FINITE AND INFINITE ERGODIC THEORY FOR LINEAR AND CONFORMAL DYNAMICAL SYSTEMS

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Finite and Infinite Ergodic Theory for Linear and Conformal Dynamical Systems.

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Abstract

The first main topic of this thesis is the thorough analysis of two families of piecewise linear maps on the unit interval, the α-Lüroth and α-Farey maps. Here, α denotes a countably infinite partition of the unit interval whose atoms only accumulate at the origin. The basic properties of these maps will be developed, including that each α-Lüroth map (denoted L_α) gives rise to a series expansion of real numbers in $[0,1]$, a certain type of Generalised Lüroth Series. The first example of such an expansion was given by Lüroth. The map L_α is the jump transformation of the corresponding α-Farey map F_α. The maps L_α and F_α share the same relationship as the classical Farey and Gauss maps which give rise to the continued fraction expansion of a real number. We also consider the topological properties of F_α and some Diophantine-type sets of numbers expressed in terms of the α-Lüroth expansion.

Next we investigate certain ergodic-theoretic properties of the maps L_α and F_α. It will turn out that the Lebesgue measure λ is invariant for every map L_α and that there exists a unique Lebesgue-absolutely continuous invariant measure for F_α. We will give a precise expression for the density of this measure. Our main result is that both L_α and F_α are exact, and thus ergodic. The interest in the invariant measure for F_α lies in the fact that under a particular condition on the underlying partition α, the invariant measure associated to the map F_α is infinite.

Then we proceed to introduce and examine the sequence of α-sum-level sets arising from the α-Lüroth map, for an arbitrary given partition α. These sets can be written dynamically in terms of F_α. The main result concerning the α-sum-level sets is to establish weak and strong renewal laws. Note that for the Farey map and the Gauss map, the analogue of this result has been obtained by Kesseböhmer and Stratmann. There the results were derived by using advanced infinite ergodic theory, rather than the strong renewal theorems employed here. This underlines the fact that one of the main ingredients of infinite ergodic theory is provided by some delicate estimates in renewal theory.

Our final main result concerning the α-Lüroth and α-Farey systems is to provide a fractal-geometric description of the Lyapunov spectra associated with each of the maps L_α and F_α. The Lyapunov spectra for the Farey map and the Gauss map have been investigated in detail by Kesseböhmer and Stratmann. The Farey map and the Gauss map are non-linear, whereas the systems we consider are always piecewise linear. However, since our analysis is based on a large family of different partitions of $\mathscr{U}$, the class of maps which we consider in this paper allows us to detect a variety of interesting new phenomena, including that of phase transitions.

Finally, we come to the conformal systems of the title. These are the limit sets of discrete subgroups of the group of isometries of the hyperbolic plane. For these so-called *Fuchsian groups*, our first main result is to establish the Hausdorff dimension of some Diophantine-type sets contained in the limit set that are similar to those considered for the maps L_α. These sets are then used in our second main result to analyse the more geometrically defined *strict-Jarník limit set* of a Fuchsian group. Finally, we obtain a "weak multifractal spectrum" for the Patterson measure associated to the Fuchsian group.

Declaration

I, Sara Munday, hereby certify that this thesis, which is approximately 25000 words in length, has been written by me, that it is the record of work carried out by me and that it has not been submitted in any previous application for a higher degree.

I was admitted as a research student in September 2006 and as a candidate for the degree of Doctor of Philosophy in September 2007; the higher study for which this is a record was carried out in the University of St Andrews between 2007 and 2011.

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I hereby certify that the candidate has fulfilled the conditions of the Resolution and Regulations appropriate for the degree of Doctor of Philosophy in the University of St Andrews and that the candidate is qualified to submit this thesis in application for that degree.

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Contents

Introduction and statement of results

The broad subject area of the research contained in this thesis is dynamical systems. For us, a dynamical system means simply a continuous map on a given topological space and the study of a dynamical system consists of an investigation into the long-term behaviour of the iterates of the map. As the title suggests, the work can be split into two parts, the first part on linear systems and the second on conformal systems.

The linear systems with which we are concerned are studied in Chapters 2 to 5. They are families of number-theoretic dynamical systems, by which we mean that an expansion (or numeration system) of real numbers can be obtained from them. An example of such an expansion is the familiar decimal expansion or the continued fraction expansion. Such examples are introduced in Chapter 1, to set the scene for the investigations to come. Also in that chapter can be found a brief recall of the definition and basic properties of the Hausdorff dimension of a set in $\mathbb{R}^n$, which will be used repeatedly throughout this thesis.

We aim to introduce two families of maps defined on partitions of the unit interval, which we denote by $\mathscr{U}$. So, let $\alpha := \{A_n : n \in \mathbb{N}\}$ denote a countably infinite partition of $\mathscr{U}$, consisting of non-empty right-closed and left-open intervals. It is assumed throughout that the elements of α are ordered from right to left, starting from A_1, and that these elements accumulate only at the origin. Let a_n denote the Lebesgue measure $\lambda(A_n)$ of $A_n \in \alpha$ and let $t_n := \sum_{k=n}^{\infty} a_k$ denote the Lebesgue measure of the n-th tail of α. Then, for a given partition α, define the map $L_\alpha : \mathscr{U} \to \mathscr{U}$ by

$$L_\alpha(x) := \begin{cases} (t_n - x)/a_n & \text{for } x \in A_n,\ n \in \mathbb{N}; \\ 0 & \text{if } x = 0. \end{cases}$$

The map L_α is referred to as the *α-Lüroth map*. This map can be thought of as a linearised generalisation of the Gauss map from elementary number theory. For each partition α the map L_α gives rise to a series expansion of numbers in the interval $\mathscr{U}$, which we refer to as the *α-Lüroth expansion*. That is, let $x \in \mathscr{U}$ be given and let the finite or infinite sequence $(\ell_k)_{k\geq 1}$ be determined by $L_\alpha^{k-1}(x) \in A_{\ell_k}$. This finite or infinite sequence gives rise to an alternating series expansion of each $x \in \mathscr{U}$, which is given by

$$x = t_{\ell_1} + \sum_{n=2}^{\infty} (-1)^{n-1} \left(\prod_{i<n} a_{\ell_i} \right) t_{\ell_n} = t_{\ell_1} - a_{\ell_1} t_{\ell_2} + a_{\ell_1} a_{\ell_2} t_{\ell_3} - \ldots$$

Let us denote finite α-Lüroth expansions by $[\ell_1, \ell_2, \ldots, \ell_k]_\alpha$, for some $k \in \mathbb{N}$, and infinite ones by $x = [\ell_1, \ell_2, \ell_3, \ldots]_\alpha$. We note that this series is a certain type of *Generalised Lüroth Series*, introduced by Barrionuevo *et al.* in [5]. For later use, let us also mention the *cylinder sets* associated to the map L_α. For each k-tuple $(\ell_1,\ldots,\ell_k)$ of positive integers, define the α-Lüroth cylinder set $C_\alpha(\ell_1,\ldots,\ell_k)$ associated with the α-Lüroth expansion to be

$$C_\alpha(\ell_1,\ldots,\ell_k) := \{[y_1,y_2,\ldots]_\alpha : y_i = \ell_i \text{ for } 1 \le i \le k\}.$$

All the details of the α-Lüroth expansion and the cylinder sets associated to it can be found in Section 2.1.

Let us now introduce a second family of maps, indexed by the same set of partitions α of $\mathscr{U}$. For a given partition $\alpha := \{A_n : n \in \mathbb{N}\}$ of $\mathscr{U}$, define the map $F_\alpha : \mathscr{U} \to \mathscr{U}$ by

$$F_\alpha(x) := \begin{cases} (1-x)/a_1 & \text{if } x \in A_1, \\ a_{n-1}(x-t_{n+1})/a_n + t_n & \text{if } x \in A_n, \text{ for } n \ge 2, \\ 0 & \text{if } x = 0. \end{cases}$$

The map F_α is referred to as the *α-Farey map*.

The first result in Section 2.2 gives the relationship between the maps F_α and L_α. We have that L_α is the jump transformation of F_α. The definition of a jump transformation is given in Definition 2.2.3. The relationship between the α-Lüroth and α-Farey maps is precisely the same relationship as that between the Gauss map and the classical Farey map (hence the name chosen for F_α). Some more basic properties of the map F_α are given and then we move on in Section 2.3 to consider the topological properties of F_α.

Recall that two dynamical systems $T : X \to X$ and $S : Y \to Y$ are said to be *topologically conjugate* if there exists a homeomorphism $h : X \to Y$, called a *conjugacy map*, such that $h \circ T = S \circ h$. Our first result in Section 2.3 is to show that for any arbitrary partition α, the map F_α is topologically conjugate to the tent map, $T : \mathscr{U} \to \mathscr{U}$, which is defined by $T(x) := 2x$ for $x \in [0,1/2)$ and $T(x) := 2-2x$ for $x \in [1/2,1]$. We also give an explicit expression for the conjugacy map in terms of the measure of maximal entropy for F_α (see Proposition 2.3.1). We finish the section by showing that this conjugacy map is both Hölder and sub-Hölder continuous.

The next section is concerned with various more specific types of partition which will be useful for the remainder of Chapter 2 and in Chapters 4 and 5. We make the following definitions. Let $\alpha := \{A_n : n \in \mathbb{N}\}$ be a countable partition of $\mathscr{U}$ of the form described above. Then:

1. The partition α is said to be *expanding* provided that

$$\lim_{n\to\infty} \frac{t_n}{t_{n+1}} = \rho, \quad \text{for some } \rho > 1.$$

2. The partition α is said to be *expansive of exponent* $\theta \ge 0$ if the tails of the partition satisfy the power law

$$t_n = \psi(n) \cdot n^{-\theta},$$

where $\psi : \mathbb{N} \to \mathbb{R}^+$ is a slowly-varying function, that is, ψ is a measurable function that satisfies $\lim_{x\to\infty} \psi(xy)/\psi(x) = 1$, for all $y > 0$.

3. The partition α is said to be *eventually decreasing* if for all sufficiently large n, we have that $a_{n+1} \le a_n$.

In Section 2.5, we describe various Diophantine-like subsets of $\mathscr{U}$ in terms of the α-Lüroth expansion. By this, we mean that the sets considered here are analogues of the sets of badly-approximable numbers and other similar sets usually defined in terms of the continued fraction expansion. Our first main result concerns α-Good sets, which are defined as follows. For each $N \in \mathbb{N}$, let the set $G_N^{(\alpha)}$ be defined by

$$G_N^{(\alpha)} := \{x = [\ell_1(x), \ell_2(x), \dots]_\alpha \in \mathscr{U} : \ell_i(x) > N \text{ for all } i \in \mathbb{N}\}.$$

Note that the name "Good" here refers to I.J. Good [31], for the similar results he proved for continued fractions, and not to any supposed nice property of these sets. We have the following result.

Theorem 1. *Suppose that α is expansive of exponent $\theta > 0$. Then,*

$$\lim_{N \to \infty} \dim_H \left(G_N^{(\alpha)} \right) = \frac{1}{1+\theta}.$$

Our second main result of this type is as follows. Define the sets F_∞ and G_∞ by setting

$$F_\infty^{(\alpha)} := \left\{ x = [\ell_1(x), \ell_2(x), \dots]_\alpha : \lim_{n \to \infty} \ell_n(x) = \infty \text{ and } \ell_n(x) \geq \ell_{n-1}(x) \right\}$$

and

$$G_\infty^{(\alpha)} := \left\{ x = [\ell_1(x), \ell_2(x), \dots]_\alpha : \lim_{n \to \infty} \ell_n(x) = \infty \right\}.$$

We obtain the following theorem.

Theorem 2. *Suppose that α is expansive of exponent $\theta > 0$. Then,*

$$\dim_H \left(F_\infty^{(\alpha)} \right) = \dim_H \left(G_\infty^{(\alpha)} \right) = \frac{1}{1+\theta}.$$

Moreover, if α is expanding, then

$$\dim_H (F_\infty^{(\alpha)}) = \dim_H \left(G_\infty^{(\alpha)} \right) = 0.$$

Our third main result concerns the following situation. Fix a sequence $(s_n)_{n \in \mathbb{N}}$ of natural numbers with the property that $\lim_{n \to \infty} s_n = \infty$. Then, let σ be given by

$$\sigma := \liminf_{n \to \infty} \frac{\log(s_1 \cdots s_n)}{(1+\theta)\log(s_1 \cdots s_n) + \theta \log(s_{n+1})} = \frac{1}{(1+\theta) + \theta \left(\limsup_{n \to \infty} \frac{\log(s_{n+1})}{\log(s_1 \cdots s_n)} \right)}.$$

Finally, let $N > 3$ and define the set

$$J_\sigma^{(\alpha)} := \{x = [\ell_1(x), \ell_2(x), \ldots]_\alpha : s_n \leq \ell_n(x) < N s_n \text{ for all } n \in \mathbb{N}\}.$$

We refer to these sets as *strict α-Jarník* sets, after V. Jarník [39], for his results on similarly defined sets in the continued fractions setting. We obtain the following result.

Theorem 3. *Suppose that α is expansive of exponent $\theta > 0$. Then,*

$$\dim_H \left(J_\sigma^{(\alpha)}\right) = \sigma.$$

In Chapter 3 we investigate various measure theoretic and ergodic theoretic properties of the maps L_α and F_α. Throughout this chapter, the partition α is again arbitrary. We first consider the map L_α. We remind the reader that a measure μ is said to be *invariant* for a dynamical system $T : X \to X$ provided that for every μ-measurable set $B \subset X$, we have $\mu \circ T^{-1}(B) := \mu(T^{-1}(B)) = \mu(B)$. It turns out that for every partition α, the Lebesgue measure λ is invariant for L_α. We provide a proof, but this result can be found in Dajani and Kraaikamp [16]. We also have that each map L_α is *exact*. Recall the definition of exactness: A non-singular transformation T of a σ–finite measure space $(\mathscr{U}, \mathscr{B}, \mu)$ is said to be exact if for each B in the tail σ-algebra $\bigcap_{n\in\mathbb{N}} T^{-n}(\mathscr{B})$ we have that either $\mu(B)$ or $\mu(\mathscr{U} \setminus B)$ vanishes. Again, we provide a proof for this result, but it follows from the result in [5] that each Generalised Lüroth map is Bernoulli. It is an immediate corollary of the exactness that each map L_α is ergodic (see also [16]). We finish Section 3.2 with a variety of results that are obtained in a straightforward way from Birkhoff's Ergodic Theorem.

We then turn to the ergodic properties of the map F_α. Before stating our first result, let us make one further definition. A partition α is said to be of *infinite type* if for the tails t_n of α we have that $\sum_{n=1}^{\infty} t_n$ diverges and is said to be of *finite type* otherwise. The following lemma details the invariant measure for the map F_α.

Lemma 4. *The λ-absolutely continuous measure ν_α, defined by the density ϕ_α which is given, up to multiplication by a constant, by*

$$\phi_\alpha := \frac{d\nu_\alpha}{d\lambda} = \sum_{n=1}^{\infty} \frac{t_n}{a_n} \cdot \mathbb{1}_{A_n},$$

is an invariant measure for the system $(\mathscr{U}, \mathscr{B}, F_\alpha)$. Moreover, ν_α is a σ-finite measure, and we have that ν_α is an infinite measure if and only if α is of infinite type.

It also turns out that this measure ν_α is the unique λ-absolutely continuous invariant measure for F_α. Our third main ergodic-theoretic result is as follows.

Theorem 5. *The α-Farey map F_α is exact.*

From this, just as in the case of the map L_α, it follows immediately that for each partition α the map F_α is ergodic. We finish Section 3.3 by stating various consequences of the ergodicity of F_α. Note that since the invariant measure for F_α can be finite or infinite, we enter the interesting realm of infinite ergodic theory here.

In Chapter 4, we are interested in the so-called *α-sum-level sets* for a given partition α. These sets are given, for each $n \in \mathbb{N}_0$, by

$$\mathscr{L}_n^{(\alpha)} := \left\{ x \in C_\alpha(\ell_1, \ell_2, \ldots, \ell_k) : \sum_{i=1}^{k} \ell_i = n, \text{ for some } k \in \mathbb{N} \right\},$$

where, for $n = 0$, we have put $\mathscr{L}_0^{(\alpha)} := \mathscr{U}$. We will use standard renewal theory to study the sequence of Lebesgue measures of these sets. Our main result is the following, where the notation $a_n \sim b_n$ means that $\lim_{n\to\infty} a_n/b_n = 1$.

Theorem 6. (1) *For the α-sum-level sets of an arbitrary given partition α of $\mathscr{U}$ we have that $\sum_{n=1}^{\infty} \lambda(\mathscr{L}_n^{(\alpha)})$ diverges, and that*

$$\lim_{n\to\infty} \lambda\left(\mathscr{L}_n^{(\alpha)}\right) = \begin{cases} 0 & \text{if } F_\alpha \text{ is of infinite type;} \\ \left(\sum_{k=1}^{\infty} t_k\right)^{-1} & \text{if } F_\alpha \text{ is of finite type.} \end{cases}$$

(2) *For a given expansive partition α which is either of exponent $\theta \in [0,1]$ or such that F_α is of finite type, we have the following estimates for the asymptotic behaviour of the Lebesgue measure of the α-sum-level sets.*

(i) *With $K_\theta := (\Gamma(2-\theta)\Gamma(1+\theta))^{-1}$ for α expansive of exponent $\theta \in [0,1]$ and with $K_\theta := 1$ for F_α of finite type, we have that*

$$\sum_{k=1}^{n} \lambda\left(\mathscr{L}_k^{(\alpha)}\right) \sim K_\theta \cdot n \cdot \left(\sum_{k=1}^{n} t_k\right)^{-1}.$$

(ii) *With $k_\theta := (\Gamma(2-\theta)\Gamma(\theta))^{-1}$ for α expansive of exponent $\theta \in (1/2,1]$ and with $k_\theta := 1$ for F_α of finite type, we have that*

$$\lambda\left(\mathscr{L}_n^{(\alpha)}\right) \sim k_\theta \cdot \left(\sum_{k=1}^{n} t_k\right)^{-1}.$$

(iii) *For an expansive partition α of exponent $\theta \in (0,1)$, we have that*

$$\liminf_{n\to\infty} \left(n \cdot t_n \cdot \lambda\left(\mathscr{L}_n^{(\alpha)}\right)\right) = \frac{\sin \pi\theta}{\pi}.$$

Moreover, if $\theta \in (0,1/2)$, then the corresponding limit does not exist in general. However, in this situation the existence of the limit is always guaranteed at least on the complement of some set of integers of zero density.

We remark that the proof of the first part makes use of the standard discrete renewal theorem due to Erdős, Pollard and Feller [20]. The second part relies upon some rather more intricate renewal results obtained by Erickson, Garsia and Lamperti [21], [29]. Note that for the Farey map and the Gauss map, an analogue of Theorem 7 has been obtained by Kesseböhmer and Stratmann [49]. In this paper the results are derived by using advanced infinite ergodic theory, rather than the strong renewal theorems employed here. This underlines the fact that one of the main ingredients of infinite ergodic theory is provided by some delicate estimates in renewal theory.

In Chapter 5, we turn to multifractal investigations of the maps L_α and F_α. This type of analysis is by now a well-established area of mathematics. It has its origins at the junction of pure mathematics and statistical physics, and can be considered as an offshoot of thermodynamic formalism. We consider the *Lyapunov spectrum* of each map. The Lyapunov spectrum of a differentiable map $S : \mathscr{U} \to \mathscr{U}$ at a point $x \in \mathscr{U}$ is defined, provided the limit exists, by

$$\Lambda(S,x) := \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \log |S'(S^k(x))|.$$

Our first main theorem gives a fractal-geometric description of the Lyapunov spectra associated with the map L_α. That is, we consider the Hausdorff dimension of the spectral sets $\{s \in \mathbb{R} : \{x \in \mathscr{U} : \Lambda(L_\alpha, x) = s\} \neq \emptyset\}$. This gives rise to the Hausdorff dimension function τ_α, which is given by

$$\tau_\alpha(s) := \dim_H(\{x \in \mathscr{U} : \Lambda(L_\alpha, x) = s\}).$$

In what follows, $p : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ denotes the *α-Lüroth pressure function*, which is defined by

$$p(u) := \log \sum_{n=1}^{\infty} a_n^u.$$

In addition, we say that L_α *exhibits no phase transition* if and only if the pressure function p is differentiable everywhere (that is, the right and left derivatives of p coincide everywhere, with the convention that $p'(u) = \infty$ if $p(u) = \infty$).

Theorem 7. *For an arbitrary given partition α, the Hausdorff dimension function of the Lyapunov spectrum associated with L_α is given as follows. For $t_- := \inf\{-\log a_n : n \in \mathbb{N}\}$ we have that τ_α vanishes on $(-\infty, t_-)$, and for each $s \in (t_-, \infty)$ we have*

$$\tau_\alpha(s) = \inf_{u\in\mathbb{R}} \left(u + s^{-1} p(u)\right).$$

Moreover, $\tau_\alpha(s)$ tends to $t_\infty := \inf\{r > 0 : \sum_{k=1}^{\infty} a_n^r < \infty\} \leq 1$ for s tending to infinity. Concerning the possibility of phase transitions for L_α, the following hold:

- *If α is expanding, then L_α exhibits no phase transition and $t_\infty = 0$.*
- *If α is expansive of exponent $\theta > 0$ and eventually decreasing, then L_α exhibits no phase transition if and only if $\sum_{n=1}^{\infty} \psi(n)^{1/(1+\theta)} (\log n)/n$ diverges. Moreover, in this situation we have that $t_\infty = 1/(1+\theta)$.*

- *If α is expansive of exponent $\theta = 0$, then L_α exhibits no phase transition if and only if $\sum_{n=1}^{\infty} a_n \log(a_n)$ diverges. Moreover, in this situation we have that $t_\infty = 1$.*

Finally, for partitions which are either expanding or expansive of exponent $\theta > 0$ and eventually decreasing, t_∞ is also equal to the Hausdorff dimension of the Good-type set $G_\infty^{(\alpha)}$ associated to L_α.

The proof of Theorem 7 relies heavily upon the general multifractal results obtained by Jaerisch and Kesseböhmer [38]. We then go on to consider the dimension spectrum associated to the map F_α. We obtain the following theorem.

Theorem 8. *Let α be a partition that is either expanding or expansive and eventually decreasing. The Hausdorff dimension function of the Lyapunov spectrum associated with F_α is then given as follows. For*

$$r_- := \inf\{-t^-(v) : v \in \mathrm{Int}(dom(t))\} \ \ and \ \ r_+ := \sup\{-t^+(v) : v \in \mathrm{Int}(dom(t))\},$$

we have that $\sigma_\alpha(s)$ vanishes outside the interval $[1/r_+, 1/r_-]$ and for each $s \in (1/r_+, 1/r_-)$, we have

$$\sigma_\alpha(s) = \inf_{v \in \mathbb{R}} \left(s^{-1} \cdot v + t(v) \right).$$

The main work involved in the proof of Theorem 8 is contained in Proposition 5.3.4, where it is shown that in order to calculate the dimension of a level set associated to the map F_α it is sufficient to calculate one connected to the map L_α. We end Chapter 5 with a section containing various examples to demonstrate the diversity of different behaviours of the spectra given by Theorem 5.3.2 and Theorem 5.3.5 in dependence on the chosen partition α. This includes a detailed discussion of the existence of phase transitions. Each partition α under consideration is eventually decreasing and either expanding or expansive of exponent $\theta > 0$.

Finally, Chapter 6 contains the work done on conformal systems. We will consider certain subsets of the limit set of a non-elementary, geometrically finite Fuchsian group G that contains at least one parabolic element. Recall that a Fuchsian group is a discrete group of isometries of the hyperbolic plane. Each Fuchsian group G has an associated Riemann surface and each limit point of G can be thought of in terms of a geodesic movement on this surface. By this, we mean that the limit point ξ is thought of as the end point of the geodesic ray s_ξ that starts at the origin and ends at the point ξ. From the assumption that G contains at least one parabolic element, it follows that the surface associated to G has at least one cusp. (All the necessary background information on hyperbolic geometry can be found in the first section of Chapter 6.) We will be interested in those geodesic movements which have some prescribed behaviour in relation to the cusp on the surface. We will begin by defining a *cusp excursion*, which, roughly speaking, is the distance traveled by a geodesic into the cusp with respect to some fixed reference point.

The first subset with which we are concerned is the set of all limit points ξ for which the ray s_ξ makes infinitely many cusp excursions, only stays outside the cusp for a fixed constant distance in between each cusp excursion and is such that each cusp excursion has hyperbolic length at least $\log(\tau)$, for some $\tau \in \mathbb{R}$. This set will be referred to as the *τ-Good set* and denoted by $\mathscr{C}_\tau(G)$. We obtain the following result, where $\dim_H$ denotes Hausdorff dimension.

Theorem 9.

$$\lim_{\tau\to\infty} \dim_H(\mathscr{C}_\tau(G)) = \frac{1}{2}.$$

We remark that this result is perhaps somewhat surprising, since it does not depend on the Hausdorff dimension of the entire limit set. It is a result of Bishop and Jones [12] that the Hausdorff dimension of the radial limit set of a non-elementary Fuchsian group G is equal to the exponent of convergence of the Poincaré series associated to the group, which we denote by $\delta := \delta(G)$. Further, by a result of Beardon [6], for a non-elementary geometrically finite Fuchsian group G which contains at least one parabolic element, we have that $\delta > 1/2$. So our result says that no matter what the dimension of the whole limit set (which must be strictly greater than $1/2$), the dimension of the τ-Good set tends to $1/2$.

The second main result of Chapter 6 concerns subsets of the limit set of G that are described by a given rate θ of traveling into a cusp. We call these sets strict θ-Jarník limit sets and denote them by $\mathscr{J}_\theta^*(G)$. More specifically, but without giving all the details here, for $\xi \in \mathscr{J}_\theta^*(G)$, we let $d_n(\xi)$ denote the length of the nth cusp excursion and (roughly speaking) let $t_n(\xi)$ denote the sum of the lengths of the first n cusp excursions. We then define $\theta \in [0,1]$ to be

$$\theta := \limsup_{n\to\infty} \frac{d_n(\xi)}{t_n(\xi)}.$$

The points in $\mathscr{J}_\theta^*(G)$ also have to satisfy the requirement that there is at most a fixed constant hyperbolic distance between each cusp excursion. We obtain the following theorem, which gives a complete description of the Hausdorff dimension spectrum $\{\dim_H\left(\mathscr{J}_\theta^*(G)\right) : \theta \in [0,1]\}$.

Theorem 10. For each $\theta \in [0,1]$, we have that

$$\dim_H(\mathscr{J}_\theta^*(G)) = \frac{1}{2}(1-\theta).$$

Our final result is an application of Theorem 10 to obtain a "weak-multifractal spectrum" for the Patterson measure. This is given in Theorem 6.5.1. We remind the reader that the Patterson measure is a $\delta(G)$-conformal measure defined on the limit set of G. (In Appendix A, a detailed description of this highly useful measure can be found.) This result should be compared with the related work of Stratmann [76] where a similar spectrum was described. The sets underlying the result in [76] are also defined by a certain rate of traveling into the cusp, but here the restriction on what happens in between excursions is dropped. The corresponding Hausdorff dimensions are given by $\delta(1-\theta)$ as opposed to the $1/2(1-\theta)$ we obtain. It is an interesting question for further research to investigate what could happen if in addition to a rate of traveling into the cusp, a rate of traveling outside the cusp was introduced. This could perhaps combine the approaches given in this thesis and in the paper [76] and we conjecture that a whole range of spectra for the Patterson measure could be obtained in this way.

Chapter 1

Preliminaries

1.1 Hausdorff measure and dimension

Felix Hausdorff (1868-1942) introduced the theory of the fractional dimension, now called the Hausdorff dimension, in his foundational paper from 1918, “Dimension und äußeres Maß” [34]. In this paper Hausdorff adapts a definition of dimension given by Carathéodory [13] so that it makes sense for non-integer values. (Hausdorff very modestly refers to this ground-breaking work as a “small contribution”.) For the necessary background in measure theory, the reader is referred to Cohn [14].

Definition 1.1.1.

1. If U is any non-empty subset of $\mathbb{R}^n$, define the *diameter* of U to be

$$|U| := \sup\{|x-y| : x, y \in U\}.$$

   In other words, the diameter of a set is the supremum of the distances between points in that set.

2. If $\{U_i\}_{i\geq 1}$ is a collection of sets of diameter at most δ with the property that $F \subseteq \bigcup_{i=1}^{\infty} U_i$, we say that $\{U_i\}$ is a δ*-cover* of F.

Definition 1.1.2. Suppose that F is a subset of $\mathbb{R}^n$. Then for any $\delta > 0$ we define

$$\mathscr{H}_\delta^s(F) := \inf\left\{\sum_{i=0}^{\infty} |U_i|^s : \{U_i\} \text{ is a } \delta\text{-cover of } F\right\}.$$

This infimum is non-decreasing as δ decreases, since fewer covers are available for smaller values of δ compared to larger ones, and so it approaches a limit as $\delta \to 0$. Thus, the following definition makes sense for any subset F of $\mathbb{R}^n$.

Definition 1.1.3. The *s-dimensional Hausdorff measure* of a set $F \subseteq \mathbb{R}^n$ is given by

$$\mathscr{H}^s(F) := \lim_{\delta\to 0} \mathscr{H}_\delta^s(F).$$

The limiting value in Definition 1.1.3 can be zero or can be infinite. One can show without too much work that $\mathscr{H}^s$ is indeed a measure. For the details, the interested reader is referred to Chapter 2 of Falconer's book [22]. For a given set F and a given $\delta < 1$, it is clear that $\mathscr{H}^s_\delta(F)$ is a non-increasing function of s. It follows that $\mathscr{H}^s(F)$ is also non-increasing. In fact, if $t > s$ and $\{U_i\}_{i=1}^\infty$ is a δ-cover of F we have that

$$\sum_{i=1}^\infty |U_i|^t = \sum_{i=1}^\infty |U_i|^{t-s+s} \leq \delta^{t-s} \sum_{i=1}^\infty |U_i|^s$$

so, taking the infimum over all δ-covers, $\mathscr{H}^t_\delta(F) \leq \delta^{t-s}\mathscr{H}^s_\delta(F)$. If we then let δ tend to zero, we see that if $\mathscr{H}^s(F)$ is finite then $\mathscr{H}^t(F) = 0$ for every $t > s$. So there is a critical value of s where $\mathscr{H}^s(F)$ jumps from $\{\infty\}$ to 0. This critical value is called the *Hausdorff dimension* of F, written $\dim_H(F)$. Explicitly,

$$\dim_H(F) := \sup\{s : \mathscr{H}^s(F) = \infty\} = \inf\{s : \mathscr{H}^s(F) = 0\}.$$

If $s = \dim_H(F)$, then $\mathscr{H}^s(F)$ may be equal to zero or may be infinite, or may satisfy $0 < \mathscr{H}^s(F) < \infty$. A set with this last property is called an *s-set*.

The following proposition collects some of the basic properties of Hausdorff dimension. For the proofs, the reader is referred again to [22]. These properties will be used throughout this thesis, often without explicit mention.

Proposition 1.1.4. *Let $F \subset \mathbb{R}^n$. Then the following hold:*

1. $0 \leq \dim_H(F) \leq n$.
2. *If $E \subseteq F$, then* $\dim_H(E) \leq \dim_H(F)$.
3. *The Hausdorff dimension is countably stable, that is, if $F_1, F_2, \ldots$ is a countable sequence of sets, then*
$$\dim_H\left(\bigcup_{i\geq 1} F_i\right) = \sup\{\dim_H(F_i) : i \in \mathbb{N}\}.$$

Although it is possible to calculate the Hausdorff dimension of a set using only the definition, it can often involve pages of complicated estimates. Of course, to obtain an upper bound for the dimension of a particular set $F \subset \mathbb{R}^n$ is usually (although by no means always), easier than obtaining the corresponding lower bound. For the upper bound it is enough to consider specific coverings of F, while for the lower bound we would have to consider *every* covering of our set F. In particular, some of the covers will consist of both very small sets and sets with relatively large diameters, making obtaining estimates more difficult. A good way around this is to use the following lemma, proved by Frostman in his doctoral thesis [26]. A *mass distribution* on F is a finite measure on $\mathbb{R}^n$ with support[1] equal to F. The proof is not complicated so we include it here for completeness.

[1]The support of a measure is defined to be the smallest (in the sense of set inclusion) closed set with complement of measure zero.

Lemma 1.1.5. *(Frostman's Lemma.) Let F be a bounded subset of $\mathbb{R}^n$. Let μ be a mass distribution on F and suppose that for some $s > 0$ there exist constants $c > 0$ and $\delta > 0$ with the property that*

$$\mu(U) \leq c|U|^s$$

for all sets U with $|U| \leq \delta$. Then $\mathscr{H}^s(F) \geq \mu(F)/c$ and so

$$s \leq \dim_H(F).$$

Proof. If $\{U_i\}$ is any δ-cover of F, then

$$0 < \mu(F) \leq \sum_{i \geq 1} \mu(U_i) \leq c \sum |U_i|^s.$$

Taking the infimum over all δ-covers of F, we obtain that $\mathscr{H}^s_\delta(F) \geq \mu(F)/c$ for all sufficiently small δ. Hence, $\mathscr{H}^s(F) \geq \mu(F)/c$.

□

1.2 Dynamical systems

For our purposes, a dynamical system is a continuous map on a metric space. In what follows, the metric space will most often be the closed unit interval with the usual Euclidean distance. Given two dynamical systems, we would like to make rigorous the notion of them being "the same", in some sense. The following definition does exactly this.

Definition 1.2.1. Two dynamical systems $T : X \to X$ and $S : Y \to Y$ are said to be *topologically conjugate* if there is a homeomorphism $h : X \to Y$, called a *conjugacy map*, such that

$$h \circ T = S \circ h.$$

In other words, T and S are topologically conjugate if there exists a homeomorphism h such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "T"] \arrow[d, "h"'] & X \arrow[d, "h"] \\ Y \arrow[r, "S"'] & Y \end{tikzcd}$$

Remark 1.2.2.

1. Topological conjugacy defines an equivalence relation on the space of all dynamical systems.

2. If two dynamical systems T and S are topologically conjugate via a conjugacy map h, then all of their corresponding iterates are topologically conjugate by means of h. That is, $h \circ T^n = S^n \circ h$ for all $n \geq 1$. Therefore there exists a one-to-one correspondence between the orbits of T and those of S. This is why two topologically conjugate systems are considered dynamically equivalent.

For later use, we now introduce a particular dynamical system, namely the shift map on a finite or countable alphabet.

Definition 1.2.3. Let I be a non-empty finite or countably infinite set, henceforth referred to as an *alphabet*. The elements of I will be called *letters* or *symbols*.

- For each $n \in \mathbb{N}$ we shall denote by I^n the set of all *words* comprising n letters from the alphabet I. For convenience, we also denote the *empty word* (that is, the word having no letters) by ε.

  For instance, if $I = \{0,1\}$ then
  $$I^1 = I$$
  and
  $$I^2 = \{00, 01, 10, 11\},$$
  whereas
  $$I^3 = \{000, 100, 010, 001, 110, 101, 011, 111\}.$$

- We will denote by $I^* := \bigcup_{n\in\mathbb{N}} E^n$ the set of all finite non-empty words over the alphabet I. The set of all infinite words will be denoted by $I^\infty := I^{\mathbb{N}}$. In other words,
  $$I^\infty := \left\{\omega = (\omega_i)_{i=1}^\infty : \omega_i \in I \text{ for all } i \in \mathbb{N}\right\}.$$

- We naturally define the length of a word to be the number of letters it consists of. For every $\omega \in I^* \cup I^\infty$, we denote by $|\omega|$ the *length* of ω, that is, the unique $n \in \mathbb{N} \cup \{\infty\}$ such that $\omega \in I^n$. By convention, $|\varepsilon| = 0$.

- If $\omega \in I^* \cup I^\infty$ and $n \in \mathbb{N}$ does not exceed the length of ω, we define the *initial block* $\omega|_n$ to be the initial n-length word of ω, that is, the subword $\omega_1\omega_2\ldots\omega_n$.

