,r}^*\}$.

Remark 1.4.9. The sets $\mathbf{B}_n$ and $\mathbf{B}_{n,r}$ form a basis of clopen sets for the topology induced by the metric d_n on $\mathfrak{C}_n$ and $\mathfrak{C}_{n,r}$ respectively. This follows since, setting $\mathcal{X}$ to be one of $\mathfrak{C}_n$ or $\mathfrak{C}_{n,r}$, for any point $\delta in \mathcal{X}$ and any number $a \in \mathbb{R}_1$, the open ball $\mathbf{B}(\delta, a)$ coincides with U_ν for $\nu \in X_n^*$ a prefix of δ of appropriate length. For a point $x \in \mathfrak{C}_n$, we shall use the phrase *open neighbourhood of* x *(or neighbourhood of* x*)* for an open set U containing x.

Notation 1.4.10. For a subset U of $\mathfrak{C}_n$ or $\mathfrak{C}_{n,r}$ and a continuous function h with domain $\mathfrak{C}_n$ or $\mathfrak{C}_{n,r}$, we shall use the notation $(U)h$ for the set $\{(x)h \mid x \in U\}$.

We shall also view $\mathfrak{C}_n$ and $\mathfrak{C}_{n,r}$ as the boundaries of the rooted n-ary tree (Definition 1.2.17) and the r-rooted n-ary forest (Definition 1.2.32) respectively. We do this by assigning a label to the edges of $\mathcal{T}_n$ and $\mathcal{T}_{n,r}$ (Section 1.2) such that the concatenation of labels of infinite geodesics beginning at the root in $\mathcal{T}_n$ and $\mathcal{T}_{n,r}$ correspond to elements of $\mathfrak{C}_n$ and $\mathfrak{C}_{n,r}$ respectively. We begin with $\mathfrak{C}_n$.

Let v be a node of $\mathcal{T}_n$. Label the n edges leaving v to nodes at level $l(v)+1$, successively with the symbols $0, 1 \ldots, n-1$. This gives a bijection between the set of edges from v to nodes at level $l(v)+1$, and the set X_n. Recall that X_n is an ordered set, thus we may assume that the edges of the graph are ordered according to the ordering induced by X_n.

Notation 1.4.11. Let $\{v, u\}$ be an edge of $\mathcal{T}_n$, we denote by $\mathtt{lab}(\{v,u\})$ the label of the edge.

Given an infinite geodesic, $v_1, v_2, \ldots,$ where v_1 is the root of $\mathcal{T}_n$, we may uniquely identify this geodesic with the element $\mathtt{lab}(\{v_1,v_2\})\mathtt{lab}(\{v_2,v_3\})\ldots \in \mathfrak{C}_n$. Moreover each element of $\mathfrak{C}_n$ corresponds to a unique infinite geodesic of $\mathcal{T}_n$ starting at the root. Thus elements of $\mathfrak{C}_n$ correspond to infinite geodesics in $\mathcal{T}_n$ beginning at the root.

The labelling on the edges of $\mathcal{T}_n$ induces a labelling on the vertices. This is because if v is an internal vertex of $\mathcal{T}_n$, we may identify v with the word $\gamma_v \in X_n^{\omega}$ labelling the unique geodesic in $\mathcal{T}_n$ from the root to v. The root may then be labelled by the empty word ϵ. Henceforth, we identify an internal vertex of the tree $\mathcal{T}_n$ with the word labelling the unique geodesic starting at the root to the vertex and we identify the root with the empty word. This labelling of the vertices of $\mathcal{T}_n$ gives a bijection from the nodes of $\mathcal{T}_n$ to the set X_n^*. Thus, we may assume that the vertices at level i, $i \in \mathbb{N}$, are ordered according to the lexicographic ordering induced from X_n^*. Figure 1.6 depicts the labelled rooted binary tree $\mathcal{T}_2$, the lexicographic ordering of the edges is indicated by drawing smaller edges to the left of larger ones; likewise the ordering of the vertices at each level is indicated by having smaller vertices appearing to the left of larger ones.

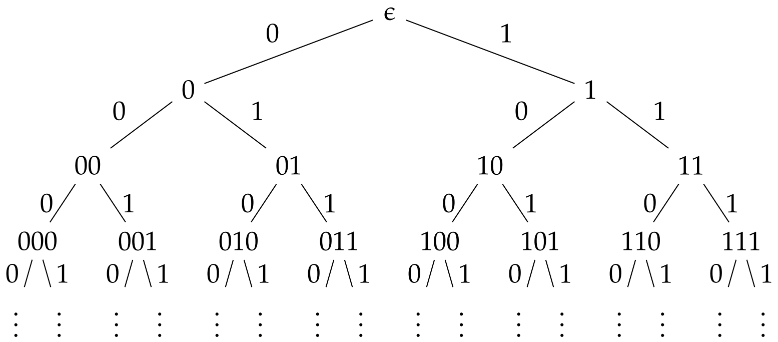


Figure 1.6: Labelled binary tree

Observe that a clopen set $U_\nu \in \mathbf{B}_n$, for $\nu \in X_n^*$, corresponds to the set of infinite geodesics beginning at the root which pass through the vertex ν of $\mathcal{T}_n$.

Now we extend the labelling of $\mathcal{T}_n$ to the r-rooted forest $\mathcal{T}_{n,r}$. Let ν be an internal vertex of one of the n-ary trees of the forest $\mathcal{T}_{n,r}$. We label, as in the case of $\mathcal{T}_n$, the n edges from ν to vertices on level $l(\nu)+1$, bijectively with the symbols $0, 1, \ldots, n-1$. Now let r_a, for $1 \leqslant a \leqslant r$, be the root of the a^{th} n-ary tree in the forest $\mathcal{T}_{n,r}$. Label all the edges leaving r_a successively with the symbols $\dot{a}j$ for $0 \leqslant j \leqslant n-1$. Thus, we have a bijection from the set of edges leaving the root r_a to the set $\dot{a}X_n$. Therefore, as in the case of $\mathcal{T}_n$, an infinite geodesic beginning at the root of an n-ary tree in $\mathcal{T}_{n,r}$, corresponds uniquely to an element of $\mathfrak{C}_{n,r}$. Moreover, any given element of $\mathfrak{C}_{n,r}$ corresponds to a unique geodesic beginning at the root of a particular n-ary tree in $\mathcal{T}_{n,r}$. Thus, whenever we depict the r-rooted n-ary forest, as in Figure 1.7, we shall have: (1.) vertices at level $i+1$, for $i \in \mathbb{N}$, appearing below vertices at level i, (2.) vertices at level i arranged so that a vertex appears to the left of vertices larger than it in the lexicographic ordering of $X_{n,r}^*$ and, (3.) edges leaving a vertex arranged so that smaller edges are to the left of larger ones.

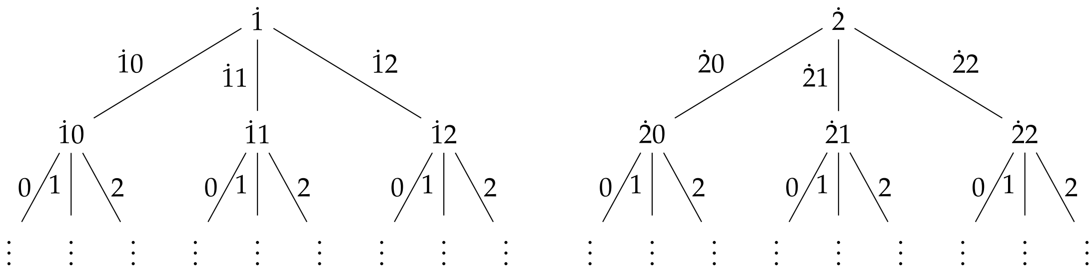


Figure 1.7: Labelled forest $\mathcal{T}_{3,2}$

This last geometric representation of the spaces $\mathfrak{C}_n$ and $\mathfrak{C}_{n,r}$, turns out to be very useful for visualising the action of certain subgroups of $H(\mathfrak{C}_n)$ and $H(\mathfrak{C}_{n,r})$ on $\mathfrak{C}_n$ and $\mathfrak{C}_{n,r}$ respectively as we will see later on.

However, we now demonstrate how transducers satisfying a certain non-degeneracy condition induce homeomorphisms of Cantor space.

1.5 Transducers and continuous functions on Cantor space

In this section, we demonstrate how finite initial transducers satisfying a certain non-degeneracy condition induce continuous functions of Cantor space $\mathfrak{C}_n$ and $\mathfrak{C}_{n,r}$. Our exposition follows those given in [30] and the author's article [10]. We begin first with $\mathfrak{C}_n$ and then make slight modifications for $\mathfrak{C}_{n,r}$.

1.5.1 Transducers and continuous functions on $\mathfrak{C}_n$

In this subsection, we demonstrate that finite initial transducers over the alphabet X_n satisfying a certain non-degeneracy condition induce continuous functions on $\mathfrak{C}_n$. We also identify those homeomorphisms of $\mathfrak{C}_n$ which may be represented by finite initial transducers satisfying the non-degeneracy condition.

All transducers in this subsection shall be over the alphabet X_n.

Definition 1.5.1. Let $A = \langle X_n, Q_A, \pi_A, \lambda_A \rangle$ be a transducer, then A is called *non-degenerate* if there is a $k \in \mathbb{N}_1$ such that for all words $\Gamma \in X_n^k$, and for any q of A, we have $\lambda_A(\Gamma, q) \neq \epsilon$.

Remark 1.5.2. Given $\delta \in \mathfrak{C}_n$, a non-degenerate initial transducer A_{q_0} over the alphabet X_n necessarily satisfies: $\lambda_A(\delta, q) \in \mathfrak{C}_n$ where q is any accessible state of A_{q_0}.

Henceforth, unless stated otherwise, all transducers introduced will be assumed non-degenerate and initial transducers are also assumed to be accessible. Therefore, we shall mostly omit these phrases in subsequent discussions; whenever we include them, we choose to do so for emphasis.

Notation 1.5.3. Let A be a transducer over the alphabet X_n, and let q be a state of A. We denote by h_{A_q}, the map on $\mathfrak{C}_n$ defined by $\delta \mapsto \lambda_A(\delta, q)$. When it is clear that q is a state of A then we shall use the symbol h_q for h_{A_q}.

We have the following result:

Proposition 1.5.4. *Let A be a transducer over the alphabet X_n and q be a state of A, then $h_q : \mathfrak{C}_n \to \mathfrak{C}_n$ is continuous.*

Proof. Let $\delta \in \mathfrak{C}_n$. Let $\rho = (\delta)h_q$ and let U be any open neighbourhood of ρ. Let $\eta \in X_n^+$ be such that U_η is an open neighbourhood of ρ contained in U. Let j be minimal such that for any $\Gamma \in X_n^j$, we have $|\lambda_A(\Gamma, q)| > |\eta|$ (such a j exists since A is non-degenerate). Now let $\Delta \in X_n^j$ be such that $\delta = \Delta\bar{\delta}$ for some $\bar{\delta} \in \mathfrak{C}_n$. Observe that $\lambda_A(\Delta, q)$ has a prefix η since Δ is a prefix of δ and an element of X_n^j, and $\rho \in U_\eta$. Therefore, for any point $\psi \in U_\Delta$, we have $(\psi)h_q \in U_\eta$. □

Notation 1.5.5. Let A be a transducer over $\mathfrak{C}_n$ and let q be a state of A. We write $\mathrm{im}(q)$ for the image of the map $h_q : \mathfrak{C}_n \to \mathfrak{C}_n$. We will also refer to $\mathrm{im}(q)$ as *the image of* q.

Proposition 1.5.4 demonstrates that an initial non-degenerate transducer A_{q_0} induces several continuous functions on $\mathfrak{C}_n$. In what follows we shall show how to construct an initial transducer A_{q_0} from a homeomorphism $h : \mathfrak{C}_n \to \mathfrak{C}_n$ such that $h_{q_0} = h$. We need a few definitions beforehand.

Notation 1.5.6. Let $U \subseteq \mathfrak{C}_n$. We set $(U)\mathrm{rt} \in X_n^* \sqcup X_n^\omega$ to be the greatest common prefix of all elements of U.

Definition 1.5.7. Let $h : \mathfrak{C}_n \to \mathfrak{C}_n$ be a continuous function. Define $\theta_h : X_n^* \to X_n^* \sqcup X_n^\omega$ by $(\nu)\theta_h \mapsto ((U_\nu)h)\mathrm{rt}$. Given a transducer A_{q_0} over $\mathfrak{C}_n$ and q a state of A then we will write θ_q for the function θ_{h_q}.

Remark 1.5.8. Note that for $\nu \in X_n^*$, and a continuous function $h : \mathfrak{C}_n \to \mathfrak{C}_n$, $(\nu)\theta_h$ is the greatest common prefix of the set $(U_\nu)h$. Further observe, since h is continuous, that for a sequence ν_i in X_n^*, $i \in \mathbb{N}$, such that for $i < j$, ν_i is a proper prefix of ν_j, then either $|(\nu_i)\theta_h|$ also tends to infinity with i, or there is some i such that for all $j \geqslant i$ $(\nu_j)\theta_h \in \mathfrak{C}_n$.

Remark 1.5.9. If h is the continuous map sending all of $\mathfrak{C}_n$ to a point $x \in \mathfrak{C}_n$, then clearly for any $\nu \in X_n^*$ $(\nu)\theta_h = x$. If $h : \mathfrak{C}_n \to \mathfrak{C}_n$ is a homeomorphism, we observe that for any point $\nu \in X_n^*$, since $(U_\nu)h$ is clopen (homeomorphisms map clopen sets to clopen sets), then $(\nu)\theta_h \in X_n^*$.

Definition 1.5.10. Let $h : \mathfrak{C}_n \to \mathfrak{C}_n$ be a continuous function and let $\nu \in X_n^*$. Define $h_\nu : \mathfrak{C}_n \to \mathfrak{C}_n$ as follows:

$$(x)h_\nu = (\nu x)h - (\nu)\theta_h.$$

We call h_ν a *local map of* h.

Remark 1.5.11. For $\nu \in X_n^*$ and $h : \mathfrak{C}_n \to \mathfrak{C}_n$ continuous, the local map h_ν is the restriction of h to U_ν where we delete the prefix $(\nu)\theta_h$ from all outputs. Observe that if h is a continuous and injective map from $\mathfrak{C}_n$ to itself, and $\nu \in X_n^+$, then h_ν is also injective by definition.

Proposition 1.5.12. *Let $h : \mathfrak{C}_n \to \mathfrak{C}_n$ be a continuous function, and let $\nu \in X_n^*$, then h_ν is also continuous.*

Proof. Let $x \in \mathfrak{C}_n$ and let U be an open neighbourhood of $y := (x)h_\nu$. Let $\mu \in X_n^+$ such that $y \in U_\mu \subseteq U$. Observe that since $(\nu x)h = (\nu)\theta_h y$ it follows that $U_{(\nu)\theta_h\mu}$ is an open neighbourhood of $(\nu x)h$. Since h is continuous, there is an open neighbourhood V of νx such that $(V)h \subseteq U_{(\nu)\theta_h\mu}$. Since V is a neighbourhood of νx, there is some $\rho \in X_n^+$ such that $U_{\nu\rho} \subset V$ is an open neighbourhood of νx. Therefore U_ρ is an open neighbourhood of x such that $(U_\rho)h_\nu \subseteq U_\mu \subseteq U$. □

We have the following fact about local maps and the function θ_h for h a homeomorphism.

Proposition 1.5.13. *Let $h : \mathfrak{C}_n \to \mathfrak{C}_n$ be a homeomorphism and let $\nu, \mu \in X_n^*$. The following holds:*

$$(\nu\mu)\theta_h = (\nu)\theta_h(\mu)\theta_{h_\nu}.$$

Proof. Observe that $(\nu)\theta_h$ is a prefix of $(U_{\nu\mu})h$. Moreover, $(\mu)\theta_{h_\nu}$ is the greatest common prefix of the set $(U_{\nu\mu})h - (\nu)\theta_h$. It therefore follows that $(\nu)\theta_h(\mu)\theta_{h_\nu} = (\nu\mu)\theta_h$. □

We now give the construction in [30] for building an initial transducer A_{q_0} from a homeomorphism $h \in H(\mathfrak{C}_n)$ such that $h_{q_0} = h$.

Construction 1.5.14. Let $h : \mathfrak{C}_n \to \mathfrak{C}_n$ be a homeomorphism. Construct an infinite transducer $A_\epsilon = \langle X_n, X_n^*, \pi_A, \lambda_A \rangle$. For ν a state of A_ϵ and $i \in X_n$, the transition and output functions of A_ϵ are defined as follows:

$$\pi_A(i, \nu) = \nu i \text{ and } \lambda_A(i, \nu) = (\nu i)\theta_h - (\nu)\theta_h.$$

Observe that for ν a state of A_ϵ and $i \in X_n$, $\lambda_A(i, \nu) = (i)\theta_{h_\nu}$.

We have the following result:

Theorem 1.5.15. *Let $h : \mathfrak{C}_n \to \mathfrak{C}_n$ be a homeomorphism and let A_ϵ be the initial transducer constructed from h as in Construction 1.5.14. For any point $x \in \mathfrak{C}_n$ we have, $\lambda_A(x, \epsilon) = (x)h$.*

Proof. Let $x \in \mathfrak{C}_n$ and suppose that $x = x_0x_1x_2\ldots$ for $x_i \in X_n$. To demonstrate that $\lambda_A(x, \epsilon) = (x)h$, it suffices, by Remarks 1.5.8 and 1.5.9, to show that for every non-empty finite prefix w of x, $\lambda_A(w, \epsilon) = (w)\theta_h$. We proceed by induction on $|w|$.

By definition $\lambda(x_0, \epsilon) = (x_0)\theta_h$ which is a (possibly empty) prefix of $(x)h$. This proves the base case.

Assume, for $m \in \mathbb{N}_1$ and for $w = x_0 \ldots x_m$, that $\lambda_A(w, \epsilon) = (w)\theta_h$. Now consider $\lambda_A(wx_{m+1}, \epsilon)$. By definition of the transition and output function we have

$$\lambda_A(wx_{m+1}, \epsilon) = \lambda_A(w, \epsilon)\lambda_A(x_{m+1}, w).$$

Notice that $\lambda_A(x_{m+1}, w) = (wx_{m+1})\theta_h - (w)\theta_h$. Therefore, by the inductive assumption, we have:

$$\lambda_A(wx_{m+1}, \epsilon) = (w)\theta_h((wx_{m+1})\theta_h - (w)\theta_h) = (wx_{m+1})\theta_h$$

as required. □

Remark 1.5.16. A similar proof to that given above demonstrates that for $x \in \mathfrak{C}_n$ and for $\nu \in X_n^*$, we have $\lambda_A(x, \nu) = (x)h_\nu$. A consequence of this is that the states of A_ϵ correspond to the local maps of h. Furthermore, A_ϵ is non-degenerate by Remarks 1.5.8 and 1.5.2. In particular Theorem 1.5.15 demonstrates that $h_\epsilon = h$.

We now modify the Construction 1.5.14 for homeomorphisms of $\mathfrak{C}_{n,r}$. To do this, we will need to adjust our definition of transducers slightly so that they induce continuous functions on $\mathfrak{C}_{n,r}$. The approach taken follows that given in the author's article [10].

1.5.2 Transducers and continuous functions on $\mathfrak{C}_{n,r}$

In order to allow for transducers to model homeomorphisms of Cantor space $\mathfrak{C}_{n,r}$ for $1 < r < n$ we shall have to adjust and impose certain additional restrictions on the transducers.

Definition 1.5.17. An *initial transducer* $A_{q_0} = \langle \dot{\mathbf{r}}, X_n, R_A, S_A, \pi_A, \lambda_A, q_0 \rangle$ *on* $\mathfrak{C}_{n,r}$ is a tuple such that:

(a) the input and output alphabets are equal to the disjoint union $\dot{\mathbf{r}} \sqcup X_n$,

(b) the set of states Q_A of A is the disjoint union $R_A \sqcup S_A$ and $q_0 \in R_A$,

(c) $\pi_A : (\dot{\mathbf{r}} \times \{q_0\}) \sqcup (X_n \times Q_A \backslash \{q_0\}) \to Q_A \backslash \{q_0\}$ is the transition function and $\lambda_A : (\dot{\mathbf{r}} \times \{q_0\}) \sqcup (X_n \times Q_A \backslash \{q_0\}) \to X_{n,r}^* \sqcup X_n^*$ is the output function.

The functions π_A and λ_A also satisfy the following restrictions:

(R.1) Whenever we transition from a state in R_A to another state in R_A we output the empty word:

$$\text{If } q_1, q_2 \in R_A \text{ and } \pi_A(x, q_1) = q_2, \text{ then } \lambda_A(x, q_1) = \epsilon.$$

(R.2) Whenever we transition from a state in R_A to a state in S_A we output a word in $X_{n,r}^+$:

$$\text{If } q \in R_A, \text{ and } x \in X_n \text{ such that } \pi_A(x, q) \in S_A, \text{ then } \lambda_A(x, q) \in X_{n,r}^+.$$

(R.3) We always transition from a state in S_A into another state in S_A, and the output is a word in X_n^*:

$$\text{If } q \in S_A, \text{ then } \forall x \in X_n, \lambda_A(x, q) \in X_n^*, \text{ and } \pi_A(x, q) \in S_A.$$

(R.4) Whenever we read a word from a state $q \in Q_A$ to the same state q the output of this transition is non-empty:

$$\text{If } q \in Q_A, \text{ and } w \in X_n^+ \sqcup X_{n,r}^+ \text{ such that } \pi_A(w, q) = q \text{ then, } \lambda_A(w, q) \neq \epsilon.$$

Remark 1.5.18. Notice that we can only read a letter from $\dot{\mathbf{r}}$ from the state q_0. Furthermore, by item (c) of Definition 1.5.17, after processing any element of $\dot{\mathbf{r}}$ from q_0, we leave the state q_0 and never return to it. We usually write $A_{q_0} = \langle \dot{\mathbf{r}}, X_n, R_A, S_A, \pi_A, \lambda_A \rangle$ when it is clear that q_0 is the initial state of A_{q_0}.

We extend the domain of π_A and λ_A to $(\dot{\mathbf{r}} \times \{q_0\}) \sqcup (X_n^+ \times Q_A \backslash \{q_0\})$ by the following rules: for $w \in X_n^+$, $i \in X_n$ and $q \in Q_A \backslash \{q_0\}$ we have,

$$\pi_A(wi, q) = \pi_A(i, \pi_A(w, q)) \text{ and } \lambda_A(wi, q) = \lambda_A(w, q)\lambda_A(i, \pi(w, q)); \tag{1.6}$$

for $w \in X_{n,r}^+$, $i \in X_n$ we have,

$$\pi_A(wi, q_0) = \pi_A(i, \pi_A(w, q_0)) \text{ and } \lambda_A(wi, q_0) = \lambda_A(w, q_0)\lambda_A(i, \pi(w, q_0)). \tag{1.7}$$

Notation 1.5.19. For a state $q \in Q_A \backslash \{q_0\}$, and a word $\rho \in X_n^{\omega}$, we set $\lambda_A(\rho, q)$ as in Notation 1.3.6. Thus, for $a \in \dot{\mathbf{r}}$ and $\delta \in X_n^{\omega}$, we set $\lambda_A(a\delta, q_0) := \lambda_A(a, q_0)\lambda_A(\delta, \pi_A(a, q_0))$.

Remark 1.5.20.

(a) Conditions (R.1) to (R.4) imply that whenever we read a word $\delta \in \mathfrak{C}_n$ through a state in S_A the output is again an infinite word in $\mathfrak{C}_n$.

(b) Conditions (R.1) to (R.4) mean that we only ever output a symbol from $\dot{\mathbf{r}}$ once. Conditions (R.1) and (R.4) mean that the transducer must exit R_A for long enough inputs. Conditions (R.2) to (R.4) and the preceding sentence imply that whenever we process an element of $\mathfrak{C}_{n,r}$ through the state q_0 of A_{q_0}, the output is also an element of $\mathfrak{C}_{n,r}$. In analogy with Definition 1.5.1, a transducer satisfying conditions (R.1) to (R.3) but which does not satisfy condition (R.4) is called a *degenerate* transducer. On the other hand, a transducer satisfying conditions (R.1) to (R.4) is called *non-degenerate*.

(c) Let $A_{q_0} = \langle \dot{\mathbf{r}}, X_n, R_A, S_A, \pi_A, \lambda_A, q_0 \rangle$ be a non-degenerate initial transducer over $\mathfrak{C}_{n,r}$ and let $q \in Q\backslash\{q_0\}$ be an accessible state of A_{q_0}. The initial transducer A_q induces a map h_{A_q} from $\mathfrak{C}_n$ defined by $(\delta)h_{A_q} = \lambda_A(\delta, q)$. The range of h_{A_q} is $\mathfrak{C}_n$ if $q \in S_A$ and $\mathfrak{C}_{n,r}$ otherwise. The initial transducer A_{q_0} induces a map $h_{A_{q_0}}$ from $\mathfrak{C}_{n,r}$ to itself.

Notation 1.5.21. Let $A_{q_0} = \langle \dot{\mathbf{r}}, X_n, R_A, S_A, \pi_A, \lambda_A, q_0 \rangle$ be a non-degenerate initial transducer over $\mathfrak{C}_{n,r}$. For a state $q \in Q_A$, we use the notation h_q for h_{A_q} whenever it is unambiguous that q is a state of A.

Remark 1.5.22. Let $A_{q_0} = \langle \dot{\mathbf{r}}, X_n, R_A, S_A, \pi_A, \lambda_A, q_0 \rangle$ be a non-degenerate initial transducer over $\mathfrak{C}_{n,r}$. For any state q of A, the induced map h_q is continuous and the argument demonstrating this is almost identical to the proof of Proposition 1.5.4.

Notation 1.5.23. Let $A_{q_0} = \langle \dot{\mathbf{r}}, X_n, R_A, S_A, \pi_A, \lambda_A, q_0 \rangle$ be a non-degenerate initial transducer over $\mathfrak{C}_{n,r}$ and let $q \in Q_)A$. As before (Notation 1.5.5), we use $\operatorname{im}(q)$ to denote the image of the map h_q. We shall refer to $\operatorname{im}(q)$ as the image of q.

Throughout this work we assume, unless otherwise stated, that all transducers over $\mathfrak{C}_{n,r}$ are non-degenerate.

In what follows we outline a procedure given in [10] for constructing a transducer A_{q_0} over $\mathfrak{C}_{n,r}$ from a homeomorphism h of $\mathfrak{C}_{n,r}$ such that $h_{q_0} = h$. We first extend the definition of the map rt of Definition 1.5.6 to $\mathfrak{C}_{n,r}$ and the definition of the map θ_g, for g a continuous function of $\mathfrak{C}_n$, given in Definition 1.5.7 to homeomorphisms of $\mathfrak{C}_{n,r}$ and we do so in the natural way. (It is possible to extend θ_g to continuous functions on $\mathfrak{C}_{n,r}$,, however, as we only focus on homeomorphisms of $\mathfrak{C}_{n,r}$ in this work, we have not done so.) For $g \in H(\mathfrak{C}_{n,r})$ and a sequence $\nu_i \in X_{n,r}^*$, $i \in \mathbb{N}$, such that $|\nu_i|$ tends to infinity as i tends to infinity, θ_g still retains the property that $|(\nu_i)\theta_g|$ tends to infinity with i also. For A_{q_0} a transducer over $\mathfrak{C}_{n,r}$ and q a state of A_{q_0} we again use the notation θ_q for θ_{h_q}. We now adjust the definition of local maps (Definition 1.5.10) for homeomorphisms of $\mathfrak{C}_n$ to homeomorphisms of $\mathfrak{C}_{n,r}$.

Notation 1.5.24. Let $h \in H(\mathfrak{C}_{n,r})$ be a homeomorphism. Let $P_h \subset X_{n,r}^*$ be the maximal set satisfying the following conditions:

(1) for all $\nu \in P_h$ there is an $\dot{a} \in \dot{\mathbf{r}}$ such that $(U_\nu)h \subseteq U_{\dot{a}}$;

(2) if μ is a proper prefix of some element of P_h then there are distinct $\dot{a}_1$ and $\dot{a}_2$ such that $(U_\mu)h \cap U_{\dot{a}_1} \neq \emptyset$ and $(U_\mu)h \cap U_{\dot{a}_2} \neq \emptyset$.

Lemma 1.5.25. *Let $h \in H(\mathfrak{C}_{n,r})$ be a homeomorphism. The set P_h exists and is a complete antichain for $X_{n,r}^*$.*

Proof. Let $\dot{a} \in \dot{\mathbf{r}}$. As the set $U_{\dot{a}}$ is clopen and h is a homeomorphism, its preimage under h, $(U_{\dot{a}})h^{-1}$ is also clopen and so compact. Therefore, there is a minimal finite subset $P_h(\dot{a}) \subset X_{n,r}^*$ such that the set $\{U_\eta \mid \eta \in P_h(\dot{a})\}$ is an open cover of $(U_{\dot{a}})h^{-1}$ and, for any $\eta \in P_h(\dot{a})$, $U_\eta \subset (U_{\dot{a}})h^{-1}$.

Minimality ensures that if $\mu \in X_{n,r}^*$ is a proper prefix of an element of $P_h(\dot{a})$, then there is a $\dot{b} \in \dot{\mathbf{r}}$ distinct from $\dot{a}$ such that $(U_\mu)h \cap \dot{b} \neq \emptyset$. From this we conclude that $P_h(\dot{a})$ is an antichain since any two elements of $P_h(\dot{a})$ must be incomparable.

For each $\dot{a} \in \dot{\mathbf{r}}$ form the set $P_h(\dot{a})$ as above. Observe that for any pair $\dot{a}$, $\dot{b}$ of distinct elements of $\dot{\mathbf{r}}$, it must be the case that $P_h(\dot{a}) \cap P_h(\dot{b}) = \emptyset$. This is because for any $\nu \in P_h(\dot{a})$ and $\eta \in P_h(\dot{b})$, we have that $(U_\nu)h \subseteq U_{\dot{a}}$ and $(U_\eta)h \subseteq U_{\dot{b}}$, therefore for any $\nu \in P_h(\dot{a})$ and $\eta \in P_h(\dot{b})$ ν and η are incomparable. This means that the set $P_h := \cup_{\dot{a}\in\dot{\mathbf{r}}} P_h(\dot{a})$ is an antichain for $X_{n,r}^*$. We now argue that it is a complete antichain.

Observe that since h is a homeomorphism and $\sqcup_{\dot{a}\in\dot{\mathbf{r}}} U_{\dot{a}} = \mathfrak{C}_{n,r}$, we must have that $\cup_{\dot{a}\in\dot{\mathbf{r}}}\{U_\nu \mid \nu \in P_h(\dot{a})\}$ is an open cover of $\mathfrak{C}_{n,r}$. Therefore, for any $\tau \in X_{n,r}^*$, either τ is a prefix of some element of P_h, or some element of P_h is a prefix of τ. Hence P_h is a complete antichain for $X_{n,r}^*$. Moreover, from this we may deduce that P_h is the maximal set satisfying the conditions (1) and (2) above. □

Remark 1.5.26. Let $g \in H(\mathfrak{C}_{n,r})$ and let $\mu \in X_{n,r}^*$ be a proper prefix of an element of P_g. Then it follows that $(\mu)\theta_g = \epsilon$. If μ has an element of P_g as a prefix, then $(\mu)\theta_g \in X_{n,r}^+$.

Definition 1.5.27. Let $h \in H(\mathfrak{C}_{n,r})$, and let $\mu \in X_{n,r}^+$. Define a map $h_\mu : \mathfrak{C}_n \to \mathfrak{C}_n \sqcup \mathfrak{C}_{n,r}$ by $(\delta)h_\mu = \rho$ for $\delta \in \mathfrak{C}_n$ and $\rho \in \mathfrak{C}_n \sqcup \mathfrak{C}_{n,r}$ such that $(\mu\delta)h = (\mu)\theta_h\rho$. For $\mu \in X_{n,r}^+$ we call h_μ the *local action* of h at μ.

Remark 1.5.28. For $h \in H(\mathfrak{C}_{n,r})$ and $\mu \in X_{n,r}^+$, we have the following observations about h_μ which should be compared with Remark 1.5.11, Proposition 1.5.12 and Proposition 1.5.13.

(a) If μ is a proper prefix of an element of P_h, then h_μ has range $\mathfrak{C}_{n,r}$, otherwise h_μ has range $\mathfrak{C}_n$,

(b) h_μ is injective and continuous ,

(c) furthermore, if $\nu \in X_n^*$, then $(\mu\nu)\theta_h = (\mu)\theta_h(\nu)\theta_{h_\mu}$. This follows, since for $\mu\nu\delta \in U_{\mu\nu}$, we have, $(\mu\nu\delta)h = (\mu)\theta_h(\nu\delta)h_\mu$.

We now outline the procedure for constructing an infinite initial transducer which represents a given homeomorphism of $\mathfrak{C}_{n,r}$.

Construction 1.5.29. Let $h \in H(\mathfrak{C}_{n,r})$ be a homeomorphism. Define an initial transducer, $A_\epsilon = \langle \dot{\mathbf{r}}, X_n, R_A, S_A, \pi_A, \lambda_A, \epsilon\rangle$, where R_A is the set of all prefixes of elements of P_h, $S_A = X_{n,r}^* \backslash R_A$, and Q_A, the set of states of A, is given by $Q_A = R_A \sqcup S_A = X_{n,r}^*$. The transition and output functions obey the following rules:

(i) for all $\dot{a} \in \dot{\mathbf{r}}$, $\pi_A(\dot{a}, \epsilon) = \dot{a}$ and $\lambda_A(\dot{a}, \epsilon) = (\dot{a})\theta_h$;

(ii) for $\nu \in X_{n,r}^+ \backslash \{\epsilon\}$, and $i \in X_n$, $\pi_A(i, \nu) = \nu i$ and $\lambda_A(i, \nu) = (\nu i)\theta_h - (\nu)\theta_h$.

We make the following observations:

Remark 1.5.30.

(1) Since for all $\nu \in X_{n,r}^*$ we have $(\nu)\theta_h = \epsilon$ if ν is a proper prefix of P_h, and $(\nu)\theta_h \in X_{n,r}^+$ if ν is equal to some element of P_h, it follows that the transducer A_ϵ satisfies restrictions (R.1) to (R.3).

(2) In analogy with Theorem 1.5.15 and Remark 1.5.16, for $\delta \in \mathfrak{C}_n$ and $\nu \in X_{n,r}^*$, we have $\lambda_A(x, \nu) = (\delta)h_\nu$. The proof is almost identical and so we omit it. This means that A_ϵ satisfies restriction (R.4) also, and so A_ϵ is a non-degenerate transducer whose states correspond to the local actions of h.

Therefore, every homeomorphism of $\mathfrak{C}_{n,r}$ can be represented by a transducer. We still have some problems to address: the transducer constructed above might be one of many representing the homeomorphism h: is there a unique minimal transducer representing h? Is there a finite one? In the next subsection, given an initial transducer A_{q_0} which induces a continuous function h_{q_0} of $\mathfrak{C}_n$ or $\mathfrak{C}_{n,r}$, we introduce several procedures for trimming off redundancies in A_{q_0} which produce a unique minimal transducer with induced continuous function h_{q_0} on $\mathfrak{C}_n$ or $\mathfrak{C}_{n,r}$.

1.6 Minimising Transducers

In this section we demonstrate how to minimise an initial transducer. The algorithms we quote below are taken from [30]; in [10] the authors indicate how to adapt the algorithms for transducers over $\mathfrak{C}_{n,r}$. We shall omit proofs of the correctness of the algorithms, giving only an indication why these algorithms work. For **this section only** we drop the standing hypothesis that all transducers are accessible. We require the following definitions:

Definition 1.6.1. Let A_{q_0} and B_{p_0} be initial transducers over $\mathfrak{C}_n$ or $\mathfrak{C}_{n,r}$. Then A_{q_0} and B_{p_0} are said to be ω-*equivalent* if $h_{q_0} = h_{p_0}$. Let q_1 and q_2 be states of a transducer C. Then q_1 is said to be ω-*equivalent* to q_2 if the initial transducers C_{q_1} and C_{q_2} are ω-equivalent.

Notation 1.6.2. Let A be a transducer, and let q_1, q_2 be a states of A. If q_1 and q_2 are ω-equivalent states of A we shall write $q_1 \sim_\omega q_2$. We shall denote by $[q_1]_\omega$ the ω-equivalence class of q_1. That is $[q_1]_\omega := \{q \in Q_A \mid q \sim_\omega q_1\}$.

Definition 1.6.3. Let A_{q_0} be an initial transducer over $\mathfrak{C}_{n,r}$ or $\mathfrak{C}_n$. Let q be a state of A_{q_0}, then q is called a *state of incomplete response* if for some $i \in X_n \sqcup \dot{\mathbf{r}}$ we have $\lambda_A(i, q)$, if defined, is a proper prefix of $(i)\theta_{h_q}$.

Definition 1.6.4. An initial accessible transducer A_{q_0} is called *minimal* if it has no states of incomplete response and no pair of ω-equivalent states, otherwise we say that A_{q_0} is *non-minimal*.

Definition 1.6.5. A, not necessarily initial, transducer A is called *weakly minimal* if it has no pair of ω-equivalent states.

Definition 1.6.6. Let A_{q_0} and B_{p_0} be two minimal initial transducers over $\mathfrak{C}_n$ or $\mathfrak{C}_{n,r}$ with state sets Q_A and Q_B respectively. We say that A_{q_0} and B_{q_0} are *isomorphic* if there is a bijection $\mathfrak{b} : Q_A \to Q_B$ such that $(q_0)\mathfrak{b} = p_0$, and, for every state $q \in Q_A$, we have A_q and $B_{(q)\mathfrak{b}}$ are ω-equivalent. Now let C and D be non-initial transducers such that for any state p of C and q of D the transducers C_p and D_q are minimal. We say that C and D are *isomorphic* if there is a bijection $\mathfrak{b} : Q_C \to Q_D$ such that for every state $q' \in Q_C$ we have $C_{q'}$ and $D_{(q')\mathfrak{b}}$ are ω-equivalent.

Notation 1.6.7. Let A_{q_0} and B_{p_0} be minimal initial transducers. We write $A_{q_0} \cong_\omega B_{p_0}$ if A_{q_0} is isomorphic to B_{p_0}.

The algorithms below remove inaccessible states of an initial transducer over $\mathfrak{C}_n$ or $\mathfrak{C}_{n,r}$, 'repair' the states of incomplete response by correcting the output function of the transducer, and identify ω-equivalent states. A package for GAP ([25]) is currently being developed which implements these algorithms.

For the remainder of the section fix A_{q_0} a, not necessarily accessible, transducer over $\mathfrak{C}_{n,r}$ or $\mathfrak{C}_n$. Recall Definition 1.5.17 that for a transducer $A_{q_0} = \langle \dot{\mathbf{r}}, X_n, R_A, S_A, \pi_A, \lambda_A \rangle$ over $\mathfrak{C}_{n,r}$, $Q_A = R_A \sqcup S_A$. We minimise A_{q_0} by successively applying the following algorithms.

M1 (Removing inaccessible states) Remove from the set of states of A_{q_0} those which are inaccessible from q_0. This does not affect the function $h_{A_{q_0}}$.

M2 (Removing incomplete response) Given a transducer A_{q_0} over $\mathfrak{C}_n$ or $\mathfrak{C}_{n,r}$, form a new initial transducer

$$A'_{q_0} := \langle \dot{\mathbf{r}}, X_n, R_A, S_A, \pi_A, \lambda'_A, q_0 \rangle$$

if A_{q_0} is an initial transducer over $\mathfrak{C}_{n,r}$, or in the case that A_{q_0} is a transducer over $\mathfrak{C}_n$,

$$A'_{q_{-1}} := \langle X_n, Q'_A, \pi'_A, \lambda'_A, q_{-1} \rangle$$

where q_{-1} is a new symbol disjoint from Q_A, and $Q'_A := Q_A \sqcup \{q_{-1}\}$.

Suppose first that A_{q_0} is a transducer over $\mathfrak{C}_{n,r}$, then the output function of A'_{q_0} is defined by the following rules:

(a) for $a \in \dot{\mathbf{r}}$, $\lambda'_A(a, q_0) = (a)\theta_{h_{A_{q_0}}}$,

(b) and for $q \in Q_A$, $q \neq q_0$, and for all $i \in X_n$, we have $\lambda'_A(i, q) = (i)\theta_{h_{A_q}} - (\epsilon)\theta_{h_{A_q}}$.

If A_{q_0} is a transducer over $\mathfrak{C}_n$, then the transition and output functions of A_{q_0} are defined by the following rules:

(a) $\pi'_A\restriction_Q = \pi_A$ and $\pi'_A(i, q_{-1}) = \pi_A(i, q_0)$ for all $i \in X_n$,

(b) for all $i \in X_n$, we have $\lambda'_A(i, q_{-1}) = (i)\theta_{h_{A_{q_0}}}$,

(c) for $q \in Q_A$ and for all $i \in X_n$, we have $\lambda'_A(i, q) = (i)\theta_{h_{A_q}} - (\epsilon)\theta_{h_{A_q}}$.

It is possible to show that $h_{A'_{q_0}} = h_{A_{q_0}}$ in the case that A_{q_0} is a transducer over $\mathfrak{C}_{n,r}$, and in the case that A_{q_0} is transducer over $\mathfrak{C}_n$, one can likewise show that $h_{A'_{q_{-1}}} = h_{A_{q_0}}$.

M3 (Identifying ω-equivalent states) Let A_{q_0} be a transducer over $\mathfrak{C}_n$ or $\mathfrak{C}_{n,r}$ without states of incomplete response. Let $[Q_A]_\omega := \{[q]_\omega \mid q \in Q_A\}$; in the case that A_{q_0} is a transducer over $\mathfrak{C}_{n,r}$, let $[R_A]_\omega := \{[q]_\omega \mid q \in R_A\}$ and $[S_A]_\omega := \{[q]_\omega \mid q \in S_A\}$. Observe that for every state of R_A there is a path into S_A with output an element of $X^+_{n,r}$ (restrictions (R.1), (R.2) and(R.4)). Since outputs of states in S_A are always in X^*_n (Remark 1.5.20), states in R_A are never ω-equivalent to states in S_A. From this it follows that $[Q_A]_\omega = [R_A]_\omega \sqcup [S_A]_\omega$. Moreover, since q_0 is the only state of A_{q_0} for which $h_{A_{q_0}}$ has domain $\mathfrak{C}_{n,r}$, then $[q_0]_\omega = \{q_0\}$.

Form a new transducer $A'_{[q_0]_\omega} = \langle \dot{\mathbf{r}}, X_n, [R_A]_\omega, [S_A]_\omega, \pi_{A'}, \lambda_{A'}\rangle$ in the case where A_{q_0} is a transducer over $\mathfrak{C}_{n,r}$, and in the case where A_{q_0} is a transducer over $\mathfrak{C}_n$, $A'_{[q_0]_\omega} = \langle X_n, [Q_A]_\omega, \pi_{A'}, \lambda_{A'}\rangle$. We describe the transition and output function first in the case where A_{q_0} is a transducer over $\mathfrak{C}_{n,r}$, and then in the case where A_{q_0} is a transducer over $\mathfrak{C}_n$. We first require the following claim.

Claim 1.6.8. *Let A_{q_0} be an initial transducer over $\mathfrak{C}_n$ or $\mathfrak{C}_{n,r}$ with no states of incomplete response, and let q_1 and q_2 be a distinct pair of ω-equivalent states of A_{q_0}. Then, for $w \in X^+_n$ and $i \in X_n$, $\lambda_A(w, q_1) = \lambda_A(w, q_2)$ and $\pi_A(i, q_1) \sim_\omega \pi_A(i, q_2)$.*

Proof. First observe that if q_1 and q_2 satisfy $\lambda_A(w, q_1) = \lambda_A(w, q_2)$ for all $w \in X^+_n$, then it must be the case that $\pi_A(\nu, q_1) \sim_\omega \pi_A(\nu, q_2)$ for all $\nu \in X^+_n$. This is because for any $\delta \in \mathfrak{C}_n$, we may write $\delta = w\rho$ for some $\rho \in \mathfrak{C}_n$ and $w \in X^*_n$, hence $\lambda_A(\rho, \pi(w, q_1)) = \lambda_A(\rho, \pi(w, q_2))$ as $q_1 \sim_\omega q_2$. From this it follows that $\pi_A(\nu, q_1) \sim_\omega \pi_A(\nu, q_2)$ for any $\nu \in X^+_n$. Thus, we only have to prove that, for all $w \in X^*_n$, $\lambda_A(w, q_1) = \lambda_A(w, q_2)$.

We proceed by contradiction. Suppose for some minimal length $w \in X^+_n$, $\lambda_A(w, q_1) \neq \lambda_A(w, q_2)$. Since $q_1 \sim_\omega q_2$, it must be the case that either $\lambda_A(w, q_1) < \lambda_A(w, q_2)$ or $\lambda_A(w, q_2) < \lambda_A(w, q_1)$. Relabelling if necessary, we assume the former inequality holds. Let $\eta = \lambda_A(w, q_2) - \lambda_A(w, q_1)$ so that $\eta \neq \epsilon$. Let $w = w_1 i$ for some $w_1 \in X^*_n$ and $i \in X_n$, and let $p = \pi_A(w_1, q_1)$. Notice that p must be a state of incomplete response. This is because, since $q_1 \sim_\omega q_2$, and $\lambda_A(w, q_1) < \lambda_A(w, q_2)$, it must be the case that $\lambda_A(i, p)\eta \leqslant (i)\theta_{h_p}$, therefore $\lambda_A(i, p) < (i)\theta_{h_p}$. However, this contradicts the assumption that A_{q_0} has no states of incomplete response. $\square$

Now we describe the transition and output function of

$$A'_{[q_0]_\omega} = \langle \dot{\mathbf{r}}, X_n, [R_A]_\omega, [S_A]_\omega, \pi_{A'}, \lambda_{A'}\rangle$$

in the case that A_{q_0} is a transducer over $\mathfrak{C}_{n,r}$.

(a) For $a \in \dot{\mathbf{r}}$, we set $\pi_{A'}(a, [q_0]_\omega) = [\pi_A(a, q_0)]_\omega$ and $\lambda_{A'}(a, [q_0]_\omega) = \lambda_A(a, q_0)$. Since $[q_0]_\omega = \{q_0\}$ both of these maps are well defined.

(b) For $i \in X_n$ and $[q]_\omega \in [Q_A]_\omega$, we set $\pi_{A'}(i, [q]_\omega) = [\pi_A(i, q)]_\omega$, and $\lambda_{A'}(i, [q]_\omega) = \lambda_A(i, q)$. By Claim 1.6.8 both of these maps are well-defined.

In the case that A_{q_0} is a transducer over $\mathfrak{C}_n$, the transition and output function of A_{q_0} are given by the following rules: for $i \in X_n$ and $[q]_\omega \in [Q_A]_\omega$, we have $\pi_{A'}(i, [q]_\omega) = [\pi_A(i, q)]_\omega$ and $\lambda_{A'}(i, [q]_\omega) = \lambda_A(i, q)$. Once more, the transition and output functions of $A'_{[q_0]_\omega}$ are well defined by Claim 1.6.8.

The following proposition is proved in [30] for transducers over $\mathfrak{C}_n$ and, with very little change, was adapted for transducers over $\mathfrak{C}_{n,r}$ in the author's article [10]:

Proposition 1.6.9. *Let A_{q_0} be an initial transducer over $\mathfrak{C}_n$ or $\mathfrak{C}_{n,r}$ and let B_{p_0} be the result after applying the procedures* **M1**, **M2**, *and* **M3** *in order, then, up to isomorphism, B_{p_0} is the unique minimal transducer ω-equivalent to A_{q_0}.*

Definition 1.6.10. We shall call the procedure which takes as an input an initial transducer A_{q_0}, then applies in order **M1**, **M2**, and **M3** to return a minimal transducer B_{q_0} ω-equivalent to A, the *minimisation procedure*.

We illustrate the minimization procedure with an example.

Example 1.6.11. Let A_{q_0} be the initial transducer shown below:

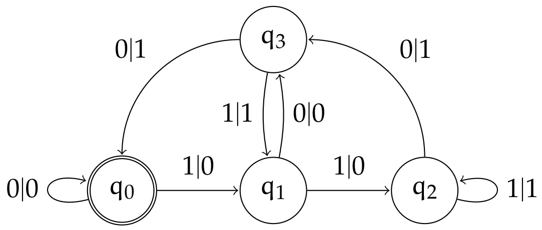


Figure 1.8: A non-minimal initial transducer A_{q_0} over $\mathfrak{C}_n$

We now apply the minimisation procedure to A_{q_0}. Firstly, observe that A_{q_0} is an accessible transducer, therefore **M1** does not change A_{q_0}. Secondly, observe that all states of A_{q_0} are states of incomplete response. For instance, consider the state state q_3: $(0)\theta_{h_{q_3}} = 10$ however $\lambda(0, q_3) = 1$. We now apply **M2**. The resulting transducer, $A_{q_{-1}}$ is as shown below:

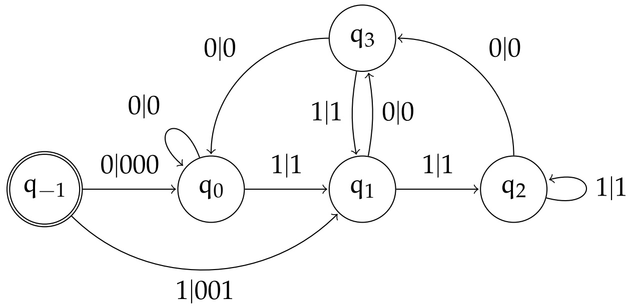


Figure 1.9: Resulting transducer $A_{q_{-1}}$ after applying **M2** to A_{q_0}

Lastly, we apply **M3** to obtain the minimal transducer B_{p_0} depicted below.

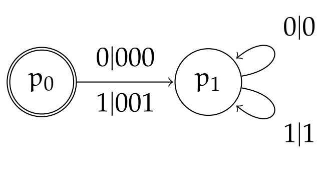


Figure 1.10: Resulting transducer B_{p_0} after minimising.

We have the following lemma about initial transducers constructed from homeomorphisms of $\mathfrak{C}_{n,r}$ or $\mathfrak{C}_n$. We state the lemma simultaneously for transducers over $\mathfrak{C}_n$ and transducers over $\mathfrak{C}_{n,r}$, we distinguish between the two readings using square brackets. For example we write: 'Let $h \in \mathfrak{C}_n$ [$h \in \mathfrak{C}_{n,r}$] and let A_{q_0} be an initial transducer over $\mathfrak{C}_n$ [$\mathfrak{C}_{n,r}$] representing...' for the two distinct sentences: 'Let $h \in \mathfrak{C}_n$ and let A_{q_0} be an initial transducer over $\mathfrak{C}_n$ representing...' and 'Let $h \in \mathfrak{C}_{n,r}$ and let A_{q_0} be an initial transducer over $\mathfrak{C}_{n,r}$ representing...' We will often make use of this convention to avoid repetition.

Lemma 1.6.12. *Let $h \in H(\mathfrak{C}_n)$ [$h \in H(\mathfrak{C}_{n,r})$] and $A_\epsilon = \langle X_n, Q_A, \pi_A, \lambda_A\rangle$ [$A_\epsilon = \langle \dot{\mathbf{r}}, X_n, R_A S_A, \pi_A, \lambda_A\rangle$] be an initial transducer representing h given by Construction 1.5.14 [Construction 1.5.29]. The transducer A_ϵ is accessible and has no states of incomplete response.*

Proof. Let $h \in H(\mathfrak{C}_n)$ [$h \in \mathfrak{C}_{n,r}$] and $a \in X_n$ [$a \in X_{n,r}$]. By construction of the output function, we have $\lambda_A(a, q_0) = (a)\theta_h$. Recall Remark 1.5.16 [Remark 1.5.30] asserting the equality $h_{A_\epsilon} = h$, from this we see that q_0 is not a state of incomplete response.

Now let $i \in X_n$ and let $\nu \in X_n^*$ [$\nu \in X_{n,r}^*$]. Once again by construction we have, $\lambda_A(i, \nu) = (\nu i)\theta_h - (\nu)\theta_h$. By Proposition 1.5.13 [Remark 1.5.28 part (c)] we have $(\nu i)\theta_h - (\nu)\theta_h = (\nu)\theta_h(i)\theta_{h_\nu} - (\nu)\theta_h = (i)\theta_{h_\nu}$. However, by Remark 1.5.16 [Remark 1.5.30] we have, $h_{A_\nu} = h_\nu$ and so ν is not a state of incomplete response.

That A_ϵ is accessible follows by construction of the transition function π_A. □

A consequence of the lemma above is the following fundamental result which was proved first in [30] for $h \in H(\mathfrak{C}_n)$ and then adapted by the authors of [10] for $h \in \mathfrak{C}_{n,r}$.

Theorem 1.6.13. *Let $h \in H(\mathfrak{C}_{n,r}) \sqcup H(\mathfrak{C}_n)$ such that the set of local actions of h is finite, then there is a* finite *minimal initial transducer A_{q_0} such that $h_{A_{q_0}} = h$.*

Proof. Let A_ϵ be the transducer representing h given by Constructions 1.5.14 and 1.5.29. By Lemma 1.6.12 A_ϵ is accessible and has no states of incomplete response therefore, in order to minimize A_{q_0}, it suffices to apply only the process **M3**.

Let B_{p_0} be the resulting transducer. Since h has only finitely many local actions and all the states of A_ϵ corresponds to a local action of h, it follows that A_ϵ has finitely many ω-equivalence classes. From this we deduce that the transducer B_{p_0} has only finitely many states. Now, as B_{p_0} is ω-equivalent to A_ϵ the result follows. □

Let

$$\mathcal{R}_n := \{h \in H(\mathfrak{C}_n) \mid h = h_{A_{q_0}} \text{ for } A_{q_0} \text{ a finite initial transducer}\}$$

and

$$\mathcal{R}_{n,r} := \{h \in H(\mathfrak{C}_{n,r}) \mid h = h_{A_{q_0}} \text{ for } A_{q_0} \text{ a finite initial transducer}\}.$$

Definition 1.6.14. We call an element $h \in H(\mathfrak{C}_n) \sqcup H(\mathfrak{C}_{n,r})$ which is an element of $\mathcal{R}_n$ or $\mathcal{R}_{n,r}$, a *rational homeomorphism*

Remark 1.6.15. We observe that $\mathcal{R}_{n,1}$ and $\mathcal{R}_n$ are equal, thus we make the identification $\mathcal{R}_{n,1} = \mathcal{R}_n$.

In the next section we shall demonstrate that both $\mathcal{R}_n$ and $\mathcal{R}_{n,r}$ are groups under the operation of composition of functions.

1.7 The groups $\mathcal{R}_n$ and $\mathcal{R}_{n,r}$

In this section we show that the sets $\mathcal{R}_n$ and $\mathcal{R}_{n,r}$ are groups under composition of functions. To demonstrate this fact, we define a multiplication of initial transducers such that the resulting initial transducer induces a map on Cantor space equal to the composition of the functions induced by the original two transducers. We shall then see that for given finite, initial transducers, the result of this multiplication is a finite transducer. From this we will deduce that $\mathcal{R}_n$ and $\mathcal{R}_{n,r}$ are closed under composition of functions which is an associative product. We then give an algorithm which constructs, given a transducer inducing a homeomorphism on the relevant Cantor space, a transducer which induces the inverse homeomorphism called the *inverse transducer*. As it turns out, if the original initial transducer is finite, then its inverse is also finite. Since the identity homeomorphism, which has a finite set of local actions, is an element of $\mathcal{R}_n$ and $\mathcal{R}_{n,r}$, we deduce from the previous points that both $\mathcal{R}_n$ and $\mathcal{R}_{n,r}$ are subgroups of $H(\mathfrak{C}_n)$ and $H(\mathfrak{C}_{n,r})$ respectively.

We begin by first describing how to multiply (initial) transducers.

1.7.1 Multiplying Transducers

In this section we outline the algorithm in Section 2.3 of [30] for multiplying transducers. We then observe that the initial transducer arising from applying this multiplication algorithm to two given initial transducers induces a function on Cantor space equal to the composition of the functions induced by the original two transducers.

We first describe how to multiply two arbitrary (not necessarily reduced or initial) transducers such that the output alphabet of the first is equal to the input alphabet of the second. We then go on to describe how to modify this procedure for transducers over $\mathfrak{C}_{n,r}$.

Let $A = \langle X, Y, Q_A, \pi_A, \lambda_A \rangle$ and $B = \langle Y, Z, Q_B, \pi_B, \lambda_B \rangle$ be transducers. We define the product of A and B to be the transducer, $A * B = \langle X, Z, Q_{A*B}, \pi_{A*B}, \lambda_{A*B} \rangle$ where the set $Q_{A*B} = \{(p, q) \mid p \in Q_A, q \in Q_B\}$ is the set of states of $A * B$, and the transition and output functions of $A * B$ are as follows: for $(p, q) \in Q_{A*B}$ and $x \in X$, the transition and output functions satisfy,

$$\pi_{A*B}(x, (p, q)) = (\pi_A(x, p), \pi_B(\lambda_A(x, p), q)) \tag{1.8}$$

$$\lambda_{A*B}(x, (p, q)) = \lambda_B(\lambda_A(x, p), q). \tag{1.9}$$

Definition 1.7.1. Let $A = \langle X, Y, Q_A, \pi_A, \lambda_A \rangle$ and $B = \langle Y, Z, Q_B, \pi_B, \lambda_B \rangle$ be transducers. Let $A * B = \langle X, Z, Q_{A*B}, \pi_{A*B}, \lambda_{A*B} \rangle$ be the transducer with state set $Q_{A*B} := Q_A \times Q_B$ and transition and output functions satisfying equations (1.8) and (1.9). We call $A * B$ the *product transducer of* A *and* B. If p_0 is a state of A, and q_0 is a state of B, then we define the product of the initial transducers A_{p_0} and B_{q_0} to be the initial transducer $(A * B)_{(q_0, p_0)}$.

Definition 1.7.2. Let $A_{p_0} = \langle X, Y, Q_A, \pi_A, \lambda_A \rangle$ and $B_{q_0} = \langle Y, Z, Q_B, \pi_B, \lambda_B \rangle$ be initial transducers. Let A and B be the underlying transducers of A_{p_0} and B_{q_0} respectively. We call the product $A * B$ the *full product transducer of* A_{q_0} *and* B_{q_0}. We shall omit the word 'full' when it is clear that A and B are not initial.

We have the following lemma:

Lemma 1.7.3. *Let* $A = \langle X, Y, Q_A, \pi_A, \lambda_A \rangle$ *and* $B = \langle Y, Z, Q_B, \pi_B, \lambda_B \rangle$ *be transducers. Let* $A * B$ *be the product transducer of* A *and* B*. Then, for* $w \in X^*$ *and for any state* (p, q) *of* $A * B$*, we have:* $\lambda_{A*B}(w, (p, q)) = \lambda_B(\lambda_A(w, p), q)$ *and* $\pi_{A*B}(w, (p, q)) = (\pi_A(w, p), \pi_B(\lambda_A(w, p), q))$.

Proof. We proceed by induction. For $w = X \sqcup \{\epsilon\}$ the lemma holds by Equations (1.8) and (1.9) and the convention that the output when the empty word is read from any state of a transducer is the empty word.

Therefore, for $m \in \mathbb{N}_1$ we assume that the lemma holds for all words of length m in X^*. Let $x \in X$, $w \in X^m$ and (p, q) be any state of $A * B$. Let $(p_1, q_1) := \pi_{A*B}(w, (p, q))$. By equation (1.3.2) We have:

$$\lambda_{A*B}(wx, (p, q)) = \lambda_{A*B}(w, (p, q))\lambda_{A*B}(x, (p_1, q_1)) \text{ and,}$$
$$\pi_{A*B}(wx, (p, q)) = \pi_{A*B}(x, (p_1, q_1)) = (\pi_A(x, p_1), \pi_B(\lambda_A(x, p_1), q_1)).$$

By the inductive assumption, $\lambda_{A*B}(w, (p, q)) = \lambda_B(\lambda_A(w, p), q)$ and $\pi_{A*B}(w, (p, q)) = (\pi_A(w, p), \pi_B(\lambda_A(w, p), q))$. Hence,

$$\lambda_{A*B}(wx, (p, q)) = \lambda_B(\lambda_A(w, p), q)\lambda_B(\lambda_A(x, p_1), q_1) \text{ and,} \tag{1.10}$$

$$\pi_{A*B}(wx, (p, q)) = (\pi_A(wx, p), \pi_B(\lambda_A(wx, p), q)). \tag{1.11}$$

Equation 1.11 demonstrates that $\pi_{A*B}(wx, (p, q)) = (\pi_A(wx, p), \pi_B(\lambda_A(wx, p), q))$ as required. We now focus on Equation 1.10.

Observe that $\lambda_B(\lambda_A(wx, p), q)$ is equal to $\lambda_B(\lambda_A(w, p)\lambda_A(x, p_1), q)$ by Equations (1.2), (1.3.2), (1.8) and the inductive assumption on π_{A*B}. We can further break this up (again by Equations (1.2),(1.3.2), (1.8) and the inductive assumption on π_{A*B}) to get:

$$\lambda_B(\lambda_A(wx, p), q) = \lambda_B(\lambda_A(w, p)\lambda_A(x, p_1), q) = \lambda_B(\lambda_A(w, p), q)\lambda_B(\lambda_A(x, p_1), q_1).$$

Hence we conclude that:

$$\lambda_{A*B}(wx, (p, q)) = \lambda_B(\lambda_A(wx, p), q)$$

as required. □

The lemma above may be viewed pictorially. Let $A = \langle X, Y, Q_A \pi_A, \lambda_A \rangle$ and $B\langle Y, Z, Q_B \pi_B, \lambda_B \rangle$ be transducers with p_0 a state of A and q_0 a state of B. Let $w = w_1 \ldots w_n$ be a word in X^+, Figure 1.11 indicates how the word w is processed through the state (p_0, q_0) of $A * B$:

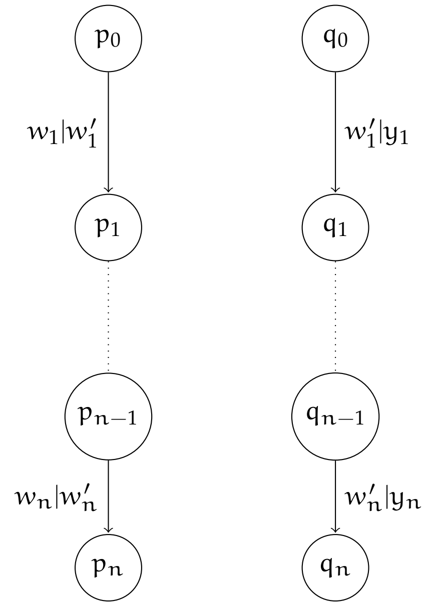


Figure 1.11: Reading a word through a state in the product transducer.

Lemma 1.7.3 indicates that instead of reading each letter w_i of w, for $1 \leqslant i \leqslant n$, successively through the states (p_i, q_i), we may also read the entire word w through the state p_0, then read the output from this process through the state q_0 as pictured above.

As a corollary of Lemma 1.7.3 we have the following:

Lemma 1.7.4. *Let* $A = \langle X, Y, Q_A, \pi_A, \lambda_A \rangle$ *and* $B = \langle Y, Z, Q_B, \pi_B, \lambda_B \rangle$ *be transducers. Let* p_0 *and* q_0 *be states of* A *and* B *respectively, and* h_{p_0} *and* h_{q_0} *be the continuous functions induced by the initial transducers* A_{p_0} *and* B_{q_0}*. We have,* $h_{(p_0,q_0)} = h_{p_0} \circ h_{q_0}$ *where* $h_{(p_0,q_0)}$ *is the continuous function induced by the initial transducer* $(A * B)_{(p_0,q_0)}$.

Proof. This follows by Lemma 1.7.3 and the definition (Notation 1.5.3) of the functions h_q for C_q an initial transducer. □

Lemma 1.7.4 demonstrates that for two non-degenerate initial transducers A_{p_0} and B_{q_0}, the resulting initial transducer $(A * B)_{(p_0,q_0)}$ is also non-degenerate.

Remark 1.7.5. The transducer product defined above is not associative. However, given three transducers A, B and C, and states p_0, q_0, and t_0 of A, B, and C respectively, Lemma 1.7.4 demonstrates that $((A * B) * C)_{((p_0,q_0),t_0)}$ is ω-equivalent to the initial transducer $(A * (B * C))_{(p_0,(q_0,t_0))}$ since composition of functions is associative.

In order to make an associative product on initial transducers, we tweak the transducer product slightly. Let $A_{p_0} = \langle X, Y, Q_A, \pi_A, \lambda_A \rangle$ and $B_{q_0} = \langle Y, Z, Q_B, \pi_B, \lambda_B \rangle$ be initial transducers, we define the product of the initial transducers A_{p_0} and B_{q_0} to be the *minimal transducer* $AB_{(p_0,q_0)}$ representing the transducer $(A * B)_{p_0,q_0}$. By uniqueness of the minimal transducer (up to isomorphism), we see that this product is now an associative product of initial transducers.

Notation 1.7.6. Let $A_{p_0} = \langle X, Y, Q_A, \pi_A, \lambda_A \rangle$ and $B_{q_0} = \langle Y, Z, Q_B, \pi_B, \lambda_B \rangle$ be initial transducers. We denote by $AB_{(p_0,q_0)}$ the minimal initial transducer ω-equivalent to $(A * B)_{(p_0,q_0)}$.

We need to be a little careful, in extending this definition to initial transducers over $\mathfrak{C}_{n,r}$, that the resulting initial transducer still satisfies all of the conditions (R.1) to (R.4). We recall that a transducer over $\mathfrak{C}_{n,r}$ is called non-degenerate if it satisfies the restrictions (R.1) to (R.4). We also recall that we only consider non-degenerate transducers; however, **from now until the end of this section we shall make the non-degeneracy assumption explicitly** as we have not yet verified that the product transducer is also non-degenerate. We proceed by first constructing the product of two initial transducers over $\mathfrak{C}_{n,r}$, then we verify that the resulting initial transducer is non-degenerate. We recall the convention (see Section 1.3) that whenever we read the empty word from a state of a transducer we remain in the same state and the output is empty. We begin with the following definition.

Definition 1.7.7. Let $A_{p_0} = \langle \dot{\mathbf{r}}, X_n, R_A, S_A, \pi_A, \lambda_A \rangle$ be an initial, non-degenerate transducer over $\mathfrak{C}_{n,r}$. Set ∂R_A to be the subset of R_A consisting of those states p for which there is some $i \in X_n$ such that $\lambda_A(i, p) \in X_{n,r}^{+}$.

Let $A_{p_0} = \langle \dot{\mathbf{r}}, X_n, R_A, S_A, \pi_A, \lambda_A \rangle$ and $B_{q_0} = \langle \dot{\mathbf{r}}, X_n, R_B, S_B, \pi_B, \lambda_B \rangle$ be two initial, non-degenerate transducers over $\mathfrak{C}_{n,r}$. Let $(A * B)_{(p_0,q_0)} = \langle \dot{\mathbf{r}}, X_n, R_{A*B}, S_{A*B}, \pi_{A*B}, \lambda_{A*B} \rangle$ be an initial transducer such that

(i) $R_{A*B} = \{(\partial R_A \sqcup \{p_0\}) \times \{q_0\}\} \sqcup \{[R_A \backslash (\partial R_A \sqcup \{p_0\})] \times R_B\} \sqcup \{S_A \times (R_B \backslash \{q_0\})\}$,

(ii) $S_{A*B} = S_A \times S_B$.

The transition function π_{A*B} and output function λ_{A*B} of $(A * B)_{(p_0,q_0)}$ are defined by the following rules: for $a \in \dot{\mathbf{r}}$ we have,

$$\pi_{A*B}(a, (p_0, q_0)) = (\pi_A(a, p_0), \pi_B(\lambda_A(a, p_0), q_0)) \tag{1.12}$$
$$\lambda_{A*B}(a, (p_0, q_0)) = \lambda_B(\lambda_A(a, p_0), q_0); \tag{1.13}$$

for $i \in X_n$ and $(p, q) \in \{R_A \backslash \{p_0\} \times R_B\} \sqcup \{S_A \times R_B\}$ we have:

$$\pi_{A*B}(i, (p, q)) = (\pi_A(i, p), \pi_B(\lambda_A(i, p), q)) \tag{1.14}$$
$$\lambda_{A*B}(i, (p, q)) = \lambda_B(\lambda_A(i, p), q). \tag{1.15}$$

By definition of the transition function, it is clear that (p_0, q_0) is the only state from which the transducer reads a symbol from $\dot{\mathbf{r}}$. Furthermore, we also observe that with very little adjustment to the proof of Lemma 1.7.3 one way prove the following analogous lemma for the transducer $(A * B)_{(p_0,q_0)}$:

Lemma 1.7.8. *Let* $A_{p_0} = \langle \dot{\mathbf{r}}, X_n, R_A, S_A, \pi_A, \lambda_A \rangle$ *and* $B_{q_0} = \langle \dot{\mathbf{r}}, X_n, R_B, S_B, \pi_B, \lambda_B \rangle$ *be initial, non-degenerate transducers over* $\mathfrak{C}_{n,r}$ *and* $(A * B)_{(p_0,q_0)}$ *be as above. For* $w \in X_{n,r}^{*} \sqcup X_n^{*}$ *and for appropriate states* (q, p) *of* $A * B$ *such that* $\lambda_{A*B}(w, (q, p))$ *and* $\pi_{A*B}(w, (q, p))$ *are defined, we have:* $\lambda_{A*B}(w, (q, p)) = \lambda_B(\lambda_A(w, q), p)$ *and* $\pi_{A*B}(w, (q, p)) = (\pi_A(w, q), \pi_B(\lambda_A(w, q), p))$.

We now verify that π_{A*B} and λ_{A*B} satisfy the restrictions (R.1) to (R.4) in the following lemma.

Lemma 1.7.9. *Let* $A_{p_0} = \langle \dot{\mathbf{r}}, X_n, R_A, S_A, \pi_A, \lambda_A \rangle$ *and* $B_{q_0} = \langle \dot{\mathbf{r}}, X_n, R_B, S_B, \pi_B, \lambda_B \rangle$ *be initial, non-degenerate transducers over* $\mathfrak{C}_{n,r}$. *The transducer* $(A * B)_{(p_0,q_0)} = \langle \dot{\mathbf{r}}, X_n, R_{A*B}, S_{A*B}, \pi_{A*B}, \lambda_{A*B} \rangle$ *with transition and output functions as defined in equations* (1.12), (1.14), (1.13) *and* (1.15), *satisfies the restrictions (R.1) to (R.4).*

Proof. We take the restrictions one at a time.

(R.1) Let $(p, q) \in R_{A*B}$ and $i \in \dot{\mathbf{r}} \sqcup X_n$ such that $\pi_{A*B}(i, (p, q))$ is well-defined and is an element of R_{A*B}.

Suppose that $\pi_{A*B}(i, (p, q)) = (p_2, q_2)$. Observe that by definition of the set R_{A*B} we must have $q_2 \in R_B$. Let $w = \lambda_A(i, p) \in X^*_{n,r} \sqcup X^*_n$.

If $w = \epsilon$, then by convention we have $\lambda_B(w, q) = \epsilon$ and so $\lambda_{A*B}(i, (p, q)) = \epsilon$.

If $w \in X^+_{n,r}$, then it must be the case that $p \in \partial R_A$. Therefore, we have $q = q_0$ and $\pi_B(w, q)$ and $\lambda_B(w, q)$ are defined. Since $\pi_B(w, q) \in R_B$, it follows that $\lambda_B(w, q) = \epsilon$ (Restriction (R.1) applied to B_{q_0}). Hence, we have $\lambda_{A*B}(i, (p, q)) = \epsilon$.

If $w \in X^+_n$, then $p \in S_A$ by (R.2) applied to A. Therefore, $q \in R_B \backslash \{q_0\}$ and $\pi_B(w, q)$ and $\lambda_B(w, q)$ are defined. Since $\pi_B(w, q) = q_2 \in R_B$, we must have $\lambda_B(w, q) = \epsilon$ by (R.1) applied to B_{p_0}.

(R.2) Let $(p, q) \in R_{A*B}$ and $i \in \dot{\mathbf{r}} \sqcup X_n$ such that $\pi_{A*B}(i, (p, q))$ is well-defined and is an element of S_{A*B}. From the last condition, we deduce that $p \in \partial R_A$ and so $q = q_0$ by definition of R_{A*B}. Now, by restriction (R.2) applied to A_{p_0}, we must have $w := \lambda_A(i, p) \in X^+_{n,r}$. However, since B_{q_0} satisfies restriction (R.2) and (R.1), and since $\pi_B(w, q_0) \in S_B$, we must also have $\lambda_B(w, q_0) \in X^+_{n,r}$.

(R.3) Let $(p, q) \in S_{A*B}$ and $i \in X_n$ then the fact that $\pi_{A*B}(i, (p, q)) \in S_{A*B}$ and $\lambda_{A*B}(i, (p, q))\mathrm{Inn}(X^*_n)$ follows from the definition of the transition function and since A_{p_0} and B_{q_0} satisfy restriction (R.3).

(R.4) Let $(p, q) \in Q_{A*B}$ and $w \in X^+_{n,r} \sqcup X^+_n$ be such that $\pi_{A*B}(w, (p, q))$ is well-defined and equal to (p, q). By Lemma 1.7.8, we must have $\pi_A(w, p) = p$ and so, since A is non-degenerate, we conclude that $\lambda_A(w, p) \neq \epsilon$ and $p \in S_A$ which further implies that $\lambda_A(w, p)$ is in fact in X^+_n. Moreover, $\pi_B(\lambda_A(w, p), q) = q$ and we once more deduce, since B is non-degenerate, that $q \in S_B$ and $\lambda_B(\lambda_A(w, p), q) \in X^+_n$. Therefore by Lemma 1.7.8 once again, we have that $\lambda_{A*B}(w, (p, q)) \neq \epsilon$. This gives the result.

□

Thus, we have now verified that for $A_{p_0} = \langle \dot{\mathbf{r}}, X_n, R_A, S_A, \pi_A, \lambda_A \rangle$ and $B_{q_0} = \langle \dot{\mathbf{r}}, X_n, R_B, S_B, \pi_B, \lambda_B \rangle$ non-degenerate, initial transducers over $\mathfrak{C}_{n,r}$, the transducer

$$(A * B)_{(p_0,q_0)} = \langle \dot{\mathbf{r}}, X_n, R_{A*B}, S_{A*B}, \pi_{A*B}, \lambda_{A*B} \rangle$$

with transition and output functions as defined by equations (1.12), (1.14), (1.13) and (1.15), is also a non-degenerate initial transducer.

We make the following definition:

Definition 1.7.10. Let $A_{p_0} = \langle \dot{\mathbf{r}}, X_n, R_A, S_A, \pi_A, \lambda_A \rangle$ and $B_{q_0} = \langle \dot{\mathbf{r}}, X_n, R_B, S_B, \pi_B, \lambda_B \rangle$ be non-degenerate, initial transducers over $\mathfrak{C}_{n,r}$. The transducer

$$(A * B)_{(p_0,q_0)} = \langle \dot{\mathbf{r}}, X_n, R_{A*B}, S_{A*B}, \pi_{A*B}, \lambda_{A*B} \rangle$$

with transition and output function as defined in equations (1.12), (1.14), (1.13) and (1.15), is called the *product transducer of* A_{p_0} *and* B_{q_0}.

With very little change to the proof of Lemma 1.7.4, one may prove the following analogous lemma for transducers over $\mathfrak{C}_{n,r}$:

Lemma 1.7.11. *Let $A_{p_0} = \langle \dot{\mathbf{r}}, X_n, R_A, S_A, \pi_A, \lambda_A \rangle$ and $B_{q_0} = \langle \dot{\mathbf{r}}, X_n, R_B, S_B, \pi_B, \lambda_B \rangle$ be non-degenerate initial transducers. Let h_{p_0} and h_{q_0} be the continuous functions induced by the initial transducers A_{p_0} and B_{q_0}. We have, $h_{(p_0,q_0)} = h_{p_0} h_{q_0}$ where $h_{(p_0,q_0)}$ is the continuous function induced the initial transducer $(A * B)_{(p_0,q_0)}$.*

The transducer product defined above for transducers over $\mathfrak{C}_{n,r}$ is not associative, however, we may once more make use of Lemma 1.7.11, as in the case of transducers over $\mathfrak{C}_n$, to obtain an associative product in the same way. Thus, we define the product of non-degenerate initial transducers A_{p_0} and B_{q_0} to be the minimal transducer $AB_{(p_0,q_0)}$ representing the transducer $(A * B)_{(p_0,q_0)}$.

Notation 1.7.12. Let A_{p_0} and B_{q_0} be non-degenerate initial transducers over $\mathfrak{C}_{n,r}$. We denote by $(A * B)_{(p_0,q_0)}$ the transducer product of A_{p_0} and B_{q_0}, whilst we denote by $AB_{(p_0,q_0)}$ the minimal initial transducer ω-equivalent to $(A * B)_{(p_0,q_0)}$. We also write $A^2_{p_0^2}$ for the transducer $(AA)_{(p_0,p_0)}$. More generally, for $i \in \mathbb{N}_2$, we write $A^i_{p_0^i}$ for the minimal transducer representing the product of the initial transducer A_{p_0} with itself i times.

For two finite, non-degenerate, initial transducers A_{p_0} and B_{q_0} over $\mathfrak{C}_n$ [$\mathfrak{C}_{n,r}$], by construction, the minimal transducer $AB_{(p_0,q_0)}$ representing the transducer product of A_{p_0} with B_{q_0} is also finite. We therefore have the following proposition:

Proposition 1.7.13. *The sets $\mathcal{R}_n, \mathcal{R}_{n,r} \subset H(\mathfrak{C}_n)$ are closed under composition of functions.*

Proof. This follows since, by definition, all elements of $\mathcal{R}_n$ and $\mathcal{R}_{n,r}$ can be represented by finite, initial, non-degenerate transducers over $\mathfrak{C}_n$ and $\mathfrak{C}_{n,r}$ respectively and, by the observation above, the composition of two such elements can also be represented by a finite, initial transducer over $\mathfrak{C}_n$ or $\mathfrak{C}_{n,r}$. $\square$

We close this section with some examples.

Example 1.7.14. We compute the product of the initial transducer from Figure 1.5 with itself. We denote this transducer by A_{q_3} and reproduce it below for convenience.

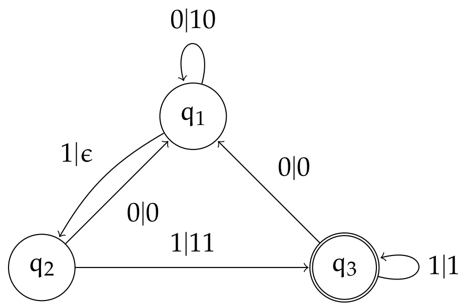


Figure 1.12: Figure 1.5 revisited

We now form the product $(A * A)_{(q_3,q_3)}$.

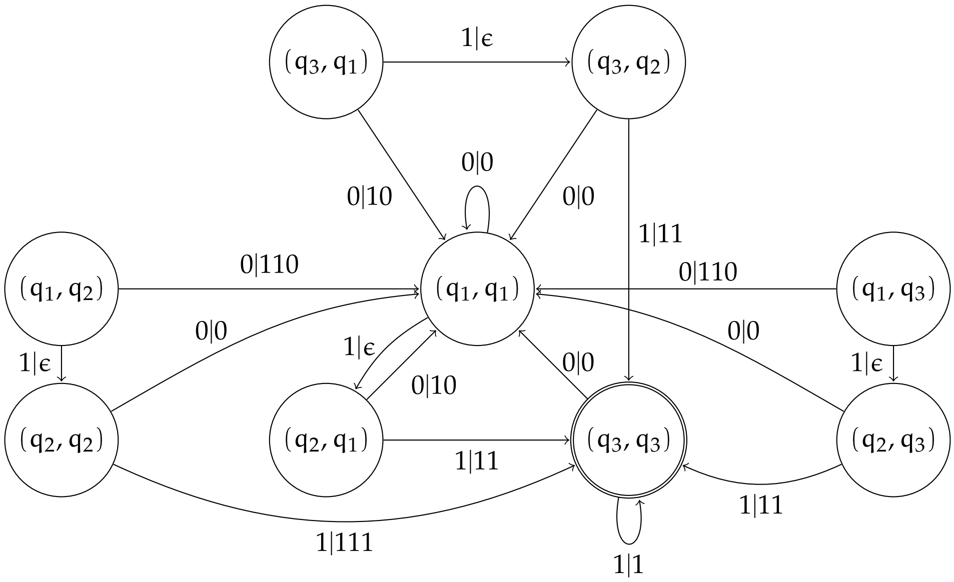


Figure 1.13: The transducer $(A * A)_{(q_3,q_3)}$

Observe that the transducer $A_{(q_3,q_3)}$ has (q_1, q_1) as a state of incomplete response and also has some inaccessible states. In order to compute $AA_{(q_1,q_1)}$ we need to apply procedures **M1**, **M2** and **M3**.

After applying **M1** and **M2** we obtain the transducer below:

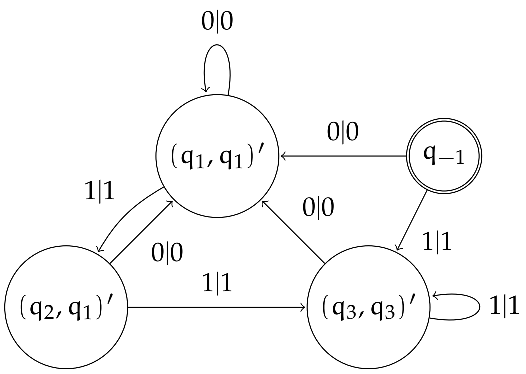


Figure 1.14: Resulting transducer after applying **M2** to $(A * A)_{(q_3,q_3)}$

After applying **M3** we see that $AA_{(q_3,q_3)}$ is the single state identity transducer. Thus the homeomorphism $h_{q_3} \in \mathcal{R}_n$ has order 2.

In the next subsection we show that $\mathcal{R}_n$ and $\mathcal{R}_{n,r}$ are closed under inverses and so conclude that they are subgroups of the group of homeomorphisms of $\mathfrak{C}_n$ and $\mathfrak{C}_{n,r}$ respectively.

1.7.2 Inverting transducers

In this section we give an algorithm for inverting transducers over $\mathfrak{C}_{n,r}$ or $\mathfrak{C}_n$ which induce homeomorphisms. As a consequence of this algorithm, we will see that the inverse of a finite transducer over $\mathfrak{C}_{n,r}$ or $\mathfrak{C}_n$ which induces a homeomorphism, is again a finite transducer. From this we will conclude that both $\mathcal{R}_{n,r}$ and $\mathcal{R}_n$ are closed under products and inverses. Since $\mathcal{R}_n$ and $\mathcal{R}_{n,r}$ both contain the identity transducer, we will thus have demonstrated that they are subgroups of $H(\mathfrak{C}_n)$ and $H(\mathfrak{C}_{n,r})$ respectively. The algorithm we produce here is based on that given in [30].

Following the structure of previous sections, we first give the algorithm for transducers over $\mathfrak{C}_n$ and then show how to modify it for transducers over $\mathfrak{C}_{n,r}$. Recall (see Section 1.5) in what follows that for any homeomorphism h of Cantor space, there is a unique, up to isomorphism, possibly infinite transducer representing h. We begin with a definition of the inverse of a transducer.

Definition 1.7.15. Let A_{q_0} be a transducer such that h_{q_0} induces a homeomorphism of $\mathfrak{C}_n$ or $\mathfrak{C}_{n,r}$. Let B_{p_0} be the minimal transducer such that $h_{p_0} = h_{q_0}^{-1}$. Then we call B_{p_0} the *inverse transducer of* A_{q_0} or just the *inverse of* A_{q_0}.

Definition 1.7.16. Let A_{q_0} be a transducer over $\mathfrak{C}_n$ or $\mathfrak{C}_{n,r}$. Then we say that A_{q_0} is invertible if h_{q_0} is a homeomorphism.

Let $A_{q_0} = \langle X_n, Q_A, \pi_A, \lambda_A \rangle$ be a transducer representing a homeomorphism h_{q_0} of $\mathfrak{C}_n$. We have the following proposition about the states of A_{q_0}.

Proposition 1.7.17. *Let A_{q_0} be a transducer which induces a homeomorphism of $\mathfrak{C}_n$. Let q be any state of Q_A, then h_q is injective and $\mathrm{im}(q)$ is clopen.*

Proof. Injectivity follows from the fact that h_{q_0} is a homeomorphism. For let q be a state of A_{q_0} and let $\nu \in X_n^*$ be such that $\pi_A(\nu, q_0) = q$. If there are distinct $\delta, \eta \in \mathfrak{C}_n$ such that $\lambda_A(\delta, q) = \lambda_A(\eta, q)$ then we must have that $\lambda_A(\nu\delta, q_0) = \lambda_A(\nu\eta, q_0)$ contradicting the injectivity of h_{q_0}.

Fixing still $\nu \in X_n^*$, such that $\pi_A(\nu, q_0) = q$, we argue that $\mathrm{im}(q)$ is clopen. Since h_{q_0} is a homeomorphism, it maps clopen sets to clopen sets. Thus, the set $(U_\nu)h_{q_0}$ is clopen. Let $\mu = \lambda_A(\nu, q_0)$, and observe that $(U_\nu)h_{q_0} = \mu\,\mathrm{im}(q)$. Recall from Definition 1.1.18 that $\mu\,\mathrm{im}(q) = \{\mu\rho \mid \rho \in \mathrm{im}(q)\}$. Now, since $\mu\,\mathrm{im}(q)$ is clopen, it follows that $\mathrm{im}(q)$ is also clopen. □

Definition 1.7.18. Let A_{q_0} be a transducer such that h_{q_0} is a homeomorphism of $\mathfrak{C}_n$, let $q \in Q_A$ and $\eta \in X_n^*$. Define $(\eta)\Theta_q = ((U_\eta)h_q^{-1})\mathrm{rt}$.

Remark 1.7.19. Let A_{q_0} be a transducer such that h_{q_0} is a homeomorphism of $\mathfrak{C}_n$. Observe that for $\eta \in X_n^*$, $(\eta)\Theta_q$ is the longest common prefix of all elements $\rho \in \mathfrak{C}_n$ such that $(\rho)h_q \in U_\eta$. Moreover, $(\epsilon)\Theta_q = \epsilon$ for any state $q \in Q_A$, since $U_\epsilon = \mathfrak{C}_n$.

We have the following lemma:

Lemma 1.7.20. *Let A_{q_0} be a finite transducer such that h_{q_0} induces a homeomorphism of $\mathfrak{C}_n$. Let $q \in Q_A$, then there are only finitely many words $\nu \in X_n^*$ such that $U_\nu \cap \mathrm{im}(q) \neq \epsilon$ and $(\nu)\Theta_q = \epsilon$.*

Proof. For each state q of A let $B_q \subset X_n^*$ be minimal such that, for all $\nu \in B_q$, $U_\nu \subset \mathrm{im}(q)$, and $U(B_q) = \{U_\nu \mid \nu \in B_q\}$ is a finite cover by basic open sets of $\mathrm{im}(q)$. Since $\mathrm{im}(q)$ is clopen for any state $q \in Q_A$, such a set exists. For $q \in Q_A$ let $m_q = \max\{|\nu| \mid \nu \in B_q\}$ and $M = \max_{q \in Q_A}\{m_q\}$.

Observe that as A_{q_0} is assumed to be non-degenerate and has only finitely many states, there is a $j \in \mathbb{N}_1$ such that for any word $\gamma \in X_n^j$ and any state $q \in Q_A$, $|\lambda_A(\gamma, q)| \geqslant M$.

Fix a state q of A_{q_0} and let $\gamma \in X_n^j$ and $i \in X_n$. Let $\nu = \lambda_A(i, q)$, $p = \pi_A(i, q)$ and $\eta = \lambda_A(\gamma, p)$. Observe that as $|\eta| \geqslant M$, we must have $U_\eta \subseteq \mathrm{im}(p)$, since there is a prefix of η which is an element of B_p. Moreover, as q is injective, $(\nu\eta)\Theta_q$ must have prefix i, since if there a word $\xi \in X_n^*$ and a letter $j \in X_n$ such that $\lambda_A(j\xi, q)$ has $\nu\eta$ as a prefix, then $(U_{j\xi})h_q \subseteq (U_{i\gamma})h_q$.

Since $\gamma \in X_n^j$ was arbitrary, it follows that the set $I_q := \{\lambda_A(\Gamma, q) \mid \Gamma \in X_n^j\} \subset X_n^+$ is such that $U(I_q) = \{U_\mu \mid \mu \in I_q\}$ is a cover for $\mathrm{im}(q)$, moreover, by the previous paragraph, for any $\mu \in I_q$ we have, $(\mu)\Theta_q \neq \epsilon$. □

Thus, given a finite transducer $A_{q_0} = \langle X_n Q_A, \pi_A, \lambda_A \rangle$, for every state $q \in Q_A$, it is possible to compute the set $S_q = \{w_1, w_2, \ldots, w_{m_q}\}$ of all words $w_i \in X_n^*$ such that $U_{w_i} \cap \mathrm{im}(q) \neq \epsilon$ and $(w_i)\Theta_q = \epsilon$.

We now construct the inverse transducer of A_{q_0}. The states of the inverse transducer will be given as a finite subset of $X_n^* \times Q_A$.

Construction 1.7.21. First we recursively construct the set of states Q'_A of the inverse of A_{q_0}. Let $Q'[1] := \{(\epsilon, q_0)\}$ and observe that $U_\epsilon = \mathrm{im}(q_0)$ (as q_0 is a homeomorphism). For $k \in \mathbb{N}_1$, set

$$Q'[k+1] := \{(wi - \lambda_A((wi)\Theta_q, q), \pi_A((wi)\Theta_q, q)) \mid i \in X_n, (w, q) \in Q'[k]\} \cup Q'[k].$$

We have the following claim about the sets $Q'[k]$.

Claim 1.7.22. *Let* $k \in \mathbb{N}_1$*, then, for all* $(w, q) \in Q'[k]$ *we have,* $U_w \subseteq$ *im*(q) *and* $(w)\Theta_q = \epsilon$.

Proof. We proceed by induction on k. For $k = 1$ this follows by the definition of the set $Q'[1]$. Assume that the claim holds for the set $Q'[k]$. We now show that it holds for the set $Q'[k+1]$.

Let $(w, q) \in Q[k]$ and $i \in X_n$. Since $U_w \subseteq \text{im}(q)$, we must have that $(wi)\Theta_q$ satisfies $\lambda_A((wi)\Theta_q, q)$ is a prefix of wi. Since if $\lambda_A((wi)\Theta_q, q) = wi\rho$ for some $\rho \in X_n^+$, then there is some $\varphi \in X_n^+$ incomparable with ρ such that $U_{wi\varphi} \not\subset \text{im}(q)$. Thus $wi - \lambda_A((wi)\Theta_q, q)$ is well defined. Let $p = \pi_A((wi)\Theta_q, q)$ and let $\nu = wi - \lambda_A((wi)\Theta_q, q)$. Observe that since $U_{wi} \subseteq \text{im}(q)$, we must have that $U_\nu \subset \text{im}(p)$. Furthermore, since $\pi_A((wi)\Theta_q, q) = p$, then $(\nu)\Theta_p = \epsilon$ otherwise $(wi)\Theta_q = (wi)\Theta_q(\nu)\Theta_p \neq (wi)\Theta_q$.

Now as

$$Q'[k+1] := \{(wi - \lambda_A((wi)\Theta, q), \pi_A((wi)\Theta, q) \mid i \in X_n, (w, q) \in Q'[k]\} \cup Q'[k]$$

it follows, by the inductive assumption and the arguments of the previous paragraph, that $Q'[k+1]$ satisfies the claim also. ☐

Remark 1.7.23. As part of the proof above, we demonstrated that for $k \in \mathbb{N}_1$, (w, q) in $Q'[k]$ and $i \in X_n$, $\lambda_A((wi)\Theta_q, q)$ is a prefix of wi. Therefore, $wi - \lambda_A((wi)\Theta_q, q)$ is well defined.

Since A_{q_0} has finitely many states, by Lemma 1.7.20, it follows that there is a $j \in \mathbb{N}_1$ such that $Q'[j] = Q'[j+1]$ and $|Q'[j]| < \infty$. Set $Q'_A := Q'[j]$.

Let $A_{(\epsilon, q_0)} = \langle X_n, Q'_A, \pi'_A, \lambda'_A \rangle$ be a transducer with transition and output functions obeying the following rules: for $i \in X_n$ and $(w, q) \in Q'_A$,

$$\pi'_A(i, (w, q)) = (wi - \lambda_A((wi)\Theta_q, q)) \text{ and } \lambda'_A(i, (w, q)) = (wi)\Theta_q.$$

By construction, $\pi'_A : Q'_A \to Q'_A$ and A_{q_0} is a finite transducer. This concludes the construction.

Let A_{q_0} be a transducer inducing a homeomorphism of $\mathfrak{C}_n$, we have the following observation about the transducer $A_{(\epsilon,q_0)} = \langle X_n, Q'_A, \pi'_A, \lambda'_A \rangle$ constructed from A_{q_0} by following Construction 1.7.21.

Lemma 1.7.24. *Let* $(w, q) \in Q'_A$ *and* $\eta \in X_n^*$*, then* $\lambda'_A(\eta, (w, q)) = (w\eta)\Theta_q$*, and* $\pi'_A(\eta, (w, q)) = (w\eta - \lambda_q((w\eta)\Theta_q, q), \pi_A((w\eta)\Theta_q, q)$.

Proof. We proceed by induction on the length of the word η. For $\eta = \epsilon$ the lemma follows by Claim 1.7.22 and the definition of the transition function π'_A. We assume, for $m \in \mathbb{N}_1$, that the lemma holds for all words $\eta \in X_n^*$ of length m.

Let $\eta \in X_n^m$ and $i \in X_n$. By Equation (1.3.2) we have:

$$\lambda'_A(\eta i, (w, q)) = \lambda'_A(\eta, (w, q))\lambda'_A(i, \pi'_A(\eta, (w, q)).$$

Let $\nu = w\eta - \lambda_A((w\eta)\Theta_q, q)$ and let $p = \pi_A((w\eta)\Theta_q, q)$. By the inductive assumption,

$$\lambda'_A(\eta, (w, q))\lambda'_A(i, \pi'_A(\eta, (w, q))) = (w\eta)\Theta_q\lambda'_A(i, (\nu, p)).$$

Thus,

$$\lambda'_A(\eta, (w, q))\lambda'_A(i, \pi'_A(\eta, (w, q))) = (w\eta)\Theta_q(\nu i)\Theta_p.$$

Let $\mu = (w\eta)\Theta_q$. By the inductive assumption on π'_A, $\nu = w\eta - \lambda_A(\mu, q)$, and $\pi_A(\mu, q) = p$. Therefore, $(w\eta i)\Theta_q = \mu(\nu i)\Theta_p$ since the greatest common prefix of all elements of $\mathfrak{C}_n$ with image in the set $U_{w\eta}$, is μ, and $\nu = w\eta - \lambda_A(\mu, q)$.

Now consider $\pi_A(\eta i, (w, q))$. Using Equation (1.2) we may have:

$$\pi'_A(\eta i, (w, q)) = \pi'_A(i, (\nu, p)) = (\nu i - \lambda_A((\nu i)\Theta_p, p), \pi_A((\nu i)\Theta_p, p)).$$

Observe that

$$\lambda_A((w\eta i)\Theta_q, q) = \lambda_A(\mu, q)\lambda_A((\nu i)\Theta_p, p)$$

using $(w\eta i)\Theta_q = \mu(\nu i)\Theta_p$ and Equation (1.3.2). Thus,

$$w\eta i - \lambda_A(\mu, q)\lambda_A((\nu i)\Theta_p, p) = \nu i - \lambda_A((\nu i)\Theta_p, p).$$

Moreover,

$$\pi_A((w\eta i)\Theta_q, q) = \pi_A(\mu(\nu i)\Theta_p, q) = \pi_A((\nu i)\Theta_p, p).$$

Thus,

$$\pi'_A(\eta i, (w, q)) = (\nu i - \lambda_A((\nu i)\Theta_p, p), \pi_A((\nu i)\Theta_p, p)) = (w\eta i - \lambda_A((w\eta i)\Theta_q, q), \pi_A((w\eta i)\Theta_q, q))$$

as required. □

Remark 1.7.25. Let A_{q_0} be a finite transducer such that h_{q_0} induces a homeomorphism of $\mathfrak{C}_n$ and let $A_{(\epsilon,q_0)}$ be the transducer constructed from A_{q_0} using Construction 1.7.21. The lemma above and Lemma 1.7.20 demonstrate that the transducer $A_{(\epsilon,q_0)}$ is non-degenerate. Moreover, for any word $\nu \in X_n^*$ we have, $\lambda'_A(\nu, (\epsilon, q_0)) = (\nu)\Theta_{q_0}$. Therefore, for any word $\delta \in \mathfrak{C}_n$ we must have, $\lambda'_A(\delta, (\epsilon, q_0)) = (\delta)h_{q_0}^{-1}$. This proves the proposition below.

Proposition 1.7.26. *Let A_{q_0} be a transducer such that h_{q_0} induces a homeomorphism of $\mathfrak{C}_n$ and let $A_{(\epsilon,q_0)}$ be the transducer constructed from A_{q_0} by Construction 1.7.21. Then $h_{(\epsilon,q_0)} = h_{q_0}^{-1}$.*

Henceforth, given a transducer A_{q_0} over $\mathfrak{C}_n$ we shall denote by $A_{(\epsilon,q_0)}$ the transducer constructed from A_{q_0} by Construction 1.7.21. The inverse of A_{q_0} is the transducer obtained by minimising the transducer $A_{(\epsilon,q_0)}$. However, the following proposition demonstrates that $A_{(\epsilon,q_0)}$ is not far from being minimal.

Proposition 1.7.27. *Let A_{q_0} be a transducer such that h_{q_0} induces a homeomorphism of $\mathfrak{C}_n$. The initial transducer $A_{(\epsilon,q_0)}$ has no states of incomplete response and is accessible.*

Proof. We proceed by contradiction. Suppose $A_{(\epsilon,q_0)}$ has a state (w, p) of incomplete response. This means that there is some $i \in X_n$ such that $\lambda'_A(i, (w, p))$ is a prefix of $(i)\theta_{(w,p)}$ (for the definition of θ, see Definition 1.5.7). Let $(\nu, q) = \pi'_A(i, (w, p))$, then, for all $j \in X_n$, $\lambda'_A(j, (\nu, q))$ has a non-empty prefix. However, for $j \in X_n$, $\lambda'_A(j, (\nu, q)) = (\nu j)\Theta_q$, thus $(\nu)\Theta_q \neq \epsilon$ which is a contradiction by Claim 1.7.22.

That A_{q_0} is accessible follows by construction (see Construction 1.7.21). □

Therefore, given A_{q_0} a transducer such that h_{q_0} induces a homeomorphism of $\mathfrak{C}_n$, in order to minimise $A_{(\epsilon,q_0)}$ we only need to apply step **M3** of the minimization procedure.

We now demonstrate how to alter the above process to account for transducers over $\mathfrak{C}_{n,r}$.

Let $A_{q_0} = \langle \dot{\mathbf{r}}, X_n, R_A, S_A, Q_A, \pi_A, \lambda_A \rangle$ be a transducer over $\mathfrak{C}_{n,r}$. Trivial modifications to the proof of Proposition 1.7.17 demonstrate that states of A_{q_0}. (Recall from Definition 1.5.17 that some states of A_{q_0} have range $\mathfrak{C}_{n,r}$ and other have range $\mathfrak{C}_n$.) We extend the definition of the function Θ as follows:

(i) $\Theta_{q_0} : X_{n,r}^* \to X_{n,r}^*$ by $\eta \mapsto ((U_\eta)h_{q_0}^{-1})\mathrm{rt}$;

(ii) for $q \in R_A\backslash\{q_0\}$, $\Theta_q : X_{n,r}^* \to X_n^*$ by $\eta \mapsto ((U_\eta)h_q^{-1})\mathrm{rt}$;

(iii) for $q \in S_A$, $\Theta_q : X_n^* \to X_n^*$ by $\eta \mapsto ((\eta)h_q^{-1})\mathrm{rt}$.

The following lemma is analogous to Lemma 1.7.20 and is proved almost identically using the fact that states of A_{q_0} are injective and have clopen image.

Lemma 1.7.28. *Let A_{q_0} be a finite transducer such that h_{q_0} induces a homeomorphism of $\mathfrak{C}_{n,r}$. Let $q \in R_A$ [$q \in S_A$], then there are only finitely many words $\nu \in X_{n,r}^*$ [$\nu \in X_n^*$] such that $U_\nu \cap$ im$(q) \neq \epsilon$ and $(\nu)\Theta_q = \epsilon$.*

It follows from this lemma that, as in the case of transducers over $\mathfrak{C}_n$, given a transducer A_{q_0} inducing a homeomorphism of $\mathfrak{C}_{n,r}$, for every state $q \in Q_A$ it is possible to compute the finite set $S_q := \{w_1, \ldots, w_{m_q}\}$ of words such that $U_{w_i} \cap \mathrm{im}(q) \neq \epsilon$ and $(w_i)\Theta_q = \epsilon$ for $1 \leqslant i \leqslant m_q$.

We now describe how to construct the inverse of a transducer A_{q_0} inducing a homeomorphism of $\mathfrak{C}_{n,r}$. This will mirror construction 1.7.21.

Construction 1.7.29. We begin with the recursive construction of the set of states of the inverse transducer. Let $Q'[0] := \{(\epsilon, q_0)\}$, for $k \in \mathbb{N}_0$ set

$$Q'[k+1] := \{ (wi - \lambda_A((wi)\Theta_q, q), \pi_A((wi)\Theta_q, q)) \\ \mid (w,q) \in Q'[k], i \in X_n \text{ if } w \neq \epsilon; \text{ if } w = \epsilon \text{ and } q \in R_A, \text{ then } i \in \dot{\mathbf{r}}\} \cup Q'[k].$$

The following claim should be compared with Claim 1.7.22:

Claim 1.7.30. *Let $k \in \mathbb{N}_1$, then, for all $(w, q) \in Q'[k]$ we have,*

(i) if $w = \epsilon$ then $q = q_0$ or $q \in S_A$,

(ii) $U_w \subseteq im(q)$,

(iii) $(w)\Theta_q = \epsilon$ and,

(iv) for $i \in X_n$ ($i \in \dot{\mathbf{r}}$ if $w = \epsilon$) $\lambda_A((wi)\Theta_q, q)$ is a prefix of wi.

Proof. We only prove that $(\epsilon, q) \in Q'[k]$ when $q = q_0$ or $q \in S_A$ since the rest of the proof follows almost exactly as in the proof of Claim 1.7.22. We proceed by induction on k. The case $k = 1$ is trivially satisfied since $Q'[1] = \{(\epsilon, q_0)\}$. Assume that the statement holds for $k \in \mathbb{N}_1$ and let $(w, q) \in Q'[k]$.

First suppose that $w \neq \epsilon$. Let $i \in X_n$, and suppose that $wi - \lambda_A((wi)\Theta_q, q) = \epsilon$. This means that $\lambda_A((wi)\Theta_q, q) \neq \epsilon$. Thus by restrictions (R.2) and (R.1) on A_{q_0} we must have that $\pi_A((wi)\Theta_q, q) \in S_A$. Hence the pair $(wi - \lambda_A((wi)\Theta_q, q), \pi_A((wi)\Theta_q, q)) \in Q'[k+1]$ satisfies (i).

Next suppose that $w = \epsilon$. By the inductive hypothesis $q = q_0$ or $q \in S_A$. We take each subcase in turn.

If $q \in S_A$ then, for $i \in X_n$, by restriction (R.3) on A_{q_0}, we have $\pi_A((wi)\Theta_q, q) \in S_A$. Therefore, the pair $(wi - \lambda_A((wi)\Theta_q, q), \pi_A((wi)\Theta_q, q)) \in Q'[k+1]$ satisfies (i).

We now suppose that $q = q_0$. For $a \in \dot{\mathbf{r}}$ if $\lambda_A((a)\Theta_{q_0}, q_0) = a$ then by restrictions (R.2) and (R.1) we must have $\pi_A((a)\Theta_{q_0}, q_0) \in S_A$. Therefore, the pair $(a - \lambda_A((a)\Theta_{q_0}, q_0), \pi_A((a)\Theta_{q_0}, q_0)) \in Q'[k+1]$ satisfies (i).

Now by definition of the set $Q'[k+1]$ and the inductive assumption, we see that $Q'[k+1]$ satisfies (i). □

Since A_{q_0} has only finitely many states, Lemma 1.7.28 guarantees that there is some $j \in \mathbb{N}_1$ such that $Q'[j] = Q'[j+1]$ and $|Q'[j]| < \infty$. Set $Q'_A := Q'[j]$.

Observe that if $(wi)\Theta_{q_0} \neq \epsilon$, then $\pi_A((wi)\Theta_{q_0}, q_0) \neq q_0$, this follows by restriction (R.4). Thus set

$$S'_A := \{(w,q) \mid (w,q) \in Q'_A, q \neq q_0\} \text{ and set } R'_A := Q'_A \backslash S'_A.$$

By Claim 1.7.30, if $(w, q) \in Q'_A$ satisfies $q \in S_A$ and $w = \epsilon$, then $q = q_0$. Moreover, if $(w, q) \in R'_A$ then $q = q_0$ by definition.

We may now construct the transducer $A_{(\epsilon,q_0)}$.

Let $A_{(\epsilon,q_0)} := \langle \dot{\mathbf{r}}, X_n, R'_A, S'_A, \pi'_A, \lambda'_A \rangle$ be the transducer with transition and output functions obeying the following rules:

(i) for $a \in \dot{\mathbf{r}}$ we have

$$\pi'_A(a, (\epsilon, q_0))) = (a - \lambda_A((a)\Theta_{q_0}, q_0), \pi_A((a)\Theta_{q_0}, q_0)) \text{ and } \lambda_A(a, (\epsilon, q_0)) = (a)\Theta_{q_0};$$

(ii) for $i \in X_n$ and $(w, q) \in Q_A \backslash \{(\epsilon, q_0)\}$ we have

$$\pi'_A(wi, q) = (wi - \lambda_A((wi)\Theta_q, q), \pi_A((wi)\Theta_q, q)) \text{ and } \lambda'_A(wi, q) = (wi)\Theta_q$$

(observe that if $w = \epsilon$ then $q \in S_A$ by Claim 1.7.30 so $(i)\Theta_q$ is well-defined).

We have the following claim for the transducer $A_{(\epsilon,q_0)}$.

Claim 1.7.31. *The transducer $A_{(\epsilon,q_0)}$ satisfies restrictions (R.1) to (R.4).*

Proof. We take each in turn.

(R.1) Let $(w, q_0) \in R'_A$. Observe that, by Claim 1.7.30, if $w = \epsilon$ then $q = q_0$. Let $a \in \dot{\mathbf{r}}$ and suppose $\pi'_A(a, (\epsilon, q_0)) \in R'_A$. If $(a)\Theta_{q_0} \neq \epsilon$, then, since $\pi'_A(a, (\epsilon, q_0)) \in R'_A$, we must have $\pi_A((a)\Theta_{q_0}, q_0) = q_0$. However, by restriction (R.1) for A_{q_0}, this means $\lambda_A((a)\Theta_{q_0}, q_0) = \epsilon$, since $q_0 \in R_A$. This contradicts restriction (R.4) on A_{q_0}. Thus, $(a)\Theta_{q_0} = \epsilon$, $\pi_A((a)\Theta_{q_0}, q_0) = q_0$ and $\pi'_A(a, (\epsilon, q_0)) = (a, q_0)$.

Now we consider the case that $(w, q_0) \in R'_A$ and $w \neq \epsilon$. By Claim 1.7.30 part (ii), it must be the case that $w \in X^+_{n,r}$, since $\operatorname{im}(q_0) = \mathfrak{C}_{n,r}$. Let $i \in X_n$ and suppose $\pi'_A(i, (w, q_0)) \in R'_A$. If $(wi)\Theta_{q_0} \neq \epsilon$, then, since $\pi'_A(i, (w, q_0)) \in R'_A$ we must have $\pi_A((wi)\Theta_{q_0}, q_0) = q_0$. However, by restriction (R.1) for A_{q_0}, this means $\lambda_A((wi)\Theta_{q_0}, q_0) = \epsilon$. This contradicts restriction (R.4) on A_{q_0}. Therefore, we conclude that $(wi)\Theta_{q_0} = \epsilon$, $\pi_A((wi)\Theta_{q_0}, q_0) = q_0$ and $\pi'_A(i, (w, q_0)) = (wi, q_0)$.

(R.2) Let $(w, q_0) \in R'_A$ and let i be an element of $X_n \sqcup X_{n,r}$ such that $\pi'_A(i, (w, q_0))$ is well defined and is an element of S'_A. This means that $\pi_A((wi)\Theta_{q_0}, q_0) \in S_A$ and so is not equal to q_0, therefore, we must have $(wi)\Theta_{q_0} \neq \epsilon$. Hence, $(wi)\Theta_{q_0} \in X^+_{n,r}$ since h_{q_0} has domain and range equal to $\mathfrak{C}_{n,r}$ and we conclude that $\lambda'_A(i, (w, q_0)) \neq \epsilon$.

(R.3) This follows by definition of the transition function π'_A, the fact that A_{q_0} is a non-degenerate transducer over $\mathfrak{C}_{n,r}$ and the fact that for all $q \in R_A \sqcup S_A$ such that $q \neq Q_0$, h_q is an injective map with domain $\mathfrak{C}_n$.

(R.4) Let $(w, q) \in Q'_A$ be any state and let $i \in X_{n,r} \sqcup X_n$ be such that $\pi'_A(i, (w, q))$ is well-defined. Suppose that $\pi'_A(i, (w, q)) = (w, q)$. This means that $\pi_A((wi)\Theta_q, q) = q$. If, furthermore, $(wi)\Theta_q = \epsilon$, then $\pi'_A(i, (w, q)) = (wi, q) \neq (w, q)$ a contradiction. Therefore we must have $(wi)\Theta_q \neq \epsilon$ as required.

□

Therefore we conclude that $A_{(\epsilon, q_0)}$ is a non-degenerate transducer. We have the following claim.

Claim 1.7.32. *Let $(w, q) \in Q'_A$ and let $\eta \in X^*_n$, then $\lambda'_A(\eta, (w, q)) = (w\eta)\Theta_q$, and $\pi'_A(\eta, (w, q)) = (w\eta - \lambda_q((w\eta)\Theta_q, q), \pi_A((w\eta)\Theta_q, q))$.*

The proof is similar to the proof of Claim 1.7.32 and so we omit it.
From Claim 1.7.32, it follows that $h_{(\epsilon, q_0)} = h^{-1}_{q_0}$. This concludes the construction.

Notation 1.7.33. Let A_{q_0} be a transducer inducing a homeomorphism of $\mathfrak{C}_{n,r}$, we denote by $A_{(\epsilon, q_0)}$ the transducer arising from Construction 1.7.29. The inverse transducer of A_{q_0} is the minimal transducer obtained by minimising $A_{(\epsilon, q_0)}$, and we denote it by $A_{q_0^{-1}}$. We use the symbol $Q_{A^{-1}}$ for the states of $A_{q_0^{-1}}$, $\pi_{A^{-1}}$ for the transition function of $A_{q_0^{-1}}$ and $\lambda_{A^{-1}}$ for the output function of $A_{q_0^{-1}}$. In the case that A_{q_0} is a transducer over $\mathfrak{C}_{n,r}$, we use the notation $S_{A^{-1}}$ and $R_{A^{-1}}$ for the disjoint subsets of Q_A such that $Q_{A^{-1}} = S_{A^{-1}} \sqcup R_{A^{-1}}$ and $A_{q_0^{-1}} = \langle \dot{\mathbf{r}}, X_n, R_{A^{-1}}, S_{A^{-1}}, \pi_{A^{-1}}, \lambda_{A^{-1}} \rangle$ satisfies restrictions (R.1) to (R.4).

As with transducers over $\mathfrak{C}_n$ (Proposition 1.7.27), the transducer $A_{(\epsilon, q_0)}$ has no states of incomplete response.

Proposition 1.7.34. *Let A_{q_0} be a transducer over $\mathfrak{C}_{n,r}$, then $A_{(\epsilon, q_0)}$ has no states of incomplete response.*

This proposition is proved in a similar way to Proposition 1.7.27 and so we omit its proof here.
Constructions 1.7.21 and 1.7.29 together with subsequent results demonstrate the following:

Proposition 1.7.35. *The sets $\mathcal{R}_n$ and $\mathcal{R}_{n,r}$ are closed under inverses.*

Since the identity homeomorphism over $\mathfrak{C}_n$ and $\mathfrak{C}_{n,r}$ is an element off $\mathcal{R}_n$ and $\mathcal{R}_{n,r}$ respectively, we have the following result:

Theorem 1.7.36. *The monoids* $\mathcal{R}_n$ *and* $\mathcal{R}_{n,r}$ *are subgroups of the groups* $H(\mathfrak{C}_n)$ *and* $H(\mathfrak{C}_{n,r})$ *respectively.*

Notation 1.7.37. We use the symbol id to represent both the single state identity transducer inducing the identity element of the group $\mathcal{R}_n$ and the two state identity transducer inducing the identity element of $\mathcal{R}_{n,r}$. We also sometimes use this symbol for the identity element of a group, however, whenever we do so, it will be clear from the context which point of view is being taken.

Remark 1.7.38. We use the term *rational group* to refer to the groups $\mathcal{R}_n$ and $\mathcal{R}_{n,r}$, and specify which group we are referring to whenever it is unclear from the context. We should perhaps mention that, for distinct $n, m \in \mathbb{N}_2$, the groups $\mathcal{R}_n$ and $\mathcal{R}_m$ are isomorphic ([30]). Furthermore the group $\mathcal{R}_{n,r}$ is isomorphic to $\mathcal{R}_n$, it is however more natural to work in the group $\mathcal{R}_{n,r}$ when discussing the automorphisms of the groups $G_{n,r}$ as will be seen in Chapter 2.

We close this section with some examples.

First we compute the inverse of a synchronous transducer A_{q_0} inducing a homeomorphism of $\mathfrak{C}_n$ or $\mathfrak{C}_{n,r}$.

Example 1.7.39. Let $A_{q_0} = \langle X_n, Q_A, \pi_A, \lambda_A \rangle$ be a possibly infinite synchronous transducer such that h_{q_0} induces a homeomorphism of $\mathfrak{C}_n$. Observe that this means, for each state $q \in Q_A$, the map $\lambda_A(\centerdot, q) : X_n \to X_n$ is a permutation. Therefore, each state q of A_{q_0} induces a homeomorphism $h_q : \mathfrak{C}_n \to \mathfrak{C}_n$. For $q \in Q_A$ let us denote by $\bar{q}$ the permutation $\lambda_A(\centerdot, q) : X_n \to X_n$. We now show that the set of states Q'_A of the inverse transducer $A_{(\epsilon, q_0)} = \langle X_n, Q'_A \pi'_A, \lambda'_A \rangle$ is precisely the set $\{(\epsilon, q) \mid q \in Q_A\}$. We also demonstrate that the transition and output functions π'_A and λ'_A of $A_{(\epsilon,q_0)}$ satisfy, $\pi'_A(i, (\epsilon, q))) = (\epsilon, p)$ and $\lambda'_A(i, (\epsilon, q)) = j$ if and only if $\pi_A(j, q) = p$ and $\lambda_A(j, q) = i$.

We begin with the set of states Q'_A. First we make the following observation: let $q \in Q_A$, for $i \in X_n$, we have $(i)\Theta_q = (i)\bar{q}^{-1}$. This is because $\lambda_A(\centerdot, q) : X_n \to X_n$ is the permutation $\bar{q}$, thus $\lambda_A((i)\Theta_q, q) = i$. Thus, we have $(i - \lambda_A((i)\Theta_q, q), \pi_A((i)\Theta_q, q)) = (\epsilon, \pi_A((i)\Theta_q, q))$. It therefore follows, by induction and Construction 1.7.21, that $Q'_A := \{(\epsilon, q) \mid q \in Q_A\}$.

Now let $(\epsilon, q) \in Q'_A$, $i \in X_n$ and $j = (i)\bar{q}^{-1}$. Observe that $\lambda'_A(i, (\epsilon, q)) = (i)\Theta_q = j$. Moreover, $\pi'_A(i, (\epsilon, q)) = (\epsilon, \pi_A(j, q))$. This demonstrates the relation stated above for the transition and output functions π'_A, λ'_A of $A_{(\epsilon,q_0)}$.

If A_{q_0} is an invertible synchronous transducer over $\mathfrak{C}_{n,r}$ then we deduce, by similar arguments, that $A_{(\epsilon,q_0)}$ has state set $Q'_A := \{(\epsilon, q) \mid q \in Q_A\}$. Moreover for $a, b \in \dot{r}$ and $p \in Q_A$ we have, $\pi'_A(a, (\epsilon, q_0)) = (\epsilon, p)$ and $\lambda'(a, (\epsilon, q_0)) = b$ if and only if $\pi_A(b, q_0) = p$ and $\lambda_A(b, q_0) = a$; for $i, j \in X_n$ and $q, p \in Q_A \backslash \{q_0\}$ we have, $\pi'_A(i, (\epsilon, q)) = (\epsilon, p)$ and $\lambda'(i, (\epsilon, q)) = j$ if and only if $\pi_A(j, q) = p$ and $\lambda_A(j, q) = i$.

We may deduce a few things about invertible synchronous transducers over $\mathfrak{C}_n$ or $\mathfrak{C}_{n,r}$ from Example 1.7.39, these constitute the remark below.

Remark 1.7.40. Let A_{q_0} be a minimal invertible synchronous transducer, then the following statements are true:

(1.) A_{q_0} has no states of incomplete response since every state q of A_{q_0} induces a permutation $\lambda_A(\centerdot, q) : X_n \to X_n$ or $\lambda_A(\centerdot, q) : \dot{r} \to \dot{r}$.

(2.) If A_{q_0} is minimal then $A_{(\epsilon,q_0)}$ is minimal.

(3.) $|Q_A| = |Q'_A|$

(4.) If A_{q_0} is a transducer over $\mathfrak{C}_{n,r}$ then $R_A = \{q_0\}$.

(5.) For $q \in Q_A$, the composition of the permutations $\lambda_A(\centerdot, q)\lambda'_A(\centerdot, (\epsilon, q))$ is the identity map on $\dot{r}$ or X_n.

The following definition arises since every state of an invertible synchronous transducer induces a self-homeomorphism on the appropriate Cantor space $\mathfrak{C}_n$ or $\mathfrak{C}_{n,r}$. First we recall Notation 1.3.9 that, for an initial transducer A_{q_0}, we denote by A the transducer A_{q_0} with no state initialised i.e A is the underlying transducer.

Definition 1.7.41. Let A_{q_0} be a synchronous transducer over $\mathfrak{C}_n$ or $\mathfrak{C}_{n,r}$ with no inaccessible states and let A be the underlying transducer of A_{q_0}. Then we say that A is *invertible* if and only if A_{q_0} is invertible. In this case we write A^{-1} for the inverse of A.

Notation 1.7.42. Let A_{q_0} be a synchronous transducer over $\mathfrak{C}_n$ or $\mathfrak{C}_{n,r}$, with underlying transducer A. For each state $q \in Q_A$ we denote by q^{-1} the state (ϵ, q_0) of A^{-1}. We thus write $A_{q_0^{-1}}$ for the inverse transducer $A_{(\epsilon,q_0)}$. We use the symbol Q_A^{-1} for the set of states of $A_{q_0^{-1}}$, $\pi_{A^{-1}}$ for the transition function of $A_{q_0^{-1}}$, and $\lambda_{A^{-1}}$ for the output function of $A_{q_0^{-1}}$. The map from Q_A to Q_A^{-1} sending a state q to the state q^{-1} is a bijection, thus we set $(q^{-1})^{-1} = q$. We also extend this notation for the underlying transducer A^{-1} of $A_{q_0}^{-1}$.

In the next example, we compute the inverse of an asynchronous transducer A_{q_0} inducing a homeomorphism of $\mathfrak{C}_n$.

Example 1.7.43. We compute the inverse of the transducer in Figure 1.5 (we reproduce it below for convenience). Let us once more denote this transducer by A_{q_3}.

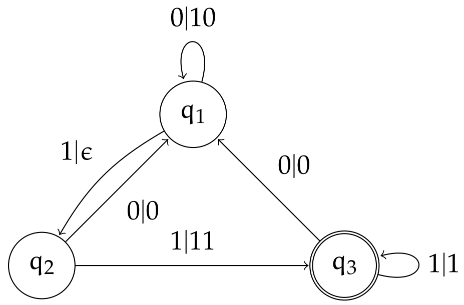


Figure 1.15: Example of an asynchronous transducer with initial state q_3

In Example 1.7.14 we showed that this transducer induces a homeomorphism of $\mathfrak{C}_n$ of order two. Thus, we expect that the minimal transducer representing the inverse of A_{q_3}, $A_{q_3^{-1}}$ to satisfy the equality: $A_{q_3^{-1}} \cong_\omega A_{q_3}$. The reader may verify that $A_{(\epsilon,q_3)}$ is the transducer below:

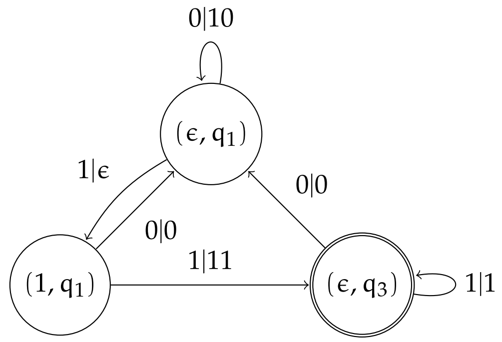


Figure 1.16: The inverse transducer of A_{q_0}

Thus since $A_{(\epsilon,q_3)} \cong_\omega A_{q_3}$ we therefore have $A_{q_3^{-1}} \cong_\omega A_{(\epsilon,q_3)} \cong_\omega A_{q_3}$.

A natural question one may ask at this stage is: given an invertible minimal transducer A_{q_0} over $\mathfrak{C}_n$ or $\mathfrak{C}_{n,r}$, is it true that $|A_{q_0^{-1}}| = |A_{q_0}|$? We have seen in Example 1.7.39 that this equality holds for synchronous transducers, however the example below demonstrates that in general this question has a negative answer.

Example 1.7.44. Consider the transducer A_{q_0} over $\mathfrak{C}_{3,2}$ below.

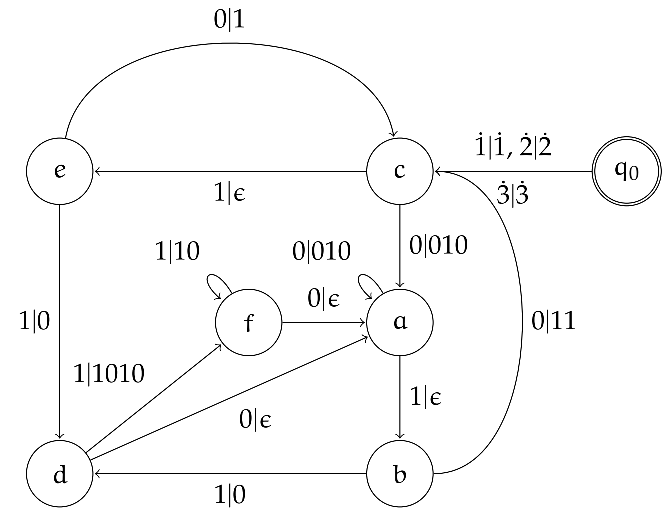


Figure 1.17: A transducer whose inverse has strictly more states

This transducer is minimal since it has no states of incomplete response and no pair of ω-equivalent states. Moreover, the initial transducer A_c induces a homeomorphism of $\mathfrak{C}_2$. In particular, since state q_0 acts as the identity on the symbols $\dot{\mathbf{3}} = \{\dot{1}, \dot{2}, \dot{3}\}$, it follows that the state (ϵ, q_0) also acts as the identity on the set $\dot{\mathbf{3}}$ and transitions to the state (ϵ, c) after reading these symbols. Therefore it suffices to show that $A_{c^{-1}}$ has strictly more states than A_c.

We leave it to the reader to verify that $A_{c^{-1}}$ is the transducer given below.

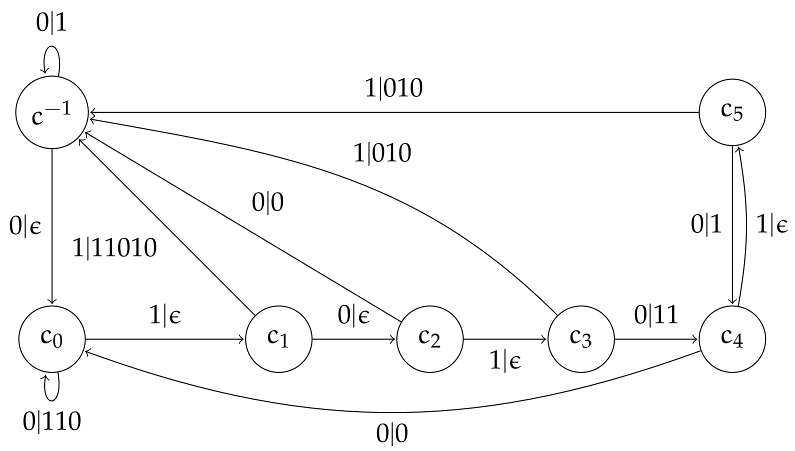


Figure 1.18: The inverse transducer of A_c has strictly more states

We should point out that the transducer $A_{(\epsilon,c)}$, constructed from A_c using Construction 1.7.21, has 11 states. Thus it is also not true, in general, that for a minimal transducer B_{p_0}, the inverse transducer $B_{(\epsilon,p_0)}$ is minimal. While these examples have been computed by hand, they have also been verified with the GAP package "aaa" for working with asynchronous transducers being developed by Collin Bleak, Fernando Flores-Brito, Plamena Minerva, the author and Angela Richardson.

In the next chapter we introduce the subgroups of $\mathcal{R}_n$ and $\mathcal{R}_{n,r}$ we shall primarily be concerned with. These subgroups may be characterised combinatorially by imposing restrictions on the transducers representing their elements.

Chapter 2

Subgroups of $\mathcal{R}_n$ and $\mathcal{R}_{n,r}$

The rational group $\mathcal{R}_n$ and the closely related group $\mathcal{R}_{n,r}$ were introduced in the papers [30] and [10]. In the paper [30] the authors demonstrate that the isomorphism class of the group $\mathcal{R}_n$ is independent of n, however we shall not require that fact here. These groups are of current topical interest in the research community: many important families of groups are subgroups of the rational group. Furthermore, as elements of $\mathcal{R}_n$ and $\mathcal{R}_{n,r}$ are homeomorphisms induced by finite transducers, this gives rise to nice representations of elements of those groups which embed as subgroups of $\mathcal{R}_n$ and $\mathcal{R}_{n,r}$. For instance, although the groups $\mathcal{R}_n$ and $\mathcal{R}_{n,r}$ are not finitely generated ([30], [5]) it is still possible to decide when a finite product results in the identity element by using the algorithm for multiplication in Chapter 1. Thus, in any finitely generated subgroup of the rational group it is possible to decide whether or not a finite products in the generators results in the trivial element of the group. If the subgroup in question is in fact finitely presented, the decision problem just stated is known as the *word problem*, and has been shown to be insoluble in general by Novikov [43] and independently by Boone [11]. However, by finding embeddings of a finitely presented group into $\mathcal{R}_n$ or $\mathcal{R}_{n,r}$ one can immediately conclude that the group in question has soluble word problem.

This chapter shall be primarily devoted to showing that the group $\mathcal{R}_n$ and $\mathcal{R}_{n,r}$ contain as subgroups the Higman-Thompson groups $G_{n,r}$ and their automorphism group $\mathrm{Aut}(G_{n,r})$. Finally, we shall focus in on a particular subgroup $\mathcal{H}_n$ of the quotient $\mathrm{Out}(G_{n,r}) = \mathrm{Aut}(G_{n,r})/\mathrm{Inn}(G_{n,r})$ of the automorphism group by the inner automorphisms. In the next chapter, we show that this group coincides exactly with a well-known and well-studied group in symbolic dynamics, namely, the group of automorphisms of the one-sided shift dynamical system.

We begin by defining a property $\mathcal{S}$ of transducers. We then observe that the set of all transducers with property $\mathcal{S}$ forms a monoid. However, when we restrict to those elements of $\mathcal{R}_n$ and $\mathcal{R}_{n,r}$ with property $\mathcal{S}$ whose inverses also have property $\mathcal{S}$, the corresponding sets yield subgroups $\mathcal{B}_n$ and $\mathcal{B}_{n,r}$. We then demonstrate that the groups $G_{n,r}$ and $\mathrm{Aut}(G_{n,r})$ are subgroups of $\mathcal{B}_{n,r}$. A consequence of this is a nice characterisation in terms of transducers of $\mathrm{Out}(G_{n,r})$. We close by homing in on a specific subgroup $\mathcal{H}_n$ of $\mathrm{Out}(G_{n,r})$, showing that this group is isomorphic to the group of automorphisms of the one-sided shift dynamical system. We shall intersperse the discussion with various consequences of the property $\mathcal{S}$ wherever it is appropriate to do so.

2.1 The synchronizing property

We now define the property $\mathcal{S}$ for an arbitrary automaton, we then extend to it to transducers by making use of the underlying automaton of a transducer.

Definition 2.1.1 (Synchronizing, [10]). An automaton $A = \langle X, Q_A, \pi_A \rangle$ is said to be *synchronizing at level* m or *synchronizing* if there a natural number m and a map $f_m : X^m \to Q_A$ such that for all $w \in X^m$ and $q \in Q_A$ we have, $\pi_A(w, q) = (w)f_m$. We call m the *synchronizing level of* A, and f_m the *synchronizing map at level* m or *a synchronizing map*. If A is synchronizing at level 1 we shall also refer to it as a *reset automaton*.

Remark 2.1.2. If A is an automaton synchronizing at level m, the definition above can be interpreted as follows: whenever the automaton processes a word of length m the new active state is *independent* of the initial state. An automaton is synchronizing at level 0 if and only if it has a single state. Observe that if A is a synchronizing automaton, then, for any state q of A, the initial automaton A_q is also synchronizing.

Definition 2.1.3. Let $A = \langle X, Q_A, \pi_A \rangle$ be an automaton. A word $w \in X^*$ is called a *synchronizing word* if the map $\pi_A(w, .) : Q_A \to Q_A$ takes only one value.

Definition 2.1.4. Let $A = \langle X, Q_A, \pi_A \rangle$ be an automaton, and $U \subset X^\omega$. A *base for* A *over* U is a finite set $\mathscr{S} \subset X^*$ consisting entirely of synchronizing words such that every element of U has a prefix in $\mathscr{S}$.

Remark 2.1.5. Let A be the single state automaton over X. For a subset $U \subset X^\omega$, we refer to a base of A over U simply as *a base for* U. For a subset U of $\mathfrak{C}_n$, a base $\mathscr{S}$ for U is simply a subset of X_n^* such that every element of U has a prefix in $\mathscr{S}$

Lemma 2.1.6. *Let $A = \langle X, Q_A, \pi_A \rangle$ be an automaton, $U \subset X^\omega$ and $\mathscr{S}$ be a base for A over U. There is a $j \in \mathbb{N}$ such that the set of all finite prefixes of elements of U of length j consists entirely of synchronizing words for A. This implies that A is synchronizing if and only if there is a base for A over X^ω (the forward implication follows from the definition of synchronization).*

Proof. Let A, U and $\mathscr{S}$ be as in the statement of the lemma Take $j = \max |\nu| \mid \nu \in \mathscr{S}$. Then the set prefixes of elements of U of length j, must have a prefix in $\mathscr{S}$. Since appending a finite suffix to a synchronizing word, results in a synchronizing word and since, by definition, $\mathscr{S}$ consists only of synchronizing words, it follows that the set of all finite prefixes of elements of U of length j consists entirely of synchronizing words for A.

For the second part of the lemma, we observe that if A is synchronizing at length k, for some $k \in \mathbb{N}$, then the set of all words of length k is a base for A over X^ω. The 'only if' direction is a consequence of the first part of the lemma. □

Remark 2.1.7. Observe that if A is a synchronizing automata then, by definition, A has only finitely many states.

Remark 2.1.8. There is weaker notion of synchronizing automata which occurs in the literature concerning the Černý Conjecture and the road colouring problem, as for instance in the articles [54] and [53]. In this definition a transducer is synchronizing if there is at least one word which is a synchronizing word for the automaton. In this work we use the word 'synchronizing' as an adjective to describe an automaton to mean precisely what is stated in Definition 2.1.1.

Notation 2.1.9. Let $A = \langle X, Q_A, \pi_A \rangle$ be an automaton which is synchronizing at level m with f_m the synchronizing map at level m. For $w \in X^m$, we denote by q_w the state $(w)f_m$ and say that q_w is the *state of* A *forced by* w.

Remark 2.1.10. Let $A = \langle X, Q_A, \pi_A \rangle$ be an automaton which is synchronizing at level m with f_m the synchronizing map at level m. Observe that for any $k \in \mathbb{N}_m$, A is also synchronizing at level k. This is because after processing a word of length k from any state of A, the resulting state depends only on the length m suffix of the word. This implies that there is a minimal synchronizing level for A. Let $j \in \mathbb{N}$ be the minimal synchronizing level of A, then, we denote by f the synchronizing map f_j and call f *the synchronizing map of* A. Clearly, for $m_1 \geqslant m_2 \geqslant j$, the image sets of the synchronizing maps f_{m_1} and f_{m_2} coincide.

Definition 2.1.11. Let $A := \langle X, Q_A, \pi_A \rangle$ be an automata which is synchronizing at level m with f_m the synchronizing map at level m. Let $\mathrm{im}(f_m) := \{(w)f_m \mid w \in X^m\} \subset Q_A$. Set $\mathrm{Core}(A) := \langle X, \mathrm{im}(f_m), \pi_A\restriction_{\mathrm{im}(f_m)} \rangle$, the subautomaton of A with state set $\mathrm{im}(f_m)$. We call $\mathrm{Core}(A)$ the core of A and, if $A = \mathrm{Core}(A)$ we say that A is *core*.

Remark 2.1.12. It follows from Remark 2.1.10 that the definition of the core of a synchronizing automaton is independent of which synchronizing map is used.

We have the following proposition about $\mathrm{Core}(A)$:

Proposition 2.1.13. *Let* $A := \langle X, Q_A, \pi_A \rangle$ *be an automaton which is synchronizing at level* m *and* f_m *the synchronizing map at level* m*. We have:*

(i) for any $i \in X$ *and* $q \in$ *im*(f_m), $\pi_A(i, q) \in$ *im*(f_m),

(ii) Core(A) *is a strongly connected subautomaton of* A.

Proof. We begin with the first point. Let $q \in \mathrm{im}(f_m)$. By definition of the map f_m, there is a word $w \in X^m$ such that $q = q_w$. Let $i \in X$, and let $\overline{w}$ be such that $w = w_1\overline{w}$ for $w_1 \in X$. Consider the word $\overline{w}i \in X^m$. For any state $p \in Q_A$, $\pi_A(wi, p) = \pi_A(i, q_w)$ (by rule (1.2)). Thus, it must be the case that for any state $p' \in Q_A$ $\pi_A(\overline{w}i, p') = \pi_A(i, q_w)$. Therefore, since $\overline{w}i \in X^m$, we conclude that $(\overline{w}i)f_m = \pi_A(i, q_w)$.

The second part follows from the definition of a synchronizing automata, since for any state q of Core(A), there is a word $w \in X^m$ such that $q_w = q$. □

We now extend the definition to transducers over $\mathfrak{C}_n$ and $\mathfrak{C}_{n,r}$. We begin with transducers over $\mathfrak{C}_n$.

Definition 2.1.14. Let $T = \langle X_n, Q_T, \pi_T, \lambda_T \rangle$ be a transducer over X_n, then T is said to be *synchronizing* if the underlying automaton $\mathcal{A}(T)$ is synchronizing. If $m \in \mathbb{N}$ is a synchronizing level of $\mathcal{A}(T)$, we say that T is *synchronizing at level* m and we call m a *synchronizing level of* T. The synchronizing map at level m of T is precisely the synchronizing map at level m of $\mathcal{A}(T)$ and the synchronizing map of T is the synchronizing map of $\mathcal{A}(T)$.

Definition 2.1.15. Let $T = \langle X_n, Q_T, \pi_T, \lambda_T \rangle$ be a transducer over X_n. A word $w \in X_n^*$ is called *a synchronizing word for* T if and only if it is a synchronizing word of $\mathcal{A}(T)$. Given a subset $U \subset \mathfrak{C}_n$ a subset $\mathscr{S}$ of X_n^* is called a *base for* T *over* U if it is a base for $\mathcal{A}(T)$ over U. As in Remark 2.1.5, if T is a single state transducer, then we shall simply refer to $\mathscr{S}$ as a *base for* U.

Notice that if T is a synchronizing transducer over X_n, then, for any state q of T, the initial transducer T_q is also synchronizing.

Definition 2.1.16. Let $T = \langle X_n, Q_T, \pi_T, \lambda_T \rangle$ be a synchronizing transducer over X_n and let f be the synchronizing map of T. Set Core$(T) := \langle X_n, \ \mathrm{im}(f), \pi_T\restriction_{\mathrm{im}(f)}, \lambda_T\restriction_{\mathrm{im}(f)} \rangle$ be the subtransducer of T consisting of the states of Core$(\mathcal{A}(T))$. We call Core(T) the *core of* T; if $T = $ Core(T), we say that T is *core*.

We now extend the definition to transducers over $\mathfrak{C}_{n,r}$. First we set up some further notation.

Notation 2.1.17. Let $T_{q_0} = \langle \dot{\mathbf{r}}, X_n, R_T, S_T, \pi_T, \lambda_T \rangle$ be an initial transducer over $\mathfrak{C}_{n,r}$. Recall (Definition 1.5.17) that $Q_T = R_T \sqcup S_T$. Set $\mathcal{A}(T)\backslash\{q_0\} := \langle X_n, Q_T\backslash\{q_0\}, \pi_T \rangle$ the automata consisting of all the states of T without the initial state q_0.

Definition 2.1.18. Let $T_{q_0} = \langle \dot{\mathbf{r}}, X_n, R_T, S_T, \pi_T, \lambda_T \rangle$ be an initial transducer over $\mathfrak{C}_{n,r}$, then T_{q_0} is said to be *synchronizing at level m* if and only if $\mathcal{A}(T)\backslash\{q_0\}$ is synchronizing at level $m - 1$. In this case call m the *synchronizing level of* T_{q_0}. If T_{q_0} is synchronizing at level m for some $m \in \mathbb{N}_1$, then we say that T_{q_0} is *synchronizing*.

Remark 2.1.19. Let $T_{q_0} = \langle \dot{\mathbf{r}}, X_n, R_T, S_T, \pi_T, \lambda_T \rangle$ be an initial transducer over $\mathfrak{C}_{n,r}$. Observe that by restriction (R.1) we must have, for any $a \in \dot{\mathbf{r}}$, $\pi_T(a, q_0) \neq q_0$. Thus, as letters from $\dot{\mathbf{r}}$ can only be processed at q_0, the synchronizing property really depends on the states of $Q_T\backslash q_0$. Therefore, it makes sense to set the synchronizing level of a transducer T_{q_0} over $\mathfrak{C}_{n,r}$ to be m precisely when $\mathcal{A}(T)\backslash\{q_0\}$ is synchronizing at level $m - 1$.

Definition 2.1.20. Let $T_{q_0} = \langle \dot{\mathbf{r}}, X_n, R_T, S_T, \pi_T, \lambda_T \rangle$ be a transducer over $\mathfrak{C}_{n,r}$. A word $aw \in X_{n,r}^+$, for $a \in \dot{\mathbf{r}}$, is called *a synchronizing word for* T_{q_0} if and only if w is a synchronizing word of $\mathcal{A}(T)\backslash\{q_0\}$. Given a subset $U \subset \mathfrak{C}_{n,r}$, a subset $\mathscr{S}$ of $X_{n,r}^*$ is called a *base for* T_{q_0} *over* U if it consists entirely of synchronizing words and every element of U has a prefix in $\mathscr{S}$. If T_{q_0} has only two states, then we call the set $\mathscr{S}$ a *base for* U.

Definition 2.1.21. Let $T_{q_0} = \langle \dot{\mathbf{r}}, X_n, R_T, S_T, \pi_T, \lambda_T \rangle$ be an initial transducer over $\mathfrak{C}_{n,r}$ which is synchronizing at level $m+1$. Let g_m and g_{m+1} be the synchronizing maps at level m and $m+1$, respectively, of $\mathcal{A}(T)$. Define a map $\overline{g}_m : X_{n,r}^{m+1} \to Q_T$ by $(aw)\overline{g}_m = (w)g_m$ where $a \in \dot{\mathbf{r}}$ and $w \in X_n^m$. Let $f_{m+1} : X_{n,r}^{m+1} \sqcup X_n^{m+1} \to Q_T$ be such that $f_{m+1}\restriction_{X_{n,r}^{m+1}} := \overline{g}_m$ and $f_{m+1}\restriction_{X_n^{m+1}} := g_{m+1}$. We call f_{m+1} the *synchronizing map at level* $m+1$ *of* T_{q_0} or *a synchronizing map of* T_{q_0}. For j the minimal synchronizing level of T_{q_0}, we denote by f the map f_j, and we call it the *synchronizing map of* T_{q_0}.

The following remark is essentially a consequence of Remark 2.1.19.

Remark 2.1.22. Let $T_{q_0} = \langle \dot{\mathbf{r}}, X_n, R_T, S_T, \pi_T, \lambda_T \rangle$ be an initial transducer over $\mathfrak{C}_{n,r}$ which is synchronizing at level m, let f_m be the synchronizing map at level m of T_{q_0} and $w \in X_{n,r}^m$, then $\pi_T(w, q_0) = (w)f_m$.

Notation 2.1.23. Let $T_{q_0} = \langle \dot{\mathbf{r}}, X_n, R_T, Q_T, \pi_T, \lambda_T \rangle$ be an initial transducer over $\mathfrak{C}_{n,r}$ which is synchronizing at level m. Let f_m be the synchronizing map at level m of T_{q_0} and $w \in X_{n,r}^m \sqcup X_n^m$, then we denote by q_w the state $(w)f_m$, and say q_w *is the state of* T_{q_0} *forced by* w.

Definition 2.1.24. Let $T_{q_0} = \langle \dot{\mathbf{r}}, X_n, R_T, Q_T, \pi_T, \lambda_T \rangle$ be an initial transducer over $\mathfrak{C}_{n,r}$ which is synchronizing at level m. Let $f_m : X_{n,r}^m \sqcup X_n^m \to Q_T$ be the synchronizing map at level m of T_{q_0} and $\operatorname{im}(f_m) := \{(w)f_m \mid w \in X_{n,r}^m \sqcup X_n^m\}$. Set $\mathrm{Core}(T_{q_0}) := \langle X_n, \operatorname{im}(f), \pi_T\restriction_{\operatorname{im}(f)}, \lambda_T\restriction_{\operatorname{im}(f)} \rangle$ the subtransducer of T_{q_0} generated by stated in $\operatorname{im}(f_m)$. We call $\mathrm{Core}(T_{q_0})$ the *core of* T_{q_0}.

Remark 2.1.25. Let $T_{q_0} = \langle \dot{\mathbf{r}}, X_n, R_T, S_T, \pi_T, \lambda_T \rangle$ be an initial transducer over $\mathfrak{C}_{n,r}$ which is synchronizing. We have the following:

(i) For m_1 and m_2 greater than the minimal synchronizing level of T_{q_0}, the synchronizing maps f_{m_1} and f_{m_2} of T_{q_0} at levels m_1 and m_2 respectively, satisfy, $\operatorname{im}(f_{m_1}) = \operatorname{im}(f_{m_2})$. Therefore, the definition of $\mathrm{Core}(T_{q_0})$ is independent of the synchronizing map used to define it.

(ii) By restriction (R.1), we must have $\mathrm{Core}(T_{q_0}) \subseteq Q_T \backslash \{q_0\}$.

(iii) As in Proposition 2.1.13, $\mathrm{Core}(T)$ is a strongly-connected subtransducer of T_{q_0}.

Below we give examples of synchronizing transducers.

Definition 2.1.26. Let T_{q_0} be an initial transducer over $\mathfrak{C}_n$ or $\mathfrak{C}_{n,r}$ which is synchronizing, then T_{q_0} is said to have *trivial core* if $\mathrm{Core}(T_{q_0}) = \mathrm{id}$.

Example 2.1.27. Let T_ϵ be the transducer below:

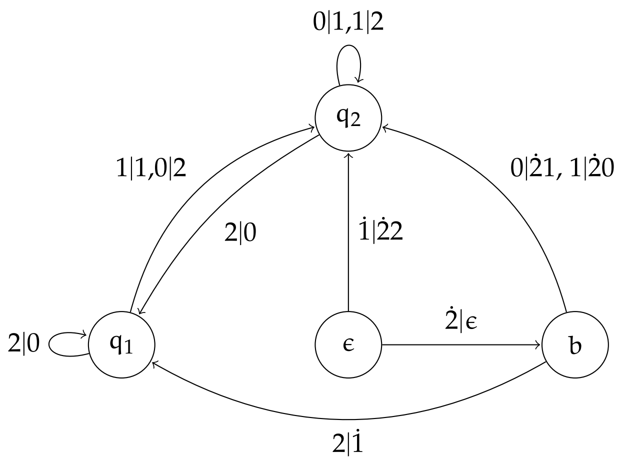


Figure 2.1: bi-synchronizing transducer

The reader can verify that T_{q_0} is synchronizing at level 3.

Example 2.1.28. The transducers A_{q_0} and $A_{c^{-1}}$ in Example 1.7.44 are both synchronizing transducers. Thus the initial transducer A_c, where c is the state of the transducer A_{q_0} in Example 1.7.44, is such that both A_c and A_c^{-1} are synchronizing.

Definition 2.1.29. Let T_{q_0} be a synchronizing transducer such that $h_{q_0} \in \mathcal{R}_n \sqcup \mathcal{R}_{n,r}$. If $h_{q_0}^{-1}$ can be represented by a synchronizing transducer then we say that T_{q_0} is *bi-synchronizing*; we say that T_{q_0} is *bi-synchronizing at level* k if T_{q_0} and the minimal transducer $T_{q_0^{-1}}$ representing the inverse of T_{q_0} are synchronizing at level k. The *minimal bi-synchronizing level* of T_{q_0} is therefore the minimal integer k such that T_{q_0} and $T_{q_0^{-1}}$ are both synchronizing at level k. If T is an invertible synchronous transducer, such T^{-1} is synchronizing then we say that T is *bi-synchronizing*.

The transducers A_c and A_{q_0} of Example 1.7.44 are both examples of bi-synchronizing transducers.

The following Proposition demonstrates that the property of being synchronizing is unaffected by the minimisation procedure.

Proposition 2.1.30 ([10]). *Let A_{q_0} be a synchronizing transducer over $\mathfrak{C}_{n,r}$ or $\mathfrak{C}_n$. Let B_{p_0} be the minimal transducer representing A_{q_0}. Then B_{p_0} is synchronizing and has minimal synchronizing level less than or equal to the minimal synchronizing level of A_{q_0}.*

Proof. Let A_{q_0} be a synchronizing transducer over $\mathfrak{C}_n$ or $\mathfrak{C}_{n,r}$. Clearly removing inaccessible states of A_{q_0} does not affect the transition function of A. Thus if B_{p_0} is the resulting transducer after removing inaccessible states from A_{q_0}, then B_{p_0} will also be synchronizing. Moreover, the minimal synchronizing level of B_{p_0} is less than or equal to the minimal synchronizing level of A_{q_0}.

Let $B'_{q_{-1}}$ be the transducer obtained from B_{p_0} by applying procedure **M2**. Observe that the set of states $Q_{B'}$ of $B'_{q_{-1}}$ can be written as $Q_{B'} = Q_B \sqcup \{q_{-1}\}$ where q_{-1} is a symbol distinct from Q_B . Moreover, $\pi_{B'}\restriction_{Q_B} = \pi_B$ and $\pi_{B'}\restriction_{\{q_{-1}\}} = \pi_B\restriction_{\{q_0\}}$. Thus, $B'_{q_{-1}}$ is synchronizing with minimal synchronizing level equal to the minimal synchronizing level of A.

Finally let $B''_{[q_{-1}]}$ be the transducer obtained from $B'_{q_{-1}}$ by applying procedure **M3**. The states of B'' are the ω-equivalence classes of states of B', moreover the transition function is defined by $\pi_{B''}(a, [q]) = [\pi_{B'}(a, q)]$ for $a \in X_{n,r} \sqcup X_n^*$ such that $\pi_{B''}(a, [q])$ is defined. Thus, it follows that $B''_{[q_{-1}]}$ is also synchronizing and has minimal synchronizing level less than or equal to the minimal synchronizing level of $B'_{q_{-1}}$. □

Notation 2.1.31. Let $\mathcal{B}_{n,r}$ [$\mathcal{B}_n$] denote the set of all elements of $\mathcal{R}_{n,r}$ [$\mathcal{R}_n$] which can be represented by bi-synchronizing transducers. Let $\widetilde{\mathcal{B}}_{n,r}$ [$\widetilde{\mathcal{B}}_n$] be the set of all elements of $\mathcal{R}_{n,r}$ [$\mathcal{R}_n$] which can be represented by synchronizing transducers.

The following proposition demonstrates that the product of two synchronizing transducers over $\mathfrak{C}_n$ or $\mathfrak{C}_{n,r}$ is again a synchronizing transducer.

Proposition 2.1.32 ([10]). *Let A_{q_0}, B_{p_0} be synchronizing transducers over $\mathfrak{C}_n$ or $\mathfrak{C}_{n,r}$. Then the product $AB_{(p_0,q_0)}$ is also synchronizing.*

Proof. First assume that $A_{q_0} = \langle X_n, Q_A, \pi_A, \lambda_A\rangle$ and $B_{p_0} = \langle X_n, Q_B, \pi_B, \lambda_B\rangle$ are transducers over $\mathfrak{C}_n$ with minimal synchronizing level j and k respectively.

Since A_{q_0} and B_{p_0} are non-degenerate transducers, there is a minimal $M \in \mathbb{N}_1$ such that $j \leqslant M$ and, for all $w \in X_n^M$ and any state $q \in Q_A$, $|\lambda_A(w, q)| \geqslant k$. Let $w_1 \in X_n^j$, $w_2 \in X_n^M$, q_{w_1} be the state of A_{q_0} forced by w_1 and $q_{w_2} = q_{w_1 w_2}$ be the state of A_{q_0} forced by w_2. Now let (q, p) be an arbitrary pair in $Q_A \times Q_B$. By assumption on M, the length of w_2, we have: $|\lambda_A(w_2, q_{w_1})| \geqslant k$. Therefore, let $\nu := \lambda_A(w_2, q_{w_1})$ and p_ν be the state of B_{p_0} forced by ν. Since $\lambda_A(w_1 w_2, q) = \lambda_A(w_1, q)\lambda_A(w_2, q_{w_1})$, then $\lambda_A(w_1 w_2, q)$ has a ν as a suffix. Thus, $\pi_{A*B}(w_1 w_2, (q, p)) = (q_{w_1 w_2}, p_\nu)$. Since $w_1 w_2 \in X_n^M$ and $(q, p) \in Q_A \times Q_B$ were chosen arbitrarily, it follows that $(A * B)_{(q_0,p_0)}$ is synchronizing at level M. It follows from Proposition 2.1.30 that $AB_{(p_0,q_0)}$ is also synchronizing.

Now let $A_{q_0} = \langle \dot{\mathbf{r}}, X_n, R_A, S_A, \pi_A, \lambda_A\rangle$ and $B_{p_0} = \langle \dot{\mathbf{r}}, X_n, R_B, S_B, \pi_B, \lambda_B\rangle$ be initial transducers over $\mathfrak{C}_{n,r}$. Let $j + 1$ and $k + 1$ be the minimal synchronizing levels of A_{q_0} and B_{p_0} respectively.

By Definition 2.1.18 and Proposition 2.1.30, it suffices to show that the automaton $\mathcal{A}((A * B)_{(q_0,p_0)})\backslash\{(p_0, q_0)\}$ is synchronizing, where $(A * B)_{(p_0,q_0)}$ is the transducer product of A_{p_0} with B_{q_0}.

Let $M \in \mathbb{N}_j$ be such that for any word $w \in X_n^M$ and any state $q \neq q_0$ of A_{q_0} we have, $|\lambda_A(w, q)| \geqslant k + 1$. Let $w_1 \in X_n^j$, $w_2 \in X_n^M$, and let $(q, p) \in Q_A * Q_B$ be such that $(q, p) \neq (q_0, p_0)$. Let q_{w_1} and $q_{w_1 w_2}$ be the states of $\mathcal{A}(A_{q_0})\backslash\{q_0\}$ forced by w_1 and $w_1 w_2$ respectively, and let $\nu := \lambda_A(w_2, q_{w_1})$. Now as $|\nu| \geqslant k + 1$, let p_ν be the state of B_{p_0} forced by ν. Thus, as before, we have $\pi_{A*B}(w_1 w_2, (q, p)) = (q_{w_1 w_2}, \pi_B(\lambda_A(w_1, q)\nu, p)) = (q_{w_1 w_2}, p_\nu)$. Since the choices of $w_1 w_2 \in X_n^{j+M}$ and $(q, p) \in Q_A * Q_B\backslash\{(q_0, p_0)\}$ were arbitrary, $(A * B)_{(p_0,q_0)}$ is synchronizing, and so $AB_{(p_0,q_0)}$ is synchronizing as well. □

Notice that if A_{q_0} and B_{p_0} above are synchronous, synchronizing transducers over $\mathfrak{C}_{n,r}$ or $\mathfrak{C}_n$, with j the minimal synchronizing level of A_{q_0} and k the minimal synchronizing level of B_{p_0} then we may take M in the proof above to be k. Thus as a corollary of Proposition 2.1.32 we have:

Proposition 2.1.33. *Let A_{q_0} and B_{p_0} be synchronizing, synchronous transducers over $\mathfrak{C}_n$ or $\mathfrak{C}_{n,r}$. Let $j, k \in \mathbb{N}$ be such that the minimal synchronizing levels of A_{q_0} and B_{q_0} are j and k respectively, then the minimal synchronizing level of $AB_{(p_0,q_0)}$ is at most $j + k$ if $h_{q_0}, h_{p_0} \in \widetilde{\mathcal{B}}_n$ and at most $j + k + 1$ if $h_{q_0}, h_{p_0} \in \widetilde{\mathcal{B}}_{n,r}$.*

We thus have the following result:

Theorem 2.1.34 ([10]). *The sets $\widetilde{\mathcal{B}}_{n,r}$ and $\widetilde{\mathcal{B}}_n$ are monoids with the multiplication inherited from $\mathcal{R}_{n,r}$ and $\mathcal{R}_n$ respectively; the sets $\mathcal{B}_{n,r}$ and $\mathcal{B}_n$ are subgroups of $\mathcal{R}_{n,r}$ and $\mathcal{R}_n$ respectively.*

Proof. The second statement follows from the first, since by definition the sets $\mathcal{B}_{n,r}$ and $\mathcal{B}_n$ are closed under taking inverses. Thus we focus only on the first statement of the theorem.

That the product is associative on $\widetilde{\mathcal{B}}_{n,r}$ and $\widetilde{\mathcal{B}}_n$ follows from the fact that it is associative on $\mathcal{R}_{n,r}$ and $\mathcal{R}_n$ respectively. Notice that the single-state identity transducer over $\mathfrak{C}_n$ is synchronizing at level 0 and the two state identity transducer over $\mathfrak{C}_{n,r}$ is synchronizing at level 1. Thus the identity element of $\mathcal{R}_n$ and $\mathcal{R}_{n,r}$ are in the sets $\widetilde{\mathcal{B}}_n$ and $\widetilde{\mathcal{B}}_{n,r}$ respectively. The closure of the product follows by Proposition 2.1.32. □

The subgroups of $\mathcal{R}_{n,r}$ and $\mathcal{R}_n$ that this thesis is concerned with are in fact subgroups of $\mathcal{B}_{n,r}$ and $\mathcal{B}_n$, though at times we consider submonoids of $\widetilde{\mathcal{B}}_{n,r}$ and $\widetilde{\mathcal{B}}_n$. It is therefore sensible at this stage to discuss how one might detect if an arbitrary element of $\mathcal{R}_n$ or $\mathcal{R}_{n,r}$ is an element of $\widetilde{\mathcal{B}}_{n,r}$ or $\widetilde{\mathcal{B}}_n$. This forms the content of the next section.

2.2 Detecting membership of $\widetilde{\mathcal{B}}_{n,r}$ [$\widetilde{\mathcal{B}}_n$] in $\mathcal{R}_{n,r}$ [$\mathcal{R}_n$]

In this subsection we present an algorithm for detecting when an arbitrary finite automata is synchronizing (see Definition 2.1.1) and which, moreover, returns the minimal synchronizing level of the automata. As a transducer is synchronizing if the underlying automata of the transducer is synchronizing, we thus are also able to detect when an arbitrary finite transducer is synchronizing. Since elements of $\mathcal{R}_{n,r}$ and $\mathcal{R}_n$ are precisely those homeomorphisms of $\mathfrak{C}_{n,r}$ and $\mathfrak{C}_n$, respectively, which can be represented by finite initial transducers, we therefore have a way of determining when an arbitrary element of $\mathcal{R}_n$ ($\mathcal{R}_{n,r}$) is an element of $\widetilde{\mathcal{B}}_{n,r}$ ($\widetilde{\mathcal{B}}_n$). By applying this algorithm to the inverse of a given transducer, we may also detect when an arbitrary element of $\mathcal{R}_n$ ($\mathcal{R}_{n,r}$) is an element of $\mathcal{B}_{n,r}$ ($\mathcal{B}_n$). The method we present below is one of several possible that may be used to detect when a transducer is synchronizing. For instance one can use a pumping lemma like argument since there are finitely many states. We conclude the section by considering some examples and non-examples and by presenting some other checks for detecting whether an element of $\mathcal{R}_n$ is synchronizing or not.

Observe that the restriction to transducers with finitely many states is not really a loss of generality. This is because by Remark 2.1.7 a synchronizing transducer has only finitely many states, thus a transducer with infinitely many states cannot be synchronizing.

We now state the algorithm for checking if a finite automata is synchronizing. First we require the following construction.

Construction 2.2.1 (Collapsing procedure)**.** Let $A = \langle X, Q_A, \pi_A \rangle$ be a finite automaton. For each state $q \in Q_A$ let $[q]$ be the set of states $p \in Q_A$ such that the functions $\pi_A(\centerdot, p) : X \to Q_A$ and $\pi_A(\centerdot, q) : X \to Q_A$ are equal. Let $Q_{A_1} := \{[q] \mid q \in Q_A\}$ and observe that Q_{A_1} is a partition of Q_A. Form a new automaton $A_1 = \langle X, Q_{A_1}, \pi_{A_1} \rangle$ where, for $i \in X$ and $[q] \in Q_{A_1}$, we set $\pi_{A_1}(i, [q]) = [\pi_{A_1}(i, q)]$.

Remark 2.2.2. Let A be an automaton and let A_1 be the automaton resulting from applying the collapsing procedure to A as above. If $|A_1| = |A|$, we must have $A_1 = A$. This is because, for a state q of A, the state $[q]$ of A_1 is the set $\{q\}$.

Let $A = \langle X, Q_A, \pi_A \rangle$ be an automaton. Form a sequence $(A_i)_{i \in \mathbb{N}}$ of automata where $A_0 = A$ and such that, for $j \in \mathbb{N}_1$, $A_j = \langle X, Q_{A_j}, \pi_{A_j} \rangle$ is the result of applying the collapsing procedure to the automaton A_{j-1}. The set of states Q_{A_j} of A_j is a partition of $Q_{A_{j-1}}$ the set of states of A_{j-1}. Since Q_{A_1} is a partition of Q_{A_0}, then, by induction, the set Q_{A_j} of states of A_j corresponds to a partition $\mathcal{P}(Q_{A_j})$ of Q_A. Thus for $j \in \mathbb{N}_1$, we identify the states of A_j with the elements of this partition so that states of Q_{A_j} correspond to subsets of Q_A. For $q \in Q_A$ and $j \in \mathbb{N}$ we fix the notation $[q]_j$ for the state of Q_{A_j} containing q; if $j = 0$, set $[q]_0 := q$. By definition of the collapsing procedure, for $j \in \mathbb{N}$, and distinct states $[p]_j$ and $[q]_j$ of A_j, $[p]_{j+1} = [q]_{j+1}$ if and only if the functions $\pi_{A_j}(\centerdot, [p]_j)$ and $\pi_{A_j}(\centerdot, [q]_j)$ are equal. In particular we have the following claim:

Claim 2.2.3. *For* $x \in X$ *and* $q \in Q_A$, $\pi_{A_j}(x, [q]_j) = [\pi_A(x, q)]_j$.

Proof. We proceed by induction. By construction the claim holds for A_1. Let $k \in \mathbb{N}_1$ and assume that the claim holds for A_i for all $1 \leqslant i < k$. Let $x \in X$ and $[q]_k$ be a state of A_k. Observe that $[q]_k = \{p \in Q_A \mid \pi_{A_{k-1}}(y, [p]_{k-1}) = \pi_{A_{k-1}}(y, [q]_{k-1}) \text{ for all } y \in X\}$. However by the inductive assumption we have: $[q]_k = \{p \in Q_A \mid [\pi_A(y, p)]_{k-1} = [\pi_A(y, q)]_{k-1}\}$. Thus for any $p \in [q]_k$ and any $y \in X$, we have $[\pi_A(y, q)]_k = [\pi_A(y, p)]_k$ since $[\pi_A(y, p)]_{k-1} = [\pi_A(y, q)]_{k-1}$. $\square$

Whenever we have an automaton A and a sequence $(A_i)_{i \in \mathbb{N}}$ of automata with $A_0 = A$ and such that each subsequent term of the sequence is obtained from the previous one by applying the collapsing procedure, we shall identify, as above, the set Q_{A_j} of states of A_j with a partition of Q_A and elements of Q_{A_j} with subsets of Q_A. Further observe that if $i, j \in \mathbb{N}$ and $i < j$, then $|A_i| \geqslant |A_j|$. Thus the sequence $(A_i)_{i \in \mathbb{N}}$ is eventually constant.

We have the following result.

Lemma 2.2.4. *Let* $A = \langle X, Q_A, \pi_A \rangle$ *be an automaton and* $(A_i)_{i \in \mathbb{N}}$ *be the sequence such that* $A_0 = A$ *and each subsequent term is the automaton resulting from applying the collapsing procedure to the previous one. Let* $p, q \in Q_A$, *then* $[p]_i = [q]_i$ *in* A_i *for some* $i \in \mathbb{N}$ *if and only if for all words* $\Gamma \in X^i$ $\pi_A(\Gamma, p) = \pi_A(\Gamma, q)$.

Proof. We proceed by induction on i. Let $i = 0$ and p, q be states of A such that $[p]_0 = [q]_0$. Since $A_0 = A$, we have $p = q$. Moreover, for any $s, t \in Q_A$ such that $\pi_A(\epsilon, s) = \pi_A(\epsilon, t)$ then since $s = \pi_A(\epsilon, s)$ and $t = \pi_A(\epsilon, t)$, we conclude that $s = t$. This establishes the base case.

Assume that for $k \in \mathbb{N}_1$ and for all $i < k$, $i \in N$, the statement of the lemma holds.

Let $p, q \in Q_A$ be such that $[p]_k = [q]_k$. Let $\Gamma \in X^{k-1}$ and $x \in X$ be arbitrary. Since $[p]_k = [q]_k$, by construction of the Collapsing procedure it follows that for all $y \in X$, $[\pi_A(y, p)]_{k-1} = [\pi_A(y, q)]_{k-1}$. Thus if $p' := \pi_A(x, p)$ and $q' := \pi_A(x, q)$, then we must have, $[p']_{k-1} = [q']_{k-1}$. Therefore by the inductive assumption we have, $\pi_A(\Gamma, p') = \pi_A(\Gamma, q')$, from this it follows that $\pi_A(x\Gamma, p) = \pi_A(x\Gamma, q)$. Since $x \in X$ and $\Gamma \in X^k$ were arbitrary, we conclude that the functions $\pi_A(\centerdot, p) : X^k \to X^k$ and $\pi_A(\centerdot, q) : X^k \to X^k$ are equal.

Now let $p, q \in Q_A$ be such that the functions $\pi_A(\centerdot, p) : X^k \to X^k$ and $\pi_A(\centerdot, q) : X^k \to X^k$ are equal. Let $x \in X^k$ be arbitrary, and $p' = \pi_A(x, p)$ and $q' = \pi_A(x, q)$. Observe that the functions $\pi_A(\centerdot, p') : X^{k-1} \to X^{k-1}$ and $\pi_A(\centerdot, q') : X^{k-1} \to X^{k-1}$ are equal since the functions $\pi_A(\centerdot, p) : X^k \to X^k$ and $\pi_A(\centerdot, q) : xX^{k-1} \to xX^{k-1}$ are equal. Thus by the inductive assumption for $k-1$ we have that $[p']_{k-1} = [q']_{k-1}$. Therefore as $x \in X$ was arbitrary, we have that for all $y \in X$, $\pi_{A_{k-1}}(y, [p]_{k-1}) = \pi_{A_{k-1}}(y, [q]_{k-1})$, and so $[p]_k = [q]_k$ as required.

$\square$

We may restate the lemma above as follows:

Lemma 2.2.5. *Let $A = \langle X, Q_A, \pi_A \rangle$ be an automaton and $(A_i)_{i \in \mathbb{N}}$ be the sequence such that $A_0 = A$ and each subsequent term is the automaton resulting from applying the collapsing procedure to the previous one. Let $p, q \in Q_A$, then $[p]_i \neq [q]_i$ in A_i for some $i \in \mathbb{N}$ if and only if there is a word $\delta \in X^i$ such that $\pi_A(\delta, p) \neq \pi_A(\delta, q)$.*

As a consequence of this lemma we have the following theorem characterising when a finite automaton is synchronizing.

Theorem 2.2.6. *Let $A = \langle X, Q_A, \pi_A \rangle$ be an automaton. Form the sequence $(A_i)_{i \in \mathbb{N}}$ where $A_0 = A$ and each subsequent term of the sequence is the result of applying the collapsing procedure to the preceding term. Let k be minimal such that $|A_k| = |A_{k+1}|$, then A is synchronizing if and only if A_k consists only of a single state. Moreover, if k is minimal such that $|A_k| = 1$, then k is the minimal synchronizing level of A.*

Proof. Observe that for all $l \in \mathbb{N}_k$ we have $A_l = A_k$ by Remark 2.2.2. Thus if $|A_k| \neq 1$, then $|A_l| \neq 1$ for any $l \in N_k$. Therefore for any $j \in N_{k-1}$, there is a pair of states $q, p \in Q_A$ such that $[q]_{j+1} \neq [p]_{j+1}$ and so, by Lemma 2.2.5, for any $j \in N_{k-1}$ there is a word $\delta \in X^{j+1}$ and states $p, q \in Q_A$ such that $\pi_A(\delta, q) \neq \pi_j(\delta, p)$. From this we conclude that A is not synchronizing.

Now suppose that $|A_k| = 1$ then by Lemma 2.2.4 we have that for any pair $p, q \in Q_A$ and any word $\Gamma \in X^k$, $\pi_A(\Gamma, p) = \pi_A(\Gamma, q)$ and so A is synchronizing. Moreover since k is minimal such that $|A_k| = 1$, Lemma 2.2.5 guarantees that it is the minimal synchronizing level of A. □

An easy consequence of the above theorem is the following:

Theorem 2.2.7. *Let $A = \langle X, Q_A, \pi_A \rangle$ be an automaton of size m. If A is synchronizing, then the minimal synchronizing level of A is at most $m - 1$*

Proof. Let $(A_i)_{i \in \mathbb{N}}$ be the sequence such that $A_0 = A$ and A_{i+1} is obtained by applying the Collapsing procedure to A_i. Notice that after each application of the collapsing procedure, the new resulting automaton, if it has size bigger than 1, must have strictly fewer states than the previous one. Thus it requires at most $m - 1$ applications of the collapsing procedure for the resulting automaton A_{m-1} to be the single-state automaton. The result now follows by applying Theorem 2.2.6. □

Remark 2.2.8. We shall later give examples of automata with m states and minimal synchronizing level $m - 1$ for each $m \in \mathbb{N}_1$.

The results above demonstrate that it is possible to decide if a given finite automaton is synchronizing. We now relate these results to statements about transducers over $\mathfrak{C}_{n,r}$ and $\mathfrak{C}_n$.

Theorem 2.2.9. *Let T_{q_0} be an initial transducer over $\mathfrak{C}_n$ [$\mathfrak{C}_{n,r}$]. Then T_{q_0} is synchronizing if and only if $\mathcal{A}(T_{q_0})$ [$\mathcal{A}(T_{q_0}) \backslash \{q_0\}$] is synchronizing. Moreover if k is the minimal synchronizing level of $\mathcal{A}(T_{q_0})$ [$\mathcal{A}(T_{q_0}) \backslash \{q_0\}$] then the minimal synchronizing level of T is k [$k + 1$].*

Proof. The proof is a consequence of Definitions 2.1.14 and 2.1.18 and Theorem 2.2.6 above. □

We consider some examples below.

Example 2.2.10. Consider the transducer T below:

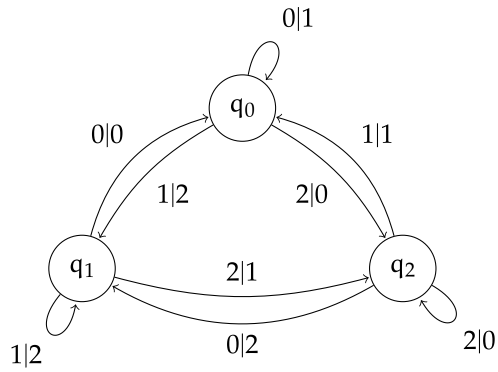


Figure 2.2: A finite transducer over $\mathfrak{C}_n$

Its underlying automaton $\mathcal{A}(T)$ is as follows:

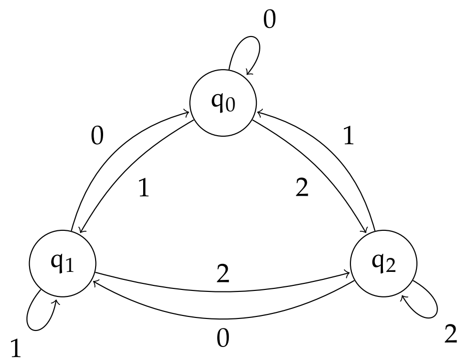


Figure 2.3: The underlying automaton of the transducer T

After one iteration of the collapsing procedure, we find that the states q_0 and q_1 are in the same equivalence class and q_2 is in a class on its own. The resulting automaton T_1 is shown below:

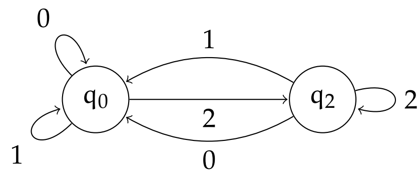


Figure 2.4: Resulting automaton after applying one step of collapsing procedure

After the second iteration of the procedure the resulting automaton is as follows:

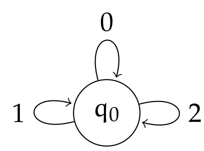


Figure 2.5: Resulting transducer after 2^{nd} application of Collapsing procedure

We can likewise perform the same process on $\mathcal{A}(A^{-1})$, which is given below:

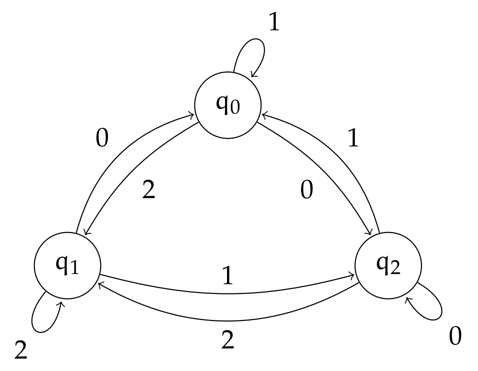


Figure 2.6: The underlying automaton of T^{-1}

This automaton can also be collapsed to a single state automaton in 2 steps. One can check that A is bi-synchronizing at level 2 and the synchronizing map is given by:

$$f : \begin{cases} 00 \mapsto q_0 & 10 \mapsto q_0 & 20 \mapsto q_1 \\ 01 \mapsto q_1 & 11 \mapsto q_1 & 21 \mapsto q_0 \\ 02 \mapsto q_2 & 12 \mapsto q_2 & 22 \mapsto q_2 \end{cases} \tag{2.1}$$

Example 2.2.11. We now illustrate a non-example. Let B_{p_0} be the following transducer over $\mathfrak{C}_{3,2}$:

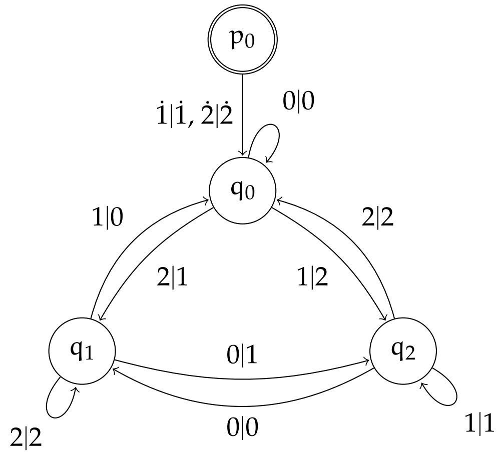


Figure 2.7: A non-example

This transducer is not synchronizing at any level. This is because for different choices of initial states, after processing a finite string of zeroes the new active state is either q_0, q_1 or q_2. Therefore we expect that after repeated applications of the collapsing procedure, all automata in the resulting sequence have size strictly greater than 1. The automaton $\mathcal{A}(B_{q_0})\backslash\{q_0\}$ is depicted below:

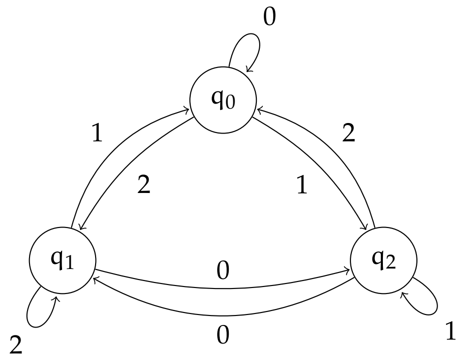


Figure 2.8: The automaton $\mathcal{A}(B_{q_0})\backslash\{q_0\}$

Observe that the automaton $\mathcal{A}(B_{p_0})\backslash\{p_0\}$ is invariant under the Collapsing procedure, since for any two distinct states q_i, q_j, $i, j \in \{0, 1, 2\}$, the functions $\pi_B(\centerdot, q_i) : X_3 \to Q_{\mathcal{A}(B_{q_0})\backslash\{q_0\}}$ and $\pi_B(\centerdot, q_j) : X_3 \to Q_{\mathcal{A}(B_{q_0})\backslash\{q_0\}}$ are distinct. Therefore B_{p_0} is not synchronizing by Theorem 2.2.6.

2.2.1 Other checks for detecting whether a transducer is bi-synchronizing

We present below lemmas which give other ways of characterising the synchronizing property. As various proofs later on demonstrate, these are, in certain situations, more helpful ways of thinking about the synchronizing property. Once more we state the lemma first for arbitrary automata and deduce application to transducers over $\mathfrak{C}_n$ or $\mathfrak{C}_{n,r}$ from these.

We need some more notions about words first.

Definition 2.2.12. Let X be a finite alphabet, and $\nu, w \in X^*$, then ν is called a *rotation* of w if there are words $u_1, u_2 \in X^*$ such that $\nu = u_1u_2$ and $w = u_2u_1$. If $\nu = w$ then we call ν a *trivial rotation* of w otherwise we call ν a *non-trivial rotation* of w. Two words $u_1, u_2 \in X^*$ are said to *commute* if $u_1u_2 = u_2u_1$. For $\nu, \mu \in X^*$, ν is called a *power of* μ if there is some $j \in \mathbb{N}_1$ such that $\mu = \nu^j$. A word $\rho \in X^*$ is called a *prime word* if there are no $\delta \in X^*$ and $j \in \mathbb{N}_1$ such that $\rho = \delta^j$.

The following is a well known result and can be found in most combinatorics text books (see for instance [50]).

Theorem 2.2.13. *Let X be a finite alphabet and $u_1, u_2 \in X^*$. Then u_1 and u_2 commute if and only if there is a word $\nu \in X^*$ such that u_1 and u_2 are both powers of ν.*

As an easy consequence we have the following corollary:

Theorem 2.2.14. *Let X be a finite alphabet. A word w is prime if and only if it is not equal to a non-trivial rotation of itself.*

Let X be a finite alphabet. The relation, $\sim$ on X^* given by $u \sim \nu$ if and only if u is a rotation of ν is an equivalence relation. We denote by $X/\sim$ the set of equivalence classes of X under $\sim$. For a word $w \in X^*$ we denote by $[w]_\sim$ the equivalence class of w under $\sim$.

We have the following lemmas which are in certain situations easier checks for determining whether an automaton is synchronizing.

Lemma 2.2.15. *Let $A = \langle X, Q_A, \pi_A \rangle$ be an automaton. Then A is not synchronizing if and only if there is a word $w \in X^+$ of length at most $|Q_A|(|Q_A| - 1) + 1$ and distinct states $p, q \in Q_A$ such that $\pi_A(w, p) = p$ and $\pi_A(w, q) = q$.*

Proof. The reverse implication follows since, for p, q and w satisfying the hypothesis of the lemma, and for any $i \in \mathbb{N}$ we have, $\pi_A(w^i, p) = p$ and $\pi_A(w^i, q) = q$. Now, if A is synchronizing at some level $j \in \mathbb{N}$ then it is synchronizing for at level k for all $k \in \mathbb{N}_j$. Therefore there is some i such that $\pi_A(w^i, p) = \pi_A(w^i, q)$ which is a contradiction, however, this contradicts the assumption that $p \neq q$ and so A is not synchronizing.

For the forward implication, assume that A is not synchronizing. This means that for any $i \in \mathbb{N}$ there is a word $w' \in X^i$ and distinct states states p' and q' such that $\pi_A(w', p') \neq \pi_A(w', q')$. Let $i \geqslant |Q_A|(|Q_A|-1)+1$. Notice we may assume that $|Q_A| > 1$ since an automaton with one state is synchronizing, thus $i > 1$. Let w, p, q be such that $w \in X^i$, p and q are distinct states of A, and $\pi_A(w, p) \neq \pi_A(w, q)$. Let $w_a \in X$, $1 \leqslant a \leqslant i$, be such that $w = w_1 w_2 \ldots w_i$, and let $p_a = \pi_A(w_1 \ldots w_a, p)$ and $q_a = \pi_A(w_1 \ldots w_a, q)$ for $1 \leqslant a \leqslant i$. Observe that $\pi_A(w, p) = p_i \neq q_i = \pi_A(w, q)$. If $p_a = q_a$ for some $1 \leqslant a \leqslant i$, then for all $a \leqslant b \leqslant i$ $q_b = p_b$ by definition of the transition function. In particular, $q_i = p_i$ which is a contradiction. Therefore $p_a \neq q_a$ for all $1 \leqslant a \leqslant i$. Since there are only at most $|Q_A|(|Q_A|-1)$ ordered pairs of elements of Q_A, there are natural numbers a and b such that $1 \leqslant a < b \leqslant i$ and $(p_a, q_a) = (p_b, q_b)$. Let $\overline{w} = w_a w_{a+1} \ldots w_b$. We thus have, $\pi_A(\overline{w}, p_a) = p_a$ and $\pi_A(\overline{w}, q_a) = q_a$, moreover $|w_a| = b - a + 1 \leqslant i$. □

A corollary of Lemma 2.2.15 is the following:

Corollary 2.2.16. *Let $A = \langle X, Q_A, \pi_A \rangle$ be an automaton. If A is synchronizing, then its minimal synchronizing level is at most $|Q_A|(|Q_A|-1)+1$.*

Proof. We may assume that $|A| > 1$ as the statement is satisfied trivially when $|A| = 1$. If A is not synchronizing at level $i := |Q_A|(|Q_A|-1)+1$, we may find a word $w \in X^i$ and distinct states p, q for which $\pi_A(w, q) \neq \pi_A(w, p)$. We then imitate the proof of Lemma 2.2.15 to find states p', q' and a word $w' \in X^+$ of length at most i such that $\pi_A(w', p') = p'$ and $\pi_A(w', q') = q$ demonstrating that A is not synchronizing. Thus if A is synchronizing, it must have minimal synchronizing level at most i. □

Remark 2.2.17. Notice that this bound is not optimal as indicated by Theorem 2.2.7.

As a further corollary of Lemma 2.2.15 we have the following:

Lemma 2.2.18. *Let $A = \langle X, Q_A, \pi_A \rangle$ be an automaton. Fix a unique representative in X^* for each equivalence class of $X \backslash \sim$ and let $\mathcal{W}$ be this set of representatives. The automaton A is synchronizing if and only if there is some $j \in \mathbb{N}$ such that the set $\mathcal{W}_{\geqslant j}$ of all elements of $\mathcal{W}$ of length greater than j consists of synchronizing words (of A).*

Proof. The forward implication follows straightforwardly, since if A is synchronizing at level j the set of all words of length greater than to j consists entirely of synchronizing words.

For the reverse implication, assume that there is a $j \in \mathbb{N}$ such that $\mathcal{W}_{\geqslant j}$ consists only of synchronizing words. Assume for a contradiction that A is not synchronizing. By Lemma 2.2.15 there is a word $w \in X^+$ and distinct states p, q such that $\pi_A(w, p) = p$ and $\pi_A(w, q) = q$. By raising w to powers if necessary we may assume that $|w| = k \geqslant j$. Let $v \in \mathcal{W}_{\geqslant j}$ be the unique choice of representative for the class $[w]_\sim$. Since $\mathcal{W}_{\geqslant j}$ consists only of synchronizing words, there is a unique state $s \in Q_A$ such that $\pi_A(v, \cdot) : Q_A \to Q_A$ takes only the value s. In particular $\pi_A(v, s) = s$. Let $w_a \in X$, $1 \leqslant a \leqslant k$, be such that $w = w_1 w_2 \ldots w_k$. If $v = w$, then $p = q = s$ and we obtain the desired contradiction. Thus we assume that $v \neq w$, therefore $v = w_i w_{i+1} \ldots w_k w_1 \ldots w_{i-1}$ for some $2 \leqslant i \leqslant k$. Let $p_a = \pi_A(w_1 \ldots w_a, p)$, and $q_a = \pi_A(w_1 \ldots w_a, q)$. Notice that $p_k = p$, $q_k = q$. Since p and q are distinct then, as in the proof of Lemma 2.2.15, we must have $p_a \neq q_a$ are distinct for all $1 \leqslant a \leqslant k$. Therefore one of p_{i-1} or q_{i-1} is not equal to s. We assume, relabelling if necessary, that $q_{i-1} \neq s$. Now observe that $\pi_A(w_i \ldots w_k w_1 \ldots w_{i-1}, q_{i-1}) = q_{i-1} \neq s$. This is because as $q_{i-1} = \pi_A(w_1 \ldots w_{i-1}, q)$ it follows that $\pi_A(w_i \ldots, w_k, q_{i-1}) = q_k = q$, thus $\pi_A(w_i \ldots w_k w_1 \ldots w_{i-1}, q_{i-1}) = q_{i-1}$. Therefore we conclude that $\pi_A(v, q_i) = q_i \neq s$ which is a contradiction since $v \in \mathcal{W}_{\geqslant j}$ is a synchronizing word. □

Using Corollary 2.2.16 we can make Lemma 2.2.18 qualitative as follows:

Lemma 2.2.19. *Let $A = \langle X, Q_A, \pi_A \rangle$ and let $k = |Q_A|(|Q_A|-1)+1$. Fix a unique representative in X^* for each equivalence class in $X/\sim$ and let $\mathcal{W}$ be this set of representatives. A is synchronizing if and only if there is a $1 \leqslant j \leqslant k$ such that for any $j \leqslant a \leqslant 2k$, the set $\mathcal{W}_a$ of all elements of $\mathcal{W}$ of length equal to a, consists of synchronizing words.*

Proof. The forward implication proceeds exactly as in the proof of Lemma 2.2.18 and applying Corollary 2.2.16 to deduce that A is synchronizing at some level $j \leqslant k$.

For the reverse implication we again imitate the proof of the reverse implication of Lemma 2.2.18. For suppose there is some $1 \leqslant j \leqslant k$ such that for all $j \leqslant a \leqslant 2k$ the set $\mathcal{W}_a$ consists only of synchronizing words. If A is not synchronizing then we may find a word $w \in X_n^*$ of length at most k and distinct states p, q such that $\pi_A(w, p) = p$ and $\pi_A(w, q) = q$. Now since $1 \leqslant |w| \leqslant k$, there is an $m \in \mathbb{N}_1$ such that $k \leqslant m|w| \leqslant 2k$. Thus we may replace w with w^m and assume that $j \leqslant k \leqslant |w|$. Therefore, as in the proof of Lemma 2.2.18, there is a word $v \in \mathcal{W}_{|w|}$ and distinct states p' and q' such that $\pi_A(v, p') = p'$ and $\pi_A(v, q') = q'$ contradicting the assumption that $\mathcal{W}_{|w|}$ consists entirely of synchronizing words. Hence we conclude that A is synchronizing. □

We now deduce from the above lemmas implications for transducers over $\mathfrak{C}_n$ and $\mathfrak{C}_{n,r}$.

Lemma 2.2.20. *Let T be a finite transducer over X_n. Let $k = |Q_T|(|Q_T| - 1) + 1$ and fix a unique representative in X^* for each equivalence class in $X/\sim$. Let $\mathcal{W}$ be this set of representatives. The transducer T_{q_0} is synchronizing if and only if there is a natural number j such that $1 \leqslant j \leqslant k$ and for any natural number a, $j \leqslant a \leqslant 2k$, the set $\mathcal{W}_a$ of all elements of $\mathcal{W}$ of length equal to a consists of synchronizing words for T_{q_0} if and only if there is some $l \in \mathbb{N}$ such that the set $\mathcal{W}_{\geqslant l}$ of all elements of $\mathcal{W}$ of length greater than l consists of synchronizing words.*

The proof follows by applying the previous results to the underlying automaton of a transducer. For T_{q_0} a finite initial transducer over $\mathfrak{C}_{n,r}$, the results apply to the automaton $\mathcal{A}(T_{q_0})\backslash\{q_0\}$.

In the next section we introduce certain important subgroups of $\mathcal{B}_{n,r}$ and $\mathcal{B}_n$ which are characterised by imposing restrictions on the core of the bi-synchronizing transducers inducing these homeomorphisms.

2.3 Higman-Thompson groups $G_{n,r}$ and their automorphism groups Aut$(G_{n,r})$

In this section we give several definitions of the Higman-Thompson groups $G_{n,r}$ for $n \in \mathbb{N}_2$ and $r \in \mathbb{N}_1$ such that $r < n$. Let us call a pair (n, r) such that $n \in N_2$ and $r \in \mathbb{N}_1$ $r < n$ an *allowable pair*. For a fixed allowable pair (n, r), the group $G_{n,r}$ is a subgroup of $H(\mathfrak{C}_{n,r})$ the group of homeomorphisms of Cantor space $\mathfrak{C}_{n,r}$. As we saw in Section 1.4 there are many different ways of viewing the Cantor space $\mathfrak{C}_{n,r}$, this leads to the different characterisations of the groups $G_{n,r}$. Our final characterisation shall be as a subgroup of $\mathcal{B}_{n,r}$ consisting of those homeomorphisms of $\mathfrak{C}_{n,r}$ which may be represented by bi-synchronizing transducers with a certain restriction on their cores. We then show that the group $\mathcal{B}_{n,r}$ is precisely the normaliser of the group $G_{n,r}$ in the rational group. By appealing to results in the author's article [10], some of which also appear in Yonah Maissel's thesis ([39]), we will deduce that $\mathcal{B}_{n,r}$ is actually isomorphic to $\mathrm{Aut}(G_{n,r})$. In the latter sections we investigate the quotient of $\mathcal{B}_{n,r}$ by $G_{n,r}$ (thought of now as a normal subgroup of $\mathcal{B}_{n,r}$), and highlight an important subgroup of this quotient. We begin by briefly recapping the history of the groups $G_{n,r}$ and why they are important in group theory.

The Thompson groups F, T and V were introduced in 1965 by Richard Thompson [52] in connection to questions in logic. He subsequently demonstrated that the groups V and T are finitely presented infinite simple groups, giving the first examples of groups of this type. The group V was later identified with the automorphism group of an algebra $V_{2,1}$. Higman generalised this construction creating algebras $V_{n,r}$ for $n, r \in \mathbb{N}$ and identified the groups $G_{n,r}$ with the automorphism group of these algebras. (The group $G_{2,1}$ is equal to Thompson's group V.) Higman then showed that for n even the group $G_{n,r}$ is simple, and when n is odd the commutator subgroup is simple and has index 2. The groups $G_{n,r}$ and $G_{m,r'}$ are isomorphic if and only if $n = m$ and $\gcd(n-1, r) = \gcd(n-1, r')$. Higman demonstrated in [34] that $G_{n,r} \cong G_{m,r'}$ implies that $n = m$ and $\gcd(n-1, r) = \gcd(n-1, r')$, the converse was shown by Pardo in [45]. This gives rise to infinitely many examples of finitely presented infinite, simple groups. Moreover, we can restrict ourselves to the groups $G_{n,r}$ where (n, r) is an allowable pair. There are two known families of finitely presented infinite simple groups: those generalising the Thompson groups and those arising like the Burger-Moses groups [18]. The recent paper [8] gives a infinite presentation of the group V highlighting some similarities to the alternating group A_n, showing, perhaps, why the group V was the first discovered finitely presented, infinite simple group.

We now present the different ways we will be thinking about the groups $G_{n,r}$. Our exposition here will be based on [10] and [20] however there are many other sources which contain a good introduction to these groups ([2], [49] and so on). Though we mentioned above connections to certain algebras we shall not make use of these connections in this thesis.

For the rest of this section, unless otherwise indicated, (n, r) shall always be allowable.

2.3.1 Higman-Thompson groups $G_{n,r}$ as prefix exchange maps on $\mathfrak{C}_{n,r}$

Our first description of $G_{n,r}$ shall be as homeomorphisms of $\mathfrak{C}_{n,r}$ which are given by prefix exchange maps.

Notation 2.3.1. Let $\nu, \mu \in X^+_{n,r}$, then we shall denote by $g_{\nu,\mu}$ the map from U_ν to U_μ given by $\nu\delta \mapsto \mu\delta$ for $\delta \in \mathfrak{C}_n$. Let $\overline{\mathbf{u}} = \{u_1, \ldots, u_m\}$ and $\overline{\mathbf{v}} = \{v_1, \ldots, v_m\}$ be two complete antichains of $X^*_{n,r}$ of equal length. We shall denote by $g_{\overline{\mathbf{u}},\overline{\mathbf{v}}}$ the map $g : \mathfrak{C}_{n,r} \to \mathfrak{C}_{n,r}$ such that for $1 \leqslant i \leqslant m$, $g{\restriction}_{U_{u_i}} = g_{u_i,v_i}$.