- Given two words $\omega, \tau \in I^* \cup I^\infty$, we define their *wedge* $\omega \wedge \tau \in \{\varepsilon\} \cup I^* \cup I^\infty$ to be their longest common initial block.

The wedge of two words is better understood via examples. If $I = \{1,2,3\}$ and we have words $\omega = 12321\ldots$ and $\tau = 12331\ldots$, then $\omega \wedge \tau = 123$. On the other hand, if $\gamma = 22331\ldots$ then $\omega \wedge \gamma = \varepsilon$. Of course, if two (finite or infinite) words ω and τ are equal, then $\omega \wedge \tau = \omega = \tau$.

Let us now introduce a metric on the space I^∞ which reflects the idea that two words are close if they share a long initial block. The longer their common initial subword, the closer two words are.

Definition 1.2.4. Let the metric $d : I^\infty \times I^\infty \to [0,1]$ be defined by $d(\omega, \tau) = 2^{-|\omega\wedge\tau|}$.

Remark 1.2.5. If ω and τ have no common initial block, then $\omega \wedge \tau = \varepsilon$. Thus, $|\omega \wedge \tau| = 0$ and $d(\omega, \tau) = 1$. On the other hand, if $\omega = \tau$ then $|\omega \wedge \tau| = \infty$ and we adopt the convention that $(1/2)^\infty = 0$.

We also have the notion of cylinder sets in this setting.

Definition 1.2.6. Given a finite word $\omega \in I^*$, the *cylinder set* $[\omega]$ generated by ω is the set of all infinite words with initial block ω, that is,

$$[\omega] := \{\tau \in I^\infty : \tau|_{|\omega|} = \omega\} = \{\tau \in I^\infty : \tau_i = \omega_i \text{ for all } 1 \le i \le |\omega|\}.$$

We now introduce the shift map, which is defined by dropping the first letter of each word and shifting all the remaining letters one place to the left.

Definition 1.2.7. The *full left-shift map* (or simply *shift map*) $\sigma : I^\infty \to I^\infty$ is defined by $\sigma(\omega) = \sigma((\omega_i)_{i\in\mathbb{N}}) = (\omega_{i+1})_{i\in\mathbb{N}}$. That is,

$$\sigma(\omega_1\omega_2\omega_3\omega_4\ldots) = \omega_2\omega_3\omega_4\ldots$$

The shift map is $|I|$-to-one on I^∞. In other words, each word has $|I|$ preimages under the shift map. In particular, if I is countably infinite, it follows that σ is countable-to-one. Indeed, given any letter $e \in I$ and any infinite word $\omega \in I^\infty$, the concatenation $e\omega = e\omega_1\omega_2\omega_3\ldots$ of e with ω is a preimage of ω under the shift map, since $\sigma(e\omega) = \omega$.

Proposition 1.2.8. *The shift map $\sigma : (I^\infty, d) \to (I^\infty, d)$ is a dynamical system.*

Proof. Recall that the definition of a dynamical system is a continuous map on a metric space. In this case, (I^∞, d) is a metric space. Also, the shift map is obviously continuous: two words that are close share a long initial block, and thus their images under the shift map, which result from dropping their first letters, will also share a long initial block. More precisely,

$$d(\sigma\omega, \sigma\tau) = 2^{-|\sigma\omega\wedge\sigma\tau|} = 2^{-|\omega\wedge\tau|+1} = 2\cdot 2^{-|\omega\wedge\tau|} = 2d(\omega,\tau)$$

whenever $d(\omega,\tau) < 1$, that is, whenever $|\omega\wedge\tau| \ge 1$. It is evident that if $\omega = \tau$, then both $d(\omega,\tau)$ and $d(\sigma\omega,\sigma\tau)$ are equal to 0.

□

Remark 1.2.9. Note that I^∞ is also the infinite product of $\mathbb{N}$ copies of the discrete space I. As such, for every finite or countable alphabet I, the space I^∞ is totally disconnected. For more details concerning this remark, see Chapter 8 of Willard [82].

1.2.1 Examples of number-theoretic dynamical systems

In this section, we introduce various dynamical systems that generate real number expansions. We refer to such systems as *number-theoretic dynamical systems*. Each example we will consider here is defined on the closed unit interval, $\mathscr{U} := [0,1]$.

For the first example, let us describe the dynamical system that generates the familiar decimal expansion of a number in the unit interval. The decimal expansion is generated by the piecewise-linear map $T : \mathscr{U} \to \mathscr{U}$, given by

$$T(x) := \begin{cases} 10x, & \text{for } x \in [0, 1/10); \\ 10x-1, & \text{for } x \in [1/10, 2/10); \\ \vdots & \vdots \\ 10x-9, & \text{for } x \in [9/10, 1]. \end{cases}$$

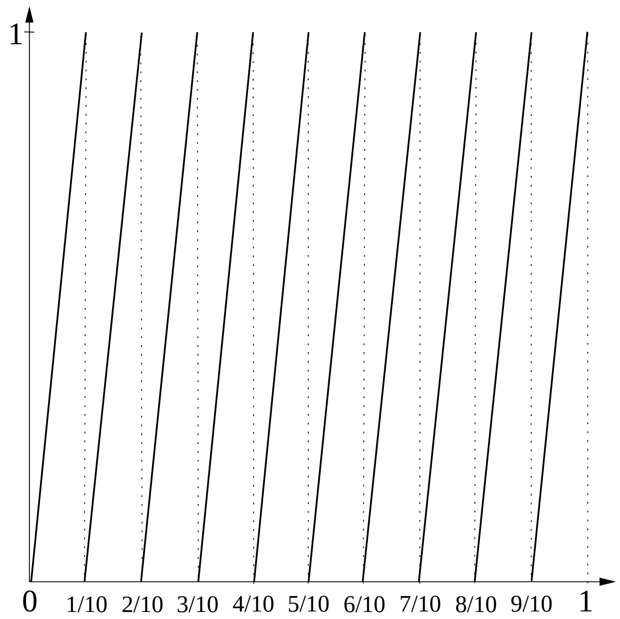


Figure 1.1: The map $T(x)$.

Here, "generated by" means that we have a finite or infinite decimal $x = 0.x_1x_2x_3\ldots$, where each $x_i \in \{0,\ldots,9\}$ and the elements x_i are given by

$$T^{i-1}(x) \in \left[\frac{x_i}{10}, \frac{x_i+1}{10}\right).$$

It is clear that $T(0.x_1x_2x_3\ldots) = 0.x_2x_3\ldots$ and so the map T can be thought of as acting on the infinite decimals as the shift map on the symbol space $E^{\mathbb{N}}$, where E is the finite alphabet $\{0,\ldots,9\}$.

Let us now discuss continued fractions. We give here the very briefest of introductions, but for more details there are several good books available, for instance the classical text by Khintchine [50] or the more modern approach given by Rockett and Szűsz [66]. For the dynamical approach we will outline below, a nice reference is Dajani and Kraaikamp [16].

An expression of the form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}},$$

where $a_0 \in \mathbb{Z}$ and $a_i \in \mathbb{N}$ for each $i \in \mathbb{N}$ is said to be a *regular continued fraction*. The numbers $a_1, a_2, \ldots$ are referred to as the *entries* of the continued fraction and the sequence of entries may be finite or infinite. A finite continued fraction is the result of a finite number of rational operations, so it represents a rational number. Every infinite continued fraction represents an irrational real number and, conversely, every real number can be represented as a continued fraction. We will always assume that $a_0 = 0$, so that we are only considering numbers in the unit interval. To simplify the notation, we write $[a_1, a_2, a_3, \ldots]$. The continued fraction expansion of a real number arises in a natural way from a dynamical system, as follows. First, recall that the notation $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.

Definition 1.2.10. Let $G : \mathscr{U} \to \mathscr{U}$ be defined by

$$G(x) := \begin{cases} \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor & \text{for } 0 < x \leq 1; \\ 0 & \text{if } x = 0. \end{cases}$$

The map G is referred to as the *Gauss map*.

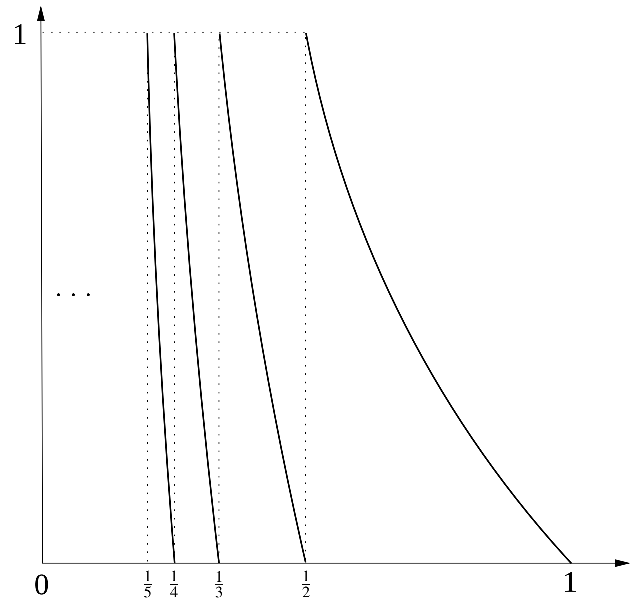


Figure 1.2: The Gauss map, $G : [0,1] \to [0,1]$.

The map G generates the continued fraction expansion of a number $x \in \mathscr{U}$, in the following way. There exists a (finite or infinite) sequence of positive integers $(a_i)_{i\geq 1}$ given by $G^{k-1}(x) \in [1/(a_k+1), 1/a_k)$. Suppose that the sequence is infinite. Then, we have that $x \in [1/(a_1+1), 1/a_1)$, so $\lfloor 1/x \rfloor = a_1$ and so $G(x) = 1/x - a_1$, or,

$$x = \frac{1}{a_1 + G(x)}.$$

Similarly, by assumption we have that $G(x) \in [1/(a_2+1), 1/a_2)$, so following the same argument gives that $G(x) = 1/(a_2 + G^2(x))$ and therefore,

$$x = \frac{1}{a_1 + \frac{1}{a_2 + G^2(x)}}.$$

Without giving every detail, it is hoped that it is clear how the continued fraction expansion of a number is obtained from the map G. It can easily be verified that the map G acts on a point $x = [a_1, a_2, \ldots]$ in the following way:

$$G(x) = [a_2, a_3, \ldots].$$

That is, the map G can be thought of as acting as the shift map on $\mathbb{N}^{\mathbb{N}}$, at least on the irrational points of $\mathscr{U}$. It should be noted, however, that we certainly do not have that G and σ are topologically conjugate. Indeed, since $\mathbb{N}^{\mathbb{N}}$ is totally disconnected and $\mathscr{U}$ is a connected space, there cannot be any continuous map from $\mathscr{U}$ to $\mathbb{N}^{\mathbb{N}}$ (see Theorem 26.3 in Willard [82]). If we define the map $h : \mathbb{N}^{\mathbb{N}} \to \mathscr{U} \setminus \mathbb{Q}$ by setting $h((a_1, a_2, a_3, \ldots)) = [a_1, a_2, a_3, \ldots]$, we do at least have that $h \circ \sigma = G \circ h$ and h is a continuous surjection onto the set of irrational numbers in $\mathscr{U}$. We thus say that G is a *factor* of σ.

Let us now define a related transformation on the unit interval.

Definition 1.2.11. The *Farey map* $F : \mathscr{U} \to \mathscr{U}$ is defined by

$$F(x) := \begin{cases} x/(1-x) & \text{for } 0 \le x \le \frac{1}{2}; \\ (1-x)/x & \text{for } \frac{1}{2} < x \le 1. \end{cases}$$

It is easily shown that the action of the map F on a point $x = [a_1, a_2, \ldots] \in \mathscr{U}$ is as follows:

$$F(x) = \begin{cases} [a_1 - 1, a_2, \ldots] & \text{if } a_1 > 1; \\ [a_2, a_3, \ldots] & \text{if } a_1 = 1. \end{cases}$$

For this reason, the Farey map is sometimes referred to as the *slow continued fraction map*, whereas the Gauss map is referred to as the *fast continued fraction map*. In Section 2.2 below, we will see maps with a similar relationship.

Remark 1.2.12. The Farey map is named for John Farey (1766-1826), who was not a mathematician, but a geologist. Farey's one contribution to Mathematics was the article *On a curious property of vulgar fractions* [25], in which he defines Farey sequences in the following way. For each $n \in \mathbb{N}$, list all the rationals between 0 and 1 which, when expressed in their lowest terms, have denominator at most equal to n. Denoting the n-th Farey sequence by $\mathscr{F}_n$, the first few are given by

$$\mathscr{F}_1 := \left\{\frac{0}{1}, \frac{1}{1}\right\}, \ \mathscr{F}_2 := \left\{\frac{0}{1}, \frac{1}{2}, \frac{1}{1}\right\}, \ \mathscr{F}_3 := \left\{\frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1}\right\},$$
$$\mathscr{F}_4 := \left\{\frac{0}{1}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{1}{1}\right\}, \ \mathscr{F}_5 := \left\{\frac{0}{1}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{1}{1}\right\}, \ldots$$

The curious property of Farey's title is that each member of the sequence is equal to the *mediant* of its two neighbours. Recall that the mediant of two rational numbers a/b and a'/b' is by definition the rational number $(a+a')/(b+b')$. Farey did not himself provide a proof of his discovered property[2] and he was doubtless not the first to notice it. Cauchy supplied the necessary proof in the same year that Farey's article appeared.

The link between this sequence and the Farey map is that if we consider the inverse branches of the Farey map, which are given for $x \in \mathscr{U}$ by

$$F_0(x) = \frac{x}{1+x} \ \text{ and } \ F_1(x) = \frac{1}{1+x},$$

[2]That Farey did not give a proof of his curious property was pointed out by Hardy [33], with the rather unfriendly comment that Farey was "at the best an indifferent mathematician".

and if we iterate the point $1/2$ under each of the two branches, each time one of the Farey fractions turns up. (In Appendix B, there is more about this connection and also how it relates to the modular group.) However, it should be noted that it is not strictly speaking the Farey sequence which appears in this manner. The difference is subtle, but important. The rationals which turn up as the endpoints of intervals in the first few levels of the backwards iteration of the intervals $\{[0,1/2],[1/2,1]\}$ under the inverse branches of the map F are really the *Stern-Brocot* sequences. The n-th Stern-Brocot sequence consists of 2^n proper fractions (not including 0) and the n-th sequence is obtained from the $(n-1)$-th by adding in the mediant of each neighbouring pair. Some of the first few are given by:

$$\mathscr{B}_3 := \left\{\frac{0}{1},\frac{1}{3},\frac{1}{2},\frac{2}{3},\frac{1}{1}\right\}, \quad \mathscr{B}_4 := \left\{\frac{0}{1},\frac{1}{4},\frac{1}{3},\frac{2}{5},\frac{1}{2},\frac{3}{5},\frac{2}{3},\frac{3}{4},\frac{1}{1}\right\},$$

$$\mathscr{B}_5 := \left\{\frac{0}{1},\frac{1}{5},\frac{1}{4},\frac{2}{7},\frac{1}{3},\frac{3}{8},\frac{2}{5},\frac{3}{7},\frac{1}{2},\frac{4}{7},\frac{3}{5},\frac{5}{8},\frac{2}{3},\frac{5}{7},\frac{3}{4},\frac{4}{5},\frac{1}{1}\right\},\ldots$$

The Stern-Brocot sequence was independently discovered by the German number-theorist Moritz Stern and the French clockmaker Achille Brocot. (Brocot used the sequences to design systems of gears. For information on these sorts of applications, see Chapter IV of Rockett and Szűsz [66].) So, it would perhaps be more reasonable to refer to the Farey map as the Stern-Brocot map. However, we choose to stick with convention on this point.

1.3 The classical Lüroth expansion

In this section, we describe a certain series expansion of real numbers and other related topics which motivate the investigation in Chapter 2. We will not give all the details of every statement in this section, deferring instead to the more general case developed in Section 2.1.

In the paper [56], J. Lüroth introduces a series representation of real numbers from the unit interval. His starting point is the observation that for every real number x in the interval $(0,1)$, either $x = 1/(\ell)$, for some positive integer $\ell \geq 2$, or, $1/x$ lies between two successive positive integers ℓ_1 and ℓ_1+1 and so

$$x = \frac{1}{\ell_1+1} + \hat{x},$$

where, since $x < 1/\ell_1$, we have that $0 < \hat{x} < 1/(\ell_1(\ell_1+1))$. Now, defining $x_1 := \hat{x}(\ell_1+1)\ell_1$ supplies the equation

$$x = \frac{1}{\ell_1+1} + \frac{x_1}{\ell_1(\ell_1+1)}.$$

Note that since $0 < \hat{x} < 1/(\ell_1(\ell_1+1))$, we also obtain the inequality $0 < x_1 < 1$. Therefore, the same argument holds for x_1 as for the original point x, which leads to the equation

$$x = \frac{1}{\ell_1+1} + \frac{1}{\ell_1(\ell_1+1)(\ell_2+1)} + \frac{x_2}{\ell_1(\ell_1+1)\ell_2(\ell_2+1)}.$$

Clearly, this process either continues until such a time as one of the x_i is equal to the reciprocal of a positive integer that is at least equal to 2, or continues indefinitely. For the special case that $x = 1$, we notice that $1 = 1/2+1/4+1/8+\ldots$. In each case, this gives the series expansion now called the *Lüroth expansion* of a real number in $\mathscr{U}$.

Each finite expansion of the form above represents a rational number. Suppose now that $x \in \mathscr{U}$ has an infinite Lüroth expansion. Since each ℓ_k is at least equal to 1, for the k-th term in the Lüroth expansion of x we have that

$$\frac{1}{\ell_1(\ell_1+1)\ldots\ell_{k-1}(\ell_{k-1}+1)(\ell_k+1)} \leq \frac{1}{2^k}.$$

Thus, it makes sense to write

$$x = \sum_{n=1}^{\infty}\left(\ell_n \prod_{k=1}^{n}(\ell_k(\ell_k+1))^{-1}\right).$$

For the time being, we will use Lüroth's original notation and write $x = S(\ell_1,\ell_2,\ldots)$ for this sum. For instance, we have that $1 = S(1,1,1,\ldots)$. The next observation in [56] is that if $x \in \mathscr{U}$ has a finite Lüroth expansion, that is, if $x = S(\ell_1,\ell_2,\ldots,\ell_k)$ for some $k \in \mathbb{N}$, then

$$x = S(\ell_1,\ell_2,\ldots,(\ell_k+1),1,1,1,\ldots).$$

Indeed, since $1/(n+2)+1/((n+1)(n+2)) = 1/(n+1)$, we have that

$$\begin{aligned}
x &= S(\ell_1,\ell_2,\ldots,\ell_k) \\
&= \frac{1}{\ell_1+1}+\cdots+\frac{1}{\ell_1(\ell_1+1)\cdots\ell_{k-1}(\ell_{k-1}+1)(\ell_k+1)} \\
&= \frac{1}{\ell_1+1}+\cdots+\frac{1}{\ell_1(\ell_1+1)\cdots\ell_{k-1}(\ell_{k-1}+1)(\ell_k+2)}+ \\
&\quad +\frac{1}{\ell_1(\ell_1+1)\cdots\ell_{k-1}(\ell_{k-1}+1)(\ell_k+1)(\ell_k+2)}\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots\right) \\
&= S(\ell_1,\ell_2,\ldots,(\ell_k+1),1,1,1,\ldots).
\end{aligned}$$

It follows that every number x in $\mathscr{U}\setminus\{0\}$ has an infinite Lüroth expansion. In fact, this infinite representation is unique (we will prove this for more general expansions in the sequel). As already mentioned, each finite Lüroth expansion represents a rational number, but it is easy to see, using only the sum of a geometric series, that each (eventually) periodic infinite Lüroth expansion is also a rational number. Of course, each finite Lüroth expansion can also be written as an eventually periodic expansion; in this case the periodic part consists of infinitely many ones. The proof of these statements are also given in [56].

It seems probable that Lüroth was thinking of a generalisation of the decimal expansion of a real number when he introduced his infinite series expansion. He states that the given expansion has many similarities with the representation through infinite decimal fractions and asks whether or not it is possible to characterise the numbers which have a finite Lüroth expansion in any other way, that is, as in the way that rational numbers with finite decimal representations are exactly those with denominators equal to $2^n 5^m$ for some positive integers n and m. As of the present moment, we are unaware of any answer to this question.

The Lüroth expansion can also be generated by a dynamical system, $L : \mathscr{U} \to \mathscr{U}$. The map L is referred to as the *Lüroth map* and it is defined by

$$L(x) := \begin{cases} n(n+1)x - n, & \text{for } x \in \left[\frac{1}{n+1}, \frac{1}{n}\right),\ n \geq 2; \\ 2x - 1, & \text{for } x \in \left[\frac{1}{2}, 1\right]; \\ 0, & \text{for } x = 0. \end{cases}$$

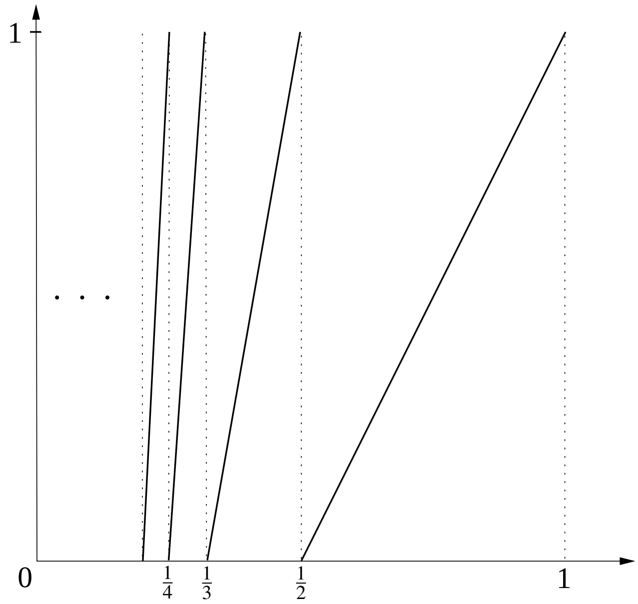


Figure 1.3: The Lüroth map, $L : [0,1] \to [0,1]$.

The Lüroth expansion of a real number in $\mathscr{U}$ is generated by the Lüroth map in the same way as the map T defined in Section 1.2.1 generates the decimal expansion. More precisely, let the real number $x \in \mathscr{U}$ be given and let the finite or infinite sequence $(\ell_k)_{k \geq 1}$ be determined by $L^{k-1}(x) \in [1/(\ell_k + 1), 1/\ell_k)$. We then use the shorthand $x = [\ell_1, \ell_2, \ell_3, \ldots]_L$. It is straightforward to check that this coincides with the definition given above, that is, we have that $S(\ell_1, \ell_2, \ldots) = [\ell_1, \ell_2, \ldots]_L$. Recall that certain rational points in $\mathscr{U}$ have both a finite and an infinite representation; the map L yields the finite representation in these cases. Referring to the graphs of the two maps (given in Figures 1.1 and 1.3), it is clear that in some sense the Lüroth map is a generalisation of the map T. Also, since

$$L([\ell_1, \ell_2, \ell_3, \ldots]_L) = [\ell_2, \ell_3, \ldots]_L,$$

we have that the map L acts as the shift map on the symbol space $\mathbb{N}^{\mathbb{N}}$, in the same sense that the Gauss map does (see Section 1.2.1).

Remark 1.3.1. The Lüroth map is sometimes defined by setting $L(x) := n(n+1)x - n$, where $x \in [1/(n+1), 1/n)$ for each $n \geq 1$, so that the point 1 is not in the domain. It makes the definition look slightly neater, but it is perhaps a little artificial to remove the point 1, since, as already mentioned, it does have a Lüroth expansion.

Recall the initial blocks of an infinite word, given in Section 1.2. It is natural to do something similar here. So, we define the n-th *Lüroth convergent* of $x = [\ell_1, \ell_2, \ell_3 \ldots]_L \in \mathscr{U}$ by setting

$$\begin{aligned}\left(\frac{p_n}{q_n}\right)_L &= [\ell_1, \ldots, \ell_n]_L \\ &= \frac{1}{\ell_1+1} + \frac{1}{\ell_1(\ell_1+1)(\ell_2+1)} + \cdots + \frac{1}{\ell_1(\ell_1+1)\cdots\ell_{n-1}(\ell_{n-1}+1)(\ell_n+1)}.\end{aligned}$$

Observe that we have

$$\left(\frac{p_1}{q_1}\right)_L < \left(\frac{p_2}{q_2}\right)_L < \cdots < \left(\frac{p_n}{q_n}\right)_L < \cdots < x.$$

In the early 1990s, S. Kalpazidou, A. Knopfmacher and J. Knopfmacher [40] introduced a related series expansion of real numbers, which they called the *alternating Lüroth expansion*. This expansion is generated in the same way as the classical Lüroth series, by a similar map, which is referred to as the *alternating Lüroth map*. Before giving the definition of this map, let us first define the *harmonic partition* α_H by setting

$$\alpha_H := \left\{ \left(\frac{1}{n+1}, \frac{1}{n} \right] : n \geq 1 \right\}.$$

The alternating Lüroth map $L_{\alpha_H} : \mathscr{U} \to \mathscr{U}$ is then given by

$$L_{\alpha_H}(x) := \begin{cases} -n(n+1)x + (n+1) & \text{for } x \in \left(\frac{1}{n+1}, \frac{1}{n}\right]; \\ 0 & \text{for } x = 0. \end{cases}$$

With respect to this map, in exactly the way outlined above, the corresponding series expansion of some arbitrary $x \in \mathscr{U}$ turns out to be

$$\begin{aligned} x &= \sum_{n=1}^{\infty} \left((-1)^{n-1}(\ell_n+1) \prod_{k=1}^{n} (\ell_k(\ell_k+1))^{-1} \right) \\ &= \frac{1}{\ell_1} - \frac{1}{\ell_1(\ell_1+1)\ell_2} + \frac{1}{\ell_1(\ell_1+1)\ell_2(\ell_2+1)\ell_3} - \ldots, \end{aligned}$$

where each $\ell_n \in \mathbb{N}$, and the expansion can be finite or infinite. We will use the notation $x = [\ell_1, \ell_2, \ell_3, \ldots]_{\alpha_H}$. The map L_{α_H} can be seen as a linearised generalisation of the Gauss map, as opposed to a generalisation of the map T that generates the decimal expansion. To see why, it is helpful to compare the graphs of these functions given in Figures 1.2 and 1.4.

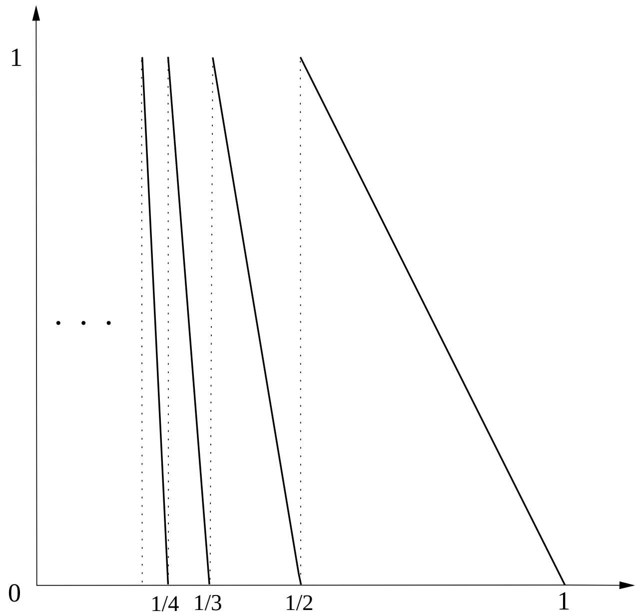
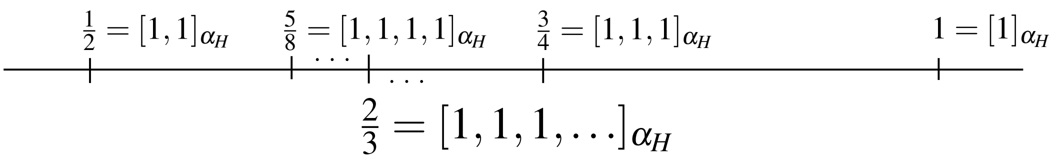


Figure 1.4: The alternating Lüroth map, L_{α_H}.

Just as for the classical Lüroth expansion, for this alternating Lüroth expansion we can form the *alternating Lüroth convergents* by setting

$$\begin{aligned}\left(\frac{p_n}{q_n}\right)_{\alpha_H} &= [\ell_1, \ell_2, \ldots, \ell_n]_{\alpha_H} \\ &= \frac{1}{\ell_1} - \frac{1}{\ell_1(\ell_1+1)\ell_2} + \cdots + \frac{(-1)^{n-1}}{\ell_1(\ell_1+1)\ldots\ell_{n-1}(\ell_{n-1}+1)\ell_n}.\end{aligned}$$

The behaviour of the alternating Lüroth convergents is not the same as the behaviour of the classical Lüroth convergents. We have the following picture, here illustrated by the convergents to the point 1:



We remark that the partition α_H is very similar to the partition that is behind the (non-alternating) Lüroth map. Let us denote the latter by $\widetilde{\alpha_H}$, so that

$$\widetilde{\alpha_H} := \{[1/2, 1], [1/(n+1), 1/n) : n \geq 2\}.$$

We will henceforth denote the Lüroth map L by $L_{\widetilde{\alpha_H}}$ and the Lüroth expansion by $[\ell_1, \ell_2, \ldots]_{\widetilde{\alpha_H}}$.

The Lüroth map and, to a lesser extent, the alternating Lüroth map have been studied by several authors. In addition to those already cited above, these works include [5], [15], [27], [28], [69], [84] and in particular [16].

Chapter 2

Introduction to α-Farey and α-Lüroth maps

In this chapter we will introduce and study the properties of two families of dynamical systems, the α-Lüroth and α-Farey systems, which are both indexed by partitions of the unit interval. These systems are linearised generalisations of the Gauss and Farey maps, respectively, and the first family includes the map L_{α_H} which was introduced in Section 1.3. In Section 2.1, we first make clear exactly what class of partitions of $\mathscr{U}$ we are interested in and then introduce the map L_α. An expansion of real numbers derived from the map L_α is described in some detail. In Section 2.2, we introduce the map F_α and establish the relationship between this new map and L_α. Section 2.3 is concerned with some topological properties of F_α, where we show that for every partition α the map F_α is topologically conjugate to the tent map and give an explicit formula for the conjugating homeomorphism θ_α. It is then shown that this θ_α is both Hölder and sub-Hölder continuous. In Section 2.4, some more specific families of partitions are defined which are used in Section 2.5 to calculate the Hausdorff dimension of various Diophantine-type sets for the expansion generated by L_α. Finally, in Section 2.6, we briefly outline the non-alternating case, that is, the map that generalises the classical Lüroth map.

2.1 Linearised generalisations of the Gauss map

In this section, we aim to introduce a generalisation of the alternating Lüroth map which was described in Section 1.3. The most natural way to generalise this map is to alter the partition α_H. So, throughout the remaining chapters, let $\alpha := \{A_n : n \in \mathbb{N}\}$ denote a countably infinite partition of the unit interval $\mathscr{U}$, consisting of non-empty right-closed and left-open intervals. It is assumed throughout that the elements of α are ordered from right to left, starting from A_1, and that these elements accumulate only at the origin. Let a_n denote the Lebesgue measure $\lambda(A_n)$ of $A_n \in \alpha$ and let $t_n := \sum_{k=n}^{\infty} a_k$ denote the Lebesgue measure of the n-th tail of α. It is clear that $t_1 := \sum_{k=1}^{\infty} a_k = 1$ for every partition α under consideration here. Notice that we can also think of these tails as the endpoints of the intervals composing the partition α, since $A_n = (t_{n+1}, t_n]$. Also, it is perhaps helpful to keep in mind that $t_{n+1} = t_n - a_n$.

Definition 2.1.1. For a given partition α of $\mathscr{U}$, define the map $L_\alpha : \mathscr{U} \to \mathscr{U}$ by

$$L_\alpha(x) := \begin{cases} (t_n - x)/a_n & \text{for } x \in A_n,\ n \in \mathbb{N}; \\ 0 & \text{if } x = 0. \end{cases}$$

The map L_α is referred to as the *α-Lüroth map*.

Remark 2.1.2. For the harmonic partition, $\alpha_H := \{(1/(n+1), 1/n] : n \in \mathbb{N}\}$, one immediately verifies that $a_n = 1/n(n+1)$ and $t_n = 1/n$. Consequently, the definition given above coincides with the definition of the alternating Lüroth map as described in Section 1.3.

In the same way as the Gauss map gives rise to the continued fraction expansion and the Lüroth map gives rise to the Lüroth expansion, for each partition α the map L_α gives rise to a series expansion of numbers in the interval $\mathscr{U}$, which we refer to as the *α-Lüroth expansion*. That is, let $x \in \mathscr{U}$ be given and let the finite or infinite sequence $(\ell_k)_{k\geq 1}$ be determined by $L_\alpha^{k-1}(x) \in A_{\ell_k}$. Note that the sequence will be finite if at some point we have that $L_\alpha^k(x) = 0$ and also note that, with the exception of the special case $x = 1$, each finite sequence has the property that the final entry is at least equal to 2. This sequence gives rise to an alternating series expansion of each $x \in \mathscr{U}$, which is given by

$$x = t_{\ell_1} + \sum_{n=2}^{\infty} (-1)^{n-1} \left(\prod_{i<n} a_{\ell_i} \right) t_{\ell_n} = t_{\ell_1} - a_{\ell_1} t_{\ell_2} + a_{\ell_1} a_{\ell_2} t_{\ell_3} - \dots$$

Let us denote finite α-Lüroth expansions by $[\ell_1, \ell_2, \dots, \ell_k]_\alpha$, for some $k \in \mathbb{N}$, and infinite ones by $x = [\ell_1, \ell_2, \ell_3, \dots]_\alpha$.

Remark 2.1.3. Note that this series expansion is a particular type of *generalised Lüroth series*, a concept which was introduced by Barrionuevo *et al.* in [5] (also see [16]).

We have given a few examples already of expansions arising from maps in this way, but without going into very explicit detail. Let us now give all the details in this case. First assume that $L_\alpha^k(x) \neq 0$ for all $k \in \mathbb{N}$. To start, we have that $x \in A_{\ell_1}$ and so

$$L_\alpha(x) = \frac{t_{\ell_1} - x}{a_{\ell_1}}.$$

Rewriting this, we obtain the relation $x = t_{\ell_1} - a_{\ell_1} L_\alpha(x)$. Then, $L_\alpha(x) \in A_{\ell_2}$ and so

$$L_\alpha^2(x) = \frac{t_{\ell_2} - L_\alpha(x)}{a_{\ell_2}}.$$

Therefore, $L_\alpha(x) = t_{\ell_2} - a_{\ell_2} L_\alpha^2(x)$ and substituting this into what has gone before, we obtain the expression

$$x = t_{\ell_1} - a_{\ell_1} t_{\ell_2} + a_{\ell_1} a_{\ell_2} L_\alpha^2(x).$$

Now suppose that for some $k \in \mathbb{N}$ we have that

$$x = t_{\ell_1} - a_{\ell_1} t_{\ell_2} + a_{\ell_1} a_{\ell_2} t_{\ell_3} - \cdots + (-1)^{k-1} a_{\ell_1} \cdots a_{\ell_{k-1}} L_\alpha^{k-1}(x).$$

Then, $L_\alpha^{k-1}(x) \in A_{\ell_k}$ and so, just as previously,

$$L_\alpha^k(x) = \frac{t_{\ell_k} - L_\alpha^{k-1}(x)}{a_{\ell_k}}.$$

Consequently, we have that $L_\alpha^{k-1}(x) = t_{\ell_k} - a_{\ell_k} L_\alpha^k(x)$ and so, by induction, we obtain the desired infinite α-Lüroth expansion.