The following result is straight-forward:

Proposition 2.3.2. *Let $\overline{\mathbf{u}} = \{u_1, \ldots, u_m\}$ and $\overline{\mathbf{v}} = \{v_1, \ldots, v_m\}$ be two complete antichains of $X^*_{n,r}$ of equal length. The map $g_{\overline{\mathbf{u}},\overline{\mathbf{v}}}$ is an element of $H(\mathfrak{C}_{n,r})$ with inverse $g_{\overline{\mathbf{v}},\overline{\mathbf{u}}}$.*

Proof. That the map $g_{\overline{\mathbf{u}},\overline{\mathbf{v}}}$ is injective and surjective follows from the fact that $\overline{\mathbf{u}}$ and $\overline{\mathbf{v}}$ are antichains of equal length. Since for every element $\delta \in \mathfrak{C}_{n,r}$ there is precisely one u_i in $\overline{\mathbf{u}}$ and $v_j \in \overline{\mathbf{v}}$ such that $u_i \leqslant \delta$ and $v_j \leqslant \delta$. In fact $g^{-1}_{\overline{\mathbf{u}},\overline{\mathbf{v}}} = g_{\overline{\mathbf{v}},\overline{\mathbf{u}}}$.

Let $\delta \in \mathfrak{C}_{n,r}$ and suppose $\delta = u_i\gamma$ for some $i \in \mathbb{N}$, $1 \leqslant i \leqslant m$, and $\gamma \in \mathfrak{C}_n$. Then $(\delta)g_{\overline{\mathbf{u}},\overline{\mathbf{v}}} = v_i\gamma$. Let U be an open neighbourhood of $v_i\gamma$. Let $\overline{\gamma}$ be a long enough prefix of γ such that $U_{v_i\overline{\gamma}} \subset U$. Then the set $U_{u_i\overline{\gamma}}$ is an open neighbourhood of δ, moreover $(U_{u_i\overline{\gamma}})g_{\overline{\mathbf{u}},\overline{\mathbf{v}}} \subseteq U_{v_i} \subseteq U$. Thus $g_{\overline{\mathbf{u}},\overline{\mathbf{v}}}$ is continuous. Since, as remarked in the previous paragraph, $g^{-1}_{\overline{\mathbf{u}},\overline{\mathbf{v}}} = g_{\overline{\mathbf{v}},\overline{\mathbf{u}}}$, we therefore have that g is a homeomorphism. $\square$

Notation 2.3.3. Set

$$G_{n,r} := \{g_{\overline{\mathbf{u}},\overline{\mathbf{v}}} \mid \overline{\mathbf{u}}, \overline{\mathbf{v}} \text{ are antichains and } |\overline{\mathbf{u}}| = |\overline{\mathbf{v}}|\}.$$

We shall show that $G_{n,r}$ is a subgroup of $H(\mathfrak{C}_{n,r})$ however to do so we require the following definitions and results.

Definition 2.3.4. Let $\overline{\mathbf{u}} = \{u_1, \ldots, u_m\}$ and $\overline{\mathbf{v}} = \{v_1, \ldots, v_m\}$ be two antichains of $X^*_{n,r}$ of the same size. Two k-fold expansions, $\overline{\mathbf{u}}'$ and $\overline{\mathbf{v}}'$, of $\overline{\mathbf{u}}$ and $\overline{\mathbf{v}}$ are called *compatible expansions of* $\overline{\mathbf{u}}$ *and* $\overline{\mathbf{v}}$ if there is a sequence $(\overline{\mathbf{u}}, \overline{\mathbf{v}}) := (\overline{\mathbf{u}}_0, \overline{\mathbf{v}}_0), \ldots, (\overline{\mathbf{u}}_k, \overline{\mathbf{v}}_k) := (\overline{\mathbf{u}}', \overline{\mathbf{v}}')$ such that:

(i) for $1 \leqslant i \leqslant k$ $\overline{\mathbf{u}}_i$ is a single expansion of $\overline{\mathbf{u}}_{i-1}$ and $\overline{\mathbf{v}}_i$ is a single expansion of $\overline{\mathbf{v}}_{i-1}$,

(ii) if $\overline{\mathbf{v}}_{i-1} = \{\nu_1, \ldots, \nu_d\}$ and $\overline{\mathbf{u}}_{i-1} = \{\mu_1, \ldots, \mu_d\}$ then $\overline{\mathbf{v}}_i$ is a single expansion over ν_j of $\overline{\mathbf{v}}_{i-1}$ if and only if $\overline{\mathbf{u}}_i$ is a single expansion over μ_j of $\overline{\mathbf{u}}_{i-1}$.

Remark 2.3.5. Let $\overline{\mathbf{u}} = \{u_1, \ldots, u_m\}$ and $\overline{\mathbf{v}} = \{v_1, \ldots, v_m\}$ be two complete antichains of the same length. Let $\overline{\mathbf{u}}'$ and $\overline{\mathbf{v}}'$ be compatible single expansions of $\overline{\mathbf{u}}$ and $\overline{\mathbf{v}}$ respectively, then $g_{\overline{\mathbf{u}},\overline{\mathbf{v}}} = g_{\overline{\mathbf{u}'},\overline{\mathbf{v}'}}$. This follows since if, for $1 \leqslant i \leqslant m$, $\overline{\mathbf{u}}'$ is an expansion over u_i of $\overline{\mathbf{u}}$ then $\overline{\mathbf{u}}' = \{u_1, \ldots, u_{i-1}, u_i0, u_i1, \ldots, u_in-1, u_{i+1}, \ldots, u_m\}$, since $\overline{\mathbf{u}}'$ and $\overline{\mathbf{v}}'$ are compatible we also have $\overline{\mathbf{v}}' = \{v_1, \ldots, v_{i-1}, v_i0, v_i1, \ldots, v_in-1, v_{i+1}, \ldots, v_m\}$. Now for $j \in X_n$, since $g_{\overline{\mathbf{u}},\overline{\mathbf{v}}}{\restriction}_{U_{u_i}} = g_{u_i,v_i}$ we have $g_{\overline{\mathbf{u}},\overline{\mathbf{v}}}{\restriction}_{U_{u_ij}} = g_{u_ij,v_ij} = g_{\overline{\mathbf{u}}',\overline{\mathbf{v}}'}{\restriction}_{U_{u_ij}}$. Thus $g_{\overline{\mathbf{u}},\overline{\mathbf{v}}}{\restriction}_{U_{u_i}} = g_{\overline{\mathbf{u}},\overline{\mathbf{v}}}{\restriction}_{\cup_{j\in X_n} U_{u_ij}} = g_{\overline{\mathbf{u}}',\overline{\mathbf{v}}'}{\restriction}_{U_{u_i}}$. Since $g_{\overline{\mathbf{u}},\overline{\mathbf{v}}}{\restriction}_{U_{u_l}} = g_{\overline{\mathbf{u}}',\overline{\mathbf{v}}'}{\restriction}_{U_{u_l}}$ for $l \in \{1, \ldots, m\}\backslash\{i\}$, we therefore conclude that $g_{\overline{\mathbf{u}},\overline{\mathbf{v}}} =_{\overline{\mathbf{u}}',\overline{\mathbf{v}}'}$.

The following lemma follows by a simple induction argument:

Lemma 2.3.6. *Let $\overline{\mathbf{u}} = \{u_1, \ldots, u_m\}$ and $\overline{\mathbf{v}} = \{v_1, \ldots, v_m\}$ be two complete antichains of the same length. Let $\overline{\mathbf{u}}'$ and $\overline{\mathbf{v}}'$ be compatible expansions of $\overline{\mathbf{u}}$ and $\overline{\mathbf{v}}$ respectively. Then $g_{\overline{\mathbf{u}},\overline{\mathbf{v}}} = g_{\overline{\mathbf{u}'},\overline{\mathbf{v}'}}$.*

We require a few additional facts about antichains.

Lemma 2.3.7. *Let $\overline{\mathbf{u}}$ be a complete antichain of X_n^*. Then there is a complete antichain $\overline{\mathbf{u}}'$, an expansion of the complete antichain $\{\epsilon\}$, such that $\overline{\mathbf{u}} = \overline{\mathbf{u}}'$ as unordered sets.*

Proof. We proceed by induction on the length of $\overline{\mathbf{u}}$. First suppose that $|\overline{\mathbf{u}}| = 1$. Observe that since $\overline{\mathbf{u}}$ is a complete antichain, then for all $i \in X_n$ i is a prefix of some element of $\overline{\mathbf{u}}$ or some element of $\overline{\mathbf{u}}$ is a prefix of i. Now suppose that $\overline{\mathbf{u}} = \{x\}$. If $x \neq \epsilon$, then there is a $j \in X_n$ such that j is not a prefix of x. Thus $x = \epsilon$ and $\overline{\mathbf{u}} = \{\epsilon\}$.

Now assume that for $m \in \mathbb{N}_1$ all complete antichains of length less than m are expansions of ϵ.

Let $\overline{\mathbf{v}}$ be a complete antichain of length m. Let $\overline{\mathbf{u}}$ be the largest complete antichain of length strictly less than m such that all elements of $\overline{\mathbf{u}}$ are prefixes of some element of $\overline{\mathbf{v}}$. We may further assume (reordering $\overline{\mathbf{u}}$ if necessary) that $\overline{\mathbf{u}}$ is an expansion of the antichain $\{\epsilon\}$.

Let $\gamma \in \overline{\mathbf{u}}\backslash\overline{\mathbf{v}}$ (such elements exists since $|\overline{\mathbf{u}}| < m$ and $\overline{\mathbf{u}}$ and $\overline{\mathbf{v}}$ are complete antichains). It must be the case that for all $a \in X_n$, γa is a prefix of some element of $\overline{\mathbf{v}}$ since $\overline{\mathbf{v}}$ is a complete antichain. Thus replace all elements of $\overline{\mathbf{u}}\backslash\overline{\mathbf{v}}$ with the elements $\gamma 0, \gamma 1, \ldots \gamma n-1$ creating a new complete antichain $\overline{\mathbf{u}}_1$ which is an expansion of $\overline{\mathbf{u}}$. If $\overline{\mathbf{u}}_1 \neq \overline{\mathbf{v}}$, then as all elements of elements of $\overline{\mathbf{u}}_1$ are prefixes of elements of $\overline{\mathbf{v}}$, then $|\overline{\mathbf{u}}_1| < |\overline{\mathbf{v}}|$, however this contradicts maximality of $\overline{\mathbf{u}}$. Therefore $\overline{\mathbf{u}}_1 = \overline{\mathbf{v}}$ as unordered sets and we are done □

As a corollary we have:

Corollary 2.3.8. *Let $\overline{\mathbf{u}}$ be a complete antichain of $X_{n,r}^*$, then there is a compete antichain $\overline{\mathbf{u}}'$ an expansion of $\{\dot{1}, \ldots, \dot{r}\}$ such that $\overline{\mathbf{u}}' = \overline{\mathbf{u}}$ as unordered sets.*

Remark 2.3.9. A consequence of the results above and Remark 1.1.22 is that the length of any complete antichain of $X_{n,r}^*$ is equal to r modulo $n-1$.

Definition 2.3.10. Let $\overline{\mathbf{u}}$ and $\overline{\mathbf{v}}$ be two antichains of $X_{n,r}^*$, then $\overline{\mathbf{u}}$ is called a *re-ordering* of $\overline{\mathbf{v}}$ if $\overline{\mathbf{u}} = \overline{\mathbf{v}}$ as unordered sets.

Lemma 2.3.11. *Let $\overline{\mathbf{v}}_1, \overline{\mathbf{v}}_2$ be complete antichains of $X_{n,r}^*$. Then there is an expansion $\overline{\mathbf{w}}$ of $\overline{\mathbf{v}}_1$, and a re-ordering $\overline{\mathbf{w}}'$ of $\overline{\mathbf{w}}$ which is an expansion of $\overline{\mathbf{v}}_2$.*

Proof. If $\overline{\mathbf{v_1}} = \overline{\mathbf{v_2}}$, we are done. Therefore we assume that $\overline{\mathbf{v_1}} \neq \overline{\mathbf{v_2}}$. Set $\overline{\mathbf{w}}_0 = \overline{\mathbf{v}}_1$ and let $\gamma \in \overline{\mathbf{w}}_0\backslash\overline{\mathbf{v}}_2$. Observe that either γ is a proper prefix of some element of $\overline{\mathbf{v}}_2$ or some element of $\overline{\mathbf{v}}_2$ is a proper prefix of γ. If the set of elements $\gamma \in \overline{\mathbf{w}}_0\backslash\overline{\mathbf{v}}_2$ such that γ is a proper prefix of some element of $\overline{\mathbf{v}}_2$ is empty, then set $\overline{\mathbf{w}} = \overline{\mathbf{w}}_0$. Otherwise, let $\gamma \in \overline{\mathbf{v}}_1\backslash\overline{\mathbf{v}}_2$ be a proper prefix of some element of $\overline{\mathbf{v}}_2$. For any $a \in X_n$, since $\overline{\mathbf{v}}_2$ is a complete antichain, γa is a prefix of some element of $\overline{\mathbf{v}}_2$. Thus replace all $\gamma \in \overline{\mathbf{w}}_0\backslash\overline{\mathbf{v}}_2$ such that γ is a proper prefix of some element of $\overline{\mathbf{v}}_2$ by $\gamma 0, \ldots, \gamma n-1$. This creates a new antichain $\overline{\mathbf{w}}_1$ which is an expansion of $\overline{\mathbf{v}}_1$.

If the set of $\gamma \in \overline{\mathbf{w}}_1\backslash\overline{\mathbf{v}}_2$ such that γ is a proper prefix of some element of $\overline{\mathbf{v}}_2$ is empty, then set $\overline{\mathbf{w}} = \overline{\mathbf{w}}_1$. Otherwise, repeat the above process with $\overline{\mathbf{w}}_1$ in place of $\overline{\mathbf{w}}_0$, to create a new antichains $\overline{\mathbf{w}}_2, \overline{\mathbf{w}}_3, \ldots$ which are an expansions of (or equal to) $\overline{\mathbf{w}}_1$. Since $\overline{\mathbf{v}}_2$ is a finite antichain, there is some $k \in \mathbb{N}_1$ minimal such that any element of $\overline{\mathbf{v}}_2$ is a prefix or equal to some element of $\overline{\mathbf{w}}_k$. Set $\overline{\mathbf{w}} = \overline{\mathbf{w}}_k$ and observe that it is an expansion of $\overline{\mathbf{v}}_1$.

We now show that there is a re-ordering $\overline{\mathbf{w}}'$ of $\overline{\mathbf{w}}$ which is an expansion of $\overline{\mathbf{v_2}}$. It suffices to show that for any element $\gamma \in \overline{\mathbf{v}}_2$, either $\gamma \in \overline{\mathbf{w}}$ or there is a complete antichain $\{u_1, \ldots, u_r\}$ of $X_{n,r}^*$ such that $\{\gamma u_1, \ldots, \gamma u_r\} \subset \overline{\mathbf{w}}$. This is because by Lemma 2.3.7, the antichain $\{\gamma u_1, \ldots, \gamma u_r\}$ is equal to a reordering of an expansion of the antichain $\{\gamma\}$.

Suppose there is an element $\gamma \in \overline{\mathbf{v}}_2\backslash\overline{\mathbf{w}}$. Let $\mu_1, \mu_2, \ldots, \mu_k$ be all elements of X_n^+ such that $\gamma\mu_a \in \overline{\mathbf{w}}$ for $1 \leqslant a \leqslant k$. Since $\overline{\mathbf{w}}$ is a complete antichain, it follows that $\{\mu_a \mid 1 \leqslant a \leqslant k\}$ is also a complete antichain. □

Remark 2.3.12. Let $\overline{\mathbf{u}}_1$, $\overline{\mathbf{u}}_2$, $\overline{\mathbf{v}}_1$, $\overline{\mathbf{v}}_2$ be complete antichains such that $|\overline{\mathbf{u}}_i| = |\overline{\mathbf{v}}_i|$ for $i \in \{1, 2\}$, then there are complete antichains $\overline{\mathbf{w}}_1$, $\overline{\mathbf{w}}_2$, $\overline{\mathbf{u}}_1'$ and $\overline{\mathbf{u}}_2'$ such that:

(i) $\overline{\mathbf{w}}_1 = \overline{\mathbf{w}}_2$ as unordered sets,

(ii) $\overline{\mathbf{w}}_1$ is an expansion of $\overline{\mathbf{v}}_1$ and $\overline{\mathbf{w}}_2$ is an expansion of $\overline{\mathbf{v}}_2$,

(iii) The antichains $\overline{\mathbf{w}}_i$ and $\overline{\mathbf{u}}_i'$, for $i \in \{1, 2\}$, are compatible expansions of $\overline{\mathbf{v}}_i$ and $\overline{\mathbf{u}}_i$ respectively.

To see this, observe that by the above lemma, there are expansions $\overline{\mathbf{w}}_i$ of $\overline{\mathbf{v}}_i$ ($i \in \{1, 2\}$) such that the $\overline{\mathbf{w}}_i$ are re-orderings of each other. Now by expanding along the $\overline{\mathbf{u}}_i$ appropriately the remark follows.

We now have the following proposition:

Proposition 2.3.13. *The set* $G_{n,r}$ *is a subgroup of* $H(\mathfrak{C}_{n,r})$.

Proof. That $G_{n,r}$ is closed under inversion and is a subset of $H(\mathfrak{C}_{n,r})$ is a consequence of Proposition 2.3.2.

Let $g_{\overline{\mathbf{u}}_1,\overline{\mathbf{v}}_1}$ and $h_{\overline{\mathbf{v}}_2,\overline{\mathbf{u}}_2}$ be two elements of $G_{n,r}$. By applying Lemma 2.3.11 we may find complete antichains $\overline{\mathbf{u}}_i'$ and $\overline{\mathbf{w}}_i$, $i \in \{1, 2\}$, such that $\overline{\mathbf{u}}_i'$ and $\overline{\mathbf{w}}_i$ are compatible expansions of $\overline{\mathbf{u}}_i$ and $\overline{\mathbf{v}}_i$, and $\overline{\mathbf{w}}_1$ is a reordering of $\overline{\mathbf{w}}_2$. By Lemma 2.3.6 we have: $g_{\overline{\mathbf{u}}_1,\overline{\mathbf{v}}_1} = g_{\overline{\mathbf{u}}_1',\overline{\mathbf{w}}_1}$ and $h_{\overline{\mathbf{v}}_2,\overline{\mathbf{u}}_2} = h_{\overline{\mathbf{w}}_2,\overline{\mathbf{u}}_2'}$. Notice that $|\overline{\mathbf{u}}_i'| = |\overline{\mathbf{w}}_i|$ for $i \in \{1, 2\}$. Let $1 \leqslant a \leqslant |\overline{\mathbf{u}}_1'|$ and let u_a and w_a be the a^{th} element of $\overline{\mathbf{u}}_1$ and $\overline{\mathbf{w}}_1$ respectively. There is a $b \in \mathbb{N}_1$ such that $b \leqslant |\overline{\mathbf{w}}_1|$ and the b^{th} element w_b of $\overline{\mathbf{w}}_2$ is equal to w_a. Let u_b be the b^{th} element of $\overline{\mathbf{u}}_2'$, then, $g_{\overline{\mathbf{u}}_1{}',\overline{\mathbf{w}}_1} \circ h_{\overline{\mathbf{w}}_2,\overline{\mathbf{u}}_2'}\restriction U_{u_a}$ is equal to $g_{u_a,w_a} \circ h_{w_b,u_b}\restriction U_{u_a}$. As a, $1 \leqslant a \leqslant |\overline{\mathbf{u}}_1'|$, was arbitrarily chosen, it follows that, $g_{\overline{\mathbf{u}}_1{}',\overline{\mathbf{w}}_1} \circ h_{\overline{\mathbf{w}}_2,\overline{\mathbf{u}}_2'} = f_{\overline{\mathbf{u}}_1',\overline{\mathbf{u}}_2'} \in G_{n,r}$. □

Therefore, the Higman-Thompson groups $G_{n,r}$ are precisely those homeomorphisms of $\mathfrak{C}_{n,r}$ which are prefix exchange maps. However, this is just one way we will be thinking about these groups. Recall that in Section 1.4 we demonstrated that Cantor space $\mathfrak{C}_{n,r}$ may be identified with the boundary of the r-rooted n-ary tree. In the next section we show that elements of the group $G_{n,r}$ can be represented by pairs of finite subforests of $\mathcal{T}_{n,r}$ with the same number of leaves, and a bijection between their leaves.

2.3.2 Higman-Thompson groups $G_{n,r}$ and their action on $\mathcal{T}_{n,r}$

In this section we demonstrate that elements of the group $G_{n,r}$ can be represented pictorially according to their action on the tree $\mathcal{T}_{n,r}$. This pictorial representation is by the so called *forest pair diagrams*. We define these first and then show how one can obtain a forest pair diagram for an element of $G_{n,r}$.

Definition 2.3.14. A *forest pair* (of $\mathcal{T}_{n,r}$) is a pair (A, B) of finite subforests of $\mathcal{T}_{n,r}$ with the same number of leaves.

Let $\mathcal{T}_{n,r}$ be the r-rooted n-ary forest labelled as in Section 1.4 and let $\overline{\mathbf{u}}$ be a complete antichain of $X_{n,r}^*$. Recall (Remark 1.2.35) that we denote by $\mathcal{T}_{n,r}^{\overline{\mathbf{u}}}$ the subforest of $\mathcal{T}_{n,r}$ with leaves the vertices in $\overline{\mathbf{u}}$.

Let $\overline{\mathbf{u}} = \{u_1, u_2, \ldots, u_k\}$ and $\overline{\mathbf{v}} = \{v_1, v_2, \ldots, v_k\}$ be two complete antichains of $X_{n,r}^*$ of equal length, and let $g_{\overline{\mathbf{u}},\overline{\mathbf{v}}} \in G_{n,r}$ be the homeomorphism of $\mathfrak{C}_{n,r}$ corresponding to these antichains. Observe that the pair $(\mathcal{T}_{n,r}^{\overline{\mathbf{u}}}, \mathcal{T}_{n,r}^{\overline{\mathbf{v}}})$ is a forest pair. Let σ be the bijection from the leaves of $\mathcal{T}_{n,r}^{\overline{\mathbf{u}}}$ to the leaves of $\mathcal{T}_{n,r}^{\overline{\mathbf{v}}}$ such that, for $1 \leqslant i \leqslant |\overline{\mathbf{u}}|$, we have $(u_i)\sigma = v_i$. The triple $(\mathcal{T}_{n,r}^{\overline{\mathbf{u}}}, \mathcal{T}_{n,r}^{\overline{\mathbf{v}}}, \sigma)$ represents the homeomorphism $g_{\overline{\mathbf{u}},\overline{\mathbf{v}}}$ by indicating how it changes finite prefixes. The bijection σ is usually indicated by a numbering of the leaves of $\mathcal{T}_{n,r}^{\overline{\mathbf{u}}}$ and the induced numbering, by σ, on the leaves of $\mathcal{T}_{n,r}^{\overline{\mathbf{v}}}$.

Definition 2.3.15. A *forest triple* (of $\mathcal{T}_{n,r}$) is a triple (A, B, ρ) where (A, B) is a forest pair and ρ is a bijection from the leaves of A to the leaves of B.

Remark 2.3.16. Every element of $G_{n,r}$ corresponds to a forest triple, and every forest triple gives rise to an element of $G_{n,r}$.

Remark 2.3.17. Let $\overline{\mathbf{u}} = \{u_1, \ldots, u_k\}$ be a complete antichain for $X_{n,r}^*$ and let $\overline{\mathbf{u}}'$ be the complete antichain arising by a single expansion of $\overline{\mathbf{u}}$ over an element $u_i \in \overline{\mathbf{u}}$. Let $\mathcal{T}_{n,r}^{\overline{\mathbf{u}}}$ and $\mathcal{T}_{n,r}^{\overline{\mathbf{u}}'}$ be the finite subforest of $\mathcal{T}_{n,r}$ with leaves which are elements of $\overline{\mathbf{u}}$ and $\overline{\mathbf{u}}'$ respectively. The leaf u_i of $\mathcal{T}_{n,r}^{\overline{\mathbf{u}}}$ is now an internal vertex of $\mathcal{T}_{n,r}^{\overline{\mathbf{u}}'}$ and has children $\{u_i a \mid a \in X_n\}$; all other leaves of $\mathcal{T}_{n,r}^{\overline{\mathbf{u}}}$ remain leaves of $\mathcal{T}_{n,r}^{\overline{\mathbf{u}}'}$. We say that $\mathcal{T}_{n,r}^{\overline{\mathbf{u}}'}$ is obtained from $\mathcal{T}_{n,r}^{\overline{\mathbf{u}}}$ by *adding a caret* to the leaf u_i of $\mathcal{T}_{n,r}^{\overline{\mathbf{u}}}$. Alternatively we say that $\mathcal{T}_{n,r}^{\overline{\mathbf{u}}}$ is obtained from $\mathcal{T}_{n,r}^{\overline{\mathbf{u}}'}$ by *deleting a caret* of $\mathcal{T}_{n,r}^{\overline{\mathbf{u}}'}$.

Definition 2.3.18. Let A, B be finite subforests of $\mathcal{T}_{n,r}$, then we say that A is an *expansion [contraction]* of B if either $B = A$ or there is a sequence $B := B_0, B_1, \ldots, B_k = A$, $k \in \mathbb{N}$ of finite subforests such that, for $1 \leqslant i \leqslant k$, B_i is obtained from B_{i-1} by adding a caret to a leaf [deleting a caret] of B_{i-1}. If $k = 1$ then we say that A is an *single expansion [contraction]* of B.

Remark 2.3.19. Observe that, by Remark 2.3.17 and a simple induction argument, if $\overline{\mathbf{u}}$ and $\overline{\mathbf{u}}'$ are complete antichains such that $\overline{\mathbf{u}}'$ is an expansion (contraction) of $\overline{\mathbf{u}}$ then, $\mathcal{T}^{\overline{\mathbf{u}}'}_{n,r}$ is an expansion (contraction) of $\mathcal{T}^{\overline{\mathbf{u}}}_{n,r}$. Therefore it follows that if $\overline{\mathbf{u}}$ and $\overline{\mathbf{v}}$ are complete antichains of the same length, then there are infinitely many forest triples representing the map $g_{\overline{\mathbf{u}},\overline{\mathbf{v}}}$.

Example 2.3.20. Consider the complete antichains $\overline{\mathbf{u}} := \{\dot{1}, \dot{2}0, \dot{2}1, \dot{2}2\}$ and $\overline{\mathbf{v}} := \{\dot{2}2, \dot{2}1, \dot{2}0, \dot{0}\}$ of $X^*_{3,2}$, and the map $g_{\overline{\mathbf{u}},\overline{\mathbf{v}}}$ of $G_{3,2}$. Below we give the forest triple representing $g_{\overline{\mathbf{u}},\overline{\mathbf{v}}}$.

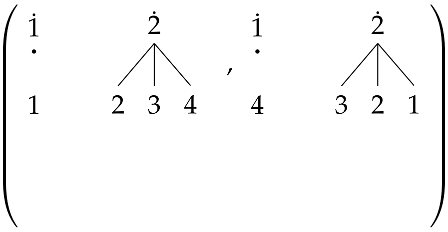


Figure 2.9: The forest triple representing the map $g_{\overline{\mathbf{u}},\overline{\mathbf{v}}}$

The bijection from the leaves of $\mathcal{T}^{\overline{\mathbf{u}}}_{n,r}$ to the leaves of $\mathcal{T}^{\overline{\mathbf{v}}}_{n,r}$ is indicated by the numbering of the leaves of both trees. We shall also use the following picture for the representation above:

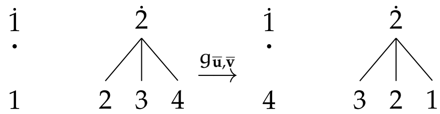


Figure 2.10: Alternative representation of Figure 2.9

Remark 2.3.21. For the case where $r = 1$, we modify the terminology above by replacing the occurrences of 'forest' with 'tree'.

We shall mainly utilise forest/tree pair diagrams in the last part of the thesis. However, these diagrams give a nice way of thinking about the group $G_{n,r}$ and the reader is encouraged to keep them at the back of their mind as a way of informing subsequent discussions.

For the moment however, we describe the final way we will be working with the groups $G_{n,r}$. One should observe that the action of elements of $G_{n,r}$ on $\mathfrak{C}_{n,r}$ utilises only finitely many local actions, in particular, after modifying a finite initial prefix, elements of $G_{n,r}$ act like the identity. This means that $G_{n,r}$ is a subgroup of $\mathcal{R}_{n,r}$. The next section characterises the transducers representing elements of $G_{n,r}$ and so gives a representation of elements of $G_{n,r}$ by finite initial transducers.

2.3.3 The Higman-Thompson groups $G_{n,r}$ as subgroups of $\mathcal{B}_{n,r}$

In this section we characterise the finite initial transducers representing elements of $G_{n,r}$. As a consequence of this characterisation, we demonstrate that $G_{n,r}$ is in fact a subgroup of $\mathcal{B}_{n,r}$ (and so a subgroup of $\mathcal{R}_{n,r}$), moreover, the core of a transducer representing an element of $G_{n,r}$ is precisely the single state identity transducer. We conclude with an example constructing a minimal initial transducer representing an element of $G_{n,r}$.

We begin with the following result:

Proposition 2.3.22. *Let $g \in G_{n,r}$, then there is a minimal transducer $A_{q_0} \in \mathcal{B}_{n,r}$ such that $h_{q_0} = g$ and* $\mathrm{Core}(A_{q_0}) = id$.

Proof. It suffices to show that for an element $g \in G_{n,r}$ there is a $k \in \mathbb{N}_1$ such that for any word $\gamma \in X_{n,r}^*$ of length k the local action g_γ is the identity map on $\mathfrak{C}_n$. It then follows, by Remark 1.5.30, that the transducer A_g' constructed using Construction 1.5.29, has the property that after processing a word of length k, the new active state induces the identity homeomorphism on $\mathfrak{C}_n$. Thus if A_{q_0} is the minimal transducer such that $h_{q_0} = g$, then A_{q_0} is synchronizing at level k and has trivial core.

Let $\overline{\mathbf{u}}$ and $\overline{\mathbf{v}}$ be complete antichains of $X_{n,r}^*$ such that $g = g_{\overline{\mathbf{u}},\overline{\mathbf{v}}}$. We may assume that $|\overline{\mathbf{u}}| \geqslant 1$ and $|\overline{\mathbf{v}}| \geqslant 1$ by Remark 2.3.5, thus $\overline{\mathbf{u}} \subset X_{n,r}^+$ and $\overline{\mathbf{v}} \subset X_{n,r}^+$. Let $k = \max_{u \in \overline{\mathbf{u}}}\{|u|\}$ and pick $\gamma \in X_{n,r}^k$. Observe that there is a word $\delta \in X_{n,r}^+$ such that for any $\rho \in \mathfrak{C}_n$ we have,

$$(\gamma\rho)g_{\overline{\mathbf{u}},\overline{\mathbf{v}}} = \delta\rho. \tag{2.2}$$

This implies that $(\gamma)\theta_g = U_\delta$. Thus for any $\rho' \in \mathfrak{C}_n$ we have $(\rho')g_\gamma = (\gamma\rho')g - (\gamma)\theta_g = \rho'$; the last equality is a consequence of Equation 2.2. $\square$

Definition 2.3.23. Let $\widetilde{\mathcal{B}}_{n,r}(\mathrm{id})$ be the subset of $\widetilde{\mathcal{B}}_{n,r}$ consisting of all those elements of $\mathcal{R}_{n,r}$ which may be represented by a synchronizing transducer with trivial core.

By Proposition 2.3.22 above, $G_{n,r}$ is a subgroup of $\mathcal{B}_{n,r}$ and is contained in $\widetilde{\mathcal{B}}_{n,r}(\mathrm{id})$. We have the following result:

Proposition 2.3.24. *Let* $g \in \widetilde{\mathcal{B}}_{n,r}(id)$*, then* $g \in G_{n,r}$.

Proof. Let $g \in \mathcal{B}_{n,r}(\mathrm{id})$, and $A_{q_0} = \langle \dot{\mathbf{r}}, X_n, Q_A, \pi_A, \lambda_A \rangle$ be a minimal transducer such that $h_{q_0} = g$. By definition of $\mathcal{B}_{n,r}(\mathrm{id})$, A_{q_0} is synchronizing and $\mathrm{Core}(A_{q_0}) = \mathrm{id}$. Let $k \in \mathbb{N}_1$ be minimal such that for any $\gamma \in X_{n,r}^k$ we have $\pi_A(\gamma, q_0) = \mathrm{id}$ (one may take k to be the minimal synchronizing level of A_{q_0}). Set $\overline{\mathbf{u}} = X_{n,r}^k$ — a complete antichain for $X_{n,r}^*$ and suppose that $\overline{\mathbf{u}}$ is ordered such that $\overline{\mathbf{u}} = \{u_1, u_2, \ldots, u_l\}$. For $1 \leqslant i \leqslant l$, set $v_i = \lambda_A(u_i, q_0)$ and let $\overline{\mathbf{v}} = \{v_i \mid 1 \leqslant i \leqslant l\}$.

We claim that $\overline{\mathbf{v}}$ is also a complete antichain for $X_{n,r}^*$. For suppose $v_i = v_j\tau$ for distinct $i, j \in \{1, 2 \ldots, l\}$, and $\tau \in X_{n,r}^+$. Let $\rho \in \mathfrak{C}_n$ and consider $\lambda_A(u_i\rho, q_0)$ and $\lambda_A(u_j\tau\rho, q_0)$. Since $\mathrm{Core}(A_{q_0}) = \mathrm{id}$, and $\pi_A(u_i, q_0) = \pi_A(u_j, q_0) = \mathrm{id}$, we have: $\lambda_A(u_i\rho, q_0) = v_j\tau\rho$ and $\lambda_A(u_j\tau\rho, q_0) = v_j\tau\rho$. However, since $u_i \neq u_j$ (as $\overline{\mathbf{u}}$ is a complete antichain for $X_{n,r}^*$), we conclude that h_{q_0} is not injective yielding the desired contradiction. Thus $\overline{\mathbf{v}}$ is an antichain. That $\overline{\mathbf{v}}$ is a complete antichain follows since h_{q_0} is surjective, and $\pi_A(u_i, q_0) = \mathrm{id}$ for all $1 \leqslant i \leqslant l$.

Consider the map $g_{\overline{\mathbf{u}},\overline{\mathbf{v}}}$. Let $\delta \in \mathfrak{C}_{n,r}$ be arbitrary, there is a $\rho \in \mathfrak{C}_{n,r} \sqcup \mathfrak{C}_n$ and $1 \leqslant i \leqslant l$ such that, $\delta = u_i\rho$. Observe that $(u_i\rho)h_{q_0} = \lambda_A(u_i\rho, q_0) = v_i\rho$, however, $(u_i\rho)g_{\overline{\mathbf{u}},\overline{\mathbf{v}}} = v_i\rho$ also. Since $\delta \in \mathfrak{C}_{n,r}$ was arbitrary, we have $h_{q_0} = g_{\overline{\mathbf{u}},\overline{\mathbf{v}}}$ as required. $\square$

Putting together Propositions 2.3.22 and 2.3.24 we have:

Theorem 2.3.25. *The subset* $\widetilde{\mathcal{B}}_{n,r}(id)$ *of* $\widetilde{\mathcal{B}}_{n,r}$ *is a subgroup of* $\mathcal{B}_{n,r}$ *and is equal to* $G_{n,r}$.

Remark 2.3.26. In keeping with the identification of $\mathcal{R}_{n,1}$ with $\mathcal{R}_n$ (Remark 1.6.15) we shall think of the group $G_{n,r}$ as a subgroup of $\mathcal{R}_n$ acting on $\mathfrak{C}_n$.

Example 2.3.27. Below we give the minimal transducer representing the element $g_{\overline{\mathbf{u}},\overline{\mathbf{v}}}$ of Example 2.3.20

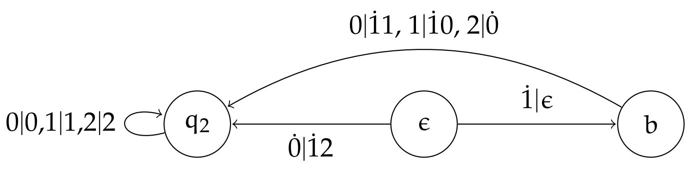


Figure 2.11: A bi-synchronizing transducer representing the element $g_{\overline{\mathbf{u}},\overline{\mathbf{v}}}$ of Example 2.3.20

For the remainder our discussion we shall alternate between viewing elements of $G_{n,r}$ as prefix exchange maps (Subsubection 2.3.1); as forest triples or pairs (Subsubection 2.3.2); and as subgroups of $\mathcal{B}_{n,r}$ (Subsubection 2.3.3). As we mentioned before (Section 2.3) the groups $G_{n,r}$ also arise as automorphisms of the Higman algebras $V_{n,r}$ (see for instance [2]). We mostly eschew this approach, though we might sometimes draw comparisons.

In the next section we explore how the group $G_{n,r}$ interacts with elements of $\mathcal{R}_{n,r}$ and $\mathcal{B}_{n,r}$. In particular we show that $N_{\mathcal{R}_{n,r}}(G_{n,r}) = \mathcal{B}_{n,r}$.

2.4 Automorphisms of the Higman-Thompson groups $G_{n,r}$

In this section we show that the group $\mathcal{B}_{n,r}$ is equal to the normaliser $N_{\mathcal{R}_{n,r}}(G_{n,r})$ of $G_{n,r}$ in $\mathcal{R}_{n,r}$. By appealing to results in [10] and a result of Rubin [47] we conclude that $\mathrm{Aut}(G_{n,r}) \cong \mathcal{B}_{n,r}$.

We first begin by illustrating the idea behind the proof with synchronous transducers over $\mathfrak{C}_n$.

Recall (Example 1.7.39) that for an invertible synchronous transducer A_{q_0} over $\mathfrak{C}_n$, $h_p \in \mathcal{R}_n$ for all states $p \in Q_A$. We also recall Notation 2.3.1 for the map $g_{\gamma,\delta} : U_\gamma \to U_\delta$, $\gamma, \delta \in X_n^*$, and Notation 1.7.42 whereby for a state r of an invertible, synchronous transducer we denote the state (ϵ, r) of A^{-1} by r^{-1}. We have the following lemma which first appears in [24]:

Lemma 2.4.1. *Let A_{q_0} be a minimal, synchronous, invertible transducer over $\mathfrak{C}_n$ and $p \in Q_A$ be a fixed state of A. Then A_{q_0} is synchronizing if and only if for every state $q \in Q_A$, $h_p^{-1}h_q \in G_{n,1}$.*

Proof. We begin with the reverse implication.

Let q be any state of A_{q_0}. Recall (Example 1.7.39) that $A_{q_0}^{-1}$ is also synchronous and minimal. By Lemma 1.7.4, the transducer product $(A^{-1} * A)_{(p^{-1},q)}$ is an initial transducer representing the homeomorphism $h_p^{-1}h_q$. Since $h_p^{-1}h_q \in G_{n,1}$, it must be the case that there is an $m \in \mathbb{N}$ such that for any $\gamma \in X_n^m$, there is a $\delta \in X_n^*$ such that $h_p^{-1}h_q\restriction_{U_\gamma} = g_{\gamma,\delta}$. We may further assume that $m \in \mathbb{N}$ is minimal with this property. Since $(A^{-1} * A)_{(p^{-1},q)}$ is a synchronous and invertible transducer, it has no states of incomplete response and so it must be the case that $|\delta| = m$. Now, using arguments similar to those in the proof of Proposition 2.3.22, after processing a word of length m from the state (p^{-1}, q) of $(A^{-1} * A)_{(p^{-1},q)}$, the resulting state must be ω-equivalent to the identity. Let $\Gamma \in X_n^m$, and suppose $\pi_{(A^{-1}*A)}(\Gamma, (p^{-1}, q)) = (r^{-1}, t)$, it follows that $h_{(r^{-1},t)}$ must be the identity homeomorphism. Since $(A^{-1} * A)_{(p^{-1},q)}$ is synchronous, we therefore have that t is ω-equivalent to the state r of A and so $t = r$ since A is minimal.

Let $\Delta = \lambda_{A^{-1}}(\Gamma, p^{-1})$. The arguments above demonstrate that $\pi_A(\Delta, q) = r$. However, since q was chosen arbitrarily, it follows that $\pi_A(\Delta, q') = r$ for any $q' \in Q_A$. Thus A is synchronizing at level m, since for any word $\Delta' \in X_n^m$, there is a word $\Gamma' \in X_n^m$ such that $\lambda_{A^{-1}}(\Gamma', p^{-1}) = \Delta'$.

For the forward implication, assume that A is synchronizing. Let k be the minimal synchronizing level of A and q be any state of A. Let $\gamma \in X_n^k$, $\Delta = \lambda_{A^{-1}}(\gamma, p^{-1})$, $\delta = \lambda_{A^{-1}*A}(\gamma, (p^{-1}, q))$, and $r^{-1} = \pi_{A^{-1}}(\gamma, p^{-1})$. Since $\pi_A(\Delta, p) = r$, it must be the case, since A is synchronizing at level k and $\Delta \in X_n^k$, that $\pi_A(\Delta, q) = r$. Therefore, it follows that $\pi_{A^{-1}*A}(\gamma, (p^{-1}, q)) = (r^{-1}, r)$. Hence $h_p^{-1}h_q\restriction_{U_\gamma} = g_{\gamma,\delta}$. □

Lemma 2.4.2. *Let A_{q_0} be a minimal, invertible, synchronous, synchronizing transducer. Fix a state p of A_{q_0} and let $k \in \mathbb{N}$ be minimal such that for every state q of A and for any $\gamma \in X_n^k$, that there is a $\delta \in X_n^*$ satisfying $h_p^{-1}h_q\restriction_{U_\gamma} = g_{\gamma,\delta}$. The minimal synchronizing level of A_{q_0} is equal to k.*

Proof. Assume that A is an invertible, synchronous, synchronizing transducer with minimal synchronizing level l. Observe that by minimality of l, there is a word $\nu \in X_n^{l-1}$ and states $q_1, q_2 \in A$ such that $\pi_A(\nu, q_1) \neq \pi_A(\nu, q_2)$. Let $\mu \in X_n^{l-1}$ be such that $\lambda_{A^{-1}}(\mu, p^{-1}) = \nu$, and $\pi_{A^{-1}}(\mu, p^{-1}) = t^{-1}$ (and so $\pi_A(\nu, p) = t$). Observe that one of $\pi_A(\nu, q_1)$, $\pi_A(\nu, q_2)$ is not equal to t. Without loss of generality we assume that $\pi_A(\nu, q_1) = s$ for s a state of A distinct from t. Since $s \neq t$, it follows that the map $\lambda_{A^{-1}*A}(\cdot, (t^{-1}, s)) : X_n \to X_n$ is not trivial and so moves at least two points. Thus, there are i, j, i', j' in X_n such that $i \neq j$, $i' \neq j'$, $\lambda_{A^{-1}*A}(i, (t^{-1}, s)) = i'$ and $\lambda_{A^{-1}*A}(j, (t^{-1}, s)) = j'$. Therefore, $\lambda_{A^{-1}*A}(\mu i, (p^{-1}, q_1)) = \nu i'$ and $\lambda_{A^{-1}*A}(\mu j, (p^{-1}, q_1)) = \nu j'$, hence we conclude that $h_p^{-1}h_q^{-1}$ does not act on U_μ as $g_{\mu,\mu'}$ for some $\mu' \in X_n^+$.

Now let q be any state of A. Let $\gamma \in X_n^l$, $\Delta = \lambda_{A^{-1}}(\gamma, p^{-1})$, $\delta = \lambda_{A^{-1}*A}(\gamma, (p^{-1}, q))$, and $r^{-1} = \pi_{A^{-1}}(\gamma, p^{-1})$. Since $\pi_A(\Delta, p) = r$, it must be the case, since A is synchronizing at level l and $\Delta \in X_n^l$, that $\pi_A(\Delta, q) = r$. Therefore, it follows that $\pi_{A^{-1}*A}(\gamma, (p^{-1}, q)) = (r^{-1}, r)$. Hence $h_p^{-1}h_q\restriction_{U_\gamma} = g_{\gamma,\delta}$. Thus l is the minimal number such that for every state q of A and for any $\gamma \in X_n^l$, that there is a $\delta \in X_n^*$ satisfying $h_p^{-1}h_q\restriction_{U_\gamma} = g_{\gamma,\delta}$. Therefore $l = k$ as required. □

We modify the previous lemma by introducing the phrase 'strictly accessible' (Definition 1.3.12).

Lemma 2.4.3. *Let A_{q_0} be a minimal, synchronous, invertible transducer over $\mathfrak{C}_n$. Fix a state $p \in Q_A$. Then A_{q_0} is synchronizing if and only if for every strictly accessible state $q \in Q_A$, $h_p^{-1}h_q \in G_{n,1}$.*

Proof. The forward implication follows exactly as in Lemma 2.4.1. For the reverse implication, let k be minimal such that for any strictly accessible state $q \in Q_A$ and for any $\gamma \in X_n^k$, there is a $\delta \in X_n^*$ such that $h_p^{-1}h_q\restriction_{U_\gamma} = g_{\gamma,\delta}$. As in the proof of Lemma 2.4.1 we once more conclude that for any pair q_1, q_2 of strictly accessible states, $\pi_A(\centerdot, q_1) : X_n^k \to Q_A$ is equal to $\pi_A(\centerdot, q_2) : X_n^k \to Q_A$. However, since any non-initial state q of A is strictly accessible, we conclude that A is synchronizing at level at most $k+1$. □

Lemma 2.4.4. *Let A_{q_0} be a minimal, synchronous, synchronizing transducer which is not strictly accessible. Fix a state p of A and let $k \in \mathbb{N}$ be minimal such that, for any strictly accessible state q of A and for any $\gamma \in X_n^k$, there is a $\delta \in X_n^*$ satisfying $h_p^{-1}h_q\restriction_{U_\gamma} = g_{\gamma,\delta}$. Then the minimal synchronizing level of A_{q_0} is either $k+1$ or k.*

Proof. It follows from Lemma 2.4.2 that A_{q_0} has minimal synchronizing level at least k. The proof of Lemma 2.4.3 demonstrates that A_{q_0} has synchronizing level at most $k+1$, thus the result follows. □

Corollary 2.4.5. *Let A_{q_0} be an invertible, minimal, synchronous transducer over $\mathfrak{C}_n$. If $h_{q_0}^{-1}G_{n,1}h_{q_0} \subseteq G_{n,1}$, then A_{q_0} is synchronizing. If $h_{q_0}^{-1}G_{n,1}h_{q_0} = G_{n,1}$ then $h_{q_0} \in \mathcal{B}_{n,r}$ and A_{q_0} is bi-synchronizing.*

Proof. Let A_{q_0} be an invertible, minimal, synchronous transducer over $\mathfrak{C}_n$ such that $h_{q_0}^{-1}G_{n,1}h_{q_0} \subseteq G_{n,1}$. In order to show that A_{q_0} is synchronizing it suffices to show that A_{q_0} satisfies the equivalent condition stated in Lemma 2.4.3.

Fix a non-trivially accessible state p of A_{q_0}. Then, by the definition of non-trivial accessibility (Definition 1.3.13), there is a word $\gamma \in X_n^+$ such that $\pi_A(\gamma, q_0) = p$ and $\lambda_A(\gamma, q_0) \neq \epsilon$. Let $\delta = \lambda_A(\gamma, q_0) \in X_n^+$ and observe that $\pi_{A^{-1}}(\delta, q_0^{-1}) = p^{-1}$ and $\lambda_{A^{-1}}(\delta, q_0^{-1}) = \gamma$. Let q be any strictly accessible state of A_{q_0}, and $\nu \in X_n^+$ be such that q is strictly accessible from q_0 by ν. Observe that there are complete antichains $\bar{\mathbf{u}}$ and $\bar{\mathbf{v}}$ of the same length such that $\gamma \in \bar{\mathbf{u}}$ and $\nu \in \bar{\mathbf{v}}$. For instance, if $|\nu| \leqslant |\gamma|$, one may take the complete antichain $X_n^{|\gamma|}$ and, since $|X_n^{|\gamma|}| \equiv |X_n^{|\nu|}| \mod n-1$, by repeatedly expanding the complete antichain $X_n^{|\gamma|}$ using elements not equal to ν, one obtains a complete antichain of the same cardinality as X_n^{γ} containing ν. Reordering if necessary, we may assume that $g_{\bar{\mathbf{u}},\bar{\mathbf{v}}}\restriction_{U_\gamma} = g_{\gamma,\nu}$.

Let C_{r_0} be the minimal initial transducer representing $g_{\bar{\mathbf{u}},\bar{\mathbf{v}}}$. Since $h_{r_0} = g_{\bar{\mathbf{u}},\bar{\mathbf{v}}}$, it follows that $\lambda_C(\gamma, r_0) = \nu$ and $\pi_C(\gamma, r_0) = \mathrm{id}$. Consider the product $(A^{-1} * C * A)_{(q_0^{-1}, r_0, q_0)}$. Observe that $\pi_{(A^{-1}*C*A)}(\delta, (q_0^{-1}, r_0, q_0)) = (p^{-1}, \mathrm{id}, q)$. This follows as $\lambda_{A^{-1}}(\delta, q_0^{-1}) = \gamma$, $\lambda_C(\gamma, p_0) = \nu$ and $\pi_A(\nu, q_0) = q$. Observe that the initial transducer $(A^{-1} * C * A)_{(p^{-1}, \mathrm{id}, q)}$ is ω-equivalent to the initial transducer $(A^{-1} * A)_{(p^{-1}, q)}$. Moreover, since $h_{(q_0^{-1}, r_0, q_0)} = h_{q_0}^{-1}g_{\bar{\mathbf{u}},\bar{\mathbf{v}}}h_{q_0} \in G_{n,1}$, it must be the case that $h_{(p^{-1}, \mathrm{id}, q)} = h_{(p^{-1}, r)}$ must be an element of $G_{n,1}$. This is because as $h_{(q_0^{-1}, r_0, q_0)}$ is a prefix replacement map, and since $\pi_{(A^{-1}*C*A)}(\delta, (q_0^{-1}, r_0, q_0)) = (p^{-1}, \mathrm{id}, q)$, we have $(U_\delta)h_{(q_0^{-1}, r_0, q_0)} = \lambda(\nu, q_0)(\mathfrak{C}_n)h_{(p^{-1}, \mathrm{id}, q)}$. Since q was an arbitrary strictly accessible state of A_{q_0} it follows that A_{q_0} satisfies the hypothesis of Lemma 2.4.3.

Now suppose that A_{q_0} is a minimal, invertible, synchronous transducer over $\mathfrak{C}_n$ such that $h_{q_0}^{-1}G_{n,1}h_{q_0} = G_{n,1}$. Multiplying on the left by h_{q_0} and on the right by $h_{q_0}^{-1}$ we therefore have that $G_{n,1} = h_{q_0}G_{n,1}h_{q_0}^{-1}$. Thus by the arguments in the preceding paragraphs we conclude that A_{q_0} and $A_{q_0^{-1}}$ are synchronizing, and so $h_{q_0} \in \mathcal{B}_{n,r}$. □

We now extend Corollary 2.4.5 to cover all elements of $\mathcal{R}_{n,r}$. The strategy remains the same, however, we need to adapt the lemmas above, in particular Lemma 2.4.3 and Lemma 2.4.1, to allow for all elements of $\mathcal{R}_{n,r}$. The proof is slightly more involved but core idea is the same.

We recall (Construction 1.7.29) that for a finite invertible transducer A_{q_0}, the states of the inverse transducer $A_{(\epsilon,q_0)} = \langle \dot{r}, X_n, R'_A, S'_A, \pi'_A, \lambda'_A \rangle$ are given by a pair (w, q) where q is a state of A, and $w \in X_n^*$ is such that $U_w \subset \operatorname{im}(q)$ and $(w)\Theta_q = \epsilon$ (Claim 1.7.30). The map Θ_q is defined in Definition 1.7.18. Further recall Notation 1.7.33 that $A_{q_0^{-1}} = \langle \dot{\mathbf{r}}, X_n, R_{A^{-1}}, S_{A^{-1}}, \pi_{A^{-1}}, \lambda_{A^{-1}} \rangle$ is the minimal transducer representing $A_{(\epsilon,q_0)}$. Observe that any non-trivially accessible state (w, p) of $A_{(\epsilon,q_0)}$, induces a continuous injection $h_{(w,p)} : \mathfrak{C}_n \to \mathfrak{C}_n$. The proposition below first appears in the author's article [10].

Proposition 2.4.6. *A minimal invertible finite transducer over $\mathfrak{C}_{n,r}$, A_{q_0}, is synchronizing if and only if there is a $k \in \mathbb{N}$ so that for any non-trivially accessible state p of $A_{q_0^{-1}}$, any state $q \in S_A$, for any $\gamma \in X_n^k$ and any $\Gamma \in \mathfrak{C}_n$, there is a $\delta \in X_n^*$ satisfying, $(\gamma\Gamma)h_p h_q = \delta\Gamma$.*

Proof. Observe that since $A_{(\epsilon,q_0)}$ has no states of incomplete response (Proposition 1.7.34), it suffices to prove the proposition with $A_{(\epsilon,q_0)}$ in place of $A_{q_0^{-1}}$. This is because, as $A_{q_0^{-1}}$ is obtained from $A_{(\epsilon,q_0)}$ by identifying ω-equivalent states, if, for any non-trivially accessible state p of $A_{q_0^{-1}}$ and any state $q \in S_A$, there is a k so that, for any $\gamma \in X_n^k$ and any $\Gamma \in \mathfrak{C}_n$, there is a $\delta \in X_n^*$ such that, $(\gamma\Gamma)h_p h_q = \delta\Gamma$, then for any non-trivially accessible state (w, p') of $A_{(\epsilon,q_0)}$, any state $q \in S_A$, any $\gamma \in X_n^k$, and any $\Gamma \in \mathfrak{C}_n$, there is a $\delta \in X_n^*$ such that, $(\gamma\Gamma)h_{(w,p')}h_q = \delta\Gamma$.

We begin with the reverse implication. For this, our strategy is to show that the subtransducer $A_{q_0}\restriction_{S_A} = \langle X_n, S_A, \pi_A\restriction_{S_A}, \lambda_A\restriction_{S_A} \rangle$ consisting of all states in S_A, is synchronizing. To do this we show that there is a base $\mathscr{S}$ for $A_{q_0}\restriction_{S_A}$ over $\mathfrak{C}_n$ (Definition 2.1.15). From this it will follow that A_{q_0} is synchronizing at a level equal to the maximum length of a word in this base plus the length of a minimal path from a state of A_{q_0} to a state in S_A (recall that by Restriction (R.1) such a path always exists). This is because after reading the initial prefix of such a word guaranteeing that the remaining suffix is processed from a state in S_A, the resulting final state is determined by this suffix.

Fix $k \in \mathbb{N}$ such that for any state $q \in S_A$, any non-trivially accessible state $(w', p') \in Q_{A_{(\epsilon,q_0)}}$, any $\gamma \in X_n^k$ and $\Gamma \in \mathfrak{C}_n$, we have, $(\gamma\Gamma)h_{(w',p')}h_q = \delta\Gamma$ for some $\delta \in X_n^*$. Let $l \in \mathbb{N}$ be minimal such that whenever there is a $\gamma \in X_n^k$, $q \in S_A$, and a non-trivially accessible state $(w', p') \in Q_{A_{\epsilon,q_0}}$ such that $(\gamma\Gamma)h_{(w',p')}h_q = \delta\Gamma$, then $|\delta| \leqslant l$. Let $\overline{k} \geqslant k$ be such that for any word $\gamma \in X_n^{\overline{k}}$, any $q \in S_A$, and non-trivially accessible $(w', p') \in Q_{A_{\epsilon,q_0}}$, we have, $|(\gamma)A_{(w',p')}A_q| \geqslant l$.

Now fix a non-trivially accessible state (w, p) of $Q_{A_{(\epsilon,q_0)}}$. Note that w satisfies that $U_w \subseteq \operatorname{im}(p)$ and $(w)\Theta_p = \epsilon$. Let $x_1x_2 \ldots x_k x_{k+1} \ldots x_{\overline{k}} \in X_n^{\overline{k}}$ and $y_1, \ldots y_t \in X_n^*$ be such that, $(x_1x_2 \ldots x_{\overline{k}})A_{(w,p)} = y_1 \ldots y_t$. Let q_1 and q_2 be any arbitrary states of S_A, and let $(y_1 \ldots y_t)A_{q_1} = z_1 \ldots z_{l_1} x_{k_1+1} \ldots x_{k_1+i}$ and $(y_1 \ldots y_t)A_{q_2} = u_1 \ldots u_{l_2} x_{k_2+1} \ldots x_{k_2+j}$. We may assume that $u_{l_2} \neq x_{k_2}$ and $z_{k_1} \neq x_{k_1}$. Here $k_a \leqslant k$ and $l_a \leqslant l$ for $a \in \{1, 2\}$, moreover we may assume that the l_a are minimal such that $(x_1x_2 \ldots x_{\overline{k}}\chi)h_{(w,p)}h_{q_a} = *_1 \ldots *_{l_a} x_{k_a+1} \ldots x_{\overline{r}}\chi$ where $(*, a) \in \{(z, 1), (u, 2)\}$.

It cannot be the case that $(y_1 \ldots y_t)A_{q_1} = z_1 \ldots z_{l_1} x_{k_1+1} \ldots x_{\overline{k}}\rho$ for some $\rho \in X_n^+$, since picking a word ρ' which is incomparable to ρ, we then have, $(x_1 \ldots x_{\overline{k}}\rho')A_{(w,p)}A_{q_1} = z_1 \ldots z_{l_1} x_{k_1+1} \ldots x_{\overline{k}}\rho\delta$ for some δ, however, by minimality of l_1, $\rho = \rho'$ which is a contradiction. A similar argument demonstrates that $(y_1 \ldots y_t)A_{q_2} \neq u_1 \ldots u_{l_2} x_{k_2+1} \ldots x_{\overline{k}}\rho$ for some $\rho \in X_n^+$.

Therefore we may assume that $k_1 + i \leqslant \overline{k}$ and $k_2 + j \leqslant \overline{k}$.

Let $\pi'_A(x_1 \ldots x_{\overline{k}}, (w, p)) = (\nu, s)$. Recall that, by Construction 1.7.29 and Lemma 1.7.32, we have, $wx_1 \ldots x_{\overline{k}} - \lambda_A(y_1 \ldots y_t, p) = \nu$ and $\pi_A(y_1 \ldots y_t, p) = s$. Since $A_{(\epsilon,q_0)}$ has only finitely many states (Lemma 1.7.28) we may choose $\overline{k}$ so that ν is a suffix of $x_1 \ldots x_{\overline{k}}$. Moreover, we also have $(wx_1 \ldots x_{\overline{k}})\Theta_p = y_1 \ldots y_t$. Let $wx_1 \ldots x_\kappa = \lambda_A(y_1 \ldots y_t, p)$ for some $\kappa \leqslant \overline{k}$. Let m be minimal such that for any state q of A and any word $\mu \in X_n^m$, we have, $|\lambda_A(\mu, q)| \geqslant \overline{k} - \kappa$. Now notice that there is a maximal set $\alpha = \{\alpha_1, \ldots \alpha_{m_1}\} \subseteq X_n^m$ such that the greatest common of prefix of the elements of α is the empty word, and such that, for $1 \leqslant a \leqslant m_1$ and $\rho_a \in X_n^*$, $\lambda(\alpha_a, s) = x_{\kappa+1} \ldots x_{\overline{k}}\rho_a$. This is because $(wx_1 \ldots x_{\overline{k}})\Theta_p = y_1 \ldots y_t$ and $\pi_A(y_1 \ldots y_t, p) = s$. Further notice that the set α depends only on $x_1 \ldots x_{\overline{k}}$, (w, p) and (ν, s), that is, α is independent of the choice of q_1 and q_2.

Fix an $\alpha_a \in \alpha$. Let $P_a = \pi_A(\alpha_a, s)$ and let $\mathrm{im}(P_a)$, as usual, represent the image of the map h_{P_a} from $\mathfrak{C}_n$ to itself. Since h_{q_0} is injective, $(\nu\rho_a \,\mathrm{im}(P_a))\Theta_s = \alpha_a$ and $(\rho_a \,\mathrm{im}(P_a))h_{(\nu,s)} = U_{\alpha_a}$. Let $T_1 := \pi_A(y_1 \ldots y_t, q_1)$ and $T_2 := \pi_A(y_1 \ldots y_t, q_2)$. Observe that $(x_1 \ldots x_{\overline{k}}\rho_a \,\mathrm{im}(P_a))h_{(w,p)} = U_{y_1 \ldots y_t \alpha_a}$. Since $(y_1 \ldots y_t)A_{q_1} = z_1 \ldots z_{l_1} x_{k_1+1} \ldots x_{k_1+i}$ and $(y_1 \ldots y_t)A_{q_2} = u_1 \ldots u_{l_2} x_{k_2+1} \ldots x_{k_2+j}$, it must be the case, for any $\delta \in \mathfrak{C}_n$, that $\lambda_A(\alpha_a\delta, T_1)$ has a prefix $x_{k_1+i+1} \ldots x_{\overline{k}}\rho_a$, likewise, $\lambda_A(\alpha_a\delta, T_2)$ has a prefix $x_{k_2+j+1} \ldots x_{\overline{k}}\rho_a$. As we assumed that A_{q_0} has no states of incomplete response, it must be the case that $\lambda_A(\alpha_a, T_*) = x_{k_*+\sharp+1} \ldots x_{\overline{k}}\rho_a$ for $(*, \sharp) \in \{(1, i), (2, j)\}$, otherwise the states $\pi_A(\alpha_a, T_*)$ will be states of incomplete response for $* \in \{1, 2\}$. Let $\delta' \in \mathrm{im}(P_a)$ be arbitrary, and let $(x_1 \ldots x_{\overline{k}}\rho_a\delta')(w, p) = y_1 \ldots y_t\alpha_a\delta$, then we must have, $(\delta)h_{\pi_A(\alpha_a, T_1)} = (\delta)h_{\pi_A(\alpha_a, T_2)} = \delta'$. Since δ' was arbitrary and $(\rho_a \,\mathrm{im}(P_a))A_{(\nu,s)} = U_{\alpha_a}$, the previous equality holds for any δ in $\mathfrak{C}_n$. This means that $\pi_A(\alpha_a, T_1)$ and $\pi_A(\alpha_a, T_2)$ are ω-equivalent.

Now as q_1 and q_2 were arbitrary states of $A_{q_0}\restriction_{S_A}$, we must have, for any pair q_1, q_2 of states of $A_{q_0}\restriction_{S_A}$, that $\pi_A(y_1 \ldots y_t\alpha_a, q_1)$ and $\pi_A(y_1 \ldots y_t\alpha_a, q_2)$ are ω-equivalent (and so equal by minimality of A_{q_0}) for all $\alpha_a \in \alpha$. Therefore, the set of words $\{y_1 \ldots y_t\alpha_a | 1 \leqslant a \leqslant m_1\}$ is a set of synchronizing words for $A_{q_0}\restriction_{S_A}$.

We now consider the set $\beta := X_n^m \backslash \alpha$. Once again β is independent of the choice of q_1 and q_2. Let $\beta := \{\beta_1, \ldots, \beta_{m_2}\}$. Let $\beta_b \in \beta$ be arbitrary. By assumption we have that $|\lambda'_A(\beta_b, (\nu, s))| \geqslant \overline{k} - \kappa$, thus let $x'_{\kappa+1} \ldots x'_{\overline{k}}\rho'_b := \lambda'_A(\beta_b, (\nu, s))$.

The word $x_1 \ldots x_\kappa x'_{\kappa+1} \ldots x'_{\overline{k}}\rho'_b$ has length greater than or equal to $\overline{k}$, therefore we may repeat the arguments above with $x_1 \ldots x_\kappa \ldots x'_{\kappa+1} x'_{\overline{k}}\rho'_b$ in place of $x_1 \ldots x_{\overline{k}}$. Notice that $\lambda'_A(x_1 \ldots x_\kappa x'_{\kappa+1} \ldots x'_{\overline{k}}\rho'_b, (w, p))$ is either a prefix of $y_1 \ldots y_t$ or contains $y_1 \ldots y_t$ as a prefix. Let $\lambda'_A(x_1 \ldots x_\kappa x'_{\kappa+1} \ldots x'_{\overline{k}}\rho'_b, (w, p)) := y'_1 \ldots y'_{t'}$ and $(\nu', s') := \pi'_A(x_1 \ldots x_\kappa x'_{\kappa+1} \ldots x'_{\overline{k}}\rho'_b, (w, p))$. After repeating the arguments above, we end up with a new set of synchronizing words for $A_{q_0}\restriction_{S_A}$. However, this new set of synchronizing words contains as a subset $\{y_1 \ldots y_t\beta_b\eta_c | 1 \leqslant c \leqslant m'_1\}$ where the η_c's all have the same size and form a maximal antichain of X_n^*. Take the union of the sets $\{y_1 \ldots y_t\beta_b\eta_c | 1 \leqslant c \leqslant m'_1\}$ and $\{y_1 \ldots y_t\alpha_a | 1 \leqslant a \leqslant m_1\}$ and let $\mathscr{S}(y_1 \ldots y_t)$ denote the union. Continuing in this way across all the $\beta_b \in \beta$, and letting $\mathscr{S}(y_1 \ldots y_t)$ denote the union at each stage, we see that $\mathscr{S}(y_1 \ldots y_t)$ is finite and is a base for $A_{q_0}\restriction_{S_A}$ over the clopen set $U_{y_1 \ldots y_t}$.

Now to finish to proof it suffices to construct a finite set $M \subset X_n^*$ satisfying the following conditions:

1.) for every element of $\nu \in X_n^*$ of long enough length there is an element of M which is a prefix of a (possibly trivial) rotation of ν,

2.) for every word $\gamma \in M$, $\mathscr{S}(\gamma)$ exists and is a base for $A_{q_0}\restriction_{S_A}$ over the clopen set U_γ.

To see that this suffices, observe that the conditions above imply that there is some $D \in \mathbb{N}$, such that for all $d \in \mathbb{N}_D$ and for any word $\nu \in X_n^d$, there is a rotation ν' of ν which has a prefix in M. Therefore, for large enough D, ν' has a prefix in $\mathscr{S}(\gamma)$ for some $\gamma \in M$, and so is a synchronizing word for A_{q_0}, we then appeal to Lemma 2.2.20 to conclude that $A_{q_0}\restriction_{S_A}$ is synchronizing.

To do this let $\mathscr{P}$ be the set of all non-trivially accessible states of $A_{(\epsilon, q_0)}$. For each $(w, p) \in \mathscr{P}$, let $\mathscr{O}((w, p)) := \{\lambda'_A(\varphi, (w, p)) \mid \varphi \in X_n^{\overline{k}}\}$. The arguments above demonstrate that for each word $\varphi \in X_n^{\overline{k}}$, and each state $(w, p) \in \mathscr{P}$, there is a set $\mathscr{S}(\lambda'_A(\varphi, (w, p)))$ of synchronizing words which is a base for $A_{q_0}\restriction_{S_A}$ over the clopen set $U_{\lambda'_A(\varphi,(w,p))}$. Taking $M := \{\nu \in X_n^* \mid \exists (w, p) \in \mathscr{P} : \nu \in \mathscr{O}((w, p))\}$ we are done if we can show that M satisfies the first condition above.

Let $D = \max_{(w,p) \in \mathscr{P}}\{|\lambda_A(\varphi, (w, p))| \mid \varphi \in X_n^{\overline{k}}\}$. Let $d \in \mathbb{N}_D$ and let $\delta \in X_n^d$. For $\dot{a} \in \dot{\mathbf{r}}$, there is a $\dot{b} \in \dot{\mathbf{r}}$ and $\rho \in \mathfrak{C}_n$ such that, $(\dot{b}\rho)h_{q_0} = \dot{a}\delta\delta\delta\ldots$. Let ρ_1 be the minimal prefix of ρ such that $\pi'_A(\rho_1, (\epsilon, q_0)) = (w, p) \in \mathscr{P}$. Let $\rho_2 \in X_n^{\overline{k}}$ be such that $\rho_1\rho_2$ is a prefix of ρ. Observe that since $\pi'_A(\rho_1, (\epsilon, q_0)) = (w, p) \in \mathscr{P}$, then, by Restriction (R.2), we must have $\lambda'_A(\rho_1, (\epsilon, q_0)) \neq \epsilon$. Therefore $\lambda'_A(\rho_2, (w, p))$ is a prefix of a (possibly trivial) rotation of δ. Since $\delta \in X_n^d$ was chosen arbitrarily, it follows that for every element $\nu \in X_n^j$, there is an element of M which is a prefix of a possibly trivial rotation of ν. This concludes the forward implication.

We now consider the reverse implication of the proposition.

We free all symbols used above. Let A_{q_0} be a minimal finite transducer over $\mathfrak{C}_{n,r}$ and suppose that A_{q_0} is synchronizing with $k \in \mathbb{N}$ the minimal synchronizing level of A_{q_0}. Let p be an arbitrary non-trivially accessible state of $A_{q_0^{-1}}$ (observe that $p \in S_{A^{-1}}$) and q be any state of S_A. There is a $j \in \mathbb{N}$ such that for any word $\gamma \in X_n^j$, $|\lambda_{A^{-1}}(\gamma, p)| \geqslant k$. Fix a word $\rho \in \mathfrak{C}_n$, and let φ be the length j prefix of ρ and $\rho' \in \mathfrak{C}_n$ be such that $\varphi\rho' = \rho$. Let $\phi = \lambda_{A^{-1}}(\varphi, p)$, and, since $|\phi| \geqslant k$, let $t \in S_A$ be the unique state of A_{q_0} forced by ϕ. Let $s = \pi_{A^{-1}}(\varphi, p)$, we now demonstrate (s,t) is actually a state of $(A_{q_0^{-1}} * A_{q_0})_{(q_0^{-1}, q_0)}$.

Let $\delta \in X_{n,r}^*$ be minimal such that $\pi_{A^{-1}}(\delta, q_0^{-1}) = p$. Observe that $\lambda_{A^{-1}}(\delta\varphi, q_0^{-1})$ has ϕ as a suffix. Thus $\pi_{(A_{q_0^{-1}} * A_{q_0})}(\delta\varphi, (q_0^{-1}, q_0)) = (s,t)$. Since $(A_{q_0^{-1}} * A_{q_0})_{(q_0^{-1}, q_0)}$ is ω-equivalent to the identity transducer, it follows that there is a fixed $\nu \in X_n^*$ such that, for any $\xi \in \mathfrak{C}_n$, $(\xi)h_s h_t = \nu\xi$. Since after after applying the algorithm **M2** for correcting the states of incomplete response in $(A_{q_0^{-1}} * A_{q_0})_{(q_0^{-1}, q_0)}$, all non-initial states of $A_{q_0^{-1}}$ are ω-equivalent to the single state identity transducer over $\mathfrak{C}_n$.

Since (s,t) is a state of $(A_{q_0^{-1}} * A_{q_0})_{(q_0^{-1}, q_0)}$, we have

$$(\rho)h_p h_q = \lambda_{(A_{q_0^{-1}} * A_{q_0})}(\varphi, (p,q))(\rho')h_s h_t.$$

Thus for ν equal to the greatest common prefix of $\mathrm{im}((s,t))$, we have, $(\rho)h_p h_q = \lambda_{(A_{q_0^{-1}} * A_{q_0})}(\varphi, (p,q))\nu\rho'$. Since $\rho \in \mathfrak{C}_n$ was arbitrary, we are done. $\square$

Remark 2.4.7. Let A_{q_0} be a transducer over $\mathfrak{C}_{n,r}$ then by Restriction (R.1) any state $q \in S_A$ is strictly accessible from q_0.

The following corollary extends Corollary 2.4.5 to any transducer A_{q_0} representing an element of $\mathcal{R}_{n,r}$.

Corollary 2.4.8. *Let A_{q_0} be a minimal, initial transducer over $\mathfrak{C}_{n,r}$ such that $h_{q_0} \in \mathcal{R}_{n,r}$. Suppose that $h_{q_0}^{-1} G_{n,r} h_{q_0} \subseteq G_{n,r}$, then A_{q_0} is synchronizing. If in fact $h_{q_0}^{-1} G_{n,r} h_{q_0} = G_{n,r}$, then $h_{q_0} \in \mathcal{B}_{n,r}$ and A_{q_0} is bi-synchronizing.*

Proof. Let A_{q_0} be a transducer over $\mathfrak{C}_{n,r}$ satisfying the hypothesis of the corollary. We demonstrate that A_{q_0} is synchronizing by showing it satisfies the hypothesis of Proposition 2.4.6.

Fix a non-trivially accessible state p of $A_{q_0^{-1}}$, and a state $q \in S_A$. Let $\delta \in X_{n,r}^*$ be such that $\pi_{A^{-1}}(\delta, q_0^{-1}) = p$, $\lambda_{A^{-1}}(\delta, q_0^{-1}) \neq \epsilon$ and $\gamma = \lambda_{A^{-1}}(\delta, q_0^{-1})$ for some $\gamma \in X_{n,r}^+$. Furthermore, since $q \in S_A$, let $\nu \in X_{n,r}^+$ be such that q is strictly accessible from q_0 by ν. As in the proof of Corollary 2.4.5, let $\bar{\mathbf{u}}$ and $\bar{\mathbf{v}}$ be complete antichains for $X_{n,r}$ of the same length such that $\gamma \in \bar{\mathbf{u}}$ and $\nu \in \bar{\mathbf{v}}$. Reordering $\bar{\mathbf{v}}$ if necessary let $g_{\bar{\mathbf{u}},\bar{\mathbf{v}}} \in G_{n,r}$ be such that $g_{\bar{\mathbf{u}},\bar{\mathbf{v}}}\restriction_{U_\gamma} = U_\nu$.

Now let C_{r_0} be the minimal transducer representing $g_{\bar{\mathbf{u}},\bar{\mathbf{v}}}$. Since $h_{r_0} = g_{\bar{\mathbf{u}},\bar{\mathbf{v}}}$, it follows that $\lambda_C(\gamma, r_0) = \nu$ and $\pi_C(\gamma, r_0) = \mathrm{id}$. Consider the product $(A_{q_0^{-1}} * C_{r_0} * A_{q_0})_{(q_0^{-1}, r_0, q_0)}$ and observe that $\pi_{(A_{q_0^{-1}} * C_{r_0} * A_{q_0})}(\delta, (q_0^{-1}, r_0, q_0)) = (p, \mathrm{id}, q)$. Since $h_{q_0}^{-1} G_{n,r} h_{q_0} \subset G_{n,r}$, it must be the case that $h_{(p,\mathrm{id},q)} = h_{(p,q)}$ satisfies the hypothesis of Proposition 2.4.6.

If $h_{q_0}^{-1} G_{n,r} h_{q_0} = G_{n,r}$, then, as in the proof of Corollary 2.4.5, we again conclude that A_{q_0} and $A_{q_0^{-1}}$ are synchronizing and so $h_{q_0} \in \mathcal{B}_{n,r}$. $\square$

We are now in position to state the main result of this section.

Theorem 2.4.9. *The normaliser of $G_{n,r}$ in the group $\mathcal{R}_{n,r}$ is precisely $\mathcal{B}_{n,r}$, that is $N_{\mathcal{R}_{n,r}}(G_{n,r}) = \mathcal{B}_{n,r}$.*

Proof. By Corollary 2.4.8 we have the inclusion $N_{\mathcal{R}_{n,r}}(G_{n,r}) \subset \mathcal{B}_{n,r}$. We now prove the inclusion $\mathcal{B}_{n,r} \subset N_{\mathcal{R}_{n,r}}(G_{n,r})$ in the lemma below.

Lemma 2.4.10. *Let A_{q_0} be a finite, minimal synchronizing transducer over $\mathfrak{C}_{n,r}$, then $h_{q_0}^{-1} G_{n,r} h_{q_0} \subset G_{n,r}$*

Proof. Let A_{q_0} be a finite, minimal synchronizing transducer. Let C_{r_0} be a transducer such that $h_{r_0} \in G_{n,r}$. By Proposition 2.3.22, $C_{r_0} \in \mathcal{B}_{n,r}(\mathrm{id})$. Let $A_{q_0^{-1}}$ be the minimal transducer representing $h_{q_0}^{-1}$ and let $k \in \mathbb{N}$ be minimal so that A_{q_0} and C_{r_0} are both synchronizing at level k. Let $j \in \mathbb{N}$ be minimal such that, for all $\gamma \in X_{n,r}^{j}$ we have $|\lambda_{A_{q_0^{-1}}}(\gamma, q_0^{-1})| \geqslant 2k$. Let $\gamma \in X_{n,r}^{j}$ be arbitrary, and δ be the length k suffix of $\lambda_{A_{q_0^{-1}}}(\gamma, q_0^{-1})$. Let $\pi_{A_{q_0^{-1}}}(\gamma, q_0^{-1}) = p$ and q be the state of A_{q_0} forced by δ. Consider $\pi_{A_{q_0^{-1}} * C_{r_0} * A_{q_0}}(\gamma, (q_0^{-1}, r_0, q_0))$ and observe that $\pi_{(A_{q_0^{-1}} * C_{r_0} * A_{q_0})}(\gamma, (q_0^{-1}, r_0, q_0)) = (p, \mathrm{id}, q)$, since after processing a word of length k, the active state of C_{r_0} is id, the single state identity transducer over $\mathfrak{C}_n$. Notice that $(p, \mathrm{id}, q) = (p, q)$ can be identified with a state of $(A_{q_0^{-1}} * A_{q_0})_{(q_0^{-1}, q_0)}$. This is because $\lambda_{A_{q_0^{-1}}}(\gamma, q_0^{-1})$ has its length k suffix equal to δ, and the state of A_{q_0} forced by δ is q. Since $\gamma \in X_{n,r}^{j}$ was arbitrary, after processing any word of $X_{n,r}^{j}$ the active state of $(A_{q_0^{-1}} * C_{r_0} * A_{q_0})_{(q_0^{-1}, r_0, q_0)}$ can be identified with a state of $(A_{q_0^{-1}} * A_{q_0})_{(q_0^{-1}, q_0)}$. Let B_{t_0} be the minimal transducer representing $(A_{q_0^{-1}} * C_{r_0} * A_{q_0})_{(q_0^{-1}, r_0, q_0)}$. After processing any word of $X_{n,r}^{j}$, the active state of B_{t_0} is id, the single state identity transducer on $\mathfrak{C}_n$. This is because after applying the algorithm **M2**, all the states of $(A_{q_0^{-1}} * C_{r_0} * A_{q_0})_{(q_0^{-1}, r_0, q_0)}$ which are equal to states of $(A_{q_0} * A_{q_0^{-1}})_{(q_0, q_0^{-1})}$ are ω-equivalent to the single state identity transducer. Thus B_{t_0} is an element of $\mathcal{B}_{n,r}$ with trivial core, and so, by Theorem 2.3.25, $h_{t_0} \in G_{n,r}$. □

Therefore if A_{q_0} is an element of $\mathcal{B}_{n,r}$, we have, by Lemma 2.4.10, $h_{q_0}^{-1} G_{n,r} h_{q_0} \subset G_{n,r}$ and $h_{q_0} G_{n,r} h_{q_0}^{-1} \subset G_{n,r}$ from which we deduce that $h_{q_0}^{-1} G_{n,r} h_{q_0} = G_{n,r}$. □

Remark 2.4.11. Notice that Lemma 2.4.10 indicates that for an element $A_{q_0} \in \mathcal{R}_{n,r}$ which is synchronizing but not bi-synchronizing, $h_{q_0}^{-1} G_{n,r} h_{q_0} < G_{n,r}$, hence $h_{q_0} G_{n,r} h_{q_0}^{-1} < G_{n,r}$. The paper [24] characterises the subgroups of $G_{n,r}$ and overgroups of $G_{n,r}$ arising in this way.

Observe that since $\mathcal{B}_{n,r}$ normalises $G_{n,r}$ in $\mathcal{R}_{n,r}$, it follows that each element of $\mathcal{B}_{n,r}$ induces an automorphism of $G_{n,r}$ by conjugation. The question thus arises if all automorphisms of $G_{n,r}$ arise in this way. This question is answered in the affirmative in the paper [10] of the author's and collaborators. We shall not reproduce the proof here but make a few observations about the strategy of the proof. A key idea of the proof is to find an isomorphism between the automorphisms of $G_{n,r}$ and the group $N_{H(\mathfrak{C}_{n,r})}(G_{n,r})$. To do this the authors' make use of Rubin's Theorem which roughly states that given a group G acting on a space X by homeomorphisms, such that the space X and the group action satisfy some conditions, then every automorphism of G is induced by conjugation by an element of $N_{H(\mathfrak{C}_n)}(G_{n,r})$. Thus there is an epimorphism from $\mathrm{Aut}(G_{n,r})$ to $N_{H(\mathfrak{C}_n)}(G_{n,r})$, it is then not too hard to show, using the fact that $G_{n,r}$ is dense in $H(\mathfrak{C}_{n,r})$ in the topology of point-wise convergence, that the kernel of this epimorphism is the trivial group. Thus, if one is able to show that $N_H(\mathfrak{C}_{n,r})(G_{n,r}) \leqslant \mathcal{R}_{n,r}$, then, by results above, we may conclude that $N_H(\mathfrak{C}_{n,r})(G_{n,r}) = \mathcal{B}_{n,r} \cong \mathrm{Aut}(G_{n,r})$. This is done in the paper [10]. We have the following result:

Theorem 2.4.12 ([10]). *The automorphism group of* $G_{n,r}$ *is isomorphic to* $\mathcal{B}_{n,r}$ *and contains* $G_{n,r}$ *as a subgroup.*

For the remainder of the chapter we explore the quotient group $\mathcal{B}_{n,r}/G_{n,r} \cong \mathrm{Out}(G_{n,r})$. We highlight a particular subgroup of $\mathrm{Out}(G_{n,r})$ which will be the focus of the next chapter.

2.5 The group Out($G_{n,r}$) and some of its subgroups

In this section we study the quotient group $\mathcal{B}_{n,r}/G_{n,r} \cong \mathrm{Out}(G_{n,r})$. We show that $\mathcal{B}_{n,r}/G_{n,r}$ is isomorphic to a group $\mathcal{O}_{n,r}$ of non-initial transducers with an appropriately defined product. We highlight a group $\mathcal{H}_n \leqslant \cap_{1 \leqslant r < n} \mathcal{O}_{n,r}$ of particular interest. The results and exposition in this section are based on the paper [10]. The set of minimal transducers inducing homeomorphisms of $\mathcal{R}_{n,r}$, by Lemma 1.7.11, is a group isomorphic to $\mathcal{R}_{n,r}$ under the binary operation which takes two elements A_{q_0}, B_{p_0} and returns the minimal transducer $AB_{(p_0,q_0)}$ representing $(A * B)_{(p_0,q_0)}$. Therefore, for convenience and to ease the exposition, we will henceforth identify elements of $\mathcal{R}_{n,r}$ with the set $\{A_{q_0} \mid A_{q_0} \text{ is minimal and } h_{q_0} \in \mathcal{R}_{n,r}\}$. We might still sometimes distinguish between the two objects for emphasis.

The following lemma demonstrates that two elements of $g_1, g_2 \in \mathcal{B}_{n,r}$ are in the same coset of $\mathcal{B}_{n,r}/G_{n,r}$ if and only if the initial transducers A_{q_0} and B_{p_0} representing g_1 and g_2 respectively have the same cores. We recall (Notation 1.7.12) for initial transducers A_{q_0}, B_{p_0} over $\mathfrak{C}_n$ or $\mathfrak{C}_{n,r}$, that $AB_{(p_0,q_0)}$ is the minimal transducer representing the product $(A * B)_{(p_0,q_0)}$.

Lemma 2.5.1 ([10]). *Let A_{q_0} and B_{p_0} be transducers representing elements $g, h \in \mathcal{B}_{n,r}$ respectively. Then* $\mathrm{Core}(A_{q_0}) \cong_\omega \mathrm{Core}(B_{p_0})$ *if and only if* $g^{-1}h \in G_{n,r}$.

Proof. Let $c = \mathrm{Core}(A_{q_0}) \cong_\omega \mathrm{Core}(B_{p_0})$, $k \in \mathbb{N}$ be such that A_{q_0}, B_{p_0} and $A_{q_0^{-1}}$ are synchronizing at level k. Since $\mathrm{Core}(A_{q_0}) = \mathrm{Core}(B_{p_0})$, we may choose k such that, if $\gamma \in X_{n,r}^k$, then the state of A_{q_0} forced by γ (a state of c) is equal to the state of B_{p_0} forced by γ. Consider the product $(A_{q_0^{-1}} * B)_{(q_0^{-1},p_0)}$. Let $j \in \mathbb{N}_{\max k,1}$ be minimal such that for any $\gamma \in X_{n,r}^j$ we have, $|\lambda_{A^{-1}}(\gamma, q_0))| \geqslant k$. Let $\gamma \in X_{n,r}^j$ be arbitrary, q be the state of $A_{q_0^{-1}}$ forced by γ, and p be the state of B_{p_0} forced by $\lambda_{A^{-1}}(\gamma, q_0)$. We observe that (q, p) is a state of $(A_{q_0^{-1}} * A_{q_0})_{(q_0^{-1},q_0)}$ since the state of A_{q_0} forced by $\lambda_{A^{-1}}(\gamma, q_0^{-1})$ is equal to the state of B_{p_0} forced by $\lambda_{A^{-1}}(\gamma, q_0^{-1})$. Therefore, as $\gamma \in X_{n,r}^j$ was arbitrary, as in the proof of Theorem 2.4.9, we conclude that $h_{q_0}^{-1}h_{p_0} \in G_{n,r}$.

For the reverse implication let A_{q_0} and B_{p_0} be transducers respectively representing homeomorphisms $g, h \in \mathcal{B}_{n,r}$ such that $g^{-1}h \in G_{n,r}$. Let $A_{q_0^{-1}}$ be the minimal transducer representing g^{-1}, $C = \mathrm{Core}(A_{q_0})$, $D = \mathrm{Core}(B_{p_0})$ and $E = \mathrm{Core}(A_{q_0^{-1}})$. Since $g^{-1}h \in G_{n,r}$, it must be the case that the minimal transducer representing the product $(A_{q_0^{-1}}B_{p_0}))_{(q_0^{-1},p_0)}$ has trivial core. Clearly $(A_{q_0^{-1}}A_{q_0})_{(q_0^{-1},q_0)}$ also has trivial core.

Let q be a state of E and notice that $\mathrm{im}(q)$ is a clopen subset of $\mathfrak{C}_n$. Fix $\Gamma \in X_n^+$ such that $U_\Gamma \subseteq \mathrm{im}(q)$ and let $j \in \mathbb{N}$ be minimal such that for all words $\nu \in X_n^j$, $|\lambda_{A^{-1}}(\nu, q)| \geqslant |\Gamma|$. Let $w_1, \ldots, w_m$ be the set of all words in X_n^j such that, for all $1 \leqslant i \leqslant m$, $\lambda_{A^{-1}}(w_i, q) = \Gamma\rho_i$ for some $\rho_i \in X_n^*$. Since $U_\Gamma \subset \mathrm{im}(q)$, we have, $\cup_{1\leqslant i\leqslant m}\Gamma\rho_i \,\mathrm{im}(\pi_{A^{-1}}(w_i, q)) = U_\Gamma$. Let w be the greatest common prefix of the set $\{w_i \mid 1 \leqslant i \leqslant m\}$ and, p_1 and p_2 be states of C and D respectively such that (q, p_1) and (q, p_2) are states in the core of $(A_{q_0^{-1}}B_{p_0}))_{(q_0^{-1},p_0)}$ and $(A_{q_0^{-1}}A_{q_0})_{(q_0^{-1},p_0)}$ respectively. Since $(A_{q_0^{-1}}B_{p_0}))_{(q_0^{-1},p_0)}$ and $(A_{q_0^{-1}}A_{q_0})_{(q_0^{-1},p_0)}$ have trivial core, it must be the case that, for any word $\nu \in X_n^*$, $(\nu)\theta_{h_q h_{p_1}} - (\epsilon)\theta_{h_q h_{p_1}} = \nu$ and $(\nu)\theta_{h_q h_{p_2}} - (\epsilon)\theta_{h_q h_{p_2}} = \nu$. Therefore, for $(i, T) \in \{(1, A), (2, B)\}$, we have, $\lambda_T(\Gamma, p_i)$ has a prefix equal to $(\epsilon)\theta_{h_q h_{p_i}} w$. This is because, as $\cup_{1\leqslant i\leqslant m}\Gamma\rho_i \,\mathrm{im}(\pi_{A^{-1}}(w_i, q)) = U_\Gamma$ and $(w_i)\theta_{h_q h_{p_i}} - (\epsilon)\theta_{h_q h_{p_i}} = w_i$, any element of $(U_\Gamma)h_{p_i}$ has prefix $(\epsilon)\theta_{h_q h_{p_i}} w$, and as A_{q_0} and B_{q_0} have no states of incomplete response, we conclude that $\lambda_T(\Gamma, p_i)$ has a prefix equal to $(\epsilon)\theta_{h_q h_{p_i}} w$. In fact it is actually the case that $\lambda_T(\Gamma, p_i) = (\epsilon)\theta_{h_q h_{p_i}} w$, since if there was some $\varphi \in X_n^+$ such that, $\lambda_T(\Gamma, p_i) = (\epsilon)\theta_{h_q h_{p_i}} w\varphi$, then choosing $\delta \in X_n^*$ and $1 \leqslant i \leqslant m$ such that $w_i\delta \perp w\varphi$, we have, $\lambda_T(\Gamma\rho_i, p_i) \perp (\epsilon)\theta_{h_q h_{p_i}} w_i\delta$ contradicting the fact that $(w_i\delta)\theta h_q h_{p_i} - (\epsilon)\theta_{h_q h_{p_i}} = w_i\delta$.

Now let $\Delta \in X_n^+$ be arbitrary, and $l \in \mathbb{N}$ be minimal such that, for all words $\mu \in X_n^l$, $|\lambda_{A^{-1}}(\mu, q)| \geqslant |\Gamma\Delta|$. By repeating the argument above with $\Gamma\Delta$ in place of Γ, we again conclude that, for $(i, T) \in \{(1, A), (2, B)\}$, $\lambda_T(\Gamma\Delta, p_i) = (\epsilon)\theta_{h_q h_{p_i}} \xi$ where ξ is the greatest common prefix of the set of all words $\mu \in X_n^l$ such that, $\lambda_{A^{-1}}(\mu, q) = \Gamma\Delta\phi$ for some $\phi \in X_n^*$. Notice that $\mu = w\mu'$ for some $\mu' \in X_n^*$, and so $\xi = w\xi'$ for some $\xi' \in X_n^*$. Therefore, since $\lambda_T(\Gamma, p_i) = (\epsilon)\theta_{h_q h_{p_i}} w$, we must have, $\lambda_T(\Delta, \pi_T(\Gamma, p_i)) = \xi'$. Hence, we conclude, since $\Delta \in X_n^*$ was arbitrary, that $\lambda_A(\Delta, \pi_A(\Gamma, p_1)) = \lambda_B(\Delta, \pi_B(\Gamma, p_2))$ for all $\Delta \in X_n^*$. This implies that, setting $p_1' := \pi_A(\Gamma\Delta, p_1)$ and $p_2' := \pi_B(\Gamma\Delta, p_2)$, $A_{p_1'}$ and $B_{p_2'}$ are ω-equivalent. Since E and D are strongly connected and have no pair of ω-equivalent states, $A_{p_1'} \cong_\omega B_{p_2'}$, and so E and D must be isomorphic, that is they are equal up to a relabelling of states. $\square$

Implicit in the proof of the reverse implication of Lemma 2.5.1 is the following result:

Lemma 2.5.2. *Let* E, C, D *be core synchronizing transducers over* $\mathfrak{C}_n$ *without states of incomplete response and no pair of* ω*-equivalent states. Let* e, c, d *be states of* E, C *and* D *respectively. If* $\mathrm{Core}(EC_{(e,c)}) = \mathrm{Core}(ED_{(e,d)}) = id$*, then* $C \cong_{\omega} D$.

Lemma 2.5.1 means that we may identify elements of $\mathcal{B}_{n,r}/G_{n,r}$ with the set $\{\mathrm{Core}(A_{q_0}) \mid A_{q_0}$ is minimal and $h_{q_0} \in \mathcal{B}_{n,r}\}$.

Notation 2.5.3. Set $\mathcal{O}_{n,r} = \{\mathrm{Core}(A_{q_0}) \mid A_{q_0}$ is minimal and $h_{q_0} \in \mathcal{B}_{n,r}\}$. For $n = 2$, there is only one choice of r, in this case we write $\mathcal{O}_2$ for the group $\mathcal{O}_{2,1}$.

Remark 2.5.4. Observe that by definition, for $T \in \mathcal{O}_{n,r}$, and any state t of T, the initial transducer T_t is minimal (it is accessible since T is synchronizing).

The following definition gives a multiplication of core synchronizing, automata over $\mathfrak{C}_n$ and is taken from [10].

Definition 2.5.5. Let C and D be core synchronizing transducers over $\mathfrak{C}_n$, and p and q be states of C and D respectively. Set $CD = \mathrm{Core}(CD_{(p,q)})$, where $CD_{(p,q)}$ is the minimal transducer representing the product $(C * D)_{p,q}$ of the initial transducers C_p and D_q. We call CD the *core product* of C and D.

Remark 2.5.6. Observe that for core, synchronizing transducers C and D over $\mathfrak{C}_n$ with states p and q, the product $CD_{(p,q)}$ is synchronizing also by Proposition 2.1.30 and Proposition 2.1.32.

The lemma below demonstrates that for core synchronizing transducers C and D over $\mathfrak{C}_n$, and transducers $A_{q_0}, B_{p_0} \in \mathcal{B}_{n,r}$ such that $\mathrm{Core}(A_{q_0}) = C$ and $\mathrm{Core}(B_{p_0}) = D$, the core product CD, for any choice of states of C and D, coincides with $\mathrm{Core}(AB_{(p_0,q_0)})$.

Lemma 2.5.7. *Let* A_{p_0} *and* B_{q_0} *be minimal transducers such that* $h_{p_0}, h_{q_0} \in \mathcal{B}_{n,r}$, $C = \mathrm{Core}(A_{p_0})$*, and* $D = \mathrm{Core}(B_{q_0})$*. For any choice of state* $p \in Q_C$ *and* $q \in Q_D$, $\mathrm{Core}(AB_{(p_0,q_0)}) = \mathrm{Core}(CD_{(p,q)})$.

Proof. Let $k \in \mathbb{N}$ be such that A_{p_0}, B_{q_0}, C and D are all synchronizing at level k. Let $j \in \mathbb{N}_k$ be minimal satisfying the following conditions for all $\gamma \in X_{n,r}^{j}$:

i.) $|\lambda_A(\gamma, p_0)| \geqslant k$ and,

ii.) for any state $p' \in Q_C$, $|\lambda_A(\gamma, p')| \geqslant k$.

For any word $\nu \in X_{n,r}^{k}$, we have $\pi_B(\nu, q_0) \in Q_D$, since B_{q_0} is synchronizing at level k with $\mathrm{Core}(B_{q_0}) = D$.