If it so happens that $L_\alpha^k(x) = 0$ for some $k \in \mathbb{N}$, since by definition $L_\alpha(0) = 0$, it follows that $L_\alpha^n(x) = 0$ for all $n \geq k$ and instead of an infinite α-Lüroth expansion we obtain a finite one. If k is the smallest positive integer for which $L_\alpha^k(x) = 0$, we infer that

$$0 = L_\alpha^k(x) = \frac{t_{\ell_k} - L_\alpha^{k-1}(x)}{a_{\ell_k}} \Rightarrow L_\alpha^{k-1}(x) = t_{\ell_k} \in A_{\ell_k}$$

and so we obtain the α-Lüroth expansion

$$\begin{aligned} x &= t_{\ell_1} - a_{\ell_1} t_{\ell_2} + \ldots + (-1)^{k-1} a_{\ell_1} \cdots a_{\ell_{k-1}} L_\alpha^{k-1}(x) \\ &= t_{\ell_1} - a_{\ell_1} t_{\ell_2} + \ldots + (-1)^{k-1} a_{\ell_1} \cdots a_{\ell_{k-1}} t_{\ell_k} = [\ell_1, \ell_2, \ldots, \ell_k]_\alpha. \end{aligned}$$

Remark 2.1.4. The final entry ℓ_k, where $k \geq 2$, in any finite α-Lüroth expansion cannot be equal to 1, since otherwise we would have that $L^{k-1}(x) = t_1 = 1$, but there exists no $x \in \mathscr{U}$ that is mapped to the point 1 by L_α. However, note that the α-Lüroth expansion of 1 is always given by $[1]_\alpha$.

From this discussion, it is clear that every element $x \in (0,1]$ has an α-Lüroth expansion. Before going any further, let us describe the action of the map L_α on the expansions it generates. For each $x = [\ell_1, \ell_2, \ell_3, \ldots]_\alpha$, we have, since $x \in A_{\ell_1}$, that

$$\begin{aligned} L_\alpha(x) &= (t_{\ell_1} - x)/a_{\ell_1} = (t_{\ell_1} - (t_{\ell_1} - a_{\ell_1} t_{\ell_2} + a_{\ell_1} a_{\ell_2} t_{\ell_3} + \ldots))/a_{\ell_1} \\ &= t_{\ell_2} + a_{\ell_2} t_{\ell_3} + \ldots = [\ell_2, \ell_3, \ell_4, \ldots]_\alpha. \end{aligned}$$

This shows that L_α, just like the Gauss map and other examples previously mentioned, can be thought of as acting as the shift map on the space $\mathbb{N}^{\mathbb{N}}$, at least for those points in $\mathscr{U}$ with infinite α-Lüroth expansions. In the next proposition we will show that every infinite expansion is unique, but that finite expansions are not.

Proposition 2.1.5.

1. *If $x \in \mathscr{U}$ has an infinite α-Lüroth expansion, then this expansion is unique.*

2. *Each $x \in (0,1)$ which has a finite α-Lüroth expansion can be expanded in exactly two ways, namely,*

$$x = [\ell_1, \ell_2, \ldots, \ell_k]_\alpha = [\ell_1, \ell_2, \ldots, (\ell_k - 1), 1]_\alpha$$

Proof. For the proof of the first statement, suppose, by way of contradiction, that we have two distinct infinite α-Lüroth expansions for a given point $x \in \mathscr{U}$, so

$$x = [\ell_1, \ell_2, \ell_3, \ldots]_\alpha = [m_1, m_2, m_3, \ldots]_\alpha.$$

Expanding this yields

$$t_{\ell_1} - a_{\ell_1} t_{\ell_2} + a_{\ell_1} a_{\ell_2} t_{\ell_3} + \ldots = t_{m_1} - a_{m_1} t_{m_2} + a_{m_1} a_{m_2} t_{m_3} + \ldots$$

It is certain that

$$x < t_{\ell_1} \text{ and } x < t_{m_1}$$

and since $[\ell_1, \ell_2, \ell_3, \ldots]_\alpha = t_{\ell_1} - a_{\ell_1}[\ell_2, \ell_3, \ldots]_\alpha$, we also obtain the inequalities

$$x > t_{\ell_1} - a_{\ell_1} \text{ and } x > t_{m_1} - a_{m_1}.$$

In other words,

$$t_{\ell_1+1} < x < t_{\ell_1} \text{ and } t_{m_1+1} < x < t_{m_1}.$$

Since these are endpoints of partition elements, that is, since x belongs simultaneously to the intervals $[t_{\ell_1+1}, t_{\ell_1})$ and $[t_{m_1+1}, t_{m_1})$, it is apparent that $\ell_1 = m_1$. However, we now have two infinite expansions $[\ell_2, \ell_3, \ell_4, \ldots]_\alpha$ and $[m_2, m_3, m_4, \ldots]_\alpha$ with the property that

$$[\ell_2, \ell_3, \ell_4, \ldots]_\alpha = [m_2, m_3, m_4, \ldots]_\alpha.$$

The same argument clearly applies again and again, allowing us to conclude that $\ell_n = m_n$ for every $n \in \mathbb{N}$, which finishes the proof of the first statement.

For the second assertion, first suppose that $x = [\ell_1, \ldots, (\ell_k - 1), 1]_\alpha$ and, for convenience, assume that k is odd (the case k even is almost identical). Then, recalling that $t_1 = 1$, we have

$$\begin{aligned}
[\ell_1, \ldots, (\ell_k - 1), 1]_\alpha &= t_{\ell_1} - a_{\ell_1} t_{\ell_2} + \ldots + a_{\ell_1} a_{\ell_2} \cdots a_{\ell_{k-1}} t_{\ell_k - 1} - a_{\ell_1} a_{\ell_2} \cdots a_{\ell_k - 1} t_1 \\
&= t_{\ell_1} - a_{\ell_1} t_{\ell_2} + \ldots + a_{\ell_1} \cdots a_{\ell_{k-1}} (t_{\ell_k - 1} - a_{\ell_k - 1}) \\
&= t_{\ell_1} - a_{\ell_1} t_{\ell_2} + \ldots + a_{\ell_1} \cdots a_{\ell_{k-1}} t_{\ell_k} = [\ell_1, \ldots, \ell_k]_\alpha.
\end{aligned}$$

To show that these are the only two possibilities, on the one hand if we suppose that $x = [\ell_1, \ldots, \ell_k]_\alpha = [m_1, \ldots, m_k]_\alpha$, then by the same argument as above we have that $\ell_i = m_i$ for each $1 \leq i \leq k$. On the other hand, if $x = [\ell_1, \ldots, \ell_k]_\alpha = [m_1, \ldots, m_k, m_{k+1}, \ldots, m_{k+n}]_\alpha$ and $m_{k+1} \neq 0$, then we have that

$$\ell_i = m_i \text{ for each } 1 \leq i \leq k-1 \text{ and } [\ell_k]_\alpha = [m_k, m_{k+1}, \ldots, m_{k+n}]_\alpha,$$

which implies that

$$t_{\ell_k} = t_{m_k} - a_{m_k}([m_{k+1}, \ldots, m_{k+n}]_\alpha).$$

There are then two possible cases, either $[m_{k+1}, \ldots, m_{k+n}]_\alpha = 1$, so $m_{k+1} = 1$ and $m_{k+j} = 0$ for $2 \leq j \leq n$, or $[m_{k+1}, \ldots, m_{k+n}]_\alpha \in (0,1)$. In the first case, it immediately follows that $m_k = \ell_k - 1$. Therefore, it only remains to show that the second case is in fact not possible. If the second case were true, we would obtain that $t_{m_k+1} = t_{m_k} - a_{m_k} < t_{\ell_k} < t_{m_k}$. This contradiction finishes the proof.

□

In exactly the way described above for the classical (and alternating) Lüroth series, for each $x = [\ell_1, \ell_2, \ell_3, \ldots]_\alpha \in \mathscr{U}$, if we truncate the α-Lüroth expansion of x after k entries we obtain the k-th *α-Lüroth convergent* of x, that is, for each $k \in \mathbb{N}$ we obtain the finite α-Lüroth expansion $r_k^{(\alpha)}(x)$, given by

$$r_k^{(\alpha)}(x) := [\ell_1, \ldots, \ell_k]_\alpha = t_{\ell_1} - a_{\ell_1} t_{\ell_2} + \ldots + (-1)^{k-1} a_{\ell_1} \cdots a_{\ell_{k-1}} t_{\ell_k}.$$

The behaviour of these convergents is described in the following proposition.

Proposition 2.1.6. *Let $x = [\ell_1, \ell_2, \ell_3, \ldots]_\alpha$. Then, the sequence of α-Lüroth convergents of x satisfies the following four properties.*

1. *The sequence $\left(r_{2n}^{(\alpha)}(x)\right)_{n \geq 1}$ of even convergents is increasing.*
2. *The sequence $\left(r_{2n-1}^{(\alpha)}(x)\right)_{n \geq 1}$ of odd convergents is decreasing.*
3. *Every convergent of odd order is greater than every convergent of even order.*
4. $\lim_{n \to \infty} \left| r_{n+1}^{(\alpha)}(x) - r_n^{(\alpha)}(x) \right| = 0.$

Proof. The proof of the first two statements is very similar, so we give only the proof for the even case. It suffices to show that $r_{2(n+1)}^{(\alpha)}(x) - r_{2n}^{(\alpha)}(x) \geq 0$, for every $n \in \mathbb{N}$. We have that

$$\begin{aligned} r_{2(n+1)}^{(\alpha)}(x) - r_{2n}^{(\alpha)}(x) &= t_{\ell_1} - a_{\ell_1} t_{\ell_2} + \ldots - a_{\ell_1} \cdots a_{\ell_{2n-1}} t_{\ell_{2n}} + a_{\ell_1} \cdots a_{\ell_{2n}} t_{\ell_{2n+1}} - a_{\ell_1} \cdots a_{\ell_{2n+1}} t_{\ell_{2n+2}} \\ &\quad - \left(t_{\ell_1} - a_{\ell_1} t_{\ell_2} + \ldots - a_{\ell_1} \cdots a_{\ell_{2n-1}} t_{\ell_{2n}}\right) \\ &= a_{\ell_1} \cdots a_{\ell_{2n}} \left(t_{\ell_{2n+1}} - a_{\ell_{2n+1}} t_{\ell_{2n+2}}\right) = a_{\ell_1} \cdots a_{\ell_{2n}} \left([\ell_{2n+1}, \ell_{2n+2}]_\alpha\right). \end{aligned}$$

It is clear that this last quantity is greater than zero for each $n \in \mathbb{N}$.

In order to prove the third statement of the proposition, first fix some $k \in \mathbb{N}$. Then, since $r_{2k+1}^{(\alpha)}(x) = r_{2k}^{(\alpha)}(x) + a_{\ell_1} \ldots a_{\ell_{2k}} t_{\ell_{2k+1}}$, we deduce that $r_{2k+1}^{(\alpha)}(x) > r_{2k}^{(\alpha)}(x)$. Then, as the sequence of odd order convergents is decreasing,

$$t_{\ell_1} = r_1^{(\alpha)}(x) > r_3^{(\alpha)}(x) > \cdots > r_{2k+1}^{(\alpha)}(x) > r_{2k}^{(\alpha)}(x).$$

Now, by way of contradiction, suppose that there exists some $n > k$ with the property that $r_{2k}^{(\alpha)}(x) \geq r_{2n+1}^{(\alpha)}(x)$. It follows, using part one, that $r_{2k}^{(\alpha)}(x) < \cdots < r_{2n}^{(\alpha)}(x)$, after which we obtain the inequality

$$r_{2n+1}^{(\alpha)}(x) \leq r_{2k}^{(\alpha)}(x) < r_{2n}^{(\alpha)}(x).$$

But now, given that $r_{2n+1}^{(\alpha)}(x) - r_{2n}^{(\alpha)}(x) = a_{\ell_1} \ldots a_{\ell_{2n}} t_{\ell_{2n+1}} > 0$, we reach a contradiction. Consequently, there exists no such n, and the proof of part three is finished.

Finally, for part four, first notice that there must exist a maximum value of a_n, since the sequence $(a_1, a_2, \ldots)$ sums to 1. Moreover, this maximum value must be strictly less than 1. Hence,

$$\lim_{n\to\infty} \left| r_{n+1}^{(\alpha)}(x) - r_n^{(\alpha)}(x) \right| = \lim_{n\to\infty} a_{\ell_1} \cdots a_{\ell_n} t_{\ell_{n+1}} \leq \lim_{n\to\infty} \left(\max_{n\in\mathbb{N}} a_n \right)^n = 0.$$

□

By analogy with continued fractions, for which a number is rational if and only if it has a finite continued fraction expansion, we say that $x \in \mathscr{U}$ is an *α-rational number* when x has a finite α-Lüroth expansion and say that x is an *α-irrational number* otherwise[1]. Of course, the set of α-rationals is a countable set. The reader should also notice that the α-rationals are not necessarily equal to actual rational numbers, unless the partition α is chosen to consist of intervals with rational endpoints. Note that the α_H-rational numbers (for the classical alternating Lüroth map), described in Section 1.3, are all rational. For a second example, consider the *dyadic partition* $\alpha_D := \left\{ \left[1/2^n, 1/2^{n-1}\right) : n \in \mathbb{N} \right\}$. In this case, one easily verifies that the α_D-rational numbers are exactly the dyadic rationals, that is, the set $\{m/2^n : 1 \leq m \leq 2^n, n \in \mathbb{N}\}$.

We will now define the cylinder sets associated with the map L_α.

Definition 2.1.7. For each k-tuple $(\ell_1, \ldots, \ell_k)$ of positive integers, define the *α-Lüroth cylinder set $C_\alpha(\ell_1, \ldots, \ell_k)$* associated with the α-Lüroth expansion to be

$$C_\alpha(\ell_1, \ldots, \ell_k) := \{[y_1, y_2, \ldots]_\alpha : y_i = \ell_i \text{ for } 1 \leq i \leq k\}.$$

Observe that these cylinder sets are closed intervals with endpoints given by $[\ell_1, \ldots, \ell_k]_\alpha$ and $[\ell_1, \ldots, (\ell_k + 1)]_\alpha$. If k is even, it follows from Proposition 2.1.6 that $[\ell_1, \ldots, \ell_k]_\alpha$ is the left endpoint of this interval. Likewise, if k is odd, $[\ell_1, \ldots, \ell_k]_\alpha$ is the right endpoint.

Suppose that k is even. Recall that if $\ell_k > 1$, then both $[\ell_1, \ldots, \ell_k]_\alpha$ and $[\ell_1, \ldots, (\ell_k - 1), 1]_\alpha$ are different representations of the same point in $\mathscr{U}$. To avoid any confusion note that the point $[\ell_1, \ldots, \ell_k]_\alpha$ is the **left** endpoint of the cylinder set $C_\alpha(\ell_1, \ldots, \ell_k)$ and the **right** endpoint of the cylinder set $C_\alpha(\ell_1, \ldots, (\ell_k - 1), 1)$. These are distinct sets, as consideration of the definition of an L_α-cylinder set given above will quickly confirm.

Directly from the values of its endpoints, for the Lebesgue measure of $C_\alpha(\ell_1, \ldots, \ell_k)$ we have that

$$\lambda(C_\alpha(\ell_1, \ldots, \ell_k)) = \prod_{i=1}^{k} a_{\ell_i}.$$

Also, from this formula, we have the relation

$$\lambda(C_\alpha(\ell_1, \ldots, \ell_k)) = \prod_{i=1}^{k} \lambda(C_\alpha(\ell_i)).$$

[1] It would also be reasonable to choose to set the α-rational numbers to be those with finite or periodic α-Lüroth expansions, in line with the classical Lüroth expansion. However, in light of results such as Proposition 2.2.9 below, we have decided against this.

Finally in this section, we give a *natural extension* for the map L_α. A natural extension is a way of creating an invertible transformation out of a non-invertible one, by adding on a "past", as it were, to each point. In order to do this, we first define the inverse branches of L_α. (For a more rigorous explanation of natural extensions, the reader is referred to [16].) Note that this is the same construction as given in [5].

Definition 2.1.8.

1. The inverse branches $L_{\alpha,n} : [0,1) \to A_n$ of L_α are given for each $n \in \mathbb{N}$ by
$$L_{\alpha,n}(x) = t_n - a_n x.$$

2. The natural extension $L_\alpha^+ : [0,1) \times \mathscr{U} \to \mathscr{U} \times [0,1)$ of L_α is given by
$$L_\alpha^+((x,y)) = (L_{\alpha,n}(x), L_\alpha(y)), \ \text{ for } y \in A_n, \text{ for all } n \in \mathbb{N}.$$

One immediately verifies that the action of this map is as follows:

$$L_\alpha^+([\ell_1,\ell_2,\ell_3,\ldots]_\alpha, [m_1,m_2,m_3,\ldots]_\alpha) = ([m_1,\ell_1,\ell_2,\ldots]_\alpha, [m_2,m_3,m_4,\ldots]_\alpha).$$

Since this map is designed to be invertible, it ought to be, and is, easy to describe the inverse. We have that $(L_\alpha^+)^{-1} : \mathscr{U} \times [0,1) \to [0,1) \times \mathscr{U}$ is given by

$$(L_\alpha^+)^{-1}((x,y)) = (L_\alpha(x), L_{\alpha,n}(y)) \ \text{ for } x \in A_n, \text{ for all } n \in \mathbb{N}.$$

2.2 Linearised Farey-like maps

Let us now introduce a second family of maps, indexed by the same set of partitions α of $\mathscr{U}$. We will soon see that these new maps are closely linked with the maps L_α introduced in the previous section.

Definition 2.2.1. For a given partition $\alpha := \{A_n : n \in \mathbb{N}\}$ of $\mathscr{U}$, define the map $F_\alpha : \mathscr{U} \to \mathscr{U}$ by

$$F_\alpha(x) := \begin{cases} (1-x)/a_1 & \text{if } x \in A_1, \\ a_{n-1}(x - t_{n+1})/a_n + t_n & \text{if } x \in A_n, \text{ for } n \geq 2, \\ 0 & \text{if } x = 0. \end{cases}$$

The map F_α is referred to as the *α-Farey map*.

Although the formula looks a bit cryptic, all that the transformation F_α does is map the set A_1 linearly onto the interval $[0,1)$ and, for each $n \geq 2$, map the interval A_n linearly onto the interval A_{n-1}. In particular, notice that $F_\alpha|_{A_1} = L_\alpha|_{A_1}$. The action of F_α on each point $x = [\ell_1,\ell_2,\ldots]_\alpha \in \mathscr{U}$ is given by

$$F_\alpha(x) := \begin{cases} [\ell_2,\ell_3,\ldots]_\alpha & \text{for } \ell_1 = 1; \\ [\ell_1 - 1,\ell_2,\ell_3,\ldots]_\alpha & \text{for } \ell_1 \geq 2. \end{cases}$$

Notice that the map F_α acts on the α-Lüroth entries of x in precisely the same way as the Farey map acts on the continued fraction entries of each point $x \in \mathscr{U}$ (see Section 1.2.1). This is the reason for the name given to F_α.

Example 2.2.2.

1. For the harmonic partition $\alpha_H := \{[1/(n+1), 1/n) : n \in \mathbb{N}\}$, we obtain the α_H-Farey map F_{α_H}, which is given explicitly by

$$F_{\alpha_H}(x) := \begin{cases} 2-2x & \text{for } x \in A_1; \\ \frac{n+1}{n-1}x - \frac{1}{n(n-1)} & \text{for } x \in A_n\ . \end{cases}$$

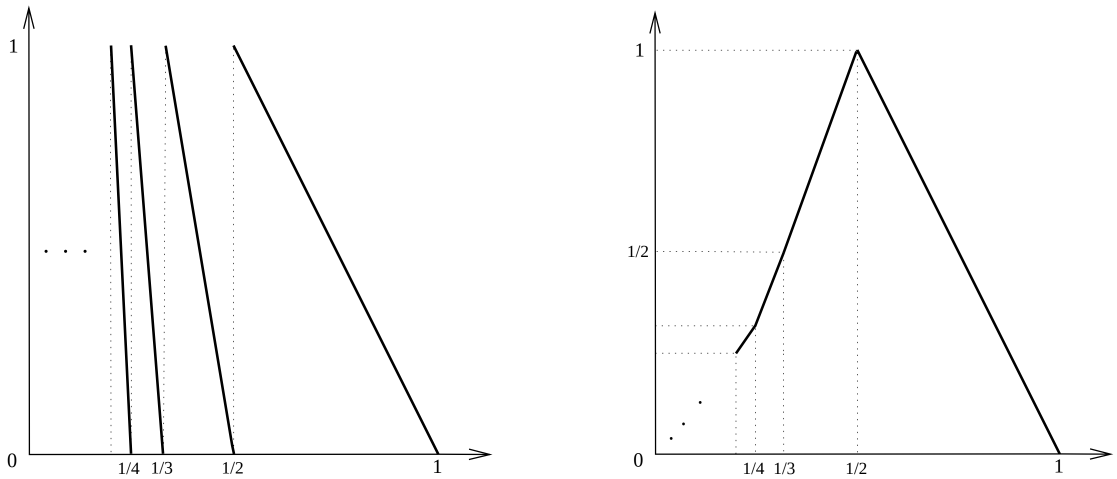


Figure 2.1: The α_H-Lüroth and α_H-Farey map, where $t_n = 1/n$, $n \in \mathbb{N}$.

2. Consider the dyadic partition $\alpha_D := \left\{\left[1/2^n, 1/2^{n-1}\right) : n \in \mathbb{N}\right\}$. One can immediately verify that the map F_{α_D} coincides with the tent map, which is given by

$$F_{\alpha_D}(x) := \begin{cases} 2x & \text{for } x \in [0, 1/2); \\ 2-2x & \text{for } x \in [1/2, 1]. \end{cases}$$

To see this, it is enough to note that for each $n \in \mathbb{N}$ we have that $a_n = 2^{-n}$ and $t_n = 2^{-(n-1)}$. We will revisit this example in Section 2.3.

3. More generally, suppose that the partition α_β is defined by the condition that $t_n = \beta^{n-1}$ for some $0 < \beta < 1$. Then, $a_n = (1-\beta)\beta^{n-1}$ and the associated α_β-Farey map F_{α_β} is defined by

$$F_{\alpha_\beta}(x) := \begin{cases} \frac{1}{\beta}x & \text{for } x \in [0, t_2); \\ \frac{1}{1-\beta}(1-x) & \text{for } x \in A_1. \end{cases}$$

In other words, the map F_{α_β} is a "skewed" tent map. An example graph is shown in Figure 2.3, below.

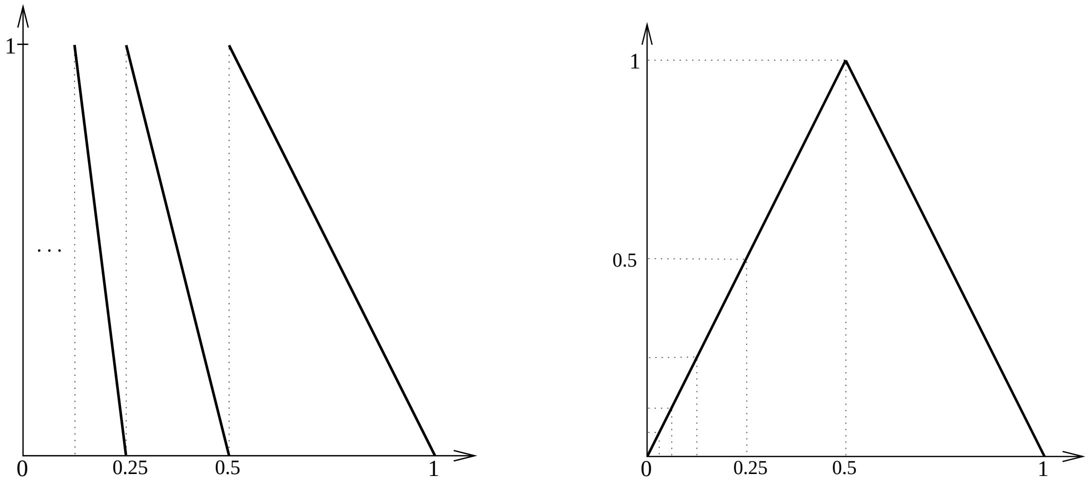


Figure 2.2: The α_D-Lüroth and α_D-Farey map, where $t_n = (1/2)^{n-1}$, $n \in \mathbb{N}$.

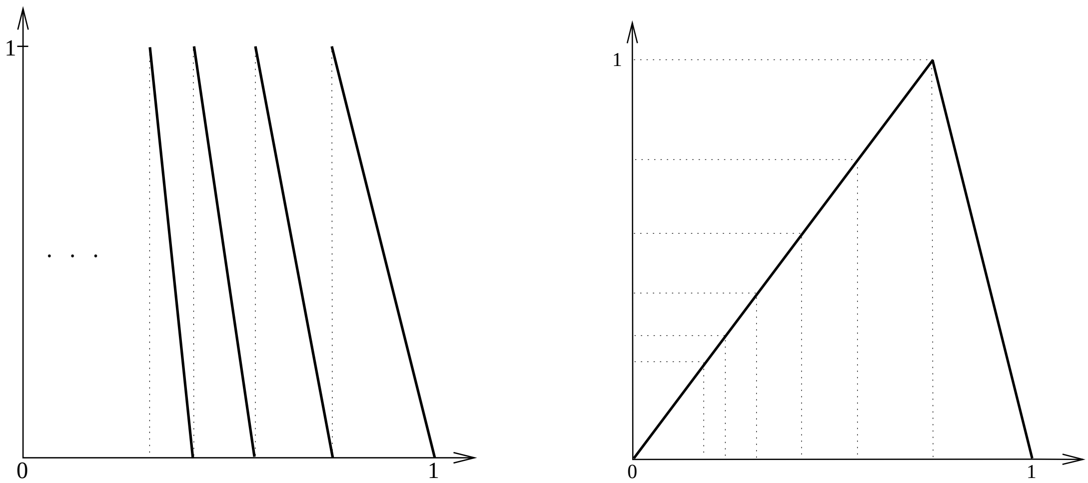


Figure 2.3: An example of an α_β-Lüroth and α_β-Farey map, where $\beta = 3/4$.

Before stating the next lemma, we must define the jump transformation of the dynamical system F_α.

Definition 2.2.3. Let the map $\rho_\alpha : \mathscr{U} \setminus \{0\} \to \mathbb{N} \cup \{0\}$ be defined by setting

$$\rho_\alpha(x) := \inf\{n \geq 0 : F_\alpha^n(x) \in A_1\}.$$

The map ρ_α is said to be the *first return to A_1 map* of F_α. Then, let the map $F_\alpha^* : \mathscr{U} \to \mathscr{U}$ be defined by

$$F_\alpha^*(x) := \begin{cases} F_\alpha^{\rho_\alpha(x)+1}(x) & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

The map F_α^* is said to be the *jump transformation on* A_1 of F_α.

Remark 2.2.4.

1. In the case of the map F_α, it is clear that the function ρ_α is well-defined for all $x \in (0,1]$. It is not always the case for any dynamical system $T : X \to X$ that such a map is well-defined. In general, some further condition on T is needed, for instance that it is a conservative map. In fact, F_α is conservative, as we shall see in Chapter 3.

2. There is some potential for confusion surrounding jump transformations. A related concept exists in the literature, which is that of the map *induced on* A_1. This is a map $F_{A_1} : \mathscr{U} \to A_1$ defined by first letting $\phi(x) := \inf\{n \geq 1 : F_\alpha^n(x) \in A_1\}$ and then setting $F_{A_1}(x) = F_\alpha^{\phi(x)}(x)$. Although these appear to be very similar, they are not the same. The most obvious difference is that the range of F_α^* is the whole of $\mathscr{U}$, whereas the range of F_{A_1} is only the partition element A_1.

Lemma 2.2.5. *The jump transformation F_α^* of F_α coincides with the α-Lüroth map L_α.*

Proof. Since $x \in A_n$ if and only if $x = [n, \ell_2, \ell_3, \ldots]_\alpha$, for each $n \geq 2$, we have that

$$F_\alpha^*(x) = F_\alpha^n([n, \ell_2, \ell_3, \ldots]_\alpha) = F_\alpha^{n-1}([n-1, \ell_2, \ell_3, \ldots]_\alpha) = \cdots = [\ell_2, \ell_3, \ldots]_\alpha = L_\alpha(x).$$

On the other hand, if $x = [1, \ell_2, \ell_3, \ldots]_\alpha \in A_1$, then $\rho_\alpha(x)$ is equal to zero and so we have that $F_\alpha^*(x) = F_\alpha(x)$, which is again equal to $L_\alpha(x)$, since $L_\alpha|_{A_1} = F_\alpha|_{A_1}$.

□

Remark 2.2.6. It is well known that the very same relationship exists between the Farey map and the Gauss map, that is, the Gauss map coincides with the jump transformation on $(1/2, 1]$ of the Farey map.

Let us now describe how to construct a partition α^* from the partition α and its associated coding for the map F_α. The partition α^* is given by $\{A, B\}$, where $A := A_1$ and $B := \mathscr{U} \setminus A_1$. Each α-irrational number in $\mathscr{U}$ has an infinite coding $x = \langle x_1, x_2, \ldots \rangle_\alpha \in \{0,1\}^{\mathbb{N}}$, which is given by $x_k = 1$ if and only if $F_\alpha^{k-1}(x) \in A$ for each $k \in \mathbb{N}$. This coding will be referred to as the α-Farey coding. The α-Farey coding is related to the α-Lüroth coding in a straightforward way. Namely, if $x \in \mathscr{U}$ has α-Lüroth coding given by $x = [\ell_1, \ell_2, \ell_3, \ldots]_\alpha$, then the α-Farey coding of x is given by $x = \langle 0^{\ell_1 - 1}, 1, 0^{\ell_2 - 1}, 1, 0^{\ell_3 - 1}, 1, \ldots \rangle_\alpha$, where 0^n denotes the sequence of n 0s. Of course, it only works in this way for α-irrational numbers, that is, those with infinite α-Lüroth codes. We can say more though. For an α-rational number $x = [\ell_1, \ell_2, \ldots, \ell_k]_\alpha$, one immediately verifies that this number has an α-Farey coding given by $\langle 0^{\ell_1 - 1}, 1, 0^{\ell_2 - 1}, 1, \ldots, 0^{\ell_k - 1}, 1, 0, 0, 0, \ldots \rangle_\alpha$. In this respect, the α-Farey coding could be considered more convenient[2], as it allows for an infinite representation of every number in $\mathscr{U}$. For example, we have that $1 = [1]_\alpha = \langle 1, 0, 0, 0, \ldots \rangle_\alpha$.

[2]The α-rational numbers still have two representations. For the point $[\ell_1, \ldots, \ell_k]_\alpha$, the other is given by $\langle 0^{\ell_1 - 1}, 1, 0^{\ell_2 - 1}, 1, \ldots, 0^{\ell_k - 2}, 1, 1, 0, 0, 0, \ldots \rangle_\alpha$.

Recall that F_α acts on $x = [\ell_1, \ell_2, \ldots]_\alpha$ in the following way:

$$F_\alpha(x) := \begin{cases} [\ell_1 - 1, \ell_2, \ell_3, \ldots]_\alpha & \text{for } \ell_1 \geq 2; \\ [\ell_2, \ell_3, \ldots]_\alpha & \text{for } \ell_1 = 1. \end{cases}$$

In particular, this means that if we instead write x in its α-Farey coding, so $x = \langle x_1, x_2, \ldots \rangle_\alpha$, then

$$F_\alpha(x) := \langle x_2, x_3, \ldots \rangle_\alpha.$$

So, again, the map F_α can be thought of as acting like the shift map σ on the shift space E^∞, where this time the alphabet $E = \{0, 1\}$ is finite. This is potentially a slight advantage of the map F_α, since $\{0, 1\}^\infty$ is a compact metric space, whereas $\mathbb{N}^\mathbb{N}$ is not.

Let us now define the cylinder sets associated with the map F_α.

Definition 2.2.7. For each n-tuple $(x_1, \ldots, x_n)$ of positive integers, define the *α-Farey cylinder set* $\widehat{C}_\alpha(x_1, \ldots, x_n)$ by setting

$$\widehat{C}_\alpha(x_1, \ldots, x_n) := \{\langle y_1, y_2, \ldots \rangle_\alpha : y_k = x_k, \text{ for } 1 \leq k \leq n\}.$$

Notice that every α-Lüroth cylinder set is also an α-Farey cylinder set, whereas the converse of this statement is not true. The precise description of the correspondence is that any α-Farey cylinder set which has the form $\widehat{C}_\alpha(0^{\ell_1 - 1}, 1, \ldots, 0^{\ell_k - 1}, 1)$ coincides with the α-Lüroth cylinder set $C_\alpha(\ell_1, \ldots, \ell_k)$ but if an α-Farey cylinder set is defined by a finite word ending in a 0, then it cannot be translated to a single α-Lüroth cylinder set. However, we do have the relation

$$\widehat{C}_\alpha(0^{\ell_1 - 1}, 1, 0^{\ell_2 - 1}, 1, \ldots, 0^{\ell_k - 1}, 1, 0^m) = \bigcup_{n \geq m+1} C_\alpha(\ell_1, \ell_2, \ldots, \ell_k, n).$$

It therefore follows that

$$\lambda(\widehat{C}_\alpha(0^{\ell_1 - 1}, 1, 0^{\ell_2 - 1}, 1, \ldots, 0^{\ell_k - 1}, 1, 0^m)) = \sum_{n \geq m+1} \lambda(C_\alpha(\ell_1, \ell_2, \ldots, \ell_k, n)) = a_{\ell_1} a_{\ell_2} \cdots a_{\ell_k} t_{m+1}.$$

In addition, we can identify the endpoints of each α-Farey cylinder set. If we consider the set $\widehat{C}_\alpha(0^{\ell_1 - 1}, 1, \ldots, 0^{\ell_k - 1}, 1)$, then we already know the endpoints of this interval (since it is also equal to an α-Lüroth cylinder set). On the other hand, if instead we have the α-Farey cylinder set $\widehat{C}_\alpha(0^{\ell_1 - 1}, 1, 0^{\ell_2 - 1}, 1, \ldots, 0^{\ell_k - 1}, 1, 0^m)$, the endpoints are given by $[\ell_1, \ldots, \ell_k, m+1]_\alpha$ and $[\ell_1, \ldots, \ell_k]_\alpha$.

Definition 2.2.8. Let the two inverse branches of the map F_α be denoted by

$$F_{\alpha,0} : \mathscr{U} \to \bigcup_{n \geq 2} A_n = \widehat{C}_\alpha(0) \;\text{ and }\; F_{\alpha,1} : [0, 1) \to A_1 = \widehat{C}_\alpha(1).$$

With the convention that $F_{\alpha,0}(0) = 0$, these are given by

$$F_{\alpha,0}(x) := \frac{a_{n+1}}{a_n}(x - t_{n+1}) + t_{n+2}, \text{ for } x \in A_n,\ n \geq 1 \;\text{ and }\; F_{\alpha,1}(x) := 1 - a_1 x, \text{ for } x \in \mathscr{U}.$$

Note that $F_{\alpha,0}$ maps the interval A_n into the interval A_{n+1}, for each $n \in \mathbb{N}$.

In preparation for the next lemma, we now describe the α-Farey decomposition of the interval $\mathscr{U}$, which is obtained by iterating the maps $F_{\alpha,0}$ and $F_{\alpha,1}$ on $\mathscr{U}$. (This is the equivalent of finding the Stern-Brocot sequences for the classical Farey map.) The first iteration gives rise to the partition $\alpha^* := \{\widehat{C}_\alpha(0), \widehat{C}_\alpha(1)\}$. Iterating a second time yields the refined partition $\alpha_1^* := \{\widehat{C}_\alpha(00), \widehat{C}_\alpha(01), \widehat{C}_\alpha(11), \widehat{C}_\alpha(10)\}$. Continuing the iteration further, we obtain successively refined partitions α_k^* of $\mathscr{U}$ consisting of 2^k α-Farey cylinder sets of the form $\widehat{C}_\alpha(x_1, \ldots, x_k)$, for every $k \in \mathbb{N}$. It is clear that exactly half of these are also α-Lüroth cylinder sets. Figure 2.4 on the opposite page illustrates the first four levels of the α-Farey decomposition. In the diagram, the α-Farey cylinder sets that correspond to α-Lüroth cylinder sets are indicated with thicker lines.

It is evident that both the endpoints of each of these so-obtained intervals are distinct α-rational numbers, moreover, every α-rational number is obtained in this way, as we show in the next proposition.