Let $\Delta \in X_n^j$ be arbitrary, p' be the state of C forced by Δ, and q' be the state of D forced by $\lambda_C(\Delta, p)$. Observe that $\pi_{C*D}(\Delta, (p, q)) = (p', q')$. We now show that (p', q') is a state of $(A * B)_{(p_0,q_0)})$. Let $\Gamma \in X_{n,r}^j$ be arbitrary, and $\Lambda \in X_n^j$ be such that the state of C forced by Λ is p (Λ exists since C is core and synchronizing). The states of A_{p_0} forced by $\Gamma\Lambda\Delta$ and $\Gamma\Lambda$ are equal to p' and p respectively, since $\pi_A(\Gamma, p_0) \in Q_C$ and C is synchronizing at level k. Thus, $\lambda_A(\Gamma\Lambda\Delta, p_0)$ has length greater than or equal to $2k$, by assumptions placed on j. Moreover, the length k suffix of $\lambda_A(\Gamma\Lambda\Delta, p_0)$ is equal to $\lambda_C(\Delta, p)$. Therefore, it must be the case that $\pi_B(\lambda_A(\Gamma\Lambda\Delta, p_0), q_0) = q'$, since reading the initial length k prefix from q_0 guarantees that the length k suffix, $\lambda_C(\Delta, p)$, is processed from a state of D to the state q'. In total we have, $\pi_{(A*B)}(\Gamma\Lambda\Delta, (p_0, q_0)) = (p', q')$.

The arguments above show that for any word $\mu \in X_n^*$ of long enough length (length at least j), $\pi_{(C*D)}(\mu, (p, q))$ is a state of $(A * B)_{(p_0,q_0)}$. From this it follows that $\mathrm{Core}(A * B)_{(p_0,q_0)} = \mathrm{Core}((C * D)_{(p,q)})$ since the core of any synchronizing transducer is strongly connected.

The conclusion of the proof follows from the following observation. Let E be a synchronizing transducer, then if E' is the transducer obtained from applying the procedure **M2** to the transducer E, then $\pi_{E'}\restriction_{(\mathrm{Core}(E'))} = \pi_E\restriction_{(\mathrm{Core}(E))}$. Moreover, the procedure **M2** modifies the transition function of a state e of E using only information about the function h_e. Therefore, after applying procedures **M2** and **M3** to $(A * B)_{(p_0,q_0)}$ and $(C * D)_{(p,q)}$ to obtain minimal transducers $AB_{(p_0,q_0)}$ and $CD_{(p,q)}$, it is still the case that $\mathrm{Core}(AB_{p_0,q_0}) = \mathrm{Core}(CD_{p,q})$ as required. □

The following Proposition follows from Lemma 2.5.7.

Proposition 2.5.8. *The set $\mathcal{O}_{n,r}$ equipped with the binary operation core product is a group. Moreover $\mathcal{O}_{n,r} \cong Out(G_{n,r})$.*

Proof. That the set $\mathcal{O}_{n,r}$ is closed under the binary operation core product is a consequence of Lemma 2.5.7. That this binary operation is associative is a consequence of the associativity of the multiplication of minimal transducers, and Lemma 2.5.7 once more. We now show that inverses exist and that they are unique.

Let $T \in \mathcal{O}_{n,r}$, then there is some $A_{q_0} \in \mathcal{B}_{n,r}$ such that $\mathrm{Core}(A_{q_0}) = T$. However, since $A_{q_0} \in \mathcal{B}_{n,r}$, there is some $B_{p_0} \in \mathcal{B}_{n,r}$ such that $AB_{(q_0,p_0)} = BA_{(p_0,q_0)} = \mathrm{id}$. Let $S = \mathrm{Core}(B_{p_0})$, then by Lemma 2.5.7 we have that $TS = ST = \mathrm{id}$. Now by Lemma 2.5.2, or the associativity of the product, it follows that S is unique.

The map from $\mathcal{B}_{n,r}/G_{n,r} \to \mathcal{O}_{n,r}$ mapping an element $[A_{q_0}] \in \mathcal{B}_{n,r}/G_{n,r}$ to $\mathrm{Core}(A_{q_0})$ is surjective, by the definition of $\mathcal{O}_{n,r}$, injective by Lemma 2.5.1, and a homomorphism by Lemma 2.5.7. Therefore we conclude that $\mathcal{O}_{n,r} \cong \mathcal{B}_{n,r}/G_{n,r} = \mathrm{Out}(G_{n,r})$. ☐

Definition 2.5.9. Let $T \in \mathcal{O}_{n,r}$ we say that T has a *homeomorphism state* if there is state u of T such that the map $h_u : \mathfrak{C}_n \to \mathfrak{C}_n$ is a homeomorphism; the state u is called a *homeomorphism state*.

The question arises if all elements of $\mathcal{O}_{n,r}$ or even of $\mathcal{O}_n$ possess homeomorphism states. Before we answer this question, we demonstrate why possessing a homeomorphism state is important.

Proposition 2.5.10. *Let $r, r' \in \{1, 2, \ldots, n-1\}$ be distinct. If $T \in \mathcal{O}_{n,r'}$ has a homeomorphism state, then $T \in \mathcal{O}_{n,r}$ also.*

Proof. Let C_{q_0} be a transducer representing an element of $G_{n,r}$ (and so $\mathrm{Core}(C_{q_0}) = \mathrm{id}$). Let T be an element of $\mathcal{O}_{n,r'}$ with a homeomorphism state t and $B_{R_0} \in \mathcal{B}_{n,r'}$ be such that $\mathrm{Core}(B_{R_0}) = T$. We form a new transducer D_{q_0} satisfying, $\mathrm{Core}(D_{q_0}) = T$ and $h_{D_{q_0}} \in \mathcal{B}_{n,r}$, by replacing $\mathrm{Core}(C_{q_0})$ with T_t.

We set $Q_D := Q_C \backslash \{\mathrm{id}\} \sqcup Q_T$. Define the transition function π_D and output function λ_D of D_{q_0} as follows: $\pi_D\restriction_{X_n \times Q_T} = \pi_T$, $\lambda_D\restriction_{X_n \times Q_t} = \lambda_T$; for $a \in X_{n,r} \sqcup X_n$ and $q \in Q_C$ such that $\pi_C(a, q)$ is defined we have:

$$\pi_D(a,q) = \begin{cases} \pi_C(a,q) & \text{if } \pi_C(a,q) \neq \mathrm{id} \\ t & \text{otherwise} \end{cases} \quad \text{and } \lambda_D(a,q) = \lambda_C(a,q).$$

If C_{q_0} is synchronizing at level k, then after processing any input of length k from any state of D_{q_0}, the active state is a state of T. Since T is synchronizing, it follows that D_{q_0} is synchronizing also. Further observe that the set $\mathcal{IN}$ of minimal paths in C_{q_0}, from q_0 to the state id is a complete antichain for $X_{n,r}^{+}$. Moreover the set of outputs $\mathcal{OUT}$ of the set $\mathcal{IN}$ when processed from q_0, is also a complete antichain for $X_{n,r}^{+}$. Notice that $\mathcal{IN}$ coincides with the set of minimal paths in D_{q_0} from q_0 to the state t, likewise $\mathcal{OUT}$ coincides with the outputs of the set $\mathcal{IN}$ when processed from q_0 in D_{q_0}. Since T_t is a homeomorphism state, we therefore conclude that $h_{D_{q_0}}$ is a homeomorphism of $\mathfrak{C}_{n,r}$.

Let $\gamma \in \mathcal{IN}$, and $\delta = \lambda_D(\gamma, q_0) \in \mathcal{OUT}$. Observe that $\delta \in X_{n,r}^{+}$ since C_{q_0} is a minimal transducer over $\mathfrak{C}_{n,r}$. Consider $(\delta)\Theta_{q_0}$, since $\pi_D(\gamma, q_0) = t$, a homeomorphism state, it follows that $(\delta)\Theta_{q_0} = \gamma$ otherwise $h_{D_{q_0}}$ is not a homeomorphism. Thus $\pi'_D(\delta, (\epsilon, q_0)) = (\epsilon, t)$. Moreover, since $\gamma \in \mathcal{IN}$ was arbitrary, we deduce that for any $\delta' \in \mathcal{OUT}$, $\pi'_D(\delta', (\epsilon, q_0)) = (\epsilon, t)$. Let $\nu \in X_{n,r'}^{*}$ satisfy $\pi_B(\nu, R_0) = t$ and let $\mu = \lambda_B(\nu, R_0) \in X_{n,r'}^{+}$. By a similar argument, we once more conclude that $\pi'_B(\mu, (\epsilon, R_0) = (\epsilon, t)$. Since t is in the core of B_{R_0}, by choosing a large enough ν we may make μ as long as we like. Therefore, if $B_{R_0^{-1}}$ is the minimal transducer representing $B_{(\epsilon, R_0)}$ and $D_{q_0^{-1}}$ is the minimal transducer representing $D_{(\epsilon, q_0)}$, then $D_{q_0^{-1}}$ is synchronizing and $\mathrm{Core}(D_{q_0^{-1}}) = \mathrm{Core}(B_{R_0^{-1}}$. This is because $\mathcal{OUT}$ is a complete antichain of $X_{n,r}^{*}$ and after processing any word of $\mathcal{OUT}$ from the state (ϵ, q_0), the resulting state is (ϵ, t). Hence, we conclude that $h_{D_{q_0}} \in \mathcal{B}_{n,r}$ as required. ☐

Notice that as part of the proof above we show the following:

Lemma 2.5.11. *Let $T \in \mathcal{O}_{n,r}$ possess a homeomorphism state t, then T_t is a bi-synchronizing transducer, in particular $T_t \in \mathcal{B}_{n,1}$.*

The example which follows is from the author's article [10] and demonstrates that elements of $\mathcal{O}_{n,r}$ need not possess a homeomorphism state:

Example 2.5.12. Consider the element $A_{q_0} \in \mathcal{B}_{4,3}$ below

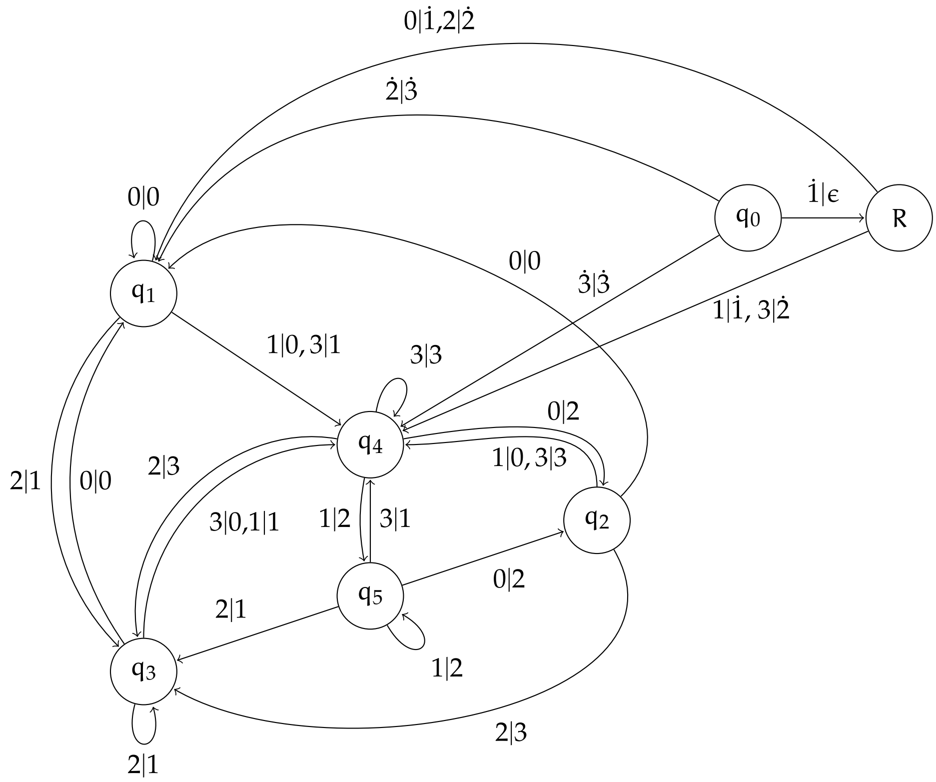


Figure 2.12: An example of an element of $\mathcal{B}_{4,3}$ whose core has no homeomorphism state

One can observe that Core(A_{q_0}) is the subtransducer induced by the state $\{q_1, q_2, q_3, q_4, q_5\}$ none of which are homeomorphism states. That $\{q_1, q_2, q_3, q_4, q_5\}$ is the Core(A_{q_0}) follows since A_{q_0} is synchronizing at level 3. That none of $\{q_1, q_2, q_3, q_4, q_5\}$ are homeomorphisms states follows since the function $\lambda_A(\centerdot, q_a) : X_4 \to X_4$ is not injective for any $1 \leqslant a \leqslant 5$.

On the other extreme, one could ask about elements of $\mathcal{O}_{n,r}$ all of whose states are homeomorphism states. These are characterised by the following lemma from [10]:

Lemma 2.5.13. *Let $T \in \mathcal{O}_{n,r}$ be such that all states of T are homeomorphism states, then $T \in \cap_{1 \leqslant r' < n} \mathcal{O}_{n,r'}$ and T is synchronous.*

Proof. Fix a state q of T, and observe that since all states of T are homeomorphism states then the set $\{\lambda_T(i, q) \mid i \in X_n\}$ must form a complete antichain for X_n^*. However, there is only one complete antichain of X_n^* consisting of n elements and that is precisely the set X_n. Thus, for any state q of T the map $\lambda_T(\centerdot, q)$ with domain X_n, is a bijection from X_n to itself. That $T \in \cap_{1 \leqslant r' \leqslant n} \mathcal{O}_{n,r'}$ follows from Proposition 2.5.10. □

Notation 2.5.14. Let $\mathcal{H}_n$ be the set of minimal, invertible, synchronous, bi-synchronizing transducers over $\mathfrak{C}_n$. From Lemma 2.5.13 and the proof of Lemma 2.5.10, it follows that $\mathcal{H}_n$ is precisely the set of all those elements $T \in \cap_{1 \leqslant r < n} \mathcal{O}_{n,r}$ all of whose states are homeomorphism states. We also denote by $\widetilde{\mathcal{H}}_n$ the set of invertible, synchronous, synchronizing (but not necessarily bi-synchronizing) transducers over $\mathfrak{C}_n$. Note that $\widetilde{\mathcal{H}}_n$ is not a subset of $\mathcal{O}_{n,r}$ for any r since it contains invertible transducers whose inverses are not synchronizing.

The set $\mathcal{H}_n$ is closed under the core product given in Definition 2.5.5, and so it is a subgroup of $\cap_{1\leqslant r<n}\mathcal{O}_{n,r}$ as it is also closed under inversion. The set $\widetilde{\mathcal{H}}_n$ is also closed under the core product, however it is not closed under inversion. The next chapter focuses on the monoid $\widetilde{\mathcal{H}}_n$ and the group $\mathcal{H}_n$. Some of the results in the next chapter also concern the monoid $\widetilde{\mathcal{P}}_n$ consisting of all synchronous, synchronizing transducers over $\mathfrak{C}_n$ with binary operation a modification of the core product. We close this chapter with a few more observations about the group $\mathcal{O}_{n,r}$, some of which may be found in the author's article [10].

Notation 2.5.15. Let $\mathcal{K}_n = \cap_{1\leqslant r<n-1}\mathcal{O}_{n,r}$; in the case $n = 2$ we have, $\mathcal{K}_n = \mathcal{O}_2$. Notice that as $\mathcal{K}_n$ is the intersection of the groups $\mathcal{O}_{n,r}$, with the binary operation core product, it is a subgroup of $\mathcal{O}_{n,r}$ for all $1 \leqslant r < n-1$.

A natural question which arises is the following: do all elements of $\mathcal{K}_n$ possess a homeomorphism state? Notice that for $n = 2$ this reduces to answering the question: do all elements of $\mathcal{O}_2$ possess a homeomorphism state? The example below demonstrates that the answer to this question is no, and shows that, in general, $\mathcal{K}_n$ need not consist only of elements with homeomorphism states. In fact this example also demonstrates that the subset of $\mathcal{K}_n$ of all elements which possess a homeomorphism state is not closed under the binary operation.

Example 2.5.16. The transducers A and B have homeomorphism states p_0 and q_0 respectively, such that A_{p_0} and B_{q_0} are bi-synchronizing. Therefore, by Proposition 2.5.10 and its proof, we have that $A, B \in \mathcal{K}_2$. However, the core product of A and B is a transducer C which, as may be verified either by hand or in GAP with the 'aaa' package, has no homeomorphism state.

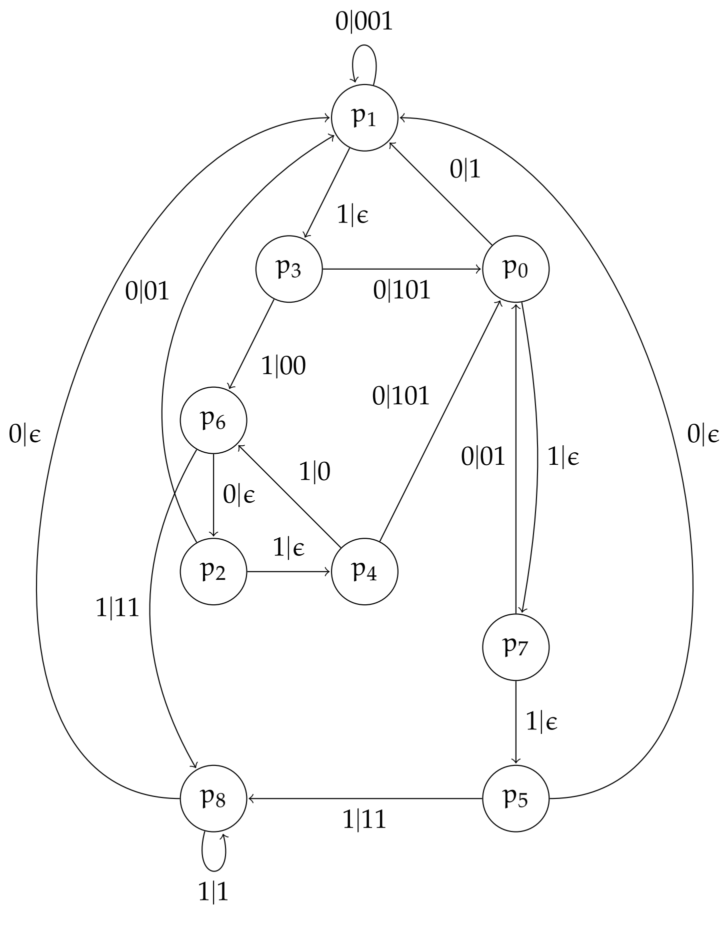


Figure 2.13: The transducer A with homeomorphism state p_0

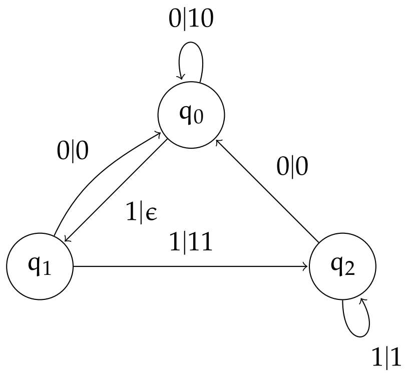


Figure 2.14: The transducer B with homeomorphism state q_0

We close with a few characteristics of elements of $\mathcal{O}_{n,r}$.

Remark 2.5.17. Let $T \in \mathcal{O}_{n,r}$, we have the following:

(1) If there is some state t of T such that T_t is minimal, then T_u is minimal for any other state u of T. This follows from the definition of states of incomplete response (Definition 1.6.3), of ω-equivalent states (Definition 1.6.1), and since T_t is accessible.

(2) For any state t of T, the map h_t is injective and has clopen image. This follows since there is a transducer $A_{q_0} \in \mathcal{B}_{n,r}$ such that $\mathrm{Core}(A_{q_0}) = T$, moreover h_{q_0} is a homeomorphism.

Chapter 3

The monoid $\widetilde{\mathcal{P}}_n$ and the group $\mathcal{H}_n$

This chapter will primarily be concerned with the monoid $\widetilde{\mathcal{P}}_n$, which contains the monoid $\widetilde{\mathcal{H}}_n$ and so the group $\mathcal{H}_n$, consisting of all core, synchronous, weakly minimal (see Definition 1.6.5), synchronizing transducers. The monoid $\widetilde{\mathcal{P}}_n$ has connections to endomorphisms of the shift dynamical system, which has implications for the group $\mathcal{H}_n$. In particular we show that $\widetilde{\mathcal{P}}_n$ is isomorphic to a submonoid F_∞ which together with the shift map generates the full monoid of endomorphisms of the shift dynamical system. The restriction to $\mathcal{H}_n$ of the isomorphism between the monoid $\widetilde{\mathcal{P}}_n$ and the monoid F_∞ gives rise to a group isomorphism between $\mathcal{H}_n$ and the group of automorphisms of the one-sided shift dynamical system. Therefore we study the group of automorphisms of the one-sided shift dynamical system by studying the group $\mathcal{H}_n$. This result is similar in spirit to the results of the paper [42] which shows that the group of automorphisms of the one-sided shift is isomorphic to a group of transducers.

This chapter explores the conjugacy and the order problem in $\mathcal{H}_n$. We present, in the context of $\mathcal{H}_n$ certain invariants of conjugacy that were already known for the automorphisms of the one-sided shift. We also describe how to construct potential or candidate conjugators and conjecture a solution to the conjugacy problem in $\mathcal{H}_n$. We apply these results to show that there are elements of $\mathcal{P}_n$ which are not conjugate to their inverses in $\mathcal{P}_n$.

The bulk of this chapter though, shall be devoted to the order problem and is based on an article of the author's ([44]). We associate to each element of $\mathcal{H}_n$ an infinite family of finite graphs. We show that if an element of $\mathcal{H}_n$ has finite order, then these graphs are eventually empty, whereas if any one of the graphs associated to an element of $\mathcal{H}_n$ has a circuit, then the element has infinite order. This therefore yields a sufficient condition for an element of $\mathcal{H}_n$ to have infinite order. We conjecture that this condition is also necessary. We also obtain results about the 'dual transducer' (this is defined in Section 3.5.3) of elements of $\mathcal{H}_n$ which have finite order. More specifically we study the semigroup consisting of powers of the dual transducer. We show that there is a natural number m such that the m^{th} power of the dual transducer of an element of $\mathcal{H}_n$ is the zero of this semigroup if and only if it is an element of finite order. It is a conjecture of Picantin in [40] that m is always equal to the number of states of the transducer minus 1. We verify Picantin's conjecture for all elements of $\mathcal{H}_n$ with only two states. By making use of a construction of Delacourt and Ollinger, we construct examples of elements of $\mathcal{H}_n$ which have finite order and such that the minimal value m for which the m^{th} power of the dual transducer is the zero of the semigroup consisting of powers of the dual, is precisely the number of states of the transducer minus 1.

As it turns out, the order problem in $\mathcal{H}_n$ is intimately connected to the question of the growth rate of certain groups of homeomorphisms of Cantor space associated to transducers in $\mathcal{H}_n$. Given an element $h \in \mathcal{H}_n$ we denote by $\mathcal{G}(h)$ the group of homeomorphisms of Cantor space associated to h. We prove that the order problem in $\mathcal{H}_n$ is equivalent to the problem of finding an algorithm which, given an element $h \in H$, determines in a finite time if the group $\mathcal{G}(h)$ is finite. Thus, we survey what was already known about the groups $\mathcal{G}(h)$ for $h \in \mathcal{H}_n$ and present results of the author's article [44]. More specifically, we present results of Silva and Steinberg ([51]) showing that the groups obtained from elements of $\mathcal{H}_n$ are always finitely generated elementary amenable groups. This together with results of Chou ([21]) and Rosset ([46]) shows that whenever the group obtained from an element of $\mathcal{H}_n$ is infinite, then it has a free subsemigroup of rank at 2. We give a different proof of this result by showing that whenever a graph in the infinite family of finite graphs associated to an element of $\mathcal{H}_n$ has a circuit, then the group associated to this element contains a free subsemigroup. Thus whenever the group $\mathcal{G}(h)$ for an element $h \in \mathcal{H}_n$ is infinite, it has exponential growth rate.

We then consider the question of how the number of states of an element $h \in \mathcal{H}_n$ grows under powers, we call this the *core growth rate*. We prove some elementary results about the core growth rate, showing amongst other things that it is invariant under taking powers and conjugation. We demonstrate that for every $n > 2$ there is an element of $\mathcal{H}_n$ with exponential core growth rate. We conjecture that it is in fact the case that every element of $\mathcal{H}_n$ which has infinite order, has exponential core growth rate.

Interspersed throughout the chapter will be certain results which do not fit under the umbrella of 'conjugacy' or 'order problem' but which naturally arise in our study of the monoid $\widetilde{\mathcal{P}}_n$. For instance we show that if $m = nd$, for $m, n, d \in \mathbb{N}_2$ then the direct sum $\mathcal{H}_n^d$ embeds as a subgroup of $\mathcal{H}_m$.

We begin by demonstrating the connection between the monoid $\widetilde{\mathcal{P}}_n$ and the endomorphisms of the one-sided and two-sided shift dynamical system. Throughout this chapter n shall be an element of $\mathbb{N}_2$.

3.1 The monoid $\widetilde{\mathcal{P}}_n$ and the group of automorphisms of the one-sided shift dynamical system

In this section we establish connections between the monoid $\widetilde{\mathcal{P}}_n$ and the group of automorphisms of the one-sided shift. The exposition here is partly based on the article [33] and the forthcoming article [6] of the author's.

We begin with a definition of the monoid $\widetilde{\mathcal{P}}_n$.

Definition 3.1.1. Let $\widetilde{\mathcal{P}}_n$ denote the set of synchronous weakly minimal synchronizing transducers. Define a product on $\widetilde{\mathcal{P}}_n$ as follows, for $T, U \in \widetilde{\mathcal{P}}_n$ set UT to be the weakly minimal transducer representing $\mathrm{Core}(U * T)$. Thus UT is again in $\widetilde{\mathcal{P}}_n$, by Theorem 2.1.32, and the set $\widetilde{\mathcal{P}}_n$ together with this product forms a monoid.

Remark 3.1.2. Observe that as elements of $\widetilde{\mathcal{H}}_n$ which are weakly minimal, are also minimal, then it follows that the product defined above for element of $\widetilde{\mathcal{P}}_n$ and the 'core product' of the previous section coincide on $\widetilde{\mathcal{H}}_n$ and so $\widetilde{\mathcal{H}}_n$ is a submonoid of $\widetilde{\mathcal{P}}_n$.

Below we define the one-sided and two-sided shift dynamical system. Recall (1.1.11) that the symbols $X_n^{\mathbb{Z}}$ and X_n^{ω} denote, respectively, the set of infinite and bi-infinite sequences over the alphabet X_n. Recall (Section 1.4) we define metrics d_n and d_∞ on the spaces X_n^{ω} and $X_n^{\mathbb{Z}}$ respectively, making each of these homeomorphic to Cantor space.

Notation 3.1.3. Let $k \in \mathbb{N}$ and $\nu \in X_n^{2k+1}$, set $U_\nu^0 := \{y \in X_n^{\mathbb{Z}} \mid y_{-k} \ldots y_{-1}y_0y_1 \ldots y_k = \nu\}$. For $\nu \in X_n^{2k+1}$, U_ν^0 is clopen, moreover the set $\cup_{k \in \mathbb{N}}\{U_\nu \mid \nu \in X_n^{2k+1}\}$ is a basis for the topology on $X_n^{\mathbb{Z}}$ induced by the metric d_∞.

Definition 3.1.4. Define a map $\sigma_n : X_n^{\omega} \sqcup X_n^{\mathbb{Z}} \to X_n^{\omega} \sqcup X_n^{\mathbb{Z}}$ by $x \mapsto y$ where y is uniquely defined by the rule $y_i = x_{i+1}$ for all $i \in \mathbb{N}$ or $i \in \mathbb{Z}$ as appropriate. Observe that $\sigma_n\restriction_{X_n^{\omega}} : X_n^{\omega} \to X_n^{\omega}$ is surjective but not injective, and $\sigma_n\restriction_{X_n^{\mathbb{Z}}} : X_n^{\mathbb{Z}} \to X_n^{\mathbb{Z}}$ is a bijection. We shall denote by σ_n the restrictions $\sigma_n\restriction_{X_n^{\omega}}$ and $\sigma_n\restriction_{X_n^{\mathbb{Z}}}$ as it will be clear from the context which is meant. The map σ_n is called the *shift on* n *letters* or simply as the *shift (map)* when the cardinality of the alphabet is clear.

Remark 3.1.5. Observe that σ_n is a continuous map on X_n^{ω} and $X_n^{\mathbb{Z}}$. Moreover the inverse of σ_n on $X_n^{\mathbb{Z}}$, we denote this map by σ_n^{-1}, is also continuous on $X_n^{\mathbb{Z}}$. Thus σ_n is a homeomorphism of $X_n^{\mathbb{Z}}$.

Definition 3.1.6. We refer to the pair (X_n^{ω}, σ_n) as the *full one-sided shift dynamical system on* n *letters* or the *full one-sided shift dynamical system*, and the pair $(X_n^{\mathbb{Z}}, \sigma_m)$ as the *full (two-sided) shift dynamical system on* n *letters* or the *full (two-sided) shift-dynamical system*.

Remark 3.1.7. There is a notion of sub-shifts (see for instance [12]) in the literature, hence the use of the word 'full' above. However, as we are only concerned with full shifts in this work, we omit the word full in subsequent discussion. We will also refer to the two-sided shift dynamical system simply as the shift-dynamical system.

Definition 3.1.8. We denote by $\mathrm{Aut}(X_n^{\mathbb{Z}}, \sigma_n)$ the group of all homeomorphisms of $X_n^{\mathbb{Z}}$ which commute with σ_n i.e the centraliser of σ_n in $H(X_n^{\mathbb{Z}})$. We denote by $\mathrm{End}(X_n^{\mathbb{Z}}, \sigma_n)$ the monoid consisting of all continuous functions, ϕ from $X_n^{\mathbb{Z}}$ to itself which commute with the shift map. Analogously, denote by $\mathrm{Aut}(X_n^{\omega}, \sigma_n)$ the group of all homeomorphisms of X_n^{ω} which commute with the shift map. The group $\mathrm{Aut}(X_n^{\mathbb{Z}}, \sigma_n)$ is called the *automorphisms of the shift dynamical system* and the group $\mathrm{Aut}(X_n^{\omega}, \sigma_n)$ is called the *automorphisms of the one-sided shift dynamical system*. The monoid $\mathrm{End}(X_n^{\mathbb{Z}}, \sigma_n)$ is called the *endomorphisms of the shift dynamical system*.

The groups of automorphisms of the one and two-sided shift dynamical system are important and well studied groups in symbolic dynamics. It is a result of Hedlund [33] that for $n \geqslant 2$, the group $\mathrm{Aut}(X_n^{\mathbb{Z}}, \sigma_n)$ has a subgroup isomorphic to any finite group. In fact the paper [37] demonstrates that for any $n, m \in \mathbb{N}_2$, $\mathrm{Aut}(X_n^{\mathbb{Z}}, \sigma_n)$ is a subgroup of $\mathrm{Aut}(X_m^{\mathbb{Z}}, \sigma_m)$ and $\mathrm{Aut}(X_n^{\mathbb{Z}}, \sigma_n)$ has a subgroup isomorphic to any countable, locally finite, residually finite group. For $n \geqslant 3$, $\mathrm{Aut}(X_n^{\omega}, \sigma_n)$ contains free groups [12], while for $n = 2$ $\mathrm{Aut}(X_n^{\omega}, \sigma_n) \cong \mathbb{Z}/2\mathbb{Z}$ [33]. Here $\mathbb{Z}/2\mathbb{Z}$ is the cyclic group of order 2.

The paper [30] demonstrates that the rational group $\mathcal{R}_2$ contains subgroups isomorphic to $\mathrm{Aut}(X_n^{\mathbb{Z}}, \sigma_n)$ and $\mathrm{Aut}(X_n^{\omega}, \sigma_n)$ for any $n \in \mathbb{N}_2$. Here we demonstrate how elements of $\mathrm{Aut}(X_n^{\mathbb{Z}}, \sigma_n)$ and $\mathrm{Aut}(X_n^{\omega}, \sigma_n)$ may be represented by non-initial transducers in $\widetilde{\mathcal{P}}_n$. Notice that this is distinct from, but related to, the embeddings of the one-sided and two-sided shift dynamical system in $\mathcal{R}_2$, as such homeomorphisms in $\mathcal{R}_2$ are represented by initial transducers. However, we shall require first a fundamental result of Hedlund, (demonstrated independently by Curtis and Lyndon) characterising elements of $\mathrm{Aut}(X_n^{\mathbb{Z}}, \sigma_n)$ and $\mathrm{Aut}(X_n^{\omega}, \sigma_n)$ by so called 'block maps'.

3.2 The Curtis, Hedlund, Lyndon theorem

In this section we present the Curtis, Hedlund, Lyndon theorem, characterising elements of $\mathrm{Aut}(X_n^{\mathbb{Z}}, \sigma_n)$ and $\mathrm{Aut}(X_n^{\omega}, \sigma_n)$ by easy to understand combinatorial data.

Notation 3.2.1. For $m \in \mathbb{N}_1$ denote by $F(X_n, m)$ the set of maps $f : X_n^m \to X_n$. A map $f \in F(X_n, m)$ will be called a *block map*.

Given a block map we may obtain an endomorphisms of the shift dynamical system.

Definition 3.2.2. Let $m \in \mathbb{N}_1$ and $f \in F(X_n, m)$. Define a map $f_\infty : X_n^{\mathbb{Z}} \to X_n^{\mathbb{Z}}$ as follows: for $x \in X_n^{\mathbb{Z}}$, $(x)f_\infty$ is the element $y \in X_n^{\mathbb{Z}}$ satisfying $y_i := (x_i x_{i+1} \ldots x_{i+m-1})f$ for all $i \in \mathbb{Z}$. In a similar way we define a map also denoted f_∞ from X_n^{ω} to itself by $x \in X_n^{\omega}$ maps to the unique $y \in X_n^{\omega}$ satisfying $y_i = (x_i x_{i+1} \ldots x_{i+m-1})f$.

Remark 3.2.3. One should think of a block map $f \in F(X_n, m)$ as a sliding window of width m which processes a string in $X_n^{\mathbb{Z}} \sqcup X_n^{\omega}$ by moving from right to left (i.e from $+\infty$ to $-\infty$ or 0), and changes the entry at the left-most point of the window according to the information that can be seen. Negative powers of the shift map may be thought of as providing access to 'future' information i.e information to the left of the current index.

We have the following:

Proposition 3.2.4 ([33]). *Let $m \in \mathbb{N}_1$ and $f \in F(X_n, m)$, then $f_\infty \in End(X_n^{\mathbb{Z}}, \sigma_n)$ and $f_\infty \in End(X_n^{\omega}, \sigma_n)$.*

Proof. Certainly, for $m \in \mathbb{N}_1$ and a given $f \in F(X_n, m)$, the map f_∞ is continuous on X_n^{ω} and $X_n^{\mathbb{Z}}$. It therefore suffices to show that f_∞ commutes with the shift map σ_n.

Let $x \in X_n^{\mathbb{Z}}$ and let $y = (x)\sigma_n$. Notice that $y_i = x_{i+1}$ for $i \in \mathbb{Z}$. Let $z = (x)f_\infty\sigma_n$ and let $t = (y)f_\infty$. Then $z_i = ((x)f)_{i+1} = (x_{i+1} \ldots x_{i+m})f = (y_i \ldots y_{i+m-1})f = t_i$ for all $i \in \mathbb{Z}$. Thus $t = z$ and $f_\infty\sigma_n = \sigma_n f_\infty$ as required.

An analogous argument shows that for any $x \in X_n^{\omega}$, $(x)f_\infty\sigma_n = (x)\sigma_n f_\infty$. □

Notation 3.2.5. Set $F_\infty = \{f_\infty \mid f \in F(X_n, m) \text{ for some } m \in \mathbb{N}_1\}$.

We have the following result:

Theorem 3.2.6 (Curtis, Hedlund, Lyndon)**.** *The following equalities are valid: End*$(X_n^{\mathbb{Z}}, \sigma_n) = \langle \sigma_n, F_\infty \rangle$ *and End*$(X_n^{\omega}, \sigma_n) = F_\infty$.

Proof. We begin with the equality $\mathrm{End}(X_n^{\mathbb{Z}}, \sigma_n) = \langle \sigma_n, F_\infty \rangle$. Let $\phi : X_n^{\mathbb{Z}} \to X_n^{\mathbb{Z}}$ be an element of $\mathrm{End}(X_n^{\mathbb{Z}})$ and $V_i = (U_i^0)\phi^{-1}$ for $i \in X_n$. For distinct $i, j \in X_n$, since $U_i^0 \cap U_j^0 = \emptyset$, then $V_i \cap V_j = \emptyset$. Therefore, as ϕ is continuous and $\{U_i^0 \mid i \in X_n\}$ is a clopen cover of $X_n^{\mathbb{Z}}$, the V_i's $i \in X_n\}$ are also pairwise disjoint and a clopen cover for $X_n^{\mathbb{Z}}$.

Now as the V_i's, for $i \in X_n$, are pairwise disjoint and clopen, there is a minimal $k \in \mathbb{N}$ such that, for distinct $i, j \in X_n$ and any pair $x \in V_i$ and $y \in V_j$, $d_\infty(x, y) \geqslant 1/(k+1)$. This follows since as each V_i is clopen there is a minimal $k_i \in \mathbb{N}$ and a subset $\mathscr{P}(V_i) \subseteq X_n^{2k_i+1}$ such that $\cup_{\nu \in \mathscr{P}(V_i)} U_\nu^0 = V_i$. We may thus set $k = \max_{i \in X_n} k_i$. Moreover, observe that the set $\mathscr{P} = \cup_{i \in X_n} \mathscr{P}(V_i)$ is a partition of $X_n^{2k_i+1}$ since the V_i are a clopen cover of $X_n^{\mathbb{Z}}$.

Define a map $f : X_n^{2k+1} \to X_n$ by $\nu \mapsto i$ if and only if $U_\nu^0 \subset V_i$. By the assumptions on k above f is well-defined since for $\nu \in X_n^{2k+1}$ there is a unique $i \in X_n$ such that $U_\nu^0 \in V_i$. We now show that $\sigma_n^{-k} f_\infty = \phi$.

Let $x \in X_n^{\mathbb{Z}}$, then $x \in V_i$ for some $i \in X_n$ and so there is some $\nu \in X_n^{2k+1}$ such that $x \in U_\nu^0$. Let $y = (x)\sigma_n^{-k}$. Observe that $y_0 \ldots y_{2k} = \nu$. Let $z = (y)f_\infty$ and let $t = (x)\phi$. Then we have that $i = z_0 = t_0$ by definition of f_∞.

Now we use the facts that both f_∞ and ϕ commute with the shift. Let $m \in \mathbb{Z}$ be arbitrary. Let $x' = (x)\sigma_n^{-m}$, $y' = (x')\sigma_n^{-k}$, $z' = (y')f_\infty$ and $t' = (x')\phi$. Observe that $z_m = z'_0$ and $t_m = t'_0$ since $(y')f_\infty = (y)f_\infty\sigma_n^m$ and $(x')\phi = (x)\phi\sigma_n^m$. However, by the computation in the paragraph above we have, $z'_0 = t'_0$ and so $z_m = t_m$. Since $m \in \mathbb{Z}$ was arbitrary, we conclude that $z = t$. Hence $\sigma_n^{-k} f_\infty = \phi$.

We free all the symbols used above.

For the equality $\mathrm{End}(X_n^{\omega}, \sigma_n) = F_\infty$ we proceed in a similar way. Let $\phi \in \mathrm{End}(X_n^{\omega}, \sigma_n)$. We set $V_i = (U_i)\phi^{-1}$ (recall the definition of U_i from Notation 1.4.6). Since the V_i for $i \in X_n$ are clopen and pairwise disjoint, then, as above, there is a minimal $k \in \mathbb{N}$ such that for distinct $i, j \in X_n$ and any pair $x \in V_i$ and $y \in V_j$, we have, $d(x, y) \geqslant 1/(k+1)$. Define $f : X_n^k \to X_n$ by $\nu \mapsto i$ if and only if $U_\nu \subseteq V_i$. As in the previous case, f is well-defined.

Let $x \in X_n^{\mathbb{Z}}$. There is a unique $i \in X_n$ such that $x \in V_i$, therefore if $t = (x)\phi$ and $z = (x)f_\infty$ then $t_0 = z_0 = i$. Again, we now use the fact that f_∞ and ϕ commute with the shift, to conclude that $t = z$. □

Remark 3.2.7.

1. Theorem 3.2.6 demonstrates that σ_n is equal to f_∞ for some $f \in F(X_n, m)$ where $m \in \mathbb{N}_1$.
2. If an element ϕ of $\mathrm{End}(X_n^{\mathbb{Z}}, \sigma_n)$ or $\mathrm{End}(X_n^{\omega}, \sigma_n)$ is given by 'finite data' e.g. it acts only on words of a fixed length occurring between certain fixed markers, as in the marker automorphisms which occur for instance in [33] and [14], then the proof of Theorem 3.2.6 gives an algorithm for computing k and the map $f \in F(X_n, 2k+1)$ such that $\phi = \sigma^{-k} f_\infty$. This is because we may build a finite clopen cover, which we identify with words in X_n^*, for the sets V_i by considering how the map ϕ acts on elements $x_{-m} \ldots x_0 \ldots x_m$ for larger and larger m. The process then terminates as soon as we find pairwise disjoint clopen covers for the V_i. Using these covers the value of k and the map $f \in F(X_n, 2k+1)$ can be computed.

Definition 3.2.8. A block map $f \in F(X_n, m)$ is called *left permutive [right permutive]* if, for any fixed $\Gamma \in X_n^{m-1}$, the map from $X_n \to X_n$ given by $i \mapsto (i\Gamma)f$ $[(\Gamma i)f]$ is a permutation. A block map $f \in F(X_n, m)$ which is both left and right permutive will be called *permutive*.

Remark 3.2.9. An element of $\phi \in F_\infty$ which is an element of $\mathrm{Aut}(X_n^{\omega}, \sigma_n)$ must be left permutive otherwise it is not injective. As we shall see later on, $\mathrm{Aut}(X_n^{\omega}, \sigma_n)$ does not coincide with the set of elements f_∞, where $f \in F(X_n, m)$, for some $m \in \mathbb{N}_1$, is a left permutive block map.

In the next section we show how elements of $\widetilde{\mathcal{P}}_n$ can be constructed from elements of $F(X_n, m)$.

3.3 From endomorphisms of the shift dynamical system to elements of $\widetilde{\mathcal{P}}_n$ via foldings of De Bruijn graphs

In this section we construct elements of $\widetilde{\mathcal{P}}_n$ from elements of $F(X_n, m)$. We then demonstrate that this yields an isomorphism from the monoid F_∞ to the monoid $\widetilde{\mathcal{P}}_n$. From this we deduce that the group $\mathcal{H}_n$ coincides with the group $\mathrm{Aut}(X_n^\omega, \sigma_n)$. This link between the monoid F_∞ and the monoid $\widetilde{\mathcal{P}}_n$ is achieved by foldings of De Bruijn graphs. The exposition in this section is based on the forthcoming article ([6]) of the author's.

We begin with a definition of the *de Bruijn graph* $G(n, m)$.

Definition 3.3.1. The *de Bruijn* graph $B(n, m)$, for $m \in \mathbb{N}_1$, is an automaton $\langle X_n, X_n^m, \pi\rangle$ such that for $i \in X_n$ and $\Gamma \in X_n^m$, we have, $\pi(i, \Gamma) = \Gamma'$ if and only if $\Gamma' = i\overline{\Gamma}$ where $\overline{\Gamma}$ is the length $m-1$ prefix of Γ'. Set $B(n, 0)$ to be the single state automaton over X_n.

Remark 3.3.2. Note that the automaton $\langle X_n, X_n^m, \pi'\rangle$ where π' satisfies, for $\Gamma \in X_n^m$, $\pi'(i, \Gamma) = \overline{\Gamma}i$ for $\overline{\Gamma}$ the length $m-1$ suffix of Γ, is isomorphic to the de Bruijn graph $B(n, m)$. However, for our purposes, it is more convenient to work with Definition 3.3.1.

Further observe that the de Bruijn graph $B(n, m)$ is synchronizing with minimal synchronizing level m since after reading a word $\gamma_1\gamma_2\ldots\gamma_m$ of length m, the resulting state of the automaton $B(n, m)$ is the state $\gamma_m\gamma_{m-1}\ldots\gamma_1$. Clearly $B(n, 0)$ is synchronizing at level 0 as it only has one state.

Notation 3.3.3. As the remark above demonstrates, for a word $\Gamma := \gamma_1\ldots\gamma_m \in X_n^m$ it is useful to have notation for the reversed word $\gamma_m\ldots\gamma_1$, and so we denote this word with the symbol $\overleftarrow{\Gamma}$.

Example 3.3.4. We depict below the de Bruijn graph $B(2, 2)$.

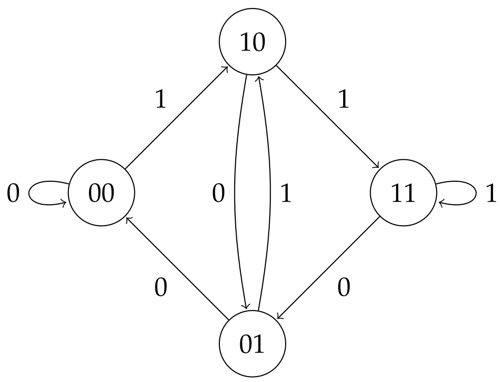
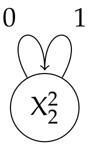


Figure 3.1: The de Bruijn graph $B(2, 2)$

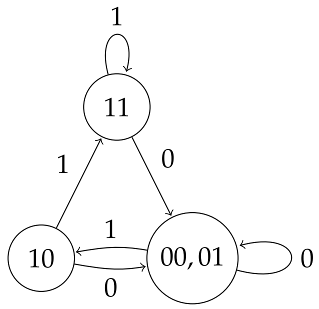
Definition 3.3.5. Consider the de Bruijn graph $B(n, m) = \langle X_n, X_n^m, \pi\rangle$. Let Q be a partition of X_n^m such that for $q \in Q$, and $i \in X_n$, there is a unique $q' \in Q$ such that for any pair $\Gamma_1, \Gamma_2 \in q$, $\pi(i, \Gamma_1), \pi(i, \Gamma_2) \in q'$. Define a map $\pi' : X_n \times Q \to Q$ by $\pi'(i, q) = q'$ where q' is the unique element of Q such that for all $\Gamma \in q$, $\pi(i, \Gamma) \in q'$. The automaton $A = \langle X_n, Q, \pi'\rangle$ is called a *folding* of the de Bruijn graph $B(n, m)$. More generally, an automaton isomorphic to a folding of $B(n, m)$ is also called a *folding* of $B(n, m)$.

Remark 3.3.6. Observe that if an automaton A is a folding of a de Bruijn graph $B(n, m)$ then, as a consequence of how the transition function of the foldings are defined, A has minimal synchronizing level at most m.

Example 3.3.7. Below we depict the foldings of $B(2, 2)$ corresponding to the partitions $\{X_2^2\}$ and $\{\{00, 01\}, \{11\}, \{10\}\}$.



(a)
Folding of $B(2,2)$ corresponding to the partition $\{X_2^2\}$



(b) Folding of $B(2,2)$ corresponding to the partition $\{\{00, 01, \{11\}, \{10\}\}$

Figure 3.2: Foldings of $B(2,2)$.

We have the following proposition relating synchronizing automata and foldings of de Bruijn graphs. The proposition below essentially states that the de Brujin graph $B(n, m)$ is universal for the property of being synchronizing at level m.

Proposition 3.3.8. *An automaton* $A = \langle X_n, Q_A, \pi_A \rangle$ *is synchronizing at level* m *if and only if* A *is a folding of the de Bruijn graph* $B(n, m)$.

Proof. The reverse implication is discussed in Remark 3.3.6.

Thus suppose that A is synchronizing at level m. For a state q of A let $W_q(m)$ denote the set of words $\Gamma \in X_n^m$ such that $\pi_A(\Gamma, \cdot) : Q_A \to Q_A$ has image set $\{q\}$. Since A is synchronizing at level m, the set $\{W_q(m) \mid q \in Q_A\}$ is a partition of X_n^m. Let $\overleftarrow{W}_q(m) = \{\overleftarrow{\Gamma} \mid \Gamma \in W_q(m)\}$, and set $Q' := \{\overleftarrow{W}_q(m) \mid q \in Q\}$, then, since the map $\leftarrow : X_n^m \to X_n^m$ by $\Gamma \mapsto \overleftarrow{\Gamma}$ is a bijection, Q' is also a partition of X_n^m.

Let $\Gamma \in W_q(m)$ and suppose that for $i \in X_n$ $\pi_A(i, q) = p$. Since A is synchronizing at level m, it follows that, for all $\Gamma := \gamma_1 \ldots \gamma_m \in W_q(m)$, $\gamma_2 \ldots \gamma_m i \in W_p(m)$. Thus for all $\Gamma := \gamma_1 \ldots \gamma_m \in W_q(m)$, $i\gamma_m \ldots \gamma_2 \in \overleftarrow{W}_p$. As i and q were arbitrary, it follows that, for any $q \in Q$, a given $i \in X_n$, for $p = \pi_A(i, q)$ and any pair $\Delta_1, \Delta_2 \in \overleftarrow{W}_q(m)$ we have, $i\overline{\Delta}_1, i\overline{\Delta}_2 \in \overleftarrow{W}_p(m)$ where, for $a \in \{1, 2\}$, $\overline{\Delta}_a$ is the length $m-1$ prefix of Δ_a.

Let $\pi' : X_n \times Q' \to Q'$ be defined by $\pi'(i, \overleftarrow{W}_q(m)) = \overleftarrow{W}_p(m)$ if and only if $p = \pi_A(i, q)$. The paragraph above demonstrates that the automaton $B = \langle X_n, Q', \pi' \rangle$ is a folding of $B(n, m) = \langle X_n, X_n^m, \pi \rangle$ since for any pair $\Gamma_1, \Gamma_2 \in \overleftarrow{W}_q(m)$, $i \in X_n$, $p = \pi_A(i, q)$, and for $a \in \{1, 2\}$, $\pi_A(i, \Gamma_a) = i\overline{\Gamma}_a \in W_p$ (again $\overline{\Gamma}_a$ denotes the length $m-1$ prefix of Γ_a).

To conclude, we observe that the map $\phi : Q' \to Q$ by $\overleftarrow{W}_q(m) \mapsto q$ is an isomorphism from B to A, by the definition of π'. $\square$

The de Bruijn graph $B(n, m)$ gives us a means of realising a 'window of length m' (see Remark 3.2.3) however, we also need to know how to transform the entry at the 'left most point of the window'. This information is provided by a block map $f \in F(X_n, m)$.

Construction 3.3.9 (From block maps to transducers)**.** Let $f \in F(X_n, m)$ be a block map. We construct a transducer $A_f = \langle X_n, X_n^{m-1}, \pi_f, \lambda_f \rangle$ as follows: the automaton $\mathcal{A}(A_f) = \langle X_n, X_n^{m-1}, \pi_f \rangle := B(n, m-1)$; the output function λ_f satisfies, for $i \in X_n$ and $\Gamma \in X_n^{m-1}$, $\lambda_f(i, \Gamma) = (i\Gamma)f$. Set T_f to be the weakly minimal transducer representing A_f. Observe that $\mathcal{A}(T_f)$ is a folding of $B(n, m-1)$ since T_f is synchronizing at level $m-1$.

Below we compute the transducer T_f arising from a map $f : X_n^2 \to X_n$ such that $f_\infty = \sigma_n$.

Example 3.3.10. We begin with the 2 letter case. Let $f : X_2 \to X_2$ be given by $(ij)f = j$ for $i, j \in X_2$ and observe that $f_\infty = \sigma_2$. We construct the transducer A_f as in Construction 3.3.9.

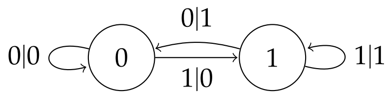


Figure 3.3: Element of $\widetilde{\mathcal{P}}_n$ corresponding to σ_2

Observe that A_f possesses no pair of ω-equivalent states and so $A_f = T_f$.

Now we consider the general case. As before let $f : X_n^2 \to X_n$ be defined by $(ij)f = j$ for $i, j \in X_n$. The transducer $T_f = A_f$ has precisely n states corresponding to the elements of X_n, since, for fixed $j \in X_n$ and any $i \in X_n$, we have $\lambda_f(i, j) = j$ and $\pi_f(i, j) = i$. Thus A_f has no pair of ω-equivalent states, and every state j of A_f, $j \in X_n$, induces a map $\lambda_f(\centerdot, j) : X_n \to X_n$ which takes only the value j.

The proposition below demonstrates that for an element $T \in \widetilde{\mathcal{P}}_n$ there is an $m \in \mathbb{N}$ and a block map $f \in F(X_n, m)$ such that $T_f = T$.

Proposition 3.3.11. *Let $T \in \widetilde{\mathcal{P}}_n$, and suppose T is synchronizing at level m for some $m \in \mathbb{N}$, then there is a block map $f_T : X_n^{m+1} \to X_n$ such that $T_{f_T} = T$.*

Proof. Let $T \in \widetilde{\mathcal{P}}_n$ be synchronizing at level m. Let $f_T : X_n^{m+1} \to X_n$ be given as follows, for $\Gamma \in X_n^m$, let q be the state of T forced by $\overleftarrow{\Gamma}$, then for $i \in X_n$, set $(i\Gamma)f_T = \lambda_T(i, q)$. Now form the non-(weakly) minimal transducer A_{f_T} as in Construction 3.3.9. We show that T_{f_T}, the weakly minimal transducer representing A_{f_T}, is isomorphic to T.

For a state q of T, let $W_q(m)$ denote the set of words $\Gamma \in X_n^m$ for which the state of T forced by Γ is q. Let $\overleftarrow{W}_q(m) = \{\overleftarrow{\Gamma} \mid \Gamma \in W_q(m)\}$. Observe that for a given $q \in T$ the states of A_{f_T} corresponding to words in $\overleftarrow{W}_q(m)$ are ω-equivalent. This follows since for any $i \in X_n$ and any $\Delta \in \overleftarrow{W}_q(m)$ we have, $\lambda_{f_T}(i, \Delta) = \lambda_T(i, q)$ (see Construction 3.3.9). Moreover $\pi_{f_T}(i, \Delta) = i\overline{\Delta}$ for $\overline{\Delta}$ the length $m-1$ suffix of Δ, however $\overleftarrow{i\overline{\Delta}} \in W_{\pi_T(i,q)}(m)$. Therefore for any $\Delta \in \overleftarrow{W}_q(m)$, $\pi_{f_T}(i, \Delta) \in \overleftarrow{W}_{\pi_T(i,q)}(m)$. By induction we therefore conclude that all the states of A_{f_T} corresponding to $\overleftarrow{W}_q(m)$ are ω-equivalent and are in fact ω-equivalent to the state q of T. Since $T \in \widetilde{\mathcal{P}}_n$, then T is weakly minimal, and we therefore conclude that $T_{f_T} \cong_\omega T$. $\square$

We have the following proposition relating left permutive block maps to the submonoid $\widetilde{\mathcal{H}}_n$ of $\widetilde{\mathcal{P}}_n$.

Proposition 3.3.12. *Let $T \in \widetilde{\mathcal{H}}_n$, and let $f_T \in F(X_n, m)$ be a block map such that $T_{f_T} \cong_\omega T$, then f_T is left permutive. Moreover, if $f \in F(X_n, m)$ is left permutive, then $T_f \in \widetilde{\mathcal{H}}_n$.*

Proof. Let $T \in \widetilde{\mathcal{H}}_n$ and $q \in Q_T$. Observe that $\lambda_T(\centerdot, q) : X_n \to X_n$ is a permutation. Let $m \in \mathbb{N}$ and $f_T \in F(X_n, m)$ be such that $T_{f_T} = T$. If f_T is not left permutive then there are $i, j \in X_n$ and $\Gamma \in X_n^m$ such that $(i\Gamma)f_T = (j\Gamma)f_T$. By Construction 3.3.9, it follows that the state Γ of A_{f_T} does not induce a permutation from X_n to itself. If T_{f_t} is the weakly minimal transducer representing A_{f_T} we therefore have that $T_{f_T} \not\cong_\omega T$ since $T_{f_t} \notin \widetilde{\mathcal{H}}_n$.

Now suppose that $f \in F(X_n, m)$ is a left permutive block map for some $m \in \mathbb{N}$. By definition of the transition function of A_f in Construction 3.3.9, every state of A_f induces a permutation from X_n to X_n. Therefore if T_f is the weakly minimal transducer representing A_f, then $T_f \in \widetilde{\mathcal{H}}_n$. $\square$

Given $T \in \widetilde{\mathcal{P}}_n$, Proposition 3.3.11 guarantees that there is an $m \in \mathbb{N}$ and a block map $f_T \in F(X_n, m)$ corresponding to T (and vice versa by Construction 3.3.9). This enables us to define an action of T on $X_n^{\mathbb{Z}}$.

Definition 3.3.13. Let $T \in \widetilde{\mathcal{P}}_n$ be synchronizing at level m. Define a map, which we also denote by T, from $X_n^{\mathbb{Z}}$ to itself by, $x \mapsto y$ where $y \in X_n^{\mathbb{Z}}$ is uniquely defined by the rule $y_i = \lambda_T(i, q_{x_{i-1} \ldots x_{i-m}})$ (recall Notation 2.1.9). Whenever it is unclear from the context that we are thinking of an element $T \in \widetilde{\mathcal{P}}_n$ as a map on $X_n^{\mathbb{Z}}$, we shall denote the induced map by h_T.

Remark 3.3.14. Let $T \in \widetilde{\mathcal{P}}_n$ be synchronizing at level m. Since $f_T \in F(X_n, m+1)$ is defined, for $i \in X_n$ and $\Gamma \in X_n^m$, by $(i\Gamma)f_T = \lambda_T(i, q_{\overleftarrow{\Gamma}})$, it follows, by definition of the map $T : X_n^{\mathbb{Z}} \to X_n^{\mathbb{Z}}$ and the map $(f_T)_\infty : X_n^{\mathbb{Z}} \to X_n^{\mathbb{Z}}$, that T and $(f_T)_\infty$ are equal. Therefore every element $T \in \widetilde{\mathcal{P}}_n$ induces an endomorphism of the shift, moreover, every element of F_∞, by Construction 3.3.9, corresponds to an element of $\widetilde{\mathcal{P}}_n$.

The following Proposition demonstrates that the monoids F_∞ and $\widetilde{\mathcal{P}}_n$ are isomorphic.

Proposition 3.3.15. *The monoid $\widetilde{\mathcal{P}}_n$ is isomorphic to the monoid F_∞.*

Proof. Let $\phi, \varphi \in F_\infty$ be distinct elements. Let f and g be block maps such that $f_\infty = \phi$ and $g_\infty = \varphi$. Let T_f and T_g be the elements of $\widetilde{\mathcal{P}}_n$ obtained from f and g as in Construction 3.3.9. Let $m \in \mathbb{N}$ be such that both T_f and T_g are synchronizing at level m. Since $\phi \neq \varphi$ there is an $x \in X_n^{\mathbb{Z}}$ such that $y = (x)\phi \neq z = (x)\varphi$ and so there is some $i \in \mathbb{Z}$ such that $y_i \neq z_i$. Let $\Gamma = x_{i+1} \ldots x_{i+m}$, we therefore have that $\lambda_{T_f}(i, q_{\overleftarrow{\Gamma}}) \neq \lambda_{T_g}(i, q_{\overleftarrow{\Gamma}})$ (since $T_f = \phi$ and $T_g = \varphi$). Thus we conclude that $T_f \not\cong_\omega T_g$ and so the map from $F_\infty \to \widetilde{\mathcal{P}}_n$ sending an element $f \in F_\infty$ to the transducer $T_f \in \widetilde{\mathcal{P}}_n$ is injective.

It remains now to show that the action of $\widetilde{\mathcal{P}}_n$ on $X_n^{\mathbb{Z}}$ is compatible with the binary operation of Definition 3.1.1. That is, that the map from $F_\infty \to \widetilde{\mathcal{P}}_n$ sending an element $f \in F_\infty$ to the transducer $T_f \in \widetilde{\mathcal{P}}_n$ is a homomorphism. Let $T, U \in \widetilde{\mathcal{P}}_n$ and consider $\mathrm{Core}(U * T)$. It suffices to show that $T \circ U : X_n^{\mathbb{Z}} \to X_n^{\mathbb{Z}}$ is equal to $\mathrm{Core}(U * T) : X_n^{\mathbb{Z}} \to X_n^{\mathbb{Z}}$ since, if UT is the weakly minimal transducer representing $\mathrm{Core}(U * T)$, then UT and $\mathrm{Core}(U * T)$ have the same action on $X_n^{\mathbb{Z}}$.

Let $m \in \mathbb{N}_1$ be minimal such that both U and T are synchronizing at level m. Let $x \in X_n^{\mathbb{Z}}$, $y = (x)U$, $z = (y)T$ and $z' = (y)\mathrm{Core}(U * T)$. Let $i \in \mathbb{Z}$ be arbitrary and consider $\Delta := \lambda_U(x_{i+m} \ldots x_i, q_{x_{i+2m} \cdots x_{i+m+1}})$. Observe that $y_i y_{i+1} \ldots y_{i+m}$ is equal to $\overleftarrow{\Delta}$ by Definition 3.3.13. Let $\overline{\Delta}$ denote the length m prefix of Δ, and observe that $z_i = \lambda_T(y_i, p_{\overline{\Delta}})$. Since, for any state q' of U, $\lambda_U(x_{i+2m} \ldots x_{i+1}, q')$ has suffix $\overline{\Delta}$, the state of $\mathrm{Core}(U * T)$ forced by $x_{i+2m} \ldots x_{i+1}$ is precisely the state $(q_{x_{i+m} \cdots x_{i+1}}, p_{\overline{\Delta}})$. Therefore, $z_i' = \lambda_{(U*T)}(x_i, (q_{x_{i+m} \cdots x_{i+1}}, q_{\overline{\Delta}})) = z_i$. Since $i \in \mathbb{Z}$ was arbitrary we conclude that $z' = z$. □

Observe that as a consequence of the proposition above, endomorphisms of the shift may be thought of as some negative power of the shift times some element of $\widetilde{\mathcal{P}}_n$. Moreover, as $F_\infty \cong \mathrm{End}(X_n^\omega, \sigma_n)$, it follows that $\widetilde{\mathcal{P}}_n \cong \mathrm{End}(X_n^\omega, \sigma_n)$. We thus deduce the following:

Proposition 3.3.16. *The group $\mathcal{H}_n$ is isomorphic to $Aut(X_n^\omega, \sigma_n)$.*

Proof. By Remark 3.2.9 elements of F_∞ which induce automorphisms of the one-sided shift are obtained from left permutive block maps. Moreover by Theorem 3.2.6, the inverse of such an element must again be an element of F_∞ obtained from a left permutive block map. Since left permutive elements correspond to $\widetilde{\mathcal{H}}_n$ (Proposition 3.3.12), it therefore follows, since elements of $\mathcal{H}_n$ correspond to left permutive block maps inducing elements of F_∞ whose inverses are again induced by left permutive block maps that $\mathcal{H}_n$ is isomorphic to $\mathrm{Aut}(X_n^\omega, \sigma_n)$. □

Remark 3.3.17. From Proposition 3.3.16 we deduce that elements of $\widetilde{\mathcal{H}}_n \backslash \mathcal{H}_n$ do not induce homeomorphisms of X_n^ω. Moreover, the set $\widetilde{\mathcal{H}}_n \backslash \mathcal{H}_n$ forms a semigroup under the product inherited from $\widetilde{\mathcal{P}}_n$ since if H_1 and H_2 do not induce homeomorphisms X_n^ω then neither does their product $H_1 H_2$.

It is result due to Hedlund [33] that for $n = 2$, $\mathrm{Aut}(X_2^\omega, \sigma_2) \cong \mathbb{Z}/2\mathbb{Z}$, thus by Proposition 3.3.16, we conclude that $\mathcal{H}_2$, apart from the identity, contains only the single state transducer which induces the permutation $0 \mapsto 1$ and $1 \mapsto 0$.

Below, following a construction of Hedlund we show that the semigroup $\widetilde{\mathcal{H}}_n \backslash \mathcal{H}_n$ for $n > 2$ is infinite.

We begin with the following Proposition.

Proposition 3.3.18. *Let $m \in \mathbb{N}_1$ and $f \in F(X_n, m)$ be permutive. Then $T_f = A_f$ and, if $m > 1$, $T_f \in \widetilde{\mathcal{H}}_n \backslash \mathcal{H}_n$. Therefore T_f has precisely n^{m-1} states.*

Proof. First observe that if $f \in F(X_n, 1)$ then as f is permutive, it induces a permutation from $X_n \to X_n$. Therefore A_f consists only of a single state which induces the permutation f in its action on X_n. Therefore $T_f = A_f \in \mathcal{H}_n$ and the proposition holds for $m = 1$. For the remainder of the proof, we assume that $f \in F(X_n, m+1)$ for $m \in \mathbb{N}_1$

Let $f \in F(X_n, m+1)$ be permutive, and form A_f as in Construction 3.3.9. Recall that states of A_f correspond to words in X_n^m. Let $\alpha, \beta \in X_n$ be distinct, and consider the words $x_1 \ldots x_{m-1}\alpha, x_1 \ldots x_{m-1}\beta$ where $x_i \in X_n$ for $1 \leqslant i \leqslant m-1$. Let $a \in X_n$ be fixed, then, since f is permutive and, for $\gamma \in \{\alpha, \beta\}$, $\lambda_f(a, x_1 \ldots x_{m-1}\gamma) = (ax_1 \ldots x_{m-1}\gamma)f$, we must have, $\lambda_f(a, x_1 \ldots x_{m-1}\alpha) \neq \lambda_f(a, x_1 \ldots x_{m-1}\beta)$. Therefore we conclude that the states $x_1 \ldots x_{m-1}\alpha$ and $x_1 \ldots x_{m-1}\beta$ are not ω-equivalent. Thus the for $x_1 \ldots x_{m-1} \in X_n^{m-1}$, the set $\{x_1 \ldots x_{m-1}b \mid b \in X_n\}$ contains no pair of ω-equivalent states.

Now let $\gamma_1 \ldots \gamma_m, \delta_1 \ldots \delta_m \in X_n^m$ be a pair of distinct words of length m. let $1 \leqslant i \leqslant m$ be minimal and $a, b \in X_n$ be such that $\gamma_i = a \neq b = \delta_i$. Let $x_1 \ldots x_{m-i} \in X_n^{m-1}$ and observe that, by construction of the transition function of A_f, $\pi_f(x_1 \ldots x_{m-i}, \nu_1 \ldots \nu_m) = x_1 \ldots x_{m-i}\nu_1 \ldots \nu_{i-1}\nu_i$ for $\nu \in \{\gamma, \delta\}$. Set $\gamma_1 \ldots \gamma_{i-1} = \delta_1 \ldots \delta_{i-1} := x_{m-i+1} \ldots x_{m-1}$. We have, $\pi_f(x_1 \ldots x_{m-i}, \gamma_1 \ldots \gamma_m) = x_1 \ldots x_{m-i}a$ and $\pi_f(x_1 \ldots x_{m-i}, \delta_1 \ldots \delta_m) = x_1 \ldots x_{m-i}b$. However by the above paragraph the states $x_1 \ldots x_{m-i}a$ and $x_1 \ldots x_{m-i}b$ are not ω-equivalent, therefore the states $\gamma_1 \ldots \gamma_m$ and $\delta_1 \ldots \delta_m$ are not equivalent either. From this we conclude that A_f has no pair of ω-equivalent states and so is weakly-minimal.

We now demonstrates that A_f is not bi-synchronizing, that is, we demonstrates that A_f^{-1} is not synchronizing since by construction A_f is synchronizing. By the Collapsing procedure 2.2.1, if $A_f^{-1} = \langle X_n, (X_n^m)^{-1}, \pi_f^{-1}, \lambda_f^{-1} \rangle$ is synchronizing, then there are a pair of states q_1, q_2 of A_f^{-1} such that $\pi_f^{-1}(i, q_1) = \pi_f^{-1}(i, q_2)$ for any $i \in X_n$. By arguments in the paragraph above, for a state $x_1 \ldots x_m$ of A_f and for letters $i, j \in X_n$, if $\lambda_f^{-1}(i, (x_1 \ldots x_m)^{-1}) = j$, then $\pi_f^{-1}(i, (x_1 \ldots x_m)^{-1}) = (jx_1 \ldots x_{m-1})^{-1}$. Therefore if q_1 and q_2 are states of A_f^{-1} such that $\pi_f^{-1}(i, q_1) = \pi_f^{-1}(i, q_2)$ for any $i \in X_n$, then there is a word $x_1 \ldots x_{m-1} \in X_n^{m-1}$ such that $q_1 = (x_1 \ldots x_{m-1}\alpha)^{-1}$ an $q_2 = (x_1 \ldots x_{m-1}\beta)^{-1}$ for distinct $\alpha, \beta \in X_n$.

Let $x_1 \ldots x_{m-1} \in X_n^m$, $\alpha, \beta \in X_n$ be distinct and consider the states $(x_1 \ldots x_{m-1}\alpha)^{-1}$ and $(x_1 \ldots x_{m-1}\beta)^{-1}$ of A_f^{-1}. Since f is permutive, there are distinct $i, j \in X_n$ such that $\lambda_f(i, x_1 \ldots x_{m-1}\alpha) = (ix_1 \ldots x_{m-1}\alpha)f = (jx_1 \ldots x_{m-1}\beta) = \lambda_f(j, x_1 \ldots x_{m-1}\beta)$. Such i and j must exist as f is left permutive, and right permutivity implies that i and j must be distinct. Therefore setting $k = \lambda_f(i, x_1 \ldots x_{m-1}\alpha) = \lambda_f(j, x_1 \ldots x_{m-1}\beta)$, we have that $\pi_f^{-1}(k, (x_1 \ldots x_{m-1}\alpha)^{-1}) = (ix_1 \ldots x_{m-1})^{-1}$ and $\pi_f^{-1}(k, (x_1 \ldots x_{m-1}\beta)^{-1}) = (jx_1 \ldots x_{m-1})^{-1}$. Since A_f is minimal, it follows that $(ix_1 \ldots x_{m-1})^{-1}$ and $(jx_1 \ldots x_{m-1})^{-1}$ are not ω-equivalent states of A_f^{-1}. Therefore, as $x_1 \ldots x_{m-1} \in X_n^m$ and $\alpha, \beta \in X_n$ were chosen arbitrarily we conclude that there are no pair of states q_1 and q_2 of A_f^{-1} such that, for any $i \in X_n$, $\pi_f^{-1}(i, q_1) = \pi_f^{-1}(i, q_2)$. By Theorem 2.2.6 we conclude that A_f^{-1} is not synchronizing. $\square$

Remark 3.3.19. It is a result of Hedlund [33] that an element f_∞ of F_∞ arising from a permutive block map $f \in F(X_n, m)$ for $m \in \mathbb{N}_1$, induces an m-to-1 map of $X_n^{\mathbb{Z}}$ — that is every element of $X_n^{\mathbb{Z}}$ has precisely m pre-images under the map f_∞.

Below we present a construction in [33] for constructing permutive block maps.

Construction 3.3.20. For $m \in \mathbb{N}_1$ define $f : X_n^m \to X_n$ as follows: $(\gamma_1 \ldots \gamma_m)f = (\gamma_1 + \gamma_m) \mod n$. For fixed $\delta_1 \ldots \delta_{m-1} \in X_n^{m-1}$, the map from $X_n \to X_n$ given by $i \mapsto i + \delta_{m-1} \mod n$ is a permutation. Moreover, for fixed $\delta_1 \ldots \delta_{m-1} \in X_n^{m-1}$, the map from $X_n \to X_n$ given by $i \mapsto \delta_1 + 1 \mod n$ is also a permutation. From this we conclude that the map f is permutive.

As consequence of Construction 3.3.20 and Proposition 3.3.18 we have the following result:

Proposition 3.3.21. *The semigroup $\widetilde{\mathcal{H}}_n \backslash \mathcal{H}_n$ is infinite.*

We identify the following submonoid of $\widetilde{\mathcal{P}}_n$.

Definition 3.3.22. Set $\mathcal{P}_n$ to be the subset of $\widetilde{\mathcal{P}}_n$ consisting of elements which induce self homeomorphisms of $X_n^{\mathbb{Z}}$.

Remark 3.3.23. Observe that if $\psi \in \mathcal{P}_n$ is such that $\sigma_n\psi = \psi\sigma_n$ then by post- and pre-multiplying by ψ^{-1} we have $\psi^{-1}\sigma_n = \sigma_n\psi^{-1}$, therefore $\mathcal{P}_n$ is a submonoid of $\mathrm{Aut}(X_n^{\mathbb{Z}}, \sigma_n)$. Moreover, all elements of $\mathrm{Aut}(X_n^{\mathbb{Z}}, \sigma_n)$ can be written in the form $\sigma_n^l\psi$ for some $\psi \in \mathcal{P}_n$ by Theorem 3.2.6.

By definition elements of $\mathcal{P}_n$ induce homeomorphisms of $X_n^{\mathbb{Z}}$ however, this does not imply that elements of $\mathcal{P}_n$ possess homeomorphism states (see Definition 2.5.9). We present such an example below.

Example 3.3.24. Below is an example of an element of $\widetilde{\mathcal{P}}_4$ which is in fact an element of $\mathcal{P}_4$ since its square, after identifying ω-equivalent states, is equal to the 4-shift, σ_4.

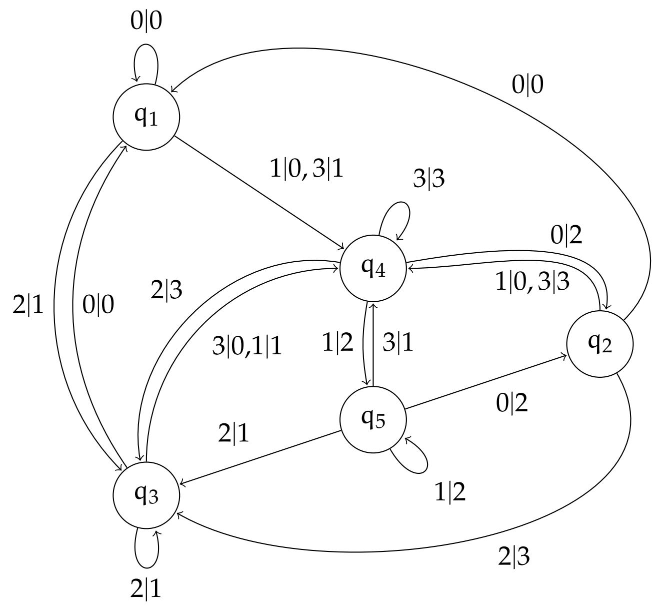


Figure 3.4: An example of an element of $\mathcal{P}_4$ with no homeomorphism state

Notice that this element is actually the core of the transducer in Example 2.5.12 and so is an element of $\mathcal{O}_{4,3}$. Further connections between the monoid $\mathcal{P}_n$ and the outer automorphisms of $G_{n,r}$ are explored in the forthcoming article [6] of the author's.

The next section shall mainly be concerned with the group $\mathcal{H}_n$, however some of the results are for the monoids $\widetilde{\mathcal{P}}_n$, $\mathcal{P}_n$, $\widetilde{\mathcal{H}}_n$.

3.4 The monoid $\widetilde{\mathcal{P}}_n$

In this section, we develop tools for working with the monoid $\widetilde{\mathcal{P}}_n$. Some of the results here may be thought of as an interpretation of known results about the group of automorphisms of the shift in the context of synchronizing transducers and we shall highlight these as we come to them.

We require first some further notation.

Notation 3.4.1. It will sometimes be convenient to work with non-weakly minimal transducers, and so given a non-initial synchronous, transducer A over $\mathfrak{C}_n$ we use the notation $\min(A)$ for the weakly minimal transducer representing A. Notice that if $A \in \widetilde{\mathcal{H}}_n$, then $\min(A)$ is the minimal transducer representing A. We also observe that for a non-core, synchronous, synchronizing transducer over $\mathfrak{C}_n$, $\min(\mathrm{Core}(A)) = \mathrm{Core}(\min(A))$, thus we write $\min\mathrm{Core}(A)$ for the weakly minimal transducer representing $\mathrm{Core}(A)$.

Notation 3.4.2. Let $\Gamma \in X_n^m$ for some $m \in \mathbb{N}_1$. We denote by $\ldots\Gamma\dot{\Gamma}\Gamma\ldots$ the bi-infinite word $x \in X_n^{\mathbb{Z}}$ such that $x_0 \ldots x_{|\Gamma|-1} = \Gamma$ and for $i \in X_n^{\mathbb{Z}}$, $x_{i|\Gamma|} \ldots x_{(i+1)|\Gamma|-1} = \Gamma$. More generally given a sequence $\Delta_i \in X_n^+$ for $i \in \mathbb{Z}$, we denote by $\ldots\Delta_{-1}\dot{\Delta}_0\Delta_1\ldots$ the bi-infinite word $x \in X_n^{\mathbb{Z}}$ such that, after disregarding indices, $x = \ldots\Delta_{-1}\Delta_0\Delta_1\ldots$ and $x_0 \ldots x_{|\Delta_0|-1} = \Delta_0$.

Notation 3.4.3. Given a set X we denote by $Sym(X)$ the symmetric group on X, that is, the group of permutations of the set X. If $|X| = m$ for some $m \in \mathbb{N}_1$ then we also write $Sym(m)$ for $Sym(X)$.

We recall, for a transducer A over X_n, a state q of A and a word $\gamma \in X_n^*$ Notation 1.3.7 $(\gamma)A_q := \lambda_A(\gamma, q)$.

We make the following results about A which will be useful later on. The first two shall apply to all elements $A \in \widetilde{\mathcal{P}}_n$.

Lemma 3.4.4. *Let A and B be elements of $\widetilde{\mathcal{P}}_n$, and let $m \in \mathbb{N}\backslash\{0\}$ be minimal such that both A and B are synchronizing at level m. Then if $A \not\cong_\omega B$, there is a word Γ, $|\Gamma| = k \geqslant m$, and states p and q of A and B, respectively, such that:*

(i) p is the state in A forced by Γ and q is the state in B forced by Γ.

(ii) p and q are not ω-equivalent.

Proof. Since $A \not\cong_\omega B$ they induce different homeomorphisms of $X_n^{\mathbb{Z}}$, and so there is a bi-infinite word $w = \ldots x_{-2}x_{-1}x_0x_1\ldots$ which they process differently.

Let $w_1 = \ldots y_{-2}y_{-1}y_0y_1y_2\ldots$ and $w_2 = \ldots z_{-2}z_{-1}z_0z_1z_2\ldots$ be the outputs from A and B respectively. Let $k \in \mathbb{N}\backslash\{0\}$ be such that A and B are synchronizing at level k. Note that $k \geqslant m$. Let $l \in \mathbb{N}$ be minimal such that $y_l \neq z_l$ or $y_{-l} \neq z_{-l}$. Then one of the words $x_{l-k}\ldots x_{l-2}x_{l-1}$ or $x_{-l-k}\ldots x_{-l-2}x_{-l-1}$ satisfies the premise of the lemma. □

Lemma 3.4.5. *Let $A \in \widetilde{\mathcal{P}}_n$ be such that $\min\mathrm{Core}(A^i) \not\cong_\omega \min\mathrm{Core}(A^j)$ for any pair $i, j \in \mathbb{N}$. Then for $i \neq j \in \mathbb{N}$ and any two states u and v of A^i and A^j respectively, the initial transducers A_u^i and A_v^j are not ω-equivalent.*

Proof. Observe that since $\min\mathrm{Core}(A^i) \not\cong_\omega \min\mathrm{Core}(A^j)$, by Lemma 3.4.4 there is a word Γ of size greater than or equal to the maximum of the minimum synchronizing levels of A and B such that the state of $\min\mathrm{Core}(A^i)$ forced by Γ is not ω-equivalent to the state $\min\mathrm{Core}(A^j)$ forced by Γ. Now since A^i and A^j are synchronizing, the initial transducers A_u^i and A_v^j are also synchronizing. Moreover $\mathrm{Core}(A_u^i) \cong_\omega \min\mathrm{Core}(A^i)$, likewise $\mathrm{Core}(A_v^j) \cong_\omega \min\mathrm{Core}(A^j)$. Therefore let Λ be a long enough word such that when read from the state u of A^i and state v of A^j the resultant state is in the core of A^i and A^j respectively. Now let u' and v' be the states of A_u^i and A_v^j respectively reached after reading $\Lambda\Gamma$ in A_u^i and A_v^j. Then u' and v' are not ω-equivalent since $\mathrm{Core}(A_u^i) \cong_\omega \min\mathrm{Core}(A^i)$, and $\mathrm{Core}(A_v^j) \cong_\omega \min\mathrm{Core}(A^j)$. Therefore there exists a word $\delta \in X_n^{\mathbb{N}}$ such that $(\delta)A_{u'}^i \neq (\delta)A_{v'}^i$, therefore we have that $(\Lambda\Gamma\delta)A_u^i \neq (\Lambda\Gamma\delta)A_v^j$. The result now follows. □

Proposition 3.4.6. *Let $A \in \mathcal{H}_n$ be bi-synchronizing at level k. Then for any non-empty word $\Gamma \in X_n^+$ there is a unique state $q_\Gamma \in Q_A$ such that $\pi(\Gamma, q_\Gamma) = q_\Gamma$. Moreover, for any $j \in \mathbb{N}_1$ the map $\overline{A}_j : X_n^j \to X_n^j$ given by $\Gamma \mapsto \lambda(\Gamma, q_\Gamma)$, where $\pi(\Gamma, q_\Gamma) = q_\Gamma$, is a permutation.*

Proof. Through out the proof let Γ be any non-empty word of length $j \geqslant 1$. We observe first that if there is a state q such that $\pi(\Gamma, q) = q$ then this state must be unique. Since if there was a state q' such that $\pi(\Gamma, q') = q'$ then $\pi(\Gamma^k, q) = q$ while $\pi(\Gamma^k, q') = q'$, and since Γ^k has length at least k we see it is a synchronizing word and so can conclude that $q = q'$.

To see that such a state q exists, consider again the word Γ^k. Since Γ is non-empty, $|\Gamma^k| \geqslant k$, so there is a unique state q such that $\pi_A(\Gamma^k, q) = q$. Now consider the state p so that $\pi(\Gamma, q) = p$. Since $\pi(\Gamma^k, q) = q$ it is the case that $\pi(\Gamma^{k+1}, q) = p$, but Γ^k and Γ^{k+1} have the same length k suffix, so that $p = q$. In particular, we have $\pi(\Gamma, q) = q$.

We now free the symbol Γ. We want to show the map defined on X_n^j (words of length exactly j), by $\Gamma \mapsto \lambda(\Gamma, q)$, where $\pi(\Gamma, q) = q$, is a bijection.

To prove this map is injective, suppose there are two words, Γ and Δ, with associated states q and r respectively, of length l, such that $\lambda(\Gamma, q) = \Gamma' = \lambda(\Delta, r)$. Now, as q is the state forced by Γ^k as above, while r is the state forced by Δ^k (again as above), we see that $\pi^{-1}((\Gamma')^k, q) = q$ while $\pi^{-1}((\Gamma')^k, r) = r$, but as $(\Gamma')^k$ is synchronizing for A^{-1} we must have that $q = r$, and then, by injectivity of A_q, that $\overline{\Gamma} = \overline{\Delta}$, so that in particular $\Gamma = \Delta$.

Therefore for each $j \in \mathbb{N}$, $j \geqslant 1$ the map induced by A, from the set of words of length j to itself, is injective. Therefore as this set of words is finite, the map is actually a bijection. □

Remark 3.4.7. Notice that we have only used the full bi-synchronizing condition in arguing invertibility. The existence and uniqueness of the state $q \in Q$ such that for $1 \leqslant j \in \mathbb{N}$ and $\Gamma \in X_n^j$, $\pi(\Gamma, q) = q$ holds for all elements of the monoid $\widetilde{\mathcal{P}}_n$. Observe that such a map indicates the action of an element of $\widetilde{\mathcal{P}}_n$ on a periodic word $\ldots\Gamma\dot{\Gamma}\Gamma\ldots$ for $\Gamma \in X_n^+$.

We illustrate the above proposition with the example below.

Example 3.4.8. Let C be the following transducer:

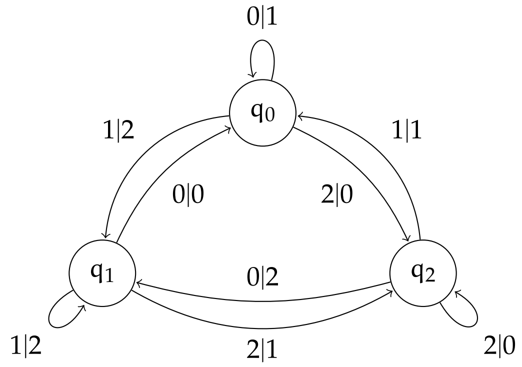


Figure 3.5: An element of $\mathcal{H}_3$

It is easily verified that this transducer is bi-synchronizing at level 2. The sets $\{00, 10, 21\}$, $\{01, 11, 20\}$ and $\{02, 12, 22\}$ are, respectively, the set of words which force the states q_0, q_1 and q_2. The permutation of words of length 2 associated to this transducer in the manner described above is given by: (00 11 22)(10 20 12)(21 01 02). The attentive reader might have observe that these disjoint cycles have an interesting structure: if we consider the states forces by each element of a cycle then the result is a cyclic permutation of $(q_0\ q_1\ q_2)$. We shall later see how such behaviour plays a role in understanding the order of an element.

We establish some further notation.

Notation 3.4.9. For $A \in \mathcal{H}_n$ bi-synchronizing at level k, and $1 \leqslant j \in \mathbb{N}$, let $\overline{A_j}$ represent the permutation of X_n^j indicated in Proposition 3.4.6. Let $\mathcal{X}_n^l$ denote the set of prime words in X_n^l and let $\overline{A}_l\restriction_{\mathcal{X}_n^l}$ denote the permutation that $\overline{A}_l$ induces on the set $\mathcal{X}_n^l$.

Remark 3.4.10. Observe that a similar proof to that given in Proposition 3.4.6 will show that we can analogously associate to each element of $\widetilde{\mathcal{P}}_n \backslash \mathcal{H}_n$ a map from $X_n^j \to X_n^j$ for every $1 \leqslant j \in \mathbb{N}$. However this map need not be invertible for every such $1 \leqslant j$, (we shall later see that for one-way synchronizing transducers there is some j, where the map so defined is not invertible). In light of this, for each $A = \langle X_n, Q, \pi, \lambda \rangle \in \widetilde{\mathcal{P}}_n$ and $1 \leqslant j \in \mathbb{N}$ let $\overline{A}_j : X_n^j \to X_n^j$ be the transformation given by $\Gamma \mapsto \lambda(\Gamma, q)$ where $q \in Q$ is the unique state such that $\pi(\Gamma, q) = q$. We observe that if $A \in \mathcal{P}_n$ then $\overline{A}_j$ is a permutation for every $j \in \mathbb{N}$. This is because $\mathcal{P}_n$ induces a homeomorphism of $X_n^{\mathbb{Z}}$. Since if for some $j \in \mathbb{N}$, $\overline{A}_j$ is not injective, then there are words Γ, Δ for which $(\Gamma)A_j = (\Delta)A_j$, this means that the bi-infinite strings $(\ldots\Gamma\dot{\Gamma}\Gamma\ldots)$ and $(\ldots\Delta\dot{\Delta}\Delta\ldots)$ are mapped to the same element of $X_n^{\mathbb{Z}}$ by A contradicting injectivity.

Remark 3.4.11. If we have found a permutation (as above) for a transducer A for words of length $j \geqslant 1$, then the disjoint cycle structure of this permutation will be present in all permutations associated to A for words of length mj, for $m \in \mathbb{N}\backslash\{0\}$. This is seen for example if $(\Gamma_1 \ \dots \ \Gamma_l)$ is a disjoint cycle in the permutation associated to words of length j , then $(\Gamma_1\Gamma_1 \ \dots \ \Gamma_l\Gamma_l)$ is a disjoint cycle in the level $2j$ permutation. This is because each Γ_i is processed from the state of A it forces and the output is Γ_{i+1}. Generalise in the obvious way for the permutation of words of length mj. For instance in the example above $(0\dots 0\,1\dots 1\,2\dots 2)$ will be present in the permutation of words of length $2m$ associated to C (where each $i\dots i$ is of length $2m$, $i \in \{0,1,2\}$). From this we deduce that for an element $A \in \widetilde{\mathcal{P}}_n$ and $l \in \mathbb{N}$, $\overline{A}_l$ depends only on $\overline{A}_j$ for $1 \leqslant j \leqslant l-1$ such that $j|l$ and the permutation $\overline{A}_l$ induces on the prime words in X_n^l. This is because any non-prime word $\Gamma \in X_n^l$ is a power of some smaller word, $\gamma \in X_n^j$ for some $1 \leqslant j < l$, and so the action of $\overline{A}_l$ on γ is determined by $\overline{A}_j$.

The following lemma shows that these maps behave well under multiplication. This is essentially the well known result about automorphisms of the shift that the action on periodic words induces an homomorphism to a symmetric group (see for example [13])

Theorem 3.4.12. *Let $A = \langle X_n, Q_A, \pi_A, \lambda_A\rangle$ and $B = \langle Q_B, X_n, \pi_B, \lambda_B\rangle$ be elements of $\widetilde{\mathcal{P}}_n$. Let $AB = \langle Q_A, S, \pi_{A*B}, \lambda_{A*B}\rangle$ be the core product of A and B, where $S \subset Q_A \times Q_B$ is the set of states in the core of $A * B$. Then $\overline{(AB)}_l = \overline{A}_l\overline{B}_l$.*

Proof. Let Γ be a word of length l in X_n, and let $p \in Q_A$ be such that $\pi_A(\Gamma, p) = p$. Let $\Delta := \lambda_A(\Gamma, p)$, and let $q \in Q_B$ be such that $\pi_B(\Delta, q) = q$. Then (p, q) is a state of AB such that $\pi_{AB}(\Gamma, (p, q)) = (p, q)$. If $\Lambda = \lambda_B(\Delta, q)$, then we have in $\overline{(AB)}_l$ that $\Gamma \mapsto \Lambda$. However $\overline{A}_l\overline{B}_l$ sends Γ to Λ also. Since Γ was an arbitrary word of length l, this gives the result. □

Let $\tau_l : \widetilde{\mathcal{P}}_n \to \mathrm{Sym}(X_n^l)$ be the map defined by $A \mapsto \overline{A}_l$ for every $l \in \mathbb{N}$. Below we demonstrate the usefulness of these maps.

Proposition 3.4.13. *Let A and B be elements of $\widetilde{\mathcal{P}}_n$. Then the following hold:*

(i) A and B commute if and only if for every $l \geqslant 1$ $\overline{A}_l$ and $\overline{B}_l$ commute.

(ii) A and B are conjugate by an invertible element of $\widetilde{\mathcal{P}}_n$ if and only if there is an invertible, $h \in \widetilde{\mathcal{P}}_n$, such that for every $l \geqslant 1$ $\overline{h}_l^{-1}\overline{A}_l\overline{h}_l = \overline{B}_l$.

(iii) A and B are equal if and only if for every $l \geqslant 1$ $\overline{A_l} = \overline{B_l}$.

Proof. The forward direction in all cases follows by Theorem 3.4.12 above which shows that the map $\tau_l : \widetilde{\mathcal{P}}_n \to \mathrm{Sym}(X_n^l)$ is a monoid homomorphism. We need only prove the reverse implications.

We proceed by contradiction.

For (i) suppose that $\overline{A}_l$ and $\overline{B}_l$ commute for every l however $\mathrm{Core}(B * A) \not\cong_\omega \mathrm{Core}(A * B)$. Let $m \in \mathbb{N}\backslash\{0\}$ be such that both $\mathrm{Core}(A * B)$ and $\mathrm{Core}(B * A)$ are bi-synchronizing at level m. Let Γ be a word of length m as in Lemma 3.4.4 such that p is the state of $\mathrm{Core}(A * B)$ forced by Γ and q is the state of $\mathrm{Core}(B * A)$ forced by Γ.

Let λ_{AB} and λ_{BA} denote, respectively, the output function of $\mathrm{Core}(A * B)$ and $\mathrm{Core}(B * A)$. Since p is not ω-equivalent to q there is a word Δ, of length $l \geqslant 1$ say, such that $\Lambda := \lambda_{AB}(\Delta, p) \neq \lambda_{BA}(\Delta, q) =: \Xi$. This now means that in $\overline{\mathrm{Core}(A * B)}_{l+m}$, $\Delta\Gamma \mapsto \Lambda W_1$ and in $\overline{\mathrm{Core}(B * A)}_{l+m}$, $\Delta\Gamma \mapsto \Xi W_2$ (for some words W_1 and W_2 of length l). Therefore we conclude that $\overline{\mathrm{Core}(A * B)}_{l+m} \neq \overline{\mathrm{Core}(B * A)}_{l+m}$ which is a contradiction.

Part (ii) proceeds in a analogous fashion. Suppose A, B and h are as in the statement of Proposition 3.4.13 (ii), but $\mathrm{Core}(A * h) \neq \mathrm{Core}(h * B)$. Let $m \in \mathbb{N}\backslash\{0\}$ be such that $\mathrm{Core}(A * h)$ and $\mathrm{Core}(h * B)$ are bi-synchronizing at level m. Let Γ be a word as in Lemma 3.4.4, and p and q be the states of $\mathrm{Core}(A * h)$ and $\mathrm{Core}(h * B)$ forced by Γ such that p and q are not ω-equivalent. Now we are able to construct a word as in part (i) demonstrating that $\overline{\mathrm{Core}(A * h)}_l \neq \overline{\mathrm{Core}(h * B)}_l$ for some l yielding a contradiction.

Part (iii) follows from Part (ii) with h the identity transducer. □

Remark 3.4.14. Notice that Proposition 3.4.13 also follows since periodic points are dense in $X_n^{\mathbb{Z}}$, and indeed the proof essentially relies on this fact. The corollary below is in some sense a qualitative version of Proposition 3.4.13.

Corollary 3.4.15. *Let* A *and* B *be elements of* $\widetilde{\mathcal{P}}_n$*, and let* $k \geqslant 1 \in \mathbb{N}$ *be such that both* A *and* B *are synchronizing at level* k*. Then the following hold:*

(i) $A = B$ *if and only if* $\overline{A}_{k+1} = \overline{B}_{k+1}$.

(ii) Let BA *and* AB *denote the minimal transducers representing the products* $\mathrm{Core}(A * B)$ *and* $\mathrm{Core}(B * A)$ *and* $l \geqslant 1 \in \mathbb{N}$ *be such that both* AB *and* BA *are synchronizing at level* l*, then* $AB = BA$ *if and only if* $\overline{A}_{l+1}\overline{B}_{l+1} = \overline{B}_{l+1}\overline{A}_{l+1}$.

(iii) A *and* B *are conjugate in* $\widetilde{\mathcal{P}}_n$ *if and only if there is an invertible* $h \in \widetilde{\mathcal{P}}_n$ *such that* $h^{-1}Ah$ *(where this is the minimal transducer representing the product) is synchronizing at level* k *and* $\overline{h}_{k+1}^{-1}\overline{A}_{k+1}\overline{h}_{k+1} = \overline{B}_{k+1}$.

Proof. Throughout the proof all products indicated shall represent the minimal transducer under ω-equivalence representing the product.

Observe that parts (ii) and (iii) are consequences of part (i). Since for part (ii) AB and BA are synchronizing at level l; for part (iii) B and $h^{-1}Ah$ are synchronizing at level k (where h is the conjugator). Therefore it suffices to prove only part (i).