Proposition 2.2.9. *Every α-rational number $[\ell_1, \ell_2, \ldots, \ell_k]_\alpha$ is an endpoint of an interval obtained in the α-Farey decomposition.*

Proof. First, one immediately verifies that for $x = [\ell_1, \ell_2, \ldots]_\alpha$ (where this expansion can be finite or infinite), the inverse branches of the map F_α act in the following way:

$$F_{\alpha,0}(x) = [\ell_1 + 1, \ell_2, \ldots]_\alpha \text{ and } F_{\alpha,1}(x) = [1, \ell_1, \ell_2, \ldots]_\alpha.$$

In particular, we have that $\mathscr{U}$ is mapped by $F_{\alpha,0}$ onto the interval $\widehat{C}_\alpha(0) = [0, [2]_\alpha]$ and $[0,1)$ is mapped by $F_{\alpha,1}(x)$ onto the interval $\widehat{C}_\alpha(1)$. In the next layer of the decomposition, two more α-rationals appear, namely $F_{\alpha,0}([2]_\alpha) = [3]_\alpha$ and $F_{\alpha,1}([2]_\alpha) = [1,2]_\alpha$. It is clear that by iterating this process, only α-rationals can turn up in each layer. Moreover, since we start in level 1 with the point $[2]_\alpha$ and either put a 1 into the first position, moving the existing entries to the right, or change the first entry to $\ell_1 + 1$, each of the added points in each level n has the form $[\ell_1, \ldots \ell_k]_\alpha$ where $\sum_{i=1}^k \ell_i = n+1$ and $\ell_k \neq 1$.

It suffices now to calculate the possible number of finite sequences $(i_1, \ldots, i_k)$ whose entries sum to n, for each $n \in \mathbb{N}$, and are such that $i_k \geq 2$. To do this, we will first prove by induction that the number of possible finite sequences $(i_1, \ldots, i_k)$ with $\sum_{j=1}^k i_j = n$ is equal to 2^{n-1}. To start, notice that there are two ways to sum to 2, namely, (1, 1) and (2). There are also four ways to sum to 3; these are (1,1,1), (1, 2), (2, 1) and (3).

We want to calculate the cardinality of the set $S_n := \{(i_1, i_2, \ldots, i_k) : \sum_{j=1}^k i_j = n\}$. First notice that we can split this up in the following way:

$$S_n = \left\{ (i_1, i_2, \ldots, i_k) : \sum_{j=1}^k i_j = n \text{ and } i_k \geq 2 \right\} \cup \left\{ (i_1, i_2, \ldots, i_k, 1) : \sum_{j=1}^k i_j = n-1 \right\}.$$

It is clear that the latter set in this union has cardinality equal to that of S_{n-1}. Call the first set in this union S_n'. We claim that the cardinality of S_n' is also equal to that of S_{n-1}. Indeed,

$$(i_1, \ldots, i_{k-1}, i_k) \in S_{n-1} \Leftrightarrow (i_1, \ldots, i_{k-1}, (i_k + 1)) \in S_n'.$$

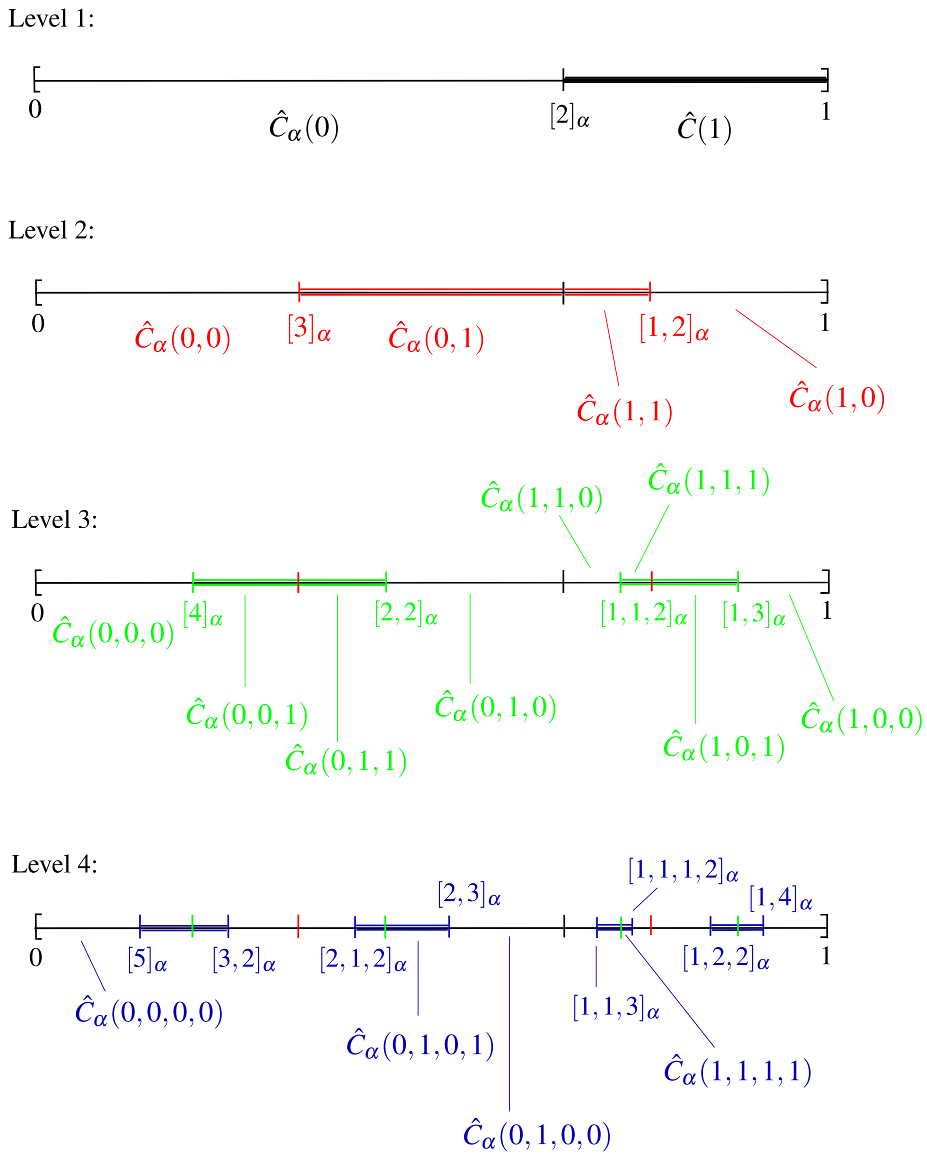


Figure 2.4: The first four levels of the α-Farey decomposition.

To finish the inductive step, suppose that we have $\#S_{n-1} = 2^{n-2}$. Then, by the argument above, we have that $\#S_n = 2 \cdot \#S_{n-1} = 2 \cdot 2^{n-2} = 2^{n-1}$.

To conclude, we have shown that the number of ways of choosing a finite sequence $(i_1, \ldots, i_k)$ with $\sum_{j=1}^{k} i_j = n$ and $i_k \geq 2$ is equal to 2^{n-2}. Recall that at level n of the Farey decomposition, we obtain 2^{n-1} distinct α-rationals with entries summing to $n+1$ and with last entry at least equal to 2. Given that there can only be 2^{n-1} such expansions, this exhausts all the possibilities. So, we do indeed come to every α-rational in this way.

□

Remark 2.2.10. Note that as the inverse branches of the actual Farey map act on the continued fraction entries in exactly the same way that the inverse branches of the map F_α act on the α-Lüroth entries, this proof also holds for the original Farey map. Of course, the fact that the Farey decomposition gives rise to all the rational numbers in the interval $\mathscr{U}$ is well known.

Finally, let us also state a natural extension for the map F_α.

Definition 2.2.11. The natural extension $F_\alpha^+ : \mathscr{U} \times \mathscr{U} \to \mathscr{U} \times \mathscr{U}$ is given by

$$F_\alpha^+(x,y) := \begin{cases} (F_{\alpha,0}(x), F_\alpha(y)) & \text{for } y \in \mathscr{U} \setminus A_1; \\ (F_{\alpha,1}(x), F_\alpha(y)) & \text{for } y \in A_1. \end{cases}$$

One immediately verifies that the action of this map is as follows.

$$F_\alpha^+(\langle x_1, x_2, x_3, \ldots \rangle_\alpha, \langle y_1, y_2, y_3, \ldots \rangle_\alpha) = (\langle y_1, x_1, x_2, \ldots \rangle_\alpha, \langle y_2, y_3, y_4, \ldots \rangle_\alpha).$$

2.3 Topological Properties

Before stating the first proposition, we remind the reader that the measure of maximal entropy μ_α for the system F_α is the measure that assigns mass 2^{-n} to each n-th level α-Farey cylinder set[3]. Also, the *distribution function* Δ_μ of a measure μ with support in $[0,1]$ is defined for each $x \in [01]$ by

$$\Delta_\mu(x) := \mu([0,x]).$$

Note that a distribution function is always non-decreasing and right-continuous (see Theorem 9.1.1 in Dudley [18]). In the case of the measure μ_α, since μ_α is non-atomic, the function Δ_{μ_α} is actually continuous. Finally, recall that the definition of topological conjugacy was given in Section 1.2.

Proposition 2.3.1. *The dynamical systems $(\mathscr{U}, F_\alpha)$ and $(\mathscr{U}, F_{\alpha_D})$ are topologically conjugate and the conjugating homeomorphism is given, for each $x = [\ell_1, \ell_2, \ldots]_\alpha$, by*

$$\theta_\alpha(x) := -2 \sum_{k=1}^{\infty} (-1)^k 2^{-\sum_{i=1}^{k} \ell_i}.$$

[3]For our purposes, this can be thought of as the definition of the measure of maximal entropy. The reader is referred to Walters [81] for more details.

Moreover, the map θ_α is equal to the distribution function of the measure of maximal entropy μ_α for the α-Farey map.

Proof. We will first show by induction that the map θ_α is indeed equal to the distribution function Δ_{μ_α} of μ_α. It is sufficient to show that these maps coincide on the set of α-rational numbers[4]. To start, observe that $\Delta_{\mu_\alpha}([1]_\alpha) = 1 = \theta_\alpha([1]_\alpha)$ and notice that for each $m \geq 2$ the α-rational number $[m]_\alpha$ appears for the first time in the $(m-1)$-th level of the α-Farey decomposition, as the right endpoint of the cylinder set $\widehat{C}_\alpha(0,\dots,0)$, with code consisting of $m-1$ zeros. By the definition of the measure of maximal entropy, we have that $\Delta_{\mu_\alpha}([m]_\alpha) = 2^{-(m-1)} = \theta_\alpha([m]_\alpha)$.

Now, suppose that $\Delta_{\mu_\alpha}([\ell_1,\dots,\ell_k]_\alpha) = \theta_\alpha([\ell_1,\dots,\ell_k]_\alpha)$ for every k-tuple of positive integers $\ell_1,\dots,\ell_k$ and each $1 \leq k \leq n$, for some $n \in \mathbb{N}$. Further suppose that n is even. (The case where n is odd proceeds similarly and is left to the reader.) Recall that the α-rational number $[\ell_1,\dots,\ell_n]_\alpha$ is, for n even, the left endpoint of the α-Lüroth cylinder set $C_\alpha(\ell_1,\dots,\ell_n)$ whose right endpoint is the α-rational number $[\ell_1,\dots,\ell_n,1]_\alpha$. This α-Lüroth cylinder set coincides with the $(\sum_{i=1}^n \ell_i)$-th level α-Farey cylinder set $\widehat{C}_\alpha(0^{\ell_1-1},1,\dots,0^{\ell_n-1},1)$ and, as such, has μ_α-measure equal to $2^{-\sum_{i=1}^n \ell_i}$.

Continuing to the next level in the α-Farey decomposition, this $(\sum_{i=1}^n \ell_i)$-th level set is split into two $((\sum_{i=1}^n \ell_i)+1)$-th level α-Farey cylinder sets, with endpoints given, in order from left to right, by

$$[\ell_1,\dots,\ell_n]_\alpha, [\ell_1,\dots,\ell_n,2]_\alpha \text{ and } [\ell_1,\dots,\ell_n,2]_\alpha, [\ell_1,\dots,\ell_n,1]_\alpha.$$

Each of these sets has μ_α-measure equal to $2^{-((\sum_{i=1}^n \ell_i)+1)}$. If we consider the α-Farey cylinder set bounded by $[\ell_1,\dots,\ell_n]_\alpha$ and $[\ell_1,\dots,\ell_n,2]_\alpha$ in the same way, we obtain two $((\sum_{i=1}^n \ell_i)+2)$-th level α-Farey cylinder sets, with endpoints given, in order from left to right, by

$$[\ell_1,\dots,\ell_n]_\alpha, [\ell_1,\dots,\ell_n,3]_\alpha \text{ and } [\ell_1,\dots,\ell_n,3]_\alpha, [\ell_1,\dots,\ell_n,2]_\alpha.$$

Each of these sets has μ_α-measure equal to $2^{-((\sum_{i=1}^n \ell_i)+2)}$. Continuing in this way $(\ell_{n+1}-1)$ more times, we arrive at the $\left(\left(\sum_{i=1}^{n+1} \ell_i\right)-1\right)$-th level α-Farey cylinder set with endpoints given by $[\ell_1,\dots,\ell_n]_\alpha$ and $[\ell_1,\dots,\ell_n,\ell_{n+1}]_\alpha$, which has μ_α-measure equal to $2^{-((\sum_{i=1}^{n+1} \ell_i)-1)} = 2 \cdot 2^{-\sum_{i=1}^{n+1} \ell_i}$. This process is illustrated in Figure 2.5.

From this calculation, we are now in a position to finish the proof by induction, as follows.

$$\begin{aligned}
\Delta_{\mu_\alpha}([\ell_1,\dots,\ell_n,\ell_{n+1}]_\alpha) &= \Delta_{\mu_\alpha}([\ell_1,\dots,\ell_n]_\alpha) + \mu_\alpha(([\ell_1,\dots,\ell_n]_\alpha,[\ell_1,\dots,\ell_n,\ell_{n+1}]_\alpha]) \\
&= \theta_\alpha([\ell_1,\dots,\ell_n]_\alpha) + 2 \cdot 2^{-\sum_{i=1}^{n+1} \ell_i} \\
&= -2\sum_{k=1}^{n} (-1)^k 2^{-\sum_{i=1}^k \ell_i} + (-2)\cdot(-1)^{n+1} 2^{-\sum_{i=1}^{n+1} \ell_i} \\
&= \theta_\alpha([\ell_1,\dots,\ell_n,\ell_{n+1}]_\alpha).
\end{aligned}$$

[4] This is because if two continuous functions coincide on a dense set of points, they coincide everywhere (see, for instance, Willard [82], Corollary 13.14). The set of α-rationals for each α is certainly a dense set. With apologies to the reader for invoking results out of sequence, that θ_α is continuous follows from Lemma 2.3.4.

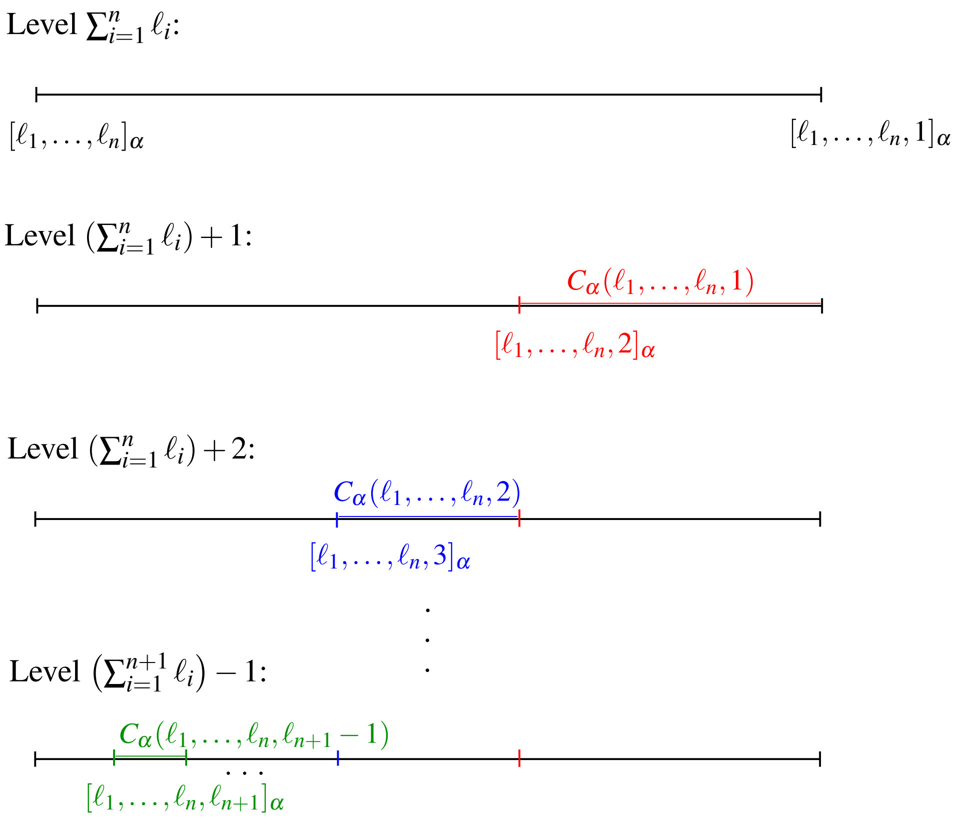


Figure 2.5: This diagram illustrates the sequence of α-Farey and α-Lüroth cylinder sets inside the set $C_\alpha(\ell_1,\dots,\ell_n)$.

It remains to show that the map θ_α is the conjugating homeomorphism from F_α to the tent system. For this, suppose first that $x \in \mathscr{U} \setminus A_1$. Then, $\theta_\alpha(x)$ is an element of $[0,1/2]$ and we have that

$$\begin{aligned} F_{\alpha_D}(\theta_\alpha(x)) &= 2\left(-2\sum_{k=1}^{\infty}(-1)^k 2^{-\sum_{i=1}^{k}\ell_i}\right) = -2\left(\sum_{k=1}^{\infty}(-1)^k 2^{-(\ell_1-1)-\sum_{i=2}^{k}\ell_i}\right) \\ &= \theta_\alpha([\ell_1-1,\ell_2,\ell_3,\ldots]_\alpha) = \theta_\alpha(F_\alpha(x)). \end{aligned}$$

Now, suppose that $x \in A_1$, that is, $x = [1,\ell_2,\ell_3,\ldots]_\alpha$. Then, it follows that $\theta_\alpha(x) \in [1/2,1]$ and we have that

$$\begin{aligned} F_{\alpha_D}(\theta_\alpha(x)) &= 2-2\left(2\cdot 2^{-1}-2\sum_{k=2}^{\infty}(-1)^k 2^{-1-\sum_{i=2}^{k}\ell_i}\right) = -2\left(\sum_{k=2}^{\infty}(-1)^k 2^{\sum_{i=2}^{k}\ell_i}\right) \\ &= \theta_\alpha([\ell_2,\ell_3,\ldots]_\alpha) = \theta_\alpha(F_\alpha(x)). \end{aligned}$$

□

Corollary 2.3.2. *For an arbitrary partition α, the topological entropy of the map F_α is equal to* $\log 2$.

Proof. Since the topological entropy of the tent map is equal to $\log 2$, this follows from the fact that topological entropy is a topological conjugacy invariant. (For details regarding topological entropy, the reader is referred to Walters [81].)

□

We remark that the proof of Proposition 2.3.1 is inspired by the proof given by Stratmann and Kesseböhmer [48] for the analogous result that the classical Farey map and the tent map are topologically conjugate. In that paper, the proof appears to be slightly different because of the fact that the Farey decomposition can be described in terms of mediants of rational numbers. (There are also some minor misprints that perhaps obscure the picture somewhat.) However, the idea is basically the same. The conjugating homeomorphism between the tent map and the Farey map is *Minkowski's question-mark function* (see [59] and [70]), the graph of which is a so-called *slippery Devil's staircase*. This means that it is a strictly increasing singular function, where "singular" means that the derivative is λ-almost everywhere equal to zero. We conjecture that (with one trivial exception, which arises from the dyadic partition α_D), the function θ_α is also a slippery Devil's staircase. This question will be the subject of future investigations.

In preparation for the next lemma, we make the following definition.

Definition 2.3.3.

1. A map $T : \mathscr{U} \to \mathscr{U}$ is said to be *Hölder continuous with exponent* κ if there exists a positive constant c such that

$$|T(x) - T(y)| \leq c|x-y|^\kappa, \text{ for all } x,y \in \mathscr{U}.$$

2. A map $T:\mathscr{U}\to\mathscr{U}$ is said to be *sub-Hölder continuous with exponent* κ if there exists a positive constant C such that

$$|T(x)-T(y)|\geq C|x-y|^{\kappa}, \text{ for all } x,y\in\mathscr{U}.$$

Our next aim is to determine the Hölder exponent and the sub-Hölder exponent of the map θ_α, for an arbitrary partition α. For this, we define $\kappa(n):=-n\log 2/(\log a_n)$ and set

$$\kappa_+ := \inf\{\kappa(n): n\in\mathbb{N}\} \text{ and } \kappa_- := \sup\{\kappa(n): n\in\mathbb{N}\}.$$

Lemma 2.3.4. *We have that the map θ_α is κ_+-Hölder continuous and κ_--sub-Hölder continuous.*

Proof. In order to calculate the Hölder exponent of θ_α, first note that

$$|\theta_\alpha(C_\alpha(\ell_1,\ell_2,\ldots,\ell_k))| = |\theta_\alpha([\ell_1,\ell_2,\ldots,\ell_k]_\alpha)-\theta_\alpha([\ell_1,\ell_2,\ldots,\ell_k+1]_\alpha)| = 2^{-\sum_{j=1}^k \ell_j}.$$

This can be seen by simply calculating the image of the endpoints of this cylinder, or by noting that every α-Lüroth cylinder $C_\alpha(\ell_1,\ell_2,\ldots,\ell_k)$ is an n-th level α-Farey cylinder, where $\sum_{j=1}^k \ell_j = n$. Suppose first that κ_+ is non-zero. In this case we have,

$$\begin{aligned}\lambda(C_\alpha(\ell_1,\ell_2,\ldots,\ell_k)) &= \prod_{i=1}^k a_{\ell_i} = \prod_{i=1}^k 2^{-\ell_i/\kappa(\ell_i)} \geq \left(\prod_{i=1}^k 2^{-\ell_i}\right)^{1/\kappa_+} = \left(2^{-\sum_{i=1}^k \ell_i}\right)^{1/\kappa_+}\\ &= |\theta_\alpha(C_\alpha(\ell_1,\ell_2,\ldots,\ell_k))|^{1/\kappa_+}.\end{aligned}$$

Or, in other words,

$$|\theta_\alpha(C_\alpha(\ell_1,\ell_2,\ldots,\ell_k))| \leq \lambda(C_\alpha(\ell_1,\ell_2,\ldots,\ell_k))^{\kappa_+}.$$

Now, let x and y be two arbitrary α-irrational numbers in $\mathscr{U}$. There must be a first time during the backwards iteration of $\mathscr{U}$ under the inverse branches of F_α in which an α-Farey cylinder set appears between the numbers x and y. Say that this cylinder set appears in the p-th stage of the α-Farey decomposition. If we iterate one more time, it is clear that there are two $(p+1)$-th level α-Farey intervals fully contained in the interval (x,y); moreover, one of these also has to be an α-Lüroth cylinder set. Let this α-Lüroth cylinder set be denoted by $C_\alpha(\ell_1,\ell_2,\ldots,\ell_k)$, where $\sum_{j=1}^k \ell_j = p+1$. This leads to the observation that, as $C_\alpha(\ell_1,\ell_2,\ldots,\ell_k)$ is contained in (x,y), we have

$$|x-y|^{\kappa_+} > \lambda(C_\alpha(\ell_1,\ell_2,\ldots,\ell_k))^{\kappa_+} \geq |\theta_\alpha(C_\alpha(\ell_1,\ell_2,\ldots,\ell_k))| = 2^{-(p+1)}.$$

Consider the interval (x,y) again. It is contained inside two neighbouring $(p-1)$-th level α-Farey intervals, and so

$$|\theta_\alpha(x)-\theta_\alpha(y)| < 2^{-(p-1)}+2^{-(p-1)} = 2^{-(p-2)} = 8\cdot 2^{-(p+1)}.$$

Combining these observations, we obtain that

$$|\theta_\alpha(x)-\theta_\alpha(y)| \le 8|x-y|^{\kappa_+}.$$

In case κ_+ is equal to zero, we have that for each $q\in\mathbb{N}$ there exists $m_0\in\mathbb{N}$ with the property that for every $m\ge m_0$,

$$\kappa(m)=\frac{m\log 2}{-\log a_m}<\frac{1}{q}, \text{ or, in other words, } a_m<2^{-qm}.$$

So we have that the sequence of partition elements are eventually exponentially decaying, and hence, the Hölder exponent of the map θ_α is necessarily equal to zero.

It remains to show that the map θ_α is κ_--sub-Hölder continuous. Suppose that κ_- is finite. Then we immediately obtain the inequality

$$|\theta_\alpha(C_\alpha(\ell_1,\dots,\ell_k))|^{1/\kappa_-}\ge\lambda(C_\alpha(\ell_1,\dots,\ell_k)).$$

Now, let $x,y\in\mathscr{U}$ be arbitrary. Let $C_\alpha(\ell_1,\dots,\ell_k)$ denote the smallest α-Lüroth interval containing both x and y. Say that $p:=\sum_{i=1}^k\ell_i$. Then there exist positive integers n and N such that $x=[\ell_1,\ell_2,\dots,\ell_k,n,x_{k+2},x_{k+3},\dots]_\alpha$ and $y=[\ell_1,\ell_2,\dots,\ell_k,N,y_{k+2},y_{k+3},\dots]_\alpha$. Thus, the interval (x,y) is contained in $\bigcup_{i=n}^N C_\alpha(\ell_1,\dots,\ell_k,i)$. Hence,

$$\begin{aligned}|x-y| &< \sum_{i=n}^N\lambda(C_\alpha(\ell_1,\dots,\ell_k,i))\le\sum_{i=n}^N|\theta_\alpha(C_\alpha(\ell_1,\dots,\ell_k,i))|^{1/\kappa_-}\\ &= 2^{-(p+n)/\kappa_-}(1+2^{-1/\kappa_-}+\dots+2^{(N-n)/\kappa_-})\le c(\kappa_-)\cdot2^{-(p+n)/\kappa_-}.\end{aligned}$$

If $|N-n|>1$, there exists a level $p+n+1$ α-Farey cylinder set fully contained in the interval (x,y) and in that case,

$$|x-y|^{\kappa_-}\ll2^{-(p+n)}\ll|\theta_\alpha(x)-\theta_\alpha(y)|.$$

The proof is finished in this case. Otherwise, if $|N-n|=1$, we must look deeper into the structure of the α-Farey cylinder sets. If $x_{k+2}\ne1$, we are again finished, as the $(p+n+2)$-th level α-Farey cylinder set $C_\alpha(\ell_1,\dots,\ell_k,n,1)$ lies entirely between x and y. So, suppose that

$$x=[\ell_1,\dots,\ell_k,n,1,s,x_{k+4},\dots]_\alpha \text{ and } y=[\ell_1,\dots,\ell_k,n+1,m,y_{k+3},\dots]_\alpha,$$

for some positive integers s and m. Thus,

$$\begin{aligned}|x-y| &< \sum_{j\ge m}\lambda(C_\alpha(\ell_1,\dots,\ell_k,n+1,j))+\sum_{i\ge s}\lambda(C_\alpha(\ell_1,\dots,\ell_k,n,1,i))\\ &\le \sum_{j\ge m}2^{-(p+n+1+j)/\kappa_-}+\sum_{i\ge s}2^{-(p+n+1+i)/\kappa_-}\\ &= 2^{-(p+n+1)/\kappa_-}\sum_{j\ge m}2^{-j/\kappa_-}+2^{-(p+n+1)/\kappa_-}\sum_{i\ge s}2^{-i/\kappa_-}\\ &= 2^{-(p+n+1)/\kappa_-}\left(\frac{2^{-m/\kappa_-}}{1-2^{-1/\kappa_-}}+\frac{2^{-s/\kappa_-}}{1-2^{-1/\kappa_-}}\right)\\ &= \left(\frac{1}{1-2^{-1/\kappa_-}}\right)\left(2^{-(p+n+1+m)/\kappa_-}+2^{-(p+n+1+s)/\kappa_-}\right).\end{aligned}$$

Note that for all $x \in \mathbb{R}^+$, we have that the function $x \mapsto x^t$ is convex if $t > 1$ and concave if $t < 1$. From the respective properties of convex and concave functions, we obtain that for all $x, y \in \mathbb{R}^+$, if $t > 1$ then $x^t + y^t < (x+y)^t$ and, if $t < 1$, then $x^t + y^t < 2^{1-t}(x+y)^t$. From these inequalities it follows on setting $t := 1/\kappa_-$ that

$$|x - y| \ll \left(2^{-(p+n+1+m)} + 2^{-(p+n+1+s)}\right)^{1/\kappa_-}.$$

Since we also have that

$$|\theta_\alpha(x) - \theta_\alpha(y)| \geq 2^{-(p+n+1+m)} + 2^{-(p+n+1+s)},$$

we finally obtain that

$$|x - y|^{\kappa_-} \ll |\theta_\alpha(x) - \theta_\alpha(y)|.$$

This finishes the proof. We have included below a diagram of the possible positions of the points x and y, which hopefully serves to make the proof somewhat clearer.

□

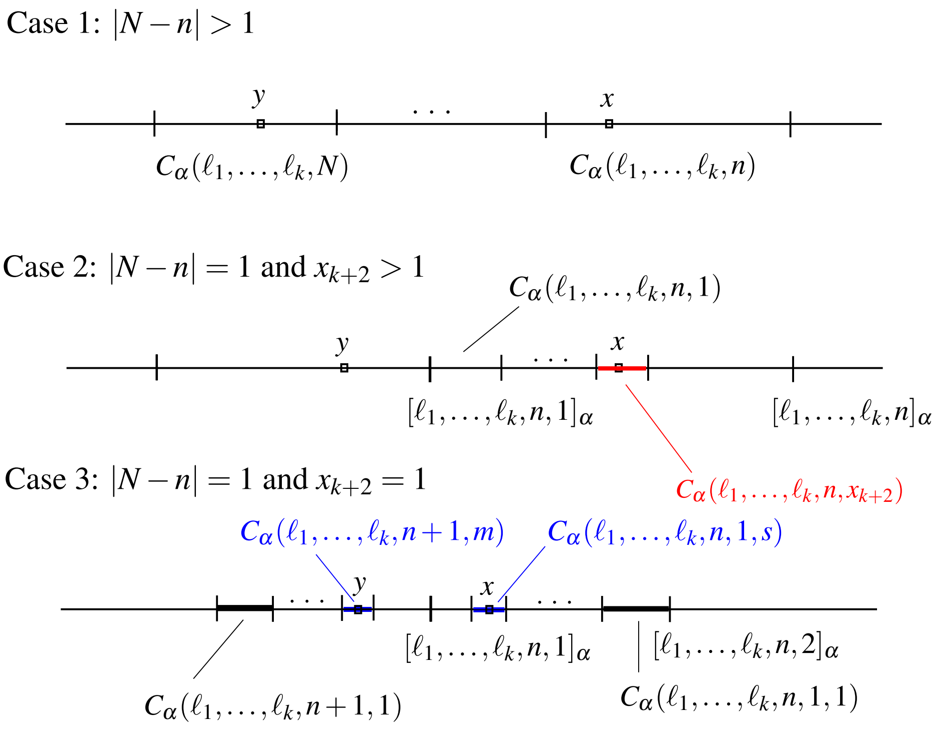


Figure 2.6: Illustration of the three cases considered in the proof that θ_α is sub-Hölder continuous.

Example 2.3.5.

1. First, for the tent map itself, F_{α_D}, we have that $a_n = 1/2^n$, so immediately we obtain that $\kappa_- = \kappa_+ = 1$. This is obviously a trivial example, since the map θ_{α_D} is the identity.

2. Recall, from from Example 2.2.2 (3), the partition α_β which is defined for each $\beta \in (0,1)$ by $a_n = (1-\beta)\beta^{(n-1)}$. Then, for the conjugating homeomorphism θ_{α_β}, we have that

$$\begin{aligned}\kappa_+ = \inf\{\kappa(n) : n \in \mathbb{N}\} &= \inf\left\{\frac{-n\log 2}{\log\left(\frac{1-\beta}{\beta}\right) + n\log\beta} : n \in \mathbb{N}\right\} \\ &= \inf\left\{\frac{\log 2}{\frac{1}{n}\log\left(\frac{\beta}{1-\beta}\right) + \log\left(\frac{1}{\beta}\right)} : n \in \mathbb{N}\right\}.\end{aligned}$$

Suppose first that $0 < \beta < 1/2$. Then, since $\log(\beta/(1-\beta)) < 0$ in this case, it is clear that the infimum in the above expression is attained as n tends to infinity. Also, the supremum (giving the value of κ_-), is achieved for $n = 1$. Thus,

$$\kappa_+ = \frac{\log 2}{-\log\beta} \text{ and } \kappa_- = \frac{\log 2}{\log(\beta/(1-\beta)) + \log(1/\beta)} = \frac{\log 2}{-\log(1-\beta)}.$$

Now, suppose that $1/2 < \beta < 1$. In this case, $\log(\beta/(1-\beta)) > 0$ and so it is immediately apparent that the values of κ_- and κ_+ are simply reversed, that is,

$$\kappa_+ = \frac{\log 2}{-\log(1-\beta)} \text{ and } \kappa_- = \frac{\log 2}{-\log\beta}.$$

3. For the conjugacy map θ_{α_H} between the maps F_{α_H} and F_{α_D}, we have that θ_{α_H} is $\log 4/\log 6$-Hölder continuous. To show this, first notice that

$$\kappa(1) = 1 > \frac{2\log 2}{\log 6} \text{ and } \frac{2\log 2}{\log 6} < \frac{3\log 2}{\log 12}$$

Then, by considering the derivative, it is easy to show that the function $x \mapsto x/\log(x^2 + x)$ is increasing for $x \geq 3$. Also, there is no real sub-Hölder continuity in this case as κ_- is infinite. This is clear from the following inequality which holds for $n \geq 2$:

$$\frac{n}{\log(n(n+1))} \geq \frac{n}{3\log n}.$$

Remark 2.3.6. The exponents κ_- and κ_+ have significance in terms of the multifractal formalism that will be developed in Chapter 5. They provide the extreme points of the region (s_-, s_+) on which the Hausdorff dimension function (which will be defined in Chapter 5) of F_α is non-zero. More precisely, we have

$$\kappa_- = \frac{\log 2}{s_-} \text{ and } \kappa_+ = \frac{\log 2}{s_+},$$

where $\kappa_- = \infty$ if and only if $s_- = 0$.

2.4 Expanding and expansive partitions

In this section, we will investigate certain types of partition which will be useful in the remainder of this chapter and in subsequent chapters. Before beginning this task, let us first make a detour into the theory of slowly-varying functions.

Definition 2.4.1. A measurable function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is said to be *slowly varying* if

$$\lim_{x\to\infty} \frac{\psi(xy)}{\psi(x)} = 1, \text{ for all } y > 0.$$

Slowly-varying functions have the following useful properties. From this list, it should be clear that the idea behind a slowly-varying function is that it behaves like a logarithmic function.

Proposition 2.4.2. *Let $\psi, \varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be two slowly-varying functions. Then the following hold:*

1. *For any $\varepsilon > 0$, we have that*

$$\lim_{x\to\infty} x^{\varepsilon} \cdot \psi(x) = \infty \text{ and } \lim_{x\to\infty} x^{-\varepsilon} \cdot \psi(x) = 0.$$

2. $$\lim_{x\to\infty} \frac{\log(\psi(x))}{\log(x)} = 0.$$

3. *For any $-\infty < a < \infty$, the functions ψ^a, $\psi \cdot \varphi$ and $\psi + \varphi$ are all slowly varying.*

Proof. See Seneta [72].

□

The following lemma, which details another property of slowly-varying functions, will prove helpful in the next section.

Lemma 2.4.3. *Let $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ be a slowly-varying function. Then,*

$$\lim_{n\to\infty} \psi(n)/\psi(n+1) = 1.$$

Proof. In order to prove this, let us first suppose by way of contradiction that

$$\lim_{n\to\infty} \psi(n)/\psi(n+1) > 1.$$

Then, directly from the definition of a slowly-varying function, we have that $\lim_{n\to\infty} \psi(cn)/\psi(n) = 1$ for all $c > 0$. Therefore, we obtain that

$$\lim_{n\to\infty} \left(\frac{\psi(n)}{\psi(2n)} \cdot \frac{\psi(2n)}{\psi(n+1)} \right) = \lim_{n\to\infty} \frac{\psi(2n)}{\psi(n+1)} > 1.$$

Thus, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, we have that

$$\frac{\psi(n)}{\psi(n+1)} > 1 \text{ and } \frac{\psi(2n)}{\psi(n+1)} > 1.$$

This implies that for $n \geq n_0$ we have

$$\psi(n) > \psi(n+1) > \psi(n+2) > \ldots > \psi(2n-1) > \psi(2n) > \psi(n+1).$$

This contradiction implies that $\lim_{n\to\infty} \psi(n)/\psi(n+1) \leq 1$. However, starting from the assumption that $\lim_{n\to\infty} \psi(n)/\psi(n+1) < 1$ leads in an analogous way to a similar contradiction. This finishes the proof.

□

Definition 2.4.4. Let $\alpha := \{A_n : n \in \mathbb{N}\}$ be a countable partition of $\mathscr{U}$ of the form described at the start of Section 2.1. Then:

1. The partition α is said to be *expanding* provided that
$$\lim_{n\to\infty} \frac{t_n}{t_{n+1}} = \rho, \text{ for some } \rho > 1.$$
2. The partition α is said to be *expansive of exponent* $\theta \geq 0$ if the tails of the partition satisfy the power law
$$t_n = \psi(n) \cdot n^{-\theta},$$
where $\psi : \mathbb{N} \to \mathbb{R}^+$ is a slowly-varying function.
3. The partition α is said to be *eventually decreasing* if for all sufficiently large n, we have that $a_{n+1} \leq a_n$.

Note that for a partition α that is expansive of exponent $\theta \geq 0$, one has that

$$\lim_{n\to\infty} \frac{t_n}{t_{n+1}} = \lim_{n\to\infty} \left(\frac{\psi(n)}{\psi(n+1)} \cdot \left(\frac{n+1}{n}\right)^{\theta} \right) = 1.$$

This follows from combining the obvious fact that $\lim_{n\to\infty}(n/(n+1))^{\theta} = 1$ with Lemma 2.4.3.

Definition 2.4.5. A partition α is said to be of *finite type* if for the sequence of tails t_n of α, we have that $\sum_{n=1}^{\infty} t_n$ converges. Otherwise, α is said to be of *infinite type*.

Notice that if α is expanding, one immediately verifies that α is of finite type. This can be seen, for instance, by applying the ratio test for series convergence. The next proposition describes the situation for expansive partitions.

Proposition 2.4.6. *Suppose that α is expansive of exponent $\theta > 0$. Then we have the following classification:*

- *If $\theta \in [0,1)$, then α is of infinite type.*

- *If $\theta > 1$, then α is of finite type.*
- *If $\theta = 1$, then α can be either of finite or infinite type.*

Proof. Suppose first that α is expansive of exponent $\theta \in [0,1)$. Then, by Proposition 2.4.2, for all $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that if $n \geq n_0$, we have that $\psi(n) \geq n^{-\varepsilon}$. Let ε be sufficiently small that $\theta + \varepsilon \in (0,1)$. Then,

$$\sum_{n=1}^{\infty} t_n = \sum_{n=1}^{n_0-1} \psi(n) \cdot n^{-\theta} + \sum_{n=n_0}^{\infty} \psi(n) \cdot n^{-\theta} \geq \sum_{n=n_0}^{\infty} n^{-(\theta+\varepsilon)} \geq \sum_{n=n_0}^{\infty} n^{-1}.$$

Consequently, $\sum_{n=1}^{\infty} t_n = \infty$ and α is of infinite type. Now suppose that α is expansive of exponent $\theta > 1$. Then, again by Proposition 2.4.2, for all $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that if $n \geq n_0$, we have that $\psi(n) \leq n^{\varepsilon}$. For ε small enough that $\theta - \varepsilon > 1$, we then have that

$$\sum_{n=n_0}^{\infty} t_n = \sum_{n=n_0}^{\infty} \psi(n) \cdot n^{-\theta} \leq \sum_{n=n_0}^{\infty} n^{-(\theta-\varepsilon)} < \infty.$$

Therefore, in this case, the partition α is of finite type. It only remains to prove the third assertion, which can be done by considering the following two examples. First, let $t_1 = 1$ and for each $n \geq 2$, let $t_n = (n \log n)^{-1}$. The partition α defined in such a way is clearly expansive of exponent 1. For this partition, we have that

$$\sum_{n=1}^{\infty} t_n = 1 + \sum_{n=2}^{\infty} \frac{1}{n \log n},$$

which diverges (by the integral test, for instance). So, in this first case, the partition is of infinite type. On the other hand, if now we set $t_1 = 1$, $t_n = n^{-1} \cdot (\log n)^{-2}$, we obtain that

$$\sum_{n=1}^{\infty} t_n = 1 + \sum_{n=2}^{\infty} \frac{1}{n(\log n)^2},$$

which is a convergent series, so in this case we have that the partition is of finite type. This finishes the proof.

□

Figure 2.7 below illustrates two α-Farey maps with α expansive. The graph on the left-hand side has α with exponent $\theta = 2$, so satisfies the condition of the second part of Proposition 2.4.6. The graph on the right-hand side has α with exponent $\theta = 1/2$, so it satisfies the condition given in the first part of Proposition 2.4.6.

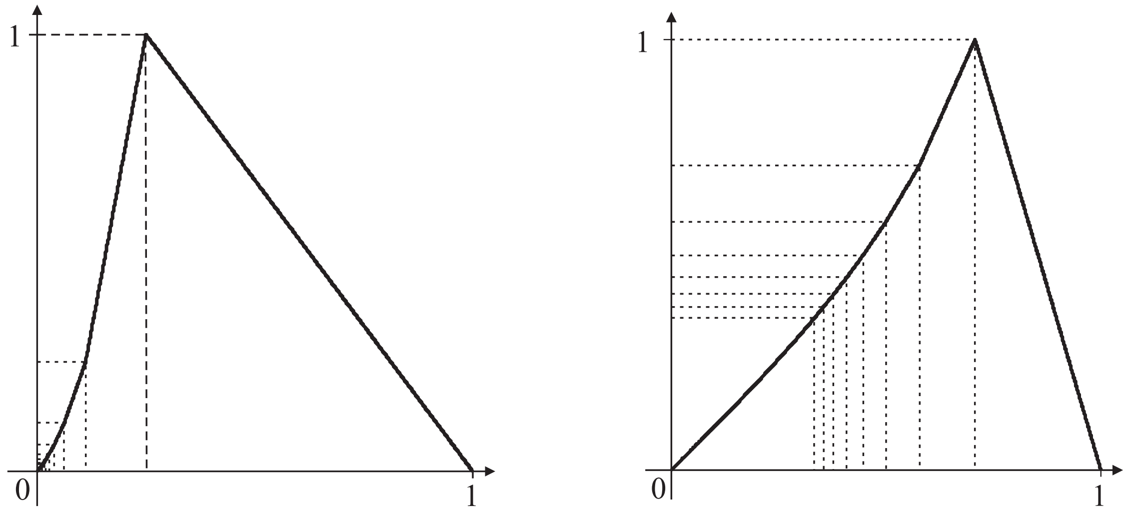


Figure 2.7: The graphs of two α-Farey maps with α expansive. The partition on the left is of finite type with $t_n = 1/n^2$, $n \in \mathbb{N}$ and the partition on the right is of infinite type with $t_n = 1/\sqrt{n}$, $n \in \mathbb{N}$.

The following proposition is basically a version of the Monotone Density Theorem, which can be found as Theorem 1.7.2 in [11].

Proposition 2.4.7. *If α is expansive of exponent $\theta > 0$ and eventually decreasing, then we have that*

$$a_n \sim \theta n^{-1} t_n.$$

Proof. Let $0 < M < N < \infty$ and, for $n \in \mathbb{N}$ large enough that the sequence $(a_k)_{k\in\mathbb{N}}$ is decreasing for all $k \geq n$, consider

$$t_{Mn} - t_{Nn} = \sum_{k=Mn}^{Nn} a_k.$$

We have that

$$\frac{n(N-M)a_{Nn}}{n^{-\theta}\psi(n)} \leq \frac{t_{Mn} - t_{Nn}}{n^{-\theta}\psi(n)} \leq \frac{n(N-M)a_{Mn}}{n^{-\theta}\psi(n)},$$

from which it follows that

$$\frac{n(N-M)a_{Nn}}{n^{-\theta}\psi(n)} \leq \frac{t_{Mn}}{(nM)^{-\theta}\psi(nM)} \cdot \frac{M^{-\theta}\psi(Mn)}{\psi(n)} - \frac{t_{Nn}}{(nN)^{-\theta}\psi(nN)} \cdot \frac{N^{-\theta}\psi(nN)}{\psi(n)} \leq \frac{n(N-M)a_{Mn}}{n^{-\theta}\psi(n)}.$$

Noting that $t_{Nn}/((nN)^{-\theta}\psi(nN)) = t_{Mn}/((nM)^{-\theta}\psi(nM)) = 1$, if we set $M = 1$, we obtain

$$\limsup_{n\to\infty} a_{Nn} \cdot \frac{n^{1+\theta}}{\psi(n)} \leq \frac{-(N^{-\theta} - 1)}{N-1}.$$

Then, letting N tend to 1, since the right-hand side is nothing other than the derivative of $-x^{-\theta}$ at the point 1, we infer that

$$\limsup_{n\to\infty} a_n \cdot \frac{n^{1+\theta}}{\psi(n)} \leq \theta.$$

Similarly, it can be shown that

$$\liminf_{n\to\infty} a_n \cdot \frac{n^{1+\theta}}{\psi(n)} \geq \theta.$$

Combining these observations, we deduce that $\lim_{n\to\infty} a_n/(\theta n^{-(1+\theta)}\psi(n)) = 1$, or, in other words,

$$a_n \sim \theta n^{-(1+\theta)}\psi(n).$$

This finishes the proof.

□

We end this section with one more useful lemma concerning partitions that are expanding or expansive.

Lemma 2.4.8. *Let α be a partition such that $\lim_{n\to\infty} t_n/t_{n+1} = \rho \geq 1$ and such that α is either expanding, or expansive of exponent $\theta > 0$ and eventually decreasing. Then:*

(1) *We have that*

$$\lim_{n\to\infty} \frac{\log a_n}{n} = \lim_{n\to\infty} \frac{\log t_n}{n} = -\log\rho.$$

(2) *We have that*

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \rho.$$

Proof. Let us first prove that if α is either expanding, or expansive of exponent $\theta > 0$ and eventually decreasing, then we have that $\lim_{n\to\infty}(\log t_n)/n = -\log\rho$. Since $\lim_{n\to\infty}(\log t_n - \log t_{n+1}) = \log\rho$, we conclude, by using Cesàro averages, that for $\rho \geq 1$ we have that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n-1}(\log t_{k+1} - \log t_k) = -\log\rho.$$

Consequently, since $\sum_{k=1}^{n-1}(\log t_{k+1} - \log t_k) = \log t_n - \log t_1 = \log t_n$, we have that

$$\lim_{n\to\infty} \frac{\log t_n}{n} = -\log\rho.$$

We will split the remainder of the proof of parts (1) and (2) into two cases; firstly, we will consider the case where α is expanding, so $\rho > 1$, and secondly we will consider the case where α is expansive of exponent $\theta > 0$ and eventually decreasing. So, first suppose that $\rho > 1$. Then, we have that

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \lim_{n\to\infty} \frac{t_n - t_{n+1}}{t_{n+1} - t_{n+2}} = \lim_{n\to\infty} \frac{\frac{t_n}{t_{n+1}} - 1}{1 - \frac{1}{t_{n+1}/t_{n+2}}} = \frac{\rho - 1}{1 - 1/\rho} = \rho.$$

This proves the statement in part (2) for the expanding case. Moreover, if $\rho > 1$, we have that $\lim_{n\to\infty}(\log a_{n+1} - \log a_n) = -\log\rho$ and exactly the same argument as that above for the t_ns yields that

$$\lim_{n\to\infty}\frac{\log a_n}{n} = \lim_{n\to\infty}\frac{\log t_n}{n} = -\log\rho.$$

Now, let us consider the case where α is expansive of exponent $\theta > 0$ and eventually decreasing. In this instance, Proposition 2.4.7 implies that $\lim_{n\to\infty}(\log a_n)/n = 0$ for the case $\rho = 1$. The proof of (2) for the expansive case is also an immediate consequence of this proposition. This completes the proof of the lemma.

□

2.5 Hausdorff dimension of Diophantine-type sets for α-Lüroth systems

We begin this section with some easily obtained results which have a Diophantine-like flavour. By this, we mean that the sets considered here are analogues of the sets of well-approximable and badly-approximable numbers usually defined in terms of the continued fraction expansion (see, for instance, [50] or [66]).

Lemma 2.5.1. *Let $\mathscr{W}_\alpha$ be the set defined by*

$$\mathscr{W}_\alpha := \{x = [\ell_1(x), \ell_2(x), \ldots]_\alpha \in \mathscr{U} : \limsup_{n\to\infty} \ell_n(x) = \infty\}.$$

Then, $\mathscr{W}_\alpha$ is of full Lebesgue measure.

Proof. We will prove this by establishing that the complement of the set $\mathscr{W}_\alpha$ has measure zero. Notice that the complement of $\mathscr{W}_\alpha$ is the set of all those $x \in \mathscr{U}$ with bounded α-Lüroth entries. In other words, if we set $\mathscr{B}_\alpha := \mathscr{U} \setminus \mathscr{W}_\alpha$, we have that

$$\mathscr{B}_\alpha = \bigcup_{N\in\mathbb{N}} \mathscr{A}_N,$$

where

$$\mathscr{A}_N := \{x = [\ell_1(x), \ell_2(x), \ldots]_\alpha \in \mathscr{U} : \ell_k(x) \le N \text{ for all } k \in \mathbb{N}\}.$$

Also, for each $N, n \in \mathbb{N}$, define

$$\mathscr{A}_N^{(n)} := \{x = [\ell_1(x), \ell_2(x), \ldots]_\alpha \in \mathscr{U} : \ell_k(x) \le N \text{ for all } 1 \le k \le n\}.$$

It is clear that $\mathscr{A}_N \subseteq \mathscr{A}_N^{(n)}$ and further that $\mathscr{A}_N^{(n+1)} \subset \mathscr{A}_N^{(n)}$ for all $N, n \in \mathbb{N}$. Notice that we may also write $\mathscr{A}_N^{(n+1)}$ in the following way:

$$\mathscr{A}_N^{(n+1)} = \bigcup_{\substack{\ell_1,\ldots,\ell_{n+1}\\ \ell_i \le N,\ 1\le i\le n+1}} C_\alpha(\ell_1,\ldots,\ell_{n+1}) = \bigcup_{\substack{\ell_1,\ldots,\ell_n\\ \ell_i \le N,\ 1\le i\le n}} \bigcup_{k\le N} C_\alpha(\ell_1,\ldots,\ell_n,k).$$

Thus, for all $n \in \mathbb{N}$, we have that

$$\lambda\left(\mathscr{A}_N^{(n+1)}\right) = \sum_{k \leq N} a_k \lambda\left(\mathscr{A}_N^{(n)}\right).$$

Hence, on applying this argument $n-1$ more times, it follows that

$$\lambda\left(\mathscr{A}_N^{(n+1)}\right) = \left(\sum_{k \leq N} a_k\right)^n \lambda\left(\mathscr{A}_N^{(1)}\right).$$

Since the last term above is simply a constant and $\sum_{k \leq N} a_k < 1$, this shows that $\lambda(\mathscr{A}_N) = 0$, for any $N \in \mathbb{N}$. Finally, we have that

$$\lambda\left(\bigcup_{N \in \mathbb{N}} \mathscr{A}_N\right) \leq \sum_{N \in \mathbb{N}} \lambda(\mathscr{A}_N) = 0.$$

This finishes the proof of the lemma.

□

Although the sets $\mathscr{A}_N$ defined in the above proof have Lebesgue measure zero for every $N \in \mathbb{N}$, we can still distinguish between their sizes by calculating their Hausdorff dimension. Luckily, this is very easy to do, as the next lemma demonstrates.

Lemma 2.5.2.

$$\dim_H(\mathscr{A}_N) = s, \text{ where } s \text{ is given by } \sum_{i=1}^N a_i^s = 1.$$

Proof. All that is required to prove this statement is to notice that for each $n \in \mathbb{N}$ the set $\mathscr{A}_N$ is an invariant set for a finite iterated function system $\{L_{\alpha,1}, \ldots, L_{\alpha,N}\}$, where $L_{\alpha,n}$ denotes the n-th inverse branch of the map L_α. Recall that these are given by $L_{\alpha,n}(x) := t_n - a_n x$. That $\mathscr{A}_N$ is an invariant set for this system means that

$$\mathscr{A}_N = \bigcup_{i=1}^N L_{\alpha,i}(\mathscr{A}_N).$$

Then, given that these inverse branches are contracting similarities, that is, they satisfy the equality $|L_{\alpha,i}(x) - L_{\alpha,i}(y)| = a_i|x-y|$ for all x and y in $\mathscr{U}$, we have that the dimension of $\mathscr{A}_N$ can be deduced directly from an application of Hutchinson's Formula (see [22], Theorem 9.3).

□

This observation can be used to calculate the Hausdorff dimension of the set $\mathscr{B}_\alpha$, as follows.

Proposition 2.5.3. *Let α be an arbitrary partition of $\mathscr{U}$ and let $\mathscr{B}_\alpha := \mathscr{U} \setminus \mathscr{W}_\alpha$, where $\mathscr{W}_\alpha$ is defined in Lemma 2.5.1. Then*

$$\dim_H(\mathscr{B}_\alpha) = 1.$$

Proof. Since $\mathscr{B}_\alpha := \bigcup_{N\in\mathbb{N}} \mathscr{A}_N$, we have that

$$\dim_H(\mathscr{B}_\alpha) = \sup_{N\in\mathbb{N}} \{\dim_H(\mathscr{A}_N)\}.$$

Then, by Lemma 2.5.2, $\dim_H(\mathscr{A}_N) = s$, where s is given by $\sum_{i=1}^{N} a_i^s = 1$ and $\dim_H(\mathscr{A}_{N+1}) = t$, where t is given by $\sum_{i=1}^{N+1} a_i^t = 1$. Therefore, $a_1^t + \cdots + a_N^t < 1$ and so $s < t$. In other words,

$$\dim_H(\mathscr{A}_N) < \dim_H(\mathscr{A}_{N+1}).$$

Furthermore, as $\sum_{i=1}^{\infty} a_i = 1$, it follows that $\dim_H(\mathscr{B}_\alpha) = 1$.

□

2.5.1 Good-type sets for the α-Lüroth system

Throughout this section, unless stated otherwise, suppose that α is expansive with exponent $\theta > 0$ and is also eventually decreasing. The first result of this section concerns α-Good sets, which are defined as follows. For each $N \in \mathbb{N}$, let the set $G_N^{(\alpha)}$ be defined by

$$G_N^{(\alpha)} := \{x = [\ell_1(x), \ell_2(x), \ldots]_\alpha \in \mathscr{U} : \ell_i(x) > N \text{ for all } i \in \mathbb{N}\}.$$

Note that the name "Good" here refers to I.J. Good [31], for the similar results he proved for continued fractions, and not to any supposed nice property of these sets. We have the following result.

Theorem 2.5.4.

$$\lim_{N\to\infty} \dim_H\left(G_N^{(\alpha)}\right) = \frac{1}{1+\theta}.$$

Proof. By assumption, α is expansive of exponent $\theta > 0$. Therefore, from Proposition 2.4.7, we have that $a_n \sim \theta\psi(n) \cdot n^{-(1+\theta)}$, where $\psi : \mathbb{N} \to \mathbb{R}^+$ is a slowly-varying function. This implies that $a_n \asymp \psi(n) \cdot n^{-(1+\theta)}$. Since ψ is slowly varying, it follows that for all positive ε, we have for sufficiently large $n \in \mathbb{N}$ that $n^{-\varepsilon} \le \psi(n) \le n^{\varepsilon}$. Thus, on combining these observations, we obtain that

$$n^{-(1+\theta+\varepsilon)} \le a_n \le n^{-(1+\theta-\varepsilon)}.$$

Let $\varepsilon > 0$ be given. Then, recalling from Section 2.1 that $\lambda(C_\alpha(\ell_1, \ldots, \ell_k)) = a_{\ell_1} \ldots a_{\ell_k}$, there exists a positive integer $N := N(\varepsilon)$ such that for each α-Lüroth cylinder set $C_\alpha(\ell_1, \ldots, \ell_k)$ with $\ell_i > N$ for each $1 \le i \le k$, we have

$$\frac{1}{(\ell_1 \cdots \ell_k)^{1+\theta+\varepsilon}} \le \lambda(C_\alpha(\ell_1, \ldots, \ell_k)) \le \frac{1}{(\ell_1 \cdots \ell_k)^{1+\theta-\varepsilon}}. \tag{2.1}$$

In order to compute the upper bound, let $\delta > 0$ and choose k large enough that

$$\mathscr{C} := \{C_\alpha(\ell_1, \ldots, \ell_k) : \ell_i > N \text{ for } 1 \le i \le k\}$$

is a δ-cover of $G_N^{(\alpha)}$. Let $s := (1+\theta-\varepsilon)^{-1}(1+\varepsilon_N)$, where ε_N is chosen to satisfy the conditions that $\varepsilon_N < 1$ and $-\varepsilon_N/\log(\varepsilon_N) > 1/\log N$. Then,

$$\begin{aligned}\mathscr{H}_\delta^s\left(G_N^{(\alpha)}\right) &\leq \sum_{\mathscr{C}} \lambda(C_\alpha(\ell_1,\ldots,\ell_k))^s \leq \sum_{\mathscr{C}} \left(\left(\frac{1}{\ell_1\cdots\ell_k}\right)^{1+\theta-\varepsilon}\right)^{(1+\theta-\varepsilon)^{-1}(1+\varepsilon_N)} \\ &= \sum_{\mathscr{C}} \left(\frac{1}{\ell_1\cdots\ell_k}\right)^{1+\varepsilon_N} = \left(\sum_{i>N}\left(\frac{1}{i}\right)^{1+\varepsilon_N}\right)^k < \left(\int_N^\infty x^{-(1+\varepsilon_N)}\,dx\right)^k \\ &= \left(\frac{1}{\varepsilon_N N^{\varepsilon_N}}\right)^k < 1.\end{aligned}$$

Thus, as this estimate is independent of δ, we have that $\dim_H\left(G_N^{(\alpha)}\right) \leq s$. Letting $\varepsilon > 0$ tend to zero and choosing the sequence $(\varepsilon_N)_{N\in\mathbb{N}}$ in such a way that $\lim\limits_{N\to\infty} \varepsilon_N = 0$, we obtain that

$$\dim_H\left(G_N^{(\alpha)}\right) \leq \frac{1}{1+\theta}.$$

In order to calculate the desired lower bound, we define a certain subset of the set $G_N^{(\alpha)}$ for each $N \in \mathbb{N}$. First, choose $M \in \mathbb{N}$ to be such that $\sum_{i=N}^M 1/i > 1$. Denote this sum by S. Then define the set

$$G_{N,M}^{(\alpha)} := \{x = [\ell_1(x), \ell_2(x), \ldots]_\alpha \in \mathscr{U} : N < \ell_i(x) \leq M \text{ for all } i \in \mathbb{N}\}.$$

Clearly, $G_{N,M}^{(\alpha)} \subseteq G_N^{(\alpha)}$ and so a lower bound for the Hausdorff dimension of the subset $G_{N,M}^{(\alpha)}$ is also a lower bound for the set $G_N^{(\alpha)}$. We aim to use Frostman's Lemma, so, to that end, define a mass distribution ν on the set $G_{N,M}^{(\alpha)}$ by setting

$$\nu(C_\alpha(\ell_1,\ldots,\ell_k)) \;:=\; \frac{1}{S^k \ell_1\cdots\ell_k}.$$

Note that from (2.1), if N is large enough, we have that

$$\nu(C_\alpha(\ell_1,\ldots,\ell_k)) \leq \left(\frac{1}{S}\right)^k \lambda(C_\alpha(\ell_1,\ldots,\ell_k))^{1/(1+\theta+\varepsilon)} < \lambda(C_\alpha(\ell_1,\ldots,\ell_k))^{1/(1+\theta+\varepsilon)},$$

where the second inequality comes from the fact that $1/S < 1$. Also note that

$$\frac{\lambda(C_\alpha(\ell_1,\ldots,\ell_k))}{\lambda(C_\alpha(\ell_1,\ldots,\ell_k,\ell_{k+1}))} = \frac{1}{a_{\ell_{k+1}}} \leq \ell_{k+1}^{1+\theta+\varepsilon} \leq M^{1+\theta+\varepsilon}.$$

Now, let $x = [\ell_1(x), \ell_2(x), \ldots]_\alpha \in G_{N,M}^{(\alpha)}$, let $r > 0$ and further let $k \in \mathbb{N}$ be such that we have

$$\lambda(C_\alpha(\ell_1(x),\ldots,\ell_{k+1}(x))) \leq r < \lambda(C_\alpha(\ell_1(x),\ldots,\ell_k(x))).$$

It is clear that $C_\alpha(\ell_1(x),\ldots,\ell_k(x),\ell_{k+1}(x)) \subset B(x,r)$, but it is possible that $B(x,r)$ intersects more than one cylinder set of length k. However, since there are at most $M-N$ possibilities and the ν-measure of each of them is comparable, without loss of generality we can assume that

$$C_\alpha(\ell_1(x),\ldots,\ell_{k+1}(x)) \subset B(x,r) \subset C_\alpha(\ell_1(x),\ldots,\ell_k(x)).$$

Then,

$$\begin{aligned}\nu(B(x,r)) &\leq \nu(C_\alpha(\ell_1(x),\ldots,\ell_k(x))) \leq \lambda(C_\alpha(\ell_1(x),\ldots,\ell_k(x)))^{1/(1+\theta+\varepsilon)} \\ &\leq M\lambda(C_\alpha(\ell_1(x),\ldots,\ell_{k+1}(x)))^{1/(1+\theta+\varepsilon)} \leq 2Mr^{1/(1+\theta+\varepsilon)}.\end{aligned}$$

Hence, by Frostman's Lemma, it follows that

$$\dim_H\left(G_{N,M}^{(\alpha)}\right) \geq \frac{1}{1+\theta+\varepsilon}.$$

Now, since for each positive integer $M > N$ we have that $G_{N,M}^{(\alpha)} \subset G_N^{(\alpha)}$, it follows that

$$\dim_H\left(G_N^{(\alpha)}\right) \geq \frac{1}{1+\theta+\varepsilon}.$$

Finally, on letting ε tend to zero, we have that

$$\lim_{N\to\infty} \dim_H\left(G_N^{(\alpha)}\right) \geq \frac{1}{1+\theta}.$$

Combining this with the upper bound given above finishes the proof of the theorem.

□

For the final main result of this section, let us consider the following sets. Let

$$F_\infty^{(\alpha)} := \left\{x = [\ell_1(x),\ell_2(x),\ldots]_\alpha : \lim_{n\to\infty} \ell_n(x) = \infty \text{ and } \ell_n(x) \geq \ell_{n-1}(x)\right\}$$

and

$$G_\infty^{(\alpha)} := \left\{x = [\ell_1(x),\ell_2(x),\ldots]_\alpha : \lim_{n\to\infty} \ell_n(x) = \infty\right\}.$$

It is immediately apparent that $F_\infty^{(\alpha)} \subset G_\infty^{(\alpha)}$, so that $\dim_H(F_\infty^{(\alpha)}) \leq \dim_H(G_\infty^{(\alpha)})$. We aim to prove the following theorem.

Theorem 2.5.5.

$$\dim_H\left(F_\infty^{(\alpha)}\right) = \dim_H\left(G_\infty^{(\alpha)}\right) = \frac{1}{1+\theta}.$$

The proof will again be split into the lower bound and the upper bound. Let us begin with the following useful lemma.

Lemma 2.5.6. *Suppose that* $x = [\ell_1(x), \ell_2(x), \ldots]_\alpha \in F_\infty^{(\alpha)}$*. Further suppose that*

$$\lambda(C_\alpha(\ell_1(x), \ldots, \ell_k(x), \ell_{k+1}(x))) \leq r < \lambda(C_\alpha(\ell_1(x), \ldots, \ell_k(x))).$$

Then, for all sufficiently large k*,*

$$B(x,r) \subset \bigcup_{i=-1}^{1} C_\alpha(\ell_1(x), \ldots, \ell_k(x)+i).$$

Proof. We consider here only the case in which k is odd, the case k even is analogous and is left to the reader. Bearing in mind that $x \in C_\alpha(\ell_1(x), \ldots, \ell_{k+1}(x))$, it is clear that if k is sufficiently large, then the right endpoint of $B(x,r)$ cannot extend past the interval $C_\alpha(\ell_1(x), \ldots, \ell_k(x)-1)$, as we are assuming that the partition α is eventually decreasing. On the other hand, the left endpoint of $B(x,r)$ cannot be smaller than the point $[\ell_1(x), \ldots, \ell_{k+1}(x)]_\alpha - a_{\ell_1(x)} \ldots a_{\ell_k(x)}$. But this point is equal to

$$\begin{aligned}
&\left(t_{\ell_1(x)} - a_{\ell_1(x)} t_{\ell_2(x)} + \ldots + a_{\ell_1(x)} \cdots a_{\ell_{k-1}(x)} t_{\ell_k(x)} - a_{\ell_1(x)} \cdots a_{\ell_k(x)} t_{\ell_{k+1}(x)}\right) - a_{\ell_1(x)} \cdots a_{\ell_k(x)} \\
&= t_{\ell_1(x)} - \ldots + a_{\ell_1(x)} \cdots a_{\ell_{k-1}(x)} \left(t_{\ell_k(x)} - a_{\ell_k(x)}\right) - a_{\ell_1(x)} \cdots a_{\ell_k(x)} t_{\ell_{k+1}(x)} \\
&= [\ell_1(x), \ldots, \ell_{k-1}(x), \ell_k(x)+1]_\alpha - a_{\ell_1(x)} \cdots a_{\ell_k(x)} t_{\ell_{k+1}(x)}.
\end{aligned}$$

Notice that the point $[\ell_1(x), \ldots, \ell_{k-1}(x), \ell_k(x)+1]_\alpha$ is the left endpoint of the the cylinder set $C_\alpha(\ell_1(x), \ldots, \ell_k(x))$, so it only remains to prove that

$$a_{\ell_1(x)} \cdots a_{\ell_{k-1}(x)} a_{\ell_k(x)} t_{\ell_{k+1}(x)} \leq a_{\ell_1(x)} \cdots a_{\ell_{k-1}(x)} a_{\ell_k(x)+1} = \lambda(C_\alpha(\ell_1(x), \ldots, \ell_k(x)+1)).$$

In other words, we must show that

$$a_{\ell_k(x)} t_{\ell_{k+1}(x)} \leq a_{\ell_k(x)+1}.$$

Recall that α is assumed to be expanding of exponent $\theta \geq 0$, so $t_n = n^{-\theta} \cdot \psi(n)$ and $a_n \asymp n^{-(1+\theta)} \cdot \psi(n)$, where $\psi : \mathbb{N} \to \mathbb{R}^+$ is a slowly varying function. Also recall that for each positive ε, if n is sufficiently large, we have that $t_n \leq n^{-(\theta-\varepsilon)}$ and $n^{-(1+\theta+\varepsilon)} \leq a_n \leq n^{-(1+\theta-\varepsilon)}$. Let $\varepsilon < \theta/6$. Then, since $x \in F_\infty^{(\alpha)}$, so $\ell_k(x) \leq \ell_{k+1}(x)$ for all k, we have that

$$\begin{aligned}
a_{\ell_k(x)} t_{\ell_{k+1}(x)} &\leq \frac{1}{\ell_k(x)^{(1+\theta-\varepsilon)}} \cdot \frac{1}{\ell_{k+1}(x)^{(\theta-\varepsilon)}} \\
&\leq \frac{1}{\ell_k(x)^{(1+2\theta-2\varepsilon)}} < \frac{1}{\ell_k(x)^{(1+(5/3)\theta)}}.
\end{aligned}$$

On the other hand, we also have that

$$a_{\ell_k(x)+1} \geq \frac{1}{(\ell_k(x)+1)^{(1+\theta+\varepsilon)}} > \frac{1}{(\ell_k(x)+1)^{(1+(7/6)\theta)}}.$$

Therefore, in order to show that $a_{\ell_k(x)}t_{\ell_{k+1}(x)} \leq a_{\ell_k(x)+1}$, it suffices to show that

$$\frac{1}{\ell_k(x)^{(1+(5/3)\theta)}} < \frac{1}{(\ell_k(x)+1)^{(1+(7/6)\theta)}},$$

or, equivalently, that

$$1-\frac{(1/2)\theta}{1+(3/5)\theta} = \frac{1+(7/6)\theta}{1+(10/6)\theta} < \frac{\log(\ell_k(x))}{\log(\ell_k(x)+1)}.$$

But, since the left-hand side is a fixed amount less than 1, depending only on θ, and the right-hand side tends to 1 as $\ell_k(x)$ increases (that is, as k increases), it follows that if k is large enough, this statement is true. Thus, the left endpoint of $B(x,r)$ lies in $C_\alpha(\ell_1(x),\ldots,\ell_k(x)+1)$ and the lemma is proved.

□

In the next lemma, we will establish the lower bound for the dimension of $F_\infty^{(\alpha)}$.

Lemma 2.5.7.

$$\frac{1}{1+\theta} \leq \dim_H\left(F_\infty^{(\alpha)}\right).$$

Proof. We will define a suitable subset of $F_\infty^{(\alpha)}$ and use Frostman's Lemma again to obtain the lower bound. So, first let $f_\varepsilon : \mathbb{N} \to \mathbb{N}$ be a slowly-varying function which satisfies the following properties:

- $\lim_{n\to\infty} f_\varepsilon(n) = \infty$.
- $f_\varepsilon(n) \leq f_\varepsilon(n+1)$ for all $n \in \mathbb{N}$.
- $f_\varepsilon(1)$ is large enough that if $\ell \geq f_\varepsilon(1)$, then $a_\ell \geq \ell^{-(1+\theta+\varepsilon)}$.