The forward implication follows by Proposition 3.4.13, so we need only show the reverse implication. Let k be as in the statement of part (i) and assume that $\overline{A}_{k+1} = \overline{B}_{k+1}$. Denote by a triple (Ξ, u, v) for $\Xi \in X_n^k$, and u and v states of A and B respectively, such that u is the state of A forced by Ξ and v is the state of B forced by Ξ. Notice that for each such $\Xi \in X_n^k$ such a triple is unique.

Let $\Gamma \in X_n^k$, belong to a triple (Γ, p, q). Let $i \in X_n$ be arbitrary. Since $\overline{A}_{k+1} = \overline{B}_{k+1}$, we must have that $(i)A_p = (i)B_q$ since $(\Gamma i)\overline{A}_{k+1} = (\Gamma i)\overline{B}_{k+1}$.

Free the symbols Γ, p, and q.

Now let $w = \ldots w_{-k} \ldots w_{-1} w_0 w_1 \ldots w_k \ldots$ be a bi-infinite word. We show that A and B process this word identically. Let w_i $i \in \mathbb{Z}$ denote the i^{th} letter of w. Then the i^{th} letter of $(w)A$ is $(w_i)A_p$ where p is the state of A forced by $\Gamma = w_{i-k} \ldots w_{i-1}$, the word of length k immediately to the left of w_i. Likewise the i^{th} letter of $(w)B$ is $(w_i)B_q$ where q is the state of B forced by Γ. Therefore (Γ, p, q) is an allowed triple. However from above we know that $(w_i)A_p = (w_i)B_q$. Since $i \in \mathbb{Z}$ was arbitrary, $(w)A = (w)B$, and $A = B$ since w was arbitrary and A and B are assumed minimal. ☐

Definition 3.4.16. Let $j \in \mathbb{N}_1$ and $\rho \in \mathrm{Sym}(X_n^{j+1})$, then we say an element $A \in \widetilde{\mathcal{P}}_n$ is *unique for the pair* (j, ρ) if A is synchronizing at level j and $\overline{A}_{j+1} = \rho$. By Corollary 3.4.15 such an A, if it exists, must be unique.

Proposition 3.4.17. *It is a result in [13] that given* $L \in \mathbb{N}$ *and a finite sequence of permutations* $(\rho_l)_{1 \leqslant l \leqslant L}$ *of* $\mathcal{X}_n^l$*, there is an element* A *of* $\mathcal{P}_n$ *such that* $\overline{A}_l\restriction_{\mathcal{X}_n^l} = \rho_l$ *for* $1 \leqslant l \leqslant L$*. Therefore given an element* $B \in \mathcal{P}_n$ *and* $L \in \mathbb{N}$*, one may find an element* $A \in \widetilde{\mathcal{P}}_n$ *such that* $\overline{A}_l = \overline{B}_l$ *for* $1 \leqslant l \leqslant L$ *and* $\overline{A}_{L+1} \neq \overline{B}_{L+1}$*. This implies that one cannot omit the synchronizing level in the uniqueness statement of Definition 3.4.16.*

Remark 3.4.18. A group is said to be *residually finite* if for any non-identity element g of the group, there is a homomorphism onto a finite group mapping g to a non-trivial element. Corollary 3.4.15 part (i) demonstrates that the group $\mathcal{P}_n$ is residually finite. This is because the map sending $A \in \mathcal{P}_n$ to $\overline{A}_{k+1}$ in the symmetric group on n^{k+1} points where k is the synchronizing level of A is a homomorphism.

Remark 3.4.19. Part (ii) of Corollary 3.4.15 demonstrates that if $B \in \widetilde{\mathcal{P}}_n$ is synchronizing at level $j \geqslant 1 \in \mathbb{N}$, and $A \in \widetilde{\mathcal{P}}_n$ is synchronizing at level $k \geqslant 1 \in \mathbb{N}$, then B commutes with A if and only if $\overline{A}_{j+k+1}$ commutes with $\overline{B}_{j+k+1}$ by Proposition 2.1.33.

Remark 3.4.20. In order to restate Corollary 3.4.15 (iii) for a non-invertible $h \in \widetilde{\mathcal{P}}_n \backslash \mathcal{P}_n$ showing that the equation $\overline{A}_{k+1}\overline{h}_{k+1} = \overline{h}_{k+1}\overline{B}_{k+1}$ holds might no longer suffice. Instead we might have to check that $\overline{A}_{j+1}\overline{h}_{j+1} = \overline{h}_{j+1}\overline{B}_{j+1}$ where $j \in \mathbb{N}$ is a level such that $\mathrm{Core}(A * h)$ and $\mathrm{Core}(B * h)$ are synchronizing at level j.

We have the following result distinguishing between elements of $\mathcal{H}_n$ and $\widetilde{\mathcal{H}}_n$. We recall the notion of prime words and rotations of words introduced in Subsection 2.2.1:

Proposition 3.4.21. *Let $A = \langle X_n, Q_A, \pi_A, \lambda_A \rangle$ be an element of $\widetilde{\mathcal{H}}_n \backslash \mathcal{H}_n$ (i.e A is one-way synchronizing) with synchronizing level k and $A^{-1} = \langle X_n, Q_{A^{-1}}, \pi_{A^{-1}}, \lambda_{A^{-1}} \rangle$ be the inverse of A. There is an $l \in \mathbb{N}$ with $0 < l \leqslant k(|Q_A|^2 + 1)$ such that $\overline{A}_l$ is not a permutation. In particular, the action of A on $X_n^{\mathbb{Z}}$ is non-injective: there exists words Δ and Λ in X_n^+ such that Δ is not a cyclic rotation of Λ and the bi-infinite strings $(\ldots\Delta\Delta\ldots)$ and $(\ldots\Lambda\Lambda\ldots)$ have the same image under A.*

Proof. We first establish some notation: for a state p of A we shall let p^{-1} denote the corresponding state in A^{-1}. We shall also apply the convention that $(p^{-1})^{-1} = p$.

Suppose A is synchronizing at level k. Since A^{-1} is not synchronizing it follows that $|Q_A| = |Q_{A^{-1}}| > 1$. Moreover, there is a pair of states (r_1, r_2) such that there is an infinite set W_1 of words $w_i \in X_n^+$ for which $\pi_{A^{-1}}(w_i, r_1) \neq \pi_{A^{-1}}(w_i, r_2)$. This follows since A^{-1} is not synchronizing at level l for any $l \in \mathbb{N}$. Therefore for each $l \in \mathbb{N}$ there is a pair states (r_1^l, r_2^l) and a word $w_l \in X_n^l$ such that $\pi_{A^{-1}}(w_l, r_1^l) \neq \pi_{A^{-1}}(w_l, r_2^l)$. Since A is a finite transducer there is a pair of states (r_1, r_2) such that for infinitely many $l \in \mathbb{N}$, $(r_1^l, r_2^l) = (r_1, r_2)$, therefore taking $W_1 := \{w_l | l \in \mathbb{N} \text{ and } (r_1^l, r_2^l) = (r_1, r_2)\}$, (r_1, r_2) and W_1 satisfy the conditions.

Now since W_1 is infinite, by an argument similar to that above, there is a pair of states (s_1, s_2) such that $\pi_{A^{-1}}(w_i, r_1) = s_1$ and $\pi_{A^{-1}}(w_i, r_2) = s_2$ and $s_1 \neq s_2$ for infinitely many $w_i \in W_1$. Let W_2 denote the set of words w_i such that $\pi_{A^{-1}}(w_i, r_1) = s_1$ and $\pi_{A^{-1}}(w_i, r_2) = s_2$.

Let $w_i \in W_2$ be such that $|w_i| \geqslant k(|Q_A|^2 + 1)$. Now since $s_1 \neq s_2$, then for any prefix φ of w_i we must have $\pi_{A^{-1}}(\varphi, r_1) \neq \pi_{A^{-1}}(\varphi, r_2)$. Moreover since $|w_i| \geqslant k(|Q_A|^2 + 1)$ there are prefixes φ_1 and φ_2 of w_i such that $||\varphi_1| - |\varphi_2|| = jk \leqslant k(|Q_A|^2 + 1)$ ($j \in \mathbb{N}\backslash\{0\}$) satisfying $\pi_{A^{-1}}(\varphi_1, r_1) = \pi_{A^{-1}}(\varphi_2, r_1) = p^{-1}$ and $\pi_{A^{-1}}(\varphi_1, r_2) = \pi_{A^{-1}}(\varphi_2, r_2) = q^{-1}$ with $p^{-1} \neq q^{-1}$, and $p^{-1}, q^{-1} \in Q_{A^{-1}}$.

Assume φ_1 is a prefix of φ_2 and let ν be the such that $\varphi_1 \nu = \varphi_2$. By construction ν satisfies $\pi_{A^{-1}}(\nu, p) = p$ and $\pi_{A^{-1}}(\nu, q) = q$ such that $p^{-1} \neq q^{-1}$. Let $\Lambda = \lambda_{A^{-1}}(\nu, p^{-1})$ and $\Delta = \lambda_{A^{-1}}(\nu, q^{-1})$. Since A is synchronizing at level k and synchronous, $\Lambda \neq \Delta$, otherwise $p = q$ and since A is synchronous $|\Lambda| = |\Delta|$.

Therefore in A we have, $\pi_A(\Lambda, p) = p$ and $\pi_A(\Delta, q) = q$ moreover, $\lambda_A(\Lambda, p) = \lambda_A(\Delta, q) = \nu$. This shows that $\overline{A}_\Lambda$ is not a permutation of $X_n^{|\Lambda|}$. We now make the assumption that Λ and Δ are the smallest words such that $\pi_A(\Lambda, p) = p$ and $\pi_A(\Delta, q) = q$ moreover, $\lambda_A(\Lambda, p) = \lambda_A(\Delta, q)$. Let $\nu \in X_n^{|\Lambda|}$ be such that $\lambda_A(\Lambda, p) = \lambda_A(\Delta, q) = \nu$.

In order to show that A represents a non-injective map on $X_n^{\mathbb{Z}}$ observe that the bi-infinite strings $(\ldots\Lambda\Lambda\ldots)$ and $(\ldots\Delta\Delta\ldots)$ are mapped to the bi-infinite string $(\ldots\nu\nu\ldots)$ under A. Therefore taking $(\ldots\Theta\dot{\Theta}\Theta\ldots)$ for $\Theta \in X_n^+$ to represent the element $y \in X_n^{\mathbb{Z}}$ defined by $y_{j|\Theta|}y_{j|\Theta|+1}\cdots y_{j|\Theta|+|\Theta|-1} := \Theta$ for any $j \in \mathbb{Z}$, we see that $(\ldots\Lambda\dot{\Lambda}\Lambda\ldots)$ and $(\ldots\Delta\dot{\Delta}\Delta\ldots)$ are distinct elements of $X_n^{\mathbb{Z}}$ which have the same image under A. This shows A is non-injective.

To conclude the proof we now need to argue that there exists words Λ' and Δ' which are not cyclic rotations of each other such that $(\ldots\Lambda'\dot{\Lambda}'\Lambda'\ldots)$ and $(\ldots\Delta'\dot{\Delta}'\Delta'\ldots)$ are mapped by A to the same word.

Suppose that Λ is a cyclic rotation of Δ, otherwise we are done.

Since $\pi_A(\Lambda, p) = p$ we must have that ν is equal to a non-trivial cyclic rotation of itself. This is the case if and only if ν is equal to some power of a third word v strictly smaller than ν (see for instance [50, Theorem 1.2.9]). In fact if $\nu = \nu'\nu'' = \nu''\nu'$ then both ν'' and ν' are powers of this word v.

We may assume that v is a prime word (that is, it cannot be written as a powers of a strictly smaller word). Let $r \in \mathbb{N}$ be such that $\mathrm{v}^r = \nu$. Notice that $r|\mathrm{v}| = |\nu| = |\Lambda|$.

First suppose that there is word $u \in X_n^{|\mathrm{v}|}$ such that $(u)A_{|\mathrm{v}|} = \mathrm{v}$ and u^r is a rotation of Λ. If a non-trivial suffix $u_1 \neq u$ of u is a prefix of Λ, then since $\lambda_A(\Lambda, p) = \mathrm{v}^r = \nu$, we must have that

ν is equal to a non-trivial cyclic rotation of itself contradicting that ν is a prime word. Therefore $\Lambda = u^r$. However, since $(\Delta)A_{|\Lambda|} = (\Lambda)A_{|\Lambda|}$ and Δ is a cyclic rotation of Λ then u^r is also a cyclic rotation of Δ. Therefore by the same argument we must have that $\Delta = u^r$. However this now implies that $\Delta = \Lambda$ yielding a contradiction since we assumed that $\Delta \neq \Lambda$.

Now since $|\nu| < |\nu|$, then either there is a word u, such that $|u| = |\nu|$ for which $(u)\overline{A}_{|\nu|} = \nu$ or $\overline{A}_{|\nu|}$ is not surjective from $X_n^{|\nu|}$ to itself, and so it is also not injective (since $X_n^{|\nu|}$ is finite). If the latter occurs, then there are strictly smaller distinct words Λ' and Δ' and states p' and q' such that $\pi_A(\Lambda', p') = p'$ and $\pi_A(\Delta', q') = q'$ so that, $\lambda_A(\Lambda', p') = \lambda_A(\Delta', q')$. Notice that since $A \in \mathcal{H}_n$ all its states are homeomorphism states, therefore p' and q' cannot be equal or A would have a non-homeomorphism state. However this is a contradiction since we assumed that Λ and Δ were the smallest such words. Therefore there is a word u so that $|u| = |\nu|$ and $(u)\overline{A}_{|\nu|} = \nu$. Notice that u^r cannot be a rotation of Λ by an argument above. Moreover the bi-infinite sequences $(\ldots \dot{u}^r u^r \ldots)$ and $(\ldots \dot{\Lambda}\Lambda \ldots)$ are mapped by A to the same bi-infinite string $(\ldots \dot{\nu}\nu \ldots)$. □

Remark 3.4.22. Let A be an element of $\widetilde{\mathcal{H}}_n \backslash \mathcal{H}_n$ which is invertible as a transducer, then A represents a surjective map from the Cantor space $X_n^{\mathbb{Z}}$ to itself. In particular as a consequence of the proposition above an element $A \in \widetilde{\mathcal{H}}_n$ is injective on $X_n^{\mathbb{Z}}$ if and only if it is a homeomorphism if and only if it is bi-synchronizing.

Proof. Our argument shall proceed as follows, we shall make use of the well known results that the continuous image of a compact topological space is compact, and that a compact subset of a Hausdorff space is closed. This means it suffices to argue that the image of A is dense in $X_n^{\mathbb{Z}}$.

Let $k \in \mathbb{N}$ be the minimal synchronizing level for A.

Notice that since A is invertible as an transducer each state of A defines an invertible map from $X_n^{\mathbb{N}}$ to itself. Therefore given an element $y \in X_n^{\mathbb{Z}}$, let p be a state of A and fix an index $i \in \mathbb{Z}$, then defining $z := y_i y_{i+1} y_{i+1} \ldots$ in $X_n^{\mathbb{N}}$, there exists $x \in X_n^{\mathbb{N}}$ such that the initial transducer $A_p : X_n^{\mathbb{N}} \to X_n^{\mathbb{N}}$ maps x to z.

Now let y, p, z and x be as in the previous paragraph, and let $\Gamma \in X_n^k$ be a word such that the state of A forced by Γ is p. Let $u \in X_n^{\mathbb{Z}}$ be defined by $u_i u_{i+1} \ldots := x$, $u_{i-k} u_{i-k+1} \ldots u_{i-1} := \Gamma$, and $u_j := 0$ for all $j < i - k$.

If $w \in X_n^{\mathbb{Z}}$ is the image of u under A, then $w_i w_{i+1} \ldots = z$. Therefore for any $y \in X_n^{\mathbb{Z}}$ we can find an element in $(X_n^{\mathbb{Z}})A$ as arbitrarily close to y with respect to the metric given in Definition 1.4.3. □

Remark 3.4.23. Given an element A of $\widetilde{\mathcal{H}}_n$, Proposition 3.4.21 gives an algorithm for determining if $A \in \mathcal{H}_n$ or if $A \in \widetilde{\mathcal{H}}_n \backslash \mathcal{H}_n$ since we have only to check if $\overline{A}_j$ is a permutation for all $1 \leqslant j \leqslant kM(A)$, where k is the synchronizing level of A and $M(A)$ is quadratic in the states of A.

Remark 3.4.24. It is a consequence of the proof of the proposition above that for $A \in \mathcal{P}_n$, $\overline{A}_l$ maps prime words to prime words for every $l \in \mathbb{N}$. This is because if, for some prime word Γ, $(\Gamma)\overline{A}_l = (\gamma)^r$ for for $|\gamma| < |\Gamma|$ and $r \in \mathbb{N}_1$, then either $\overline{A}_l : X_n^{|\gamma|}$ is not surjective and so it is not injective either, or there is a word $\delta \in X_n^{\gamma}$ such that $(\delta)\overline{A}_l = \gamma$. Since Γ is a prime word it follows in either case, as in the proof of Proposition 3.4.21, that A does not induce a homeomorphism of $X_n^{\mathbb{Z}}$. An alternative proof of this fact can be found in [10].

Proposition 3.4.13 indicates that if two elements A and B in $\mathcal{H}_n$ are such that $\overline{A}_j$ and $\overline{B}_j$ have the same disjoint cycle structure for all $j \in \mathbb{N}$ then A and B are likely to be conjugate. This however need not be the case as will be seen in Theorem 3.4.35. First we make the following definitions.

Definition 3.4.25. Let $\Gamma = \gamma_0 \gamma_2 \ldots \gamma_k - 1$ be a word in X_n^k for some natural number $k > 0$. Define the i^{th} rotation of Γ to be the word: $\Gamma' = \gamma_{k-i} \gamma_{k-i+1} \cdots \gamma_0 \gamma_1 \cdots \gamma_{k-i-1}$.

Remark 3.4.26. One can think of Γ as decorating a circle divided into k intervals (counting from zero), and Γ' is the result of rotating the circle clockwise by i. Then the 0^{th} rotation of Γ is simply Γ.

Definition 3.4.27 (Rotation)**.** Let $A \in \mathcal{P}_n$ and let $l \in \mathbb{N}$. Given a prime word $\Gamma \in X_n^l$, let C be the disjoint cycle of $\overline{A}_l$ containing Γ. Notice that C consists only of prime words by Remark 3.4.24. Let $1 \leqslant s \leqslant \text{length}(C)$ be minimal in $\mathbb{N}$ such that $(\Gamma)\overline{A}_l^s$ is a rotation of Γ and $0 \leqslant i < l$ be minimal such that $(\Gamma)\overline{A}_l^s$ is the i^{th} rotation of Γ. We say that C has minimal *rotation* i of Γ. We call the triple $(\text{length}(C), s, i)_\Gamma$ *the triple associated to* C *for* Γ.

Lemma 3.4.28. *Let* $C \in \overline{A}_l$ *be a disjoint cycle with associated triple* $(\text{length}(C), s_C, r_C)_{\Gamma_0}$ *for* Γ_0 *a prime word belonging to* C*. Then we have the following:*

(i) for any other word Γ *belonging to* C *we have:*

$$(\text{length}(C), s_C, r_C)_{\Gamma_0} = (\text{length}(C), s'_C, r'_C)_{\Gamma},$$

(ii) and $\text{Length}(C) = o \cdot s_C$ *where* o *is the order of* r_C *in the additive group* $\mathbb{Z}_l$*, if* $r_C = 0$ *then take* $o = 1$*.*

Proof. Let $C = (\Gamma_0 \dots \Gamma_j)$ and let $(\text{length}(C), s_C, r_C)_{\Gamma_0}$ be the triple associated to C for Γ_0, where Γ_0 is a prime word . Then s_C is minimal such that Γ_{s_C} is the $r_C{}^{\text{th}}$ rotation of Γ_0. Now since Γ_1 is the output of the unique loop of A labelled by Γ_0, then Γ_{s_C+1} is also a $r_C{}^{\text{th}}$ rotation of Γ_1. This is because the unique loop of A labelled by Γ_{s_C+1} is the $r_C{}^{\text{th}}$ rotation of the loop labelled by Γ_0. We can now replace C with the disjoint cycle $(\Gamma_1 \dots \Gamma_j \Gamma_1)$ and repeat the argument, until we have covered all rotations of C. This shows that the triple $(\text{length}(C), s_C, r_C)_{\Gamma_1}$ is independent of the choice of Γ_1.

For the second part of the lemma, first observe that if $s_C = \text{length}(C)$, then $r_C = 0$ and we are done. Therefore we may assume that $1 \leqslant s_c < \text{length}(C)$.

Now observe that by minimality of s_C and the above argument, $\Gamma_{s_C+s_C}$ is the $2r_C{}^{\text{th}}$ rotation of Γ_0, moreover no Γ_k for $s_C < k < 2s_C$ is a rotation of Γ_0. Notice that r_C has finite order in the additive group $\mathbb{Z}_l$. Let o be the order of r_C. Then Γ_{os_C} is the $or_C{}^{\text{th}}$ rotation of Γ which is just Γ. Moreover by minimality of s_C, and repetitions of the argument in the previous paragraph, o is minimal such that $\Gamma_{os_C} = \Gamma_0$. However by the first part of the lemma, we must also have $(\Gamma_k)\overline{A}_l^{os_C} = \Gamma_k$ $1 \leqslant k \leqslant j$. Minimality now ensures that $os_C = j$. □

Definition 3.4.29. As a consequence of the remark above, for a given disjoint cycle $C \in \overline{A}_l$ we call $(\text{length}(C), s_C, r_C)$ *the triple associated to* C.

Remark 3.4.30. We observe that for a cycle C of $\overline{A}_l$, $A \in \mathcal{P}_n$, the number r_C is what is called the 'return number' in [13], although we arrived at this notion independently. From the 'return numbers' the authors of [13] derive what they call the 'gyration function'. This has proven to be a very important and useful function, however we shall not require it for this work.

Definition 3.4.31 (Spectrum)**.** Let $A \in \mathcal{P}_n$, and let $k \in \mathbb{N}$. For each triple (L_C, S_C, T_C) associated to a disjoint cycle of prime words in the disjoint cycle structure of $\overline{A}_k$, let d_C denote the multiplicity with which it occurs as we consider all such triples associated to the disjoint cycles of $\overline{A}_k$. Then define $\text{Sp}_k(A) := \{(k, d_C, (L_C, S_C, T_C))\}$ as C runs over all disjoint cycles of $\overline{A}_k$. Define $\text{Sp}(A) := \bigcup_{k \in \mathbb{N}} \text{Sp}_k(A)$.

Theorem 3.4.32. *Let* $A \in \mathcal{P}_n$*, and let* $k \in \mathbb{N}$*, then* $\text{Sp}_k(A)$ *is a conjugacy invariant of* A *in* $\mathcal{P}_n$*.*

Proof. Let C be a cycle in the disjoint cycle structure of $\overline{A}_k$ and let (L_C, S_C, T_C) be its associated triple. Let $J \in \mathcal{P}_n$ be arbitrary and invertible.

That L_C is preserved under conjugation by J follows from Proposition 3.4.13, and standard results about permutation groups.

That S_C is preserved under conjugation is a consequence of the fact that $J \in \mathcal{P}_n$. To see this first suppose that $C = (\Gamma_1 \dots \Gamma_j)$ for some $j \in \mathbb{N}$. Let $\Delta_i = (\Gamma_i)\overline{J}_k$. Then $(\Delta_1 \dots \Delta_j)$ is a cycle of $\overline{J}_k^{-1}\overline{A}_k\overline{J}_k$. Since Δ_i is the output of the unique loop of J labelled by Γ_i $(1 \leqslant i \leqslant j)$, and since S_C is minimal so that Γ_{S_C} is a rotation of Γ_1, then S_C is also the minimal position so that Δ_{S_C} is a rotation of Δ_1.

That T_C is preserved under conjugation is once more a consequence of the fact that $J \in \mathcal{P}_n$. Let Γ_i and Δ_i for $1 \leqslant i \leqslant j$ be as in the previous paragraph. Since Γ_{S_C} is the $T_C{}^{\text{th}}$ rotation of Γ_1, then as Δ_1 is the output of the unique loop of J labelled by Γ_1, Δ_{S_C} is the $T_C{}^{\text{th}}$ rotations of Δ_1. □

Corollary 3.4.33. *Let* $A \in \mathcal{P}_n$*, then* $\text{Sp}(A)$ *is a conjugacy invariant of* A *in* $\mathcal{P}_n$*.*

Remark 3.4.34. Theorem 3.4.32 and Corollary 3.4.33 are known already in the literature centering around the automorphisms of the shift, in particular they appear in [14] in the language of 'return numbers'. However we arrived at our results independently and only later learned about the results of Boyle and Krieger.

Let M be the element of $\mathcal{H}_n$ below. By computing $Sp_3(M)$ and $Sp_3(M^{-1})$ and using Theorem 3.4.32 we shall show that M is not conjugate to its inverse in $\mathcal{P}_n$ and therefore in $\mathcal{H}_n$.

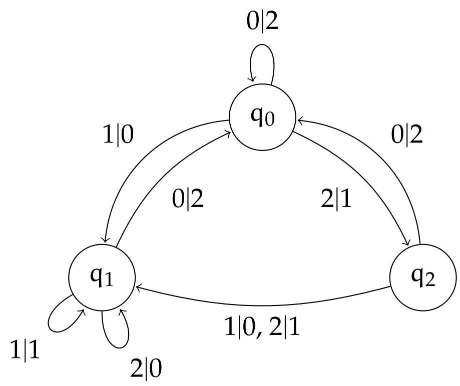


Figure 3.6: An element of $\mathcal{H}_n$ which is not conjugate to its inverse

$$\begin{aligned}\overline{M}_3 =&(000\ 222)(111)\\ &(001\ 220\ 112\ 110\ 012\ 200\ 122\\ &\ 100\ \ 022\ 211\ 011\ 201\ 020\ 212\\ &\ 010\ \ 202\ 121\ 101\ 120\ 002\ 221)\\ &(021\ 210\ 102)\end{aligned}$$

and

$$\begin{aligned}\overline{M}_3^{-1} =&(000\ 222)(111)\\ &(001\ 221\ 002\ 120\ 101\ 121\ 202\\ &\ 010\ \ 212\ 020\ 201\ 011\ 211\ 022\\ &\ 100\ \ 122\ 200\ 012\ 110\ 112\ 220)\\ &(021\ 102\ 210)\end{aligned}$$

From this we see that $Sp_3(M) = \{(3, 1, (21, 7, 1)), (3, 1, (3, 1, 2))\}$, and $Sp_3(M^{-1}) = \{(3, 1, (21, 7, 2)), (3, 1, (3, 1, 1))\}$. Since $Sp_3(M) \neq Sp_3(M^{-1})$, then M is not conjugate to M^{-1} by Theorem 3.4.32.

Theorem 3.4.35. *There are elements $M \in \mathcal{P}_n$ such that M and M^{-1} are not conjugate in $\mathcal{P}_n$.*

Note that the above theorem is false in the group of automorphisms of the n-ary rooted tree. The author was unable to find Theorem 3.4.35 in the literature on automorphisms of the shift dynamical system.

We now return to the question of conjugacy in $\mathcal{P}_n$. In light of Corollary 3.4.15 we describe below a method of constructing candidate conjugators. We first begin by identifying a subset of $Sym(X_n^j)$ for $j \in \mathbb{N}$.

Definition 3.4.36 (Allowable Permuations)**.** Let $k \geqslant 1 \in \mathbb{N}$ and ρ be a permutation of X_n^{k+1}, then ρ is called an *allowable permutation* if the following hold:

(1) there do not exist $i, j, l \in X_n$ and $\Gamma \in X_n^k$ such that $(i\Gamma)\rho = j\Delta$ and $(l\Gamma)\rho = j\Lambda$ for some Λ and Δ in X_n^k;

(2) let $\Gamma = g_1 g_2 \dots g_{k+1}$, and set $g_0 = \epsilon$. Let $\Delta = d_1 d_2 \dots d_{k+1}$ and $\Lambda_i \in X_n^k$ $(1 \leqslant i \leqslant k+1)$ be such that $(g_i g_{i+1} \dots g_{k+1} g_1 \dots g_{i-1})\rho = d_i \Lambda_i$ for $1 \leqslant i \leqslant k$ and $(g_{k+1} g_1 \dots g_k)\rho = d_{k+1}\Lambda_{k+1}$, then $\Delta = (\Gamma)\rho$.

Definition 3.4.37. Let $j \in \mathbb{N}$ and let $\rho : X_n^{j+1} \to X_n^{j+1}$ be any transformation of the set X_n^{j+1}, if ρ satisfies part (2) of Definition 3.4.36, then we call ρ an *allowable transformation*.

The following lemma is immediate from the definition of allowable transformations and the definition of $\overline{A}$ for $A \in \widetilde{\mathcal{P}}_n$ (Remark 3.4.10) so we omit its proof.

Lemma 3.4.38. *Let $A \in \widetilde{\mathcal{H}}_n$ ($A \in \widetilde{\mathcal{P}}_n$) be synchronizing at level j, then $\overline{A}_{j+1}$ is an allowable permutation (transformation).*

Proposition 3.4.39. *Let ρ be an allowable permutation [transformation] of X_n^{j+1} for $j \in \mathbb{N}_1$, then there is an element $A \in \widetilde{\mathcal{H}}_n$ [$A \in \widetilde{\mathcal{P}}_n$] which is synchronizing at level j, which is unique for (j, ρ).*

Proof. Let ρ and j be as in the statement of the proposition. We construct $A = \langle X_n, Q, \lambda, \pi \rangle$ as follows. The state set Q of A will be the set X_n^j. Fix a state Γ of A, let $\overline{\Gamma}$ denote the length $j-1$ suffix of Γ and $i \in X_n$, then the following equations determine the transition of state Γ on input i:

$$\pi(i, \Gamma) = \overline{\Gamma} i$$
$$\lambda(i, \Gamma) = j, \text{ where } (i\Gamma)\rho = j\Delta \text{ for } \Delta \in X_n^j$$

Observe that the resulting transducer A is synchronizing at level j, since for two words Δ, Γ in X_n^j, regarding Δ as a state of A, we have $\pi(\Gamma, \Delta) = \Gamma$. This argument also shows that A is equal to its core. Moreover, by Definition 3.4.36 part (1), we have that $A \in \widetilde{\mathcal{H}}_n$ since all states of A induce a bijection from $X_n \to X_n$.

We now argue that $\overline{A}_{j+1} = \rho$ from which it will follow that A is unique for (j, σ) by Corollary 3.4.15. First we establish some notation: for a word $\Xi \in X_n^+$ set $\Xi\restriction_1$ to be the first letter of Ξ.

Let $\Gamma = \gamma_1 \dots \gamma_{j+1} \in X_n^{j+1}$ and set $\overline{\Gamma} := \gamma_2 \dots \gamma_{j+1}$. Observe that $\overline{\Gamma}$ is the unique state of A with a loop labelled Γ, since $\pi(\Gamma, \boldsymbol{.}) : Q_A \to Q_A$ takes only the value $\overline{\Gamma}$. Moreover, for $1 \leqslant i \leqslant j+1$ we have

$$\pi(\gamma_i, \gamma_{i+1} \dots \gamma_j \gamma_1 \dots \gamma_{i-1}) = \gamma_{i+2} \dots \gamma_j \gamma_1 \dots \gamma_i$$

and

$$\lambda(\gamma_i, \gamma_{i+1} \dots \gamma_j \gamma_1 \dots \gamma_{i-1}) = (\gamma_i \gamma_{i+1} \dots \gamma_j \gamma_1 \dots \gamma_{i-1})\rho\restriction_1$$

if $i = j+1$ take $i+1 = \gamma_1$, and if $i = 1$ take $\gamma_{i-1} = \epsilon$. Set $\delta_i := \lambda(\gamma_i, \gamma_{i+1} \dots \gamma_j \gamma_1 \dots \gamma_{i-1}) = (\gamma_i \gamma_{i+1} \dots \gamma_j \gamma_1 \dots \gamma_{i-1})\rho\restriction_1$ and $\Delta = \delta_1 \dots \delta_{j+1}$. Observe that $\Delta = (\Gamma)\overline{A}_{j+1}$. Moreover, by Definition 3.4.36 part (2), we have that $\Delta = (\Gamma)\rho$. Since $\Gamma \in X_n^{j+1}$ was chosen arbitrarily we have, $\rho = \overline{A}_{j+1}$. Set B to be the weakly minimal transducer (and so minimal since $A \in \widetilde{\mathcal{H}}_n$) representing A, then B is synchronizing at level j, by Proposition 2.1.30, moreover $\overline{B}_{j+1} = \overline{A}_{j+1} = \rho$ since the map $\overline{A}_{j+1}$ is unaffected by minimization.

The other reading of the proposition is proved analogously. □

Remark 3.4.40. Observe that given any transformation ρ of the set X_n^{j+1}, one can construct an element $f \in F(X_n, j+1)$, by setting $(\Gamma)f = (\Gamma)\rho\restriction_1$ for $\Gamma \in X_n^j$. Thus from any transformation ρ of X_n^{j+1} one can obtain an element A of $\widetilde{\mathcal{P}}_n$ which is synchronizing at level j. However, it is not always the case that $\overline{A}_{j+1} = \rho$. If we further insist that ρ is an allowable transformation of X_n^j, Proposition 3.4.39 guarantees that we can find a unique element A of $\widetilde{\mathcal{P}}_n$ which is synchronizing at level j for which $\overline{A}_j = \rho$.

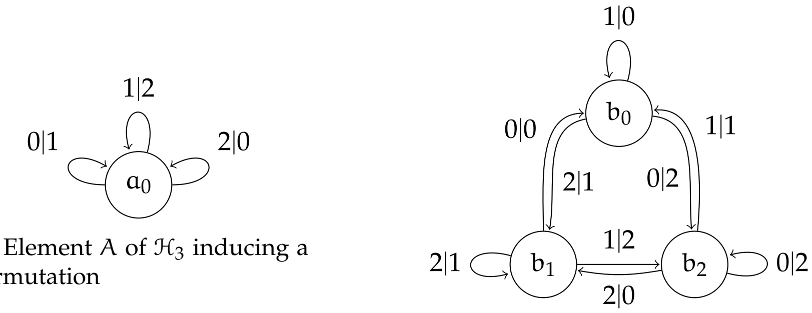
Let $A, B \in \mathcal{H}_n$ be transducers and let $k \in \mathbb{N}_1$ such that A and B are synchronizing at level k. Suppose $\overline{A}_{k+1}$ and $\overline{B}_{k+1}$ have the same disjoint cycle structure. If $\overline{h}$ is a permutation of X_n^k conjugating $\overline{A}_k$ to $\overline{B}_k$, then Proposition 3.4.39 allows us to construct candidate conjugators for A and B where $\overline{h}$ is an allowable permutation. We describe the process in detail in the following construction.

Construction 3.4.41. Let $A, B \in \mathcal{H}_n$ and suppose that A and B are conjugate in $\mathcal{H}_n$. Let $k \in \mathbb{N}_1$ be such that both A and B are synchronizing at level k. Consider the permutation $\overline{A}_{k+1}$ and $\overline{B}_{k+1}$. Pick a representative for each disjoint cycle of $\overline{A}_{k+1}$ and $\overline{B}_{k+1}$ the cycle. For instance if $(\Gamma_1 \ldots \Gamma_l)$ is a cycle of A then we may chose $(\Gamma_1 \ldots \Gamma_l)$ or any cycle in the set $\{(\Gamma_i \ldots \Gamma_l \Gamma_1 \ldots \Gamma_{i-1}) \mid 1 < i \leqslant l\}$. We identify the disjoint cycles of $\overline{A}_{k+1}$ and $\overline{B}_{k+1}$ with their representatives. Set $C(\overline{A}_{k+1})$ and $C(\overline{B}_{k+1})$ to be the set of disjoint cycle of $\overline{A}_{k+1}$ and $\overline{B}_{k+1}$. Let $\mathfrak{c} : C(\overline{A}_{k+1}) \to C(\overline{B}_{k+1})$ be a bijection from the disjoint cycle of $\overline{A}_{k+1}$ to the disjoint cycles of $\overline{B}_{k+1}$. Such a bijection exists since A and B are conjugate. We may further insist that for $C \in C(\overline{A}_{k+1})$ and $D \in C(\overline{B}_{k+1})$ such that $(C)\mathfrak{c} = D$, the triple (L_C, s_C, r_C) associated to C (see Definition 3.4.29) is equal to the triple (L_D, s_D, r_D) associated to D by Theorem 3.4.32. We now construct an allowable transformation ρ of X_n^j.

Fix a cycle $C \in C(\overline{A}_{k+1})$ and a cycle $D \in C(\overline{B}_{k+1})$ such that $(C)\mathfrak{c} = D$. Suppose $C = (\Gamma_1 \ldots \Gamma_{l_C})$ and $D = (\Gamma_1 \ldots \Gamma_{L_D})$, fix $1 \leqslant i \leqslant l_C$, and suppose that $\Gamma_i = \gamma_{i,1} \ldots \gamma_{i,k+1}$ and $\Delta_i = \delta_{i,1} \ldots \delta_{i,k+1}$. Set $(\Gamma_i)\rho = \Delta_i$, and for $2 \leqslant j \leqslant k+1$ set $(\gamma_{i,j} \ldots \gamma_{i,k+1}\gamma_1 \ldots \gamma_{i,j-1})\rho = \delta_{i,j} \ldots \delta_{i,k+1}\delta_1 \ldots \delta_{i,j-1}$. Observe that this requirement means that ρ satisfies part (2) of Definition 3.4.36. Repeating this process across all disjoint cycles of A, we see that the transformation ρ so constructed is in fact a permutation of X_n^j, since every element $\Gamma \in X_n^j$ is in at most one disjoint cycle of $C(\overline{A}_{k+1})$ and of $C(\overline{B}_{k+1})$. Moreover by construction ρ is an allowable transformation. Furthermore, observe that since, for a disjoint cycle $C = (\Gamma_1 \ldots \Gamma_{l_C}) \in C(\overline{A}_{k+1})$ and $D = (\Delta_1 \ldots \Delta_{l_D}) \in C(\overline{B}_{k+1})$ such that $(C)\mathfrak{c} = (D)\mathfrak{c}$, we have that $(\Gamma_i)\rho = \Delta_i$ for all $1 \leqslant i \leqslant l_C$, it thus follows, from well known results about conjugacy in the symmetric group, that $\rho^{-1}\overline{A}_{k+1}\rho = \overline{B}_{k+1}$. Therefore we may construct a transducer $H \in \widetilde{\mathcal{P}}_n$ such that $\overline{H}_{k+1} = \rho$. If moreover $H \in \mathcal{H}_n$ (which can be checked in finite time) and is such that $\min \mathrm{Core}(H^{-1}AH)$ is synchronizing at level k, then Theorem 3.4.12 and Corollary 3.4.15 indicate that $\min \mathrm{Core}(H^{-1}AH) \cong_\omega B$.

We illustrate Construction 3.4.41 with a example below.

Example 3.4.42. Consider the conjugate transducers $A, B \in \mathcal{H}_n$ below. We construct a conjugator $H \in \mathcal{H}_n$ as in Construction 3.4.41.



(a) Element A of $\mathcal{H}_3$ inducing a permutation

(b) Element $B \in \mathcal{H}_3$

Figure 3.7: Conjugate elements of $\mathcal{H}_3$.

In order to construct a candidate conjugator we need only consider disjoint cycles of $\overline{A}_3$ and $\overline{B}_3$ for each rotation class of a word $\Gamma \in X_n^3$. We give these below.

For $\overline{A}_3$ we need only consider the permutation:

(000 111 222) (001 112 220)
(002 110 221) (012 120 201)
(021 102 210)

For $\overline{B}_3$ we need only consider:

$$(000\ 222\ 111)\ (102\ 220\ 110)$$
$$(001\ 221\ 012)\ (020\ 200\ 002)$$
$$(112\ 211\ 121)$$

This is because, for instance, one may deduce from these that the cycle containing 010 in $\overline{A}_3$ is (010 121 202), thus the conjugator H is determined by understanding what it does to elements on the cycles given above.

Consider the permutation ρ of X_n^3 given as follows:

$$(111\ 222)\ (111\ 222)$$
$$(001\ 102\ 211\ 022\ 011\ 122\ 201\ 002)$$
$$(010\ 021\ 112\ 220\ 110\ 221\ 012\ 020)$$
$$(100\ 210\ 121\ 202\ 101\ 212\ 120\ 200)$$

One can check $\rho^{-1}\overline{A}_3\rho = \overline{B}_3$, and that ρ is an allowable permutation. The element $H \in \mathcal{H}_3$ which is unique for $(2, \rho)$ is given below:

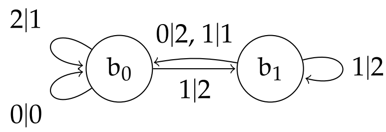


Figure 3.8: The conjugator H unique for $(2, \rho)$.

The following question is natural to ask at this stage:

Question 3.4.43. *For two conjugate elements* $A, B \in \mathcal{H}_n$ *and* $k \in \mathbb{N}$ *minimal such that both* A *and* B *are synchronizing at level* k*, is there always an element* $H \in \mathcal{H}_n$ *synchronizing at level* k *such that* $H^{-1}AH = B$*?*

Of course an answer in the affirmative yields a solution to the conjugacy problem in $\mathcal{H}_n$.

In the next section we focus on the order problem in $\mathcal{H}_n$ and the related finiteness problem for groups generated by transducers in $\mathcal{H}_n$.

3.5 The order problem, finiteness problem and groups and semigroups generated by transducers in $\widetilde{\mathcal{P}}_n$

This section shall deal mainly with the order problem and the related finiteness problem for groups generated by transducers in $\widetilde{\mathcal{H}}_n$. We shall also be concerned with the growth rates of the groups generated by transducers in $\widetilde{\mathcal{H}}_n$, and also the rate at which the number of states in the core increases with raising an element of $\widetilde{\mathcal{H}}_n$ to powers. As in the previous section some of the results in this section shall concern the semigroups generated by transducer in $\widetilde{\mathcal{P}}_n$. All transducers in this section shall be over the alphabet X_n unless otherwise stated.

We begin by making relevant definitions and stating known results.

Definition 3.5.1. Let A be a synchronous transducer over the alphabet X_n. For each state $q \in Q_A$ the initial transducer A_q induces a continuous function $h_q : \mathfrak{C}_n \to \mathfrak{C}_n$ (see Subsection 1.5.1). Thus set $\mathcal{S}(A)$ to be the semigroup generated by the set $\{h_q \mid q \in Q_A\}$. If A is also invertible, then for each state $q \in Q_A$, the map $h_q : \mathfrak{C}_n \to \mathfrak{C}_n$ is a homeomorphism, thus set $\mathcal{G}(A)$ to be the group generated by the set $\{h_q \mid q \in Q_A\}$. We call the semigroup $\mathcal{S}(A)$ the *automaton semigroup generated by* A and the group $\mathcal{G}(A)$ the *automaton group generated by* A.

Remark 3.5.2. Synchronous transducers are elsewhere in the literature referred to as Mealy-automata (see for instance [30]) hence the phrases 'automaton group' and 'automaton semigroup'. For a (semi)group G, we shall use the phrase 'G is a (semi)group generated by a transducer' to indicate that there is a synchronous (invertible) transducer A such that $G = \mathcal{G}(A)$. We will suppress the adjective 'synchronous' as this section deals only with synchronous transducers.

Given a set M of self-homeomorphisms of $\mathfrak{C}_n$, we will sometimes be concerned with both the group and semigroup generated by M, we establish the following notation to distinguish between the two.

Notation 3.5.3. Given a set M of continuous functions $m : \mathfrak{C}_n \to \mathfrak{C}_n$, we denote by $\langle M \rangle_+$ the semigroup generated by the elements of M. If M consists of self-homeomorphisms of $\mathfrak{C}_n$ then we denote by $\langle M \rangle$ the group generated by M.

Groups and semigroups generated by transducers are a well studied class of groups and have proven to be a source of groups with interesting properties. For instance the first example of a group of intermediate growth, the first Grigorchuk group, is a group generated by a transducer. The following algorithmic questions are in some sense natural to ask about this class of groups:

The finiteness problem: Given a finite, synchronous, transducer A is there an algorithm which decides in finite time if the automaton semigroup (or group if A is invertible) generated by A is finite?

The order problem: Given a finite, synchronous, invertible transducer A is there an algorithm which, given an element $g \in \mathcal{G}(A)$, decides in finite time if g has finite order?

As we shall see, for groups generated by transducers in $\widetilde{\mathcal{H}}_n$ the order problem and finiteness problem are equivalent.

The finiteness problem for semigroups generated by transducers has been demonstrated to have a negative answer in general by Pierre Gillibert ([27]). However, the finiteness problem for groups generated by transducers remains open. The order problem, has also recently been shown to be undecidable by Gillibert ([26]) and, independently by Bartholdi and Mirtofanov ([3]). However, it remains open in certain classes of automaton groups including those generated by transducers in $\widetilde{\mathcal{H}}_n$.

In the literature around automaton groups, it is also a normal procedure to investigate the growth rate of a group or semigroup G generated by a transducer A with respect to the word metric. It turns out that, in the semigroup case, this is equivalent to the growth rate of the transducer A itself. We introduce these two notions of growth below.

Definition 3.5.4. Let $G = \langle M \rangle$ be a finitely generated group, that is, $|M| < \infty$. Given an element $g \in G$ we say that g has *length* l, if l is minimal in $\mathbb{N}$ such that g can be written as a product $g = m_1 m_2 \dots m_l$ for $m_i \in M$ and $1 \leqslant i \leqslant l$. Write $l(g)$ for the length of g.

We now define the growth function.

Definition 3.5.5. Let $G = \langle M \rangle$ be a finitely generated group. Define a function $\gamma_G : \mathbb{N} \to \mathbb{N}$ by $\gamma_G(l) = |\{g \in G \mid l(g) \leqslant l\}|$. We call γ_G the *growth function of* G.

Definition 3.5.6. A finitely generated group $G = \langle M \rangle$ is said to have *exponential growth (rate)* if there is a $C \in \mathbb{R}^+$ such that $\gamma_G(l) \geqslant e^{Cl}$; G is said to have *polynomial growth* if there are $C, d \in \mathbb{R}^+$ such that $\gamma_G(l) \leqslant Cl^d$ for all $l \in \mathbb{N}$; G is said to have *intermediate growth* if γ_G is greater than any polynomial function on $\mathbb{N}$ and less then any exponential function on $\mathbb{N}$.

Definition 3.5.7. Given two non-decreasing functions $g_1 : \mathbb{N} \to N$ and $g_2 : \mathbb{N} \to \mathbb{N}$, we write $g_1 \preceq g_2$ if there is a constant $C \in \mathbb{R}^+$ such that $(l)g_1 \leqslant (Cl)g_2$. If $g_1 \preceq g_2$ and $g_2 \preceq g_1$ then we write $g_1 \simeq g_2$.

Remark 3.5.8. It is a standard result that the growth rate of the group is independent of the choice of generating set. Observe also that one may make analogous definitions for a finitely generated semigroup $S = \langle M \rangle_+$ for a finite set M. Thus we obtain analogous notions of a semigroup of exponential, intermediate and polynomial growth.

The following result is standard in the literature around the growth rates of group and semigroups.

Theorem 3.5.9. *If a finitely generated group or semigroup* G *contains a non-abelian free subsemigroup, then* G *has exponential growth.*

For a semigroup G generated by a transducer A, the growth rate of the group is connected to the growth rate of the transducer A which we define below.

Definition 3.5.10. Let A be a synchronous transducer over the alphabet X_n. Define a function $\gamma_A : \mathbb{N} \to \mathbb{N}$ by $\gamma_A(l) = |\min(A^l)|$. We call γ_A the *growth function of the transducer* A. We say that A has *exponential growth (rate)* if there is a $C \in \mathbb{R}^+$ such that $\gamma_A(l) \geqslant e^{Cl}$; A is said to have *polynomial growth* if there are $C, d \in \mathbb{R}^+$ such that $\gamma_A(l) \leqslant Cl^d$ for all $l \in \mathbb{N}$; A is said to have *intermediate growth* if γ_A is greater than any polynomial function on $\mathbb{N}$ and less then any exponential function on $\mathbb{N}$.

Remark 3.5.11. Notice that for a synchronous transducer A over the alphabet X_n and $l \in \mathbb{N}$, the value $\gamma_A(l)$ is precisely the number of distinct elements of $\mathcal{S}(A)$ which can be written as a product of length l with respect to the generating set $\{h_q | q \in Q_A\}$.

The following proposition is standard in the literature on automaton groups and can be found in [28].

Proposition 3.5.12. *Let* S *be a semigroup generated by a transducer* A*, then* $\gamma_S \simeq \gamma_A$ *where the growth function* γ_S *of* S *is with respect to the generating set* $\{h_q \mid q \in Q_A\}$.

The remainder of this chapter shall be devoted mainly to investigating the finiteness problem, the order problem for groups generated by transducers in $\widetilde{\mathcal{H}}_n$ and investigating the growth rate of groups and semigroups generated by transducers in $\widetilde{\mathcal{H}}_n$. We begin by considering what initially seems to be a special case, that is we consider the case of level one synchronizing transducers. Such transducers are called *reset automata* elsewhere in the literature and the groups generated by reset automata were studied by Silva and Steinberg in [51]. We present some of their results relating to the structure of the groups generated by reset automata and demonstrate the relation to groups generated by synchronizing transducers.

3.5.1 Level one synchronizing transducers

In this section we present some known results about level one synchronizing, synchronous, transducers. Such transducers are elsewhere in the literature called reset automata and groups generated by reset automata have been studied by Silva and Steinberg in [51]. Reset automata also have connections to tilings of the plane ([36]) and making use of these connections Gillibert [27] was able to demonstrate that the finiteness problem for semigroups generated by reset automata is undecidable. However we shall not make use of the connections to tillings of the plane in this work.

Our first result will be to demonstrate that making the reduction to reset automata makes no difference when analysing the group or semigroup generated by synchronizing, synchronous transducers. We begin with the following construction.

Construction 3.5.13. Let $A = \langle X_n, Q_A, \pi_A, \lambda_A \rangle$ be a synchronizing, synchronous transducer, and let $k \in \mathbb{N}_1$ be the minimal synchronizing level of A. Form a new transducer $A_k = \langle X_n^k, Q_A, \pi_{A_k}, \lambda_{A_k} \rangle$ with the same set of states as A and input and output alphabet the set of words of length k. The transition and output function of A are defined as follows, for $\Gamma \in X_n^k$ and $p \in Q_A$ set $\pi_{A_k}(\Gamma, p) = \pi_A(\Gamma, p)$ and $\lambda_{A_k}(\Gamma, p) = \lambda_A(\Gamma, p)$. Observe that since A_k is synchronous, then λ_A is a function from $X_n^k \times Q_A$ to X_n^k.

Remark 3.5.14. Let $A = \langle X_n, Q_A, \pi_A, \lambda_A \rangle$ be a synchronizing, synchronous transducer, and let $k \in \mathbb{N}_1$ be the minimal synchronizing level of A. Since A is synchronizing at level k, it follows that A_k is a reset automata, furthermore if A is invertible then A_k is also invertible, moreover $(A^{-1})_k = (A_k)^{-1}$ by definition of the transition and output functions of A_k.

We have the following result:

Proposition 3.5.15. *Let A be a synchronous, synchronizing transducer which is synchronizing at level k. Form the reset automaton A_k as in Construction 3.5.13, then $\mathcal{S}(A) \cong \mathcal{S}(A_k)$ and, if A is invertible, $\mathcal{G}(A) \cong \mathcal{G}(A_k)$.*

Proof. We begin with the semigroup case. Let A be a synchronous, synchronizing transducer and let $k \in \mathbb{N}_1$ be the synchronizing level of A. In order to distinguish between states of A and A_k, for a state p of A we shall denote by p' the corresponding state of A_k.

Let $p_1, p_2, \ldots, p_l$ and $q_1, q_2, \ldots, q_m$ for $l, m \in \mathbb{N}_1$ be states of A. Suppose that the composition $h_{p_1} \ldots h_{p_l} \neq h_{q_1} \ldots h_{q_m}$. Observe that $h_{p_1} \ldots h_{p_l}$ is equal to $h_{(p_1,\ldots,p_l)}$ for $(p_1, \ldots, p_l)$ a state of A^l, likewise $h_{q_1} \ldots h_{q_m}$ is equal to $h_{(q_1,\ldots,q_m)}$. Since $h_{(p_1,\ldots,p_l)} \neq h_{(q_1,\ldots,q_m)}$, there is a word $\Gamma \in X_n^{mk}$ for some $m \in \mathbb{N}_1$ such that $\lambda_{A^l}(\Gamma, (p_1, \ldots, p_l)) \neq \lambda_{A^m}(\Gamma, (q_1, \ldots, q_m))$. However, since $\Gamma \in (X_n^k)^m$, by definition of the output function λ_{A_k}, we have, $\lambda_{A_k^l}(\Gamma, (p'_1, \ldots, p'_l)) \neq \lambda_{A_k^m}(\Gamma, (q'_1, \ldots, q'_m))$. Therefore $h_{p'_1} \ldots h_{p'_l} \neq h_{q'_1} \ldots h_{q'_m}$.

Let $\phi' : Q_A \to Q_{A_k}$ be defined by $(p)\phi' = p'$ for all $p \in Q_A$. Then ϕ' extends to a homomorphism $\phi : \mathcal{S}(A) \to \mathcal{S}(A_k)$. Since ϕ is onto the generators of $\mathcal{S}(A)$ it is surjective, and by the preceding paragraph it is injective. Therefore, ϕ is an isomorphism.

For the case where A is invertible, we likewise define the map $\phi' : Q_A \to Q_{A_k}$ by $p \mapsto p'$. We then observe that for states $p_1, \ldots, p_l$ of Q_A, $l \in \mathbb{N}_1$, and $\epsilon_i \in \{-1, +1\}$, if the composition $h_{p_1}^{\epsilon_1} \ldots h_{p_l}^{\epsilon_l}$ is not trivial, then there is a word $\Gamma \in X_n^{mk}$, $m \in \mathbb{N}_1$ such that $(\Gamma)A_{p_1}^{\epsilon_1} \ldots A_{p_l}^{\epsilon_l} \neq \Gamma$. However, since A_{p_i} and $(A_k)_{p'_i}$ coincide on X_n^{mk} for $m \in \mathbb{N}_1$ by definition of the output function and transition function of A_k, we have, $(\Gamma)(A_k)_{p_1}^{\epsilon_1} \ldots (A_k)_{p_l}^{\epsilon_l} \neq \Gamma$. Therefore, the map ϕ' once more extends to an isomorphism ϕ from $\mathcal{G}(A) \to \mathcal{G}(A_k)$. □

Proposition 3.5.15 implies that, from the standpoint of the finiteness and order problem, the reduction to reset automata is without loss of generality.

The following result is due to Silva and Steinberg [51].

Theorem 3.5.16. *For an invertible, synchronous, reset automaton A the group $\mathcal{G}(A)$ is infinite if and only if there is a state $p \in Q_A$ such that $\langle h_p \rangle \cong \mathbb{Z}$. Moreover, in the case that $\mathcal{G}(A)$ is infinite, we have $\mathcal{G}(A) = N \ltimes \langle h_p \rangle$ for N a locally finite group.*

In order to prove this result, we first require the following lemma.

Lemma 3.5.17. *Let A be a synchronous, synchronizing transducer with synchronizing level k, then $A^{-1}A$ is synchronizing at level k, and* Core$(A^{-1}A) = $ *id.*

Proof. This follows from the forward implication of Lemma 2.4.1 and its proof. □

Observe that Lemma 3.5.17 implies that given a synchronous transducer A synchronizing at level k, then for any pair p, q of states of A, $h_p^{-1}h_q$ is an element of $G_{n,1}$ which, for every $j \in \mathbb{N}$, induces a map from X_n^j to itself.

We are now ready to prove Theorem 3.5.16.

Proof of Theorem 3.5.16. Fix a state $p \in Q_A$. For ease of notation, for a state $q \in Q_A$, we identity q with the map h_q. Set $G = \langle p^{-1}q \mid q \in Q_A \rangle$. By Lemma 3.5.17, G is a subgroup of $G_{n,1}$. Let $N = \langle p^{-i}Gp^i \mid i \in \mathbb{Z} \rangle$. Observe that for $i_1, i_2, \ldots, i_l, j \in \mathbb{Z}$ and $g_1, g_2, \ldots g_l \in G$, $(g_1^{p^{i_1}} \ldots g_l^{p^{i_l}})^{p^j} = g_1^{p^{i_1+j}} \ldots g_l^{p^{i_l+j}} \in N$, and so for $\nu \in N$ and $j \in \mathbb{Z}$, $\nu^{p^j} \in N$. Further observe that for any state $q \in Q_A$ we have, $q = p(p^{-1}q)$ and $q^{-1} = p^{-1}(p(q^{-1}p)p^{-1})$, therefore $q, q^{-1} \in \langle p \rangle N$ for any $q \in Q_A$. Moreover given $j_1, j_2 \in \mathbb{Z}$ and $\nu_1, \nu_2 \in N$ we have, $p^{j_1}\nu_1 p^{j_2}\nu_2 = p^{j_1+j_2}\nu_1^{p^{j_2}}\nu_2 \in \langle p \rangle N$ since $\nu_1^{p^{j_2}} \in N$. Therefore, $\mathcal{G}(A) = \langle p \rangle N$.

Now we demonstrate that N is locally-finite. Let $l \in \mathbb{N}_1$ and $\nu_1, \ldots, \nu_l \in N$. Consider the subgroup $\langle \nu_i \mid 1 \leqslant i \leqslant l \rangle \leqslant N$. Since, for all $1 \leqslant i \leqslant l$, ν_i is a product of elements of the form $p^j g p^{-j}$, $j \in \mathbb{Z}$ and $g \in G$, there exist numbers $m, M \in \mathbb{N}$, such that for all $1 \leqslant i \leqslant l$, $p^{-m}\nu_i p^m \in \langle p^{-t}Gp^t \mid 1 \leqslant t \leqslant M \rangle$. Set $H = \langle p^{-t}Gp^t \mid 1 \leqslant t \leqslant M \rangle$ and observe that $\langle \nu_i \mid 1 \leqslant i \leqslant l \rangle$ is a subgroup of H. Since p is synchronizing, p^t is synchronizing for $1 \leqslant t \leqslant M$

by Lemma 2.1.32, and since G is a subgroup of $G_{n,1}$, it follows by Lemma 2.4.10, that $H \subset G_{n,1}$. Moreover, as every element of H preserves the length of words (since A is synchronous), it follows that H is a finite group, and so $\langle v_i \mid 1 \leqslant i \leqslant l \rangle$ is also a finite group. Therefore, N is locally finite.

If $\langle p \rangle$ is infinite, then we have that $\langle p \rangle \cap N = \{\mathrm{id}\}$, and $G = \langle p \rangle \ltimes N$. If p has finite order $o(p)$, then $N = \{p^{-i} G p^{i} \mid 1 \leqslant i \leqslant o(p)\}$ is a finitely generated locally-finite group and so finite, from which we deduce that G is finite. $\square$

Theorem 3.5.16 extends to invertible, synchronous, synchronizing transducers by applying Proposition 3.5.15, thus we have the following Corollary.

Corollary 3.5.18. *Let A be a synchronizing, invertible, synchronous transducer, then $\mathcal{G}(A)$ is either finite or isomorphic to $N \rtimes \mathbb{Z}$ for a locally finite group N. Moreover $\mathcal{G}(A)$ is infinite if and only if there is a state $p \in Q_A$ such that h_p has infinite order.*

Remark 3.5.19. Observe that if a state p of a reset automaton has infinite order, then by Theorem 3.5.16, all states have infinite order. Moreover, by Theorem 3.5.16, the finiteness problem for a group generated by a reset automaton is equivalent to the order problem. This is because if the finiteness problem is soluble in this class of groups, then given a reset automaton A it is possible to determine if the states have finite order or not. If the states have infinite order, then using the semi-direct product decomposition of $\mathcal{G}(A) = N \ltimes \langle h_p \rangle$ for N a locally finite group and p a state of A, the order problem is soluble in $\mathcal{G}(A)$. If the states have finite order, then $\mathcal{G}(A)$ is finite by Theorem 3.5.16, and so all elements of $\mathcal{G}(A)$ have finite order. Making use of Proposition 3.5.15, it follows that the finiteness and order problem for groups generated by invertible, synchronous, synchronizing transducers are equivalent.

It follows from Theorem 3.5.16, that given a synchronizing, invertible, synchronous, transducer A the group $\mathcal{G}(A)$ is elementary amenable. This is a consequence of the definition ([22]) of elementary amenable groups as those which may be built from all finite groups and abelian groups by taking subgroups, direct unions, quotients and extensions.

In the next section we show the equivalence of the finiteness and order problem for groups generated by synchronous, synchronizing, transducers to the order problem in the group $\mathcal{H}_n$ by considering what happens to the core of elements of such transducers when raised to powers.

3.5.2 The equivalence of the finiteness and order problems for groups generated by synchronizing, synchronous transducers to the order problem in $\mathcal{H}_n$

In this section we demonstrate that the finiteness problem and order problem for groups generated by synchronous, synchronizing, transducers is equivalent to the order problem in $\mathcal{H}_n$. Notice that by Remark 3.5.19, it suffices to show that the finiteness problem for groups generated by synchronous, synchronizing transducers is equivalent to the finiteness problem for groups generated by transducers in $\widetilde{\mathcal{H}}_n$. We then show that the finiteness problem for groups generated by transducers in $\widetilde{\mathcal{H}}_n$ is equivalent to the order problem in $\mathcal{H}_n$.

We begin with the following proposition, but first observe that given a minimal, invertible, synchronous, synchronizing transducer A, $\min \mathrm{Core}(A) \in \widetilde{\mathcal{H}}_n$ by definition.

Proposition 3.5.20. *Let A be an invertible, synchronous, synchronizing transducer, and let $B = \mathrm{Core}(A)$. Then $\mathcal{G}(A)$ is finite if and only if $\mathcal{G}(B)$ is finite.*

Proof. Clearly $\mathcal{G}(B)$ is a subgroup of $\mathcal{G}(A)$, hence if $\mathcal{G}(A)$ is finite, then $\mathcal{G}(B)$ is finite. Thus, suppose that $\mathcal{G}(B)$ is finite. We now demonstrate that $\mathcal{G}(A)$ is finite as a consequence.

Let $k \in \mathbb{N}$ be the minimal synchronizing level of A, then as $B = \mathrm{Core}(A)$, for any word $\Gamma \in X_n^k$ and any state $p \in Q_A$ we have, $\pi_A(\Gamma, p) \in Q_B$. Let $p_1, \ldots, p_l \in Q_A$ for $l \in \mathbb{N}_1$ and consider the product $h_{p_1} \ldots h_{p_l}$. This product is equivalent to $h_{(p_1,\ldots,p_l)}$ where $(p_1, \ldots, p_l)$ is a state of A^l. Let $\Gamma \in X_n^k$ be arbitrary and observe that $\pi_{A^l}(\Gamma, (p_1, \ldots, p_l))$ is a state of B^l. Thus we may represent $h_{p_1} \ldots h_{p_l}$ as a pair $(\rho, (g_1, g_2, \ldots, g_{n^k}))$ where ρ is a permutation of X_n^k and $g_i \in \mathcal{G}(B)$ for $1 \leqslant i \leqslant n^k$. This is because after processing a word of length k through a state of A^l the resulting output is a word of length k and resulting active state is a state of B^l. Now since the state $(p_1, \ldots, p_l)$ of A^l was arbitrary, and the set of permutations of X_n^k and $\mathcal{G}(B)$ are finite, it follows that $\mathcal{G}(A)$ is also finite. $\square$

Thus the finiteness and order problem for groups generated by synchronous, invertible, synchronizing transducers, is equivalent to the finiteness and order problem for group generated by transducers in $\widetilde{\mathcal{H}}_n$.

Remark 3.5.21. Let $A \in \widetilde{\mathcal{H}}_n$ and suppose that for a state $p \in Q_A$ the initial transducer A_p has finite order (this is equivalent to the map h_p having finite order). Suppose $o(p) \in \mathbb{N}$ is such that $(A_p)^{o(p)} \cong_\omega \text{id}$, notice here that we are taking a product of initial transducers and so $A_p^{o(p)}$ corresponds to the initial transducer $A^{o(p)}_{(p,p,\ldots,p)}$. Since A_p is synchronizing, then by Proposition 2.1.32, $A_p^{o(p)-1}$ is also synchronizing. Moreover, since A is synchronizing, it is strongly connected, and since $A_p A_p^{o(p)-1} = \text{id}$, it follows that $A^{-1} = A_p^{o(p)-1}$, thus A is bi-synchronizing. Hence an element of $\widetilde{\mathcal{H}}_n$ which generates a finite group must be an element of $\mathcal{H}_n$ and all elements of $\widetilde{\mathcal{H}}_n \backslash \mathcal{H}_n$, for $n > 2$ generate infinite groups.

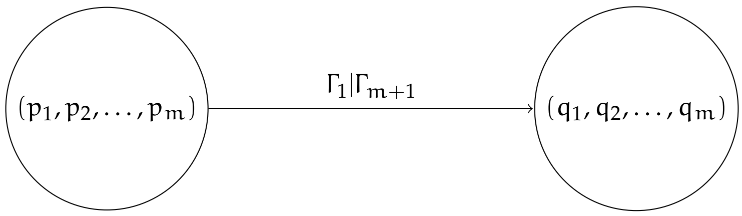
We have the following lemma for elements of $\mathcal{H}_n$ which have finite order.

Lemma 3.5.22. *Let $A \in \mathcal{H}_n$ be bi-synchronizing at level k and have finite order m. Let $\{(q_1, q_2, \ldots, q_m)\} \subseteq Q^m$ be the states of* $\text{Core}(A^m)$*. For each state $(q_1, q_2, \ldots, q_m)$ of* $\text{Core}(A^m)$*, let $W_{(q_1,q_2,\ldots,q_m)}$ be the set of words $\Gamma \in X_n^{km}$ for which the state of A^m forced by Γ is $(q_1, q_2, \ldots, q_m)$. Then, for a fixed state $(q_1, q_2, \ldots, q_m)$ of* $\text{Core}(A^m)$*, a word $\Gamma \in W_{(q_1,q_2,\ldots,q_m)}$ and any state $(p_1, \ldots, p_m) \in Q_{A^m}$, we have $\lambda_{A^m}(\Gamma, (p_1, \ldots, p_m)) \in W_{(q_1,q_2,\ldots,q_m)}$.*

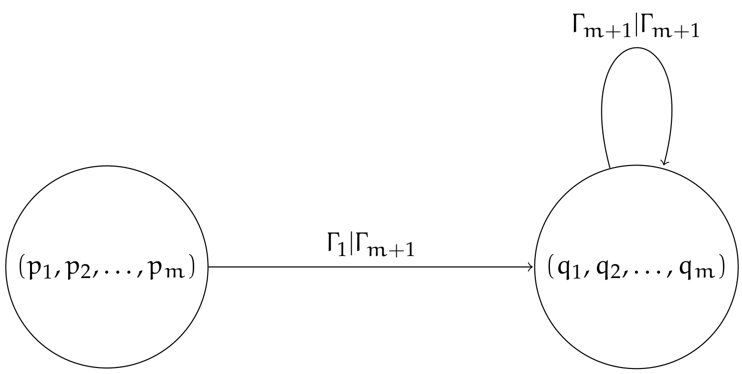
Proof. Fix a state $(q_1, q_2, \ldots, q_m)$ of $\text{Core}(A^m)$, and let Γ_1 be a word of length km in $W_{(q_1,q_2,\ldots,q_m)}$. Let $(p_1, p_2, \ldots, p_m)$ be any element of Q^m. Let A_{p_i} represent A initialised at state p_i. Suppose we have the following transition:

$$\Gamma_1 \xrightarrow{A_{p_1}} \Gamma_2 \xrightarrow{A_{p_2}} \Gamma_3 \xrightarrow{A_{p_3}} \ldots \xrightarrow{A_{p_m}} \Gamma_{m+1} \tag{3.1}$$

Since $\Gamma_1 \in W_{q_1,q_2,\ldots,q_m}$ we must have that $\Gamma_i \in W_{q_i}$ for $1 \leqslant i \leqslant m$ (here W_{q_i} are the set of words in A of length greater than or equal to k such that the state of A forced by an element of W_{q_i} is q_i). Let us now consider what happens in A^m. Here we have the following transition taking place:



Now observe that since A^m is bi-synchronizing at level km (Proposition 2.1.33) and since $(q_1, q_2, \ldots, q_m)$ is a state in $\text{Core}(A^m)$ and all states in $\text{Core}(A^m)$ are locally the identity map then the following must happen:



This means that Γ_{m+1} is again in $W_{(q_1,q_2,\ldots,q_m)}$. Now since Γ_1 was arbitrary in $W_{(q_1,q_2,\ldots,q_m)}$, then the same holds for every word in this set. □

The following result links the size of the group generated by an element of $\mathcal{H}_n$ to the order of the element in the group $\mathcal{H}_n$.

Theorem 3.5.23. *Let $A \in \mathcal{H}_n$. Then $\mathcal{G}(A)$ is finite if and only if A has finite order as an element of $\mathcal{H}_n$.*

Proof. Suppose $\mathcal{G}(A)$ is a finite group. It follows from Theorem 3.5.16 that there is a state $p \in Q_A$ and $o(p) \in \mathbb{N}$ such that $A_p^{o(p)} \cong_\omega \text{id}$. Now observe that $\text{Core}(A^{o(p)})$ is a subtransducer of $A_p^{o(p)}$ since $A^{o(p)}$ and $A_p^{o(p)}$ are synchronizing and $A_p^{o(p)}$ is a subtransducer of $A^{o(p)}$. Therefore we have that $\text{Core}(A^{o(p)}) \cong_\omega \text{id}$ since $A_p^{o(p)}$ is synchronous and ω-equivalent to the single state identity transducer. Thus we have that A has order dividing $o(p)$ as an element of $\mathcal{H}_n$.

Now suppose that A has finite order m as an element of $\mathcal{H}_n$. Observe that this means all states of $\text{Core}(A^m)$ must induce the identity transformation on X_n. The fact that A generates a finite group is a consequence of Lemma 3.5.22.

For let $j \in \mathbb{N}$ and suppose that $j = rm + s$ for some $r \in \mathbb{N}$ and $1 \leqslant s \leqslant m-1$. For a state $(t_1, t_2, \ldots, t_m)$ of A^m, let $W_{(t_1,t_2,\ldots,t_m)}$ be the set of all words in X_n^{km} such that the state of A^m forced by these words is $(p_1, p_2, \ldots, p_[m])$. Lemma 3.5.22 above means that if $p_1 \ldots p_j$ is any word in the input alphabet of $A^\vee$, then reading any word in $W_{(q_1,q_2,\ldots,q_m)}$ from the state $(p_1 \ldots p_j)$ of A^j, the resulting state will be $((q_1, q_2, \ldots, q_m)^r, q_1, \ldots, q_s)$. Since $\text{Core}(A^m) \cong_\omega \text{id}$ and $(q_1, q_2, \ldots q_m)$ is a state of $\text{Core}(A^m)$, we have that the state $(q_1 q_2 \ldots q_m)^r q_1 \ldots q_s$ of A^j is ω equivalent to the state $(q_1, \ldots, q_s)$ of A^s. Thus it follows that the map $h_{p_1} \ldots h_{p_j}$ can be represented as a pair $(\rho, (g_1, g_2, \ldots, g_{n^k}))$ where $g_i = h_{q_{i,1}} \ldots h_{q_{i,s}}$ for $q_{i,1}, \ldots, q_{i,s}$ states of A and $1 \leqslant i \leqslant n^k$, and ρ is the permutation of X_n^{km} induced by the state $(p_1, \ldots, p_j)$ of A^j on X_n^{km}. Since $j \in N$ was arbitrary and $(p_1, \ldots, p_j)$ was any state of A^j, it follows that such a decomposition holds for any $j \in N$. Now as there are only finitely many permutations of X_n^{km} and, $A^{s'}$, for $1 \leqslant s' \leqslant m-1$, has finitely many states, it follows that $\mathcal{G}(A)$ is finite. □

Remark 3.5.24. Putting together the results of this section and the previous one we deduced that the following are equivalent:

(1) the finiteness problem is soluble for groups generated by synchronizing, synchronous transducers,

(2) (Theorem 3.5.16 and Corollary 3.5.18) the order problem is soluble for groups generated by synchronous, synchronizing transducers

(3) (Proposition 3.5.20) the finiteness (or equivalently order) problem for groups generated by transducers in $\widetilde{\mathcal{H}}_n$ is soluble,

(4) (Remark 3.5.21) the finiteness or order problem for groups generated by transducers in $\mathcal{H}_n$ is soluble,

(5) (Theorem 3.5.23) the order problem in $\mathcal{H}_n$ is soluble.

We should point out that the paper [23] gives a different proof that the finiteness problem for groups generated by reset automata is equivalent to the order problem in the group of automorphisms of the one-sided shift (which is isomorphic to $\mathcal{H}_n$).

The remainder of this chapter shall be devoted mainly to the order problem in $\mathcal{H}_n$, though there shall be other results which do not fit directly under this heading. For instance we also investigate the growth rate of groups generated by transducers in $\widetilde{\mathcal{H}}_n$, and the rate at which the number of states in the core of elements of $\widetilde{\mathcal{H}}_n$ grows with powers. In the course of investigating the order problem in $\mathcal{H}_n$ we shall give a different proof of Lemma 3.5.22 which generalises in a natural way to the monoid $\widetilde{\mathcal{P}}_n$.

3.5.3 The order problem in $\mathcal{H}_n$

In this section we shall start to introduce the tools needed to understand the order problem in the group $\mathcal{H}_n$. However, as the techniques are applicable to $\widetilde{\mathcal{P}}_n$ and will be relevant in later sections, we shall work with the monoid $\widetilde{\mathcal{P}}_n$ for majority of this section. It is standard in the literature to tackle the order problem by investigating the structure of the dual transducer, see for instance [38, 1], and this is what we do below. The exposition and results in this paper are from the paper [44].

The dual at level k

Let $A = \langle X, Q, \pi, \lambda \rangle$ be a synchronous transducer and let $k \in \mathbb{N}$.

We form the level k dual,

$$A_k^{\vee} = \langle X_k^{\vee}, Q_k^{\vee}, \pi_k^{\vee}, \lambda_k^{\vee} \rangle$$

of A as follows. The state set $Q_k^{\vee}$ of $A_k^{\vee}$ is the set of all words of length k in the input alphabet X. This dual transducer has its input alphabet equal to its output alphabet and they are both equal to $X_k^{\vee} := Q$ the set of states of A. The transition function $\pi_k^{\vee}$ is defined as follows: for states $q, q' \in Q$, and $\Gamma, \Gamma' \in Q_k^{\vee}$ we have:

1. $\pi_k^{\vee}(q, \Gamma) = \Gamma'$ if and only if $\lambda(\Gamma, q) = \Gamma'$, and
2. $\lambda_k^{\vee}(q, \Gamma) = q'$ if and only if $\pi(\Gamma, q) = q'$.

We observe that $A_{k+1}^{\vee} = A_k^{\vee} * A_1^{\vee}$. For suppose that Γi a word of length $k+1$ is a state in $A_{k+1}^{\vee}$, and q is any state symbol of A such that after reading q from Γi in $A_{k+1}^{\vee}$ we are in state Δj and the output is p. Then in A we have $\pi(\Gamma i, q) = p$ and $\lambda(\Gamma i, q) = \Delta j$. We can break up this transition into two steps. Suppose $\pi(\Gamma, q) = p'$, then we have $\lambda(\Gamma, q) = \Delta$, $\lambda(i, p') = j$ and $\pi(i, p') = p$. Hence in $A_k^{\vee}$ we read q from Γ and transition to Δ and p' is the output p'. Moreover in $A_1^{\vee}$ we read p' from i and transition to j with output p. Therefore the state (Γ, i) of $A_k^{\vee} * A_1^{\vee}$, is such that we read q from this state and transition to the state (Δ, j) and the output produced is p.

Notation 3.5.25. As we shall sometimes work simultaneously with a transducer A and its dual, given $i \in \mathbb{N}_1$ we shall, occasionally write a state $(p_1, \ldots, p_i)$ of A^i as a word $p_1 \ldots p_i$.

The following definition gives a tool which connects the synchronizing level of powers of an element of $\widetilde{\mathcal{P}}_n$ to a property of the dual transducer.

Definition 3.5.26 (Splits)**.** Let A be an element of $\widetilde{\mathcal{P}}_n$, with synchronizing level k. Then we say that $A_r^{\vee}$ $(r \geqslant k)$ *splits* if we have the following picture in $A_r^{\vee}$:

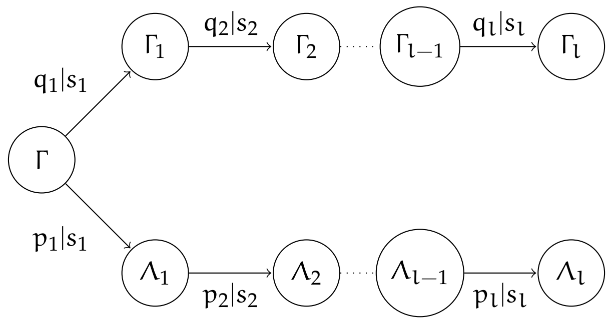


Figure 3.9: A split

where $\Gamma_l \in W_{t_1}$ and $\Lambda_l \in W_{t_2}$ for distinct states t_1 and t_2 of A, and for all other pairs (Γ_i, Λ_i), $1 \leqslant i \leqslant l-1$, Γ_i and Λ_i are in the same W_{u_i}. We say that the l-tuples $(p_1, \ldots, p_l)$ and $(q_1, \ldots, q_l)$ *split* $A_r^{\vee}$. We shall call $\{p_1, q_1\}$ the *top of the split*, $\{t_1, t_2\}$ the *bottom of the split*, and the triple $((q_1 \ldots, q_l), (p_1, \ldots, p_l), \Gamma)$ a *split* of $A_r^{\vee}$.

Definition 3.5.27. Let A be an element of $\widetilde{\mathcal{P}}_n$, with synchronizing level k. Let $r \geqslant k$ and let $((q_1 \ldots, q_l), (p_1, \ldots, p_l), \Gamma)$ be a split of $A_r^{\vee}$ for $\Gamma \in X_n^r$ and $(q_1 \ldots, q_l), (p_1, \ldots, p_l) \in Q_A^l$. Let $\{t_1, t_2\}$ be the bottom of this split. Then we say that *the bottom of the split* $((q_1 \ldots, q_l), (p_1, \ldots, p_l), \Gamma)$ *depends only on the top* if for any other tuples $U_1, U_2 \in Q_A^{l-1}$ we have that $((q_1, U_1), (p_1, U_2), \Gamma)$ is also a split with bottom $\{t_1, t_2\}$ and we have for any $u, u' \in Q$, $\pi_{A^{l+1}}(\Gamma, (p_1, \ldots, p_l, u)) = \pi_{A^{l+1}}(\Gamma, (p_1, U_2, u'))$ and $\pi_{A^{l+1}}(\Gamma, (q_1, \ldots, q_l, u)) = \pi_{A^{l+1}}(\Gamma, (q_1, q_2, u'))$. The last condition means that if $\lambda_{A^l}(\Gamma, (q_1, \ldots, q_l)) \in W_{t_1}$ then so also is $\lambda_{A^l}(\Gamma, (q_1, U_1))$ and likewise for $(p_1, \ldots, p_l)$ and (P_1, U_2).

Definition 3.5.28. For a transducer A, we define the *r splitting length of* A (for r greater than or equal to the minimal synchronizing length) to be minimal l such that there is a pair of l-tuples of states which split $A_r^{\vee}$. If there is no such pair the we set the r splitting length of A to be ∞.

Remark 3.5.29. Let A be a transducer with minimal r splitting length $l < \infty$. By minimality of l it follows that for a given pair in $Q^l \times Q^l$ which splits $A_r^{\vee}$, then the bottom of the split depends only on the top. Therefore the top and bottom of the split have cardinality two. In particular for any split whose bottom depends only on its top, the top and bottom of the split both have cardinality two.

Remark 3.5.30. Let A be a transducer such that the minimal r splitting length of A is infinite for some r, then the minimal $r+1$ splitting length of A is also infinite.

The following lemma demonstrates that for $A \in \widetilde{\mathcal{P}}_n$ and $r > 2$, the r splitting length of A is bigger than the $r-1$ splitting length of A.

Lemma 3.5.31. *Let* $A \in \widetilde{\mathcal{P}}_n$ *be synchronizing at level* k*, and suppose that the* mk *splitting length of* A *is finite for* $m \in \mathbb{N}$, $m > 0$*, then the* $(m+1)k$ *splitting length of* A *is strictly greater than the* mk *splitting length of* A.

Proof. Suppose that A has mk splitting length l. It suffices to show that for any word $\Gamma \in X_n^{(m+1)k}$, and any $l+1$-tuple P in Q_A^{l+1}, the output of P through Γ depends only on Γ.

First we set up some notation. Let $A^j := \langle X_n, Q_A^j, \lambda_j, \pi_j \rangle$ and let $A_j^{\vee} := \langle Q_A, X_n^j, \lambda_j^{\vee}, \pi_j^{\vee} \rangle$ for $j \in \mathbb{N}$. For a word $\gamma \in X_n^k$ let q_γ denote the state of A forced by Γ.

Now since A has mk splitting length l, it follows that for any $P := (p_1, \ldots, p_l)$ and $T := (t_1, \ldots, t_l)$ in Q_A^l and $\Gamma \in X_n^{mk}$ we have that $\lambda_{mk}^{\vee}(P, \Gamma) = \lambda_{mk}^{\vee}(T, \Gamma)$. By definition of the dual, $\lambda_{mk}^{\vee}(P, \Gamma) = \pi_l(\Gamma, P)$.

Now let $\gamma \in X_n^k$ be arbitrary and let $p \in Q_A$ and $P \in Q_A^l$ also be arbitrary. Consider $\lambda_{(m+1)k}^{\vee}(Pp, \Gamma\gamma)$, we have:

$$\lambda_{(m+1)k}^{\vee}(Pp, \Gamma\gamma) = \pi_{l+1}(\Gamma\gamma, Pp) = \pi_l(\gamma, \pi_l(\Gamma, P))\pi_1(\lambda_l(\gamma, \pi_l(\Gamma, P)), \pi_1(\lambda_l(\Gamma, P), p))$$

However observe that since A is synchronizing at level k that the suffix

$$\pi_1(\lambda_l(\gamma, \pi_l(\Gamma, P)), \pi_1(\lambda_l(\Gamma, P), p))$$

depends only on $\lambda_l(\gamma, \pi_l(\Gamma, P))$. However since $A_{mk}^{\vee}$ has minimal splitting length l we have that $\pi_l(\Gamma, P)$ depends only on Γ. Therefore we have that $\lambda_{(m+1)k}^{\vee}(Pp, \Gamma\gamma)$ depends only on $\Gamma\gamma$. $\square$

Remark 3.5.32. It follows from the lemma above that if $A \in \widetilde{\mathcal{P}}_n$ is synchronizing at level k, then the mk splitting length of A, if it is finite, is at least m for $m \in \mathbb{N}$, $m > 0$.

The following lemma shows that the minimal splitting length is connected with the synchronizing level of powers of a transducer.

Lemma 3.5.33. *Let* A *be a transducer with synchronizing level less than or equal to* k*. If* A *has* k *splitting length* l*, then* $\min(\mathrm{Core}(A^{l+1}))$ *has minimal synchronizing level* $m \geqslant k+1$.

Proof. Let l be as in the statement of the lemma. Now consider $\mathrm{Core}(A^l)$. The states of $\mathrm{Core}(A^l)$ will consist of all length l outputs of $A_k^{\vee}$. Moreover by choice of l, $\mathrm{Core}(A^l)$ is also synchronizing at level k.

Let Γ be the word which achieves the minimal l, and suppose the picture is exactly as given in Figure 3.9, where $\Gamma_l \in W_{t_1}$ and $\Lambda_l \in W_{t_2}$ for distinct states t_1 and t_2.

We now consider $B := \mathrm{Core}(A * \mathrm{Core}(A^l))$. It is easy to see that there are states in B of the form (p_1, P), (q_1, Q) for appropriate $P, Q \in \mathrm{Core}(A^l)$. Therefore in B we have that when we have read Γ through (p_1, P), we are in state $(s_1, \ldots, s_l, t_1)$, and when we have read Γ through state (q_1, Q) we go to state $(s_1, \ldots, s_l, t_2)$. Since $t_1 \neq t_2$ these states are not ω equivalent. This concludes the proof. □

Lemma 3.5.34. *An element $A \in \mathcal{H}_n$ either has finite order or for all $k \in \mathbb{N}$ there is an $N \in \mathbb{N}$ such that for all $m \in \mathbb{N}$ we have that A^{Nm} is bi-synchronizing at level greater than k, moreover N is depends only on A and k.*

Proof. Suppose that A does not have finite order. Since A has infinite order then $A_j^{\vee}$ splits for every $j \in \mathbb{N}$, therefore there is an $m_0 \in \mathbb{N}$ and $N_1 \in \mathbb{N}$ such that $\min \mathrm{Core}(A^{N_1})$ is bi-synchronizing at level m_0. In order to simplify the notation we shall identify A^{N_1} with the minimal, core transducer $\min \mathrm{Core}(A^{N_1})$.

Now consider the permutation $\overline{A}_{m_0}^{N_1}$, we shall assume that it is written as a product of disjoint cycles. Let d be the order of this permutation. Let $\Gamma \in X_n^{m_0}$, then Γ belongs to a cycle $(\Gamma_1\Gamma_2 \ldots \Gamma_{d_\Gamma})$ where $\Gamma_1 := \Gamma$ and $d_\Gamma | d$. Let $e_\Gamma = d/d_\Gamma$. To this cycle there is associated a tuple of states $(q_{\Gamma_1}, q_{\Gamma_2}, \ldots, q_{\Gamma_{d_\Gamma}})$ where q_{Γ_i} is the state of A^{N_1} forced by Γ_i for $1 \leqslant i \leqslant d_\Gamma$. Now observe that $(q_{\Gamma_1}, q_{\Gamma_2}, \ldots, q_{\Gamma_{d_\Gamma}})$ is a state of $A^{N_1 d_\Gamma}$, moreover since $\lambda_{N_1}(\Gamma_i, q_{\Gamma_i}) = \Gamma_{i+1}$ for $1 \leqslant i < d_\Gamma$, and $\lambda_{N_1}(\Gamma_{d_\Gamma}, q_{\Gamma_{d_\Gamma}}) = \Gamma_1$, then we have that

$$\pi_{N_1 d_\Gamma}(\Gamma_1, (q_{\Gamma_1}, q_{\Gamma_2}, \ldots, q_{\Gamma_{d_\Gamma}})) = (q_{\Gamma_1}, q_{\Gamma_2}, \ldots, q_{\Gamma_{d_\Gamma}}).$$

Now let t_Γ be the order of the permutation induced by $(q_{\Gamma_1}, q_{\Gamma_2}, \ldots, q_{\Gamma_{d_\Gamma}})^{e_\Gamma}$ on $X_n^{m_0}$.

Let t be the lowest common multiple of the set $\{t_\Gamma | \gamma \in X_n^{m_0}\}$. In order to keep the notation concise let P_Γ represent the state $(q_{\Gamma_1}, q_{\Gamma_2}, \ldots, q_{\Gamma_{d_\Gamma}})^{e_\Gamma t}$. Notice that P_Γ acts locally as the identity for all $\Gamma \in X_n^{m_0}$. Moreover P_Γ is the state of $A^{N_1 dt}$ such that $\pi_{N_1 dt}(\Gamma, P_\Gamma) = P_\Gamma$.

Now let $N = N_1 dt$, and let $m \geqslant N$. Suppose that A^m, (where again A^m is identified with the minimal core transducer of A^m) is synchronizing at level m_0. Since $m \geqslant N$ we may write $m = rN$ for some $r \in \mathbb{N}$ and $0 \leqslant s < N$. Therefore states of A^m look like P for P a state of A^N.

Now observe once more that all the states P_Γ are locally the identity for all $\Gamma \in X_n^{m_0}$ and $\pi_m(\Gamma, P_\Gamma) = P_\Gamma$. Now since A^m is synchronizing at level N, we must have that the state of A^m forced by Γ is precisely P_Γ. Therefore the states of A^m can be identified with the states P_Γ. Now as all of these states are locally identity it follows that A^m is the identity. Which is a contradiction of our initial assumption that A does not have finite order. Therefore A^m must be synchronizing at level greater than m_0. □

Lemma 3.5.35. *Let $A \in \mathcal{P}_n$ be a core, minimal transducer such that $|A| > n(n+1)$. Let B be any transducer synchronizing at level 1. Then $\min \mathrm{Core}(AB)$ is synchronizing at level strictly greater than 1.*

Proof. For each $i \in X_n$ let $\mathcal{I}_i := \{\pi_A(i, p) | p \in Q_A\}$. Notice since A is synchronizing it is also strongly connected, therefore for all $p \in Q_A$ there is a set $\mathcal{I}_i$ for some $i \in X_n$ such that $p \in \mathcal{I}_i$. It now follows that $\cup_{i \in X_n} \mathcal{I}_i = Q_A$.

Now if $|\mathcal{I}_i| < n+1$ for all i then we have that:

$$|A| = |\cup_{i \in X_n} \mathcal{I}_i| \leqslant \sum_{i=1}^{n} |\mathcal{I}_i| < n * (n+1) < |A|$$

which is a contradiction. Therefore there must be an $i \in X_n$ such that $|\mathcal{I}_i| > n+1$. Fix such an $i \in X_n$.

Now since $|\mathcal{I}_i| > n+1$, there must be states $p_1', p_2', p_1, p_2 \in Q_A$ such that $p_1 \neq p_2$ and $p_1' \neq p_2'$ and such that the following transitions are valid:

$$p_1' \xrightarrow{i|j} p_1 \qquad p_2' \xrightarrow{i|j} p_2$$

for some $j \in X_n$.

Now observe that there are states (p_1', q_1') and (p_2', q_2') in the core of AB where q_1 and q_2 are states of B. Let $\pi_B(j, q_1') = q_j$ and $\pi_B(j, q_2') = q_j$ (since B is synchronizing at level 1).

Therefore the following transitions are valid:

$$(p_1', q_1') \xrightarrow{i|l_1} (p_1, q_j) \qquad (p_2', q_2') \xrightarrow{i|l_1} (p_2, q_j)$$

where $l_1 = \lambda_B(j, q_1')$ an $l_2 = \lambda_B(j, q_2')$ Now if $\min \mathrm{Core}(AB)$ is synchronizing at level 1, then (p_1, q_j) an (p_2, q_j) would be ω-equivalent, since (p_1', q_1') and (p_2', q_2') are states in the core of AB. However $(p_1, q_j) \cong_\omega (p_2, q_j)$ implies that $p_1 \cong_\omega p_2$, but by assumption p_1 and p_2 are distinct and A is minimal and so $p_1 \cong_\omega p_2$ is a contradiction.

Therefore $\min \mathrm{Core}(AB)$ is not synchronizing at level 1. □

Now suppose that $A \in \widetilde{\mathcal{P}}_n$ and the semigroup $\langle A \rangle_+ := \{A^i | i \in \mathbb{N}\}$ is finite. Notice that if $A \in \widetilde{\mathcal{H}}_n$ and the semigroup $\langle A \rangle_+$ is finite then it coincides with the group generated by A. The next result demonstrates that in the case where the semigroup $\langle A \rangle_+$ is finite, there must be some $j \in \mathbb{N}$, $j > 0$ for which the j splitting length of A is infinite. From this result one may deduce Lemma 3.5.22.

Lemma 3.5.36. *Let $A \in \widetilde{\mathcal{P}}_n$ be synchronizing at level k. Suppose that the semigroup $\langle A \rangle_+$ is finite, and that j is the maximum of the minimal synchronizing level of the elements of $\langle A \rangle_+$. Then A has infinite j splitting length.*

Proof. This is a consequence of Lemma 3.5.33. Since if A has j splitting length l, then by Lemma 3.5.33 $\min(\mathrm{Core}(A^{l+1}))$ has minimal synchronizing level $j+1$, which is a contradiction. □

Remark 3.5.37. The above means that we can partition $A_j^\vee$ into components $D_1, \ldots, D_i$ such that to each component there is a pair of words $W_{i,1}$ and $W_{i,2}$ in the states of A such that the only possible outputs from the component D_i for any input have the form $u(W_{i,2})^l v$ where u is any suffix of $W_{i,1}W_{i,2}$, including the empty suffix, and v is a prefix of $W_{i,2}$ including the empty prefix.

Below are some examples of finite order bi-synchronizing, synchronous transducers witnessing Lemma 3.5.36.

Example 3.5.38. Consider the transducer C below. This is a transducer of order 3, in particular, it is a conjugate of the single state transducer which can be identified with the permutation $(0, 1, 2)$.

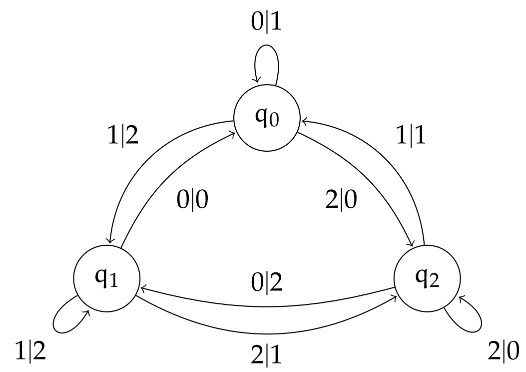


Figure 3.10: An element of order 3 in $\mathcal{H}_n$

This transducer, as noted before, is bi-synchronizing at the second level. The level 3 dual has 27 nodes and so we shall not give this below. However utilising either the AAA package or the Automgrp package [41] in GAP [25], together with (in AutomGrp) the function "MinimizationOfAutomaton( )" which returns an ω-equivalent transducer, applied to the third power of the dual transducer, we get the following result:

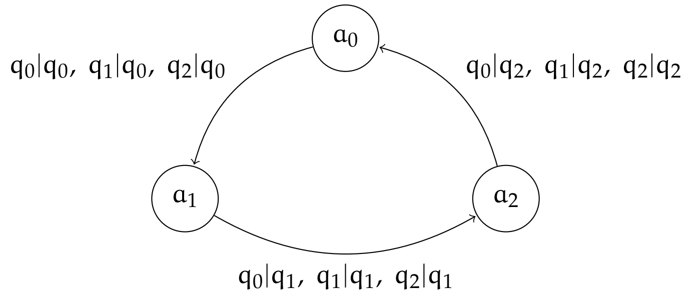


Figure 3.11: The level 3 dual of C.

Since the original transducer C has order 3 we can see from its level 3 dual above that the states in the core will be cyclic rotations of (q_0, q_1, q_2) all of which are locally identity.

We illustrate another example below, but now with an element of order 2.

Example 3.5.39. Consider the transducer of order two given below constructed based on an example in [10].

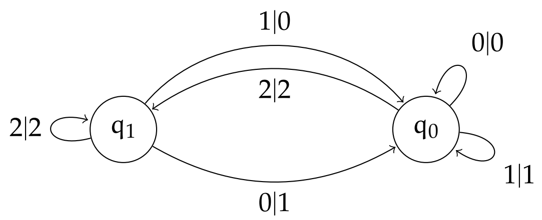


Figure 3.12: An element of order 2

This transducer is synchronizing on the first level. We give the dual below.

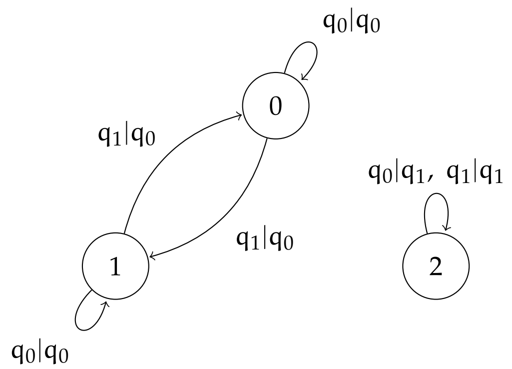


Figure 3.13: The level 1 dual.