Now, define a second function $g : \mathbb{N} \to \mathbb{N}$ by setting $g(n)$ to be the least integer such that

$$S_n := \sum_{i=f_\varepsilon(n)}^{g(n)} \frac{1}{i} > 1.$$

Note that the function g is also slowly varying. Indeed, for any $k \in \mathbb{N}$, if $f_\varepsilon(n) \in \{2^k+1,\ldots,2^{k+1}\}$ it follows that $2^{k+1} \leq g(n) \leq 2^{k+3}$. Hence, $f_\varepsilon(n) < g(n) \leq 8f_\varepsilon(n)$. Finally, define the set

$$F_{f_\varepsilon,g}^{(\alpha)} := \{x = [\ell_1(x),\ell_2(x),\ldots]_\alpha : f_\varepsilon(n) \leq \ell_n(x) \leq g(n) \text{ and } \ell_n(x) \geq \ell_{n-1}(x) \text{ for all } n \in \mathbb{N}\}.$$

It is clear that $F_{f_\varepsilon,g}^{(\alpha)} \subset F_\infty^{(\alpha)}$. So, it suffices to show that $\dim_H(F_{f_\varepsilon,g}^{(\alpha)}) \geq 1/(1+\theta)$. To that end, define a mass distribution on $F_{f_\varepsilon,g}^{(\alpha)}$ by setting

$$\nu(C_\alpha(\ell_1(x),\ldots,\ell_k(x))) := \frac{1}{S_1\cdots S_k}\cdot\frac{1}{\ell_1(x)\cdots\ell_k(x)}.$$

Note that due to the choice of f_ε and g, we have that

$$\nu(C_\alpha(\ell_1(x),\dots,\ell_k(x))) \leq \lambda(C_\alpha(\ell_1(x),\dots,\ell_k(x)))^{1/(1+\theta+\varepsilon)}.$$

In addition, observe that

$$\frac{\lambda(C_\alpha(\ell_1(x),\dots,\ell_k(x)))}{\lambda(C_\alpha(\ell_1(x),\dots,\ell_{k+1}(x)))} = \frac{1}{a_{\ell_{k+1}(x)}} \leq \ell_{k+1}(x)^{1+\theta+\varepsilon} \leq g(k+1)^{1+\theta+\varepsilon}.$$

As in the proof of Theorem 2.5.4, let $r > 0$ and choose k such that

$$\lambda(C_\alpha(\ell_1(x),\dots,\ell_{k+1}(x))) \leq r < \lambda(C_\alpha(\ell_1(x),\dots,\ell_k(x))).$$

Again, it is clear that $C_\alpha(\ell_1(x),\dots,\ell_k(x),\ell_{k+1}(x)) \subset B(x,r)$, but it is possible that $B(x,r)$ intersects more than one interval in level k. There are no longer a fixed finite set of possibilities, but for large enough k (that is, for small enough r), we can apply Lemma 2.5.6 to conclude that

$$C_\alpha(\ell_1(x),\dots,\ell_{k+1}(x)) \subset B(x,r) \subset \bigcup_{i=-1}^{1} C_\alpha(\ell_1(x),\dots,\ell_k(x)+i).$$

Now, let $\delta > 0$ be arbitrary. Then, recall that g is slowly varying, so that if k is large enough, $g(k+1) \leq (k+1)^\delta < (1/r)^\delta$. Then, the proof of the lemma follows from the following calculation.

$$\begin{aligned}
\nu(B(x,r)) &\ll \nu(C_\alpha(\ell_1(x),\dots,\ell_k(x))) \leq \lambda(C_\alpha(\ell_1(x),\dots,\ell_k(x)))^{1/(1+\theta+\varepsilon)} \\
&\leq g(k+1)\lambda(C_\alpha(\ell_1(x),\dots,\ell_k(x),\ell_{k+1}(x)))^{1/(1+\theta+\varepsilon)} \\
&\ll g(k+1)\cdot r^{1/(1+\theta+\varepsilon)} \\
&\leq r^{1/(1+\theta+\varepsilon)-\delta}.
\end{aligned}$$

Since this is true for all $\delta > 0$, an application of Frostman's Lemma yields that

$$\frac{1}{1+\theta+\varepsilon} \leq \dim_H(F^{(\alpha)}_{f_\varepsilon,g}). \tag{2.2}$$

Finally, (2.2) shows that for every $\varepsilon > 0$ we have that $\dim_H(F^{(\alpha)}_\infty) \geq 1/(1+\theta+\varepsilon)$, so letting ε approach zero completes the proof.

□

All that remains for the proof of Theorem 2.5.5 is to give the upper bound for the dimension of $G^{(\alpha)}_\infty$. For this, first observe that if we consider the set

$$G^{(\alpha)}_{N,k} := \{x = [\ell_1(x),\ell_2(x),\dots]_\alpha : \ell_n(x) > N \text{ for all } n \geq k\},$$

we can easily see that for all $k \in \mathbb{N}$ this set has the same dimension as the set $G^{(\alpha)}_N$. To prove this claim, first let $\underline{b} = (b_1,\dots,b_k) \in \mathbb{N}^k$, where for simplicity we further assume that k is even, and define the set

$$G^{(\alpha)}_{N,\underline{b}} := \{x = [\ell_1(x),\ell_2(x),\dots]_\alpha : \ell_1(x) = b_1,\dots,\ell_k(x) = b_k,\ \ell_{k+i}(x) > N, \text{ for all } i \in \mathbb{N}\}.$$

Then, notice that the function $f : \mathscr{U} \to \mathscr{U}$ defined by $f(x) := [b_1, \dots, b_k]_\alpha + (a_{b_1} \dots a_{b_k})x$ maps the set $G_N^{(\alpha)}$ onto $G_{N,\underline{b}}^{(\alpha)}$. This is a similarity mapping (since it is linear) and in particular it is bi-Lipschitz. It is well known that bi-Lipschitz mappings preserve Hausdorff dimension (see Corollary 2.4 in [22], for instance), so we obtain that $\dim_H(G_{N,\underline{b}}^{(\alpha)}) = \dim_H(G_N^{(\alpha)})$. Now, observe that for any $k \in \mathbb{N}$ we can write

$$G_{N,k}^{(\alpha)} = \bigcup_{\substack{\underline{b}=(b_1,\dots,b_k) \\ \in \mathbb{N}^k}} G_{N,\underline{b}}^{(\alpha)}$$

and so

$$\dim_H\left(G_{N,k}^{(\alpha)}\right) = \sup\left\{\dim_H\left(G_{N,\underline{b}}^{(\alpha)}\right) : \underline{b} = (b_1, \dots, b_k) \in \mathbb{N}^k\right\} = \dim_H\left(G_N^{(\alpha)}\right).$$

It is also clear that for all $N \in \mathbb{N}$ there exists some $k \in \mathbb{N}$ such that $G_\infty^{(\alpha)} \subset G_{N,k}^{(\alpha)}$. Therefore, it follows from Theorem 2.5.4 that

$$\dim_H\left(G_\infty^{(\alpha)}\right) \leq \frac{1}{1+\theta}.$$

Taking this observation together with Lemma 2.5.7, we have proved Theorem 2.5.5.

□

For the final result of this section, let α be an expanding partition. Recall that this means that α satisfies the property that $\lim_{n\to\infty} t_n/t_{n+1} = \rho$, where $\rho > 1$. In this case, we have the following result.

Proposition 2.5.8. *Suppose that α is an expanding partition. Then,*

$$\lim_{N\to\infty} \dim_H\left(G_N^{(\alpha)}\right) = 0.$$

Proof. First note that by Lemma 2.4.8 we have that for any expanding partition α,

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1}{\rho} < 1.$$

This implies that for all $\varepsilon > 0$, if n is sufficiently large, then $a_n \ll (1/(\rho-\varepsilon))^n$. Let ε be small enough that $\tau := (\rho - \varepsilon) > 1$ and let $n_0 \in \mathbb{N}$ be such that $a_n \ll (1/\tau)^n$ for all $n > n_0$. Now, fix $N \in \mathbb{N}$ with $N > n_0$. We can cover the set $G_N^{(\alpha)}$ by cylinder sets of the form $C_\alpha(\ell_1, \dots, \ell_n)$ where each $\ell_i > N$, for $1 \leq i \leq n$. For any $\delta > 0$, we can always choose n sufficiently large that the cover consisting of the family of all such cylinder sets is a δ-cover. Let ε_N be defined implicitly by the equation $\tau^{N\varepsilon_N} \cdot \log(\tau^{\varepsilon_N}) > 1$. Then,

$$\begin{aligned}
\mathscr{H}_\delta^{\varepsilon_N}\left(G_N^{(\alpha)}\right) &\leq \sum_{\substack{(\ell_1,\dots,\ell_n)\\ \in\{N+1,N+2,\dots\}^n}} (a_{\ell_1}\cdots a_{\ell_n})^{\varepsilon_N} \\
&\ll \sum_{\substack{(\ell_1,\dots,\ell_n)\\ \in\{N+1,N+2,\dots\}^n}} \tau^{-(\ell_1+\dots+\ell_n)\varepsilon_N} \\
&= \left(\sum_{\ell>N} \tau^{-\ell\varepsilon_N}\right)^n \leq \left(\int_N^\infty \tau^{-\varepsilon_N x}\,dx\right)^n \\
&= \left(\left[-\frac{1}{\log\tau^{\varepsilon_N}}\cdot(\tau^{\varepsilon_N})^{-x}\right]_N^\infty\right)^n = \frac{1}{(\tau^{N\varepsilon_N}\cdot\log(\tau^{\varepsilon_N}))^n}.
\end{aligned}$$

Given the choice of ε_N above, this implies that $\mathscr{H}^{\varepsilon_N}(G_N^{(\alpha)}) \leq 1$ and consequently we obtain that $\dim_H(G_N^{(\alpha)}) \leq \varepsilon_N$. By letting ε_N tend to zero as N increases, we obtain the desired result, namely,

$$\lim_{N\to\infty} \dim_H\left(G_N^{(\alpha)}\right) = 0.$$

Note that it is certainly possible to choose a sequence $(\varepsilon_N)_{N\in\mathbb{N}}$ satisfying $\lim_{N\to\infty}\varepsilon_N = 0$. For example, if N is chosen large enough that $N\log(\tau) > -n\log\log(\sqrt[n]{\tau}) = n\log n - n\log\log(\tau)$, it suffices to let $\varepsilon_N = 1/n$.

□

Corollary 2.5.9. *For any expanding partition α, we have that*

$$\dim_H\left(G_\infty^{(\alpha)}\right) = 0.$$

Proof. This follows from exactly the same argument as the upper bound for the case of α expansive. Once an upper bound of zero is established, the lower bound is automatically the same.

□

2.5.2 Strict Jarník sets for the α-Lüroth system

Again, suppose throughout this section that we have a partition α behind our map L_α that is expansive with exponent $\theta > 0$. Let us now consider the following situation. Fix a sequence $(s_n)_{n\in\mathbb{N}}$ of natural numbers with the property that $\lim_{n\to\infty} s_n = \infty$. Then, let σ be given by

$$\sigma := \liminf_{n\to\infty} \frac{\log(s_1\cdots s_n)}{(1+\theta)\log(s_1\cdots s_n) + \theta\log(s_{n+1})} = \frac{1}{(1+\theta)+\theta\left(\limsup_{n\to\infty}\frac{\log(s_{n+1})}{\log(s_1\cdots s_n)}\right)}.$$

Finally, let $N > 3$ and define the set

$$J_\sigma^{(\alpha)} := \{x = [\ell_1(x), \ell_2(x), \dots]_\alpha : s_n \leq \ell_n(x) < Ns_n \text{ for all } n \in \mathbb{N}\}.$$

We refer to these sets as *strict* α*-Jarník* sets, after V. Jarník [39], for his results on similarly defined sets in the continued fractions setting. The reason for the word "strict" here is that in Jarník's work, the condition given for the continued fraction entries only has to be satisfied for infinitely many entries, whereas we require our condition to be met for all α-Lüroth entries. We will prove the following theorem.

Theorem 2.5.10.

$$\dim_H \left(J_\sigma^{(\alpha)}\right) = \sigma.$$

Remark 2.5.11. Before beginning the proof of Theorem 2.5.10, notice that for each $\sigma \in \mathbb{R}^+$ the set $J_\sigma^{(\alpha)}$ is contained in the set $G_\infty^{(\alpha)}$. Therefore the dimension can be at most $1/(1+\theta)$. This is consistent with the result given here, since we have that $\sigma = 1/((1+\theta)+\theta\cdot\tau)$, where $\tau := \limsup_{n\to\infty} \log(s_{n+1})/\log(s_1 \dots s_n) \geq 0$.

Proof of Theorem 2.5.10. Let us begin by establishing the upper bound. The set $J_\sigma^{(\alpha)}$ can be covered by sets of the form

$$\widetilde{C}_\alpha(\ell_1,\dots,\ell_k) := \bigcup_{m\geq s_{k+1}} C_\alpha(\ell_1,\dots,\ell_k,m),$$

where $s_i \leq \ell_i < Ns_i$ for each $1 \leq i \leq k$. We have that

$$\lambda(\widetilde{C}_\alpha(\ell_1,\dots,\ell_k)) = a_{\ell_1}\dots a_{\ell_k} t_{s_{k+1}}.$$

Recall that since α is expansive of exponent θ and eventually decreasing, for each positive ε, there exists $k\in\mathbb{N}$ such that $\ell^{-(1+\theta+\varepsilon)} \leq a_\ell \leq \ell^{-(1+\theta-\varepsilon)}$ for all $\ell \geq k$. Since the sequence $(s_n)_{n\in\mathbb{N}}$ tends to infinity, we may assume without loss of generality that if $x \in J_\sigma^{(\alpha)}$, then

$$(\ell_n(x))^{-(1+\theta+\varepsilon)} \leq a_{\ell(x)} \leq (\ell_n(x))^{-(1+\theta-\varepsilon)} \text{ for all } n\in\mathbb{N}.$$

For each $x \in J_\sigma^{(\alpha)}$, these observations lead to the estimate

$$\frac{1}{(\ell_1\cdots\ell_k)^{(1+\theta+\varepsilon)}(s_{k+1})^{(\theta+\varepsilon)}} \leq \lambda(\widetilde{C}_\alpha(\ell_1(x),\dots,\ell_k(x))) \leq \frac{1}{(\ell_1\cdots\ell_k)^{(1+\theta-\varepsilon)}(s_{k+1})^{(\theta-\varepsilon)}}.$$

In turn, this yields

$$\frac{1}{(N^k s_1\cdots s_k)^{(1+\theta+\varepsilon)}(s_{k+1})^{(\theta+\varepsilon)}} \leq \lambda(\widetilde{C}_\alpha(\ell_1(x),\dots,\ell_k(x))) \leq \frac{1}{(s_1\cdots s_k)^{(1+\theta-\varepsilon)}(s_{k+1})^{(\theta-\varepsilon)}}. \quad (2.3)$$

Now, define

$$\sigma_\varepsilon := \liminf_{n\to\infty} \frac{\log(s_1\cdots s_n)}{(1+\theta-\varepsilon)\log(s_1\cdots s_n)+(\theta-\varepsilon)\log(s_{n+1})}.$$

Directly from this definition, we have that if $\sigma' \in (\sigma_\varepsilon, 3\sigma_\varepsilon)$ and n is sufficiently large, then

$$\frac{\sigma' - \sigma_\varepsilon}{2} \leq \frac{\log(s_1 \cdots s_n)}{\log\left((s_1 \cdots s_n)^{(1+\theta-\varepsilon)}(s_{n+1})^{(\theta-\varepsilon)}\right)}.$$

Thus,

$$\begin{aligned}\left(\frac{1}{(s_1 \cdots s_n)^{(1+\theta-\varepsilon)}(s_{n+1})^{\theta-\varepsilon}}\right)^{\frac{\sigma'-\sigma_\varepsilon}{2}} &\leq \left(\frac{1}{(s_1 \cdots s_n)^{(1+\theta-\varepsilon)}(s_{n+1})^{\theta-\varepsilon}}\right)^{\frac{\log(s_1 \cdots s_n)}{\log(s_1 \cdots s_n)^{(1+\theta-\varepsilon)}(s_{n+1})^{(\theta-\varepsilon)}}} \\ &= \frac{1}{s_1 \cdots s_n}.\end{aligned}$$

It follows that $s_1 \cdots s_n \leq \left((s_1 \cdots s_n)^{(1+\theta-\varepsilon)}(s_{n+1})^{\theta-\varepsilon}\right)^{\frac{\sigma'-\sigma_\varepsilon}{2}}$. Now, since $\lim_{n\to\infty} s_n = \infty$, we have that $\lim_{n\to\infty} \log(s_n) = \infty$ and this in turn implies that $\lim_{n\to\infty}(\log(s_1 \cdots s_n))/n = \infty$. Therefore, for large enough $n \in \mathbb{N}$ we deduce that $\log(N-1) < \log(s_1 \ldots s_n)/n$. From this, we obtain the inequality

$$(N-1)^n \leq \left((s_1 \cdots s_n)^{(1+\theta-\varepsilon)}(s_{n+1})^{\theta-\varepsilon}\right)^{\frac{\sigma'-\sigma_\varepsilon}{2}}. \tag{2.4}$$

On the other hand, again from the definition of σ_ε, there exists a sequence $(n_k)_{k\in\mathbb{N}}$ of positive integers such that for all $\sigma' > \sigma_\varepsilon$, we have

$$\frac{\log(s_1 \cdots s_{n_k})}{\log\left((s_1 \cdots s_{n_k})^{(1+\theta-\varepsilon)}(s_{n_k+1})^{(\theta-\varepsilon)}\right)} \leq \frac{\sigma' + \sigma_\varepsilon}{2}.$$

Thus,

$$s_1 \cdots s_{n_k} \leq \left((s_1 \cdots s_{n_k})^{(1+\theta-\varepsilon)}(s_{n_k+1})^{\theta-\varepsilon}\right)^{\frac{\sigma'+\sigma_\varepsilon}{2}}. \tag{2.5}$$

Consequently, if we neglect any terms of the sequence (n_k) that are too small and rename the sequence accordingly, by combining the estimates in (2.4) and (2.5), we obtain for all $k \geq 1$ that

$$(N-1)^{n_k} s_1 \cdots s_{n_k} \leq \left((s_1 \cdots s_{n_k})^{(1+\theta-\varepsilon)}(s_{n_k+1})^{\theta-\varepsilon}\right)^{\sigma'}.$$

Thus,

$$\begin{aligned}\mathscr{H}^{\sigma'}\left(J_\sigma^{(\alpha)}\right) &\leq \liminf_{k\to\infty} \sum_{\substack{(\ell_1,\ldots,\ell_{n_k}) \\ s_i \leq \ell_i < N s_i}} \lambda(\widetilde{C}_\alpha(\ell_1,\ldots,\ell_k))^{\sigma'} \\ &\leq (N-1)^{n_k} s_1 \cdots s_{n_k} \cdot \left((s_1 \cdots s_{n_k})^{-(1+\theta-\varepsilon)}(s_{n_k+1})^{-(\theta-\varepsilon)}\right)^{\sigma'} \leq 1.\end{aligned}$$

Hence, for all $\varepsilon > 0$ and all $\sigma' > \sigma_\varepsilon$, we have that $\dim_H\left(J_\sigma^{(\alpha)}\right) \leq \sigma'$ and so, $\dim_H(J_\sigma^{(\alpha)}) \leq \sigma_\varepsilon$. It therefore follows, on letting ε tend to zero, that

$$\dim_H\left(J_\sigma^{(\alpha)}\right) \leq \sigma.$$

Let us now provide the lower bound. For this, as usual, we will use Frostman's Lemma. To that end, define a mass distribution m on $J_\sigma^{(\alpha)}$ by setting $m(C_\alpha(\ell_1,\ldots,\ell_k)) = 1/(\ell_1\cdots\ell_k)$. Let $x \in J_\sigma^{(\alpha)}$, $r > 0$ and choose k such that

$$\lambda(\widetilde{C}_\alpha(\ell_1(x),\ldots,\ell_{k+1}(x))) \leq r < \lambda(\widetilde{C}_\alpha(\ell_1(x),\ldots,\ell_k(x))).$$

There are now two possibilities. Either,

$$\lambda(\widetilde{C}_\alpha(\ell_1(x),\ldots,\ell_{k+1}(x))) \leq r < \lambda(C_\alpha(\ell_1(x),\ldots,\ell_k(x),\ell_{k+1}(x))), \tag{2.6}$$

or,

$$\lambda(C_\alpha(\ell_1(x),\ldots,\ell_{k+1}(x))) \leq r < \lambda(\widetilde{C}_\alpha(\ell_1(x),\ldots,\ell_k(x))). \tag{2.7}$$

Suppose we are in the situation of (2.6) and, for simplicity, assume that k is odd. It is clear that if k is large enough, the left endpoint of the ball $B(x,r)$ cannot extend past the cylinder set $C_\alpha(\ell_1(x),\ldots,\ell_{k+1}(x)-1)$ (since α is assumed to be eventually decreasing). On the other hand, the right endpoint cannot be larger than $[\ell_1(x),\ldots,\ell_{k+1},1]_\alpha + a_{\ell_1(x)}\cdots a_{\ell_{k+1}(x)}t_{s_{k+2}}$. We claim that as long as k is chosen large enough, this point lies inside $C_\alpha(\ell_1(x),\ldots,\ell_{k+1}(x)+1)$. To prove this claim, we are required to show that

$$a_{\ell_1(x)}\cdots a_{\ell_{k+1}(x)}t_{s_{k+2}} < a_{\ell_1(x)}\cdots a_{\ell_{k+1}(x)+1},$$

or, in other words, that

$$a_{\ell_{k+1}(x)}t_{s_{k+2}} < a_{\ell_{k+1}(x)+1}.$$

Note that by choosing k sufficiently large, the value of $t_{s_{k+2}}$ can be made as small as we like, so it is enough to show that there exists some constant K with the property that for all large enough $n \in \mathbb{N}$,

$$\frac{a_n}{a_{n+1}} \leq K.$$

Since α is expansive of exponent θ, we have that there exists a constant c such that

$$\frac{a_n}{a_{n+1}} \leq \frac{c(n+1)^{1+\theta}\psi(n)}{n^{1+\theta}\psi(n+1)}.$$

It follows that we can set the sought-after constant K equal to the above constant c, because we obviously have that $\lim_{n\to\infty}((n+1)/n)^{1+\theta} = 1$, and we have shown in Lemma 2.4.3 that $\lim_{n\to\infty}\psi(n)/\psi(n+1) = 1$.

In a slight abuse of notation, let us redefine the quantity σ_ε used above in the following way:

$$\sigma_\varepsilon := \liminf_{n\to\infty} \frac{\log(s_1\cdots s_n)}{(1+\theta+\varepsilon)\log(s_1\cdots s_n)+(\theta+\varepsilon)\log(s_{n+1})}.$$

We have shown that $B(x,r)\subset\bigcup_{i=-1}^{1} C_\alpha(\ell_1(x),\ldots,\ell_{k+1}(x)+i)$. Therefore, if we let $\sigma'<\sigma_\varepsilon$ and bear in mind that $r\geq a_{\ell_1(x)}\ldots a_{\ell_{k+1}(x)}t_{s_{k+2}}$, we obtain, via (2.3) and the definition of σ_ε, that

$$\begin{aligned} m(B(x,r)) &\leq 3m(C_\alpha(\ell_1(x),\ldots,\ell_{k+1}(x)))\leq \frac{3}{s_1\cdots s_{k+1}} \\ &\leq 3\left(\frac{1}{(s_1\cdots s_{k+1})^{(1+\theta+\varepsilon)}(s_{k+2})^{(\theta+\varepsilon)}}\right)^{\sigma'} \\ &\leq 3r^{\sigma'}. \end{aligned}$$

In this case, then, an application of Frostman's Lemma yields that for all $\varepsilon>0$ and all $\sigma'<\sigma_\varepsilon$, we have that

$$\dim_H\left(J_\sigma^{(\alpha)}\right)\geq\sigma'.$$

Let us now consider the second case, that of (2.7). Again, suppose for the sake of argument that k is odd. Then, it is clear once more that if k is large enough, the right endpoint of $B(x,r)$ cannot extend past the cylinder set $C_\alpha(\ell_1(x),\ldots,\ell_k(x)-1)$, since α is eventually decreasing. On the other hand, the left endpoint of $B(x,r)$ is not less than $[\ell_1(x),\ldots,\ell_k(x)]_\alpha-2a_{\ell_1(x)}\cdots a_{\ell_k(x)}t_{s_{k+1}}$. If k is sufficiently large, it is clear that $2a_{\ell_1(x)}\cdots a_{\ell_k(x)}t_{s_{k+1}}<a_{\ell_1(x)}\cdots a_{\ell_k(x)}$ (as $t_{s_{k+1}}$ can be made arbitrarily small by choosing large enough k). This implies that the left endpoint of $B(x,r)$ is contained within the cylinder set $C_\alpha(\ell_1(x),\ldots,\ell_k(x))$ and consequently $B(x,r)$ can only intersect the sets $\widetilde{C}_\alpha(\ell_1(x),\ldots,\ell_k(x))$ and $\widetilde{C}_\alpha(\ell_1(x),\ldots,\ell_k(x)-1)$ in this level.

Also, note that the smallest size that a cylinder set in the $(k+1)$-th level can have is at least equal to $(N^{k+1}s_1\cdots s_{k+1})^{-(1+\theta+\varepsilon)}$. Consequently, at most $2r(N^{k+1}s_1\cdots s_{k+1})^{(1+\theta+\varepsilon)}$ of these cylinder sets can intersect $B(x,r)$. Taking these observations together, we have that[5]

$$\begin{aligned} m(B(x,r)) &\leq \min\left\{2m(\widetilde{C}_\alpha(\ell_1,\ldots,\ell_k)),\left(2r(N^{k+1}s_1\cdots s_{k+1})^{(1+\theta+\varepsilon)}\right)m\left(C_\alpha(\ell_1,\ldots,\ell_{k+1})\right)\right\} \\ &\leq \min\left\{\frac{2}{s_1\cdots s_k},\frac{2(N^{k+1}s_1\cdots s_{k+1})^{(1+\theta+\varepsilon)}\cdot r}{s_1\cdots s_k s_{k+1}}\right\} \\ &= \frac{2}{s_1\cdots s_k}\left\{1,\left((N^{k+1}s_1\cdots s_k)^{(1+\theta+\varepsilon)}(s_{k+1})^{(\theta+\varepsilon)}\right)\cdot r\right\}. \end{aligned}$$

Note that $\min\{a,b\}\leq a^{1-s}b^s$ for all $s\in(0,1)$ and let $\sigma'<\sigma_\varepsilon$. It follows from this that

$$m(B(x,r))\leq\frac{2}{s_1\cdots s_k}\left((N^{k+1}s_1\cdots s_k)^{(1+\theta+\varepsilon)}(s_{k+1})^{(\theta+\varepsilon)}\right)^{\sigma'}r^{\sigma'}.$$

[5]We have left out the dependence on x of the entries $\ell_i(x)$, but this is only for lack of space.

By definition of σ_ε, we have for all $\sigma' < \sigma_\varepsilon$ and all large enough k that

$$\frac{1}{s_1 \cdots s_k} \leq \left((N^{k+1} s_1 \cdots s_k)^{(1+\theta+\varepsilon)} (s_{k+1})^{(\theta+\varepsilon)} \right)^{\sigma'}.$$

Thus,

$$m(B(x,r)) \leq 2r^{\sigma'}.$$

Therefore, as in the case of (2.6) described above, for all $\varepsilon > 0$ and all $\sigma' < \sigma_\varepsilon$, we have that

$$\dim_H \left(J_\sigma^{(\alpha)} \right) \geq \sigma'.$$

Finally, since this holds in both cases for all $\sigma' < \sigma_\varepsilon$, we first obtain that $\dim_H \left(J_\sigma^{(\alpha)} \right) \geq \sigma_\varepsilon$ and then, by letting ε tend to zero, we obtain that

$$\dim_H \left(J_\sigma^{(\alpha)} \right) \geq \sigma.$$

Combining this lower bound with the upper bound given above completes the proof of the theorem.

□

Remark 2.5.12. A similar situation for continued fractions is considered by Fan *et al.* in [24]. We will comment further on their paper in Chapter 6.

2.6 Appendix to Chapter 2: The non-alternating case

Just as for the alternating Lüroth map L_{α_H} and the classical (non-alternating) Lüroth map $L_{\widetilde{\alpha_H}}$, we could consider a non-alternating version of the map L_α. To do this, define the partition $\widetilde{\alpha} := \{[t_2, t_1], [t_{n+1}, t_n) : n \geq 2\}$, where the t_ns are the same as for the partition α, and then define the map $L_{\widetilde{\alpha}} : \mathscr{U} \to \mathscr{U}$ by setting

$$L_{\widetilde{\alpha}}(x) := \begin{cases} (x - t_{n+1})/a_n & \text{for } x \in [t_{n+1}, t_n), \text{ for } n \geq 2; \\ (x - t_2)/a_1 & \text{for } x \in [t_2, t_1]; \\ 0 & \text{for } x = 0. \end{cases}$$

In other words, the map $L_{\widetilde{\alpha}}$ has all positive slopes instead of all negative slopes.

The non-alternating $\widetilde{\alpha}$-Lüroth map $L_{\widetilde{\alpha}}$ also generates a series expansion of the numbers in $\mathscr{U}$. In this case, the expansion is given by

$$x = \sum_{n=1}^{\infty} \left(\prod_{i<n} a_{\ell_i} \right) t_{\ell_n+1} = t_{\ell_1+1} + a_{\ell_1} t_{\ell_2+1} + a_{\ell_1} a_{\ell_2} t_{\ell_3+1} + \dots.$$

We write $[\ell_1, \ell_2, \ell_3, \dots]_{\widetilde{\alpha}}$. This expansion can be finite or infinite, but just as for the classical Lüroth map, every finite expansion can alternatively be written as an eventually periodic expansion with periodic part consisting of infinitely many 1's. The convergents and cylinder sets in this non-alternating case are defined in the obvious way. It is clear that the Lebesgue measure of the cylinder set $C_{\widetilde{\alpha}}(\ell_1,\ldots,\ell_n)$ is equal to its L_α counterpart.

It is also possible to consider a non-alternating version of the α-Farey map. The map $F_{\widetilde{\alpha}}$ would again be defined on the partition $\widetilde{\alpha}$, the difference being that the right-hand branch would have a positive slope. The definition of the left-hand remains basically the same. That is,

$$F_{\widetilde{\alpha}}(x) := \begin{cases} (x-t_2)/a_1 & \text{for } x \in [t_2, 1]; \\ a_{n-1}(x-t_{n+1})/a_n + t_n & \text{for } x \in [t_{n+1}, t_n). \end{cases}$$

Concerning the topological properties of $F_{\widetilde{\alpha}}$, if we replace the tent map F_{α_D} by the binary expansion map $F_{\widetilde{\alpha_D}} : x \mapsto 2x \pmod 1$, then the maps $F_{\widetilde{\alpha}}$ and $F_{\widetilde{\alpha_D}}$ are topologically conjugate, via the conjugating homeomorphism $\theta_{\widetilde{\alpha}}$ which is given by

$$\theta_{\widetilde{\alpha}}(x) := \sum_{k=1}^{\infty} 2^{-\sum_{i=1}^{k} \widetilde{\ell_i}},$$

where the $\widetilde{\ell_i}$ are now the entries of the expansion of x with respect to $L_{\widetilde{\alpha}}$. The function $\theta_{\widetilde{\alpha}}$ is equal to the distribution function of the measure of maximal entropy of the system $(\mathscr{U}, \mathscr{B}, F_{\widetilde{\alpha}})$.

Chapter 3

Ergodic theoretic properties of F_α and L_α

In this chapter we will investigate various measure theoretic and ergodic theoretic properties of the maps L_α and F_α. It turns out that the Lebesgue measure is invariant for every map L_α and that there exists a unique Lebesgue-absolutely continuous invariant measure for F_α. We will give an exact expression for the density of this measure. Also, we prove that both L_α and F_α are exact, and thus ergodic. First, we give some background material useful for the results of the following sections.

3.1 Measure and ergodic theoretic preliminaries

In this section, we give an outline of the main results we need in this and the following chapters. Whilst there are many available references dealing with various aspects of ergodic theory, the main reference used here is Walters [81]. Throughout, let $(X, \mathscr{B})$ be a measurable space.

Definition 3.1.1. A measure μ is said to be *invariant* for a map $T : X \to X$ provided that for every μ-measurable set $B \subset X$, we have $\mu \circ T^{-1}(B) := \mu(T^{-1}(B)) = \mu(B)$. We also say that the map T *preserves* the measure μ.

In practice, it could be difficult to check that a map preserves a given measure using only this definition, as it is often the case that no specific information is known about a general measurable set. However, it is enough to have knowledge of a particular semi-algebra that generates $\mathscr{B}$, as the following theorem shows.

Theorem 3.1.2. *Suppose that $(X, \mathscr{B}, \mu)$ is a measure space, $T : X \to X$ is a map and also suppose that $\mathscr{S}$ is a generating semi-algebra for $\mathscr{B}$. Then, if $\mu \circ T^{-1}(B) = \mu(B)$ for every set $B \in \mathscr{S}$, we have that the map T preserves the measure μ.*

Proof. See Theorem 1.1 in [81]. □

Note that if the given measurable space is the unit interval $\mathscr{U}$, then the collection $\mathscr{S}$ of all sets of the form $[0, b]$ and $(a, b]$ with $0 \le a < b \le 1$ is a generating semi-algebra for the Borel σ-algebra of subsets of $\mathscr{U}$.

Definition 3.1.3. Let $T : X \to X$ be a transformation and let μ be a T-invariant Borel probability measure. Then T is said to be *ergodic* with respect to μ provided that whenever $A \in \mathscr{B}$ is such that $T^{-1}(A) = A$ we have that $\mu(A) = 1$ or $\mu(A) = 0$. In other words, an ergodic transformation only has trivial invariant subsets.

The first major result in ergodic theory was proved in 1931 by G.D. Birkhoff. There are now various proofs available, the reader is referred to either [81], [16] or, for the proof usually known as the "non-standard" one, the paper of Kamae and Keane [41]. We shall state it here in the simplest case of an ergodic finite measure-preserving system.

Theorem 3.1.4. Birkhoff's Ergodic Theorem. *Suppose that $T : (X, \mathscr{B}, \mu) \to (X, \mathscr{B}, \mu)$ is an ergodic measure-preserving transformation and also suppose that $\mu(X) < \infty$. Let f be a μ-integrable function. Then, for μ-a.e. $x \in X$, we have*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} f \circ T^j(x) = \int f \, d\mu.$$

Definition 3.1.5. A subset W of X is said to be a *wandering set* for a map $T : X \to X$ if the collection $\{T^{-n}(W) : n \geq 0\}$ is pairwise disjoint. In particular, this implies that

$$\sum_{n=0}^{\infty} \mathbb{1}_W \circ T^n \leq 1.$$

Definition 3.1.6. A map $T : X \to X$ is said to be *conservative* if every wandering set for T has measure zero.

The following theorem is a useful way of determining that a given transformation is conservative. The proof is not too long or complicated, so we include it here for completeness. Throughout, we will use the notation "mod μ" to indicate that two sets are equal up to a set of μ-measure zero.

Theorem 3.1.7. Maharam's Recurrence Theorem. *([1], p.19): Let T be a measure-preserving transformation on a measure space $(X, \mathscr{B}, \mu)$ and suppose that there exists a set $A \in \mathscr{B}$ of finite measure such that $\bigcup_{n=0}^{\infty} T^{-n}(A) = X \mod \mu$. Then T is conservative.*

Proof. First observe that $T^{-N}(X) = X$ for every positive integer N and so, for each N, we have that

$$\bigcup_{n=N}^{\infty} T^{-n}(A) = X \text{ (mod } \mu).$$

Hence, almost every $x \in X$ must belong to infinitely many sets $T^{-k}(A)$, or, in other words,

$$\sum_{n=0}^{\infty} \mathbb{1}_A \circ T^n = \infty \ \ \mu\text{-almost everywhere on } X.$$

Now, let W be a wandering set for T, so $\sum_{n=0}^{\infty} \mathbb{1}_W \circ T^n \leq 1$. Then, for all positive integers n, the T-invariance of μ implies that

$$\begin{aligned}
\infty &> \mu(A) = \mu(T^{-n}(A)) \geq \int_{T^{-n}(A)} \sum_{k=0}^{n} \mathbb{1}_W \circ T^k \, d\mu \\
&= \sum_{k=0}^{n} \int_{T^{-n}(A)} \mathbb{1}_W \circ T^k \, d\mu = \sum_{k=0}^{n} \int_X \mathbb{1}_{T^{-n}(A)} \cdot \mathbb{1}_W \circ T^k \, d\mu \\
&= \sum_{k=0}^{n} \int_X (\mathbb{1}_{T^{k-n}(A)} \circ T^k)(\mathbb{1}_W \circ T^k) \, d\mu = \sum_{k=0}^{n} \int_X (\mathbb{1}_{T^{k-n}(A)} \cdot \mathbb{1}_W) \circ T^k \, d\mu \\
&= \sum_{k=0}^{n} \int_X \mathbb{1}_{T^{k-n}(A)} \cdot \mathbb{1}_W \, d\mu = \int_W \sum_{k=0}^{n} \mathbb{1}_A \circ T^k \, d\mu.
\end{aligned}$$

Thus, $\mu(W) = 0$ and T is conservative.

□

Definition 3.1.8. A transformation $T : X \to X$ of a measure space $(X, \mathscr{B}, \mu)$ is said to be *non-singular* if whenever $\mu(B) = 0$, then $\mu(T^{-1}(B)) = 0$. That is, the map T preserves sets of measure zero.

Note here that if we have an invariant measure for a map T, then the map T is automatically non-singular with respect to this invariant measure.

Definition 3.1.9. A non-singular transformation T of a σ–finite measure space $(\mathscr{U}, \mathscr{B}, \mu)$ is said to be *exact* if for each B in the tail σ-algebra $\bigcap_{n\in\mathbb{N}} T^{-n}(\mathscr{B})$ we have that either $\mu(B)$ or $\mu(\mathscr{U} \setminus B)$ vanishes.

Remark 3.1.10.

1. It is immediately clear that this definition only makes sense for non-invertible transformations. Indeed, if $T : X \to X$ is invertible, it follows that $T^{-n}(\mathscr{B}) = \mathscr{B}$ for every $n \in \mathbb{N}$.

2. The tail σ-algebra is not a completely transparent object. It helps to remember that it is an intersection of sets of sets. In particular, this means that if $B \in \bigcap_{n\in\mathbb{N}} T^{-n}(\mathscr{B})$, then $B \in T^{-n}(\mathscr{B})$ for all $n \in \mathbb{N}$. Thus, there exists a sequence of sets $(B_1, B_2, B_3, \ldots)$ such that $B = T^{-n}(B_n)$ for every $n \in \mathbb{N}$.

3. It is easy to see that an exact transformation must be ergodic, for if $T : X \to X$ is exact and B is a measurable subset of X such that $T^{-1}(B) = B$, then $T^{-n}(B) = B$ for all $n \in \mathbb{N}$ and so the set B belongs to the tail σ-algebra.

An equivalent formulation of the definition of an invariant measure comes from the transfer operator, which we now define.

Definition 3.1.11. The *transfer operator* $\mathscr{T} : L^1(\mu) \to L^1(\mu)$ associated with a map $T : X \to X$ on a measure space $(X, \mathscr{B}, \mu)$ is a positive linear operator given by

$$\int_B \mathscr{T}(f) \, d\mu = \int_{T^{-1}(B)} f \, d\mu, \ \text{ for all } f \in L^1(\mu) \text{ and all } B \in \mathscr{B}.$$

Lemma 3.1.12. *The measure μ is T-invariant if and only if $\mathscr{T}(\mathbb{1}_X) = \mathbb{1}_X$.*

Proof. First notice that from the definition of the transfer operator we obtain

$$\int_B \mathscr{T}(\mathbb{1}_X)\, d\mu = \int_{T^{-1}(B)} \mathbb{1}_X\, d\mu = \mu(T^{-1}(B)). \tag{3.1}$$

Now suppose that $\mathscr{T}(\mathbb{1}_X) = \mathbb{1}_X$. It follows that

$$\int_B \mathscr{T}(\mathbb{1}_X)\, d\mu = \int_B \mathbb{1}_X\, d\mu = \mu(B).$$

Consequently, the condition that $\mathscr{T}(\mathbb{1}_X) = \mathbb{1}_X$ implies that the measure μ is T-invariant. Conversely, if μ is T-invariant, we have from (3.1) that

$$\mu(B) = \mu(T^{-1}(B)) = \int_B \mathscr{T}(\mathbb{1}_X)\, d\mu \;\Rightarrow\; \mathscr{T}(\mathbb{1}_X) = \mathbb{1}_X.$$

□

3.2 Ergodic theoretic properties of L_α

Let us begin this section by showing that the Lebesgue measure is invariant under L_α, for any arbitrary partition α.

Lemma 3.2.1. *The invariant measure for L_α is equal to the Lebesgue measure λ.*

Proof. This follows directly from Proposition 2.3.1 in [16], since, as previously noted, the map L_α is a particular type of Generalised Lüroth map. However, we include a proof for completeness.

Recall from Definition 2.1.8 the inverse branches $L_{\alpha,n} : [0,1) \to A_n$ of L_α. These branches are given by $L_{\alpha,n}(x) := t_n - a_n x$, for all $n \in \mathbb{N}$. In order to show that L_α is λ-invariant, by Theorem 3.1.2 it suffices to show that

$$\lambda([a,b)) = \lambda(L_\alpha^{-1}[a,b)),$$

for every interval $[a,b)$ with $0 \le a < b < 1$ contained in $\mathscr{U}$. A straightforward calculation shows that

$$\begin{aligned}
\lambda(L_\alpha^{-1}[a,b)) &= \lambda\left(\bigcup_{n=1}^{\infty} L_{\alpha,n}([a,b))\right) = \sum_{n=1}^{\infty} \lambda(L_{\alpha,n}([a,b))) \\
&= \sum_{n\in\mathbb{N}} |(t_n - a_n \cdot a) - (t_n - a_n \cdot b)| \\
&= \sum_{n\in\mathbb{N}} a_n(b-a) = b - a = \lambda([a,b)).
\end{aligned}$$

This gives the L_α-invariance of λ.

□

We can show much more than this, namely, that the map L_α is an exact transformation with respect to Lebesgue measure. Recall that the definition of exactness is given above, in Definition 3.1.9. In fact, that L_α is exact follows from the result proved in [5] that each Generalised Lüroth System is Bernoulli, but we provide a direct proof here for completeness.

Lemma 3.2.2. *The map L_α is exact with respect to λ.*

Proof. The proof is an adaptation of the proof of Kolmogorov's zero-one law for the one-sided Bernoulli shift (see [51]). To start, let $B \in \bigcap_{n\in\mathbb{N}} L_\alpha^{-n}(\mathscr{B})$ be given such that $\lambda(B) > 0$. We aim to prove that $\lambda(B) = 1$. As noted in Remark 3.1.10, there exists a sequence of Borel sets $(B_n)_{n\in\mathbb{N}}$ such that $B = L_\alpha^{-n} B_n$, for all $n \in \mathbb{N}$. We first claim that for every finite union $\mathscr{C}$ of L_α-cylinder sets we have that

$$\lambda(B \cap \mathscr{C}) = \lambda(B)\lambda(\mathscr{C}).$$

To prove the claim, first consider a single α-Lüroth cylinder set $C_\alpha(\ell_1,\ldots,\ell_m)$. Then, for this set, by the translation invariance and scaling properties of Lebesgue measure and by the fact that the Lebesgue measure is invariant for the map L_α, we have that

$$\begin{aligned}
\lambda(B \cap C_\alpha(\ell_1,\ldots,\ell_m)) &= \lambda(C_\alpha(\ell_1,\ldots,\ell_m) \cap L_\alpha^{-m}(B_m)) \\
&= \lambda(\{[\ell_1,\ldots,\ell_m,x_1,x_2,\ldots]_\alpha : [x_1,x_2,\ldots]_\alpha \in B_m\}) \\
&= \lambda([\ell_1,\ldots,\ell_m]_\alpha + a_{\ell_1}\ldots a_{\ell_m} B_m) = a_{\ell_1}\cdots a_{\ell_m}\lambda(B_m) \\
&= \lambda(C_\alpha(\ell_1,\ldots,\ell_m))\lambda(L_\alpha^{-m}(B_m)) = \lambda(C_\alpha(\ell_1,\ldots,\ell_m))\lambda(B).
\end{aligned}$$

One immediately verifies that this also holds for a finite union $\mathscr{C}$ of L_α-cylinder sets. From this, because the finite unions of cylinder sets generate the σ-algebra $\mathscr{B}$, we deduce that

$$\lambda(B \cap C) = \lambda(B)\lambda(C), \text{ for all } C \in \mathscr{B}.$$

Therefore, by choosing C to be equal to $\mathscr{U} \setminus B$, we conclude that

$$0 = \lambda(B \cap (\mathscr{U} \setminus B)) = \lambda(B)\lambda(\mathscr{U} \setminus B).$$

This shows that $\lambda(\mathscr{U} \setminus B) = 0$ and so $\lambda(B) = 1$, and hence finishes the proof. □

Since exactness implies ergodicity, the following list of properties of the system $(\mathscr{U},\mathscr{B},L_\alpha,\lambda)$ can be derived quite simply from Birkhoff's ergodic theorem.

Proposition 3.2.3. *For λ-almost every $x \in \mathscr{U}$, the following statements hold:*

(i) $\lim_{n\to\infty} \frac{1}{n}\#\{j \le n : \ell_j(x) = k\} = a_k.$

(ii) $\lim_{n\to\infty} \frac{1}{n}\log\left(\prod_{j=1}^{n} \ell_j(x)\right) = \sum_{k=1}^{\infty} a_k \log k.$

(iii) $\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} \ell_j(x) = \sum_{k=1}^{\infty} t_k.$

(iv) *Every finite block $\ell_1,\ldots,\ell_k \in \mathbb{N}^k, k \in \mathbb{N}$ appears infinitely often in the α-Lüroth expansion of x.*

(v) *With the additional assumption on the partition α that $a_n \leq t_{n+1}$ for sufficiently large $n \in \mathbb{N}$, we obtain that*

$$\lim_{n\to\infty} \frac{1}{n} \log \left| x - r_n^{(\alpha)} \right| = \sum_{k=1}^{\infty} a_k \log a_k.$$

Proof. Each of the above statements follow directly on application of Birkhoff's ergodic theorem to a specific λ-integrable function f. For the first assertion, choose f to be the characteristic function $\mathbb{1}_{A_k}$. Then, for any positive integer j, it follows that

$$f \circ L_\alpha^j(x) = \begin{cases} 1 & \text{if } L_\alpha^j(x) = k; \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 1 & \text{if } \ell_{j+1}(x) = k; \\ 0 & \text{otherwise.} \end{cases}$$

In light of this, we have $\sum_{j=0}^{n-1} f \circ L_\alpha^j(x) = \#\{1 \leq j \leq n : \ell_j(x) = k\}$ and so

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} f \circ L_\alpha^j(x) = \lim_{n\to\infty} \frac{1}{n} \#\{1 \leq j \leq n : \ell_j(x) = k\} = \int_{\mathscr{U}} \mathbb{1}_{A_k} \, d\lambda = a_k.$$

The second, third and fourth statements follow similarly by choosing, in turn, the function f to be given by $f(x) = \log(\ell_1(x))$, $f(x) = \ell_1(x)$ and $f(x) = \mathbb{1}_{C_\alpha(\ell_1,\ldots,\ell_k)}$.

For part (v), first notice that under the stated condition on α, we have for sufficiently large n that

$$a_{\ell_1} \ldots a_{\ell_n} a_{\ell_{n+1}} \leq \left| x - r_n^{(\alpha)} \right| \leq a_{\ell_1} \ldots a_{\ell_n}. \tag{3.2}$$

Let f be given by $f(x) = \log(a_{\ell_1(x)})$. Then, by inequality (3.2), we have that

$$\begin{aligned} \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} f \circ L_\alpha^j(x) &= \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \log(a_{\ell_j(x)}) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \log\left(\left| x - r_n^{(\alpha)} \right| \right) \\ &= \int_{\mathscr{U}} \log(a_{\ell_1(x)}) \, d\lambda = \sum_{k=1}^{\infty} \int_{A_k} \log(a_{\ell_1(x)}) \, d\lambda \\ &= \sum_{k=0}^{\infty} a_k \log(a_k). \end{aligned}$$

This finishes the proof. $\square$

Remark 3.2.4.

1. The above list is only a small sample of the possible results available by using Birkhoff's ergodic theorem in this manner. The reader can no doubt think of others.

2. Similar results for the specific example of the alternating Lüroth series were obtained by Kalpazidou, Knopfmacher and Knopfmacher [40].

3. The above argument for part (v) of Proposition 3.2.3 also shows that
$$\lim_{n\to\infty}\frac{1}{n}\log\left(\lambda\left(C_\alpha(\ell_1(x),\ldots,\ell_n(x))\right)\right)=\sum_{k=0}^{\infty}a_k\log(a_k).$$
By the Shannon-McMillan-Breiman Theorem (see Billingsley [10], Section 13), we then have that the measure-theoretic entropy of L_α is equal to $\sum_{k=0}^{\infty}a_k\log(a_k)$, for suitable partitions α.

4. The extra condition on α given in part (iv) is equivalent to $a_n/t_n \le 1/2$. For the example of the alternating Lüroth map, the condition is met. It is also satisfied for any expansive partition of exponent $\theta > 0$ and for expanding partitions with $\rho < 2$.

5. Part (iv) of Proposition 3.2.3 implies the result in Lemma 2.5.1.

Finally in this section, we describe an invariant probability measure for the natural extension of L_α, which was introduced in Definition 2.1.8.

Proposition 3.2.5. *The 2-dimensional Lebesgue measure $\lambda\times\lambda$ is an invariant measure for L_α^+.*

Proof. This again follows from the corresponding result given in [16] for the Generalised Lüroth situation, which is Proposition 4.4.1.

□

3.3 Ergodic properties of F_α

We now turn our attention to the ergodic theoretical properties of the α-Farey system. The first of these that we immediately obtain is that F_α is a conservative transformation. This can be seen by observing that $\bigcup_{n=0}^{\infty}F_\alpha^{-n}(A_1)=\mathscr{U}\setminus\{0\}$, and hence, Maharam's Recurrence Theorem (see Theorem 3.1.7) applies, giving that F_α is conservative.

In the following lemma, we give a Lebesgue absolutely continuous invariant measure for the system F_α.

Lemma 3.3.1. *The λ-absolutely continuous measure ν_α defined by the density ϕ_α which is given, up to multiplication by a constant, by*
$$\phi_\alpha := \frac{d\nu_\alpha}{d\lambda}=\sum_{n=1}^{\infty}\frac{t_n}{a_n}\cdot\mathbb{1}_{A_n},$$
is an invariant measure for the system $(\mathscr{U},\mathscr{B},F_\alpha)$. Moreover, ν_α is a σ-finite measure, and we have that ν_α is an infinite measure if and only if α is of infinite type.

Proof. Let us first show that ν_α is an invariant measure for F_α. In order to do this, it suffices to show that

$$\nu_\alpha(I) = \nu_\alpha \circ F_\alpha^{-1}(I),$$

for each interval I belonging to the generating semi-algebra $\mathscr{S} := \{(a,b] : 0 < a < b \leq 1\} \cup \{[0,b] : 0 \leq b \leq 1\}$. In other words, recalling that $F_{\alpha,0}$ and $F_{\alpha,1}$ denote the inverse branches of F_α, we must show that

$$\int_I \phi_\alpha \, d\lambda = \int_{F_{\alpha,0}(I)} \phi_\alpha \, d\lambda + \int_{F_{\alpha,1}(I)} \phi_\alpha \, d\lambda. \tag{3.3}$$

Each interval $(a,b]$ is contained in a finite number of elements of α, say $\{A_n, A_{n-1}, \ldots, A_m\}$. So each of the integrals in (3.3) can be split up into a sum over the sets $(a,t_n], A_{n-1}, \ldots, A_{m+1}, (t_{m+1}, b]$. We will only show that

$$\int_{(a,t_n]} \phi_\alpha \, d\lambda = \int_{F_{\alpha,0}(a,t_n]} \phi_\alpha \, d\lambda + \int_{F_{\alpha,1}(a,t_n]} \phi_\alpha \, d\lambda,$$

but it is clear that the same calculation extends to each of the other parts in a similar way. So first note that

$$\int_{(a,t_n]} \phi_\alpha \, d\lambda = \int_{(a,t_n]} \frac{t_n}{a_n} \, d\lambda = \frac{t_n}{a_n}(t_n - a).$$

Then, $F_{\alpha,0}$ maps $(a,t_n]$ linearly into A_{n+1}, with slope a_{n+1}/a_n, and $F_{\alpha,1}$ maps $(a,t_n]$ linearly into A_1, with slope $-a_1$. Therefore,

$$\begin{aligned}
\int_{F_{\alpha,0}(a,t_n]} \phi_\alpha \, d\lambda + \int_{F_{\alpha,1}(a,t_n]} \phi_\alpha \, d\lambda &= \frac{t_{n+1}}{a_{n+1}} \cdot \frac{a_{n+1}}{a_n}(t_n - a) - \frac{t_1}{a_1} \cdot a_1(t_n - a) \\
&= \frac{t_n + a_n}{a_n}(t_n - a) - (t_n - a) = \frac{t_n}{a_n}(t_n - a).
\end{aligned}$$

It remains to show that the same relation holds for intervals of the form $[0,b]$. Without loss of generality, we assume that $b = t_n$, for some $n \in \mathbb{N}$. Then,

$$\int_0^{t_n} \phi_\alpha \, d\lambda = \int_0^{t_{n+1}} \phi_\alpha \, d\lambda + \int_{A_n} \phi_\alpha \, d\lambda = \int_0^{t_{n+1}} \phi_\alpha \, d\lambda + t_n,$$

whereas

$$\begin{aligned}
\int_{F_{\alpha,0}([0,t_n])} \phi_\alpha \, d\lambda + \int_{F_{\alpha,1}([0,t_n])} \phi_\alpha \, d\lambda &= \int_0^{t_{n+1}} \phi_\alpha \, d\lambda + \int_{[1,t_n]_\alpha}^{1} \frac{1}{a_1} \, d\lambda \\
&= \int_0^{t_{n+1}} \phi_\alpha \, d\lambda + \frac{1}{a_n}(1 - (1 - a_1 t_n)) = \int_0^{t_{n+1}} \phi_\alpha \, d\lambda + t_n.
\end{aligned}$$

This finishes the proof of the first assertion of the lemma.

Regarding the second statement of the lemma, it is clear that the measure ν_α is σ-finite and a simple calculation shows that for each $n \in \mathbb{N}$ we have that

$$\nu_\alpha\left(\bigcup_{k=1}^{n} A_k\right) = \sum_{k=1}^{n} \nu_\alpha(A_k) = \sum_{k=1}^{n} \int_{A_k} \phi_\alpha \, d\lambda = \sum_{k=1}^{n} \frac{t_k}{a_k} \cdot a_k = \sum_{k=1}^{n} t_k.$$

Recalling that α is of infinite type provided that $\sum_{k=1}^{\infty} t_k$ diverges, the proof is finished.

□

Remark 3.3.2.

1. It is clear that ν_α is absolutely continuous with respect to λ; indeed, it is defined that way. The converse is also true, since the density ϕ_α is strictly positive. Hence, the measures ν_α and λ are in the same measure class.

2. The reader may recall the Bogolyubov-Krylov Theorem, which states that there always exists an invariant probability measure for each continuous map $T : X \to X$ on a compact metric space X. However, this theorem does not say anything about whether the measure is absolutely continuous with respect to Lebesgue measure. There may be several invariant measures, but we will shortly see that in our case the λ-absolutely continuous one given by ϕ_α is unique.

3. The proof given above is only one possible way to show that ν_α is an invariant measure for F_α. Another method is by considering the *Ruelle operator*. This is an operator $\mathscr{R}_\alpha : L^1(\mu) \to L^1(\mu)$ defined with respect to a measure μ on $(\mathscr{U}, \mathscr{B})$ by

   $$\mathscr{R}_\alpha(f) = \left|F_{\alpha,0}{}'\right| \cdot (f \circ F_{\alpha,0}) + \left|F_{\alpha,1}{}'\right| \cdot (f \circ F_{\alpha,1}), \text{ for all } f \in L^1(\mu).$$

   If the measure μ is λ-absolutely continuous with density $\psi := d\mu/d\lambda$ and if ψ is strictly positive μ-almost everywhere, it is easy to verify that $\mathscr{R}_\alpha$ and the transfer operator $\mathscr{F}_\alpha$ for F_α (defined in Section 3.1) are related in the following way:

   $$\mathscr{F}_\alpha(f) = \frac{1}{\psi} \cdot \mathscr{R}_\alpha(\psi \cdot f), \text{ for all } f \in L^1(\nu).$$

   So, recalling that μ is F_α invariant if and only if $\mathscr{F}_\alpha(\mathbb{1}_{\mathscr{U}}) = \mathbb{1}_{\mathscr{U}}$, in order to verify that a particular function ψ is a density which gives rise to an invariant measure for the map F_α, it is sufficient to show that ψ is an eigenfunction of $\mathscr{R}_\alpha$, that is,

   $$\psi = \mathscr{R}_\alpha(\psi \cdot \mathbb{1}_{\mathscr{U}}).$$

   For the inverse branches $F_{\alpha,1}$ and $F_{\alpha,0}$ and the density ϕ_α given in Lemma 3.3.1, a straightforward computation shows that we have

   $$\phi_\alpha \circ F_{\alpha,1} = t_1/a_1 \cdot \mathbb{1}_{\mathscr{U}} \text{ and } \phi_\alpha \circ F_{\alpha,0} = \sum_{n=1}^{\infty} t_{n+1}/a_{n+1} \cdot \mathbb{1}_{A_n}.$$

Moreover, one immediately verifies that

$$|F'_{\alpha,1}| = a_1 \cdot \mathbb{1}_{\mathscr{U}} \text{ and } |F'_{\alpha,0}| = \sum_{n=1}^{\infty} a_{n+1}/a_n \cdot \mathbb{1}_{A_n}.$$

Using these two observations, it follows that

$$\begin{aligned}\mathscr{R}_\alpha(\phi_\alpha) &= |F_{\alpha,0}{}'| \cdot (\phi_\alpha \circ F_{\alpha,0}) + |F_{\alpha,1}{}'| \cdot (\phi_\alpha \circ F_{\alpha,1}) = \sum_{n=1}^{\infty} \left(\frac{a_{n+1}}{a_n} \frac{t_{n+1}}{a_{n+1}} \right) \cdot \mathbb{1}_{A_n} + t_1 \cdot \mathbb{1}_{\mathscr{U}} \\ &= \sum_{n=1}^{\infty} \left(\frac{t_{n+1}}{a_n} + 1 \right) \cdot \mathbb{1}_{A_n} = \sum_{n=1}^{\infty} \frac{t_n}{a_n} \cdot \mathbb{1}_{A_n} = \phi_\alpha.\end{aligned}$$

Our next aim is to prove that the map F_α is exact. A major ingredient in the proof is the Lebesgue Density Theorem, which we need in a slightly different version to the one usually found in the literature. Almost always, the Lebesgue Density Theorem is stated in terms of balls converging onto a point. We need a similar statement, but with the balls replaced by a sequence of sets that shrink to x in a particular way.

Definition 3.3.3. Suppose that $x \in \mathbb{R}$. A sequence $(E_i)_{i\in\mathbb{N}}$ of Borel sets is said to *shrink to x nicely* if there exists a number $a > 0$ with the following property: There is a sequence of balls $(B(x,r_i))_{i\in\mathbb{N}}$, with $\lim_{i\to\infty} r_i = 0$, such that for each $i \in \mathbb{N}$ we have $E_i \subset B(x,r_i)$ and

$$\lambda(E_i) \geq a \cdot \lambda(B(x,r_i)).$$

We then have the following theorem.

Theorem 3.3.4. *To each $x \in \mathbb{R}$ associate a sequence of sets $(E_i(x))_{i\in\mathbb{N}}$ that shrinks to x nicely and let $f \in L^1(\lambda)$. Then,*

$$f(x) = \lim_{i\to\infty} \frac{1}{\lambda(E_i(x))} \int_{E_i(x)} f \, d\lambda, \ \textit{for a.e. } x.$$

Proof. This can be found as Theorem 7.10 in Rudin [68].

□

Theorem 3.3.5. Lebesgue Density Theorem. *If A is a Lebesgue measurable set, then there is a subset E of A with $\lambda(E) = 0$ such that every point $x \in A \setminus E$ satisfies*

$$\lim_{n\to\infty} \lambda(A|E_i(x)) := \lim_{i\to\infty} \frac{\lambda(A \cap E_i(x))}{\lambda(E_i(x))} = 1,$$

where $(E_i(x))_{i\in\mathbb{N}}$ is a sequence of sets that shrink to x nicely.

Proof. This follows immediately on setting $f := \mathbb{1}_A$ in Theorem 3.3.4 above.

□

It is clear that for any $x \in \mathscr{U}$ the sequence of cylinder sets $(C_\alpha(\ell_1(x),\ldots,\ell_k(x)))_{k\in\mathbb{N}}$ shrinks to x nicely. Indeed, if we define $B(x,r_k)$ to be the ball centred around x with radius r_k given by $\max\{[\ell_1(x),\ldots,\ell_k(x)]_\alpha - x, [\ell_1(x),\ldots,\ell_k(x),1]_\alpha - x\}$, then $C_\alpha(\ell_1(x),\ldots,\ell_k(x)) \subset B(x,r_k)$ and the number a in the above definition can be taken to be equal to $1/2$. So, the Lebesgue Density Theorem is valid in particular for these sequences.

Theorem 3.3.6. *The α-Farey map F_α is exact.*

Proof. Since ν_α and λ are absolutely continuous with respect to each other, it is sufficient to show the exactness of F_α with respect to λ. Let $B_0 \in \bigcap_{n\in\mathbb{N}} F_\alpha^{-n}\mathscr{B}$ be given such that $\lambda(B_0) > 0$. The aim is, therefore, to show that $\lambda(B_0^c) = 0$. For this, first recall that there must exist a sequence $(B_n)_{n\in\mathbb{N}}$ in $\mathscr{B}$ such that $B_0 = F_\alpha^{-n}B_n$, for all $n \in \mathbb{N}_0$. Clearly, we then have that $B_{k+m} = F_\alpha^k B_m$, for all $k,m \in \mathbb{N}_0$.

Secondly, recall from Section 2.2 that the first return map ρ_α is defined by $\rho_\alpha(x) := \inf\{n \geq 0 : F_\alpha^n(x) \in A_1\}$ and that it is finite for every point x in $\mathscr{U} \setminus \{0\}$. Further, define the function $\rho^{(n)}$ by setting

$$\rho^{(n)}(x) := \sum_{k=0}^{n-1} \left(\rho_\alpha\left(L_\alpha^k(x)\right) + 1\right).$$

Note that for $x = [\ell_1, \ell_2, \ldots]_\alpha$, we have that $\rho^{(n)}(x) = \sum_{k=1}^n \ell_k$.

Using the facts that λ is L_α–invariant and Bernoulli with respect to L_α (just as in the proof that L_α is exact), we obtain for λ-almost every $x = \langle x_1, x_2, \ldots\rangle_\alpha = [\ell_1, \ell_2, \ldots]_\alpha$,

$$\begin{aligned}
\lambda\left(B_0|\widehat{C}_\alpha(x_1,\ldots,x_{\rho^{(n)}(x)})\right) &:= \frac{\lambda\left(F_\alpha^{-\rho^{(n)}(x)} B_{\rho^{(n)}(x)} \cap \widehat{C}_\alpha(x_1,\ldots,x_{\rho^{(n)}(x)})\right)}{\lambda\left(\widehat{C}_\alpha(x_1,\ldots,x_{\rho^{(n)}(x)})\right)} \\
&= \frac{\lambda\left(L_\alpha^{-n} B_{\rho^{(n)}(x)} \cap C_\alpha(\ell_1,\ldots,\ell_n)\right)}{\lambda\left(C_\alpha(\ell_1,\ldots,\ell_n)\right)} \\
&= \frac{\lambda\left(L_\alpha^{-n} B_{\rho^{(n)}(x)}\right)\lambda\left(C_\alpha(\ell_1,\ldots,\ell_n)\right)}{\lambda\left(C_\alpha(\ell_1,\ldots,\ell_n)\right)} \\
&= \lambda\left(B_{\rho^{(n)}(x)}\right).
\end{aligned}$$

Also, by the Lebesgue Density Theorem, we have for λ-almost every $x = \langle x_1, x_2, \ldots\rangle_\alpha$,

$$\lim_{n\to\infty} \lambda\left(B_0|\widehat{C}_\alpha(x_1,\ldots,x_{\rho^{(n)}(x)})\right) = \mathbb{1}_{B_0}(x) = \begin{cases} 1 & \text{if } x \in B_0; \\ 0 & \text{if } x \notin B_0. \end{cases}$$

Combining these observations, it follows that B_0 coincides up to a set of measure zero with the set Ω, where Ω is defined by

$$\Omega := \left\{x \in \mathscr{U} : \lim_{n\to\infty} \lambda\left(B_{\rho^{(n)}(x)}\right) > 0\right\}.$$

(In fact, it is equivalent to put the limit in the definition of Ω equal to 1.)

Since, by assumption, $\lambda(B_0) > 0$, we now have that $\lambda(\Omega) > 0$. Hence, to finish the proof, we are left to show that $\lambda(\Omega) = 1$. For this recall that λ is L_α-invariant and ergodic. Thus, it is sufficient to show that $L_\alpha^{-1}\Omega = \Omega$, up to a set of measure zero.

Recalling that $B_{k+m} = F_\alpha^k B_m$, for all $k,m \in \mathbb{N}_0$, we have the relation

$$B_{\rho^{(n+1)}(x)} = B_{\rho(x)+\rho^{(n)}(L_\alpha(x))} = F_\alpha^{\rho(x)} B_{\rho^{(n)}(L_\alpha(x))}. \tag{3.4}$$

Suppose that $x \in \Omega$. From the above relation, we have that $B_{\rho^{(n)}(L_\alpha(x))} = F_\alpha^{-\rho(x)}(B_{\rho^{(n+1)}(x)})$. Since λ and ν_α are in the same measure class, we have that if $\lambda(B_{\rho^{(n+1)}(x)}) > 0$, then $\nu_\alpha(B_{\rho^{(n+1)}(x)}) > 0$. Therefore, as ν_α is invariant under F_α, it follows that $\nu_\alpha(F_\alpha^{-\rho(x)}(B_{\rho^{(n+1)}(x)})) > 0$ and, finally, we obtain that $\lambda(F_\alpha^{-\rho(x)}(B_{\rho^{(n+1)}(x)})) > 0$. Therefore,

$$\lim_{n\to\infty} \lambda(B_{\rho^{(n)}(L_\alpha(x))}) > 0.$$

Thus, $\Omega \subset L_\alpha^{-1}\Omega$. Therefore, in order to complete the proof, we are left to show that $L_\alpha^{-1}\Omega \subset \Omega$, or, in other words, that

$$\lim_{n\to\infty} \lambda(B_{\rho^{(n)}(L_\alpha(x))}) > 0 \text{ implies } \lim_{n\to\infty} \lambda(B_{\rho^{(n)}(x)}) > 0.$$

By the relation in (3.4) again, the above assertion would hold if we establish that for each $\varepsilon > 0$ and $\ell \in \mathbb{N}$ there exists $\kappa > 0$ such that for all $C \in \mathscr{B}$ with $\lambda(C) > \varepsilon$ we have $\lambda(F_\alpha^\ell C) > \kappa$. Therefore, assume that $\lambda(C) > \varepsilon$ and let α_ℓ^* denote the ℓ-th refinement of the partition α^* for the map F_α. Also, one clearly can remove an open neighbourhood of the boundary points of the intervals in α_ℓ^* to obtain a closed set $U \subset \mathscr{U}$ such that $\lambda(U) > 1 - \varepsilon/2$. Since there are 2^ℓ elements in α_ℓ^*, this immediately implies that $\lambda(C \cap B \cap U) > \varepsilon 2^{-\ell-1}$ for some $B \in \alpha_\ell^*$. Indeed, if this were not the case, we would have that

$$\lambda(C\cap U) = \lambda\left(\bigcup_{B\in\alpha_\ell^*} C\cap B\cap U\right) = \sum_{B\in\alpha_\ell^*} \lambda(C\cap B\cap U) \le 2^\ell \cdot \varepsilon 2^{-l-1} = \frac{\varepsilon}{2}.$$

This contradicts the facts that $\lambda(C) > \varepsilon$ and $\lambda(U) > 1 - \varepsilon/2$, so proving the claim.

Now, keeping in mind that $F_\alpha^\ell|_B : B \to \mathscr{U}$ is bijective (and F_α^ℓ maps B linearly onto the whole of $\mathscr{U}$), by the choice of the set U there exists a constant $c > 0$ such that $\lambda(F_\alpha^\ell(B\cap U\cap C)) > c\lambda(B\cap U\cap C)$. It now follows that $\lambda(F_\alpha^\ell(C)) \ge \lambda(F_\alpha^\ell(C\cap B\cap U)) > c2^{-\ell-1}\varepsilon$. Hence, by setting $\kappa := c2^{-\ell-1}\varepsilon$, the proof follows.

□

We then have the following immediate corollary.

Corollary 3.3.7. *The map F_α is ergodic.*

The next theorem appears as Theorem 1.5.6 in the book [1] by Aaronson. We will use it to show that the invariant measure ν_α for the system $(\mathscr{U}, F_\alpha)$, identified in Lemma 3.1.10, is unique.

Theorem 3.3.8. *Let $T : X \to X$ be a conservative, ergodic, non-singular transformation of $(X, \mathscr{B}, \mu)$. Then, up to multiplication by a constant, there is at most one μ-absolutely continuous σ-finite T-invariant measure.*

Proposition 3.3.9. *Up to multiplication by a constant, the invariant measure ν_α is unique.*

Proof. First, F_α is non-singular with respect to λ, since ν_α and λ are in the same measure class. Then, as F_α is ergodic and (as previously noted) conservative, an application of the theorem above gives that ν_α is unique.

□

We end this section by stating the following applications of some general results from infinite ergodic theory to the system $(\mathscr{U}, \mathscr{B}, F_\alpha, \nu_\alpha)$. Note that the first, but only the first, is also valid for α of finite type.

- *A consequence of Hopf's Ergodic Theorem* [37]:
  For each non-negative $f \in L^1(\nu_\alpha)$ with $\int_{\mathscr{U}} f \, d\nu_\alpha > 0$, we have that

$$\lim_{n\to\infty} \sum_{k=0}^{n-1} f(F_\alpha^k(x)) = \infty, \text{ for } \nu_\alpha\text{-almost every } x \in \mathscr{U}.$$

- *A consequence of Krengel's Theorem* [52]:
  If α is of infinite type, then we have, for each $\varepsilon > 0$,

$$\lim_{n\to\infty} \lambda\left(\left\{x \in \mathscr{U} : \left|1/n \sum_{k=0}^{n-1} f(F_\alpha^k(x))\right| \geq \varepsilon\right\}\right) = 0, \text{ for all } f \in L^1(\lambda), f \geq 0.$$

- *A consequence of Aaronson's Theorem* [1, Theorem 2.4.2]:
  If α is of infinite type, then we have, for each $f \in L^1(\lambda)$ such that $f \geq 0$ and for each sequence$(c_n)_{n\in\mathbb{N}}$ of positive integers, that either

$$\liminf_{n\to\infty} \frac{\sum_{j=0}^{n-1} f(F_\alpha^j(x))}{c_n} = 0,$$

  or, there exists a subsequence $(c_{n_k})_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} \frac{\sum_{j=0}^{n_k-1} f(F_\alpha^j(x))}{c_{n_k}} = \infty.$$

- *A consequence of Lin's Criterion for exactness* [54]:
  For α of infinite type, since F_α is exact, we have that

$$\lim_{n\to\infty} \int |\mathscr{F}_\alpha^n(f)| \, d\nu_\alpha = 0, \text{ for all } f \in L^1(\nu_\alpha) \text{ such that } \int f \, d\nu_\alpha = 0.$$

We will have occasion to use Lin's criterion for exactness in Chapter 4.

Chapter 4

Renewal Theory

In this chapter, we aim to use some classical results from renewal theory to study the sequence of α-sum-level sets for the map L_α. These sets are defined in Section 4.2. We begin the chapter with a section describing the renewal theorems that we will later apply.

4.1 Classical renewal results

In this section, we state and prove the original discrete renewal theorem due to P. Erdős, H. Pollard and W. Feller [20]. For the treatment of this result we follow the book by Krengel [53]. We also state, this time without proof, some more in-depth renewal results.

Let us begin by giving a lemma needed for the proof of the standard renewal theorem.

Lemma 4.1.1. *If d is the greatest common divisor of the numbers $n_1, n_2, n_3, \ldots \in \mathbb{N}$, then there exist numbers K and M with the property that every integer dm, where $m \geq M$, can be written in the form*

$$dm = \sum_{k=1}^{K} c_k n_k, \quad \textit{with each } c_k \in \mathbb{N}.$$

Proof. We can assume that $d = 1$ (otherwise just divide each of the n_k by d), and also that d is the greatest common divisor of the first K of the given numbers, that is, $(n_1, n_2, \ldots, n_K) = 1$. It is well known that there then exist integers $a_1, a_2, \ldots, a_k$ with the property that[1]

$$a_1 n_1 + \ldots + a_K n_k = 1.$$

Letting $a := \max\{|a_1|, |a_2|, \ldots, |a_K|\}$ and $M := a n_1(n_1 + \ldots + n_K)$, we have that each $m \geq M$ can be written in the form

$$m = a n_1(n_1 + \ldots + n_K) + i n_1 + r(a_1 n_1 + \ldots + a_K n_k),$$

[1] This can be seen, for instance, by considering moduli of integers. The interested reader is referred to Section 2.9 of The Theory of Numbers, by Hardy and Wright [33].

where $i \geq 0$ and $0 \leq r < n_1$ come from the division algorithm applied to $m - M$. Therein lie the factors c_k and (since $an_1 > a_k r$), they are clearly positive integers.