It is easy to see that the states 0 and 1 are ω-equivalent, and can be identified to a single node that produces q_0 for all inputs. The states in the core of the square of the original transducer will be (q_0, q_0) and (q_1, q_1).

For a transducer of finite order, A, as above, we have the following result about the semigroup $\langle A^{\vee}\rangle_{+}$.

Theorem 3.5.40. *Let $A \in \widetilde{\mathcal{P}}_n$ be synchronizing at level k. Suppose that the semigroup $\langle A\rangle_{+}$ is finite with $j \in \mathbb{N}_1$ the maximum of the minimal synchronizing levels of the elements of $\langle A\rangle_{+}$, then $A_j^{\vee} = (A^{\vee})^j$ is a zero in $\langle A^{\vee}\rangle_{+}$, the semigroup generated by $A^{\vee}$.*

Proof. It suffices to show that $A_j^{\vee}$ is a right zero of the semigroup since the semigroup $\langle A^{\vee}\rangle_{+}$ is commutative.

Our strategy shall be to to show that for any state Γ of $A_j^{\vee}$ and any state x of $A^{\vee}$, that the state Γx of $A_{j+1}^{\vee}$ is ω-equivalent to a state of $A_j^{\vee}$. To this end let $\Gamma \in X_n^j$ be a word of length $j+1$. By Lemma 3.5.31 and Remark 3.5.37 there is a pair of words W_{1,Γ_1} and W_{2,Γ_1} such that any input read from Γ_1 has output of the form $W_1(W_2)^l\nu$ for $l \in \mathbb{N}$ and ν a prefix of W_2, otherwise the output is a prefix of W_1. Let $\gamma \in X_n^j$ be the length j suffix of Γ_1. Observe that the outputs of the state γ of $A_j^{\vee}$ must also all be of the form $W_1(W_2)^l\nu$ for $l \in \mathbb{N}$ and ν a prefix of W_2, otherwise the output is a prefix of W_1 and the output depends only on the length of the input word. Therefore we must have that Γ and γ are ω-equivalent.

On the other hand, given a word $\gamma \in X_n^j$, then a similar argument demonstrates that the state $x\gamma$ for any $x \in X_n^j$ is ω-equivalent to γ. □

The next result observes that Lemma 3.5.36 gives a complete characterisation of elements of $\mathcal{H}_n$ with finite order.

Proposition 3.5.41. *Let A be an element of $\widetilde{\mathcal{P}}_n$ and suppose A is synchronizing at level k. Then the semigroup $\langle A\rangle_{+}$ generated by A is finite if and only if there is some $m \in \mathbb{N}$ such that the following hold:*

(i) $A_m^{\vee}$ *is a zero of the semigroup* $\langle A^{\vee}\rangle$,

(ii) $A_m^{\vee}$ *is ω-equivalent to a transducer with r components D_i $1 \leqslant i \leqslant r$. For each component D_i there is a fixed pair of words $w_{i,1}, w_{i,2}$ (in the states of A) associated to D_i such that whenever we read any input from a state in the D_i, the output is of the form $w_{i,1}w_{i,2}^l\nu$ for $l \in \mathbb{N}$ and ν a prefix of $w_{i,2}$ or has the from u for some prefix U of $w_{i,1}$. Moreover the output depends only on which state in the component D_i we begin processing inputs.*

Proof. $\Rightarrow$: This direction follows from Lemma 3.5.36, Remark 3.5.37 and Theorem 3.5.40.

$\Leftarrow$: Assume that $A_m^{\vee}$ has r components and let $w_{i,1}$ and $w_{i,2}$ $1 \leqslant i \leqslant r$ be the pair of words in the states of A associated with each component D_i. To see that the semigroup $\langle A\rangle_{+}$ is finite observe that the assumptions that $A_m^{\vee}$ is a zero of the semigroup $\langle A^{\vee}\rangle_{+}$ and that the output depends only on which state in the component D_i of $A_m^{\vee}$ we begin processing inputs means that A^l is synchronizing at level m for all $l \in \mathbb{N}$. Therefore the set $\{A^l | l \in \mathbb{N}\}$ is finite, since there are only finitely many transducers which are synchronizing at level l. □

Remark 3.5.42. In the case where A is an element of $\mathcal{H}_n$ in the above proposition, then each component D_i is a strongly connected component. In particular one of $w_{i,1}$ or $w_{i,2}$ will be the empty string for any component D_i. We should point out that Picantin in his habilitation thesis [40] conjectures that a level one synchronizing transducer A in $\mathcal{H}_n$ is finite if and only if $(A^{\vee})^{|Q_A|-1}$ is a zero of $\langle A^{\vee}\rangle_{+} = \{(A^{\vee})^i \mid i \in \mathbb{N}\}$. Proposition 3.5.41 gives a partial answer to this conjecture. Picantin's conjecture generalises to elements $A \in \mathcal{H}_n$ which are synchronizing at level k as follows:

Conjecture 3.5.43 (Picantin)**.** *Let $A \in \mathcal{H}_n$ be synchronizing at level k, then A generates a finite group if and only if $(A^{\vee})^{k(|Q_A|-1)}$ is the zero of $\langle A^{\vee}\rangle_{+} = \{(A^{\vee})^i \mid i \in \mathbb{N}\}$.*

If Picantin's conjecture is true, then the order problem in $\mathcal{H}_n$ is soluble. The author is grateful to Laurent Bartholdi for drawing the results of Picantin's habilitation thesis to his attention.

Remark 3.5.44. Given a transducer $A \in \widetilde{\mathcal{P}}_n$ which is synchronizing at level k, by Construction 3.5.13, one can identify A with an element of A_k of $\widetilde{\mathcal{P}}_{n^k}$. It is an easy exercise to verify that $(A_k^{\vee})^{\vee} = A_k$, therefore, $(A_k)^{\vee} = A_k^{\vee}$, hence by Proposition 3.5.41, A_k has finite order if and only if A has finite order. In order to simplify calculations, we shall often assume that our transducers are synchronizing or bi-synchronizing at level 1.

Applications to the order problem

On the surface Proposition 3.5.41 appears to reduce the order problem in $\mathcal{H}_n$ to an equivalent problem of deciding whether the semigroup generated by the dual has a zero. However a consequence of the above lemmas (in particular Lemma 3.5.36), is that for certain transducers where the dual at the bi-synchronizing level has some property, we are able to conclude that this transducer will be an element of infinite order. We shall need a few definitions first. Once more we shall make these definitions for elements of $\widetilde{\mathcal{P}}_n$, we then apply the results to $\mathcal{H}_n$ as a special case.

Definition 3.5.45 (Bad pairs)**.** Let $A \in \widetilde{\mathcal{P}}_n$ be a transducer which is synchronizing at level k, and let $r \geqslant k$. Let l be the minimal splitting length of $A_r^{\vee}$. Let $\mathscr{B}_r$ be the set of tops of those pairs $((q_1 \ldots, q_m), (p_1, \ldots, p_m))$ of m tuples, $m \geqslant l$, which split $A_r^{\vee}$ and for which there is a split $((q_1 \ldots, q_m), (p_1, \ldots, p_m), \Gamma)$ such that the bottom of the split depends only on the top. Then we call $\mathscr{B}_r$ the set of *bad pairs* associated to $A_r^{\vee}$. Notice that if $B \in \mathscr{B}_r$ then $B \subset Q$ and $|B| = 2$. Furthermore observe that by minimality of l, $\mathscr{B}_r$ contains the tops of all splits consisting of a pair of l tuples and a word in X_n^r. Let $B_r \subset \mathscr{B}_r$ be this subset. We call B_r the *minimal bad pairs* associated to $A_r^{\vee}$.

Definition 3.5.46 (Graph of Bad pairs)**.** For a transducer $A \in \widetilde{\mathcal{P}}_n$, and for r greater than or equal to the minimal synchronizing level, such that $A_r^{\vee}$ has minimal splitting length l, form a directed graph $G_r(A)$ associated to $A_r^{\vee}$ as follows:

(i) The vertex set of $G_r(A)$ is the set $\mathscr{B}_r$ of bad pairs.

(ii) Two elements $\{x_1, x_2\}$, and $\{y_1, y_2\}$ of $\mathscr{B}_r$ are connected by an arrow going from $\{x_1, x_2\}$ into $\{y_1, y_2\}$, if there are pairs $(T_1, T_2) \in Q^m \times Q^m$, for some $m \geqslant l$, splitting $A_r^{\vee}$, with top $\{x_1, x_2\}$ and bottom $\{y_1, y_2\}$ and such that the bottom depends only on the top. By Remark 3.5.29 this definition makes sense.

We call $G_r(A)$ the *graph of bad pairs* associated to $A_r^{\vee}$. This graph possesses an interesting subgraph $\overline{G}_r(A)$ whose vertices are elements of B_r the set of minimal bad pairs, with an edge from $\{x_1, x_2\}$ to $\{y_1, y_2\}$, $\{x_1, x_2\}, \{y_1, y_2\} \in B_r$ if there is a split of minimal length l, with top $\{x_1, x_2\}$ and bottom $\{y_1, y_2\}$. We call $\overline{G}_r(A)$ the *minimal graph of bad pairs*.

Remark 3.5.47. There is a much larger graph containing $G_r(A)$ which we do not consider here. This graph has all subsets of $Q_A \times Q_A$ with size exactly two, and there is a directed edge between two such vertices if there is a split of $A_r^{\vee}$ with top the initial vertex and bottom the terminal vertex. The reason we do not consider this larger graph is due to the existence of elements of finite order in $\widetilde{\mathcal{H}}_n$ whose dual at the synchronizing level splits (see Example 3.5.56). This means that in the larger graphs contain information which is not carried by powers of the transducers.

The following results link graph theoretic properties of $G_r(A)$ and the order of A when $A \in \widetilde{\mathcal{H}}_n$. All of these results apply also to the minimal graph of bad pairs $\overline{G}_r(A)$. In most cases the information given by $G_r(A)$ can already be seen in $\overline{G}_r(A)$, however this is not always the case as we will see in the examples to follow.

Lemma 3.5.48. *Let $A \in \widetilde{\mathcal{H}}_n$ be a transducer, and suppose that k is the minimal synchronizing level of A. Let $r \geqslant k$ and let $G_r(A)$ be the graph of bad pairs associated to $A_r^{\vee}$. If $G_r(A)$ is non-empty and contains a circuit i.e there is a vertex which we can leave and return to, then A has infinite order.*

Proof. Let l be the minimal splitting length of $A_r^{\vee}$. The proof will proceed as follows, for every $m \geqslant 1$, we construct a word, $w(rm)$, of length rm, such that there are two distinct elements of Q^{ml+1} which have different outputs when processed through $w(rm)$ (in $A_{rm}^{\vee}$). This will contradict

A having finite order, since by Lemma 3.5.36 above, if A has finite order, then there will be a j such that any two sequences of states of any length will have the same output when processed through a word of length rj (see Remark 3.5.37).

Since $G_r(A)$ is non-empty, and has a circuit, there exists a circuit: $\{x_1, y_1\} \to \{x_2, y_2\} \to \ldots \to \{x_j, y_j\} \to \{x_1, y_1\}$. For $i \in \mathbb{N}$ let $A^i = \langle X_n, Q^i, \pi_i, \lambda_i \rangle$.

Now for $m = 1$, since $\{x_1, y_1\}$ is a vertex of $G_r(A)$ with at least one incoming edge, there is a state Γ_1 of $A_r^\vee$, and a pair $(S_1, T_1) \in Q^{l_1} \times Q^{l_1}$ (for $l_1 \geqslant l$) such that (S_1, T_1, Γ_1) is a split of $A_r^\vee$ with bottom $\{x_1, y_1\}$ and such that the bottom depends only on the top. We may assume that the top of this split is $\{x_j, y_j\}$. Therefore for any $p \in Q$, the output of $S_1 p$ when processed through Γ_1 is not equal to the output of $T_1 p$ when processed through Γ_1. However the output of S_1 and T_1 are equal when processed through Γ_1 since the bottom of the split depends only on its top. Hence the following picture is valid, for appropriate $U \in Q^{l_1}$.

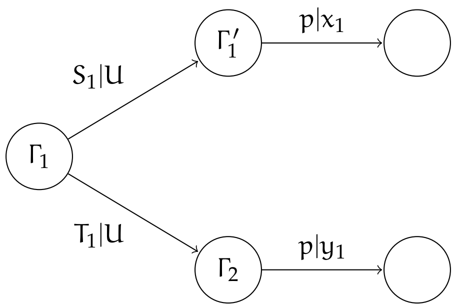


Figure 3.14: Stage 1 of construction

Now since $\{x_1, y_1\}$ is connected to $\{x_2, y_2\}$ there is a word $\Lambda_1 \in X_n^r$ such that there is a pair $(S_2, T_2) \in Q^{l_2} \times Q^{l_2}$ and (S_2, T_2, Λ_1) is a split with top $\{x_1, y_1\}$ and bottom $\{x_2, y_2\}$ such that the bottom depends only on the top. Let Λ_1' be the word of length r such that $\lambda_{l_1}(\Lambda_1', U) = \Lambda_1$ (such a word exists since A is invertible). Since the bottom of the split depends only on the top there is $V \in Q^{l_2}$ such that for any $P \in Q^{l_2 - 1}$ we have $\pi_{l_2}(\Lambda_1, (x_1, P)) = V = \pi_{l_2}(\Lambda_1, (y_1, P))$. Let V' be the state of A^{l_1} such that $\pi_{l_1}(\Lambda_1', U) = V'$. Then let $w(k2) = \Gamma_1 \Lambda_1'$. Now by the Remark 3.5.29, we have the following transition

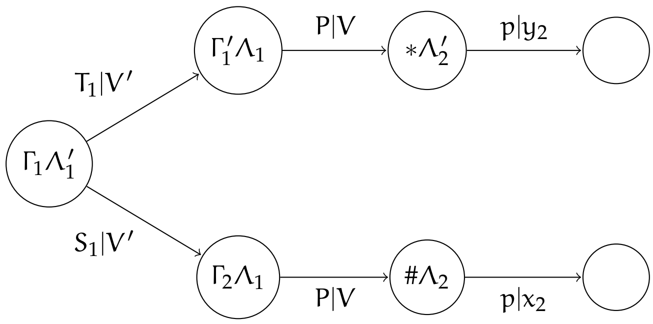


Figure 3.15: Stage 2 of construction.

for some $P \in Q^{l_2}$ and $p \in Q$. Now we can iterate the above process, since $\{x_2, y_2\}$ is a vertex of $G_r(A)$ with an outgoing edge to another vertex of $G_r(A)$, and the output of $S_1 P$ and $T_1 P$ when processed through $\Gamma_1 \Lambda_1'$ are the same.

Label the levels of the above picture by $1, 2, 3, 4$.We grow our word from bottom to top. Let Δ_1 be the word such that there is a pair (S_3, T_3) in $Q^{l_3} \times Q^{l_3}$ and (S_3, T_3, Δ_1) is a split of $A_r^\vee$ with top $\{x_2, y_2\}$ and bottom $\{x_3, y_3\}$ such that the bottom depends only on the top. Attach Δ_1 to right end of both words representing the states of level 3. There is a word Δ_1' such that $\lambda_{l_1 + l_2}(\Delta_1', V'V) = \Delta_1$. Hence $W(r3) = \Gamma_1 \Lambda_1' \Delta_1'$. Moreover since the bottom of the split (S_3, T_3, Δ_1) depends only on its top we see that $A^{\vee}_{3r}$ has a split of length $l_1 + l_2 + l_3$ with bottom $\{x_3, y_3\}$ whose bottom depends only on its top. We repeat the above process and in this way construct the words $w(rm)$ demonstrating that $A^{\vee}_{rm}$ has a split with bottom $\{x_i, y_i\}$ where $1 \leqslant i \leqslant j$, and $i \equiv m \mod j$ such that the bottom depends only on the top. □

Remark 3.5.49. The proof above in fact demonstrates that if $A \in \widetilde{\mathcal{H}}_n$ is such that for some r bigger than or equal to its synchronizing level, the graph $G_r(A)$ of bad pairs has a circuit, then there is an $r' > r$ depending on the length of the circuit in $G_r(A)$, such that $G_{r'}(A)$ has a loop.

The above gives a sufficient condition for determining when an element of $\widetilde{\mathcal{H}}_n$ has infinite order, although it does not produce a witness. The next results show that when the graph of bad pairs contains a circuit for an element of $\widetilde{\mathcal{H}}_n$ then we are also able to generate a rational word on an infinite orbit under the action of the transducer.

We start with the case where the graph of bad pairs contains a loop and prove the general case by reducing to the loop case. First we recall that rational word is a word which is accepted by an automaton in the language theoretic sense.

Proposition 3.5.50. *Let $A \in \widetilde{\mathcal{H}}_n$ be synchronizing at level k, and let $r \geqslant k$. If the graph $G_r(A)$ of bad pairs has a loop, then there is rational word in $X_n^{\mathbb{Z}}$ in an infinite orbit under the action of A.*

Proof. Let $\{p, q\}$ be a vertex of $G_r(A)$ with a loop. Furthermore assume that $m \in \mathbb{N}$ is the minimum splitting length of $A^{\vee}_r$. Let $\Gamma_1 \in X_n^r$, and $P, Q \in Q_A^{l-1}$ be such that (pP, qQ, Γ_1) is a split with top and bottom equal to $\{p, q\}$ such that the bottom depends only on the top.

Let $A^l = \langle X_n Q^l, \pi_l, \lambda_l \rangle$, then since (pP, qQ, Γ_1) is a split with top and bottom equal to $\{p, q\}$ such that the bottom depends only on the top, there is a state S_0 such that $\pi_l(pP, \Gamma_1) = S_0 = \pi_l(qQ, \Gamma_1)$ for any P and Q in Q_A^{l-1}. Therefore let $S_1 := \pi_l(\Gamma_1, S_0)$. Let $\Gamma_2 \in X_n^r$ be the unique word such that $\lambda_l(\Gamma_2, S_1) = \Gamma_1$. Assume that Γ_i is defined and S_i is equal to $\pi_l(\Gamma_i, S_{i-1})$, then let Γ_{i+1} be the unique word in X_n^r such that $\lambda_l(\Gamma_{i+1}, S_i) = \Gamma_i$. Eventually we find there are $i \leqslant j \in \mathbb{N}$ such that $\Gamma_i = \Gamma_{j+1}$.

Suppose that $\lambda_l(\Gamma_1, pP) = \Delta$ and $\lambda_l(\Gamma_1, qP) = \Lambda$. Consider the bi-infinite word:

$$\ldots \Delta\dot{\Delta}\Gamma_1 \ldots \Gamma_{i-1}(\Gamma_i \ldots \Gamma_j)(\Gamma_i \ldots \Gamma_j) \ldots$$

where '$\dot{\Delta}$' indicates that Δ starts at the zero position. There are two cases to be considered.

Case 1: $\Delta \in W_p$ and $\Lambda \in W_q$. We consider how powers of A^l act on this word. Since $\Delta \in W_p$ and for any $T \in pQ^{l-1}$, $\lambda_l(\Gamma_1, T) \in W_p$, the bottom of the split (pP, qQ, Γ_1) depends only on the top, we must have that:

$$\ldots \Delta\dot{\Delta}\Gamma_1 \ldots \Gamma_{i-1}(\Gamma_i \ldots \Gamma_j)(\Gamma_i \ldots \Gamma_j) \ldots \xrightarrow{A^l} \ldots \dot{*}_0\, \Delta_1\Gamma_1 \ldots \Gamma_{i-1}(\Gamma_i \ldots \Gamma_j)(\Gamma_i \ldots \Gamma_j) \ldots$$

Now since $\Delta_1 \in W_p$, we can repeat the above

$$\ldots \dot{*}_0\, \Delta_1\Gamma_1 \ldots \Gamma_{i-1}(\Gamma_i \ldots \Gamma_j)(\Gamma_i \ldots \Gamma_j) \ldots \xrightarrow{A^l} \ldots \dot{*}_0'\, *_1\Delta_2\Gamma_1 \ldots \Gamma_{i-1}(\Gamma_i \ldots \Gamma_j)(\Gamma_i \ldots \Gamma_j) \ldots$$

Therefore after applying A^l t times for some $t \in \mathbb{N}$ we see that from the position i onwards the output is of the form $\Delta_r\Gamma_1 \ldots \Gamma_{i-1}(\Gamma_i \ldots \Gamma_j)(\Gamma_i \ldots \Gamma_j) \ldots$, and $\Delta_m \in W_p$. Therefore if $\Gamma_i \neq \Gamma_1$, $\ldots \Delta\dot{\Delta}\Gamma_1 \ldots \Gamma_{i-1}(\Gamma_i \ldots \Gamma_j)(\Gamma_i \ldots \Gamma_j) \ldots$ is on an infinite orbit under the action of A^l. This follows for if $t, t' \in \mathbb{N}$ such that $t \neq t'$, then we have:

$$(\ldots \Delta\dot{\Delta}\Gamma_1 \ldots \Gamma_{i-1}(\Gamma_i \ldots \Gamma_j)(\Gamma_i \ldots \Gamma_j) \ldots)A^{l*t} \neq (\ldots \Delta\dot{\Delta}\Gamma_1 \ldots \Gamma_{i-1}(\Gamma_i \ldots \Gamma_j)(\Gamma_i \ldots \Gamma_j) \ldots)A^{l*t'}$$

otherwise:

$$\ldots \Delta\dot{\Delta}\Gamma_1 \ldots \Gamma_{i-1}(\Gamma_i \ldots \Gamma_j)(\Gamma_i \ldots \Gamma_j) \ldots = (\ldots \Delta\dot{\Delta}\Gamma_1 \ldots \Gamma_{i-1}(\Gamma_i \ldots \Gamma_j)(\Gamma_i \ldots \Gamma_j) \ldots)A^{l*|t-t'|}$$

However by minimality of i, we have that $\Gamma_1 \neq \Gamma_t$ for $1 < t \leqslant j$, yielding a contradiction.

If $\Gamma_i = \Gamma_1$ then our original word $\ldots \Delta\dot{\Delta}\Gamma_1 \ldots \Gamma_{i-1}(\Gamma_i \ldots \Gamma_j)(\Gamma_i \ldots \Gamma_j)\ldots$, becomes

$$\ldots \Delta\dot{\Delta}(\Gamma_1 \ldots \Gamma_{i-1})(\Gamma_1 \ldots \Gamma_{i-1})\ldots.$$

Notice that the infinite word corresponding to the coordinates $\mathbb{N}\backslash\{0\}$ is periodic with period $i-1$. Now if $\Gamma_{i-1} \notin W_p$, Then for any $t \in \mathbb{N}$ such that $t > 0$ we have:

$$(\ldots \Delta\dot{\Delta}(\Gamma_1 \ldots \Gamma_{i-1})(\Gamma_1 \ldots \Gamma_{i-1})\ldots)A^{lt(i-1)} \neq \ldots \Delta\dot{\Delta}(\Gamma_1 \ldots \Gamma_{i-1})(\Gamma_1 \ldots \Gamma_{i-1})\ldots.$$

since $\Delta_{t(i-1)} \neq \Gamma_{t(i-1)} = \Gamma_{i-1}$. Therefore for any $t, t' \in \mathbb{N}$ such that $t \neq t'$, we must have that:

$$(\ldots \Delta\dot{\Delta}(\Gamma_1 \ldots \Gamma_{i-1})(\Gamma_1 \ldots \Gamma_{i-1})\ldots)A^{lt(i-1)} \neq (\ldots \Delta\dot{\Delta}(\Gamma_1 \ldots \Gamma_{i-1})(\Gamma_1 \ldots \Gamma_{i-1})\ldots)A^{lt'(i-1)}.$$

If $\Gamma_i = \Gamma_1$ and $\Gamma_{i-1} \in W_p$, then consider the bi-infinite word $\ldots \Lambda\dot{\Lambda}\Gamma_1 \ldots \Gamma_{i-1}(\Gamma_1 \ldots \Gamma_{i-1})$, since $q \neq p$ and $\Lambda \in W_q$ and $\Gamma_{i-1} \in W_p$ we have $\Gamma_{t(i-1)} = \Gamma_{i-1} \neq \Lambda_{t(i-1)}$ for any $m \in \mathbb{N}$. Here $\Lambda_{t(i-1)}$ is defined analogously to $\Delta_{t(i-1)}$. Therefore by the argument above $\ldots \Lambda\dot{\Lambda}\Gamma_1 \ldots \Gamma_{i-1}(\Gamma_1 \ldots \Gamma_{i-1})\ldots$ is on an infinite orbit under the action of A^l.

Case 2 : We assume now that $\Delta \in W_q$ and $\Lambda \in W_p$. As in the previous case we consider how powers of A^l act on the word $\ldots \Delta\dot{\Delta}\Gamma_1 \ldots \Gamma_{i-1}(\Gamma_i \ldots \Gamma_j)(\Gamma_i \ldots \Gamma_j)\ldots$.

Making an argument similar to case 1, we have that:

$$\ldots \Delta\dot{\Delta}\Gamma_1 \ldots \Gamma_{i-1}(\Gamma_i \ldots \Gamma_j)(\Gamma_i \ldots \Gamma_j)\ldots \xrightarrow{A^l} \ldots \dot{*}_0\, \Delta_1\Gamma_1 \ldots \Gamma_{i-1}(\Gamma_i \ldots \Gamma_j)(\Gamma_i \ldots \Gamma_j)\ldots$$

However, in this case $\Delta_1 \in W_p$. Applying A^l again we have:

$$\ldots \dot{*}_0\, \Delta_1\Gamma_1 \ldots \Gamma_{i-1}(\Gamma_i \ldots \Gamma_j)(\Gamma_i \ldots \Gamma_j)\ldots \xrightarrow{A^l} \ldots \dot{*}_0 *_1 \Delta_2\Gamma_1 \ldots \Gamma_{i-1}(\Gamma_i \ldots \Gamma_j)(\Gamma_i \ldots \Gamma_j)\ldots$$

where $\Delta_2 \in W_q$. Therefore given $t \in \mathbb{N}$ we know that after applying A^l t times, the resulting word, from the t^{th} position onwards is of the form: $\Delta_t\Gamma_1 \ldots \Gamma_{i-1}(\Gamma_i \ldots \Gamma_j)(\Gamma_i \ldots \Gamma_j)\ldots$ where $\Delta_t \in W_q$ if t is even, and $\Delta_t \in W_p$ if t is odd.

By considering the bi-infinite word: $\ldots \Lambda\dot{\Lambda}\Gamma_1 \ldots \Gamma_{i-1}(\Gamma_i \ldots \Gamma_j)(\Gamma_i \ldots \Gamma_j)\ldots$, and similarly defining the Λ_t's $t \in \mathbb{N}$, we see that after applying A^l t-times to this word, the output, from the t^{th} position onwards is of the form: $\Lambda_t\Gamma_1 \ldots \Gamma_{i-1}(\Gamma_i \ldots \Gamma_j)(\Gamma_i \ldots \Gamma_j)\ldots$ where $\Lambda_t \in W_q$ if t is odd, and $\Lambda_t \in W_p$ if t is even.

Now we go through the subcases as in Case 1. If $\Gamma_i \neq \Gamma_1$, then the argument proceeds exactly as in Case 1.

Hence consider the case $\Gamma_i = \Gamma_1$. Again we split into subcases. Now either $\Gamma_{i-1} \in W_p \sqcup W_q$ or it is disjoint from these two sets. We assume $\Gamma_{i-1} \in W_q$ (the other case is proved analogously). Since $\Lambda_{2t(i-1)} \in W_p$, for $t \in \mathbb{N}$ then by similar arguments to Case 1 above we conclude that $\ldots \Lambda\dot{\Lambda}\Gamma_1 \ldots \Gamma_{i-1}(\Gamma_i \ldots \Gamma_j)(\Gamma_i \ldots \Gamma_j)\ldots$ is on an infinite orbit under the action of A^{2l} and so under the action of A^l.

If $\Gamma_{i-1} \cap (W_p \sqcup W_q) = \emptyset$ then $\ldots \Delta\dot{\Delta}\Gamma_1 \ldots \Gamma_{i-1}(\Gamma_1 \ldots \Gamma_{i-1})\ldots$ and $\ldots \Lambda\dot{\Lambda}\Gamma_1 \ldots \Gamma_{i-1}(\Gamma_1 \ldots \Gamma_{i-1})\ldots$ are on infinite orbits under the action of A^l by repeating the argument in Case 1.

□

Remark 3.5.51. In the proof above, in showing that the witness is in an infinite orbit under the action of the transducer, our argument has made use only of the right infinite portion corresponding to the coordinates $\mathbb{N} \sqcup \{-1, \ldots, -r\}$. In particular this means that we can replace the left infinite portion corresponding to the coordinates $\{\ldots, -r-3, -r-2, -r-1\}$ by any infinite word in $X_n^{\mathbb{N}}$.

Corollary 3.5.52. *Let $A \in \widetilde{\mathcal{H}}_n$ be synchronizing at level k, and let $r \geqslant k$. If the graph $G_r(A)$ of bad pairs has a circuit, then there is rational word in $X_n^{\mathbb{Z}}$ on an infinite orbit under the action of A.*

Proof. This is a consequence of the proposition above and Remark 3.5.49. $\square$

As a further corollary of Proposition 3.5.50 we are able to prove Picantin's conjecture for all two state transducers in $\mathcal{H}_n$. We observe first of all that a two state synchronizing transducer is necessarily a reset automaton by Construction 2.2.1 and Theorem 2.2.6

Corollary 3.5.53. *Let $A \in \widetilde{\mathcal{H}}_n$ be a two state transducer then either $\langle A^{\vee} \rangle_+ = \{A^{\vee}\}$ or A has infinite order and there is a rational word in $X_n^{\mathbb{Z}}$ on an infinite orbit under the action of A.*

Proof. Since A is a two-state element of $\widetilde{\mathcal{H}}_n$, then A is reset. Further observe that as A has only two states, then by construction, either the graph $G_1(A)$ is empty or it consists of a single vertex with a loop. If $G_1(A)$ is empty, then by definition $A^{\vee}$ does not split, thus it follows by Proposition 3.5.41 that A has finite order as an element of $\mathcal{H}_n$. If $G_1(A)$ is non-empty, then it has a loop and by Corollary 3.5.52 A has infinite order and there is an rational word in $X_n^{\mathbb{Z}}$ which is on an infinite orbit under the action of A. $\square$

Below is an example of a transducer B whose graph of bad pairs at level 1, $G_1(B)$ satisfies the conditions of Lemma 3.5.48. We also construct a witness as in Proposition 3.5.50 which demonstrates that the transducer has infinite order.

Example 3.5.54. Let B be the transducer in Figure 3.16. Its dual is given by Figure 3.17.

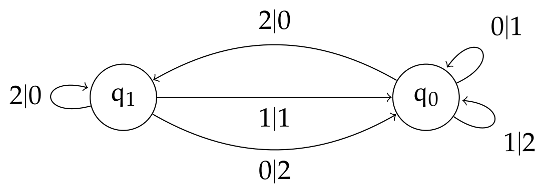


Figure 3.16: An element of infinite order in $\mathcal{H}_n$

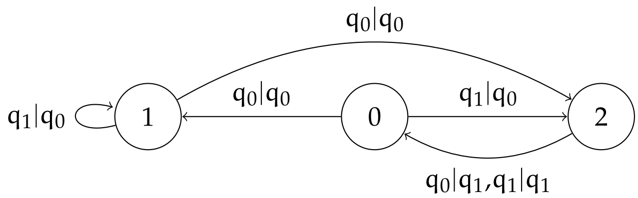


Figure 3.17: The level 1 dual of B

From this it is easy to see that the pair $\{q_0, q_1\}$ is a vertex of G_1 and there is a directed edge with initial and terminal vertex $\{q_0, q_1\}$. Therefore the conditions of Lemma 3.5.48 are satisfied and B has infinite order. Going through the construction in the proof of Proposition 3.5.50, we see that $\dots 11\dot{1}(02)(02)\dots$ is on an infinite orbit under the action of B.

Example 3.5.55. The transducer A shown in Figure 3.18 demonstrates that though the graph of bad pairs may contain a circuit at some level, the minimal graph of bad pairs at the same level may not do so.

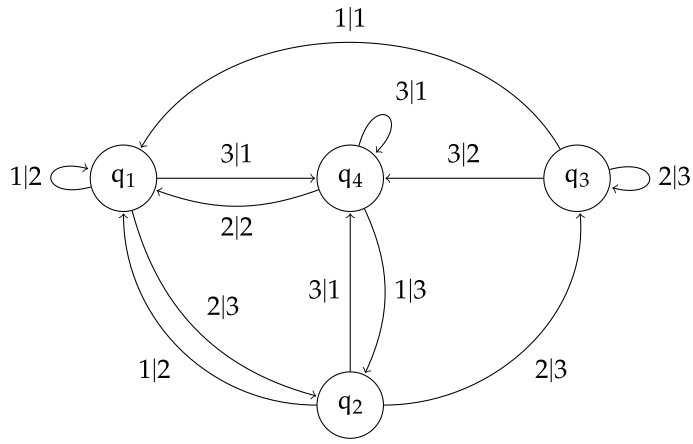


Figure 3.18: An element of infinite order whose minimal graph of bad pairs has no circuits.

It is easy to see that this transducer is bi-synchronizing at level 3 using the Collapsing procedure (Construction 2.2.1), or by direct computation in GAP. The graph $G_3(A)$ of bad pairs has a loop at the vertex $\{q_1, q_2\}$. The minimal graph of bad pairs $\overline{G}_3(A)$ is as shown in Figure 3.19:

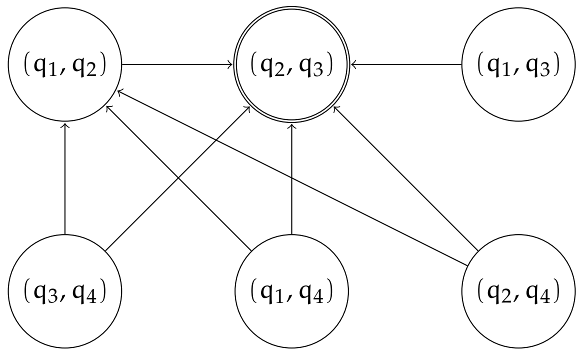


Figure 3.19: The graph $\overline{G}_3(A)$ of minimal bad pairs

Here the the double-circled state pair (q_2, q_3) is not a minimal bad pair, and in fact reads any word of length 3 into a pair of the form (p, p) (it acts like a sink through which we escape the minimal bad pairs).

Example 3.5.56. The transducer $H \in \mathcal{H}_5$ shown in Figure 3.20 is an element of finite order whose dual at its minimal bi-synchronizing level splits. However the next power of its dual is the zero of the semigroup generated by the dual. This means that the splits can be fixed by taking powers of the dual.

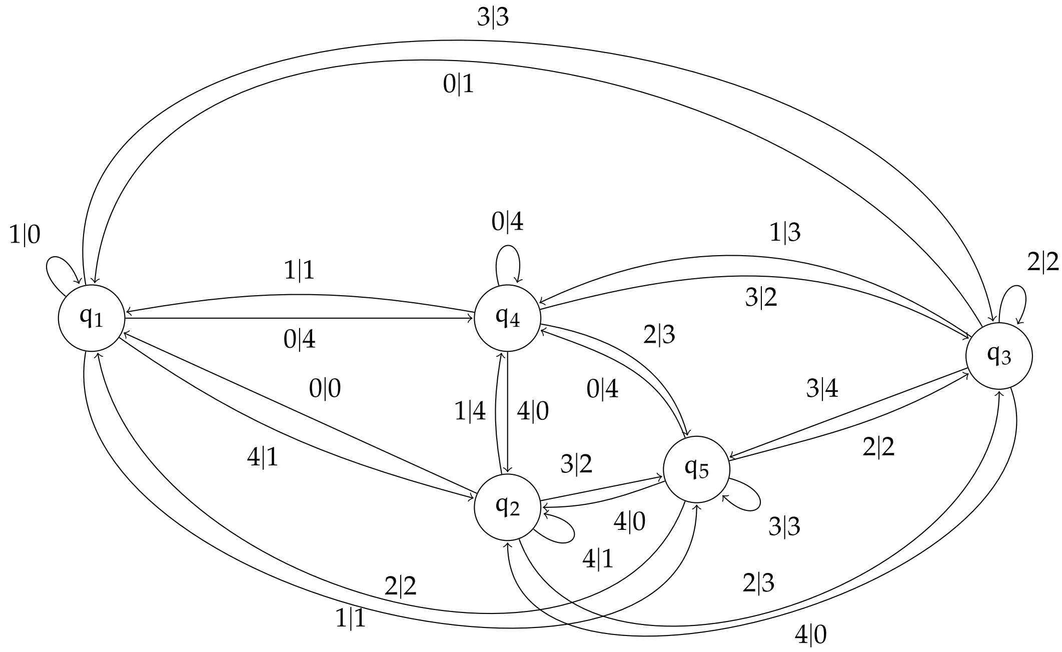


Figure 3.20: An element of finite order whose graph of bad pairs splits.

We have the following conjecture, which might in some sense be thought of as complementing Picantin's conjecture.

Conjecture 3.5.57. *Let $A \in \mathcal{H}_n$ be an element of infinite order which is synchronizing at level k, then there is an $r \in \mathbb{N}_k$ such that $G_r(A)$ has a loop.*

If Conjecture 3.5.57 is true, then the order problem in $\mathcal{P}_n$ is soluble and every element in $\mathcal{H}_n$ of infinite order has a rational word in $X_n^{\mathbb{Z}}$ on an infinite orbit.

Alternative conditions for having infinite order

In this section we give an algebraic condition which implies that the graph of bad pairs for a transducer for some power of its dual has a loop.

To this end let $A \in \widetilde{\mathcal{H}}_n$, and let $r \in \mathbb{N}$ be greater than or equal to the minimal synchronizing level of A. Let $l \in \mathbb{N}$ be the minimal splitting length of $A_r^{\vee}$.

To each state Γ of $A_r^{\vee}$ we associate a transformation σ_Γ of the set Q_A of states of A. We do this as follows. For each state $q \in Q_A$ and for any $l-1$ tuple $S \in Q^{l-1}$, there is a unique state $p \in Q_A$, such that if Δ is the output when Γ is processed through qS in A^l, then $\Delta \in W_p$. This is because as l is the minimal splitting length of $A_r^{\vee}$, the state p is independent of which $l-1$ tuple S we chose. Therefore define σ_Γ such that $q \overset{\sigma_\Gamma}{\mapsto} p$.

For $j \in \mathbb{N}$ let $\mathfrak{S}_{r,j}$ be the set of all products of length j of elements of the set $\{\sigma_\Gamma | \Gamma \in X_n^r\}$. We have the following result:

Proposition 3.5.58. *Let $A \in \widetilde{\mathcal{H}}_n$, and $r \in \mathbb{N}$ be greater than or equal to the minimal synchronizing level of A. Let $l \in \mathbb{N}$ be the minimal splitting length of $A_r^{\vee}$. Then $\mathfrak{S}_{r,|Q_A|^2+1}$ contains a transformation of Q_A which is not a right zero if and only if the graph $\overline{G}_r(A)$ of minimal bad pairs contains a circuit.*

Proof. Let A, r, and l be as in the statement of the proposition.

Now $\mathfrak{S}_{r,|Q_A|^2+1}$ contains a transformation of Q_A which is not a right zero if and only if there is a product $\sigma_{\Gamma_1}\sigma_{\Gamma_2}\ldots\sigma_{\Gamma_{|Q_A|^2+1}}$, for $\Gamma_i \in X_n^r$, $1 \leqslant i \leqslant |Q_A|^2+1$, whose image set has size at least 2. This occurs if and only if there are $p_0, q_0 \in Q_A$ which map to distinct elements under $\sigma_{\Gamma_1}\sigma_{\Gamma_2}\ldots\sigma_{\Gamma_{|Q_A|^2+1}}$. Let $p_i := (p_0)\sigma_{\Gamma_1}\sigma_{\Gamma_2}\ldots\sigma_{\Gamma_i}$ and $q_i := (q_0)\sigma_{\Gamma_1}\sigma_{\Gamma_2}\ldots\sigma_{\Gamma_i}$ for $1 \leqslant i \leqslant |Q_A|^2+1$. Notice that $p_i \neq q_i$ since p_0 and q_0 have distinct images under $\sigma_{\Gamma_1}\sigma_{\Gamma_2}\ldots\sigma_{\Gamma_{|Q_A|^2+1}}$.

By the pigeon hole principle there exists $1 \leqslant i, j \leqslant |Q_A|^2+1$ such that $\{p_i, q_i\} := \{p_j, q_j\}$.

This implies that in the graph $\overline{G}_r(A)$ of minimal bad pairs we have: $\{p_i, q_i\} \to \{p_{i+1}, q_{i+1}\} \to \ldots \to \{p_j, q_j\} \to \{p_i, q_i\}$. This follows by definition of the σ_Δ, $\Delta \in X_n^r$ and of the graph $\overline{G}_r(A)$.

Now suppose the graph $\overline{G}_r(A)$ contains a circuit. Let $j \in \mathbb{N}$ be the length of the circuit, and let $\{p_i, q_i\}$ $1 \leqslant i \leqslant j$ be the vertices on the circuit.

Let $1 \leqslant i < j$ be arbitrary. Now an edge $\{p_i, q_i\} \to \{p_{i+1}, q_{i+1}\}$ corresponds to the existence of some $\Gamma_i \in X_n^r$ and $S_i, T_i \in Q_A^{l-1}$ such that $(p_iS_i, q_iT_i, \Gamma_i)$ is a split of $A_r^\vee$ with bottom $\{p_{i+1}, q_{i+1}\}$. It then follows that the product $\sigma_{\Gamma_1}\ldots\sigma_{\Gamma_j}$ maps $\{p_1, q_1\} \to \{p_1, q_1\}$. This means that $\mathfrak{S}_{r,|Q_A|^2+1}$ contains a transformation of Q_A which is not a right zero. □

Corollary 3.5.59. *Let $A \in \widetilde{\mathcal{H}}_n$, and let $r \in \mathbb{N}$ be greater than or equal to the minimal synchronizing level of A. Let $l \in \mathbb{N}$ be the minimal splitting length of $A_r^\vee$. If $\mathfrak{S}_{r,|Q_A|^2+1}$ contains a transformation of Q_A which is not a right zero then A has infinite order. Moreover there is a rational word in $X_n^{\mathbb{Z}}$ on an infinite orbit under the action of A.*

Remark 3.5.60. The above now implies that if $A \in \widetilde{\mathcal{H}}_n$ has infinite order, but none of its graph of minimal bad pairs, $\overline{G}_r(A)$, for $r \in \mathbb{N}$ greater than or equal to the minimal synchronizing length, has a loop, then $\mathfrak{S}_{r,|Q_A|^2+1}$ consists entirely of right zeroes.

We have already seen that given an element $A \in \widetilde{\mathcal{P}}_n$ we can associate a transformation $\overline{A}_j$ of the set X_n^j to A. We shall now introduce a new transformation, which is defined only for elements of $\widetilde{\mathcal{H}}_n$.

Definition 3.5.61. Let $H \in \widetilde{\mathcal{H}}_n$, and let $j \in \mathbb{N}$, we shall define a transformation $\underline{H}_j$ of X_n^j by

$$\Gamma \mapsto (\Gamma)q_\Gamma^{-1}$$

where q_Γ is the unique state of H forced by Γ, and $(\Gamma)q_\Gamma^{-1}$ is the unique element of X_n^j such that $\lambda_H((\Gamma)q_\Gamma^{-1}, q_\Gamma) = \Gamma$. If j is zero, then $\underline{H}_j$ is simply the identity map on the set containing the empty word.

Remark 3.5.62. Given an element $H \in \widetilde{\mathcal{H}}_n$ and a $j \in \mathbb{N}$ such that $j > 1$, then $\underline{H}_j$ is not injective in general. One can check that for the transducer B of Figure 3.16, $\underline{B}_1$ is not injective. The map ϕ_j from $\widetilde{\mathcal{H}}_n$ to the full transformation semigroup on X_n^j which maps H to $\underline{H}_j$ is not a homomorphism.

Lemma 3.5.63. *Let $H \in \widetilde{\mathcal{H}}_n$ and let $j \in \mathbb{N}$ be greater than or equal to the minimal synchronizing level of H, then $\underline{H}_j$ is not injective if and only if $H_j^\vee$ has a split of length one such that the top and bottom of the split are equal.*

Proof. ($\Rightarrow$): suppose that, for $j \in \mathbb{N}$ and H as in the statement of the lemma, $\underline{H}_j$ is not injective. This means that there are two distinct elements Γ and Δ of X_n^j such that $(\Gamma)\underline{H}_j = (\Delta)\underline{H}_j$. Let $\Lambda := (\Gamma)\underline{H}_j$. Let q_Γ and q_Δ be the states of H forced by Γ and Δ respectively. Then by definition of $\underline{H}_j$ we have: $\lambda(\Lambda, q_\Gamma) = \Gamma$ and $\lambda(\Lambda, q_\Delta) = \Delta$.

Observe that as consequence it must be the case that $q_\Gamma \neq q_\Lambda$. Therefore it follows that we have the following split in $H_j^\vee$:

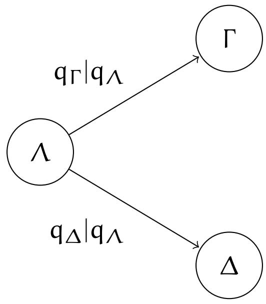


Figure 3.21: Split in $A_j^{\vee}$ with top equal to bottom

($\Leftarrow$) Suppose that $A_j^{\vee}$ has a split of length 1 such that the bottom of the split is equal to its top. This means that there exists Γ, Δ and Λ in X_n^j so that if q_Δ is the state of H forced by Δ and q_Γ is the state of H forced by Γ, then we have $\lambda(\Lambda, q_\Delta) = \Delta$ and $\lambda(\Lambda, q_\Gamma) = q_\Gamma$. This now means that $\Lambda = (\Gamma)\underline{H}_j = (\Delta)\underline{H}_j$ and $\underline{H}_j$ is not injective. $\square$

Corollary 3.5.64. *Let $H \in \widetilde{\mathcal{H}}_n$ and let $j \in \mathbb{N}$ be greater or equal to the minimal synchronizing level of H. If $\underline{H}_j$ is not injective then H has infinite order and there is a rational word in $X_n^{\mathbb{Z}}$ on an infinite orbit under the action of H.*

We segue briefly to present those results about the group $\mathcal{H}_n$ and the monoid $\widetilde{\mathcal{P}}_n$ which do not sit under the headings of 'order problem' or 'growth', but which are still related. Following this segue, we shall turn to the question of the growth rate of groups generated by transducers in $\widetilde{\mathcal{H}}_n$ and the growth rate of the core of elements of $\widetilde{\mathcal{H}}_n$ with powers.

3.6 Combining elements of $\widetilde{\mathcal{P}}_n$ and some embedding results

This section shall be concerned mainly with describing methods for combining elements of $\mathcal{P}_n$ to create an element of $\mathcal{P}_m$ for large enough m. As a consequence we shall prove some embedding results for the group $\mathcal{H}_n$ and the monoid $\widetilde{\mathcal{P}}_n$.

3.6.1 Two element case

In this section we describe a several ways of combining two elements of $\widetilde{\mathcal{P}}_n$ and $\widetilde{\mathcal{P}}_m$, into a single element of $\widetilde{\mathcal{P}}_{m+n}$. As we will see, this has interesting consequences when we restrict to the group $\mathcal{H}_n$.

Let $A = \langle X_n, Q_A, \pi_A, \lambda_A \rangle$ and $B = \langle X_m, Q_B, \pi_B, \lambda_B \rangle$ be elements of $\widetilde{\mathcal{P}}_n$ and $\widetilde{\mathcal{P}}_m$. It is a consequence of Proposition 3.4.6 that for all $1 \leqslant i \leqslant n$ an $1 \leqslant j \leqslant m$ there is a state, q_i of A and P_j of A and B respectively such that $\pi_A(i, q_i) = q_i$ and $\pi_B(j, p_j) = p_j$.

Form a new transducer $\widetilde{B} = \langle \{n, \ldots, n+m-1\}, Q_{\widetilde{B}}, \pi_{\widetilde{B}}, \lambda_{\widetilde{B}} \rangle$ with input and output alphabet $\{n, \ldots, n+m-1\}$ such that the states of $\widetilde{B}$ are in bijective correspondence with the states of B; we denote them by $\widetilde{q}$, where q is a state of B.

The transition and rewrite function of $\widetilde{B}$ are defined by the following rules for $i, j \in X_m$:

$$\pi_{\widetilde{B}}(n+i, \widetilde{q}) = \widetilde{p} \iff \pi_B(i, q) = p$$
$$\lambda_{\widetilde{B}}(n+i, \widetilde{q}) = n+j \iff \lambda_B(i, q) = j$$

Now we form a new transducer, $A \sqcup B = \langle X_{n+m}, Q_{A \sqcup B}, \pi_{A \sqcup B}, \lambda_{A \sqcup B} \rangle$ as follows:

$Q_{A \sqcup B} = Q_A \sqcup Q_{\widetilde{B}}$; the states of Q_A transition exactly as in A for all inputs in X_n and the states of $Q_{\widetilde{B}}$ transition as in $\widetilde{B}$ for all inputs in $X_{n+m} \backslash X_n$. Finally for all $i \in X_n$, and for any $\widetilde{q} \in Q_{\widetilde{B}}$, we have $\pi_{A \sqcup B}(i, \widetilde{q}) = p_i$, where p_i is the state of Q_A such that $\pi_A(i, p_i) = p_i$, furthermore,

$\lambda_{A\sqcup B}(i,\widetilde{q}) = i$. An analogous condition holds in $A \sqcup B$ for the states of A on inputs in $X_{2n}\backslash X_n$. We shall demonstrate this in Example 3.6.4.

Now further assume that $A \in \widetilde{\mathcal{P}}_n$ and $B \in \widetilde{\mathcal{P}}_m$ both have finite order (notice that this implies $A \in \mathcal{H}_n$ and $B \in \mathcal{H}_m$). We argue that the order of $A \sqcup B$ is at least $\mathrm{lcm}(O(A), O(B))$ — the lowest common multiple of the orders of A and B.

First we show that $A \sqcup B$ has finite order.

Let k be the maximal bi-synchronizing level of A and B. We shall give two different proofs showing that $A \sqcup B$ is synchronizing; similar arguments show that $(A \sqcup B)^{-1}$ is also synchronizing at level $k+1$.

Claim 3.6.1. $A \sqcup B$ *is bi-synchronizing at level* $k+1$

Proof 1. Observe that as soon we read i, $i \in X_n$, we must be processing from a state of A and if we read an $n+i$, $i \in X_n$ we must be processing from a state of $\widetilde{B}$.

Let $\Gamma \in X_{2n}^{k+1}$ a word of length $k+1$.

If $\Gamma \in X_n^{k+1}$ or $\Gamma \in \{n, \ldots, 2n-1\}^{k+1}$, then the state of $A \sqcup B$ forced by Γ is the state of A or $\widetilde{B}$ forced by Γ. Since reading the first letter guarantees, by the observation in the first paragraph, that the active state is a state of A (or B if $\overline{\Gamma} \in \{n, \ldots, 2n-1\}^{k+1}$), and A and B are bi-synchronizing at level k.

Hence we need only consider the case that $\overline{\Gamma}$ contains at least one letter from X_n and one letter from $X_{2n}\backslash X_n$.

Let $\Gamma = g_0 \ldots g_k$. Let $g_i \in X_n$ and assume $g_0 \in X_{2n}\backslash X_n$ (the other case follows by a similar argument) and suppose that $0 < j$ and j is minimal bigger than i such that $g_j \in X_{2n}\backslash X_n$. By the observation in the first paragraph, regardless of the starting state, after processing the g_0, the active state must be some state of A. By the minimality of j, after processing g_{j-1}, the active state is still some state of A. Now, notice that every state of A will read g_j to a fixed state $\widetilde{q}_{g_j}$ of B. Therefore regardless of the starting position, we always process the final $k-j$ inputs from the state q_{g_j}.

To see that $(A \sqcup B)^{-1}$ is also synchronizing at level $k+1$ observe that the states corresponding to states A^{-1} in $(A \sqcup B)^{-1}$ process words in X_n^* exactly as A^{-1} does. Moreover all states of $(A \sqcup B)^{-1}$ corresponding to states of A^{-1} read a fixed letter j in $\{n, \ldots, 2n-1\}$ to a unique state $\widetilde{q}_j^{-1}$ corresponding to the state $\widetilde{q}_j^{-1}$ of $\widetilde{B}^{-1}$. Analogously for the states of $(A \sqcup B)^1$ corresponding to the states of $\widetilde{B}^{-1}$ and elements of $\{n, \ldots, 2n-1\}^*$ and letters in X_n. Therefore we may repeat the argument already given for $(A \sqcup B)$ to show that $(A \sqcup B)^{-1}$ is synchronizing at level $k+1$ also. □

Proof 2. We apply the Collapsing procedure (Construction 2.2.1). Since k is the maximal bi-synchronizing level of A and B, then after k steps of the procedure, the copies of A and B have both been reduced to singletons. Now since every input in X_n is read into A and every input in $X_{n+m}\backslash X_n$ is read into B, we only need at most one additional step to reduce $A \sqcup B$ to a singleton. The result now follows. The same argument shows that $(A \sqcup B)^{-1}$ is synchronizing at level $k+1$. □

We now free the symbol k. To show that $A \sqcup B$ has finite order, we prove the following claim.

Claim 3.6.2. *Let* k *be minimal such that both* $A_k^{\vee}$ *and* $B_k^{\vee}$ *are the zero of* $\langle A^{\vee}\rangle_+$ *and* $\langle B^{\vee}\rangle_+$ *respectively. Then* $A \sqcup B_{k+1}^{\vee}$ *is* ω*-equivalent to a disjoint union of cycles as in Proposition 3.5.41.*

Proof. First we consider the case where $\Gamma \in X_n^{k+1}$ or $\Gamma \in \{n+1, \ldots, n+m-1\}^{k+1}$. Let $\Gamma = g_0 \ldots g_k$. By the assumption that both A and B (hence $\widetilde{B}$) have finite order, this implies that there is a state q of A (or $\widetilde{B}$ if $\Gamma \in \{n+1, \ldots, n+m-1\}^{k+1}$) such that the image of Γ is in the set W_q. This is because after reading g_0 we are enter a state of A (or $\widetilde{B}$ if $\Gamma \in \{n+1, \ldots, n+m-1\}^{k+1}$), and using the fact that $g_1 \ldots g_k$ belongs to a cycle of states as in Proposition 3.5.41 in $A_k^{\vee}$ (or $\widetilde{B}^{\vee}$ if $\Gamma \in \{n+1, \ldots, n+m-1\}^{k+1}$). Moreover if $\Gamma \in X_n^k$, then the image of Γ through any state of $A \sqcup B$ is also in X_n^k and analogously if $\Gamma \in \{n+1, \ldots, n+m-1\}^{k+1}$. Therefore the fact that Γ belongs to such a cycle of states is a consequence of the fact that A and B have finite order.

Now we consider the case where Γ contains a letter in X_n and a letter in $X_{n+m}\backslash X_n$. Similarly to the proof of Claim 3.6.1, let $\Gamma = g_0 g_1 \ldots g_k$. Suppose that a letter from X_n (the other case being analogous) occurs first and let j be minimal such that $j > 0$ and $g_j \in X_{n+m} \backslash X_n$. The proof of Claim 3.6.1 shows that the state of $A \sqcup B$ forced by Γ depends only on the suffix $g_j g_{j+1} \ldots g_k$. Let q_{g_j} be the state of $\widetilde{B}$ such that $\pi_{\widetilde{B}}(g_j, q_{g_j}) = Q_{g_j}$. Since every state of A acts as the identity on $X_{n+m} \backslash X_n$, it is the case that processing Γ from any state of $A \sqcup B$, the first letter of the output is an element of X_n and the j^{th} letter is the minimal element in $X_{n+m} \backslash X_n$ and is in fact equal to g_j, moreover since we process the length $j - k$ suffix from the state q_{g_j}, the length $j - k$ suffix is independent of the state we begin processing from. However we can now repeat this argument to show that the output through any state of the set of images of Γ, have the same $j - k$ suffix and the j^{th} letter equal to g_j. It now follows from the observation above that the state of $A \sqcup B$ forced depends only on the length $j - k$ suffix, and by induction, that Γ belongs to a cycle of states as in Proposition 3.5.41.

The above two paragraphs show that it is possible to decompose $A \sqcup B$ into a disjoint union of cycles as in Proposition 3.5.41 and so $A \sqcup B$ has finite order. □

The following alternative way of combining elements of $\widetilde{\mathcal{P}}_n$ also has the property that combining finite order elements results in elements of finite order by mechanical substitutions in the arguments above.

Let $A \in \widetilde{\mathcal{P}}_n$ and $B \in \widetilde{\mathcal{P}}_m$ be as above and form $\widetilde{B}$ as before. For each $1 \leqslant i \leqslant n$, let p_i be the state of A such that $\pi_A(i, p_i) = p_i$, by the definition of the transformation (in the case where A and B have finite order a permutation), $\overline{A}_1$, we have $\lambda_A(i, p_i) = \overline{A}_1(i)$, likewise there is a state q_j of B such that $\lambda_B(j, q_j) = \overline{B}_1(j)$ for $1 \leqslant j \leqslant m$.

We form $A \oplus B$ analogously to $A \sqcup B$. The set of states, and the transition function, $\pi_{A \oplus B}$ are identical but we make some adjustments to the rewrite function. For any letter $i \in X_n$, and any state $\widetilde{q}$ of $\widetilde{B}$, we take $\lambda_{A \oplus B}(i, \widetilde{q}) = \overline{A}_1(i)$, likewise for any letter $n + j \in \{n, \ldots n + m - 1\}$ and any state p of A, we have $\lambda_{A \oplus B}(j, p) = n + \overline{B}_1(j)$.

We have the following claim analogous to Claim 3.6.2 and proved in a similar way.

Claim 3.6.3. *Let k be minimal such that both $A_k^{\vee}$ and $B_k^{\vee}$ are the zero of $\langle A^{\vee} \rangle_+$ and $\langle B^{\vee} \rangle_+$ respectively. Then $A \oplus B_{k+1}^{\vee}$ is ω-equivalent to a disjoint union of cycles as in Proposition 3.5.41.*

The methods described above of combining elements of $\widetilde{\mathcal{P}}_n$ do not exhaust all possibilities, for instance we could fix a state of $\widetilde{B}$ such that reading any letter $X_{n+m} \backslash X_n$ from A goes into this state, and likewise we could fix such a state of $\widetilde{B}$. Similar arguments to those given above will show that these methods also give rise to elements of finite order whenever the initial elements have finite order, in fact we may prove versions of Claim 3.6.2 for each by making mechanical substitutions in the original proof.

We remark also that as there are new cycles of states introduced in $(A \sqcup B)_{k+1}^{\vee}$ and $(A \oplus B)_{k+1}^{\vee}$ that are not present in $A_k^{\vee}$ or $B_k^{\vee}$ it might be that the order of $A \sqcup B$ is strictly greater than $\mathrm{lcm}(O(A), O(B))$ in some cases.

We give an example below.

Example 3.6.4. Consider the elements of $\mathcal{H}_3$ and $\mathcal{H}_2 \cong \mathbb{Z}/2\mathbb{Z}$ of order three and order 2 respectively,

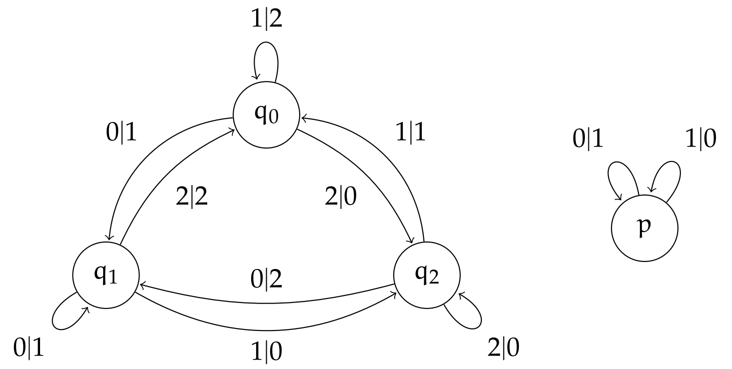


Figure 3.22: Elements of $\mathcal{H}_3$ and $\mathcal{H}_2$.

we now combine them to give an element of order 6 in $\mathcal{H}_5$. Using any of the methods describe above will yield an element of order 6, we only illustrate one such method.

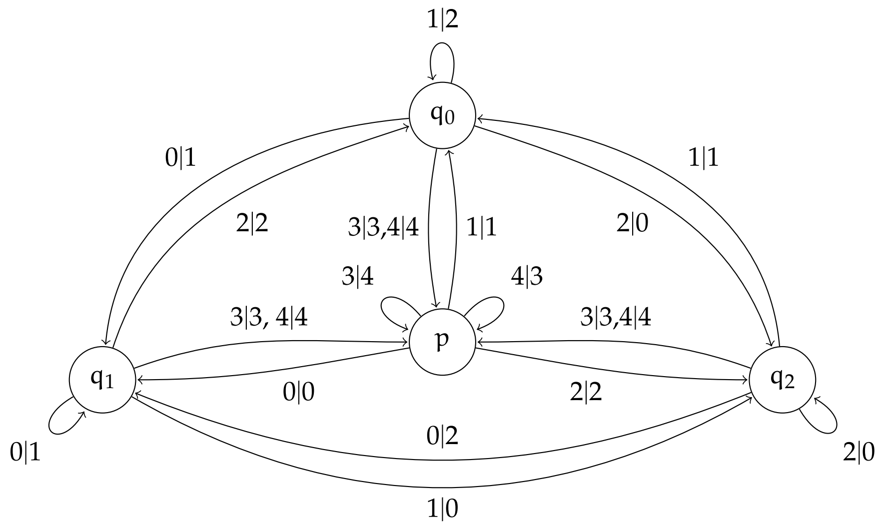


Figure 3.23: The result of combining the elements in Figure 3.22.

3.6.2 Embedding direct sums of $\widetilde{\mathcal{P}}_m$ in $\widetilde{\mathcal{P}}_n$ for n large enough

The aim of this section is to show that for given $n \in \mathbb{N}$ and for a sequence of non-zero natural numbers $d_1 \leqslant d_2 \leqslant \ldots \leqslant d_l$ such that $\sum_{i=1}^{l} d_i = n$, the semigroup $\widetilde{\mathcal{P}}_n$ contains a subsemigroup isomorphic to $\widetilde{\mathcal{P}}_{d_1} \times \widetilde{\mathcal{P}}_{d_2} \times \ldots \times \widetilde{\mathcal{P}}_{d_l}$. This will then yield, as a corollary, that $\mathcal{H}_n$ contains a subgroup isomorphic to $\mathcal{H}_{d_1} \times \mathcal{H}_{d_2} \times \ldots \times \mathcal{H}_{d_l}$. In order to do this, we first extend the results of the previous section. Essentially we shall simultaneously merge the elements of $\mathcal{H}_{d_i}$, as opposed to inductively applying the construction in the previous section, this allows us to better control the synchronizing level of the resulting transducers.

Let $n \in \mathbb{N}$ and let d_i, $1 \leqslant i \leqslant l$, be as in the previous paragraph. Let $X_0 := \{0, 1, \ldots, d_1 - 1\}$ and for $2 \leqslant i \leqslant l$ let $X_i := \{\sum_{j=1}^{i-1} d_j, \sum_{j=1}^{i-1} d_j + 1, \ldots, \sum_{j=1}^{i} d_j - 1\}$. Furthermore let A_i $1 \leqslant i \leqslant l$ be synchronous synchronizing transducers on the alphabet X_i. We shall now describe how to form $\sqcup_{i=1}^{l} A_i \in \widetilde{\mathcal{P}}_n$, which will simply be an extension of the 2 element case described in Subsection 3.6.1.

For each $i \in \mathbb{N}$ let $\overline{A_i}$ denote the transformation of the words of length 1 induced by A_i as in Remark 3.4.10. The transducer $\sqcup_{i=1}^{l} A_i := \langle X_n, \sqcup_{i=1}^{l} Q_{A_i}, \pi_{\sqcup}, \lambda_{\sqcup} \rangle$ will consists of the disjoint union of copies of the A_i which are connected in a specific way. Fix a j such that $1 \leqslant j \leqslant l$, and consider the copy of A_j in $\sqcup_{i=1}^{l}A_i$. Then the copy of A_j in $\sqcup_{i=1}^{l}A_i \in \widetilde{\mathcal{P}}_n$ transitions precisely as A_j does when restricted to X_j; we now describe how A_j acts for inputs not in X_j.

Let $1 \leqslant i \leqslant l$ be arbitrary, then for any $x_i \in X_i$, there is a unique state $q_{x_i} \in Q_{A_i}$ such that $\pi_{A_i}(x_i, q_{x_i}) = q_{x_i}$ and $\lambda_{A_i}(x_i, q_{x_i}) = (x_i)\overline{A_i}$. Therefore in $\sqcup_{i=1}^{l}A_i \in \mathcal{P}_n$ we set $\pi_{\sqcup}(x_i, A_j) = q_{x_i}$ and $\lambda_{\sqcup}(x_i, A_j) = (x_i)\overline{A_i}$. Hence we have now described how the copy of A_j acts on all inputs in $\sqcup_{i \neq j, 1 \leqslant i \leqslant l} X_i$.

Repeating the above for each A_j, $1 \leqslant j \leqslant l$, we now have that $\sqcup_{i=1}^{l}A_i$ is connected and all states are defined on $X_n := \{0, 1 \ldots n-1\}$.

The proof that $\sqcup_{i=1}^{l}A_i \in \widetilde{\mathcal{P}}_n$ (i.e that the resulting transducer is synchronizing) requires only the obvious amendments to the 2 element case proven in the previous Section. Therefore the following theorem is valid:

Theorem 3.6.5. *Let $n \in \mathbb{N}$ and let d_i $1 \leqslant i \leqslant l$ be an increasing sequence of non-zero natural numbers such that $\sum_{i=1}^{l} d_i = n$. Let $X_1 := \{0, 1, \ldots, d_1 - 1\}$ and for $2 \leqslant i \leqslant l$ let $X_i := \{\sum_{j=1}^{i-1} d_j - 1, \sum_{j=1}^{i-1} d_j, \ldots, \sum_{j=1}^{i} d_j - 1\}$. Furthermore let A_i $1 \leqslant i \leqslant l$ be synchronous synchronizing transducers on the alphabet X_i. Then $\sqcup_{i=1}^{l}A_i$ is an element of $\widetilde{\mathcal{P}}_n$. If k_i is the synchronizing level of each A_i, $1 \leqslant i \leqslant l$ then the synchronizing level of $\sqcup_{i=1}^{l}A_i$ is at most $\max_{1 \leqslant i \leqslant l}\{k_i\} + 1$.*

Remark 3.6.6. If we begin with elements $A_i \in \mathcal{P}_{d_i}$ acting on the alphabet X_i, such that one of the A_i does not possess a homeomorphism state, then the resulting transducer $\sqcup_{k=1}^{l}A_k$ does not represent a homeomorphism of $X_n^{\mathbb{Z}}$.

Proof. To see this let $i \in \{1 \ldots l\}$ be such that A_i does not posses a homeomorphism state. Then since A_i is a synchronous transducer there is a state q_i of A_i and $x_i, y_i \in X_i$ such that $\lambda_{A_i}(x_i, q_i) = \lambda_{A_i}(x_i', q_i)$. Let $p_i := \pi_{A_i}(x_i, q_i)$ and $p_i' = \pi_{A_i}(x_i', q_i)$. Let q_{z_i} be the state of A_i such that $\pi_{A_i}(z_i, q_{z_i}) = q_{z_i}$, and let Γ_i in X_i^* be a path from q_{z_i} to q_i. Furthermore let $x_j \in X_j$ for $j \in \{1, \ldots, l\} \backslash \{i\}$, and let q_{x_j} be the state of A_j such that $\pi_{A_j}(x_j, q_{x_j}) = q_{x_j}$.

Now observe that by definition of $\sqcup_{k=1}^{l}A_k$, the words $x_j z_i \Gamma_i x_i x_j$ and $x_j z_i \Gamma_i x_i' x_j$ are such that $\pi_{\sqcup}(x_j z_i \Gamma_i x_i x_j, q_{x_j}) = q_{x_j}$ and $\pi_{\sqcup}(x_j z_i \Gamma_i x_i' x_j, q_{x_j}) = q_{x_j}$. Moreover $\lambda_{\sqcup}(x_j z_i \Gamma_i x_i x_j, q_{x_j}) = \lambda_{\sqcup}(x_j z_i \Gamma_i x_i' x_j, q_{x_j})$.

Therefore $\sqcup_{k=1}^{l}A_k$ maps the bi-infinite strings $\ldots(x_j z_i \Gamma_i x_i x_j)\ldots$ and $\ldots(x_j z_i \Gamma_i x_i x_j)\ldots$ to the same element of $X_n^{\mathbb{Z}}$, and so it is not injective. □

Remark 3.6.7. If we restrict instead to $\mathcal{H}_{d_i}$ instead of $\mathcal{P}_{d_i}$ then the resulting transducer $\sqcup_{k=1}^{l}A_k$ will be in $\mathcal{H}_n$ as we will see below.

It was shown in the 2 element case, that for A and B acting on alphabets X_1 and X_2 such that $X_1 \sqcup X_2 := \{0, 1, \ldots, n-1\}$ then $A \sqcup B$ has finite order if and only if A and B have finite order. In this more general setting we prove the following stronger result.

Theorem 3.6.8. *Let $n \in \mathbb{N}$ and let d_i $1 \leqslant i \leqslant l$ be an increasing sequence of non-zero natural numbers such that $\sum_{i=1}^{l} d_i = n$. Let $X_1 := \{0, 1, \ldots, d_1 - 1\}$ and for $2 \leqslant i \leqslant l$ let $X_i := \{\sum_{j=1}^{i-1} d_j, \sum_{j=1}^{i-1} d_j, \ldots, \sum_{j=1}^{i} d_j - 1\}$. By an abuse of notation let $\widetilde{\mathcal{P}}_{d_i}$ denote the monoid of synchronous, synchronizing transducers on the alphabet X_i. Then the map $\phi : \bigoplus_{i=1}^{l} \widetilde{\mathcal{P}}_{d_i} \to \widetilde{\mathcal{P}}_n$, $(A_1, \ldots, A_l) \mapsto \sqcup_{i=1}^{l}A_i$ is a monomorphism.*

Proof. That this map is injective follows from that the fact that the action of each A_i on $X_i^{\mathbb{Z}}$ is replicated exactly when we restrict $\sqcup_{i=1}^{l}A_i$ to $X_i^{\mathbb{Z}}$. Therefore we need only prove that ϕ is a homomorphism.

Let $(A_1, \ldots, A_l)$ and $(B_1 \ldots, B_l)$ be elements of $\phi : \bigoplus_{i=1}^{l} \widetilde{\mathcal{P}}_{d_i}$, and let $(C_1, \ldots, C_l)$ be their product, hence $C_i = \mathrm{Core}(A_i * B_i)$. We shall show that the $D := \mathrm{Core}(\sqcup_{i=1}^{l}A_i * \sqcup_{i=1}^{l}B_i) = \sqcup_{i=1}^{l}C_i$.

First notice that for $q_i \in Q_{A_i}$ and $p_j \in Q_{B_j}$ the pair (q_i, p_j) is not a state of D. This is because for any word $\Gamma \in X_n^*$ such that the state of $\sqcup_{r=1}^{l}A_r$ forced by Γ is q_i, then Γ must have a non-empty suffix in X_i^*, and hence so also must its output through any state of $\sqcup_{r=1}^{l}A_r$ by construction. Therefore the output of Γ through any state of $\sqcup_{r=1}^{l}B_r$ will synchronise to a state in Q_{B_i}. Therefore the states of D are precisely a subset of $\sqcup_{i=1}^{l}Q_{A_i} \times Q_{B_i}$.

Now since the states of D intersecting $Q_{A_i} \times Q_{B_i}$ arising from the transducer product $A_i * B_i$ form precisely the sub-transducer C_i, therefore to conclude the proof it suffices (by the injectivity of ϕ) to show two things. Firstly, that for $j \neq i$ all states of $A_j \times B_j$ act on X_i precisely as $\overline{C_{i1}} = \overline{A_{i1}} \times \overline{B_{i1}}$ (the final equality follows from Theorem 3.4.12). Secondly, that all states (q_j, p_j) of $A_j \times B_j$ read an $x_i \in X_i$ into the unique state of C_i with a loop labelled by x_i.

The first part follows from the following observation. By construction for any $j \neq i$ the copy of A_j in $\sqcup_{i=1}^{l} A_i$ acts on X_i precisely as $\overline{A_{i1}}$ does, similarly in $\sqcup_{i=1}^{l} B_i$. Now by Theorem 3.4.12, $\overline{C_{i1}} := \overline{A_{i1}} \times \overline{B_{i1}}$. Therefore the first part is proved.

For the second part consider the following. Notice that for any $j \neq i$ and for any state q_j, a state of the copy of A_j in $\sqcup_{i=1}^{l} A_i$, and for any $x_i \in X_i$ we have that $\pi_{\sqcup A}(x_i, q_j) = q_{x_i}$, where $\pi_{\sqcup A}$ is the transition function of $\sqcup_{i=1}^{l} A_i$, and q_{x_i} is the unique state of A_i such that $\pi_{A_i}(x_i, q_{x_i}) = q_{x_i}$. An analogous statement holds for $\sqcup_{i=1}^{l} B_i$. Therefore given $(q_j, p_j) \in A_j \times B_j$, and $x_i \in X_i$, we have $\pi_{\sqcup D}(x_i, q_j, p_j) = (\pi_{\sqcup A}(x_i, q_j), \pi_{\sqcup B}((x_i)\overline{A_{i1}}, p_j))$, however this is simply the state $(q_{x_i}, p_{(x_i)\overline{A_{i1}}})$ of $A_i \times B_i$. However, since by definition of $\overline{A_{i1}}$, $(x_i)\overline{A_{i1}} = \lambda_{A_i}(x_i, q_{x_i})$, then $(q_{x_i}, p_{(x_i)\overline{A_{i1}}})$ is precisely the unique state of C_i with a loop labelled by x_i.

□

Remark 3.6.9. It is straight-forward to see from the above that ϕ maps $\bigoplus_{i=1}^{l} \mathcal{H}_{d_i}$ to a subgroup of $\mathcal{H}_n$ since $\bigoplus_{i=1}^{l} \mathcal{H}_{d_i}$ is a subgroup of $\widetilde{\mathcal{P}}_n$.

Corollary 3.6.10. *Let $n \in \mathbb{N}$ and let d_i $1 \leqslant i \leqslant l$ be an increasing sequence of non-zero natural numbers such that $\sum_{i=1}^{l} d_i = n$. Let $X_1 := \{0, 1, \ldots, d_1 - 1\}$ and for $2 \leqslant i \leqslant l$ let $X_i := \{\sum_{j=1}^{i-1} d_j - 1, \sum_{j=1}^{i-1} d_j, \ldots, \sum_{j=1}^{i} d_j - 1\}$. By an abuse of notation let $\widetilde{\mathcal{P}}_{d_i}$ denote the monoid of synchronous synchronizing transducers on the alphabet X_i. Given $(A_1, \ldots, A_l) \in \bigoplus_{i=1}^{l} \widetilde{\mathcal{P}}_{d_i}$, $\sqcup_{i=1}^{l} A_i$ has finite order if and only if each of the A_i's have finite order. Moreover the order of $\sqcup_{i=1}^{l} A_i$ is precisely the lowest common multiple of the orders of the A_i.*

Proof. This is a consequence of Theorem 3.6.8 and well known results about direct sums of groups. □

The following result generalises Claim 3.6.2.

Proposition 3.6.11. *Let $n \in \mathbb{N}$ and let d_i $1 \leqslant i \leqslant l$ be an increasing sequence of non-zero natural numbers such that $\sum_{i=1}^{l} d_i = n$. Let $X_1 := \{0, 1, \ldots, d_1 - 1\}$ and for $2 \leqslant i \leqslant l$ let $X_i := \{\sum_{j=1}^{i-1} d_j, \sum_{j=1}^{i-1} d_j, \ldots, \sum_{j=1}^{i} d_j - 1\}$. By an abuse of notation let $\widetilde{\mathcal{P}}_{d_i}$ denote the monoid of synchronous, synchronizing transducers on the alphabet X_i. Given $(A_1, \ldots, A_l) \in \bigoplus_{i=1}^{l} \widetilde{\mathcal{P}}_{d_i}$ and $k \in \mathbb{N}$ minimal such that $(A_i)_k^{\vee}$ is the zero of $\langle A_i^{\vee} \rangle_+$, $1 \leqslant i \leqslant l$, then $(\sqcup_{i=1}^{l} A)_{k+1}^{\vee}$ is the zero of $\langle \sqcup_{i=1}^{l} A^{\vee} \rangle_+$.*

Proof. The proof is the natural generalisation of the proof of Claim 3.6.2 and so we omit it. □

It is a result by Boyle, Franks and Kitchens [12] that for $n \geqslant 3$ $\mathcal{H}_n$ contains free groups. Therefore we have the following corollary:

Corollary 3.6.12. *Let $n \geqslant 3$ and let m and l be natural numbers such that $n = 3m + l$ where $0 \leqslant l \leqslant 3$. Then $\mathcal{H}_n$ contains a subgroup isomorphic to $\Pi_{i=1}^{m} F_2$ where F_2 is the free group on two generators.*

Notice that since $F_2 \times F_2$ has undecidable subgroup membership problem, it follows that for $n \geqslant 6$ $\mathcal{H}_n$ has undecidable subgroup membership problem.

Remark 3.6.13. We can modify the construction above. Let n, d_i and A_i, $1 \leqslant i \leqslant l$, be as before. For each A_i fix a permutation σ_i of X_i and an element $S_i \in Q_{A_i}^{d_i}$. Then we may form the transducer $\sqcup_{i=1,\sigma_i,S_i}^{l} A_i = \langle X_n, \sqcup_{i=1}^{l} A_i, \lambda_{\sqcup}, \pi_{\sqcup} \rangle$. For a given $j \in \{1, \ldots l\}$ the copy of A_j in $\sqcup_{i=1,\sigma_i,S_i}^{l} A_i$ is precisely A_j when restricted to X_j. However for $i \neq j$, and any state q_j of A_j, then given any $x_i \in X_i$ we have $\lambda_{\sqcup}(x_i, q_j) = (x_i)\sigma_i$; $\pi_{\sqcup}(x_i, q_j)$ is the entry of S_i corresponding to the position of x_i when the elements of X_i are ordered according to the natural ordering induced from $\mathbb{N}$. Then once more the resulting element of $\widetilde{\mathcal{P}}_n$ is synchronizing and has finite order if and only if all the A_i's have finite order. Moreover we may prove a version of Proposition 3.6.11 in each case.

3.7 The difference between the synchronizing and bi-synchronizing level

Using the techniques developed in Section 3.6 we now construct a class of examples of finite order elements which are all synchronizing at level 1, but whose inverses are synchronizing at the maximum possible level for the given number of states. A side-effect of the construction is that the alphabet size increases with the gap in the size of the synchronizing and bi-synchronizing level.

Our base transducer B is the transducer in Figure 3.24 to the left. Let A be the transducer on the right.

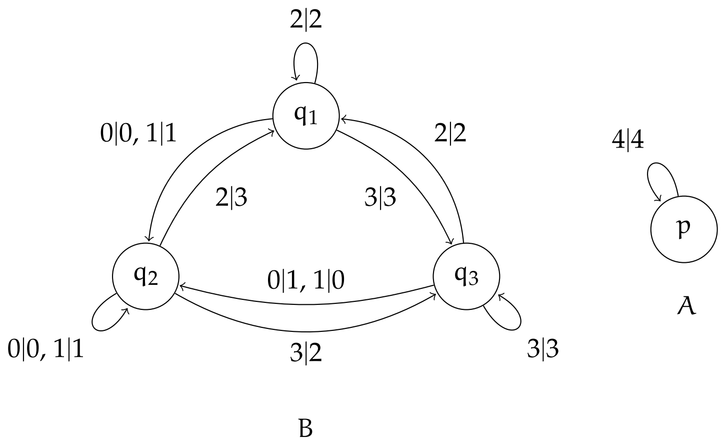


Figure 3.24: The base transducer B and an element of $\mathcal{H}_1$

Notice that B is synchronizing at level 1 but bi-synchronizing at level 2. Observe that by Theorem 2.2.7 a transducer with j states is synchronizing at level at most $j-1$, since we must identity two states at each step of the algorithm. Therefore B^{-1} attains the maximum synchronizing level for a 3 state transducer. One can check that B has order 4, moreover 2 is minimal such that $B_2^{\vee}$ is the zero of $\langle B^{\vee}\rangle_+$.

Now we attach A to B using the construction described in Remark 3.6.13. Let σ_2 be any permutation of $\{0,1,2,3\}$ that maps 0 to 3. There is only one permutation of $\{4\}$; form $\sqcup_{i=1,\sigma_i,S_i}^{2} C_i = \langle X_n, \sqcup_{i=1}^{2} C_i, \lambda_{\sqcup}, \pi_{\sqcup}\rangle$ where $C_1 = B$, $C_2 = A$, and σ_1 is the identity map. The resulting transducer is as shown in Figure 3.25 to the left:

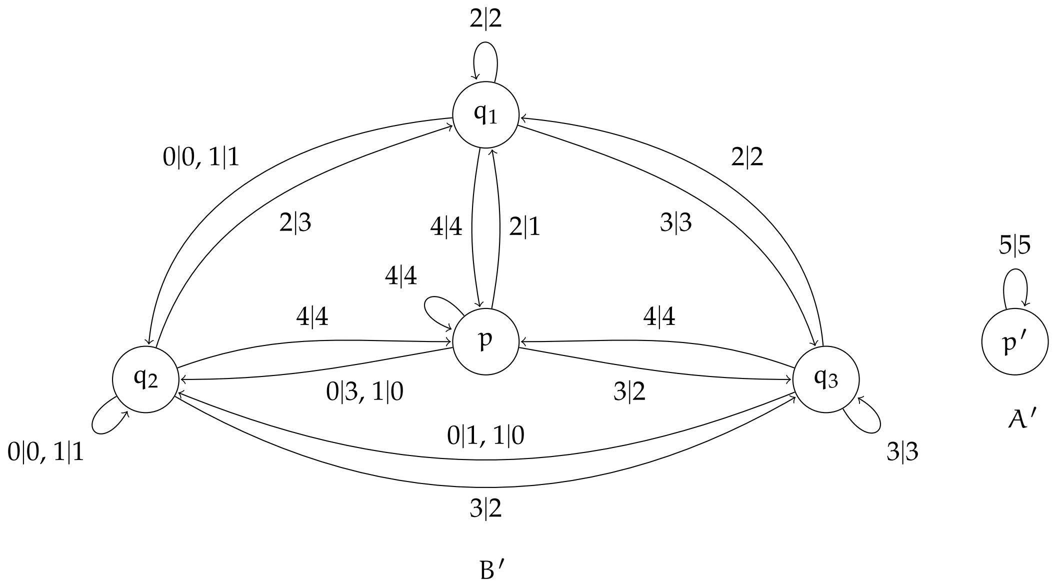


Figure 3.25: The resulting transducer B' which is a merge of B and A and an element A' of $\mathcal{H}_1$

Since all the states B map $\{0,1\} \to \{0,1\}$, p is not ω-equivalent to any state of $B' := \sqcup_{i=1,\sigma_i,S_i}^{2} C_i$. Moreover notice that since there is a path $p \overset{0|3}{\to} q_2 \overset{2|3}{\to} q_1 \overset{3|3}{\to} q_4$, in order to identify p^{-1} with q_1^{-1}, q_2^{-1} and q_3^{-1} in B' we must first have identified q_1^{-1}, q_2^{-1} and q_3^{-1}. Therefore since B^{-1} is synchronizing at level 2, it takes 3 steps to collapse B'^{-1} to a single state. It follows by Theorem 2.2.6 that B' is bi-synchronizing at level 3 and synchronizing at level 1. Moreover since 2 is minimal such that both $B_2^{\vee}$ and $A_2^{\vee}$ are zeros in $\langle B^{\vee} \rangle_+$ and $\langle A^{\vee} \rangle_+$ respectively, it follows by the appropriate generalisation of Proposition 3.6.11, that 3 is minimal such that $B_3'^{\vee}$ is the zero of $\langle B'^{\vee} \rangle_+$ since, $B_3'^{\vee}$ is zero and 3 is the minimal bi-synchronizing level of B'.

Now since all the states of B' fix 4, we can repeat the process. Let A' be the transducer to the right of Figure 3.25, and let σ_2' be any permutation of $\{0,1,2,3,4\}$ that maps 4 to 3. Then by the repeating the arguments above the transducer $B'' := \sqcup_{i=1,\sigma_i',S_i}^{2} C_i = \langle X_n, \sqcup_{i=1}^{2} C_i, \lambda_{\sqcup}, \pi_{\sqcup} \rangle$ where $C_1 = B'$ $C_2 = A'$, and σ_1' is the identity map, is bi-synchronizing at level 4 and synchronizing at level 1. Moreover, again by the appropriate generalisation of Proposition 3.6.11, 4 is minimal such that $B_4''^{\vee}$ is the zero of the semigroup $\langle B''^{\vee} \rangle_+$. We may continue on in this way, to construct transducers D of finite order with r states, $r \geqslant 3$ such that D has minimal bi-synchronizing level $r-1$ and $r-1$ is minimal such that $D_{r-1}^{\vee}$ is the zero of $\langle D^{\vee} \rangle_+$.

In the next section we again consider Pincantin's conjecture.

3.8 Level one synchronizing transducers and Picantin's conjecture

In this section we make use of the result of the previous section and a construction of Delacourt and Ollinger in [23], to construct elements A of $\mathcal{H}_n$ of finite order with m states, $m \in \mathbb{N}_3$, which are synchronizing at level 1 and such that $m-1$ is minimal so that $A_{m-1}^{\vee}$ is the zero of $\langle A^{\vee} \rangle_+ = \{A_i^{\vee} \mid i \in \mathbb{N}\}$. Recall that Picantin's conjecture states that a level 1 synchronizing transducer in $\mathcal{H}_n$ of finite order, with m states, satisfies $A_{m-1}^{\vee}$ is the zero of $\langle A^{\vee} \rangle_+$.

We begin by first describing the construction of Delacourt and Ollinger. This construction embeds elements of $\mathrm{End}(X_n, \sigma_n)$ of the form f_{∞} for a left permutive map $f \in F(X_n, 2)$ into $\mathrm{Aut}(X_n, \sigma_m)$ for m large enough, and is given in terms of the action on $X_n^{\mathbb{Z}}$. We describe the construction below in terms of level 1 synchronizing transducers in $\widetilde{\mathcal{H}}_n$, and as $\widetilde{\mathcal{H}}_n$ is isomorphic to the monoid of left permutive elements of $\mathrm{End}(X_n, \sigma_n)$ (Proposition 3.3.12), we get an equivalent construction to that of Delacourt and Ollinger. The author is once more indebted to Bartholdi for drawing his attention to this construction and for helpful discussions.

Construction 3.8.1 (Construction of Delacourt and Ollinger)**.** Let $A \in \widetilde{\mathcal{H}}_n$, be synchronizing at level 1 and let $m \in \mathbb{N}_2$ and $1 \leqslant l \leqslant m-1$. Let p be a symbol distinct from Q_A, set $Q_{A_{m,l}} = Q_A \sqcup \{p\}$ and $X^{(m)} = \{x^{(i)} \mid x \in X_n, 1 \leqslant i \leqslant m\}$. Let $\pi_{A_{m,l}} : X^{(m)} \times Q_{A_{m,l}} \to Q_{A_{m,l}}$ be defined as follows: for $q \in Q_{A_{m,l}}$ and $x \in X_n$ we have, $\pi_{A_{m,l}}(x^{(l)}, q) = q_x$ (see Notation 2.1.9). For $i \in \{1, 2, \ldots, m\} \backslash \{l\}$, any $q \in Q_{A_{m,l}}$ and any $x \in X_n$ we have, $\pi_{A_{m,l}}(x^{(i)}, q) = p$. Also define a function $\lambda_{A_{m,l}} : X^{(m)} \times Q_A \to X^{(m)}$ as follows: for $q \in Q_A$ and for $x \in X_n$ we have, $\lambda_{A_{m,l}}(x^{(m)}, q) = (\lambda_A(x, q))^{(1)}$ and $\lambda_{A_{m,l}}(x^{(m)}, p) = x^{(1)}$. For $q \in Q_{A_{m,l}}$, $x \in X_n$, and $1 \leqslant i \leqslant m-1$, we have $\lambda_{A_{m,l}}(x^{(i)}, q) = x^{(i+1)}$. Set $A_{m,l} := \langle X^{(m)}, Q_{A_{m,l}}, \pi_{A_{m,l}}, \lambda_{A_{m,l}} \rangle$ a transducer.

Remark 3.8.2. Let $m \in \mathbb{N}_2$ and $1 \leqslant l \leqslant m-1$ and let $A \in \widetilde{\mathcal{H}}_n$ be synchronizing at level l. Observe that $A_{m,l}$ is also reset. This is because for $i \in \{1, \ldots, m\} \backslash \{l\}$, $\pi_{A_{m,l}}(x^{(i)}, .)$ takes only the value p for any $x \in X_n$, whereas $\pi_{A_{m,l}}(x^{(l)}, .)$ takes only the value q_x. Moreover, if $A \in \widetilde{\mathcal{H}}_n$ has a state q such that $\lambda_A(., q) : X_n \to X_n$ is the identity map, then, since A is reset and the maps $\lambda_{A_{m,l}}(., q) : X^{(m)} \to X^{(m)}$ and $\lambda_{A_{m,l}}(., p) : X^{(m)} \to X^{(m)}$ are equal, the state $q \in Q_{A_{m,l}}$ and the state p are ω-equivalent. Therefore, whenever A has a state q which is the identity, we identify the state q with the state p in $A_{m,l}$ and for convenience we shall use the symbols p and q interchangeably to denote this state.

Definition 3.8.3. Let $A = \langle X, Q_A, \pi_A, \lambda_A \rangle$ be a synchronous transducer. We say that A possesses a *local identity state* if there is a state $q \in Q_A$ such that $\lambda_A(., q) : X \to X$ is the identity map.

Remark 3.8.4. Observe that if A is a minimal, reset automata, the A has exactly one local identity state, otherwise A is not minimal.

We have the following claim:

Claim 3.8.5. *Let $m \in \mathbb{N}_2$ and $1 \leqslant l \leqslant m-1$. Let $A \in \widetilde{\mathcal{H}}_n$ be a level one synchronizing transducer, then $A_{m,l}$ is bi-synchronizing at level 1.*

Proof. That A is reset demonstrated in Remark 3.8.2. That A is invertible follows from the fact that for each $q \in Q_{A_{m,l}}$, $\lambda_{A_{m,l}}(., q)$ is a permutation. This is because for $i \neq m$, $\lambda_{A_{m,l}}(x^{(i)}, q) = x^{(1+i \mod m)}$, and for $i = m$, we have , for $q \in Q_A$, $\lambda_{A_{m,l}}(x^{(m)}, q) = (\lambda_A(x, q))^{(1)}$ and $\lambda_{A_{m,l}}(x^{(m)}, p) = x^{(1)}$. Thus for $q \in Q_{A_{m,l}}$, $\lambda_{A_{m,l}}(., q)$ is surjective from $X^{(m)}$ to $X^{(m)}$ and so bijective. To see that A is reset observe that by construction of $\lambda_{A_{m,l}}$, for any $x \in X_n$, and $i, j \in \{1, 2, \ldots, m-1\}$ such that $j \neq l+1$ and $\lambda_{A_{m,l}}(x^{(i)}, q) = x^{(j)}$, we must have that $\pi_{A_{m,l}}(x^{(i)}, q) = p$. On the other hand for any $x \in X_n$, $i \in \{1, 2 \ldots, m-1\}$ such that $\lambda_{A_{m,l}}(x^{(i)}, q) = x^{(l+1)}$ we must have that $i = l$ and $\pi_{A_{m,l}}(x^{(i)}, q) = q_x$. From this it follows that $A_{m,l}$ is bi-synchronizing at level 1. □

Proposition 3.8.6. *Let $m \in \mathbb{N}_2$ and $1 \leqslant l \leqslant m-1$. Let $A \in \widetilde{\mathcal{H}}_n$ be synchronizing at level 1, then $A_{m,l}$ is finite if and only if A has finite order. In the case that A has finite order, let $k \in \mathbb{N}_1$ be minimal such that $A^{\vee}_k$ is the zero of $\langle A^{\vee} \rangle_+$, then, if A has a local identity state, k is minimal such that $(A_{m,l})^{\vee}_k$ is the zero of $\langle (A_{m,l})^{\vee} \rangle_+$, otherwise $(A_{m,l})^{\vee}_{k+1}$ is the zero of $\langle (A_{m,l})^{\vee} \rangle_+$.*

Proof. We begin with the following observation. Let $x_0 \in X_n$ and $q_0 \in Q_A$. Suppose in $A^{\vee}$ there is a path:

$$x_0 \xrightarrow{q_0 | q_{x_0}} x_1 \ldots \xrightarrow{q_r | q_{x_r}} x_{r+1}.$$

Consider a word $a_{2,0} \ldots a_{m,0} q_0 a_{2,1} \ldots a_{m,1} q_1 \ldots a_{2,r} \ldots a_{m,r} q_r \in Q^+_{A_{m,l}}$ where $a_{i,j} \in Q_{A_{m,l}}$ for $2 \leqslant i \leqslant m$ and $0 \leqslant j \leqslant r$. When $a_{2,0} \ldots a_{m,0} q_0 a_{2,1} \ldots a_{m,1} q_1 \ldots a_{2,r} \ldots a_{m,r} q_r$ is read from the state $x^{(1)}_0$ in $(A_{m,l})^{\vee}$ the output is,

$$b_{2,0} \ldots b_{l,0} q_{x_0} b_{2,1} \ldots b_{m,1} q_{x_1} \ldots b_{2,r} \ldots b_{m,r} q_{x_r} b_{2,r+1} \ldots b_{m-l+1,r+1},$$

where $b_{i,j} = p$ for all valid $i, j \in \mathbb{N}$, and the resulting state is $x^{(1)}_{r+1}$. Observe that that the resulting output is of the same form as the input, but with a shorter length word preceding the q term. We fix this deficit by 'backtracking' and 'padding' the input word appropriately as follows. Let $w \in Q^{m-l}_{A_{m,l}}$ and consider what happens when we read the word $w a_{2,0} \ldots a_{m,0} q_0 a_{2,1} \ldots a_{m,1} q_1 \ldots a_{2,r} \ldots a_{m,r} q_r$ from the state $x_0^{(l+1)}$ i.e we pad the input with a word of length $m - l$ and, correspondingly, backtrack from x_0^1 to x_0^{l+1}. The resulting output is $b_{2,0} \ldots b_{m,0} q_{x_0} b_{2,1} \ldots b_{m,1} q_{x_1} \ldots b_{2,r} \ldots b_{m,r} q_{x_r} b_{2,r+1} \ldots b_{m-l+1,r+1}$, where $b_{i,j} = p$ for all valid $i, j \in \mathbb{N}$, and the resulting state is $x^{(1)}_{r+1}$.

Now suppose for $y_0 \in X_n$ we have, $y_0 \xrightarrow{q_{x_0}|q_{y_0}} y_1 \ldots \xrightarrow{q_{x_r}|q_{y_r}} y_{r+1}$. In $(A_{m,l})^\vee$ after reading the word $b_{2,0} \ldots b_{m,0} q_{x_0} b_{2,1} \ldots b_{m,1} q_{x_1} \ldots b_{2,r} \ldots b_{m,r} q_{x_r} b_{2,r+1} \ldots b_{m-l+1,r+}$ through the state $y_0^{(1)}$, the output has a prefix

$$b_{2,0} \ldots b_{l,0} q_{y_0} b_{2,1} \ldots b_{m,1} q_{y_1} \ldots b_{2,r} \ldots b_{m,r} q_{y_r} b_{2,r+1} \ldots b_{m-l+1,r+1},$$

where $b_{i,j} = p$ for all valid $i, j \in \mathbb{N}$, and the resulting state is $y^{(1)}_{r+1}$. Since $w \in Q^{m-l}_{A_{m,l}}$ was arbitrary, we may backtrack once more and consider what happens when $w_1 w_2 a_{2,0} \ldots a_{m,0} q_0 a_{2,1} \ldots a_{m,1} q_1 \ldots a_{2,r} \ldots a_{m,r} q_r$, for $w_1, w_2 \in Q^{m-l}_{A_{m,l}}$, is read from the state $x_0^{(1+(2l-m) \mod m)} y_0^l$. We see that the output has a prefix

$$b_{2,0} \ldots b_{m,0} q_{y_0} b_{2,1} \ldots b_{m,1} q_{y_1} \ldots b_{2,r} \ldots b_{m,r} q_{y_r} b_{2,r+1} \ldots b_{m-l+1,r+},$$

where $b_{i,j} = p$ for all valid $i, j \in \mathbb{N}$. Since the states $q_0, \ldots, q_r \in Q_A$ were chosen arbitrarily, it follows by induction that given a word $\Gamma \in X_n^k$, there is a word $(\Gamma) \in (X^{(m)})^k$ and $M(r) \in \mathbb{N}$, such that whenever a word $q_0 q_1 \ldots q_r \in Q_A^r$ is read from Γ with output $q'_0 \ldots q'_r$ in $A_k^\vee$, then there is a word $w(q_0, \ldots, q_r) \in Q^{M(r)}_{A_{m,l}}$ such that when $w(q_0, \ldots, q_r)$ is read from (Γ) the output has a prefix $(b_{2,0} \ldots b_{m,0} q'_0 \ldots b_{2,r} \ldots b_{m,r} q'_r)$ in $(A_{m,l})_k^\vee$. Therefore whenever there is a split in $A_k^\vee$ for $k \in \mathbb{N}$, there is also a split in $(A_{m,l})_k^\vee$. Thus if A has infinite order $A_{m,l}$ also has infinite order by Proposition 3.5.41. Moreover if A has finite order and $k \in \mathbb{N}_1$ is minimal such that $A_k^\vee$ is the zero of $\langle A^\vee \rangle_+$, then if $A_{m,l}$ has finite order, the minimal $k' \in \mathbb{N}_1$ such that $(A_{m,l})^\vee_{k'}$ is the zero of $\langle A^\vee_{m,k} \rangle_+$ is greater than or equal to k.

Now suppose that A has finite order, we show that $A_{m,l}$ also has finite order. We may assume that A contains a local identity state. This is because if A does not have a local identity state, then by picking a symbol p distinct from Q_A and setting $\pi_{A'}(x, p) = q_x \in Q_A$, $\lambda_{A'}(x, p) = x$ for all $x \in X_n$, and $Q_{A'} = Q_A \sqcup \{p\}$, we may form a new transducer $A' = \langle X_n, Q_{A'}, \pi_{A'}, \lambda_{A'} \rangle$. Now since for all states $q \in Q_{A'}$ and all $x \in X_n$, $\pi_{A'}(x, q) \in Q_A$ it follows that if $A_k^\vee$ is the zero of $\langle A^\vee \rangle_+$, then $A'^\vee_{k+1}$ is the zero of $\langle A'^\vee \rangle_+$. Thus we may replace A' with A if necessary.

Let $k \in \mathbb{N}$ be such that $A_k^\vee$ is the zero of $\langle A^\vee \rangle_+$. We now show that $(A_{m,l})_k^\vee$ is also the zero of $\langle A^\vee \rangle_+$.

Let $\gamma_0^{(1)} \in X^{(m)}$. As in the first half of the proof, the output when

$$a_{2,0} \ldots a_{m,0} q_0 a_{2,1} \ldots a_{m,1} q_1 \ldots a_{2,r} \ldots a_{m,r} q_r$$

is read from $\gamma_0^{(1)}$ is of the form

$$b_{2,0} \ldots b_{l,0} q_{x_0} b_{2,1} \ldots b_{m,1} q_{x_1} \ldots b_{2,r} \ldots b_{m,r} q_{x_r} b_{2,r+1} \ldots b_{m-l+1,r+1},$$

where $b_{i,j} = p$ for all valid $i, j \in \mathbb{N}$, $x_0 = \gamma_0$, and $q_{x_0} \ldots q_{x_r}$ is the output when $q_1 \ldots q_r$ is read from γ_0 in $A^\vee$. Once more for any $w \in Q^{m-l}_{A_{m,l}}$, the output when $w a_{2,0} \ldots a_{m,0} q_0 a_{2,1} \ldots a_{m,1} q_1 \ldots a_{2,r} \ldots a_{m,r} q_r$ is read from $\gamma_0^{(l+1)}$ has a prefix equal to $b_{2,0} \ldots b_{m,0} q_{x_0} b_{2,1} \ldots b_{m,1} q_{x_1} \ldots b_{2,r} \ldots b_{m,r} q_{x_r} b_{2,r+1} \ldots b_{m-l+1,r+1}$, where $b_{i,j} = p$ for all valid $i, j \in \mathbb{N}$. Let $\gamma_1^{(d)} \in X^{(m)}$ be arbitrary and suppose that $d \in \{2, \ldots, m\}$. Consider what happens when $b_{2,0} \ldots b_{m,0} q_{x_0} b_{2,1} \ldots b_{m,1} q_{x_1} \ldots b_{2,r} \ldots b_{m,r} q_{x_r} b_{2,r+1} \ldots b_{m-l+1,r+1}$ is read from

$\gamma_1^{(d)}$. We observe that the output in $Q^+_{A_{m,l}}$ is a fixed word independent of $q_{x_0}, \ldots, q_{x_r}$. This is because the state p acts by adding one to a superscript and correcting modulo m so superscripts remain between 1 and m. Thus after reading $b_{2,0} \ldots b_{m,0}$ through $\gamma_1^{(d)}$ in $A^\vee_{m,l}$, the resulting state is $\gamma_1^{(d-1)}$. Since $d \neq 1$, for any state $q \in Q_{A_{m,l}}$ we have, $\lambda_A(\gamma_1^{(d-1)}, q) = \gamma_1^{(d)}$, and we are back to the start state. Hence, by induction, we see that the output when $b_{2,0} \ldots b_{m,0} q_{x_0} b_{2,1} \ldots b_{m,1} q_{x_1} \ldots b_{2,r} \ldots b_{m,r} q_{x_r} b_{2,r+1} \ldots b_{m-l+1,r+1}$ is read from $\gamma_1^{(d)}$ in $A^\vee_{m,l}$ is a fixed word independent of $q_{x_0}, \ldots, q_{x,r}$ and depending only on $\gamma_1^{(d)}$. Therefore if $d \in \{2, \ldots, m-1\}$ then, since we may chose r as large as we wish and $(A_{m,l})^\vee_k$ is finite, we see that for any input, the output when the input is read through $\gamma_0^{(l+1)}\gamma_1^{(d)}$ is a fixed word independent of $q_{x_0}, \ldots, q_{x,r}$ and depending only on $\gamma_1^{(d)}$. Hence for any $(\Delta) \in (X^{(m)})^{k-2}$, there is a fixed periodic word in $Q^\omega_{A_{m,l}}$ such that the output of any input read through the state $\gamma_0^{(l+1)}\gamma_1^{(d)}(\Delta)$ is a prefix of this word.

Hence, we may assume that $d = 1$. In this case, the output when

$$b_{2,0} \ldots b_{m,0} q_{x_0} b_{2,1} \ldots b_{m,1} q_{x_1} \ldots b_{2,r} \ldots b_{m,r} q_{x_r} b_{2,r+1} \ldots b_{m-l+1,r+1}$$

is read from $\gamma_1^{(1)}$ in $A^\vee_{m,l}$ has a prefix equal to

$$b_{2,0} \ldots b_{l,0} q_{x'_0} b_{2,1} \ldots b_{m,1} q_{x'_1} \ldots b_{2,r} \ldots b_{m,r} q_{x'_r} b_{2,r+1} \ldots b_{m-l+1,r+1},$$

where $b_{i,j} = p$ for valid i, j, $x'_0 = \gamma_1$ and $q_{x'_0} \ldots q_{x'_r}$ is the output when $q_{x_0} \ldots q_{x_r}$ is read from γ_1 in $A^\vee$ (notice that this is the same as the output when $q_0 q_1 \ldots q_r$ is read from $\gamma_0\gamma_1$). As in the first half of the proof, choosing $w_1, w_2 \in Q^{m-l}_{A_{m,l}}$, and backtracking from $\gamma_0^{(l+1)}\gamma_1^{(1)}$ to $\gamma_0^{(1+(2l-m \mod m))}\gamma_1^{(l+1)}$, we see that the output when $w_1 w_2 a_{2,0} \ldots a_{m,0} q_0 a_{2,1} \ldots a_{m,1} q_1 \ldots a_{2,r} \ldots a_{m,r} q_r$ is read from $\gamma_0^{(1+(2l-m \mod m))}\gamma_1^{(l+1)}$ is, $b_{2,0} \ldots b_{m,0} q_{x'_0} b_{2,1} \ldots b_{m,1} q_{x'_1} \ldots b_{2,r} \ldots b_{m,r} q_{x'_r} b_{2,r+1} \ldots b_{m-l+1,r+1}$. Let $\gamma_2^{(d')} \in X^{(m)}$. Once again if $d' \neq 1$, then the output when

$$b_{2,0} \ldots b_{m,0} q_{x'_0} b_{2,1} \ldots b_{m,1} q_{x'_1} \ldots b_{2,r} \ldots b_{m,r} q_{x'_r} b_{2,r+1} \ldots b_{m-l+1,r+1}$$

is read from $\gamma_2^{(d')}$ is a unique fixed word depending only on $\gamma_2^{(d)}$. Therefore, for any $(\Delta) \in (X^{(m)})^{k-3}$ and all states accessible from $\gamma_0^{(1+(2l-m \mod m))}\gamma_1^{(l+1)}\gamma_2^{(d)}(\Delta)$ in $(A_{m,l})^\vee_k$, the output of any input read through such a word is a unique fixed word independent of the input and depending only on $\gamma_2^{(d)}$. Therefore, we may assume that $d = 1$. By backtracking to the state $\gamma_0^{(1+(3l-m \mod m))}\gamma_1^{(1+(2l-m \mod m))}\gamma_2^{(l+1)}$, and padding the input word $w_1 w_2 a_{2,0} \ldots a_{m,0} q_0 a_{2,1} \ldots a_{m,1} q_1 \ldots a_{2,r} \ldots a_{m,r} q_r$ with a word of length $m - l$ at the front, we see that the output when the new input is processed from the state $\gamma_0^{(1+(3l-m \mod m))}\gamma_1^{(1+(2l-m \mod m))}\gamma_2^{(l+1)}$ has a prefix equal to

$$b_{2,0} \ldots b_{l,0} q_{x''_0} b_{2,1} \ldots b_{m,1} q_{x''_1} \ldots b_{2,r} \ldots b_{m,r} q_{x''_r} b_{2,r+1} \ldots b_{m-l+1,r+1},$$

where $b_{i,j} = p$ for valid i, j, $x''_0 = \gamma_2$, and $q_{x''_0} \ldots q_{x''_r}$ is the output when $q_{x'_0} \ldots q_{x'_r}$ is read from γ_1 in $A^\vee$ (notice that this is the same as the output when $q_0 q_1 \ldots q_r$ is read from $\gamma_0\gamma_1\gamma_2$). Continuing on in this way, and choosing, at each step the letter $\gamma_i^{(1)}$ arbitrarily, we see that for any word $(\Delta) \in (X^{(m)})^k$ there are two possibilities. Either there is a word $\widetilde{\Delta} \in X^k_n$ such that if, for any word $\bar{t} = t_1 \ldots t_r$, the output when $\bar{t}$ is read from $\widetilde{\Delta}$ is $s_1 s_2 \ldots s_r$, then, for any word $w \in Q^+_{A_{m,l}}$ of appropriately large length, the output when this word is read from the state (Δ) has a prefix equal to $b_{2,0} \ldots b_{l,0} s_1 b_{2,1} \ldots b_{m,1} s_2 \ldots b_{2,r} \ldots b_{m,r} s_r b_{2,r+1} \ldots b_{m-l+1,r+1}$, where $b_{i,j} = p$ for valid i, j. Otherwise, there is a fixed periodic word in $Q^\omega_{A_{m,l}}$ such that the output of any input processed through the word (Δ) is a prefix of this word. In particular, since $A^\vee_k$ is the zero $\langle A^\vee_k \rangle_+$, and since we may chose r as large as we wish, we see that $(A_{m,l})^\vee_k$ is also the zero of $\langle (A_{m,l})^\vee \rangle_+$. $\square$

Applying the Proposition 3.8.6 to the examples of Section 3.7 gives examples of elements $A \in \mathcal{H}_n$, for large enough n, with r states which are bi-synchronizing at level 1 and such that $r-1$ is the minimal value for which $A^{\vee}_{r-1}$ is the zero of $\langle A^{\vee} \rangle_{+}$. We construct one such example below.

Example 3.8.7. We apply Construction 3.8.1 for the case $m = 2$ and $l = 1$ to the transducer B in Figure 3.24. To simplify the transducer we let x represent any element of X_4 and, for $q \in Q_B$, we set $(x)q := \lambda_B(x, q)$.

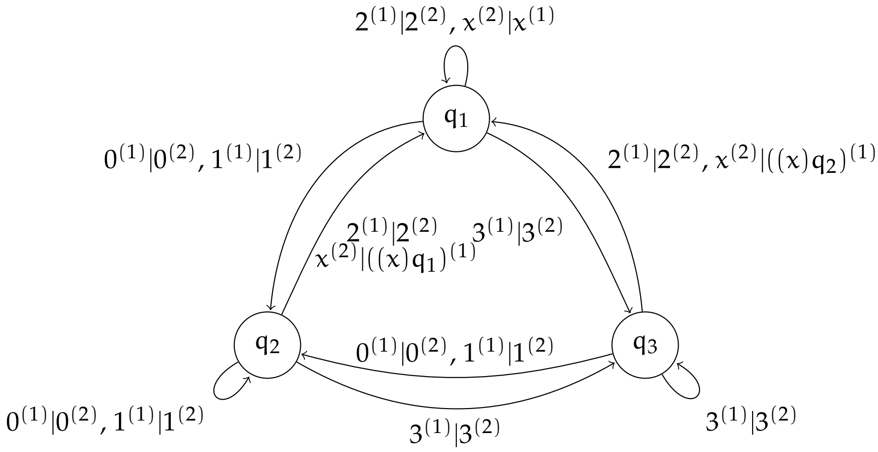


Figure 3.26: The transducer $B_{2,1}$

We now turn in the next section to the question of the growth rates of groups generated by synchronizing transducers.

3.9 The automaton group generated by a synchronizing transducer has exponential growth

In this section we study the growth rate of groups generated by elements of $\widetilde{\mathcal{H}}_n$. Recall that by Theorem 3.5.16 all such groups are finitely generated elementary amenable groups, more specifically, they are finitely generated locally finite-by-cyclic groups. It is a result Chou ([21]) that such groups are either have polynomial growth (and so are virtually nilpotent by Gromov's result ([32])) or they contain a free subsemigroup of rank at least 2 and so have exponential growth. Unfortunately, Chou's original has a gap, however this is fixed by a result of Rosset [46] stating that for a finitely generated group G of subexponential word growth, and a normal subgroup N of G such that G/N is soluble, N is finitely generated. Thus we deduce that groups generated by transducers in $\widetilde{\mathcal{H}}_n$ are either finite (as $\mathbb{Z}$ is soluble and a finitely generated locally finite group is finite) or contain a free subsemigroup of rank at least 2 and so have exponential growth rate. In this section we give a different proof of this result. Our approach is to link graph theoretic properties of the graph of bad pairs to the existence of free sub-semigroups in the automaton semigroup generated by an element of $\mathcal{H}_n$. We should also mention that Silva and Steinberg give some sufficient conditions (but not necessary) for when the (semi)group generated by a reset automata contains a free subsemigroup of rank 2. We begin with the following proposition:

Proposition 3.9.1. *Let $A \in \widetilde{\mathcal{H}}_n$ be an element of infinite order. Either there is a $j \in \mathbb{N}$ such that the minimal graph of bad pairs $\overline{G}_j(A)$ has a loop or the automaton (semi)group generated by A contains a free semigroup of rank at least 2.*

Proof. We may assume, by changing the alphabet size that A is bi-synchronizing at level 1.

Since A has infinite order then for each $j \in \mathbb{N}$, $(A^{\vee})^j$ splits. Fix $j \in \mathbb{N}$ and let Top_j be the set of pairs of states $\{p_1, p_2\}$ such that there exits $(p_1, s_1, \ldots, s_r)$ and $(p_2, s'_1, \ldots, s'_r)$ which split $(A^{\vee})^j$

where r is the minimal splitting length of $(A^\vee)^j$. By definition Top_j is the set of tops of minimal length splits of $A_j^\vee$. Analogously, for fixed $j \in \mathbb{N}$ let Bottom_j be the set of bottoms of minimal length splits. That is Bottom_j consists of sets $\{t_1, t_2\}$ such that there exists a split of minimal length $(\Gamma, (s_1, \ldots, s_r), (s_1', \ldots, s_r'))$ of $A_j^\vee$ with bottom $\{t_1, t_2\}$.

Since A has finitely many states there exists an infinite subset $\mathcal{J}' \subset \mathbb{N}$ and a fixed set of pairs $\{t_1, t_2\}$ such that $\{t_1, t_2\} \in \mathrm{Bottom}_j$ for all $j \in \mathcal{J}'$. Now consider the set of tops of all splits of $A_j^\vee$, $j \in \mathcal{J}'$ with bottom $\{t_1, t_2\}$. Since $|\mathcal{J}'| = \infty$ and $\{t_1, t_2\} \in \mathrm{Bottom}_j$ for all $j \in \mathcal{J}'$, there exists an infinite subset $\mathcal{J} \subset \mathcal{J}'$ and a fixed set of pairs $\{p_1, p_2\}$ such that $\{p_1, p_2\} \in \mathrm{Top}_j$ for all $j \in \mathcal{J}$ and there exists splits of $A_j^\vee$ with top $\{p_1, p_2\}$ and bottom $\{t_1, t_2\}$ for all $j \in \mathcal{J}$.

If $\{p_1, p_2\} = \{t_1, t_2\}$ we are done, since $\overline{G}_j(A)$ has a loop for any $j \in \mathcal{J}$. Therefore assume that this is not the case. Under this assumption, we have two cases to consider.

Case 1: Suppose that there are $i, i' \in \mathbb{N}$, $i, i' \geqslant 1$ and $S_1, S_1' \in Q^i$ and $S_2, S_2' \in Q^{i'}$ such that (p_1, S_1) is ω-equivalent to (t_1, S_1') and (p_2, S_2) is ω-equivalent to (t_2, S_2'). We may assume that $i = i'$ by padding out one of the pairs (p_i, S_i) and (t_i, S_i'), $i = 1, 2$.

Let $j \in \mathcal{J}$ be such that $j > i + 1$. Consider $(A^\vee)^j$, it has minimal splitting length, r, greater than or equal to j. Now by choice of $\{p_1, p_2\}$, there exists $(p_1, s_1, \ldots, s_{r-1})$, $(p_2, s_1', \ldots, s_{r-1}')$ elements of Q^r and Γ a state of $(A^\vee)^j$ such that $((p_1, s_1, \ldots, s_{r-1}), (p_2, s_1', \ldots, s_{r-1}'), \Gamma)$ is a split of $(A^\vee)^j$ with bottom $\{t_1, t_2\}$. Hence, by minimality of r, it now follows that $((p_1, S_1, s_{i+1}, \ldots, s_{r-1}), (p_2, S_2, s_{i+2}', \ldots, s_{r-1}', \Gamma)$ is also a split of $(A^\vee)^j$ with bottom (t_1, t_2). However this now implies, again by minimality of r and since (t_1, S_1') and (t_2, S_2') are ω-equivalent to (p_1, S_1) and (p_2, S_2) respectively, that $((t_1, S_1', s_{i+1}, \ldots, s_{r-1}), (t_2, S_2, s_{i+2}', \ldots, s_{r-1}'), \Gamma)$ is also a split of $(A^\vee)^j$ with bottom (t_1, t_2). Therefore $(A^\vee)^j$ has a loop.

Case 2: We assume that Case 1 does not hold, that is for all $i, i' \in \mathbb{N}$ there does not exist a choice of $S_1, S_1' \in Q^i$ and $S_2, S_2' \in Q^{i'}$ such that $(p_1, S1)$ is ω-equivalent to (t_1, S_1') and (p_2, S_2) is ω-equivalent to (t_2, S_2') . We may also assume that none of the graph of bad pairs $\overline{G}_j(A)$ has a loop for any j greater than the minimal synchronizing level of A, since otherwise we are done.

The latter assumption implies that $\mathfrak{S}_{j, |Q_A|^2+1}$ consists of transformations with image size 1 by Remark 3.5.60. However since $(A^\vee)^j$ splits for every $j \in \mathbb{N}$, then for j larger than the minimal synchronizing level, there are elements $\Gamma \in X_n^j$, such that σ_Γ has image size at least 2. Fix an arbitrary such Γ. Since $\sigma_\Gamma^{|Q_A|^2+1}$ has image size 1, then there is a state $p \in Q_A$ such that $(p)\sigma_\Gamma = p$. Therefore there is a pair of states $p_1, p_2 \in Q_A$ such that there is a split of $(A^\vee)^j$ with top $\{p_1, p_2\}$, and bottom $\{t_1, p_2\}$.

The above argument now implies that we may chose $\{p_1, p_2\}$ and $\{t_1, t_2\}$ above so that $p_2 = t_2$, and there exists $\Gamma \in X_n^j$, $j \in \mathcal{J}$, such that $(p_1)\sigma_\Gamma = t_1$ and $(p_2)\sigma_\Gamma = p_2$.

Now since case one does not hold, and $p_2 = t_2$, therefore it follows that for any $m \in \mathbb{N}\backslash\{0\}$ and any $S_1, S_2 \in Q^m$ that $p_1 S_1$ is not ω-equivalent to $t_1 S_2$. We now argue that the sub-semigroup $\langle p_1, t_1 \rangle_+$ of $\mathcal{S}(A)$ (the automaton semigroup generated by A) is free.

Now as A has infinite order, it follows that $\mathrm{Core}(A^i) \not\cong_\omega \mathrm{Core}(A^j)$ for any $i \neq j \in \mathbb{N}$. Therefore given two words v and w in $\langle p_1, t_1 \rangle_+$ such that v and w are ω-equivalent it follows that $|v| = |w|$.

Therefore consider the case of words $v, w \in \langle p_1, t_1 \rangle_+$ such that $|v| = |w|$. Suppose $v = v_1 \ldots v_l$ and $w = w_1 \ldots w_l$, where $|v| = |w| = l$. Let $1 \leqslant i \leqslant l$ be the minimal index so that $v_i \neq w_i$. We may assume that $v_i = p_1$ and $w_i = t$. Therefore $v = v_1 \ldots v_i p_1 v_{i+2} \ldots v_l$ and $w = v_1 \ldots v_i t_1 w_{i+2} \ldots w_l$. Hence v is ω-equivalent to w if and only if $p_1 v_{i+2} \ldots v_r$ is ω-equivalent to $t_1 w_{i+2} \ldots w_r$. However by assumption this is not the case. Therefore given any two distinct words in $\{p_1, t_1\}^*$, they represent distinct automorphisms of the rooted n-ary tree hence we conclude that $\langle p_1, t_1 \rangle_+$ is a free semigroup. □

Corollary 3.9.2. *Let $A \in \widetilde{\mathcal{H}}_n$ be an element of infinite order. Either there is a $j \in \mathbb{N}$ such that the graph of bad pairs $G_j(A)$ has a loop or the automaton (semi)group generated by A contains a free semigroup of rank at least 2.*

In the proposition below we introduce a condition on the graph of bad pairs $G_j(A)$ which guarantees the existence of free subsemigroups of certain rank in the automaton semigroup generated by an element of $\widetilde{\mathcal{H}}_n$. This condition at first glance appears to be very strong, however we shall introduce a large class of examples which satisfy the hypothesis of the Proposition. In particular whenever the graph of bad pairs has a loop the hypothesis is immediately satisfied.

Proposition 3.9.3. *Let* $A \in \widetilde{\mathcal{H}}_n$ *and suppose that* A *is synchronizing at level* k *and is minimal. Let* $G_j(A)$ *be the graph of bad pairs for some* $j \geqslant k \in \mathbb{N}$. *Suppose there is a subset* $\mathcal{S}$ *of the set of states of* A, *such that the following things hold:*

(i) $|\mathcal{S}| \geqslant 2$,

(ii) the set $\mathcal{S}(2)$ *of two element subsets of* $\mathcal{S}$ *is a subset of the vertices of* $G_j(A)$,

(iii) for each element of $\mathcal{S}(2)$ *there is a vertex accessible from it which belongs to a circuit,*

then the automaton (semi)group generated by A *contains a free semigroup of rank at least* $|\mathcal{S}|$. *In particular the automaton (semi)group generated by* A *has exponential growth.*

Proof. First observe that since $G_j(A)$ is assumed to have a circuit, by Lemma 3.5.48 A has infinite order. Now let U and V be distinct non-empty words in $\mathcal{S}^*$, if $|U| \neq |V|$ then since A has infinite order A_U cannot be ω-equivalent to A_V by Lemma 3.4.5. Therefore we may assume that $|U| = |V|$.

Let $U = u_1 \dots u_r$ and $V = v_1 \dots v_r$ and let $1 \leqslant i \leqslant r$ be the minimal index so that $u_i \neq v_i$. If $i = r$ then we are done, since $U = Sq$ and $V = Sp$ (or vice versa) for some $S \in \mathcal{S}^{r-1}$ and $q \neq p \in \mathcal{S}$.

Therefore assume that $i \leqslant r$ and that $U = SqT_1$ and $V = SpT_2$ for $S \in \mathcal{S}^{i-1}$, $T_1, T_2 \in \mathcal{S}^{r-i}$ and $q, p \in \mathcal{S}$. If U and V are not ω-equivalent, we are done. Therefore assume that U and V are ω-equivalent.

Since $p, q \in \mathcal{S}(2)$, there is a path in $G_j(A)$ from $\{p, q\}$ to a vertex which belongs to a circuit. Therefore we may assume that there is path in the graph $G_j(A)$ as follows :

$$\{p, q\} := \{p_0, q_0\} \to \{p_1, q_1\} \to \dots \to \{q_l, p_l\} \to \{q_{l+1}, p_{l+1}\}$$

where for each $\{p_a, q_a\}$, $0 \leqslant a \leqslant l$ there is a split of length m_a with top $\{p_a, q_a\}$ such that the bottom depends only on the top, and the bottom is $\{p_{a+1}, q_{a+1}\}$ and $\{p_{l+1}, q_{l+1}\}$ is a vertex on a circuit in $G_j(A)$. Notice that $m_a \geqslant 1$ for all $1 \leqslant a \leqslant l$. Therefore by travelling along this circuit in $G_j(A)$ as long as required, we may also assume that $m_0 + m_1 + \dots + m_l + 1 \geqslant r - i + 1$.

By appending a common suffix to U and V, thus preserving ω-equivalence, if necessary we may further assume that $r - i + 1 = |qT_1| = |pT_2|$ is equal to $m_1 + m_1 \dots + m_l + 1$. Redefining T_1 and T_2 we assume that $U = SqT_1t_1$ and $V = SpT_2t_2$ where $|qT_1| = |pT_2| = m_0 + m_1 \dots + m_l + 1$ and t_1 and t_2 are possibly distinct elements of Q_A. Since $|qT_1| = |pT_2| = m_0 + m_1 \dots + m_l + 1$, write $qT_1 = R_1R_2 \dots R_l$ and $pT_2 = P_1P_2 \dots P_l$ where $R_a, P_a \in Q_A^{m_a}$ for $1 \leqslant a \leqslant l$, moreover R_1 begins with q and P_1 begins with p.

Since $\{q, p\}$ is a vertex of $G_j(A)$, there is a word Γ of length j belonging to a split of length m_0, whose bottom depends only on the top $\{q, p\}$, and with bottom $\{q_1, p_1\}$. Let Λ be the word such that the output when processed through A_S is Γ. Let S_Γ be the state of A^{m_0} such that $\pi_{A^{m_0}}(\Gamma, P_1) = \pi_{A^{m_0}}(\Gamma, Q_1)$. Such an S_Γ exists by definition of what it means for the bottom of a split to depend only on its top (see Definition 3.5.27). Then we have, on reading Λ through A_U and A_V respectively that we transition to the states $S'S_\Gamma Q_1'Q_3' \dots Q_l't_1'$ and $S'S_\Gamma P_1'P_3' \dots P_l't_2'$. Moreover Q_1' begins with q_1 and P_1' begins with p_1. Once more $Q_a' \in Q_A^{m_a}$ for $1 \leqslant a \leqslant l$.

Since, by assumption, each $\{p_a, q_a\}$ for $1 \leqslant a \leqslant l$ has an outgoing edge corresponding to a split of length m_a whose bottom, $\{p_{a+1}, q_{a+1}\}$, depends only on its top, we can now repeat the argument of the above paragraph until the last letters of the final pair of state are a vertex of $G_r(A)$. Therefore we are in the situation that $i = r$ at which point we conclude that the final pair of states are not ω-equivalent.

Now since U and V are ω-equivalent, then the final pair of states should also be ω-equivalent, since we read the same word from A_U and A_V into this pair. This yields the desired contradiction. Therefore we conclude that A_U and A_V are not ω-equivalent.

The above now means that the semigroup $\langle A_p | p \in \mathcal{S} \rangle_+$ satisfies no relations and so is a free semigroup. In fact this argument actually demonstrates that for any word $W \in Q^*$ (Q being the set of states of A), the semigroup $\langle A_{Wp} | p \in \mathcal{S} \rangle_+$ is a free semigroup. $\square$

There are a few ways of extending the argument. One can also show that for a subset $\mathcal{S} \subset Q$ satisfying the conditions of the proposition, and for any vertex on a path from a vertex of G_k to a vertex accessible from $\mathcal{S}(2)$, then the pair of states making up this vertex generate a free semigroup. Notice that if the graph of bad pairs has a circuit then the conditions of the Proposition 3.9.3 are satisfied.

Remark 3.9.4. In proving Propositions 3.9.1 and 3.9.3 we have made use of the cancellative property of automata groups generated by elements of $\mathcal{H}_n$, in particular the above arguments can be extended to elements of $\widetilde{\mathcal{P}}_n$ where we still retain this cancellative property.

Corollary 3.9.5. *For an element $A \in \widetilde{\mathcal{H}}_n$ the automaton (semi)group generated by A contains a free semigroup of rank at least 2.*

Proof. This follows from Propositions 3.9.1 and 3.9.3. □

Theorem 3.9.6. *Let $A \in \widetilde{\mathcal{H}}_n$ then the automaton (semi)group generated by A has exponential growth.*

Proof. This follows from standard results in the literature on the growth rates of groups and semigroups and the fact that the automaton semigroup generated by A contains a free semigroup. □

We return briefly to the order problem in the next section, before considering a special class of transducers in $\widetilde{\mathcal{H}}_n$ introduced in [51].

3.9.1 Further conditions for having infinite order: avoiding loops

In this subsection we outline a method for detecting when an element of $\widetilde{\mathcal{H}}_n$ has infinite order which does not depend on detecting loops. This turns out to be particularly effective when $n = 3$. Our approach is to deduce implications on the local action of states of the transducer from a power of the dual transducer being a zero.

First we need the following notion.

Let $A \in \widetilde{\mathcal{H}}_n$. For each letter $i \in X_n$ let $[i] := \{\pi_A(i, p) \mid p \in Q_A\}$ and let $[i]^{-1} := \{\pi_{A^{-1}}(i, p^{-1}) \mid p^{-1} \in Q_A^{-1}\}$. Note that it is not necessarily the case that if $p \in [i]$ then $p^{-1} \in [i]^{-1}$. Let $\mathfrak{P}(A)_1 := \{[i] \mid i \in X_n\}$. Now refine $\mathfrak{P}_1$ as follows: whenever $i, j \in X_n$ are such that if $[i] \cap [j] \neq \emptyset$, then let $[i, j] := [i] \cup [j]$, let $\mathfrak{P}(A)_2$ be the result of this process. An element of $\mathfrak{P}(A)_2$ is either of the form $[i, j]$ for $i, j \in X_n$ or just $[i]$ for some $i \in X_n$. Repeat the process: whenever two elements of $\mathfrak{P}_2$ have non-empty intersection, we take their union, and let $[i_1, i_2, \ldots, i_m]$ denote the resulting set, where the i_l's are distinct for $1 \leqslant l \leqslant m$, $[i_l] \subset [i_1, i_2, \ldots, i_m]$ and m is at most 4. Recursively form sets $\mathfrak{P}_j$ for $j \in \mathbb{N}$. Since $|X_n| = n$ there is a $j \in \mathbb{N}$ such that $\mathfrak{P}(A)_j = \mathfrak{P}(A)_{j+1}$. Let $\mathfrak{P}(A) = \mathfrak{P}(A)_j$ for this j. Notice that $\mathfrak{P}(A)$ is a partition of the states of A, and we call $\mathfrak{P}(A)$ the *letter induced partition of* A.

Lemma 3.9.7. *Let $A \in \mathcal{H}_n$ and let $\mathfrak{P}(A)$ be the letter induced partition of A, then there exists $P \in \mathfrak{P}(A)$ and distinct letters i and j in X_n such that $[i] \cup [j] \subset P$.*

Proof. Let $A \in \mathcal{H}_n$. Since A^{-1} is synchronizing, it follows (see Construction 2.2.1) that there are distinct states p^{-1} and q^{-1} of A^{-1} such that for all $l \in X_n$ we have $\pi_{A^{-1}}(l, p) = \pi_{A^{-1}}(l, q)$. Now since A is minimal and synchronous, A^{-1} is also minimal, therefore there is an $i' \in X_n$ such that $i = \lambda_{A^{-1}}(i', p^{-1}) \neq \lambda_{A^{-1}}(i', q^{-1}) = j$. Hence in A we have, $\pi(i, p) = \pi(j, q)$. It follows by definition that there is some $P \in \mathfrak{P}(A)$ such that $[i] \cup [j] \subset P$. □

Lemma 3.9.8. *Let $A \in \mathcal{H}_n$ and let $\mathfrak{P}(A)$ be the letter induced partition of A. Let $k \in \mathbb{N}$ be greater than or equal to the synchronizing level of A. Suppose that $A_k^{\vee}$ splits and that if $A_{k+1}^{\vee}$ splits then it has minimal splitting length strictly greater than the minimal splitting length of $(A^{\vee})^k$. We have the following:*

(i) $\mathfrak{P}(A) \neq \{[0, 1 \ldots, n-1]\}$

(ii) For any $\Gamma \in X_n^k$ the transformation σ_Γ has image size strictly less than n . In particular given $\Gamma \in X_n^k$ then for a given $P \in \mathfrak{P}(A)$ then there exists a $q \in Q$ such that for all $t \in P$ we have $q = (t)\sigma_\Gamma$.

Proof. Let $A \in \mathcal{H}_n$ be as in the statement of the lemma and let l be the minimal splitting length of $(A^\vee)^k$. Let $m \in \mathbb{N}$ be minimal such that $\mathfrak{P}(A)_m = \mathfrak{P}(A)$.

For part (i) let Γ be a word in X_n^k such that σ_Γ has image size at least 2. Let $q_1q_2\ldots q_l \in Q_A^l$ be any l tuple of states, then since $A_k^\vee$ has minimal splitting length l then the l^{th} letter of the output when $q_1q_2\ldots q_l$ is processed from the state Γ of $(A^\vee)^k$ depends only on q_1. In particular for $i \in X_n$, and for any word $p_1 \ldots p_l$ in the state of A the l^{th} letter of the output when $p_1 \ldots p_l$ is processed from $i\Gamma$ depends only on the state $\pi(i, p_1)$. In particular this letter is equal to $(\pi(i, p_1))\sigma_\Gamma$. However since the splitting length of $(A^\vee)^{k+1}$, if it splits, is strictly greater than l then it must be the case that $(\pi(i, p_1))\sigma_\Gamma = (\pi(i, p))\sigma_\Gamma$ for any state p of Q_A.

Now if $j \in X_n$ is such that $[i] \cap [j] \neq \emptyset$ i.e $[i, j]$ is an element of $\mathfrak{P}(A)_2$ then there are states q_1 and q_2 of A such that $\pi(i, q_1) = \pi(j, q_2)$. It therefore follows that $(\pi(i, q_1))\sigma_\Gamma = (\pi(j, q_2))\sigma_\Gamma$. By the previous paragraph we therefore have that for any q in $[i]$ and $p \in [j]$, $(q)\sigma_\Gamma = (p)\sigma_\Gamma$.

Now assume that for all $1 \leqslant r < m$ for any set $P_1 \in \mathfrak{P}(A)_r$ and any pair of states q_1 and q_2 in P_1 we have that $(\pi(i, q_1))\sigma_\Gamma = (\pi(j, q_2))\sigma_\Gamma$. Now let P_1 and P_2 in $\mathfrak{P}(A)_r$ such that $P_1 \cap P_2 \neq \emptyset$. This means that there is a pair $i, j \in X_n$ such that $[i] \subset P_1$ an $[j] \subset P_2$ such that $[i] \cap [j] \neq \emptyset$. Therefore by repeating the argument in the previous paragraph we have that $(\pi(i, q_1))\sigma_\Gamma = (\pi(j, q_2))\sigma_\Gamma$ for any pair of state $q_1 \in [i]$ and $q_2 \in [j]$. By the inductive assumption we therefore have that $(\pi(i, q_1))\sigma_\Gamma = (\pi(j, q_2))\sigma_\Gamma$ for any pair of states $q_1 \in P_1$ and $q_2 \in P_2$.

Now since σ_Γ has image size at least 2, there are states t_1 and t_2 of A such that $(t_1)\sigma_\Gamma \neq (t_2)\sigma_\Gamma$. Now as A is core and synchronizing, there are elements P_1 and P_2 of $\mathfrak{P}(A)$ such that $t_1 \in P_1$ and $t_2 \in P_2$. Now by observations in the previous paragraph it follows that $P_1 \neq P_2$. This demonstrates ((i)).

The second part of the lemma now follows since as demonstrated above for $P \in \mathfrak{P}(A)$, there is a fixed $q \in Q_A$ such that $q = (t)\sigma_\Gamma$ for all $t \in P$. Therefore, by Lemma 3.9.7, we have $2 \leqslant |\operatorname{im}(\sigma_\Gamma)| = |\mathfrak{P}(A)| < n$. ☐

Remark 3.9.9. Notice that if $A \in \mathcal{H}_n$ has infinite order, then there are infinitely many numbers $k \in \mathbb{N}$, where k is greater than or equal to the minimal synchronizing level of A, such $(A^\vee)^k$ has splitting length strictly less than $(A^\vee)^{k+1}$. For each such k, any $\Gamma \in X_n^k$ and a given $P \in \mathfrak{P}(A)$ all elements of P have the same image under σ_Γ. In particular since $(A^\vee)^k$ splits there elements P_1 and P_2 such that for $t_1 \in P_1$ and $t_2 \in P_2$ there is some $\Delta \in X_n^k$ such that $(t_1)\sigma_\Delta \neq (t_2)\sigma_\Delta$. Furthermore we may insist that there is an infinite subset $\mathcal{J} \subset \mathbb{N}$ such that for all $j \in \mathcal{J}$ there is a $\Delta \in X_n^j$ such that $(t_1)\sigma_\Delta \neq (t_2)\sigma_\Delta$ for $t_1 \in P_1$ and $t_2 \in P_2$. This follows since there are infinitely many numbers $k \in \mathbb{N}$, where k is greater than or equal to the minimal synchronizing level of A, such $(A^\vee)^k$ has splitting length strictly less than $(A^\vee)^{k+1}$. Now by repeating, with slight modifications, the proof of Proposition 3.9.1, if no power of $A^\vee$ has a loop then for any pair $t_1 \in P_1$ and $t_2 \in P_2$ the semigroup generated by t_1 and t_2 is free.

Remark 3.9.10. Notice that if $A \in \mathcal{H}_3$ has infinite order, then $\mathfrak{P}(A)$ has only two elements, P_1 and P_2. Moreover for all numbers $k \in \mathbb{N}$, where k is greater than or equal to the minimal synchronizing level of A and the splitting length of $A_{k+1}^\vee$ is strictly greater than the splitting length of $A_k^\vee$, there is some $\Gamma \in X_n^k$ such that $(t_1)\sigma_\Gamma \neq (t_2)\sigma_\Gamma$ for any pair $t_1 \in P1$ and $t_2 \in P_2$.

Now consider the case that $n = 3$. By Lemmas 3.9.7 and 3.9.8, if an element $A \in \mathcal{H}_n$ has infinite order then $\mathfrak{P}(A)$ contains only the elements $[i_1, i_2]$ and $[i_3]$ where $\{i_1, i_2, i_3\} := X_3$.

Lemma 3.9.11. *Let $A \in \mathcal{H}_3$. Let i_1, i_2, and i_3 be distinct elements of X_3. Suppose that $[i_1] \cap [i_2] \neq \emptyset$. If for some $l \in \mathbb{N}$ there is an $i_a \in X_3$ and a pair of states S_1, S_2 of $A^l = \langle X_n, Q^l, \pi_{Al}, \lambda_{Al}\rangle$ such that:*

(i) $\pi_{Al}(i_a, S_1) = \pi_{Al}(i_a, S_2)$,

(ii) $\lambda_{Al}(i_a, S_1) \in \{i_1, i_2\}$ *and,*

(iii) $\lambda_{Al}(i_a, S_2) = i_3$

then either $(A^\vee)^k = (A^\vee)^{k+1}$ where k is the minimal synchronizing level of A, or A has infinite order.

Proof. It suffices to show by Proposition 3.5.41 that if $A \in \mathcal{H}_3$ with minimal synchronizing level k, has finite order and satisfies the conditions of the lemma then $A_{k+1}^\vee = A_k^\vee$.

Therefore let $A \in \mathcal{H}_3$ be an element of finite order which satisfies the conditions of the lemma. Let $k \in \mathbb{N}$ be the minimal synchronizing level of A. Let $\mathfrak{P}_A$ be the letter induced partition of A. If $A^{\vee}_{k+1} = A^{\vee}_{k}$ we are done. Thus suppose that there is some $m > k$, $m \in \mathbb{N}$ such that $A^{\vee}_{m+2} = A^{\vee}_{m+1}$ but $A^{\vee}_{m}$ splits.

Since $A^{\vee}_{m}$ splits it follows from Lemma 3.9.8 and the condition that $[i_1] \cap [i_2] \neq \emptyset$ that the letter induced partition of A consists of the elements $[i_1, i_2]$ and $[i_3]$. Moreover there are states q_1 and q_2, and some $\Gamma \in X_n^m$ such that $q_1 = (t_1)\sigma_\Gamma$ and $q_2 = (t_2)\sigma_\Gamma$ for any pair $t_1 \in [i_1, i_2]$ and $t_2 \in [i_3]$.

Now let $S3 = \pi_{Al}(i_a, S_1) = \pi_{Al}(i_a, S_2)$, and let $\Delta \in X_n^m$ be such that $\lambda_{Al}(\Delta, S_3) = \Gamma$. Consider the word $i_a\Delta$ a state of $A^{\vee}_{m+1}$. Now after processing the words S_1 and S_2 from the state $i_a\Delta$ of $A^{\vee}_{m+1}$ the active states are $i_b\Gamma$ and $i_3\Gamma$ $i_b \in \{i_1, i_2\}$. Now since $[i_b] \subset [i_1, i_2]$, it follows from the previous paragraph that for any input of length l processed from the state $i_b\Gamma$ the l^{th} letter of the output must be q_1. Likewise for any input of length l processed from the state $i_3\Gamma$ the l^{th} letter of the output must be q_2. Therefore we see that $A^{\vee}_{m+1}$ splits also which is a contradiction. □

As a corollary we have:

Corollary 3.9.12. *Let $A \in \mathcal{H}_3$ and suppose that $\mathfrak{P}(A) \neq \{[0, 1, 2]\}$. Let i_1, i_2 be distinct elements of X_3 such that $[i_1, i_2]$ is in $\mathfrak{P}(A)$, then if $[i_1]^{-1} \cap [i_2]^{-1} = \emptyset$ either $A^{\vee}_{k+1} = A^{\vee}_{k}$ where k is the minimal synchronizing level of A or A has infinite order.*

Proof. Let $A \in \mathcal{H}_3$ satisfy the conditions of the lemma and let k be the minimal synchronizing level of A. Furthermore assume that $A^{\vee}_{k}$ splits and so $A^{\vee}_{k+1} \neq A^{\vee}_{k}$.

Since A is synchronizing it follows that there are states q_1 and q_2 of A such that for all $i \in X_n$ $\pi_A(i, q_1) = \pi_A(i, q_2)$ (see Construction 2.2.1). Since A is minimal there is a $j \in X_n$ such that $\lambda(j, q_1) \neq \lambda(j, q_2)$.

Now the condition that $[i_1]^{-1} \cap [i_2]^{-1} = \emptyset$ implies by Lemma 3.9.8 that, since $A^{\vee}_{k}$ splits, either $[i_1]^{-1} \cap [i_3]^{-1} \neq \emptyset$ or $[i_2]^{-1} \cap [i_3]^{-1} \neq \emptyset$. We assume by relabelling if necessary that $[i_1]^{-1} \cap [i_3]^{-1} \neq \emptyset$. This means that, since $A^{\vee}$ splits, $\mathfrak{P}(A^{-1})$ consists of the elements $[i_1, i_3]^{-1} := [i_1]^{-1} \cup [i_3]^{-1}$ and $[i_2]^{-1}$. Hence we have that $\lambda(j, q_1) = i_1$ and $\lambda(j, q_2) = i_3$ or $\lambda(j, q_1) = i_3$ and $\lambda(j, q_2) = i_1$. In either case we have that A satisfies the conditions of Lemma 3.9.11 and we are done.

□

Remark 3.9.13. The above corollary implies that if $A \in \mathcal{H}_3$ is such that $\mathfrak{P}(A) = \{[i_1, i_2], [i_3]\}$ for $\{i_1, i_2, i_3\} = X_n$, then either $\mathfrak{P}(A^{-1}) = \{[i_1, i_2]^{-1}, [i_3]^{-1}\}$ where $[i_1, i_2]^{-1} := [i_1]^{-1} \cup [i_2]^{-1}$ or $A^{\vee}_{k}$ does not split.

We conclude the section with the following lemma:

Lemma 3.9.14. *Let $A \in \mathcal{H}_3$. Assume that $\mathfrak{P}(A) = \{[i_1, i_2], [i_3]\}$ and $\mathfrak{P}(A^{-1}) = \{[i_1, i_2]^{-1}, [i_3]^{-1}\}$. Let q_1 and q_2 be distinct states of A such that for all $i \in X_n$, $\pi_A(i, q_1) = \pi_A(i, q_2)$ and $\{q_1, q_2\}$ is a subset of some $P \in \mathfrak{P}(A)$. If there are (not necessarily distinct) states p_1, p_2 of A and (not necessarily distinct letters j_1 and j_2 in X_n such that $\pi(j_1, p_1) = q_1$, $\pi(j_2, p_2) = q_2$, $\lambda(j_1, p_1) \in \{i_1, i_2\}$, and $\lambda(j_2, p_2) = i_3$, then there is a conjugate B of A such that $|Q_B| < |Q_A|$.*

Proof. Let $A \in \mathcal{H}_3$ satisfy the conditions of the lemma. Observe that the condition $\mathfrak{P}(A) = \{[i_1, i_2], [i_3]\}$ and $\mathfrak{P}(A^{-1}) = \{[i_1, i_2]^{-1}, [i_3]^{-1}\}$ implies that whenever a state q of A is such that there is some state p of A and an $i \in X_n$ with, $\pi_A(i, p) = q$ and $\lambda_A(i, p) = i_3$, then for any other state p' and any letter i' such that $\pi_A(i', p') = q$ we must have that $\lambda_A(i', p') = i_3$. Thus if $i \in X_n$ is such that $\lambda_A(i, q_1) = i_3$ then $\lambda_A(i, q_2) = i_3$. Therefore if $\overline{q_2^{-1}q_1}$ is the permutation of X_n induced by the state $q_2^{-1}q_1$ of $A^{-1}A$, $\overline{q_2^{-1}q_1}$ fixes i_3. This is because if $i = \lambda_{A^{-1}}(i_3, q_2^{-1})$ then $\lambda(i, q_1) = i_3$. Likewise let $\overline{q_1^{-1}q_2}$ be the permutation of X_n induced by the state $q_1^{-1}q_2$ of $A^{-1}A$, $\overline{q_2^{-1}q_1}$ this also fixes i_3 by a similar argument. Moreover for any state t of Q_A such that there is some $j \in X_n$ and $\pi_A(j, t) = q_1$, we must also have that $\lambda_A(j, t) \in \{i_1, i_2\}$. Now since q_1 and q_2 are states of A then j_1 and j_2 are either equal, or $\{j_1, j_2\} = \{i_1, i_2\}$, by an abuse of notation write $[j_1, j_2]$ for the element of $\mathfrak{P}(A)$ containing q_1 and q_2.

Let $C = \langle X_3, Q_C, \pi_C, \lambda_C \rangle$ where $Q_C := \{c_1, c_2\}$ be defined as follows. $\pi_C(i, \centerdot) : Q_C \to \{c_1\}$ if $i \in \{j_1, j_2\}$ otherwise $\pi_C(i, \centerdot) : Q_C \to \{c_2\}$. The map $\lambda_C(\centerdot, c_1) : X_3 \to X_3$ is the identity permutation.

Set the map $\lambda_C(\cdot, c_2) : X_3 \to X_3$ to be the permutation $\overline{q_2^{-1}q_1}$ if $q_1, q_2 \in [i_1, i_2]$ otherwise set $\lambda_C(\cdot, c_2) : X_3 \to X_3$ to be the permutation $\overline{q_1^{-1}q_2}$. Notice that both c_1 and c_2 map i_3 to i_3. Moreover since q_1 and q_2 are distinct states of A and A is a minimal transducer we also have that the state c_2 induces the transposition swapping i_2 and i_2. Therefore C is a minimal transducer. Furthermore since whenever we read i_1 and i_2 the active state is c_a for some $a \in \{1, 2\}$ and the output is an element of the set $\{i_1, i_2\}$ we also have that C is bi-synchronizing at level 1 and has order 2.

Now consider $\mathrm{Core}(CAC)$. Since A is synchronizing it follows that $\mathrm{Core}(CAC)$ is synchronizing. Let k be greater than maximum of the minimal synchronizing length of $\mathrm{Core}(CAC)$ and the minimal synchronizing length of A. Using the conditions that $\pi(j_1, p_1) = q_1$ and $\lambda(j_1, p_1) \in \{i_1, i_2\}$, there is a string $\Gamma \in X_n^k$ with last letter equal to j_1 such that the state of A forced by Γ is q_1. This means by an observation in the first paragraph that the output of Γ when processed from any state has last letter in the set $\{i_1, i_2\}$. Likewise there is a word $\Delta \in X_n^k$ with last letter j_2 such that the state of A forced by Δ is q_2 and the output of Δ when processed from any state has last letter equal to i_3. Now since q_1 and q_2 belong to the same element $[j_1, j_2]$ of $\mathfrak{P}(A)$, it follows that any word of length k in W_{q_1} or W_{q_2} must have last letter in the set $\{j_1, j_2\}$.

Now all states of C map $\{j_1, j_2\}$ to the set $\{j_1, j_2\}$ (since they all fix i_3), therefore for any word $\Lambda \in X_n^k$ and any state c of C such that $\lambda_C(\Lambda, c) \in W_{q_1}$ we must that the state of C forced by Λ is c_1 and the last letter of Λ is in the set $\{j_1, j_2\}$. Therefore reading such a word Λ from any state of CAC beginning with c the active state will be $(c_1, q_1, c_{[i_1,i_2]})$, where $c_{[i_1,i_2]} = c_1$ if $\{j_1, j_2\} = \{i_1, i_2\}$ otherwise $c_{[i_1,i_2]} = c_2$. This is because by an observation in the first paragraph all single letter inputs to the state q_1 have output in the set $\{i_1, i_2\}$. Therefore $(c_1, q_1, c_{[i_1,i_2]})$ is a state of $\mathrm{Core}(CAC)$. Likewise for any word $\Lambda' \in X_n^k$ and any state c' of C such that $\lambda_C(\Lambda', c') \in W_{q_2}$ we must have that the state of C forced by Λ' is c_1 and the last letter of Λ' is in the set $\{j_1, j_2\}$. Therefore reading such a word Λ' from any state of CAC beginning with c' the active state will be $(c_1, q_2, c_{[i_3]})$, where $c_{[i_3]} = c_1$ if $\{j_1, j_2\} = \{i_3\}$ otherwise $c_{[i_3]} = c_2$. This is because by an observation in the first paragraph all single letter inputs to the state q_2 have output equal to i_3. Therefore $(c_1, q_2, c_{[i_3]})$ is also a state in $\mathrm{Core}(CAC)$.

Now the above arguments are actually independent of q_1 and q_2 and demonstrate that if dqd' is a state of $\mathrm{Core}(CAC)$ then d depends only the set S of $\mathfrak{P}(A)$ such that $q \in S$ and d' depends only on the set S' of $\mathfrak{P}(A^{-1})$ such that $q^{-1} \in S'$. Therefore $\mathrm{Core}(CAC)$ has as many states as A.

We now demonstrate that $(c_1, q_2, c_{[i_3]})$ and $(c_1, q_1, c_{[i_1,i_2]})$ are ω-equivalent. Since C is synchronizing at level 1, since both states of CAC begin with c_1, since q_1 and q_2 satisfy $\pi_A(i, q_1) = \pi_A(i, q_2)$ for all $i \in X_n$, and since all states of C read i_1 and i_2 to the same location, it follows that for any word $i \in X_n$ we have $\pi_{CAC}(i, (c_1, q_2, c_{[i_3]})) = \pi_{CAC}(i, (c_1, q_2, c_{[i_1,i_2]}))$. This is because for any $i \in X_n$, $\{\lambda_A(i, q_1), \lambda_A(i, q_2)\} = \{i_1, i_2\}$ or $\{\lambda_A(i, q_1), \lambda_A(i, q_2)\} = \{i_3\}$. Thus, it suffices to show that $(c_1, q_2, c_{[i_3]})$ and $(c_1, q_1, c_{[i_1,i_2]})$ induce the same permutation on X_n. However this follows by construction, since if $\{j_1, j_2\} = \{i_1, i_2\}$ we have that $c_{[i_3]} = c_2$, the permutation of X_n induced by c_2 is $\overline{q_2^{-1}q_1}$ and $c_{\{}[i_1, i_2]\} = c_1$ (recall c_1 induces the identity permutation on X_n). Therefore the permutation of X_n induced by the states $(c_1, q_2, c_{[i_3]})$ and $(c_1, q_1, c_{[i_1,i_2]})$ coincide and is equal to $\overline{q_1}$. On the other hand if $\{j_1, j_2\} = \{i_3\}$ we have that $c_{[i_3]} = c_1$, $c_{[i_1,i_2]} = c_2$ and the permutation of X_n induced by c_2 is equal to $\overline{q_1^{-1}q_2}$. Therefore the permutation of X_n induced by the states $(c_1, q_2, c_{[i_3]})$ and $(c_1, q_1, c_{[i_1,i_2]})$ coincide and is equal to q_2. Therefore setting B to be the minimal transducer representing $\mathrm{Core}(CAC)$ we see that $B \in \mathcal{H}_3$ is a conjugate of A with $|Q_A| - |Q_B| \geqslant 1$. □

3.9.2 The growth rate of Cayley machines

In this section we show that for a finite group G, the automaton semigroup generated by the Cayley machine, $\mathcal{C}(G)$ has growth rate, $|G|^n$. To this end, we begin by describing the construction of the Cayley machine which were introduced in the paper [51]. We note that the Cayley machine of a finite group satisfies the sufficient conditions for containing a free subsemigroup of rank at least 2 given in [51].

Let M be a finite monoid (e.g. a finite group), then one can form the transducer $\mathcal{C}(M) := \langle M, M, \pi, \lambda \rangle$ called its *Cayley machine*, with input and output alphabet, M and state set M. The transition and rewrite function satisfy the following rules for $l, m \in M$:

(1.) $\pi(l, m) := ml$

(2.) $\lambda(l, m) := ml$

In each case ml is the evaluation of the product of m and l in the monoid M. If M is a finite group G then by Cayley's Theorem no two states of $\mathcal{C}(G)$ are ω-equivalent, and the functions $\pi(\centerdot, m) : M \to M$ and $\lambda(\centerdot, m) : M \to M$ are bijections. Hence $\mathcal{C}(G)$ is reduced and invertible. It is not hard to see that $(\mathcal{C}(G))^{-1}$ is synchronizing at level 1 (or is a reset automaton).

Remark 3.9.15. With a little work it can be shown that $(\mathcal{C}(G))^{-1}$ satisfies the conditions of Proposition 3.9.3 where, in this case, $\mathcal{S} = G$. This shows that the automaton semigroup generated by $\mathcal{C}(G)$ is free. Silva and Steinberg give a proof of this in [51].

We have the following lemma for synchronizing transducers:

Lemma 3.9.16. *Let $A = \langle X_n, Q, \pi, \lambda \rangle \in \widetilde{\mathcal{P}}_n$ be a transducer, which is synchronizing at level k. Furthermore assume that for every $\Gamma \in X_n^k$ and for all states $q \in Q$, there is a state $p \in Q$ such that $\lambda(\Gamma, p) \in W_q$. Then under this condition, A has the property that for all $m \in \mathbb{N}$, $\mathrm{Core}(A^m) = A^m$.*

Proof. We may assume, by increasing the alphabet size, that A is synchronizing at level 1.

We proceed by induction on m. For $m = 1$ it holds that $A = \mathrm{Core}(A)$ by assumption that $A \in \widetilde{\mathcal{P}}_n$.

Assume $\mathrm{Core}(A^j) = A^j$ for all $j \leqslant m - 1$.

Consider $A^{m-1} = \mathrm{Core}(A^{m-1})$. Fix an arbitrary state $b_1 \dots b_{m-1} \in A^{m-1}$. There is a state $a_1 \dots a_{m-2}a_{m-1}$ and letters x and y in X_n such that

$$a_1 \dots a_{m-2}a_{m-1} \xrightarrow{x|y} b_1 \dots b_{m-1}$$

Let y' be the output when x is read from $a_1 \dots a_{m-2}$. Notice that since A is synchronizing at level 1 the state of A forced by y' must be b_{m-1}. By assumption, for every state $q \in Q$ there is a state p such that $\lambda(y', p) \in W_q$. Therefore given an arbitrary $q \in Q$, by setting $a_{m-1} := p$ we may assume that $y \in W_q$ moreover, the inductive hypothesis guarantees that $a_1 \dots a_{m-1}$ is a state of $\mathrm{Core}(A^{m-1})$.

Observe that A^{m-1} is synchronizing at level $m - 1$ and so there is a word Δ of length $m - 1$ labelling a loop based at $a_1 \dots a_{m-2}a_{m-1}$. Let Λ be the output of this loop. Then reading Δx in $(A)^{m-1}$ from the state $a_1 \dots a_{m-2}a_{m-1}$ the output is Λy. Now, the state of A forced by Λy is q, therefore reading Δx through any state $a_1 \dots a_{m-2}a_{m-1}s$ for any $s \in Q$, the active state becomes $b_1 \dots b_{m-1}q$.

The above paragraph now implies that $b_1, \dots b_{m-1}q$ is a state of $\mathrm{Core}(A^m)$, since A^m is synchronizing at level m, hence the state of A^m forced by Δx is $b_1, \dots b_{m-1}q$. Therefore for any $q \in Q$, $b_1 \dots b_{m-1}q$ is a state of $\mathrm{Core}(A^m)$. Moreover $b_1 \dots b_{m-1}$ was arbitrary, so we conclude that $\mathrm{Core}(A^m) = A^m$ as required. □

In our next result we apply Lemma 3.9.16 to the transducer $(\mathcal{C}(G))^{-1}$ for a finite group G by showing that $(\mathcal{C}(G))^{-1}$ satisfies the condition of the lemma.

Theorem 3.9.17. *Let G be a finite group, then $|(\mathcal{C}(G))^n| = |G|^n$, hence the transducer $\mathcal{C}G$ has growth rate $|G|^n$. Moreover every state of $\mathcal{C}(G)^n$ is accessible from every other state.*

Proof. Since, either by Remark 3.9.15 or a result in [51], the automaton semigroup generated by $\mathcal{C}(G)$ is free it suffices to show that $(\mathcal{C}(G)^{-1})$ satisfies the conditions of Lemma 3.9.16.

Since the states of $(\mathcal{C}(G))^{-1}$ are in bijective correspondence with the states of $\mathcal{C}(G)$ we shall let g' be the state of $(\mathcal{C}(G))^{-1}$ corresponding to the state g of $\mathcal{C}(G)$.

Let $g, h \in G$. We shall show that there is a state m' of $(\mathcal{C}(G))^{-1}$ such that $\lambda'_G(g, m') = h$ (here λ'_G represents the rewrite function of $(\mathcal{C}(G))^{-1}$).

By definition of $\mathcal{C}(G)$ it suffices to take $m' = (gh^{-1})'$. □

In next section we introduce the notion of 'core growth rate' and investigate the core growth of elements of $\widetilde{\mathcal{P}}_n$ and $\mathcal{H}_n$.

3.10 Growth rates of the core of elements of $\widetilde{\mathcal{P}}_n$

In this section we explore how the core of elements of $\widetilde{\mathcal{P}}_n$ grow with powers of the transducer.

Definition 3.10.1 (Core growth rate)**.** Let $A \in \mathcal{P}_n$ be a transducer, and let χ be one of 'logarithmic', 'polynomial', and, 'exponential', then we say that A has *core χ growth (rate)* if the core of powers of A grows at a rate χ with powers of A.

Lemma 3.9.16 of the previous section indicates that there are many examples of elements of $\widetilde{\mathcal{P}}_n$ and $\widetilde{\mathcal{H}}_n$, $n \in \mathbb{N}$ and $n \geqslant 2$ which have core exponential growth. In particular the Cayley machine of any finite group. Notice moreover that Lemma 3.9.16 applies to transducers without homeomorphism states. The transducer in Figure 3.27 is a non-minimal synchronizing transducer whose action on $X_2^{\mathbb{Z}}$ induces the shift-homeomorphism. We call this the 2-shift transducer. This transducer satisfies the hypothesis of Lemma 6.9 and so has core exponential growth rate.

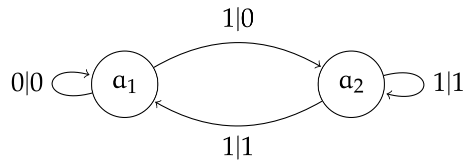


Figure 3.27: The shift map has core exponential growth rate

If we restrict to $\mathcal{H}_n$, then it is a result due to Hedlund [33] that $\mathcal{H}_2$ is the cyclic group of order 2. However using Lemma 3.9.16 one can verify that the element H of $\mathcal{H}_4$ shown in Figure 3.28 has core exponential growth rate.

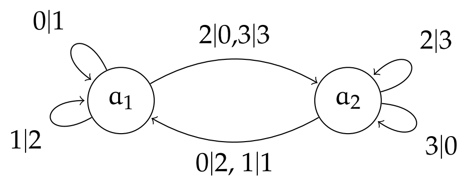


Figure 3.28: An element of $\mathcal{H}_4$ with core exponential growth rate

Now for, $n \in \mathbb{N}$ let P be the single state transducer which acts as the identity on the symbols $i \in X_n \backslash \{0, 1, 2, 3\}$. Then by Theorem 3.6.8 $H \sqcup P$ is an element of $\mathcal{H}_n$. Furthermore, by the same result, we have $\min \mathrm{Core}((H \sqcup P)^m) = \min \mathrm{Core}(H^m) \sqcup P$. Therefore $H \sqcup P$ is an element of $\mathcal{H}_n$ with core exponential growth, since H has core exponential growth. We have now shown that for $n \geqslant 4$ $\mathcal{H}_n$ contains elements with core exponential growth, which leaves $\mathcal{H}_3$.

The transducer G shown in Figure 3.29 is an element of $\mathcal{H}_3$, we shall show that this element has core exponential growth rate. Our argument for demonstrating this is somewhat convoluted.

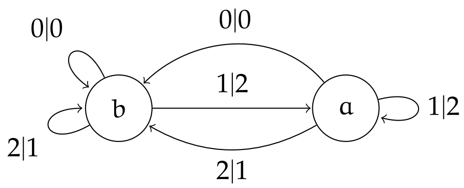


Figure 3.29: An element of $\mathcal{H}_3$ with core exponential growth

The graph of bad pairs of G at level 1 has a loop it then follows by Proposition 3.9.3 the automaton semigroup generated by G has exponential growth and is in fact a free semigroup. This means that different words in $\{a, b\}$ of the same length represent inequivalent states of some power of G. Since no reductions can be made, we will denote by $\mathtt{Core}(G^i)$ the transducer representing the core of G^i for some $i \in \mathbb{N}$.

Observe that b^i is a state of $\mathtt{Core}(G^i)$ for all $i \in \mathbb{N}$, since $\pi(0, b) = b$ and $G_b(0) = 0$. Therefore we can treat G as an initial transducer with start state b.

To keep the analysis simple we shall reduce to the case of transducer on a two letter alphabet which will serve as a 'dummy' variable for G in a sense that will be made precise. To do this, consider the binary tree in Figure 3.30 representing how the initial transducer $G_{bb} := \mathtt{Core}(G^2)$ transitions on certain inputs. The left half of tree corresponds to transitions from the set $\{1\} \times \{0, 2\} \times \{0, 1\} \times \{0, 2\} \times \{0, 1\} \ldots$ the right half of the tree corresponds to transitions from the set $\{2\} \times \{0, 1\} \times \{0, 2\} \times \{0, 1\} \times \{0, 2\} \ldots$. Let $T_1 := \{0, 2\} \times \{0, 1\} \times \{0, 2\} \times \{0, 1\} \ldots$ and $T_2 := \{0, 1\} \times \{0, 2\} \times \{0, 1\} \times \{0, 2\} \ldots$.

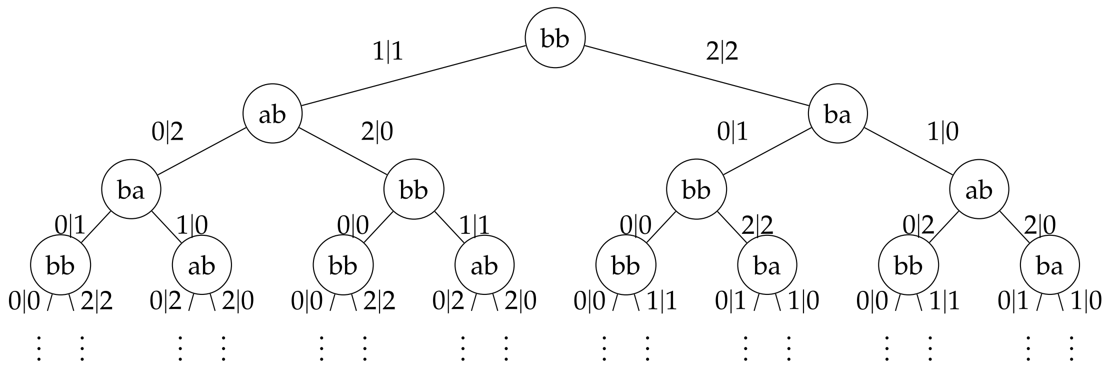


Figure 3.30: Binary tree depicting the transitions of G_{bb}

Using Figure 3.30 we form a dummy transducer which mimics the transitions of G_{bb} as follows. We shall only be interested in the transitions of this dummy transducer and so whenever we take powers of the dummy transducer we will not minimise it. First form new states $B \sim bb$, $\sigma_1^1 \sim ab$, $\sigma_0^1 \sim ba$, σ_0^0 and σ_1^0. Here σ_0^0 corresponds to the state bb whenever we read an element of $\{0, 1\}$ from bb and σ_1^0 corresponds to the state bb whenever we read an element of $\{0, 2\}$ from bb. Now notice that all states on the left half of below the root, at odd levels map $\{0, 2\}$ into $\{0, 2\}$ and at all states at even levels map $\{0, 1\}$ into $\{0, 1\}$. Analogously all states on the right half of the tree below the root map $\{0, 1\}$ into $\{0, 1\}$ at odd levels and $\{0, 2\}$ into $\{0, 2\}$ at even levels. Since we only care about transitions we may transform the tree into a binary tree by replacing all the 2's with 1's so long as we still encode the information about which side of the tree we are on, and about parity, even or odd, of the level of the tree we are acting on. This is achieved by the states σ_0^0 and σ_1^0 which represent the occurrence of bb on the left half of the tree at even levels and on the right half of the tree at odd levels. The resulting initial transducer $\widetilde{G}_B = \langle \{0, 1\}, \widetilde{\pi}, \widetilde{\lambda} \rangle$ on a two-letter alphabet now transitions similarly to G_{bb}, and has states corresponding to states of G. In particular, by construction, any state of $\widetilde{G}^i$ (we do not minimise this transducer as we are interested only in transitions) accessible from B (in $\widetilde{G}$) will correspond to a state in G^i (where we replace σ_i^j, $i, j = 0, 1$ by the corresponding state of G) accessible from bb (in G) by reading either a 1 or 2 then, in the first case alternating between reading an element of $\{0, 2\}$ and an element of $\{0, 1\}$ and in the second between an element of $\{0, 1\}$ and an element of $\{0, 2\}$.

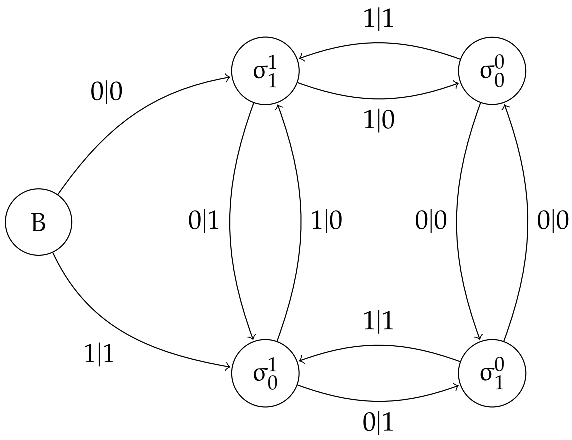


Figure 3.31: The dummy transducer $\widetilde{G}_B$

The point of building the transducer $\widetilde{G}$ is that it encodes the transitions of G in a fashion which is much easier to describe. One should think of $\widetilde{G}$ as a dummy transducer for G in which it is much easier to read transitions as we shall see.

Since we transition from B^i to $(\sigma_1^1)^i$ by reading 0 it suffices to show that the initial transducer $\widetilde{G}_{\sigma_1^1}$ has exponential growth. Recall that here we are interested in how the number of states of $\widetilde{G}$ grow without considering the ω-equivalence of these states. We shall then argue from this fact that G has core exponential growth since the automaton semigroup generated by G is free and the states of G correspond nicely to the states of powers of $\widetilde{G}$ (without minimising).

First we argue that the number of states of $\widetilde{G}^i_{(\sigma_1^1)^i}$ is at least $2^{\lceil i/2 \rceil}$. We stress once more that we are not concerned with the ω-equivalence of some of these states, they merely act as dummy variables for the states of $G^i_{(bb)^i}$. In particular whenever we raise $\widetilde{G}_{(\sigma_1^1)}$ to some power, we shall not minimise it.

Notice that for $x, i, j = 0, 1$,

$$\widetilde{\pi}(x, \sigma_i^j) = \sigma_{i+1}^{x+ij} \tag{3.2}$$

$$\widetilde{\lambda}(x, \sigma_i^j) = x + j \mod 2. \tag{3.3}$$

In (3.2) and (3.3) subscripts and exponents are taken modulo 2. Since $a + b \mod 2 = ((a \mod 2) + (b \mod 2)) \mod 2$ and $ab \mod 2 = ((a \mod 2)(b \mod 2)) \mod 2$, we can iterate the above formulae.

We shall require the following notation in order to simplify the discussion that follows. Set, for $i, j \in \mathbb{Z}$, $i \geqslant 1$.

$$\Sigma(i, j) := \sum_{l_1=1}^{j} \sum_{l_2=1}^{l_1} \cdots \sum_{l_i=1}^{l_{i-1}} l_i$$

If $j = 0$ or is negative then take $\Sigma(i, j) = 0$. Notice that the $\Sigma(1, j)$ is simply the sum of the first j numbers $j \geqslant 1$. Furthermore observe that

$$\sum_{k=1}^{j} \Sigma(i, k) = \Sigma(i + 1, j) \tag{3.4}$$

Remark 3.10.2. It is straight-forward to show either by finite calculus or by induction making use of the identity $\sum_{k=m}^{j} \binom{k}{m} = \binom{j+1}{m+1}$ that $\Sigma(i, j) = \binom{j+i}{i+1}$. We shall not require this fact.

Freeing the symbol k, let $k \geqslant 1 \in \mathbb{N}$ and let $x_1 x_2 \dots x_k \in \{0,1\}^k$. In what follows below whenever we have an x_i for $i \in \mathbb{Z}$ and $i < 0$, we shall take x_i to be 0 and $x_0 = 1$. We have the following claim:

Claim 3.10.3. *For i even and bigger than or equal to 1 after reading the first i terms the j^{th} term of the active state is*

$$\sigma_1^{x_i+(j-1)x_{i-1}+\Sigma(1,j-1)x_{i-2}+\Sigma(2,j-2)x_{i-3}+\Sigma(3,j-2)x_{i-4}+\ldots+\Sigma(i-2,j-i/2)x_1+\Sigma(i-1,j-i/2)\cdot 1}. \tag{3.5}$$

After reading the first $i+1$ terms of the sequence $x_1 \dots x_k$ through $\widetilde{G}^k_{(\sigma_1^1)^k}$ the j^{th} term of the active state is

$$\sigma_0^{x_{i+1}+jx_i+\Sigma(1,j-1)x_{i-1}+\Sigma(2,j-1)x_{i-2}+\Sigma(3,j-2)x_{i-3}+\Sigma(4,j-2)x_{i-4}+\ldots+\Sigma(i-1,j-i/2)x_1+\Sigma(i,j-i/2)\cdot 1} \tag{3.6}$$

exponents are taken modulo 2.

Proof. The proof follows by induction and a mechanical calculation making use of (3.4), (3.3).

We first establish the base cases $i = 1$ and $i = 2$. The top row of array (3.7) consists of k copies of the state σ_1^1 of $\widetilde{G}_B$. The first column of the second row indicates the we are reading the letter x_1 through state σ_1^1. In the second column, the symbol $x_1 + 1$ is the input to be read through the second copy of σ_1^1, and $\sigma_0^{1+x_1}$ is equal to $\widetilde{\pi}(x_1, \sigma_1^1)$. The remaining columns are to be read in a similar fashion.

$$\begin{array}{ccccccc} & \sigma_1^1 & & \sigma_1^1 & & \sigma_1^1 & \ldots & \sigma_1^1 \\ x_1 & x_1+1 & \sigma_0^{1+x_1} & x_1+2 & \sigma_0^{x_1+2} & x_1+3 & \sigma_0^{x_1+3} & \ldots\ x_1+k\ \sigma_0^{x_1+k} \end{array} \tag{3.7}$$

Therefore after reading x_1 from the state $(\sigma_1^1)^k$ the active state of the transducer $\widetilde{G}^k_{(\sigma_1)^k}$ is

$$\sigma_0^{x_1+1}\sigma_0^{x_1+2}\sigma_0^{x_1+3}\ldots\sigma_0^{x_1+k}$$

which is as indicated by the formula (3.6).

Now we read x_2 through the active state $\sigma_0^{x_1+1}\sigma_0^{x_1+2}\sigma_0^{x_1+3}\ldots\sigma_0^{x_1+k}$ to establish the case $i = 2$. We shall make use of an array as in (3.7) to do demonstrate this.

$$\begin{array}{cccccc} & \sigma_0^{x_1+1} & & \sigma_0^{x_1+2} & & \sigma_0^{x_1+3} \\ x_2 & x_2+x_1+\Sigma(1,1)\ \sigma_1^{x_2} & & x_2+2x_1+\Sigma(1,2)\ \sigma_1^{x_2+x_1+\Sigma(1,1)} & & x_2+3x_1+\Sigma(1,3)\ \sigma_1^{x_2+2x_1+\Sigma(1,2)} \end{array} \tag{3.8}$$

A simple induction shows that the $(k+1)^{st}$ entry of the second row is:

$$x_2 + kx_1 + \Sigma(1,k)\ \sigma_1^{x_2+(k-1)x_1+\Sigma(1,k-1)}$$

and so all the terms of the active state are as indicated by the formula (3.5).

Now assume that i is even and $2 \leqslant i \leqslant k-1$ and that the j^{th} of the active state after reading the first i terms of $x_1 \dots x_k$ is as given by the formula (3.5). We now show that after reading x_{i+1} through the active state the j^{th} term of the active state is as given in (3.6). We shall proceed by induction on j.

By assumption the first term of the active state is $\sigma_1^{x_i}$. Therefore $\widetilde{\pi}(x_{i+1}, \sigma_1^{x_i}) = \sigma_0^{x_{i+1}+x_i}$ and $\widetilde{\lambda}(x_{i+1}, \sigma_1^{x_i}) = x_{i+1} + x_i$. Therefore the first term $\sigma_0^{x_{i+1}+x_i}$ of the new active state satisfies the formula (3.6) with $j = 1$.

By assumption the second term of the current active state is $\sigma_1^{x_i+x_{i-1}} + \Sigma(1,1)x_{i-2}$. Therefore

$$\widetilde{\pi}\left(x_{i+1}+x_i, \sigma_1^{x_i+x_{i-1}+\Sigma(1,1)x_{i-2}}\right) = \sigma_0^{x_{i+1}+2x_i+x_{i-1}+\Sigma(1,1)x_{i-2}}$$

and

$$\widetilde{\lambda}\left(x_{i+1}+x_i, \sigma_1^{x_i+x_{i-1}+\Sigma(1,1)x_{i-2}}\right) = x_{i+1}+2x_i+x_{i-1}+\Sigma(1,1)x_{i-2}.$$

Now we may rewrite $x_{i+1}+2x_i+x_{i-1}+\Sigma(1,1)x_{i-2}$ as $x_{i+1}+xx_i+\Sigma(1,1)x_{i-1}+\Sigma(2,1)x_{i-2}$ since $\Sigma(2,1) = \Sigma(1,1)$ and $\Sigma(1,1) = 1$. Therefore the 2nd term of the new active state $\sigma_0^{x_{i+1}+2x_i+\Sigma(1,1)x_{i-1}+\Sigma(2,1)x_{i-2}}$ satisfies the formula (3.6) with $j=2$.

Now assume that for $2 \leqslant j \leqslant k$ the $j-1^{\text{st}}$ term of the new active state is given by:

$$\sigma_0^{x_{i+1}+(j-1)x_i+\Sigma(1,j-2)x_{i-1}+\Sigma(2,j-2)x_{i-2}+\Sigma(3,j-3)x_{i-3}+\Sigma(4,j-3)x_{i-4}+\ldots+\Sigma(i-1,j-1-i/2)x_1+\Sigma(i,j-1-i/2)\cdot 1}$$

and the output when x_i is read through the first $j-1$ terms of the current active state is

$$x_{i+1}+(j-1)x_i+\Sigma(1,j-2)x_{i-1}+\Sigma(2,j-2)x_{i-2}+\Sigma(3,j-3)x_{i-3}+\Sigma(4,j-3)x_{i-4}+\ldots \\ +\Sigma(i-1,j-1-i/2)x_1+\Sigma(i,j-1-i/2)\cdot 1.$$

Therefore the j^{th} term of the new active state will be the active state after $x_{i+1}+(j-1)x_i+\Sigma(1,j-2)x_{i-1}+\Sigma(2,j-2)x_{i-2}+\Sigma(2,j-3)x_{i-3}+\Sigma(3,j-3)x_{i-4}+\ldots+\Sigma(i-1,j-1-i/2)x_1+\Sigma(i,j-1-i/2)\cdot 1$ is read from the current active state. By assumption the current active state is:

$$\sigma_1^{x_i+(j-1)x_{i-1}+\Sigma(1,j-1)x_{i-2}+\Sigma(2,j-2)x_{i-3}+\Sigma(3,j-2)x_{i-4}+\ldots+\Sigma(i-2,j-i/2)x_1+\Sigma(i-1,j-i/2)\cdot 1}.$$

Making use of the (3.2) and (3.3) and the rule (3.4), the new active state is given by

$$\sigma_0^{x_{i+1}+jx_i+\Sigma(1,j-1)x_{i-1}+\Sigma(2,j-1)x_{i-2}+\Sigma(3,j-2)x_{i-3}+\Sigma(4,j-2)x_{i-4}+\ldots+\Sigma(i-1,j-i/2)x_1+\Sigma(i,j-i/2)\cdot 1}$$

which is exactly the formula given in (3.6).

The case where i is odd is proved in an analogous fashion. □

Observe that for all $i>0$ we have $\Sigma(i,1)=1$. Now for i even and $j=i/2+1$ consider $\widetilde{G}^{i+1}_{(\sigma_1^1)^{i+1}}$, the following formulas determine the exponents of the first j terms of the active state after reading the first $i+1$ terms of the sequence $x_1,\ldots x_k$. The subscripts of these states are all 0.

$$\begin{aligned}
&x_{i+1}+x_i\\
&x_{i+1}+2x_i+\Sigma(1,1)x_{i-1}+\Sigma(2,1)x_{i-2}\\
&x_{i+1}+3x_i+\Sigma(1,2)x_{i-1}+\Sigma(2,2)x_{i-2}+\Sigma(3,1)x_{i-3}+\Sigma(4,1)x_{i-4}\\
&\ \vdots\\
&x_{i+1}+(j-1)x_i+\Sigma(1,j-2)x_{i-1}+\Sigma(2,j-2)x_{i-2}++\Sigma(i-3,j-i/2+1)x_3+\Sigma(i-2,j-i/2+1)x_2\\
&x_{i+1}+jx_i+\Sigma(1,j-1)x_{i-1}+\Sigma(2,j-1)x_{i-2}+\ldots\Sigma(i-1,j-i/2)x_1+\Sigma(i,j-i/2)
\end{aligned}$$

Let $y_1,\ldots y_j$ in $\{0,1\}^j$ be any sequence. Since the coefficients of the last two terms of all the equations above is 1, there is a choice of $x_1\ldots x_{i+1}$ such that the exponent of the l^{th} term ($1 \leqslant l \leqslant j$) of the active state after reading $x_1\ldots x_{i+1}$ in $\widetilde{G}^{i+1}_{(\sigma_1^1)^{i+1}}$ is y_l. This is achieved inductively, first we solve $x_{i+1}+x_i = y_1$ in $\mathbb{Z}_2$. This determines x_{i+1} and x_i. Next we pick x_{i-1} so that $x_{i+1}+2x_i+\Sigma(1,1)x_{i-1}=0 \mod 2$, and set $x_{i-2}=y_2$. This determines x_{i-1} and x_{i-2}. Therefore we may now pick x_{i-3} so that $x_{i+1}+3x_i+\Sigma(1,2)x_{i-1}+\Sigma(2,2)x_{i-2}+\Sigma(3,1)x_{i-3} = 0$ and set $x_{i-4}=y_3$. We carry on in this way until we have determined x_l for $i+1 \leqslant l \leqslant 2$. Then we solve the equation

$$x_{i+1}+jx_i+\Sigma(1,j-1)x_{i-1}+\Sigma(2,j-1)x_{i-2}+\ldots+(\Sigma(i,j-i/2)-y_j)+\Sigma(i-1,j-i/2)x_1=0$$

for x_1 in $\mathbb{Z}_2$.

That is for any sequence $y_1\ldots y_j \in \{0,1\}^j$, there is a state of $\widetilde{G}^{i+1}_{(\sigma_1^1)^{i+1}}$ whose first j terms are $\sigma_0^{y_1}\ldots\sigma_0^{y_j}$.

Now for $\widetilde{G}^i_{\sigma_1^1}$, a similar argument shows for any such sequence $y_1\ldots y_j$, there is a state of $\widetilde{G}_{(\sigma_1^1)^i}$ whose first j terms is $\sigma_1^{y_1}\ldots\sigma_1^{y_j}$.

Now using the correspondence stated above that $\sigma_1^1 \sim ab$, $\sigma_0^1 \sim ba$ and $\sigma_0^0 \sim bb$ and $\sigma_1^0 \sim bb$, the states of $\widetilde{G}^{i+1}_{\sigma_1^1}$ and $\widetilde{G}^{i}_{\sigma_1^1}$ correspond to states of G^i and G^{i+1} accessible from the state $(bb)^i$ and $(bb)^{i+1}$. Since the automaton semigroup generated by G is free, then two different words in $\{a, b\}^{2(i+1)}$ will correspond to distinct states of $G^{(2(i+1))}$. Now by the arguments above we have that for every element $y_1 \ldots y_j$ in the set $\{0, 1\}^j$ $\widetilde{G}^{i+1}_{\sigma_1^1}$ and $\widetilde{G}^{i}_{\sigma_1^1}$ have states beginning with $\sigma_0^{y_1} \ldots \sigma_0^{y_j}$ and $\sigma_1^{y_1} \ldots \sigma_1^{y_j}$ respectively. Now using the fact that the automaton semigroup generated by G is free, it follows that for $y_1 \ldots y_j$ and $y'_1 \ldots y'_j$ in $\{0, 1\}^j$, the states $\sigma_l^{y_1} \ldots \sigma_l^{y_j}$ and $\sigma_l^{y'_1} \ldots \sigma_l^{y'_j}$ for $l \in \{0, 1\}$ correspond to distinct states of G^j. Therefore $G^{i+1}_{(bb)^{i+1}}$ has at least $2^{i/2\ +1} = 2^{\lceil (i+1)/2 \rceil}$ states $G^{i}_{(bb)^i}$ has at least $2^{\lceil (i+1)/2 \rceil}$ states. It now follows that for arbitrary $i \in \mathbb{N}$, G_{b^i} has at least $2^{\lfloor i/2 \rfloor}$ states for any $i \geqslant 1 \in \mathbb{N}$.

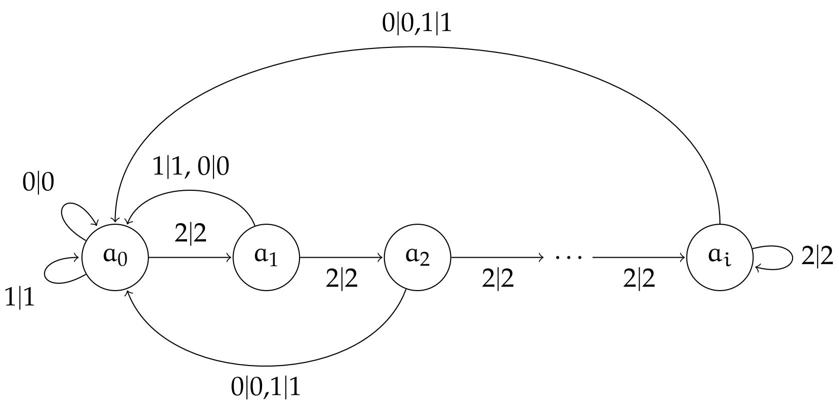
The above all together now means that G is an element of $\mathcal{H}_3$ with core exponential growth. Therefore we have:

Theorem 3.10.4. *For any* $n > 2$ *there are elements of* $\mathcal{H}_n$ *which have core exponential growth.* □

Remark 3.10.5. for $i \in \mathbb{N}$, the maximum difference in the size of elements of $\mathcal{H}_n$ which are bi-synchronizing at level i grows exponentially with i.

Proof. For each $i \in \mathbb{N}$ it is possible to construct an element of $\mathcal{H}_n$ which is bi-synchronizing at level i, see Figure 3.32 for an indication of how to do so. On the other hand there are elements of $\mathcal{H}_n$ which are bi-synchronizing at level 1, and which have core exponential growth (for instance the example in Figure 3.29). Let G be such an element. Then $\min \mathrm{Core}(G^i)$ is bi-synchronizing at level i by Proposition 2.1.33 and has at least e^{ci} states for some positive constant c. Therefore the maximum difference in the size of elements of $\mathcal{H}_n$ which are bi-synchronizing at level i is at least $e^{ci} - i - 1$. □

Figure 3.32: An element of $\mathcal{H}_3$ bi-synchronizing at level i



In the subsequent discussion we explore some of the elementary properties of the core growth rate, and state a conjecture about the core growth rates of elements of $\widetilde{\mathcal{H}}_n$ which have infinite order.

Lemma 3.10.6. *Let* $A \in \mathcal{P}_n$ *be an element of infinite order. If* B *is conjugate to* A *in* $\mathcal{P}_n$ *then core growth rate of* B *is equivalent to the core growth rate of* A.

Proof. Let $C \in \mathcal{P}_n$ be such that B is the minimal transducer representing the core of $C^{-1}AC$.

Since $\mathcal{P}_n$ restricting to the core is a part of multiplication in $\mathcal{P}_n$. It follows that $\min(\mathrm{Core}(C^{-1}A^mC)) \cong_\omega B^m$, where A^m and B^m are here identified with the minimal transducer representing the core of A^m and B^m respectively.

This readily implies:

$$|C||A^m||C| = |B^m|$$

as required. □

The next lemma shows that the core growth rate is invariant under taking powers.

Lemma 3.10.7. *Let $A \in \mathcal{P}_n$ of infinite order, and let χ be one of 'exponential', 'polynomial', or 'logarithmic'. Then if there is some $m \in \mathbb{N}$ such that $\mathrm{Core}(A^m)$ has core χ growth rate, then A also has core χ growth rate.*

Proof. This is a straight-forward observation. Let $m \in \mathbb{N}$ be fixed such that Core(A^m) has exponential growth.

Let $i \in \mathbb{Z}_m$ and let $k \in \mathbb{N}$. Now notice that $|\min(A^{km+i})| \geqslant |\min \mathrm{Core}(A^m)^{k+1}|/|A|^{m-i} \geqslant \exp^{c(k+1)}/|A|^{m-i} \geqslant \exp^{c(km+i)/m}/|A|^{m-i}$, for a positive constant c. Now as every positive integer can be written at some $qm + i$, $0 \leqslant q \in \mathbb{Z}$ and $i \in \mathbb{Z}_m$ we are done.

If Core(A^m) has polynomial growth rate, then there are positive numbers C and d such that $|\min \mathrm{Core}(\mathrm{Core}(A^m)^k)| \leqslant Cn^d$. Now consider the following inequalities:

$$\begin{aligned}|A|^i C(km+i)^d &\geqslant |A|^i Ck^d \geqslant |A|^i |\min \mathrm{Core}(\mathrm{Core}(A^m)^k)| \\ &\geqslant |A^i \min \mathrm{Core}(\mathrm{Core}(A^m)^k)| \geqslant |\min \mathrm{Core}(A^{km+i})|\end{aligned}$$

An Analogous argument shows that if Core(A^m) has core logarithmic growth rate then so does A. □

As a corollary of the lemma above we are able to reduce the question of determining the core growth rates for non-initial automata to the question of determining the growth rate of initial automata.

Corollary 3.10.8. *Let $A \in \widetilde{\mathcal{H}}_n$ then the core growth rate of A is equivalent to the growth rate of some initial transducer B_{q_0}.*

Proof. By Remark 3.4.10 we can associate to A a transformation $\overline{A}$ of X_n. Now observe that there is an $i \in \mathbb{N}$ such that there is an $x \in X_n$ such that $(x)\overline{A}^i = x$.

This means, by Theorem 3.4.12, there is a state of q_0 of $\min \mathrm{Core}(A^i)$ with a loop labelled $x|x$ based at q_0. This readily implies that for any power A^{ki} of A^i the state q_0^k is in the core, since this is the unique state of A^{ki} with loop labelled $x|x$. Therefore we may take $B = \min(A^i_{q_0})$. □

We have the following conjecture about the growth rates of elements of $\widetilde{\mathcal{H}}_n$:

Conjecture 3.10.9. *Let $A \in \widetilde{\mathcal{H}}_n$ be an element of infinite order, then the core growth rate of A is exponential.*

A strategy for verifying this conjecture is to show that in reducing to the core we do not lose too many states. To this end we make the following definition:

Definition 3.10.10. Let A be a finite synchronous transducer. We say has *core distance* k if there is a natural number k such that for any $\Gamma \in X_n^k$ and any $q \in A$, $\pi_A(\Gamma, q)$ is a state of Core(A). Let CoreDist(A) be the minimal k such that A has core distance k. If $A = \mathrm{Core}(A)$ then CoreDist(A) = 0.

The lemma below explores how the function CoreDist behaves under taking products.

Lemma 3.10.11. *Let $A, B \in \widetilde{\mathcal{P}}_n$ and let k_A and k_B be minimal so that A is synchronizing at level k_A and B is synchronizing at level k_B then* CoreDist$(A * B) \leqslant k_B$.

Proof. Indeed observe that given a state U of A such that the transition $U \xrightarrow{x|y} V$ for $x, y \in X_n^{k_B}$ and V a state of A holds in A, then since U is in the core of A (as $A = \mathrm{Core}(A)$) there is a word, z of length k_A such that there is a loop labelled $z|t'$ based at U. Let p be the state of B forced by y.

Observe that since A is synchronizing at level k_A, there is a path $V \xrightarrow{z|t} U \xrightarrow{x|y} V$. Therefore there is a loop labelled $zx|ty$ based at V. Therefore in $A * B$ there is a loop labelled zt based at Vp, since the state of B forced by y is p. Hence for any state Uq of $A * B$, we read an x into a state Vp which is in $\mathrm{Core}(A * B)$.

□

We have as a corollary:

Lemma 3.10.12. *Let $A \in \widetilde{\mathcal{P}}_n$ be synchronizing at level 1. Let A^m represent the minimal transducer representing the core of A^m, then* $\mathrm{CoreDist}(A^m * A) \leqslant 1$.

Notice that by Lemma 3.9.16 there are elements $A \in \widetilde{\mathcal{P}}_n$ for which $\mathrm{CoreDist}A^m = 0$ for all $m \in \mathbb{N}$.

Lemma 3.10.13. *Let $A \in \mathcal{H}_n$ by bi-synchronizing at level k. Then* $\mathrm{CoreDist}(A^m) \leqslant \lceil mk/2 \rceil$.

Proof. First notice that $A^m = A^{\lfloor m/2 \rfloor} * A^{\lceil m/2 \rceil}$. Furthermore both $A^{\lfloor m/2 \rfloor}$and $A^{\lceil m/2 \rceil}$ are bi-synchronizing at level $\lceil m/2 \rceil$.

Let U and V be states respectively of $A^{\lfloor m/2 \rfloor}$ and $A^{\lceil m/2 \rceil}$. Let $\Gamma \in X_n^{\lceil m/2 \rceil}$. Suppose we have the transition:

$$U \xrightarrow{\Gamma|\Delta} U'.$$

Since $A^{\lfloor m/2 \rfloor}$ is bi-synchronizing at level $\lceil m/2 \rceil$, then the state of $A^{-\lfloor m/2 \rfloor}$ forced by Δ is U'^{-1} (the state of $A^{-\lfloor m/2 \rfloor}$ corresponding to U'). Therefore there is a loop labelled $\Delta|\Gamma'$ based at U'^{-1} in $A^{-\lfloor m/2 \rfloor}$, hence there is a loop labelled $\Gamma'|\Delta$ based at U' in $A^{\lfloor m/2 \rfloor}$.

Let T' be the state of $A^{\lceil m/2 \rceil}$ forced by Δ, then $U'T'$ is in $\mathrm{Core}(A^m)$.

Hence we have shown that for any state T of $A^{\lceil m/2 \rceil}$ then the state UT is at most $\lceil m/2 \rceil$ steps from $\mathrm{Core}(A^m)$. Since U was chosen arbitrarily this concludes the proof. □

Lemma 3.9.16 once again shows that the lemma above is an over-estimate in some cases.

If we are able to obtain good bounds on the function CoreDist for a given transducer $A \in \mathcal{H}_n$ of infinite order, then it is possible to prove core exponential growth. In particular it is not hard to show that if there is an $M \in \mathbb{N}$ such that $\mathrm{CoreDist}(A^m) \leqslant M$ for all $m \in \mathbb{N}$ then A has core exponential growth rate if it has infinite order.

We have seen above that there are elements of $\mathcal{P}_n$ which attain the maximum core growth rate possible. The proposition below establishes a lower bound for the core growth rate of those elements A of $\mathcal{H}_n$ of infinite order such that their graph G_r of bad pairs possesses a loop for some $r \in \mathbb{N}$.

We have the following result:

Proposition 3.10.14. *Let $A \in \mathcal{H}_n$ be an element of infinite order, and suppose that for some $r \in \mathbb{N}$ the graph $G_r(A)$ of bad pairs of A has a loop, then A has at least core polynomial growth.*

Proof. By Lemma 3.5.33 we know that the synchronizing level of A grows linearly with powers of A

By the collapsing procedure (Construction 2.2.1), a transducer with minimal synchronizing level i must have at least i states, since at each step of this procedure we must be able to perform a collapse.

Therefore we conclude that the core growth rate of A is at least linear in powers of A. □

In the next chapter we once more view $\mathcal{H}_n$ as a subgroup of $\mathcal{O}_{n,r}$ for $1 \leqslant r < n$. In particular we consider the ϕ-twisted conjugacy problem for an element $\phi \in \mathrm{Aut}(G_{n,r})$ with core an element of $\mathcal{H}_n$.

Chapter 4

ϕ-Twisted Conjugacy and R_∞ in $G_{n,r}$ for certain $\phi \in \mathrm{Aut}(G_{n,r})$

In this chapter, we consider the ϕ-twisted conjugacy problem in $G_{n,r}$ for ϕ an automorphism of $G_{n,r}$ with core in $\mathcal{H}_n$. We show that there are infinitely many ϕ-twisted conjugacy classes for this choice of ϕ. Recall (Section 2.3) that $\mathrm{Aut}(G_{n,r})$ may be identified with elements of $\mathcal{B}_{n,r}$ (which we identify with the transducers inducing the homeomorphisms of $\mathfrak{C}_{n,r}$), thus it makes sense to speak of $\mathrm{Core}(\phi)$ for $\phi \in \mathrm{Aut}(G_{n,r})$. In the latter half of this chapter we hone in on the case $n = 2$. In this case the group $\mathcal{H}_2$ is the group of order 2 and consists only of the identity transducer and the single state transducer which induces the permutation swapping 0 and 1 by a result of Hedlund [33]. By adapting the arguments for the conjugacy problem in $V = G_{2,1}$ outlined in the paper [49], we show that for $\phi \in \mathrm{Aut}(V)$ with core in $\mathcal{H}_2$, the ϕ-twisted conjugacy problem is soluble in V. We begin by briefly introducing the ϕ-twisted conjugacy problem for a finitely generated group G and an automorphism ϕ of G.

4.1 Introduction

Let G be a group given by a finite presentation, and let $\mathrm{Aut}(G)$ denote the automorphism group of G. Then for $\rho \in \mathrm{Aut}(G)$, the ρ-twisted conjugacy problem, is the problem of deciding whether for two elements $f, g \in G$, there exists an element $h \in G$ such that:

$$g = h^{-1}f(h)\rho \tag{4.1}$$

and in the case where such an h exists we say that f, g are ρ-twisted conjugated to each other. Furthermore we say that a group has soluble twisted conjugacy problem if the ρ-twisted conjugacy problem is soluble for any $\rho \in \mathrm{Aut}(G)$. This means that there is an algorithm, terminating in a finite time, which given the two elements f, and g and an element $\rho \in \mathrm{Aut}(G)$ decides if f and g are ρ-twisted conjugate to one another. If moreover, G has infinitely many ρ-twisted conjugacy classes then we say that G has the R_∞ property.

In this chapter we shall be concerned with tackling these questions for the family of groups $G_{n,r}$ which are introduced in Section 2.3 (note that $G_{2,1}$ is also denoted by V in the literature). Recall (Section 2.3) that the group V has subgroup F and T with T simple and F possessing a simple derived subgroup. It is shown in [19]that Thompson's group F has soluble twisted conjugacy problem and Thompson's group T and F have the R_∞ property. The paper [7] gives a different proof that Thompson's group F has the R_∞ property.

In Section 2.4 we give a description of $\mathrm{Aut}(G_{n,r})$ as the group homeomorphisms induced by finite initial invertible bi-synchronizing transducers. Using this classification of $\mathrm{Aut}(G_{n,r})$ we are able to demonstrate that, for automorphisms $\rho \in \mathrm{Aut}(G_{n,r})$ with core in $\mathcal{H}_n$, the Higman-Thompson groups $G_{n,r}$, of which $V = G_{2,1}$, have infinitely many twisted conjugacy classes.

We introduce notation for this class of automorphisms of $G_{n,r}$.

Notation 4.1.1. Let $\mathcal{BH}_{n,r}$ denote the subgroup of those elements of $\mathcal{B}_{n,r}$ with core an element of $\mathcal{H}_n$. For $r = 1$ we shall denote this subgroup by $\mathcal{BH}_n$.

For the case of $V = G_{2,1}$, it is a result of Hedlund that $\mathcal{H}_2 \cong \mathbb{Z}/2\mathbb{Z}$. In this case we are able to say a little bit more, and shall demonstrate, building on techniques in [49], that V has soluble ϕ-twisted conjugacy problem for $\phi \in \mathcal{BH}_2$.

4.2 $G_{n,r}$ has infinitely many ϕ-twisted conjugacy classed for $\phi \in \mathcal{BH}_{n,r}$

Recall that in Chapter 2 Section 2.3 the automorphism group of $G_{n,r}$ was classified as consisting of those homeomorphisms of Cantor space $\mathfrak{C}_{n,r}$ which can be represented by a certain finite reduced initial transducers that are bi-synchronizing at level k for some $k \in \mathbb{N}$. Using this information we demonstrate that the family of groups $G_{n,r}$ has the infinitely many ϕ-twisted conjugacy classed for $\phi \in \mathcal{BH}_{n,r}$. Throughout this section we mainly view elements of $G_{n,r}$ as maps between r-rooted n-ary forests (Subsection 2.3.2) since this fits in well with the language used in [49]. There is an alternative way of looking at elements of $G_{n,r}$ which is as automorphisms of the Higman algebras $V_{n,r}$ (see [2, section 3] for example). The two views are equivalent, however as the first is more suited for discussing dynamics we cast our discussion entirely in this language.

4.2.1 Twisted Conjugacy in $G_{n,r}$

Let $\hat{\tau}$ be an element of $\mathrm{Aut}(G_{n,r})$ (i.e $\hat{\tau}$ acts by topological conjugation by an element τ of $\mathcal{B}_{n,r}$), the $\hat{\tau}$-twisted conjugacy problem is the algorithmic problem of deciding, given two elements $f, g \in G_{n,r}$ if there exists an element $h \in G_{n,r}$ such that the following equation holds:

$$h^{-1}g\hat{\tau}(h) = h^{-1}g\tau^{-1}h\tau = f \tag{4.2}$$

Rearranging slightly this yields:

$$h^{-1}g\tau^{-1}h = f\tau^{-1} \tag{4.3}$$

Therefore the above is equivalent to deciding if there exists an $h \in G_{n,r}$ such that $g\tau$ and $f\tau$ (elements of the group of homeomorphisms of $\mathfrak{C}_{n,r}$) are conjugate by h. We remark that by Lemma 2.5.1 if two reduced transducers A_q and A_p have equivalent cores then we can take one to the other by multiplying by an element of $G_{n,r}$. Hence the above question is equivalent to determining whether the homeomorphism induced by two elements of $\mathcal{BH}_{n,r}$ with equivalent core are conjugate by an element of $G_{n,r}$. For $n = 2$, $r = 1$ there is only one non-trivial element of $\mathcal{BH}_{2,1}$ which acts on an infinite strings of zeroes and ones by swapping ones and zeroes. In the next section we develop a way of representing elements of $\mathcal{BH}_{n,r}$ using forest pairs which generalises the forest pairs representation of $G_{n,r}$ of Subsection 2.3.2.

4.2.2 The elements $g\tau$

Here we describe how to denote the elements $\psi = g\tau$ for $g \in G_{n,r}$ and $\tau \in \mathcal{BH}_{n,r}$ generalising the approach for denoting elements of $g\mathcal{R}$ given in Section 4.3. Assume that τ is a reduced transducer such that $\hat{\tau} \in \mathrm{Aut}(G_{n,r})$(where $\hat{\tau}$ sends $g \in G_{n,r}$ to $\tau^{-1}g\tau$). Since $\tau \in \mathcal{BH}_{n,r}$, it is bi-synchronizing and so there is a k such that after we have processed a word of length k the transducer is in the synchronous core. Therefore form the set of minimal paths $\mathcal{P}_{in}$ from the start state to any state in the core, and the corresponding output paths $\mathcal{P}_{out}$ as in the previous section. These form complete anti-chains. Form the corresponding forest triple as in Subsection 2.3.2, but now we attach states in the core to the leaves of the range forest. That is, if (A, B, σ) is the resulting forest triple, we modify the map σ to a map $\overline{\sigma}$ as follows. If u is a leaf of A, and p is the state in the core of τ such that after we have read u from the start state we are in p, then $(u)\overline{\sigma} = \nu_p$. Inductively, if $(u)\overline{\sigma} = \nu_p$, then $(ui)\overline{\sigma} = \nu\lambda_\tau(i,p)_{\pi_\tau(i,p)}$. Note that for a node x of $\mathcal{T}_{n,r}$ which lies at or beneath a leaf of A we also write $(x)g\tau$ for the leaf $(x)\overline{\sigma}$ of B. The forest triple representing $g\tau$ is now $(A, B, \overline{\sigma})$. In other words the elements $g\tau$, $g \in G_{n,r}$, look like elements of $G_{n,r}$ with transducers attached to the leaves in the range forest. Notice that if τ was a transducer representing an element of $G_{n,r}$ the forest triple (A, B, σ) is exactly as in Subsection 2.3.2.

Definition 4.2.1. Define an elementary contraction (and so expansion) as follows. Let $(A, B, \overline{\sigma})$ be a forest for an element $g\tau$ as described above. Let $c_1 x\alpha_1, \dots c_1 x\alpha_n$ be leaves beneath a node $c_1 x$ of A. Let $c_2 y\beta_{1p_1} \dots c_2 y\beta_{np_n}$ be the corresponding leaves in B such that $(c_1 x\alpha_i)\overline{\sigma} = c_2 y\beta_{ip_i}$. If there is a single state q in the synchronous core of τ such that for all α_i, $1 \leqslant i \leqslant n$, we have $\pi_\tau(\alpha_i, q) = p_i$ and $\lambda_\tau(\alpha_i, q) = \beta_i$, then we delete the leaves beneath $c_1 x$ and $c_2 y$ and form a new map ρ, such that $\rho|_{A\backslash\{c_1 x\alpha_1, \dots c_1 x\alpha_n\}} = \overline{\sigma}|_{A\backslash\{c_1 x\alpha_1, \dots c_1 x\alpha_n\}}$ and $(c_1 x)\rho = c_2 y_q$. Since we assumed our transducer is reduced, such a state q is unique. This is because any other state satisfying the above, will process elements of $\mathfrak{C}_n$ identically to q. A forest triple (C, D, ρ) is called an *expansion [contraction]* of $(A, B, \overline{\sigma})$ if it can be obtained from $(A, B, \overline{\sigma})$ by applying a sequence of elementary expansions [contractions].

This definition makes sense since the states of the core are synchronous and represent homeomorphisms of $\mathfrak{C}_{n,r}$. Moreover, by definition of the minimal forest all such contractions must happen on leaves strictly below the leaves of the minimal forest, since otherwise we are not yet in the core.

Given an element $g\tau$, any expansion $(A, B, \overline{\sigma})$ (including the trivial expansion) of the minimal forest pair representing $g\tau$ together with a bijection $\overline{\sigma}$ between the leaves of the forest pair, is called a *representative forest triple*. The rationale behind this being that we want to be able to guarantee processing from the core which will be useful later on. Below we illustrate such a forest triple representing the element of $\mathcal{BH}_{3,2}$ given in Figure 2.1.

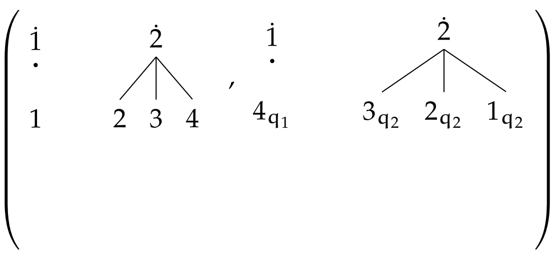


Figure 4.1: The forest triple representing the transducer from Figure 2.1

Notice that if τ represents an element of $G_{n,r}$ then this definition of contraction and expansion is exactly the same as that given in Subsection 2.3.2, since the core is a single state transducer which acts trivially on all inputs.

Olga-Salazar Diaz introduces in [49] revealing pairs for examining the dynamics of elements of R. Thompson group V, the techniques introduced there can in fact be generalised to the Higman-Thompson groups $G_{n,r}$ as demonstrated by Bleak et al in [9, section 4]. We now further extend these techniques to the elements $g\tau$ in the section below. The exposition which follows is based on [15, section 10] which gives a very clear exposition of revealing pairs.

4.2.3 Revealing Pairs

For the discussion below, unless otherwise specified, whenever we use the term *forest* we shall mean precisely an n-ary r-rooted forest. We shall fix throughout this section $\tau \in \mathcal{BH}_{n,r}$ and $g \in G_{n,r}$.

Definition 4.2.2. Let A be a finite forest. The leaves of A shall be defined, in the usual way (See Section 1.2), as those nodes in the full n-ary forest $\mathcal{T}_{n,r}$ whose children are not in A. Those nodes of A which are not leaves will be called *interior nodes* of A. Given a node ν of T_n, the $n+1$-tuple of ν and the children of ν, $(\nu, \nu_0, \nu_1, \dots, \nu_{n-1})$, is called a *caret*.

Let (A, B, π) be a representative forest triple for $g\tau$ (noting that the leaves of B will have states in the core of our transducer attached), we shall denote by $A - B$ the carets in A which are not carets of B and likewise define $B - A$. Finally $A \cap B$ shall be those carets of A which are also carets of B, therefore all leaves of A which are also leaves of B will be contained in $A \cap B$. We have the following definition:

Definition 4.2.3. Let (A, B, π) be a forest triple representing $g\tau$. We define an *augmentation* of (A, B, π) as follows. Let U be any finite subtree of $\mathcal{T}_n$, and a any leaf of A and b, with p the attached state of the core, be the leaf representing its image under $g\tau$, that is $b_p = (a)\pi$. Take an expansion of (A, B, π) until we have the subtree U rooted at the leaf a. Simultaneously, we have rooted at b a tree U' which we call the *image of* U *through the the state* p. Since the states in the core are all synchronous, and represent homeomorphisms of $\mathfrak{C}_n$, the number of leaves of U and the number of leaves of U' are the same. We also have a new bijection π' (corresponding to this expansion) between the leaves of the new forest pair. Notice that all the leaves of U' will have states in the core attached to them.

Clearly the new augmented pair is also a representative forest pair for $g\tau$, since it is simply a sequence of elementary expansions of the original forest triple. Following [15] we now go on to 'iterate augmentations', but first we make the following definition which is similar to the definition of an X-component in [2, section 4.1].

Definition 4.2.4. Let (A, B, π) be a representative triple for $g\tau$ and let $u_1, \ldots, u_r$ be a sequence of leaves of A. Such a sequence is called an *iterated augmentation chain* if:

(1) All the u_i are distinct and all the $(u_i)\pi$ in B are distinct (thought of simply as leaves of B).

(2) $(u_i)\pi = u_{i+1_p}$ ($1 \leqslant i \leqslant r-1$ and where p is a state in the core)

Notice in the above definition that the second condition implies that the leaves u_i, $2 \leqslant i \leqslant r$ (ignoring the states attached to them) are leaves of $A \cap B$.

Definition 4.2.5. Let $g\tau$ be represented by the forest triple (A, B, π) and let $u_1, \ldots, u_r$ be an iterated augmentation chain. Let p_i, a state of Core(τ), be the state attached to π_{u_i} for $1 \leqslant i \leqslant r$. We define two types of iterated augmentations of (A, B) as follows.

(1) *Forward iterated augmentation*: Let U be any finite subtree of $\mathcal{T}_n$. Take an augmentation (A', B', π') of (A, B, π) at node u_1 using subtree U. We now have a subtree U' attached to the leaf u_2 in B the range tree. Now perform an augmentation of (A', B', π') at leaf u_2 in A' using the subtree U'. We repeat this process until we have performed an augmentation using the leaf u_r and an appropriate finite tree.

   We call the alterations described above to the forest triple (A, B, π), a *forward iterated augmentation by* U *along* $u_1, , \ldots, u_r$.

(2) *Backward iterated augmentation*: Let U be any finite subtree of $\mathcal{T}_n$. Take an augmentation (A', B', π') of (A, B, π) at node u_r using a subtree U' such that the image of U' through state p_r is U. Such a U' exists since p_r represents a homeomorphism of $\mathfrak{C}_n$. We now have a subtree U attached to the leaf $(u_r)\pi$ in B, the range tree, and a subtree U' attached to u_r in the domain tree. Now perform an augmentation of (A', B', π') at leaf u_{r-1} of A using the subtree U'' such that the image of U'' through the state p_{r-1} is U'. We repeat this process until we have performed an augmentation using the leaf u_1 and appropriate finite tree.

   We call the alterations described above to the forest triple (A, B, π), a *backward iterated augmentation by* U *along* $u_1, \ldots, u_r$.

Remark 4.2.6. Backward and forward iterated augmentations are equivalent, however we distinguish between the two in order to highlight, in the forward case, that the focus is on the first leaf in A, and in the backward case, the focus is on the leaf $(u_r)\pi$ of B.

We now define an imbalance of a tree pair exactly as in [15, p. 10.7]

Definition 4.2.7. Let (A, B, π) be a forest triple representing $g\tau$. Since A and B have the same number of leaves, they have the same number of carets, therefore $A - B = A - (B \cap A)$ and $B - A = B - (A \cap B)$ have the same number of carets. We call this number the *imbalance* of the forest triple (A, B, π) .

Definition 4.2.8. Let (A, B, π) be a forest triple for an element $g\tau$. A leaf of A which is also a leaf of B is called a *neutral leaf*

Let (A, B) be a representative pair which has the minimal number of components of $B - A$ amongst those pairs which have minimal imbalance and minimal number of components of $A - B$. Now using the definition of iterated augmentation in Definition 4.2.5, it is easy to see that lemmas 10.2 to 10.5 of [15] hold in this context. The proof is modified only by applying either a forward or a backward iterated augmentation, depending on if we are working with components of $R - D$ or $D - R$. In order to illustrate how we apply the iterated augmentations as defined above we include both the statements of lemmas and their proofs.

Lemma 4.2.9. *It is impossible to have an iterated augmentation chain $u_1, \ldots, u_r$ so that u_1 is an interior node of B and $(u_r)\pi$ is an interior node of A.*

Proof. Suppose for a contradiction the lemma is false. Since $(u_r)\pi$ is a leaf of B and an interior node of A, then it is the root of a finite n-ary tree of the forest $A - B$. Let U denote the component with root $(u_r)\pi$. Perform a backward iterated augmentation by U along $u_1, \ldots, u_r$. By construction of the backward iterated augmentation, after this process all the leaves of U are now neutral leaves; for $2 \leqslant i \leqslant r$, the subtree $U^{(i)}$ attached to u_i in A is precisely the same subtree attached to u_i in B and so the leaves of these subtrees are now also neutral leaves. Notice that as the states in the core are synchronous all the subtrees $U^{(i)}$ have the same number of carets as U. Finally, if s is the number of carets of U, since u_1 is an interior node of B, the subtree $U^{(1)}$ contributes strictly fewer than s carets to the new caret difference. Since all other components of $A - B$ and $B - A$ are unchanged we have a forest triple with imbalance strictly less than (A, B, π) which is a contradiction. ☐

Lemma 4.2.10. *It is impossible to have an iterated augmentation chain $u_1, \ldots, u_r$ so that u_1 is not a node of B, so that $(u_r)\pi$ is an interior node of A and so that the component of $A - B$ containing u_1 is not the component of $A - B$ whose root is at $(u_r)\pi$).*

Proof. Suppose for a contradiction that the statement of the lemma is false. Let U be the component of $A - B$ rooted at $(u_r)\pi$. Take a backward iterated augmentation by U along $u_1, \ldots, u_r$. By construction, and analogously to the proof the previous lemma, the process is such that the leaves of U become neutral leaves in the resulting forest triple; introduces new neutral leaves which are the children of the u_i $2 \leqslant i \leqslant r$; and adds a subtree with the same number of leaves as U rooted at u_1 to the component of $A - B$ containing u_1. In particular, the resulting forest triple has the same imbalance as (A, B, π), since u_1 is not a node of B, but reduces the number of components of $A - B$ which is a contradiction. ☐

Lemma 4.2.11. *It is impossible to have an iterated augmentation chain $u_1, \ldots, u_r$ so that $(u_r)\pi$ is not a node of A, so that u_1 is an interior node of B and so that the component of $B - A$ containing $(u_r)\pi$ is not the component of $A - B$ whose root is at u_1.*

Proof. The proof is analogous to that of Lemma 4.2.10, only here A and B swap roles, and we perform a forward iterated augmentation by the component of $B - A$ rooted at u_1 along $u_1, \ldots, u_r$. ☐

Lemma 4.2.12. *For each non-trivial component U of $A - B$ there is a unique leaf $\lambda(U)$ of U so that if $r(U)$ is the root of U, then there is an iterated augmentation chain $\lambda(U) = u_1, \ldots, u_s$ with $(u_s)\pi = r(U)$.*

For each non-trivial component V of $B - A$ there is a unique leaf $\lambda(V)$ of V so that if $r(V)$ is the root of V, then there is an iterated augmentation chain $r(V) = u_1, \ldots, u_s$ with $(u_s)\pi = \lambda(V)$.

Proof. We prove only the first statement, since the second is proved similarly.

Since $r(U)$ is the root of a component of $A - B$, it must be a leaf of B, therefore there is a leaf u_1 of A such that $(u_1)\pi = r(U)$ (we shall ignore the state in the core we are processing from for the moment.) Now by Lemmas 4.2.9 and 4.2.10, since u_1 is a leaf of A such that $(u_1)\pi$ is an interior leaf of A, then either u_1 is a leaf of U or it is a leaf of B. If u_1 is a leaf of U we are done, otherwise there is a leaf u_2 of A such that $(u_2)\pi = u_1$. Relabel $u_1 := u_2$ and $u_2 = u_1$.

Inductively assume we have an iterated augmentation chain $u_1, \ldots, u_s$ such that $(u_s)\pi = r(U)$. Once more, by Lemma 4.2.9 and 4.2.10, either u_1 is a leaf of U or it is a leaf of B. If u_1 is a leaf of U we are done.

Assume that u_1 is a leaf of B. Increase the subscript of every element of the iterated augmentation chain by 1, and redefine $u_1 = (u_2)\pi^{-1}$. Arguing as in the previous paragraph either u_1 is a leaf of B or a leaf of U, and so we have increased the length of our chain.

If we enter into a cycle of neutral leaves, since (A, B, π) represents a homeomorphism of $\mathfrak{C}_{n,r}$ and all the states in the core are homeomorphisms of $\mathfrak{C}_n$, this will be a contradiction. Therefore we may exclude this case. Since A is a finite forest, the process must stop eventually. □

We now define repellers, attractors, sources and sinks.

Let $g\tau$ be represented by a forest triple (A, B, π) such that (A, B, π) is a representative pair which has the minimal number of components of $B - A$ amongst those pairs which have minimal imbalance, we have the following definitions:

Definition 4.2.13.

(i) A leaf of A (B) which is $\lambda(U)$ [$\lambda(V)$], according to the notation of Lemma 4.2.12, for some component U [V] of $A - B$ ($B - A$) is called a *repeller* [*attractor*]. The *period* of a repeller u is the value s such that $(u)\pi^s = r(U)$. The period of an attractor, v, is the value, s, such that $(v)\pi^{-s} = r(V)$.

(ii) A leaf of A [B] is called a *source* [*sink*] if it is a leaf of a component U of $A - B$ [V of $B - A$] not equal to the repeller [attractor].

Finally we define what it means for a forest pair to be a revealing pair for an element $g\tau$:

Definition 4.2.14. A forest triple (A, B, π) representing an element $g\tau$ is called a *revealing pair* if every component of $A - B$ has a repeller and every component of $B - A$ has an attractor.

We have the following lemma which is very similar to lemma 4.18 of [2] and is a consequence of the results above.

Lemma 4.2.15. *Let (A, B, π) be a revealing pair for $g\tau$, and let u be a leaf either of A or of B then one of the following holds:*

(i) u is an attractor or repeller.

(ii) u is in an orbit of neutral leaves.

(iii) u is in the iterated augmentation chain of a source or a sink.

(iv) u is a source or a sink.

Remark 4.2.16. It is a consequence of above lemmas that if u is a source then there is an iterated augmentation, $u = u_1, u_2, \ldots, u_s$ such that $(u_s)\pi$ is a leaf of $B - A$ and, since entering into a cycle of neutral leaves will yield a contradiction, is a sink in particular.

Let (A, B, π) be a revealing triple for $g\tau$ and let x a leaf of A be in a finite cycle of neutral leaves of length m. Suppose the orbit is as follows:

$$x := x_1 \to x_{2_{p_1}} \to \ldots \to x_{m_{p_{m-1}}} \to x_{1_{p_m}}$$

Since A_{p_i} is a synchronous transducer, it induces a permutation in $Sym(X_n)$ for inputs of length 1. Let σ_{p_i} be the associated permutation. Let σ be the product $\sigma_{p_1}\sigma_{p_2}\ldots\sigma_{p_m}$, written as a product of disjoint cycles. Let $j \in \{0, 1, \ldots, n-1\}$ and suppose the disjoint cycle containing j (in the cycle decomposition of σ) has length l. This means that $(j)\sigma^l = j$ and l is the minimal value for which this holds. Observe that $(x_1 j)(g\tau)^m = x_1(j)\sigma$. Therefore $(xj)(g\tau)^{lm} = x_1(j)\sigma^l = xj$. Therefore xj is also in a cycle of neutral leaves. Applying induction, we therefore have, for $\Gamma \in \{0, \ldots, n-1\}^*$, that $x\Gamma$ is in a cycle of neutral leaves whenever x is.

We now focus on elements ψ with revealing pairs (A, B, π) such that $A = B$. That is all the leaves of A (and so of B) are in finite orbits of neutral leaves. It is not true that such an element has finite order in general, since the argument above demonstrates that the length of the finite orbits usually increase when we make an elementary expansion. The order of such an element is therefore tied to the order of the synchronous core.

4.2.4 $G_{n,r}$ has infinitely many ϕ-twisted conjugacy classes for $\phi \in \mathcal{BH}_{n,r}$

Let ψ be an element of $\mathcal{BH}_{n,r}$ with revealing pair (A, B, π) such that $A = B$. Let $h \in G_{n,r}$ be arbitrary such that h is represented by the revealing pair (K, L, θ). We can assume that K is a subset of A since it is not hard to come up with algorithms to take an arbitrary pair to a revealing pair. Therefore, by taking expansions, (A, A', θ') is a representative pair for h. Let us now consider what happens when we conjugate ψ by h.

First observe that as h has a trivial core, then by Lemma 2.5.7, we know that the core of $h^{-1}\psi h$ is equivalent to the core of ψ. Let $\varphi := h^{-1}\psi h$. Let x be a leaf of A and let $x' = (x)\psi$ a leaf of A', then:

$$x\psi h = xh\varphi = x'h. \tag{4.4}$$

This shows that φ permutes the leaves of A'. Now suppose that the state attached to x' is q. We show that φ necessarily has the same state q attached to $x'h$. Let $\Gamma \in \mathfrak{C}_n$, then

$$x\Gamma h\varphi = x\Gamma\psi h = x'(\Gamma)A_q h = x'h(\Gamma)A_q = xh\varphi(\Gamma)A_q \tag{4.5}$$

The transducer representing φ after processing the word xh outputs the string $x'h$ and is in some state. Equation 4.5 now shows that this state is equivalent to the state q. This means that (A', A', ρ) is a revealing pair for φ, with ρ defined according to Equation 4.4 above. Moreover if $x \in A$ is in a complete finite orbit of length d under ψ then xh is in a complete finite orbit of length d under φ and vice-versa.

Let ψ be a periodic element as above, that is all revealing pairs for ψ are such that all leaves are in finite cycle of neutral leaves. Let (A, A, π) be a revealing pair for ψ with the additional property that no proper contraction (using the definition of contraction established earlier) of A is a revealing pair for ψ. Assume now that (C, C, θ) $(C \neq A)$ is any other revealing pair which also has this property. Let u be a root of a component, U of $A - C$ (relabelling if necessary). Then u is a leaf of C and is in a finite cycle of neutral leaves of C. However this means that there is a contraction of A such that all the leaves are in a finite cycle of neutral leaves. This is a contradiction. Therefore there is only one revealing pair with this property and so all revealing pairs for ψ must be expansions of it. Let us call this revealing pair the *minimal revealing pair*. We frame the above observations in the proposition below:

Proposition 4.2.17. *Let ψ and φ be elements of $\mathcal{BH}_{n,r}$ with minimal revealing pairs (A, A, π) and (B, B, θ) respectively. Let $C(A) = \{C(x_i) : x_i$ is a leaf of $A\}$ be the collection of finite cycles of neutral leaves of A (one for each cycle, that is if x_i and x_j are in the same orbit we include only $C(x_i)$), likewise define $C(B) = \{C(y_i) : y_i$ is a leaf of $B\}$. Then ψ and φ are conjugate if and only if there exists partitions $\mathcal{P}$ of $C(A)$ and $\mathcal{Q}$ of $C(B)$ such that the following hold:*

(i) there is a bijection, f, between $\mathcal{P}$ and $\mathcal{Q}$ such that $f(P) = Q$ if and only if

- *(a) the sum of the cycle lengths in P is congruent to the corresponding sum in Q modulo $n - 1$,*
- *(b) for all $C(x_i)$ in P there exists $C(y_i)$ in Q such that if $l(x_i)$ and $l(y_i)$ are the lengths of $C(x_i)$ and $C(y_i)$ respectively, then there are positive integers δ_1 and δ_2 which can be factored as products of elements in the set $\{1, 2, \ldots, n\}$ such that $\delta_1 l(x_i) = \delta_2 l(y_i)$; likewise for each $C(y_i)$ there exists such a $C(x_i)$,*

(ii) these partitions are realisable, meaning that there are expansions of (A, A, π) and (B, B, θ) such that for each cycle under the leaves of an element $P \in \mathcal{P}$, there is a unique cycle under the leaves of an element $Q \in \mathcal{Q}$ having the same length and same labelling of states (up to cyclic reshuffling), moreover the number of cycles under the leaves of P is equal to the number of cycles under the leaves of Q.

Proof. The forward implication follows immediately from the observations above. Let h be the conjugator such that $h^{-1}\psi h = \varphi$, and let (K, L, ρ) be a pair representing h. Let (A', A', π') be a revealing pair for ψ (and so an expansion of (A, A)) such that $K \subset A'$. Now by expanding appropriately we may obtain a pair (A, B', ρ') representing h. As observed earlier the pair (A', B', ρ') representing h is such that (B', B', θ') is a revealing pair for φ, with θ' defined according to Equation 4.4. Note that (B', B', θ') is an expansion of (B, B, θ). Take a cycle $C(x_1)$ in A and take all the cycles underneath it (in (A', A', π')), these correspond under h, to cycles (in (B', B', θ')

under a set of leaves $C(y_1), C(y_2), \ldots, C(y_k)$ of B. The last sentence holds since by Equation 4.4, φ permutes the elements $B' = A'h$ and (B, B, θ) is a minimal revealing pair for φ. Now as all the cycles in (B', B', θ') under the leaves $C(y_1), C(y_2), \ldots, C(y_k)$, correspond by h^{-1} to cycles in (A', A', π'), we may now cycle back and forth until we have a set P of cycles in A and corresponding set Q of cycles in B such that the cycles under P (in (A', A', π')) are in one-to-one correspondence to the cycles under Q (in (B', B', θ')) by h. Note that since we can expand along P and Q such that they have the same number of leaves, then the sum of cycle lengths in P must be congruent to the corresponding sum in Q modulo $n-1$. Moreover, observe that when we take a simple expansion along the leaves of a cycle, the new cycles obtained have lengths which are multiples by $1 \leqslant k \leqslant n$ of the original by the discussion following Remark 4.2.16. Now we may repeat this process beginning with the next cycle, $C(x_2) \in C(A)$ not already in P. Observe that the cycles under $C(x_2)$ in (A', A', π') cannot correspond under h to cycles in Q, as all cycles in Q correspond under h^{-1} to a cycle in P. We may thus repeat the above process to obtain a set of cycles P' and Q' such that the cycles under P (in (A', A', π')) are in one-to-one correspondence to the cycles under Q (in (B', B', θ')) by h. Carrying on in this way we obtain a partition $\mathcal{P}$ of $C(A)$ and $\mathcal{Q}$ of $C(B)$ and a map $f : \mathcal{P} \to \mathcal{Q}$ which maps $P \in \mathcal{P}$ to the corresponding $Q \in \mathcal{Q}$, satisfying Parts (i)a and (i)b. Moreover these partitions are realised by the expansions (A', A', π') and (B', B', θ') of (A, A, π) and (B, B, θ) respectively.

The backward implication is likewise straightforward. Take expansions (A', A', π') of (A, A, π) and (B', B', θ') of (B, B, θ) which realise the partitions $\mathcal{P}$ and $\mathcal{Q}$. Then as every cycle of leaves of A' corresponds to a cycle of leaves in B' of the same length and with the same labelling of states in B' we can define a bijection $\rho : A' \to B'$ by mapping leaves in a cycle of leaves of A' to their corresponding leaves of B' ensuring the states match. It is easy to check that $h = (A', B', \rho)$ is such that $h^{-1}\psi h = \varphi$. □

The above proposition is sufficient to show that $G_{n,r}$ has the infinitely many ϕ-twisted conjugacy classes for $\phi \in \mathcal{BH}_{n,r}$.

First recall Example 1.7.39 stating that for a synchronous finite transducer $A_{q_0} = \langle X_I, X_O, Q, \lambda, \pi \rangle_{q_0}$ the inverse transducer is given as follows, whenever there is a label on arrow $i|j$ replace it with the label $j|i$. We shall use a " $'$ " on the states and transitions of the inverse transducer to distinguish them from the states of the original in the synchronous case. Now let $\tau = \langle \dot{\mathbf{r}}, X_n, R_\tau, S_\tau, \pi_\tau, \lambda_\tau \rangle \in \mathcal{BH}_{n,r}$ a transducer representing a homeomorphism of $\mathfrak{C}_{n,r}$. By definition $B = \mathrm{Core}(\tau) \in \mathcal{H}_n$. Let p and q be any two states in B. Consider the product of the initial transducers B_p and $B_{q'} := B_q{}^{-1}$. These two initial transducers are synchronizing at the same level as the core, say the synchronizing level is m. Observe that by the proof of Lemma 2.4.1, after reading a word of length m trough the state (p, q') of $B_p * B_{q'}$, the resulting state is ω-equivalent to the identity map and so $B_p * B_g$ is an element of $G_{n,1}$.

Notice that $\mathrm{Core}(\tau^{-1}) = \mathrm{Core}(\tau)^{-1}$, this follows since the map from $\mathrm{Aut}(G_{n,r})$ to $\mathcal{O}_{n,r}$ mapping $\tau \to \mathrm{Core}(\tau)$ is a homomorphism. Alternatively, we can see this by considering a forest triple for τ as in Subsection 4.2.2. Recall that the range forest has attached to its leaves initial transducers initialised at some state in the core of τ. The inverse map is then given in the usual way by swapping the range and the domain forests, and whenever we had A_q attached to a leaf, we replace it by $A_q{}^{-1}$.

Let s be a prime number, we construct an element $\psi \in \mathcal{BH}_{n,r}$ with a revealing pair which contains a cycle of length s and all other cycles of length 1. Let $\tau \in \mathcal{BH}_{n,r}$ and take an element $\psi' \in G_{n,r}$ with a revealing pair which contains a cycle of length s and with all other cycles having length 1. Suppose ψ' has j leaves. Decorate the j leaves of the range forest with states in the core of τ. Let (A, A, ρ) be the resulting forest triple, and let $\psi \in \mathcal{BH}_{n,r}$ be the resulting homeomorphism of $\mathfrak{C}_{n,r}$, we show that $\psi = g\tau$, for some $g \in G_{n,r}$, by showing the product $\psi\tau^{-1} \in G_{n,r}$. We compute this product as we would in $G_{n,r}$ using the forest triple for τ as in Subsection 4.2.2 and the definition of expansion and contractions (Definition 4.2.1). Take expansions so that the range forest of ψ and domain forest of τ^{-1} match up. Suppose the forest triple for ψ and τ^{-1} are given by (A', A', ρ') and (A', B, σ). The forest triple for $\psi\tau^{-1}$ is given by (A', B, δ) where δ is defined as follows. If a leaf y of B is mapped to a leaf $z_{\tau_q{}^{-1}}$ by σ then, if x is the leaf of A such that $(x)\rho = y_p$, we have $(x)\delta = z_{\tau_p * \tau_q{}^{-1}}$. Since we have demonstrated that $\tau_p * \tau_q{}^{-1}$ is an element of $G_{n,1}$, we see that $\psi\tau^{-1}$ is an element of $G_{n,r}$. Therefore $\psi = g\tau$ for some $g \in G_{n,r}$.

Let t be a prime distinct from s. We may likewise construct an element $\varphi = h\tau$, $h \in G_{n,r}$, with revealing pair (C, C, γ) containing a cycle of length t and all other cycles of length 1. Now for s and t larger than n, by Proposition 4.2.17 there is no $f \in G_{n,r}$ satisfying the equation:

$$f^{-1}h\tau f\tau^{-1} = g$$

Since this would mean that

$$f^{-1}h\tau f = g\tau$$

However, any revealing pair of ψ has a cycle of length a multiple of s, but all revealing pairs for φ have cycle lengths equal to 1 or divisible by some element of $\{2, \dots, n\} \sqcup \{t\}$. Therefore ψ and φ are in different $\hat{\tau}$-twisted conjugacy classes (where $\hat{\tau}$ denotes conjugation by τ^{-1}). Now, as there are infinitely many primes there are infinitely many $\hat{\tau}$-twisted conjugacy classes. Since $\hat{\tau}$ was arbitrary in $\mathcal{BH}_{n,r}$, this holds for every $\hat{\tau}$ in $\mathcal{BH}_{n,r}$. Therefore we have the following result.

Corollary 4.2.18. *The group $G_{n,r}$ has infinitely many ϕ-twisted conjugacy classes for $\phi \in \mathcal{BH}_{n,r}$.*

Now fix $\tau = \langle X_n, Q_\tau, \pi_\tau, \lambda_\tau \rangle \in \mathcal{H}_n$ an element of finite order m. By the discussion in Section 3.5.3 we know that the level k dual of τ, for some k greater than the synchronizing level of τ, is a disjoint union of cycles such that the output of each cycle on any input word in the states of τ, is a cyclic rotation of some word in Q_τ^+ independent of the input. Fix $k \in \mathbb{N}$ such that $\tau_k^\vee$ is the zero of $\langle \tau^\vee \rangle_+$. We assume that we are working with the minimal transducer under ω-equivalence representing the level k dual of τ. We have the following lemma.

Lemma 4.2.19. *Let ψ be an element of $\mathcal{BH}_{n,r}$ with revealing pair (A, B, θ) and $\mathrm{Core}(\psi) = \tau$. Let $\{C(x_i) | 1 \leqslant i \leqslant r\}$ be the cycles of neutral leaves of (A, B, θ), then there is a number $M \in \mathbb{N}$, which is computable and depends on (A, B, θ) and τ, such that for any expansion (A', B', θ') of (A, B, θ) the length of any cycle of neutral leaves of is bounded by M.*

Proof. First observe that since m is the order of τ, then ψ^m is an element of $G_{n,r}$. Moreover since $\psi^m \in G_{n,r}$, there is a number l, which is computable, such that for any representative pair of ψ^m, all cycle of neutral leaves of ψ^{ml} have length exactly 1. Therefore all cycles of neutral leaves of any expansion of (A', B', θ') must have length bounded by ml.

□

The rest of the discussion will be focused on the group $G_{2,1} = V$, as noted above here we have a much simpler description for $\mathcal{H}_2$ and so we are able to go further: we solve the ϕ-twisted conjugacy problem for $\phi \in \mathcal{BH}_{2,1}$. The overlap with Subsection 4.2.3 above, serves to illustrate the ideas of this subsection for a given element of $\mathcal{BH}_{2,1}$.

4.3 Thompson's group $V = G_{2,1}$

Let $\mathcal{R}$ be the element of $\mathcal{H}_2$, which is the single state transducer swapping 0 and 1. Note that as $\mathcal{R}$ is also an element of $\mathcal{BH}_{2,1}$. We consider elements of V as acting on Cantor space $\mathfrak{C}$. In order the solve the ϕ-twisted conjugacy problem in V for $\phi \in \mathcal{BH}_{2,1}$, given $g, f \in V$ we need to find an element $h \in V$ such that:

$$g = h^{-1}f(k^{-1}hk), \text{ if } \mathrm{Core}(\phi) = \mathrm{id} \tag{4.6}$$

$$g = h^{-1}f(k^{-1}\mathcal{R}h\mathcal{R}k), \text{ otherwise.} \tag{4.7}$$

Rearranging slightly, the above becomes:

$$gk^{-1} = h^{-1}(fk^{-1})h \tag{4.8}$$

$$gk^{-1}\mathcal{R} = h^{-1}(fk^{-1}\mathcal{R})h \tag{4.9}$$

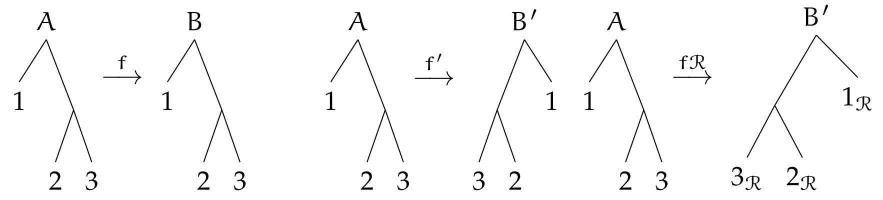
Equation (4.8) is solved by the methods given in [49]. Therefore to solve the ϕ-twisted conjugacy problem in V for $\phi \in \mathcal{BH}_{2,1}$, it suffices to show that there is an algorithm which determines whether two elements $f\mathcal{R}$, $g\mathcal{R}$, $f, g \in V$ are conjugate by an element of V.

4.3.1 Revealing pairs for elements of $V\mathcal{R}$

Here we illustrate Subsection 4.2.3 with the specific instance of $V\mathcal{R}$.

Let $f \in V$ and let (A, B, σ) be a representation of f by a forest triple where σ is a bijection between the leaves of A and B. Let $f' = (A, B', \sigma')$ be a new element of V, where B' is the reflection of B about the root such that each leaf of B' preserves its labelling of the leaves. Let σ' be the bijection from the leaves of A to the leaves of B with this relabelling. Then $f\mathcal{R}$ is given by appending $\mathcal{R}$ to every leaf of B', by which we mean that the full binary tree beneath that node is being acted upon by $\mathcal{R}$. We call the tree pair (A, B') the tree pair associated to $f\mathcal{R}$. This is illustrated in the example below:

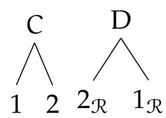
Example 4.3.1. Constructing the product $f\mathcal{R}$ given an element $f \in V$.



Definition 4.3.2. For an arbitrary finite binary tree A we shall call the reflection of A about the root, the *flip* of A.

Definition 4.3.3. In this context, an *elementary expansion* of a triple (A, B, σ) representing an element $f\mathcal{R}$, is an addition of a caret to a leaf of A and to its image point such that when we renumber the leaves of A, if i, j are the labels for the leaves of the added caret (ordering left to right), then $j_{\mathcal{R}}$ and $i_{\mathcal{R}}$ are the labels for the added caret in B. We call a tree pair (C, D, ρ) a *contraction* of (A, B, σ) if there is an elementary expansion taking (C, D, ρ) to (A, B, σ). This is simply a restatement of the definition we had earlier. A forest triple (A', B', σ') is called an *expansion (contraction)* of (A, B, σ) if it can be obtained from (A, B, σ) by a sequence of elementary expansions (contractions).

Considering Example 4.3.1, we can see the triple (A, B', σ) (where σ is given by the numbering on the leaves) is an elementary expansion of the following tree pair:



We now discuss methods for moving from one revealing pair to another as in section 3.5 of [49].

4.3.2 New Rollings

Following [49] we make the following definitions.

Definition 4.3.4. Let U be a binary tree, and let ν be a node of U. Let U_ν be the subtree of U rooted at ν, then we denote by U^ν the caret difference $U - U_\nu$, that is U^ν is the subtree U with the nodes under the vertex ν deleted. Notice that ν is a leaf of U^ν.

The definition above should be compared with Definition 1.2.22.

Definition 4.3.5. Let (A, B) be a revealing pair associated to an element $f\mathcal{R}$. A sequence of leaves is called *cancelling chain* if it satisfies one of the following criteria:

(i) the sequence of leaves is the set of all leaves of A in the forward orbit of a single repeller,

(ii) the sequence of leaves is the set of all leaves in the forward orbit of a single source,

(iii) the sequence of leaves is a finite cycle of neutral leaves,

(iv) the sequence of leaves is the set of all leaves in the backward orbit of a single attractor.

Definition 4.3.6. Given a cancelling chain of type (i) [(iv)] we define a *cancelling tree* as follows. Let U be a component of $A - B$ [$B - A$], with root u and repeller [attractor] s, let ν be a node in the path from the root u to the repeller [attractor] s. Then U^{ν} is a cancelling tree. If $\nu \neq s$, U^{ν} is called *a proper cancelling tree*.

For a chain of type (iii), we identify two cases. If our finite cycle of leaves is odd, then a cancelling tree will be any finite tree which is symmetric about the root. If the cycle is even then, a cancelling tree is any finite tree.

Finally for a cancelling chain of type (ii) a cancelling tree is any finite tree.

Definition 4.3.7. Given a cancelling tree for a chain of type (i) & (iv), where ν is chosen to be the first node in the path from the root u to the repeller/attractor s, then we call such a tree a *small cancelling tree* for s.

For a cancelling chain of type (ii) or (iii), a small cancelling tree is a single caret (which is symmetric with respect to the root).

We now define a new set of rollings.

Definition 4.3.8. We say that a tree pair (A', B') is a *single rolling* of type E of (A, B), if it is obtained from (A, B) by one of the following ways:

(a) Adding a cancelling tree to each of the leaves $u_1, u_2, \dots, u_{2k+1}$ of a cancelling chain of type (iii) in A, where the cycle of leaves has odd length, and to B at $(u_1)f\mathcal{R}, (u_2)f\mathcal{R}, \dots, (u_r)f\mathcal{R}$.

(b) For $u_1, u_2, \dots, u_{2k}$ a cancelling chain of type (iii), adding a cancelling tree to each of the leaves $u_1, u_3, \dots, u_{2k-1}$ and the flip of the cancelling tree to each of the leaves $u_2, u_4, \dots, u_{2k}$ in A. Furthermore for each tree added to a leaf u_i we add the flip of that tree to the leaf $(u_i)f\mathcal{R}$ of B.

(c) Adding a cancelling tree to all the odd numbered leaves and the flip of the cancelling tree to all the even numbered leaves of a cancelling chain $u_1, u_2, \dots, u_n$ of type (ii). Furthermore for each tree added to a leaf u_i we add the flip of that tree to the leaf $(u_i)f\mathcal{R}$ of B.

If the cancelling tree in all the above cases is small, then the rolling is called a *small single rolling of type E*.

We say that (A', B') is a *single rolling of type I* if it is obtained from (A, B) in one of the following ways:

(a) By adding the flip of a proper cancelling tree to the last leaf in A of a chain of type (i), and working backwards through the leaves alternating between adding the flip of the cancelling tree and the cancelling tree. i.e if we attach the proper cancelling tree to u_i, then we add the flip of the tree to u_{i-1}. Furthermore for each tree added to a leaf u_i we add the flip of that tree to the leaf $(u_i)f\mathcal{R}$ of B.

(b) By adding a proper cancelling tree to the first leaf in A of a cancelling chain of type (iv), moving forwards through the leaves, alternating between adding the flip of the cancelling tree and the cancelling tree i.e if we attach the proper cancelling tree to u_i, then we add the flip of the tree to u_{i+1}. Furthermore for each tree added to a leaf u_i we add the flip of that tree to the leaf $(u_i)f\mathcal{R}$ of B.

Once more if the cancelling tree is small, then the rolling is called a *small single rolling of type I*.

The tree pair (A', B') is called a *single rolling of type II*, if it is obtained from (A, B) in one of the following ways:

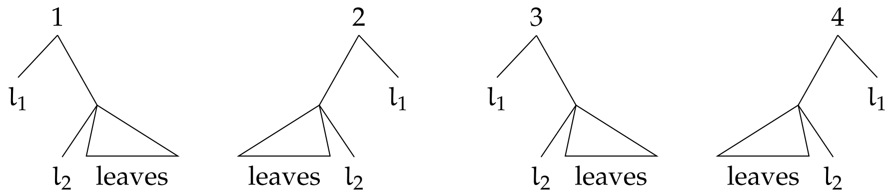
(a) By adding a component U of $A - B$ to the leaf $(u_n)f\mathcal{R}$ and the flip of this component to the leaf u_n, where $u_1, u_2, \dots, u_n$ is a cancelling chain of type (i) and u_1 is the unique repelling leaf of U.

(b) By adding a component W of $B - A$ to the leaf u_1 of A and the flip of the component W to the leaf $(u_1)f\mathcal{R}$, where $u_1, u_2, \dots, u_n$ is a cancelling chain of type (iv) and where $(u_n)f\mathcal{R}$ is the unique attracting leaf of W.

In this case a single rolling of type II will also be called a *small single rolling of type II*.

Remark 4.3.9. Notice that the tree pair obtained by applying a rolling, is by definition an expansion of the original tree pair, therefore we do not change the element $f\mathcal{R}$, $f \in V$, defined by a tree pair by applying a rolling.

Below we illustrate an elementary rolling of type E to a finite cycle of leaves of length 4 using an asymmetric finite tree.



The fact that an application of these rollings produce a revealing pair, follows by making slight modifications to the proofs given in Section 3.5 of [49] as we see below.

Lemma 4.3.10. *If* (A, B, σ) *is such that* (A, B) *a revealing pair for an element* $f\mathcal{R}$, $f \in V$, *and* (A', B', σ') *is obtained from* (A, B) *by a small single rolling, then* (A', B') *is a also a revealing pair for* $f\mathcal{R}$.

Proof. We consider each rolling separately.

(i) For a rolling of type E applied to a cycle of neutral leaves of odd or even length. By definition this has no effect on the components of $A - B$ or $B - A$ and so these still have unique attractors and repellers. Furthermore, in both cases, after the rolling is performed, by construction, the new leaves introduced are all neutral leaves.

For a rolling of type E applied along a chain of type (ii), the leaves added along the neutral leaves of this chain, are, by construction leaves of A' and B'. A caret is added to the source in A and to the sink in B; attractors and repeller are unaffected as are the number of components of $A - B$.

(ii) For a rolling of type I applied along a cancelling chain of type (i). Let U be the component of $A - B$ containing the repeller, u be the root of U and s, the repelling leaf and $\Gamma = \alpha\overline{\Gamma}$ denote the path from u to S in U, here $\alpha = 0, 1$. Let $s = u_1, u_2, \ldots, u_n$ be the iterated augmentation chain such that $(u_n)\sigma' = u$. By construction of this rolling, the leaves of the cancelling tree attached to the neutral leaves of this iterated augmentation chain, are still neutral leaves of (A', B'). Therefore $A - B = A' - B'$. After this rolling the component U is transformed into a new component U' as follows: U' is the subtree of U now rooted at $u\alpha$ but with the cancelling tree U^{α} or its flip attached to the leaf s. This is because the subtree U^{α} of U rooted at u is now a component of A' and B', and $U - U^{\alpha} = U_{\alpha}$ and the definition of the rolling.

Finally consider the leaf $s\overline{\alpha}$ (where $\overline{\alpha} \in \{0, 1\}$ is either equal to or not equal to alpha depending on whether a U^{α} or its flip is attached to s). By construction of the rolling o type I, and definition of the map σ', we must have that $(s\overline{\alpha})(\sigma')^{(n)} = u\alpha$. Therefore $s\overline{\alpha}$ is the unique repeller of the component U' of $A' - B'$.

We now consider the case of a small single rolling of type I applied along a chain of type (iv) . This is exactly the dual of the previous case and is proved analogously. In this case also, if w is the root of the component W, and a is the attractor of this component such that $\Lambda = \beta\overline{\Lambda}$ is the path from w to a and $\beta \in \{0, 1\}$ is the first letter of Λ, then $a\overline{\beta}$ is the new attractor of the component W' constructed as in the previous paragraph. We note once more that $\overline{\beta} = \beta$ precisely when we attach W^{α} and not its flip to a.

(iii) Finally we consider the case of a small single rolling of type II. First consider the case of a small single rolling of type II applied to a cancelling chain of type (i).

Let U be a component of $A - B$ with repelling leaf s, and iterated augmentation chain $s = x_1, x_2, , \ldots, x_r$ such that $(x_r)\sigma$ is the root of U. Assume that the rolling of type II is performed along this iterated augmentation chain using the component U. By definition of a small single rolling of type II, the leaves of the subtree rooted at $(x_r)\sigma$ are now neutral leaves of A' and B'. However since x_r is a neutral leaf of A, there is a component U', the flip of U, rooted at x_r of $A' - B'$. All other components of $A' - B'$ and $B'' - A'$ are precisely as in $A - B$ and $B - A$. If Γ is the path from the root of U to the repelling leaf s, then the leaf s' of the path Γ' (which is Γ with the zeroes and ones swapped) from the root of U', is the repelling leaf of U'. Since, by construction, $(s')\sigma' = s$.

The case of a small single rolling of type II applied to a cancelling chain of type (iv) is analogous to the previous case.

□

4.3.3 Effects Of New Rollings

We consider the effects of the new rollings to the length of finite cycles, source-sink chains, and attracting and repelling orbits. We also consider the effect the rollings have on the path Γ from the root of a component of $A - B$ or $B - A$ to the attractor or repeller in that component.

The following lemma follows easily from the definition of the rollings.

Lemma 4.3.11. *Let (A', B') a revealing pair obtained from a revealing pair (A, B) associated to an element $f\mathcal{R}$ by an application of a small single rolling of one of the three types given above, then the following hold:*

Type E.(i) *If (A', B') is obtained by a small single rolling of type E applied to a finite cycle of odd leaves $a_1 \ldots a_{2k+1}$, then each child of the a_i is in a finite cycle of leaves of length $2(2k+1)$. i.e we double the length of the cycle and increase by 2 the number cycles of length $2(2k+1)$, and decrease the number of cycle of length $2k+1$ by 1.*

If the finite cycle of leaves is even of length $2k$, then each child of a leaf in the cycle yields a new cycle of leaves of the same length i.e we simply increase the number cycles of length $2k$ by 1.

Type E.(ii) *If (A', B') is obtained by a small single rolling of type E applied to the cancelling chain of a source, then we increase the number of sources (and so sinks) of that period by 1.*

Type I. (i) *If we obtain (A', B') by applying a small single rolling of type I to the cancelling chain of a repelling leaf u of a component U of $A - B$ with root r, then we have two cases.*

If the length of the chain is odd and $\Gamma = \alpha\Lambda$ ($\alpha = 0, 1$) is the path from the root r to the leaf u. The new repeller path is $\Lambda\overline{\alpha}$ ($\overline{\alpha} = (\alpha)\mathcal{R}$ is zero if α is 1 and vice-versa) and the length of the repeller path stays the same. Sources in the new component U' of $A - B$ which were also sources of U are unaffected by the rolling. If z is a source of U which is not in U' then we can assume that $z = r\rho$ where ρ does not begin with α. Let $\overline{\rho} = (\rho)\mathcal{R}$, then $z' = u\overline{\rho}$ is a source of U'. Moreover the length of the source sink chain of z' is the sum of the length of the repelling path and the length of the source sink chain of z.

If the length of the chain is even and $\Gamma = \alpha\Lambda$ (α is 0 or 1) is the repelling path. The new repelling path is given by $\Lambda\alpha$, and the length of the repeller path stays the same. Sources in the new component U' of $A - B$ which were also sources U are unaffected by the rolling. If z is a source of U which is not in U' then we can assume that $z = r\rho$ where ρ does not begin with α. Then $z' = \rho u$ is a source of U'. Moreover the length of the source sink chain of z' is the sum of the length of the repelling path and the length of the source sink chain of z.

Type I. (ii) *If we obtain (A', B') by applying a small single rolling of type I to the cancelling chain of an attracting leaf $(u)f\mathcal{R}$ of a component W of $B - A$ with root r, then this case is analogous to the previous one.*

Type II.(i) *Let U be the component of $A - B$ with which the small rolling of type II is applied to obtain (A', B'). Let u be the unique repelling leaf of U and Γ be the path from the root r of U to u. The new component U' of $A' - B'$ is the flip of U, with path $\overline{\Gamma}$ from the root to the new repelling leaf u'. Furthermore, the length of the new iterated chain corresponding to the repelling leaf u' is the same as before. All sources U' corresponding to the sources in U, have lengths of their chains one greater than their counterparts in U.*

Type II.(ii) *Let W be the component of $B - A$ with which the small rolling of type II is done to obtain (A', B'). Let u be the unique attracting leaf in U, and Γ be the path from the root r of U to u. The new* *component W' of $B' - A'$ is the flip of W, with path $\overline{\Gamma}$ from the root to the new attracting leaf $(u')f\mathcal{R}$. Furthermore, the length of the new iterated chain corresponding to the attracting leaf $(u')f\mathcal{R}$ is the same as before. All sinks W' corresponding to the sinks in W have lengths of their chains one greater than their counterparts in W.*

Proof.

Type E.(i) Let $u_1, \dots, u_{2k+1}$ be the finite cycle of neutral leaves to which we apply the small single rolling of type E. Notice that in B all these leaves have the transducer $\mathcal{R}$ attached to them. Moreover observe that $\mathcal{R}^{2k+1} = \mathcal{R}$ and $\mathcal{R}^{2(2k+1)} = \mathrm{id}$. Therefore for $i = 0, 1$ $u_1 i$ belongs to the following cycle of neutral leaves in (A', B') $v_1 := u_1 i, v_2 := u_2(i)\mathcal{R}, v_3 := u_3 i, \dots v_{4k+2} = u_{2k+1}(i)\mathcal{R}$. Hence the number of cycles of neutral leaves of length $2k+1$ is one less than in (A, B) and we have a new cycle of neutral leaves of length $(4k+2)$ as required.

By a similar argument, since $\mathcal{R}$ to an even power is the identity transducer, applying a small single rolling of type E to a cycle of neutral leaves of even length, will produce two cycles of neutral leaves of equal length to the first.

Type E.(ii) Let $u_1, , \dots, u_k$ be a cancelling chain of type (ii) with which we perform a small single rolling of type E where u_1 is a source and u_k is a sink. Let $i = 0, 1$, then in (A', B'), $u_1 i$ is still a source and belongs to the cancelling chain, $v_1 := u_1 i, v_2 := u_2(i)\mathcal{R}, \dots v_k := u_k(i)\mathcal{R}^{k-1}$, since u_k is a leaf of a component of $B - A$ not equal to the attractor, then $u_k i$ is a leaf of the corresponding component in $B' - A'$ and is also not equal to the attractor. This demonstrates the lemma.

Type I. (i) Let $u := u_1, u_2, \dots, u_{2k+1}$ be the cancelling chain of a repelling leaf to which we apply a rolling of type I. Let U be the component of $A - B$ containing the repeller u, and let $\Gamma = \alpha\Lambda$ ($\alpha = 0, 1$) be the path from the root r of U to u. Let $T := U^{\alpha}$, notice that $r\alpha$ is a leaf of T. By definition of a small single rolling of Type I, in (A', B'), the component U' corresponding to U, has the flip of T rooted at u in A', and in B' the leaf r of B now has a copy of T rooted at r. Consider the leaf $u\overline{\alpha}$ of U', this belongs to a cancelling chain $v_1 = u\overline{\alpha}, v_2 := u_2\alpha, \dots v_{2k+1} = u_{2k+1}\overline{\alpha}$. Note that since u_{2k+1} under $f\mathcal{R}$ is mapped to r, then $(u_{2k+1}\overline{\alpha})f\mathcal{R} = r(\overline{\alpha})f\mathcal{R} = r\alpha$. Therefore since $r\alpha$ is a parent of u, $u\overline{\alpha}$ is the repeller of U'. The repeller path is now $\Lambda\overline{\alpha}$, and is of the same length as Γ, moreover the length of the cancelling chain is unchanged.

Sources of U' which were also sources in U, by definition are not affected by a small single rolling of Type I. Let $z = r\rho$ where the first letter of ρ is not equal to α, then we claim that z belongs to the cancelling chain of the source $z' = u\overline{\rho}$. Notice that since the flip of T is rooted at u, then z' is a leaf of U' not equal to the source. Consider the first $2k+1$ members of the cancelling chain of z', these are $z_1 = u\overline{\rho}, z_2 = u_2\rho, \dots z_{2k+1} = u\overline{\rho}$. Now observe that $(u\overline{\rho})f\mathcal{R} = r\rho$. Hence after the first $2k+1$ members, the remaining members of the cancelling chain are precisely the cancelling chain of z in (A, B). This proves the lemma in this case.

For a cancelling chain of type II of even length the proof proceeds analogously. The differences arising since in the component U' of $A' - B'$, corresponding to U, we have a copy of $T = U^{\alpha}$ rooted at u and not its flip.

Type I. (ii) The attractor case is proved analogously to the repeller case.

Type II.(i) Let U be the component of $A - B$ with which the small single rolling of type II is performed. Let u be the unique repelling leaf of U, and let $u = u_1, \dots, u_n$ be the iterated augmentation chain of U such that $(u_n)f\mathcal{R}$ is the root r of U. Now there is a component U' of $A' - B'$ rooted at U which is the flip of U, moreover all the leaves of the component U of $A - B$ are now neutral leaves of (A', B'). Let Γ be the path from r to the repelling leaf u. We show that the leaf $v = u_n\overline{\Gamma}$ is the repelling of U'. Observe that by construction $(u_n\overline{\Gamma})f\mathcal{R} = u_1$, hence the iterated augmentation chain for v is $v = v_1, u_1, u_2, \dots, u_{n-1}$, note that u_1 is a neutral leaf of (A', B'). Now $(u_{n-1})f\mathcal{R} = u_n$ is the root of U'. For the second half of this point, let z be a source of U in (A, B). Then there is a path $\Delta \neq \Gamma$ from r to z. The leaf z is a neutral leaf of (A', B'), however the leaf $z' = u_n\overline{\Delta}$ is a source. Observe that $(z')f\mathcal{R} = z$, thus if

$z = z_1, z_2, \ldots, z_m$ is the iterated augmentation chain of z, then $z', z_1, \ldots, z_m$ is the iterated augmentation chain of z_m. Notice that z_m remains a sink in (A', B').

Type II.(ii) This case is analogous to the previous one so we omit its proof.

□

The lemma below follows again from the definitions, and gives a relation between rollings of type I and type II. It is identical to claim 17 of [49].

Lemma 4.3.12. *Let (A, B) be a revealing pair for $f\mathcal{R}$, and U be a component of $A - B$ with root r and repeller u. Let n be the length of the iterated augmentation chain containing u and let Γ be a finite non-empty word in 0's and 1's of length k such that $(u)(f\mathcal{R})^n\Gamma = u$. Then an application of k small rollings of type I on U has the same result as an application of n small rollings of type II on U.*

Proof. This is a straight-forward induction argument using the definition of the rollings and Lemma 4.3.11. □

4.3.4 Conjugacy in $V\mathcal{R}$

Let $f\mathcal{R}$ and h be elements of $V\mathcal{R}$ and V respectively with associated tree pairs, (A, B), (K, L) such that $K \subseteq A$ and $K \subseteq B$. Making elementary expansions to the tree pair (K, L) we have (A, A') and (B, B') are also tree pairs associated to h. In order distinguish between tree pairs for elements of V and $V\mathcal{R}$, for a finite tree D we shall let $D_{\mathcal{R}}$ denote the same finite tree with $\mathcal{R}$ attached to each leaf to represent its action on the tree beneath those leaves. Observe that for an element $g \in V$ with associated tree pair (D, E), for a leaf $z_{\mathcal{R}} \in D_{\mathcal{R}}$ there is a corresponding leaf $z \in D$ such that $(z)g_{\mathcal{R}} = (z_{\mathcal{R}})g$.

The following results are proved in detail in [49] for elements f, h of V. The proofs we give below for elements of $V\mathcal{R}$ are modifications of these.

Claim 4.3.13. *The pair $(A', B'_{\mathcal{R}})$ is a tree pair associated with $hf\mathcal{R}h^{-1}$*

Proof. Let x be a leaf of A' then there is a leaf y of A such that $(y)h = x$. Since y is a leaf of A, $(y)f\mathcal{R} = z_{\mathcal{R}}$ is a leaf of $B_{\mathcal{R}}$. Hence, there is a leaf z of B corresponding to $z_{\mathcal{R}}$ ($z \leftrightarrow z_{\mathcal{R}}$), for which $(z)h$ is a leaf of B' and so $(z)h_{\mathcal{R}} = (z_{\mathcal{R}})h$ is a leaf of $B'_{\mathcal{R}}$. Therefore $(x)hf\mathcal{R}h^{-1}$ is a leaf of $B'_{\mathcal{R}}$

Let $z_{\mathcal{R}}$ a leaf of $B'_{\mathcal{R}}$. There is a leaf $z \leftrightarrow z_{\mathcal{R}}$ of B'. Let y be the leaf of B such that $(y)h = z$, then $y_{\mathcal{R}}$ of $B_{\mathcal{R}}$ is such that $(y_{\mathcal{R}})h = z_{\mathcal{R}}$. Let x a leaf of A be such that $(x)f\mathcal{R} = y_{\mathcal{R}}$. Since $x \in A$, there is a leaf w of A' such that $(w)h = x$. We now have: $z_{\mathcal{R}} = (y_{\mathcal{R}})h = (x)hf\mathcal{R} = (w)hf\mathcal{R}h^{-1}$. □

Claim 4.3.14. *If (A, B) is a revealing pair associated to $f\mathcal{R}$ then (A', B') constructed above is a revealing pair associated to $hf\mathcal{R}h^{-1}$.*

Proof. Let W be a component of $A' - B'$. Let w be the root of W and let $w_1, \ldots, w_m$ be the leaves of W. Notice that $w_1, \ldots, w_m$ are all leaves of A', and w is a leaf of B'. Let $w_i = w\Gamma_i$ for some $\Gamma_i \in X_2^+$ and $1 \leqslant i \leqslant m$. Since w_i's are leaves of A' and (A', A) is a tree pair for h, there are leaves u_i of A such that $(u_i)h = w_i$, $1 \leqslant i \leqslant m$. Since w is leaf of B' and (B, B') is an tree pair associated to h, there is a leaf u of B such that $(u)h = w$. Now consider $(w_i)h^{-1} = (w\Gamma_i)h^{-1}$, since (B', B) is an associated tree pair for h^{-1}, we have $(w_i)h^{-1} = (w)h^{-1}\Gamma_i = u\Gamma_i$. However we also have $(w_i)h^{-1} = u_i$ a leaf of A, hence we see that u is an internal node of A and a leaf of B. Moreover the component U of $A - B$ rooted at u has leaves $u_1, u_2, \ldots, u_r$. Therefore we have that the component W of $A' - B'$ is taken by h^{-1} to the isomorphic (as rooted trees) component U of $A - B$.

Now since (A, B) is a revealing pair for $f\mathcal{R}$, there is an iterated augmentation chain $\mu = \mu_1, \mu_2, \ldots, \mu_n$, where μ_1 is a leaf of U, and $(\mu_n)f\mathcal{R} = u$. Let ν be the leaf of W such that $(\mu)h = \nu$. Observe that $(\nu)h^{-1}(f\mathcal{R})^n h = (u)h = w$. Thus ν is a repelling leaf of W. Since W was an arbitrary component of $A' - B'$ we see that all components of $A' - B'$ contain a repelling leaf.

In a similar way one can demonstrate that all components of $B' - A'$ contain an attracting leaf. Thus (A', B') is a revealing tree pair for $f\mathcal{R}$. □

Proposition 4.3.15. *If $f\mathcal{R}, g\mathcal{R} \in V\mathcal{R}$ are conjugate, then there exists revealing pairs (A, B), (C, D) for $f\mathcal{R}$ and $g\mathcal{R}$ respectively, and $h \in V$ which is given by a bijection from the leaves of $A \cap B$ to $C \cap D$ such that*

(1) *If* $(a_1, \ldots, a_p)$ *is a cycle of neutral leaves of* A *then*$((a_1)h, \ldots, (a_p)h)$ *is a cycle of neutral leaves of* C.

(2) *Given a component* U *of* $A - B$ *with a cancelling chain,* $u = u_1, u_2, \ldots, u_n$*, of type (i), where* u *is the repeller in* U *and* Γ *is the path from the root of* U *to* u *i.e* $(u)(f\mathcal{R})^n\Gamma = (u_n)f\mathcal{R}\Gamma = u$*, there is a component* U' *of* $C - D$ *with a repeller* u' *in an iterated augmentation chain* $u' = u'_1, (u_2)h, \ldots, (u_n)h$ *and* $g\mathcal{R}((u_n)h)\Gamma = u'$.

(3) *Given a component* W *of* $B - A$ *with a cancelling chain,* $t_1, t_2, \ldots, t_n = t$*, of type (iv), where* $(t)f$ *is the attractor in* W *and* Λ *is the path from the root of* W *to* $(t)f$ *i.e* $(t_1)(f\mathcal{R})^n = (t_n)f\mathcal{R} = t_1\Lambda$*. Then there is a component* W' *of* $D - C$ *with an attractor* $(t')g\mathcal{R}$ *in an iterated augmentation chain* $(t'_1)h, (t_2)h, \ldots, (t_n)h = t'$ *and* $(t')g\mathcal{R} = ((t_n)h)g\mathcal{R} = (t_1)h\Lambda$.

(4) *The components* U *of* $A - B$ *and* U' *of* $C - D$ *are isomorphic as trees.*

(5) *The components* W *of* $B - A$ *and* W' *of* $D - C$ *are isomorphic as trees.*

(6) *Let* s *be a source in* A *i.e* $s = (u_n)f\mathcal{R}\Delta$ *where* u_n *is as in (2), and there is a cancelling chain,* $s = s_1, s_2, \ldots, s_m$ *of type (ii) where* $(s)f\mathcal{R}^m = (s_m)f\mathcal{R} = t_1\Theta$ *with* t_1 *as in (3). Then, the source* $s' = ((u_n)h)g\mathcal{R}\Delta$ *in* C *has cancelling chain* $s', (s_2)h, \ldots, (s_m)h$ *and satisfies* $(s')(g\mathcal{R})^m = ((s_m)h)g\mathcal{R} = (t_1)h\Theta \in D$.

Proof. The proof is a consequence of Claim 4.3.14 and its proof. □

Proposition 4.3.16. *If* (A, B) *and* (C, D) *are revealing pairs associated to elements* $f\mathcal{R}$ *and* $g\mathcal{R}$ *such that all the conditions Proposition 4.3.15 are satisfied, then* $f\mathcal{R}$ *and* $g\mathcal{R}$ *are conjugate. Furthermore the element* $h \in V$ *with associated tree pair* $(A \cap B, C \cap D)$ *is a conjugator.*

Proof. It suffices to show that $h^{-1}f\mathcal{R}h = g\mathcal{R}$, however this is a consequence of the fact that h satisfies the hypothesis of Proposition 4.3.15. □

4.3.5 Criterion for Conjugacy

Let (A, B) and (C, D) be revealing pairs associated with elements $f\mathcal{R}$ and $g\mathcal{R}$ of $V\mathcal{R}$ respectively, moreover, by Lemma 4.3.11, we can assume that all finite cycle of neutral leaves have an even period.

In what follows we shall give an all but identical construction, modifications warranted by the introduction of $\mathcal{R}$, as in section 4.5 of [49].

Let $U_1, U_2, \ldots, U_p$ be the components of $A - B$ reading left to right, likewise let $W_1, W_2, \ldots, W_q$ be the components of $B - A$. Each component U_k has a repelling leaf x^k_1 which is the first leaf in an iterated augmentation chain $x^k_1, x^k_2, \ldots, x^k_{n_k}$ and Γ_k is the path from the root to the repelling leaf. Each component W_l has an attracting leaf, $(y_{n_l})f\mathcal{R} = w_l$ such that y_{n_l} is the last leaf of an iterated augmentation chain $y^l_1, y^l_2, \ldots, y_{n_l}$ and Δ_l is the path from the root to the attracting leaf. Moreover, leaves of A which are not sources belong to one of r finite cycle of leaves: $(a^1_1 a^1_2, \ldots, a^1_{t_1}), \ldots, (a^r_1 a^r_2, \ldots, a^r_{t_r})$ By assumption all the t_j's, $1 \leqslant j \leqslant r$, are even.

Analogously let $U'_1, U'_2, \ldots, U'_{p'}$ and $W'_1, W'_2, \ldots, W'_{q'}$ be the components of $C - D$ and $D - C$ respectively with corresponding repellers $x^{k'}_1$ and attractors w'_l in iterated augmentation chains of length n'_k and n'_l respectively, and Γ'_k and Δ'_l the respective paths from the root to the repelling/attracting leaves. The leaves of C which are not sources belong one of r' finite cycle of leaves: $(c^1_1 c^1_2, \ldots, c^1_{t'_1}), \ldots, (c^{r'}_1 c^{r'}_2, \ldots, c^{r'}_{t'_{r'}})$ By assumption all the t'_j's are even.

By Proposition 4.3.15 and the fact that any two revealing pairs for $f\mathcal{R}$ have the same number of attracting and repelling leaves [49, section 3], we know that for $f\mathcal{R}$ and $g\mathcal{R}$ to be conjugate by an element of V then we must have that $p' = p$ and $q' = q$.

Let $\mathcal{T}_n$ be the finite rooted tree with n leaves and with $n - 1$ the length of the geodesic (see Section 1.2) from the root to the rightmost leaf .

Add copies of $U_1, \ldots, U_p$ to the leaves $\mathcal{T}_p$ and copies of $W_1, \ldots, W_q$ to the leaves of $\mathcal{T}_q$. To each component associate the period of the corresponding repeller/attractor. For a component U_i map each leaf representing a source of (A, B) to the corresponding sink in some component U_j. No mapping is assigned to the repellers/attractors. Denote the pair of trees with the mapping above $(A, B)_\star$. We shall refer to leaves of $(A, B)_\star$ corresponding to sources/sinks/attractors/repellers of (A, B) as sources/sinks/attractors/repellers.

Define a $(p + q + 4)$-tuple $X := (\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_q, \Pi_1, \Pi_2, \Theta_1, \Theta_2)$ such that $0 \leqslant \alpha_i < |\Gamma_i|$ and $0 \leqslant \beta_i < |\Delta_i|$, Π_1 is a permutation in $\mathrm{Sym}(p)$ (see Notation 3.4.3), $\Pi_2 \in \mathrm{Sym}(q)$,$\Theta_1$ is a tuple of the form $(\sigma_1, \ldots, \sigma_p)$ $\sigma_i \in \{0, 1\}$, and Θ_2 is a tuple of length q with entries 0 or 1. For such a tuple X construct $(A, B)^X_\star$, by replacing U_i by the new component obtained when we apply a rolling of type I using the prefix of length α_i of Γ and, if $\Theta_1(i) = 1$, applying a rolling of type II using the newly created component and adjoining the resulting component to the leaf $\Pi_1(i)$ of $\mathcal{T}_p$. If $\Theta_1(i) = 0$ then we do not perform a rolling of type II. We adjust the W_j's in an analogous manner (depending on the value of $\Theta_2(i)$) and adjoining it to the leaf in position $\Pi_2(j)$ of $\mathcal{T}_q$.

Define $\Xi_{f\mathcal{R}}$ as the set of all $(A, B)^X$ as X runs over all possible $p + q + 4$ tuples.

For each source s of $(A, B)_\star$ let, $L(s)$ be the length of the source sink chain. Let $L(A, B)_\star$ be the (length) vector with entries $L(s)$ for all s (we take the left to right ordering induced by the leaves of A once more).

Given two length vectors L and L' of the same size above, let $D = L - L'$. Define vectors R_i and K_j as follows: R_i has an entry one for each source in the component U_i and an entry zero for each source not in U_i, similarly K_j has an entry one for each source whose sink is in the component W_j and an entry zero for sources with sink not in W_j. Define an equivalence relation of length vectors by $L \sim L'$ if $D = L - L'$ has even entries and $D/2$ can be given as a linear combination with integer coefficients of the vectors R_i and K_j. Observe that by construction this linear combination of vectors R_i and K_j gives an indication of which components rollings of type II should be done in order for the length vectors to match up.

Given a pair $(E, F) \in \Xi_{f\mathcal{R}}$, let (G, H) be the pair obtained by adding a caret to a source u in E and to F at its corresponding sink v. Suppose the added carets are labelled (left to right) α_0, α_1 and β_0 β_1. If $L(u)$ is even, we map α_i to β_i and if $L(u)$ is odd we map α_i to β_{i+1} (addition of indices mod 2), the mappings for the remaining sources are unaffected. We write $(E, F) \to (G, H)$ and associate a length vector $L(G, H)$ to (G, H) where $L(\alpha_i) = L(u)$. Notice that $\to$ corresponds to a single rolling of type E applied to the iterated augmentation chain of the affected source.

The following lemma is proved almost identically to [49, claim 21].

Lemma 4.3.17. *Let (A', B') be a revealing pair for $f\mathcal{R}$ which is a rolling (not necessarily a single rolling) of another revealing pair (A, B) of $f\mathcal{R}$, then there is a pair $(E, F)_\star \in \Xi_{f\mathcal{R}}$ such that $(E, F)_\star \to^* (E', F')$ and $L(A', B')_\star \sim L(E', F')$. (Note that $\to^*$ denotes an application of none or finitely many $\to$).*

Proof. First observe that applying rollings of Type E commutes with applying rollings of Type I or Type II and rollings of Type I commute with rollings of Type II. Thus we may assume that all the rollings of Type E are performed last, rollings of type I second and rollings of type II first. Now observe that after performing all the rollings of type I and Type II to (A, B) we obtain some tree pair (E, F) such that $(E, F)_\star \in \Xi_{f\mathcal{R}}$ by Lemma 4.3.11. This is because the number of rollings of type I per component of $A - B$ or $B - A$ has to be less than the length of the path from the root of that component to the repelling or attracting leaf by Lemma 4.3.12. Let (A'', B'') the revealing pair obtained after applying all type I rollings. Now observe that an even number rolling of type II do not change the structure of the components of $A'' - B''$ or $B'' - A''$ but might change their left to right order, whereas an odd number of rollings of type II will replace a component of $A - B$ or $B - A$ with its flip as well as changing the left right order. Thus we may find a $(p + q + 4)$-tuple X such that $(A, B)^X_\star = (E, F)_\star$. Moreover, after performing all the rollings of type I and type II, we see that $L(E, F)_\star = L(A', B')_\star$. Now as rollings of type E do not affect the length of source-sink chains, but the number of source-sink chains of a given length, we see that after applying the rollings of type E, $(E, F)_\star \to^* (E', F')$ and $L(A', B')_* \sim L(E', F')$. $\square$

The next result is a corollary of Proposition 4.3.15.

Proposition 4.3.18. *If $f\mathcal{R}$ and $g\mathcal{R}$ are conjugate then there are revealing pairs (A, B) and (C, D) for $f\mathcal{R}$ and $g\mathcal{R}$ respectively such that $(A, B)_\star = (C, D)_\star$ and $L(A, B) = L(C, D)$*

The theorem below, which is a modification of [49, Theorem 2], gives a necessary and sufficient criterion for when two elements $f\mathcal{R}$ and $g\mathcal{R}$ are conjugate.

Theorem 4.3.19. *Let $f\mathcal{R}$ and $g\mathcal{R}$ be elements of $V\mathcal{R}$ with revealing pairs (A, B) and (C, E) respectively. Furthermore assume that all finite cycle of leaves in both pairs have an even period (by Lemma 4.3.11 we can make this assumption without losing generality). Let $C_{f\mathcal{R}}$ and $C_{g\mathcal{R}}$ denote the set of periods of finite cycles of leaves. Let p be the number of components of $A - B$, q be the number of components of $B - A$, p' be the number of components of $C - E$ and q' be the number of components of $E - C$, then $f\mathcal{R}$ and $g\mathcal{R}$ are conjugate by an element $h \in V$ if and only if the following hold:*

(i) $p' = p$ *and* $q' = q$,

(ii) *as sets we have* $C_{f\mathcal{R}} = C_{g\mathcal{R}}$,

(iii) *there exists* $(A, B)_\star^X \in \Xi_{f\mathcal{R}}$ *and* $(C, E)_\star^{X'} \in \Xi_{g\mathcal{R}}$ *with periods on repellers and attractors the same, such that there is pair* (G, H) *with* $(A, B)_\star^X \to^* (G, H)_{f\mathcal{R}} = (G, H)_{g\mathcal{R}} \,{}^*\!\leftarrow (C, E)_\star^{X'}$,*and* $L(G, H)_{f\mathcal{R}} \sim L(G, H)_{g\mathcal{R}}$.

Proof. $(\Rightarrow)$ follows by Proposition 4.3.18, and Proposition 4.3.15.

$(\Leftarrow)$ Carry out rollings of Type I and Type II (note that we only perform at most one rolling of Type II per component) to (A, B) and (C, E) to get new revealing pairs (A', B') and (C', E') for $f\mathcal{R}$ and $g\mathcal{R}$ respectively, and such that $(A', B')_\star = (A, B)_\star^X$ and $(C', E')_\star = (C, E)_\star^{X'}$. By assumption, we know that $(A, B)_\star^X$ and $(C, E)_\star^{X'}$ can be taken under applications of $\to^*$ to pairs $(G, H)_{f\mathcal{R}}$ and $(G, H)_{g\mathcal{R}}$. Apply the rollings of type E indicated by $\to^*$ to the appropriate sources in (A', B') and (C', E').

Notice that as finite cycles of neutral leaves do not feature in our construction of $(A', B')_\star$ and $(C', E')_\star$, we can apply rollings of type E to the finite cycles of neutral leaves in (A', B') or (C', E') so that the number of cycles of each period coincide and not affect $(A', B')_\star$ and $(C', E')_\star$.

Let (A'', B'') and (C'', E'') be the revealing pairs obtained after all rollings of type E have been performed.

By assumption we have that $D = L(G, H)_{f\mathcal{R}} - L(G, H)_{g\mathcal{R}}$ can be written:

$$D = L(G, H)_{f\mathcal{R}} - L(G, H)_{g\mathcal{R}} = \sum 2c_i R_i + \sum 2d_j K_j$$

Rewrite the above as follows:

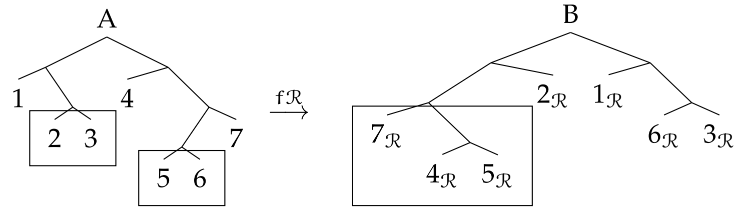
$$\sum_{c_i > 0} 2c_i R_i + \sum_{d_j > 0} 2d_j K_j + L(G, H)_{g\mathcal{R}} = \sum_{c_i < 0} (-2c_i) R_i + \sum_{d_j < 0} (-2d_j) K_j + L(G, H)_{f\mathcal{R}}$$

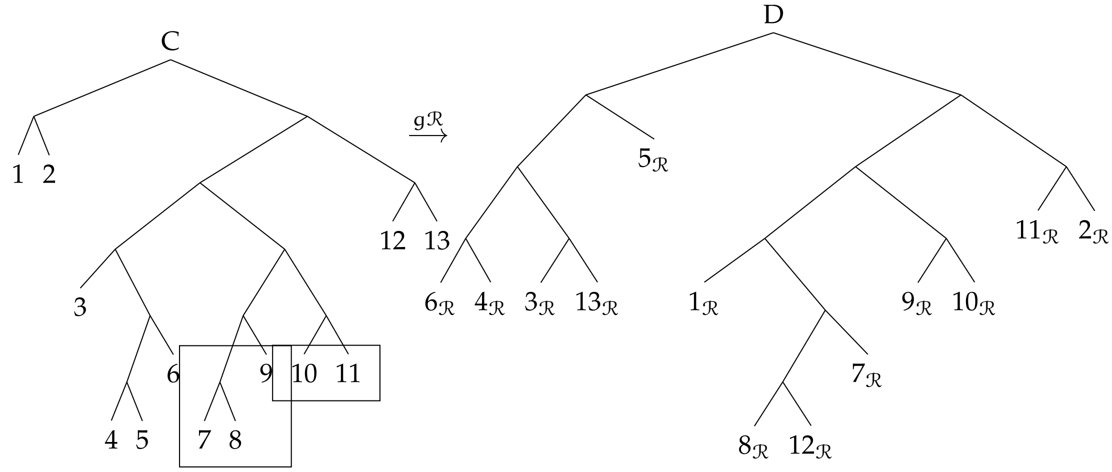
Let (A_h, B_h) and (C_h, E_h) be the revealing pairs obtained after applications of $2c_i$ rollings of type II using component U'_i of $C'' - D''$ for $i \in \mathbb{N}$ such that $c_i > 0$, $-2c_j$ rollings of type II with component U_i of $A'' - B''$, for $j \in \mathbb{N}$ such that $c_i < 0$, $2d_{i'}$ rollings of type II with component $W'_{i'}$ of $D'' - C''$, for $i' \in \mathbb{N}$ such that $d_{i'} > 0$ and $-2d_{j'}$ rollings of type II with component $W_{j'}$ of $B'' - A''$, for $j' \in \mathbb{N}$ such that $d_{j'} < 0$. By Lemma 4.3.11, and since an application of an even number of rollings of type II leaves a component unchanged, we have that $(A_h, B_h)_\star = (G, H)_{f\mathcal{R}}$ after permuting the components, and $(G, H)_{g\mathcal{R}} = (C_h, E_h)_\star$. Moreover, by Lemma 4.3.11, the chain length of sources and sinks, the length of iterated augmentation chains of repeller and attractors, and the number of finite cycles of neutral leaves of each period coincide in (A_h, B_h) and (C_h, E_h). Therefore, we must have that the number of leaves in all the trees A_h, B_h, C_h and E_h are the same. In particular, $|A_h \cap B_h| = |C_h \cap E_h|$. Let $h \in V$ be the map with representative tree pair $(A_h \cap B_h, C_h \cap E_h)$, and a bijection σ between the leaves such that, finite cycles map to finite cycles, the root of a component $U_{\Pi_1(i)}$ of $A_h - B_h$ gets sent to the root of the component $U'_{\Pi_1(i)}$ of $C_h - E_h$. Neutral leaves of A_h in the iterated augmentation chain of any leaf in a component $U_{\Pi_1(i)}$ get sent to the neutral leaves of B_h in the iterated augmentation chain of the corresponding source in $U'_{\Pi_1(i)}$. Analogously for neutral leaves in the iterated augmentation chain of the attractor in a component $W_{\Pi_2(j)}$. The map h thus defined is such that (A_h, B_h), (C_h, E_h) satisfies Proposition 4.3.16. $\square$

Remark 4.3.20. Note that there is an algorithm for computing the solution of the equation $\Sigma c_i R_i + \Sigma d_j K_j = D/2$, where $D/2$ is an integer matrix using standard linear algebra techniques.

We now go through a detailed example to illustrate the ideas of the proof.

Example 4.3.21. Let $f\mathcal{R}$ and $g\mathcal{R}$ be as follows:

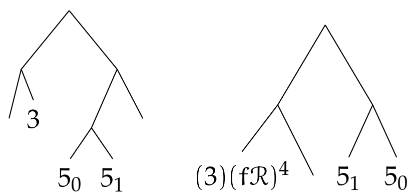




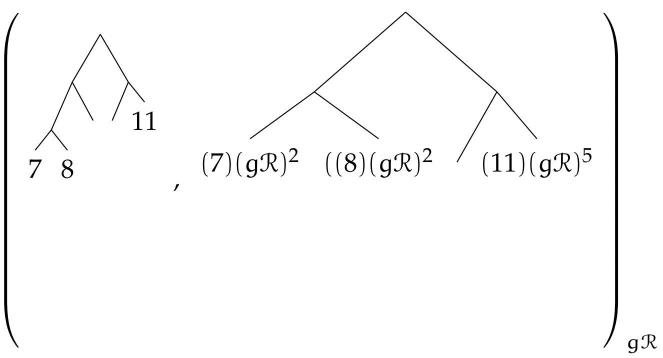
The element $f\mathcal{R}$ has 2 repellers of period 1, and an attractor of period 2, and no finite cycle of neutral leaves (Note that boxed components are the non-neutral leaves). The element $g\mathcal{R}$ has two repellers of period 1, and an attractor of period 2. The length matrices are as follows:

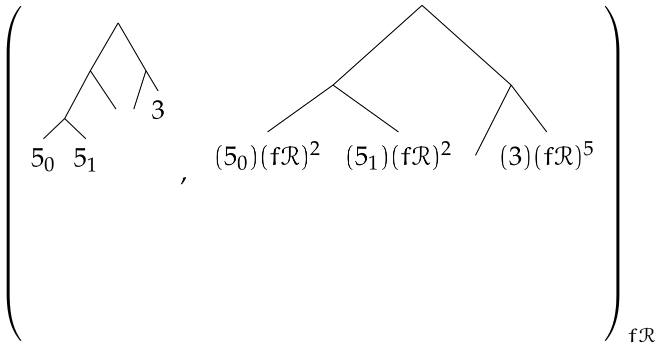
$$L(A,B)_\star = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad L(C,D)_\star = \begin{bmatrix} 2 \\ 2 \\ 5 \end{bmatrix}$$

Carry out a single rolling of type I using the attracting leaf 4 of B, and a single rolling of type E using the source 5 of A, to get a new revealing pair (A', B') for $f\mathcal{R}$. Such that $(A', B')_\star$ is as follows:



Let $X = (1, 0, 0, \Pi_1, \mathrm{id}, (0,0), (1))$, where Π_1 simply swaps the first and second component then we have that $(A,B)_\star^X \to (G,H)_{f\mathcal{R}}$ which is just $(A', B')_\star$ with components of $A - B$ swapped and the component of $B - A$ flipped. $(G,H)_{g\mathcal{R}} = (C,D)_\star$.



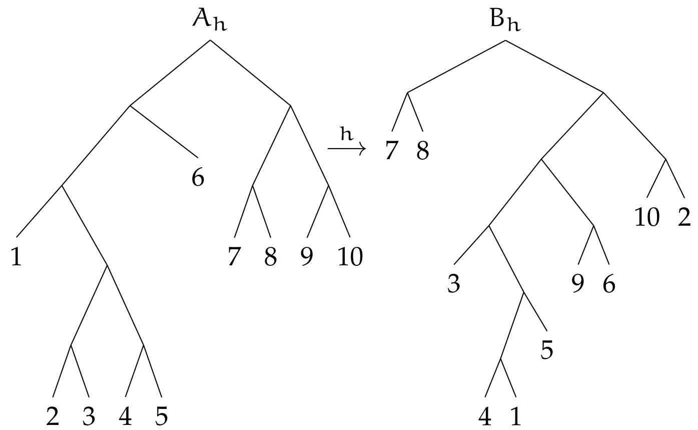


$$L(G,H)_{f\mathcal{R}} = \begin{bmatrix}2\\2\\5\end{bmatrix}, \quad L(G,H)_{g\mathcal{R}} = \begin{bmatrix}2\\2\\5\end{bmatrix}, D = L(G,H)_{f\mathcal{R}} - L(G,H)_{g\mathcal{R}} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

Moreover we have

$$R_1 = \begin{bmatrix}1\\1\\0\end{bmatrix}, \ R_2 = \begin{bmatrix}0\\0\\1\end{bmatrix}, \ K_1 = \begin{bmatrix}1\\1\\1\end{bmatrix}$$

Therefore $D = 0R_1 + 0R_2 + 0K_1$. Hence after an application of single rolling of type 1 to the attracting component of $A' - B'$ we get the tree pair (A_h, B_h) . Carefully following the algorithm set out in the proof of Theorem 4.3.19, gives us, h as follows:



Corollary 4.3.22. *Thompson's Group V has soluble $\mathcal{R}$-twisted conjugacy problem.*

Proof. Given $f\mathcal{R}$ and $g\mathcal{R}$ in $V\mathcal{R}$ with revealing pairs (A, B) and (C, E) respectively, there are only finite many elements of $\Xi_{f\mathcal{R}}$ and $\Xi_{g\mathcal{R}}$. For each pair $(A, B)^X_\star \in \Xi_{f\mathcal{R}}$ and $(C, E)^{X'}_\star \in \Xi_{g\mathcal{R}}$, one can determine in finite time if $(A, B)^X_\star \to^* (G, H)_{f\mathcal{R}} = (G, H)_{g\mathcal{R}} \ {}^*\!\!\leftarrow (C, E)^{X'}_\star$, and $L(G, H)_{f\mathcal{R}} \sim L(G, H)_{g\mathcal{R}}$. □

Remark 4.3.23. Theorem 4.3.19 and the corresponding Theorem in [49] solving the conjugacy problem in V, gives a solution to the conjugacy problem in $\mathcal{BH}_2$. Note that every element of $\mathcal{BH}_2$ can be written in the form $g\mathcal{R}$ for some $g \in V$, since $\mathcal{R} \in \mathcal{BH}_2$. Now let ψ, ρ be elements of $\mathcal{BH}_2$. Consider the equations $h^{-1}\psi h = \rho$ and $\mathcal{R}h^{-1}\psi h\mathcal{R} = \rho$ for $h \in V$. We can rearrange the second equation so it reads: $h^{-1}\psi h = \mathcal{R}\rho\mathcal{R}$. Therefore in order to decide if two elements ψ and ρ are conjugate in $\mathcal{BH}_2$, reduces to deciding if ψ is conjugate to ρ by an element $h \in V$ or if ψ is conjugate to $\mathcal{R}\rho\mathcal{R}$ by an element $h \in V$.

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