□

Definition 4.1.2. Let $(v_n)_{n\in\mathbb{N}}$ be an infinite probability vector, that is, a sequence of non-negative real numbers for which $\sum_{k=1}^{\infty} v_n = 1$. Assume that associated to this vector there exists a sequence $(w_n)_{n\in\mathbb{N}_0}$ such that $w_0 = 1$ and such that (w_n) satisfies the *renewal equation*

$$w_n = \sum_{m=1}^{n} v_m w_{n-m}, \quad \text{for all } n \in \mathbb{N}.$$

A pair $((v_n),(w_n))$ of sequences with these properties is referred to as a *renewal pair*.

For a given renewal pair $((v_n),(w_n))$, we make the following definitions:

$$d_v := \gcd\{n \geq 1 : v_n > 0\} \text{ and } d_w := \gcd\{n \geq 1 : w_n > 0\}.$$

Then, for all n with $w_n = 0$ we also have that $v_n = 0$, since using the renewal equation gives that $w_n = 0 = \sum_{m=1}^{n} v_m w_{n-m}$ and so each term in this sum must be equal to zero. In particular, $v_n w_0 = 0$, but since $w_0 = 1$, it follows that $v_n = 0$. This implies that d_w is a factor of d_v. It is also possible to show, using a fairly straightforward but somewhat ungainly induction argument, that d_v is a factor of d_w. Thus, these two quantities are always equal.

Finally, before stating the theorem, we also need the following simple analytical lemma. We include the proof for completeness.

Lemma 4.1.3. *Let $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ be two sequences with the property that $\lim_{n\to\infty}(a_n + b_n)$ exists. Then, provided that we do not have $\liminf_{n\to\infty} a_n = -\infty$ and $\limsup_{n\to\infty} b_n = \infty$, or vice versa, it follows that*

$$\lim_{n\to\infty}(a_n + b_n) = \liminf_{n\to\infty} a_n + \limsup_{n\to\infty} b_n.$$

Proof. Let $a := \liminf_{n\to\infty} a_n$ and $b := \limsup_{n\to\infty} b_n$ and then pick two sequences $(n_i)_{i\geq 1}$ and $(n_j)_{j\geq 1}$ such that

$$\lim_{i\to\infty} a_{n_i} = a \text{ and } \lim_{j\to\infty} b_{n_j} = b.$$

Then, on the one hand we have

$$\begin{aligned}\lim_{n\to\infty}(a_n + b_n) &= \limsup_{i\to\infty}(a_{n_i} + b_{n_i}) \leq \limsup_{i\to\infty} a_{n_i} + \limsup_{i\to\infty} b_{n_i} \\ &\leq a + \limsup_{n\to\infty} b_n = a + b.\end{aligned}$$

But, on the other hand,

$$\begin{aligned}\lim_{n\to\infty}(a_n + b_n) &= \liminf_{j\to\infty}(a_{n_j} + b_{n_j}) \geq \liminf_{j\to\infty} a_{n_j} + \liminf_{j\to\infty} b_{n_j} \\ &\geq \liminf_{n\to\infty} a_n + b = a + b.\end{aligned}$$

Combining these two inequalities, the lemma is proved.

□

Theorem 4.1.4. (Discrete Renewal Theorem.) *Let $((v_n),(w_n))$ be a renewal pair and suppose that $d_v = 1$. Then*

$$\lim_{n\to\infty} w_n = \frac{1}{\sum_{m=1}^{\infty} m \cdot v_m},$$

where the limit is equal to zero if the series in the denominator diverges.

Proof. For ease of notation, throughout we denote $S := \sum_{m=1}^{\infty} m \cdot v_m$. First, we show by induction that $0 \le w_n \le 1$, for each $n \in \mathbb{N} \cup \{0\}$. To start, notice that $w_1 = v_1 \cdot w_0 = v_1 \le 1$. Now suppose that $0 \le w_k \le 1$ for all $0 \le k \le n-1$. Then

$$w_n = v_1 \cdot w_{n-1} + v_2 \cdot w_{n-2} + \ldots + v_n \cdot w_0 \le \sum_{k=1}^{n} v_k \le 1.$$

Let $\omega := \limsup_{n\to\infty} w_n$ and pick a subsequence $(w_{n_k})_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty} w_{n_k} = \omega$. Then, for all $m \ge 1$, we have, via Lemma 4.1.3, that

$$\begin{aligned}
\omega &= \lim_{k\to\infty} w_{n_k} = \lim_{k\to\infty} \left(v_m \cdot w_{n_k-m} + \sum_{\substack{1\le s\le n_k \\ s\ne m}} v_s \cdot w_{n_k-s} \right) \\
&= \liminf_{k\to\infty} v_m \cdot w_{n_k-m} + \limsup_{k\to\infty} \sum_{\substack{1\le s\le n_k \\ s\ne m}} v_s \cdot w_{n_k-s} \\
&\le v_m \liminf_{k\to\infty} w_{n_k-m} + \sum_{s\ne m} v_s \limsup_{k\to\infty} w_{n_k-s} \\
&\le v_m \liminf_{k\to\infty} w_{n_k-m} + (1-v_m)\omega.
\end{aligned}$$

From this it follows that $v_m\omega \le v_m \liminf_{k\to\infty} w_{n_k-m}$ and so, provided that $v_m > 0$, we obtain

$$\limsup_{n\to\infty} w_n =: \omega \le \liminf_{k\to\infty} w_{n_k-m}.$$

Therefore,

$$\lim_{k\to\infty} w_{n_k-m} = \omega. \tag{4.1}$$

Applying this argument many times over, one can say that equation (4.1) holds for all m such that there exist positive integers $m_1, \ldots m_j$ with each $v_{m_i} > 0$ so that $m = m_1 + \ldots + m_j$. Given that $d_v = 1$, from Lemma 4.1.1 it follows that every large enough m has this form (where we can do without the factors c_i as there is no reason that all the m_i have to be distinct). In other words, there exists some $M \in \mathbb{N}$ such that (4.1) holds for every $m \ge M$.

Now, for each $n \in \mathbb{N}$, set

$$r_n := \sum_{m=n+1}^{\infty} v_m.$$

Then $r_0 = 1$ and

$$\sum_{m=1}^{\infty} m \cdot v_m = \sum_{m=1}^{\infty} v_m + \sum_{m=2}^{\infty} v_m + \sum_{m=3}^{\infty} v_m + \ldots = \sum_{n=0}^{\infty} r_n.$$

From the renewal equation and the fact that $r_m - r_{m-1} = -v_m$, we deduce that

$$r_0 \cdot w_n = w_n = -\sum_{m=1}^{n} (r_m - r_{m-1}) w_{n-m}$$

and, bringing the negative terms to the left-hand side, we can write this in the following way:

$$r_0 \cdot w_n + r_1 \cdot w_{n-1} + \ldots + r_n \cdot w_0 = r_0 \cdot w_{n-1} + \ldots + r_{n-1} \cdot w_0. \tag{4.2}$$

Denote the left-hand side of Equation (4.2) by A_n, so the right-hand side is then A_{n-1}. Note that $A_0 = r_0 \cdot w_0 = 1$. Thus, in light of Equation (4.2), we have that $A_n = 1$ for all $n \in \mathbb{N}$. In particular,

$$\sum_{i=0}^{n_k - M} r_i \cdot w_{n_k-(M+i)} = 1. \tag{4.3}$$

We will now show that $\omega = 1/S$. First, suppose that S is finite. In that case, for all positive ε there exists $N \in \mathbb{N}$ with

$$r_0 + r_1 + \ldots r_N \geq S - \varepsilon$$

If k is sufficiently large that $n_k - M \geq N$, then by (4.3) we have that

$$1 \geq \sum_{i=0}^{N} r_i \cdot w_{n_k-(M+i)}.$$

It then follows from (4.1) that

$$1 \geq \omega(r_1 + \ldots r_N) \geq \omega(S - \varepsilon).$$

Since ε was an arbitrary positive number, we obtain the inequality $\omega \leq 1/S$.

On the other hand, from (4.3) and from the inequalities $w_n \leq 1$ and $(r_{N+1} + r_{N+2} + \ldots) \leq \varepsilon$, we deduce that

$$1 \leq \varepsilon + \sum_{i=0}^{N} r_i \cdot w_{n_k-(M+i)}.$$

Letting k tend to infinity, from the above equation we obtain that $1 \leq \varepsilon + \omega \sum_{m=1}^{\infty} m \cdot v_m$ and so we also have the opposite inequality, namely, $\omega \geq 1/S$.

If we are instead in the situation that S is infinite, we have that for all positive numbers C, there exists an $N \in \mathbb{N}$ with

$$r_0 + r_1 + \ldots r_N > C,$$

from which, in a similar way to the above, we obtain the inequality $1 \geq C\omega$. Since C can be arbitrarily large, it follows that $\omega = 0$. Notice that if $\limsup_{n\to\infty} w_n = 0$, we must in fact have that $\lim_{n\to\infty} w_n = 0$, as these are all positive numbers. Therefore, in the case where S is infinite, the proof is finished.

In the case where S is finite, we would also have to show that $\liminf_{n\to\infty} w_n = 1/S$. This proceeds analogously, starting by setting $\omega' := \liminf_{n\to\infty} w_n$ and choosing a subsequence that achieves this lower limit.

□

Remark 4.1.5. In the above proof, if it so happens that $v_m > 0$ for every $m \in \mathbb{N}$, we could dispense with the slight complication of having to use Lemma 4.1.1, since in this situation we would have that Equation (4.1) would hold for every $m \in \mathbb{N}$.

We will now state some stronger renewal results obtained by Erickson, Garsia and Lamperti, which we will apply in the next section. Let the sequences $((v_n)_{n\in\mathbb{N}}, (w_n)_{n\in\mathbb{N}\cup\{0\}})$ be a given renewal pair, and let the two associated sequences $(V_n)_{n\in\mathbb{N}}$ and $(W_n)_{n\in\mathbb{N}}$ be defined by

$$V_n := \sum_{k=n}^{\infty} v_k \text{ and } W_n := \sum_{k=1}^{n} w_k,$$

for all $n \in \mathbb{N}$. The principal assumption in these results is that

$$V_n = \psi(n) n^{-\theta},$$

for all $n \in \mathbb{N}$, for some $\theta \in [0,1]$ and for some slowly-varying function ψ.

Strong renewal results by Garsia/Lamperti and Erickson [29, Lemma 2.3.1], [21, Theorem 5]:

For $\theta \in [0,1]$, we have that

$$W_n \sim (\Gamma(2-\theta)\Gamma(1+\theta))^{-1} \cdot n \cdot \left(\sum_{k=1}^{n} V_k\right)^{-1}.$$

Also, if $\theta \in (1/2,1]$, then

$$w_n \sim (\Gamma(2-\theta)\Gamma(\theta))^{-1} \cdot \left(\sum_{k=1}^{n} V_k\right)^{-1}.$$

Finally, for $\theta \in (0,1/2)$ we have that the limit in the latter formula does not have to exist in general. However, for $\theta \in (0,1/2]$ it is shown in [29, Theorem 1.1] that one at least has

$$\liminf_{n\to\infty} n \cdot w_n \cdot V_n = \frac{\sin \pi\theta}{\pi},$$

and that if we replace the above "$\liminf$" by "$\lim$", then this limit exists on the complement of some set of integers of zero density[2].

[2]The density of a set of integers A is given, where the limit exists, by $d(A) = \lim_{n\to\infty} \#A(n)/n$, where $A(n) := \{1,\dots,n\} \cap A$. For example, if $A := \{n^2 : n \in \mathbb{N}\}$, then since $\#A(n) \le \sqrt{n}$ we have $d(A) = 0$.

4.2 Renewal results applied to α-sum-level sets

In this section we use standard renewal theory in order to study the sequence of the Lebesgue measures of the α*-sum-level sets* for a given partition α. These sets are given, for each $n \in \mathbb{N}_0$, by

$$\mathscr{L}_n^{(\alpha)} := \left\{ x \in C_\alpha(\ell_1, \ell_2, \ldots, \ell_k) : \sum_{i=1}^{k} \ell_i = n, \text{ for some } k \in \mathbb{N} \right\},$$

where, for $n = 0$, we have put $\mathscr{L}_0^{(\alpha)} := \mathscr{U}$. The first members of this sequence are as follows:

$$\begin{gathered} C_\alpha(1) \\ C_\alpha(2) \cup C_\alpha(1,1) \\ C_\alpha(3) \cup C_\alpha(1,2) \cup C_\alpha(2,1) \cup C_\alpha(1,1,1) \\ C_\alpha(4) \cup C_\alpha(3,1) \cup C_\alpha(2,2) \cup C_\alpha(2,1,1) \cup C_\alpha(1,3) \cup C_\alpha(1,2,1) \cup C_\alpha(1,1,2) \cup C_\alpha(1,1,1,1) \end{gathered}$$

In order to obtain the precise rate of decay of the Lebesgue measure of the α-sum-level sets $\mathscr{L}_n^{(\alpha)}$, we will now employ the renewal theorems detailed in the previous section. We begin our discussion with the following crucial observation, which shows that the sequence of the Lebesgue measure of the α-sum-level sets satisfies the renewal equation. Here, the role of the probability vector is filled by the sequence of Lebesgue measures of the partition elements of α, that is, the sequence $(a_m)_{m \in \mathbb{N}}$.

Lemma 4.2.1. *For each $n \in \mathbb{N}$, we have that*

$$\lambda(\mathscr{L}_n^{(\alpha)}) = \sum_{m=1}^{n} a_m \lambda(\mathscr{L}_{n-m}^{(\alpha)}).$$

Proof. Since $\lambda(\mathscr{L}_0^{(\alpha)}) = 1$ and $\lambda(\mathscr{L}_1^{(\alpha)}) = a_1$, the assertion certainly holds for $n = 1$. For $n \geq 2$, the following calculation finishes the proof.

$$\begin{aligned} \lambda(\mathscr{L}_n^{(\alpha)}) &= \lambda(C_\alpha(n)) + \sum_{m=1}^{n-1} \sum_{\substack{C_\alpha(\ell_1,\ldots,\ell_k,m) \in \mathscr{L}_n^{(\alpha)} \\ k \in \mathbb{N}}} \lambda(C_\alpha(\ell_1, \ldots, \ell_k, m)) \\ &= \lambda(C_\alpha(n)) + \sum_{m=1}^{n-1} a_m \sum_{\substack{C_\alpha(\ell_1,\ldots,\ell_k) \in \mathscr{L}_{n-m}^{(\alpha)} \\ k \in \mathbb{N}}} \lambda(C_\alpha(\ell_1, \ldots, \ell_k)) \\ &= a_n \lambda(\mathscr{L}_0^{(\alpha)}) + \sum_{m=1}^{n-1} a_m \lambda(\mathscr{L}_{n-m}^{(\alpha)}) = \sum_{m=1}^{n} a_m \lambda(\mathscr{L}_{n-m}^{(\alpha)}). \end{aligned}$$

□

We are now in the position to state our main result. The first part of our main result is valid for arbitrary partitions, but for the second part we must restrict ourselves to partitions that are either expansive of exponent $\theta \in [0,1]$ (recall that these were introduced in Definition 2.4.4) or of finite type.

Theorem 4.2.2. (1) *For the α-sum-level sets of an arbitrary given partition α of $\mathscr{U}$ we have that $\sum_{n=1}^{\infty} \lambda(\mathscr{L}_n^{(\alpha)})$ diverges, and that*

$$\lim_{n\to\infty} \lambda\left(\mathscr{L}_n^{(\alpha)}\right) = \begin{cases} 0 & \text{if } F_\alpha \text{ is of infinite type;} \\ \left(\sum_{k=1}^{\infty} t_k\right)^{-1} & \text{if } F_\alpha \text{ is of finite type.} \end{cases}$$

(2) *For a given expansive partition α which is either of exponent $\theta \in [0,1]$ or such that F_α is of finite type, we have the following estimates for the asymptotic behaviour of the Lebesgue measure of the α-sum-level sets.*

(*i*) *With $K_\theta := (\Gamma(2-\theta)\Gamma(1+\theta))^{-1}$ for α expansive of exponent $\theta \in [0,1]$ and with $K_\theta := 1$ for F_α of finite type, we have that*

$$\sum_{k=1}^{n} \lambda\left(\mathscr{L}_k^{(\alpha)}\right) \sim K_\theta \cdot n \cdot \left(\sum_{k=1}^{n} t_k\right)^{-1}.$$

(*ii*) *With $k_\theta := (\Gamma(2-\theta)\Gamma(\theta))^{-1}$ for α expansive of exponent $\theta \in (1/2,1]$ and with $k_\theta := 1$ for F_α of finite type, we have that*

$$\lambda\left(\mathscr{L}_n^{(\alpha)}\right) \sim k_\theta \cdot \left(\sum_{k=1}^{n} t_k\right)^{-1}.$$

(*iii*) *For an expansive partition α of exponent $\theta \in (0,1)$, we have that*

$$\liminf_{n\to\infty} \left(n \cdot t_n \cdot \lambda\left(\mathscr{L}_n^{(\alpha)}\right)\right) = \frac{\sin \pi\theta}{\pi}.$$

Moreover, if $\theta \in (0,1/2)$, then the corresponding limit does not exist in general. However, in this situation the existence of the limit is always guaranteed at least on the complement of some set of integers of zero density.

Proof of Theorem 4.2.2 (1). The general form of the discrete renewal theorem given in Theorem 4.1.4 above can be applied directly to our specific situation, namely, the sequence of the Lebesgue measure of the α-sum-level sets. For this, fix some partition $\alpha = \{A_n : n \in \mathbb{N}\}$, and set $v_n := \lambda(A_n) = a_n$, for each $n \in \mathbb{N}$. Notice that this is certainly a probability vector. Then, put $w_n := \lambda(\mathscr{L}_n^{(\alpha)})$, for each $n \in \mathbb{N}_0$. In light of Lemma 4.2.1 and the observation that $w_0 = \lambda(\mathscr{L}_0^{(\alpha)}) = 1$, we then have that these particular sequences (v_n) and (w_n) are indeed a renewal pair. Consequently, an application of the discrete renewal theorem immediately implies that

$$\lim_{n\to\infty} \lambda\left(\mathscr{L}_n^{(\alpha)}\right) = \left(\sum_{k=1}^{\infty} k \cdot a_k\right)^{-1} = \left(\sum_{k=1}^{\infty} t_k\right)^{-1},$$

where this limit is equal to zero if $\sum_{k=1}^{\infty} t_k$ diverges. Note that, by Lemma 3.3.1, the divergence of the latter series is equivalent to the statement that the α-Farey map F_α is of infinite invariant type.
For the remaining assertion in (1), let us consider the two generating functions a and ℓ, which are given by $a(s) := \sum_{n=1}^{\infty} a_n s^n$ and $\ell(s) := \sum_{m=0}^{\infty} \lambda\left(\mathscr{L}_m^{(\alpha)}\right) s^m$. Using Lemma 4.2.1 and the fact that $\lambda\left(\mathscr{L}_0^{(\alpha)}\right) = 1$, the following calculation shows that for each $s \in (0,1)$ we have that $\ell(s) - 1 = \ell(s)a(s)$. On the one hand, we have that

$$\ell(s)a(s) = \sum_{n=1}^{\infty} a_n s^n + s\lambda\left(\mathscr{L}_1^{(\alpha)}\right) \sum_{n=1}^{\infty} a_n s^n + s^2\lambda\left(\mathscr{L}_2^{(\alpha)}\right) \sum_{n=1}^{\infty} a_n s^n + \dots$$

On the other hand, we also have that

$$\begin{aligned}
\ell(s) - 1 &= \sum_{m=1}^{\infty} \lambda\left(\mathscr{L}_m^{(\alpha)}\right) s^m \\
&= s\lambda\left(\mathscr{L}_1^{(\alpha)}\right) + s^2\lambda\left(\mathscr{L}_2^{(\alpha)}\right) + s^3\lambda\left(\mathscr{L}_3^{(\alpha)}\right) + \dots \\
&= a_1 s + s^2\left(a_1\lambda\left(\mathscr{L}_1^{(\alpha)}\right) + a_2\right) + s^3\left(a_1\lambda\left(\mathscr{L}_2^{(\alpha)}\right) + a_2\lambda\left(\mathscr{L}_1^{(\alpha)}\right) + a_3\right) + \dots \\
&= \sum_{n=1}^{\infty} a_n s^n + s\lambda\left(\mathscr{L}_1^{(\alpha)}\right) \sum_{n=1}^{\infty} a_n s^n + s^2\lambda\left(\mathscr{L}_2^{(\alpha)}\right) \sum_{n=1}^{\infty} a_n s^n + \dots
\end{aligned}$$

Hence, $\ell(s) = 1/(1 - a(s))$. Since $a(1) = 1$, this gives that $\lim_{s\nearrow 1} \ell(s) = \infty$, which shows that $\sum_{n=0}^{\infty} \lambda(\mathscr{L}_n^{(\alpha)})$ diverges. This finishes the proof of Theorem 4.2.2 (1).

Proof of Theorem 4.2.2 (2) (i), (ii) and (iii). The statements in part (2) concerning partitions α such that F_α is of finite type follow easily from Theorem 4.2.2 (1). Indeed, given that

$$\lim_{n\to\infty} \left(\frac{\lambda\left(\mathscr{L}_n^{(\alpha)}\right)}{\left(\sum_{k=1}^{n} t_k\right)^{-1}} \right) = \lim_{n\to\infty} \lambda\left(\mathscr{L}_n^{(\alpha)}\right) \cdot \lim_{n\to\infty} \sum_{k=1}^{n} t_k = 1,$$

the statement in part (2) (ii) follows immediately. The corresponding claim in part (2) (i) follows directly on considering the Cesàro average of the sequence of Lebesgue measures of α-sum-level sets. Similarly to the proof of part (1), the statements in Theorem 4.2.2 (2) (i), (ii) and (iii) concerning partitions that are expansive of exponent θ follow from straightforward applications of the strong renewal results of Garsia/Lamperti and Erickson to the setting of the α-sum-level sets. For this we have to put $v_n := a_n$, $V_n := t_n$ and $w_n := \lambda(\mathscr{L}_n^{(\alpha)})$, and to recall that the pair $((a_n)_{n\in\mathbb{N}}, (\lambda(\mathscr{L}_n^{(\alpha)}))_{n\in\mathbb{N}_0})$ satisfies the conditions of a renewal pair.

□

Let us now briefly discuss the α-sum-level sets in a more dynamical way. It would appear, on first sight, that the sequence of sets $(\mathscr{L}_n^{(\alpha)})_{n\in\mathbb{N}}$ is not in the least dynamical in character. For instance, consider the set $\liminf \mathscr{L}_n^{(\alpha)} := \bigcup_{m\geq 1}\bigcap_{n\geq m}\mathscr{L}_n^{(\alpha)}$, which contains all those $x \in \mathscr{U}$ that lie in all but finitely many of the sets $\mathscr{L}_n^{(\alpha)}$. If $x \in \liminf \mathscr{L}_n^{(\alpha)}$, there must exist some $n_0 \in \mathbb{N}$ such that $x = [\ell_1, \dots, \ell_m, 1, 1, 1, \dots]_\alpha$ where $\sum_{i=1}^m \ell_i = n_0$. Otherwise, if infinitely often there appeared an entry not equal to 1 in the α-Lüroth expansion of x we would not be able to sum the entries to $n_0 + i$ for all $i \in \mathbb{N}$. Therefore, the set $\liminf \mathscr{L}_n^{(\alpha)}$ consists of all those $x \in \mathscr{U}$ whose α-Lüroth expansion terminates in infinitely many 1s. This is evidently a countable set. Now, consider the set $\limsup \mathscr{L}_n^{(\alpha)} := \bigcap_{m\geq 1}\bigcup_{n\geq m}\mathscr{L}_n^{(\alpha)}$. This set consists of all those $x \in \mathscr{U}$ that lie in infinitely many of the sets $\mathscr{L}_n^{(\alpha)}$. In this case, it is clear that all that is required is that the elements of $\limsup \mathscr{L}_n^{(\alpha)}$ have infinite α-Lüroth expansions. In other words, this set contains every α-irrational number. This is most definitely not a countable set. However, as the following lemma makes clear, the sequence $(\mathscr{L}_n^{(\alpha)})_{n\in\mathbb{N}}$ can be described in a surprisingly concise dynamical way.

Lemma 4.2.3.

$$\mathscr{L}_n^{(\alpha)} = F_\alpha^{-(n-1)}\left(\mathscr{L}_1^{(\alpha)}\right).$$

Proof. Recall that the inverse branches of F_α act on $x = [\ell_1, \ell_2, \dots]_\alpha$ in the following way:

$$F_{\alpha,0}(x) = [\ell_1 + 1, \ell_2, \dots]_\alpha \text{ and } F_{\alpha,1}(x) = [1, \ell_1, \ell_2, \dots]_\alpha.$$

From this, it is easy to directly calculate that $\mathscr{L}_2^{(\alpha)} = F_\alpha^{-1}(\mathscr{L}_1^{(\alpha)})$. Now, suppose that we have $\mathscr{L}_{n-1}^{(\alpha)} = F_\alpha^{-(n-2)}(\mathscr{L}_1^{(\alpha)})$ for some $n \in \mathbb{N}$. For a proof by induction, it suffices to show that

$$\mathscr{L}_n^{(\alpha)} = F_\alpha^{-(n-1)}\left(\mathscr{L}_1^{(\alpha)}\right) = F_\alpha^{-1}\left(\mathscr{L}_{n-1}^{(\alpha)}\right).$$

To that end, first suppose that $x \in \mathscr{L}_{n-1}^{(\alpha)}$. Then $x = [\ell_1, \ell_2, \ell_3, \dots]_\alpha$, where $\sum_{i=1}^k \ell_i = n-1$. Since $F_{\alpha,0}(x) = [\ell_1+1, \ell_2, \dots]_\alpha$ and $F_{\alpha,1}(x) = [1, \ell_1, \ell_2, \dots]_\alpha$, we see that $F_\alpha^{-1}(\mathscr{L}_{n-1}^{(\alpha)}) \subset \mathscr{L}_n^{(\alpha)}$.

Conversely, recalling from the proof of Proposition 2.2.9 that there are 2^{n-1} ways to sum a finite sequence of positive integers $(i_1, \dots, i_k)$ to n, we have that there are 2^{n-1} α-Lüroth cylinder sets contained in the set $\mathscr{L}_n^{(\alpha)}$. These can be split into the family of 2^{n-2} sets given by

$$\left\{C_\alpha(1, \ell_1, \dots, \ell_k) : \sum_{i=1}^k \ell_i = n-1\right\} = \left\{F_{\alpha,1}(C_\alpha(\ell_1, \dots, \ell_k)) : \sum_{i=1}^k \ell_i = n-1\right\}$$

and the family of 2^{n-2} sets given by

$$\left\{C_\alpha(\ell_1, \dots, \ell_k) : \ell_1 > 1 \text{ and } \sum_{i=1}^k \ell_i = n\right\} = \left\{F_{\alpha,0}(C_\alpha(\ell_1 - 1, \dots, \ell_k)) : \ell_1 > 1 \text{ and } \sum_{i=1}^k \ell_i = n\right\}.$$

Thus, we have the opposite inclusion, $\mathscr{L}_n^{(\alpha)} \subset F_\alpha^{-1}(\mathscr{L}_{n-1}^{(\alpha)})$, and the proof is finished. □

We can use the above lemma to give an alternative proof of the first part of Theorem 4.2.2, using some infinite ergodic theory. The result we need is Lin's criterion for exactness, which was stated at the end of Chapter 3. This second proof can be deduced from the following general fact.

Lemma 4.2.4. *For all $C \in \mathscr{B}$ with $\nu_\alpha(C) < \infty$, we have that*

$$\lim_{n\to\infty} \lambda\left(F_\alpha^{-n}(C)\right) = 0.$$

Proof. Let C be as in the statement of the lemma and let $A \in \mathscr{B}$ be such that $0 < \nu_\alpha(A) < \infty$. Then,

$$\begin{aligned}
\lambda\left(F_\alpha^{-n}(C)\right) &= \int_{F_\alpha^{-n}(C)} \frac{1}{\phi_\alpha}\, d\nu_\alpha \\
&= \int \mathbb{1}_{F_\alpha^{-n}(C)} \cdot \left(\frac{1}{\phi_\alpha} - \frac{\mathbb{1}_A}{\nu_\alpha(A)} + \frac{\mathbb{1}_A}{\nu_\alpha(A)}\right) d\nu_\alpha \\
&= \int \mathbb{1}_{F_\alpha^{-n}(C)} \cdot \left(\frac{1}{\phi_\alpha} - \frac{\mathbb{1}_A}{\nu_\alpha(A)}\right) d\nu_\alpha + \frac{1}{\nu_\alpha(A)} \int \mathbb{1}_{F_\alpha^{-n}(C)} \cdot \mathbb{1}_A\, d\nu_\alpha \\
&\leq \int \left|\mathscr{F}_\alpha^n\left(\frac{1}{\phi_\alpha} - \frac{\mathbb{1}_A}{\nu_\alpha(A)}\right)\right| d\nu_\alpha + \frac{\nu_\alpha(F_\alpha^{-n}(C) \cap A)}{\nu_\alpha(A)} \\
&\leq \int \left|\mathscr{F}_\alpha^n\left(\frac{1}{\phi_\alpha} - \frac{\mathbb{1}_A}{\nu_\alpha(A)}\right)\right| d\nu_\alpha + \frac{\nu_\alpha(C)}{\nu_\alpha(A)}.
\end{aligned}$$

Now, notice that

$$\int \frac{1}{\phi_\alpha} - \frac{\mathbb{1}_A}{\nu_\alpha(A)}\, d\nu_\alpha = \int \frac{1}{\phi_\alpha}\, d\nu_\alpha - \frac{1}{\nu_\alpha(A)} \int_A d\nu_\alpha = \int_0^1 d\lambda - 1 = 0.$$

So, we can apply Lin's criterion, which yields that

$$\lim_{n\to\infty} \lambda\left(F_\alpha^{-n}(C)\right) \leq \lim_{n\to\infty} \left(\int \left|\mathscr{F}_\alpha^n\left(\frac{1}{\phi_\alpha} - \frac{\mathbb{1}_A}{\nu_\alpha(A)}\right)\right| d\nu_\alpha\right) + \frac{\nu_\alpha(C)}{\nu_\alpha(A)} = \frac{\nu_\alpha(C)}{\nu_\alpha(A)}.$$

Then, on choosing A to have arbitrarily large ν_α-measure, the proof is finished.

□

Proposition 4.2.5. *Let F_α be of infinite type. Then*

$$\lim_{n\to\infty} \lambda\left(\mathscr{L}_n^{(\alpha)}\right) = 0.$$

Proof. This follows directly from Lemma 4.2.3 and Lemma 4.2.4, by setting $C = \mathscr{L}_1^{(\alpha)}$.

□

Finally, let us now discuss the outcomes of Theorem 4.2.2 with respect to the particular examples of the alternating Lüroth map, L_{α_H}, and the classical Lüroth map, $L_{\widetilde{\alpha_H}}$. The reader might like to see that the Lebesgue measures of the first members of the sequence $\left(\mathscr{L}_n^{(\alpha_H)}\right)$ are as follows:

$$\lambda(\mathscr{L}_0^{(\alpha_H)}) = 1,\ \lambda(\mathscr{L}_1^{(\alpha_H)}) = \tfrac{1}{2},\ \lambda(\mathscr{L}_2^{(\alpha_H)}) = \tfrac{5}{12},\ \lambda(\mathscr{L}_3^{(\alpha_H)}) = \tfrac{3}{8},\ \lambda(\mathscr{L}_4^{(\alpha_H)}) = \tfrac{251}{720}.$$

Since the Lebesgue measure of the sum-level set $\mathscr{L}_n^{(\widetilde{\alpha_H})}$ associated with the map $L_{\widetilde{\alpha_H}}$ coincides with the Lebesgue measure of the sum-level set $\mathscr{L}_n^{(\alpha_H)}$, Theorem 4.2.2 gives the following corollaries.

Corollary 4.2.6. $\lim\limits_{n\to\infty} \lambda\left(\mathscr{L}_n^{(\alpha_H)}\right) = \lim\limits_{n\to\infty} \lambda\left(\mathscr{L}_n^{(\widetilde{\alpha_H})}\right) = 0.$

Proof. It is immediately clear that both of the partitions α_H and $\widetilde{\alpha_H}$ are of infinite type. The result then follows directly from Theorem 4.2.2 (1). $\square$

Corollary 4.2.7. *For the classical and for the alternating Lüroth map the following hold, for n tending to infinity.*

1. $\displaystyle\sum_{k=1}^{n} \lambda\left(\mathscr{L}_n^{(\widetilde{\alpha_H})}\right) = \sum_{k=1}^{n} \lambda\left(\mathscr{L}_n^{(\alpha_H)}\right) \sim n\left(\sum_{k=1}^{n}\frac{1}{k}\right)^{-1} \sim \frac{n}{\log n};$

2. $\displaystyle\lambda\left(\mathscr{L}_n^{(\widetilde{\alpha_H})}\right) = \lambda\left(\mathscr{L}_n^{(\alpha_H)}\right) \sim \left(\sum_{k=1}^{n}\frac{1}{k}\right)^{-1} \sim \frac{1}{\log n}.$

Chapter 5

Lyapunov spectra for F_α and L_α

Calculating the Lyapunov spectrum of a given dynamical system is a species of multifractal analysis. This type of analysis is by now a well-established area of mathematics. It has its origins at the junction of pure mathematics and statistical physics, and can be considered an offshoot of thermodynamic formalism. We aim in this chapter to describe the Lyapunov spectra of the maps L_α and F_α. These systems can be described by an infinite iterated function system. For such systems, which we will soon describe in some detail, there are powerful results available to us. First, we will outline the classical multifractal analysis situation, where it is assumed that all alphabets are finite. There are several good references available for this, for instance Falconer [23] or Pesin [65]. Then, we will describe a very general result due to Jaerisch and Kesseböhmer which we will go on to apply in the third section to obtain our main results. Finally, we present a section consisting of examples of various of the behaviours that can occur for the α-Lüroth and α-Farey systems.

5.1 Introduction to multifractal analysis

Let us first introduce the multifractal formalism in as simple a setting as possible, that of a finite *iterated function system* consisting of two linear contractions on the unit interval. By an iterated function system, all that is meant is a collection of (possibly countably many) maps on a subset of some Euclidean space $\mathbb{R}^d$, $d \geq 1$. So, suppose that $\Phi := \{\phi_1, \phi_2 : \mathscr{U} \to \mathscr{U}\}$, where ϕ_1 and ϕ_2 are linear contracting similarities with *contraction ratios* r_1 and r_2, respectively. To avoid any trivial cases later on, we assume that $r_1 \neq r_2$. At this point, we assume that the reader recalls from Section 1.2 the basics of symbolic dynamics described there, including the definitions of the symbolic space $I^{\mathbb{N}}$ and the shift map $\sigma : I^{\mathbb{N}} \to I^{\mathbb{N}}$. Here, our symbolic space is $\{1,2\}^{\mathbb{N}}$. For each $\omega \in \{1,2\}^{\mathbb{N}}$, write

$$\phi_{\omega|_n} := \phi_{\omega_1} \circ \cdots \circ \phi_{\omega_n} : \mathscr{U} \to \mathscr{U}.$$

One immediately verifies (by applying Cantor's Intersection Theorem, for instance) that the intersection $\bigcap_{n\in\mathbb{N}} \phi_{\omega|_n}(\mathscr{U})$ is a singleton. This gives rise to a canonical *coding map* $\pi_\Phi : \{1,2\}^{\mathbb{N}} \to \mathscr{U}$. The image of this map, the set $\Lambda_\Phi := \pi_\Phi(\{1,2\}^{\mathbb{N}})$, is said to be the *limit set* of the iterated function system Φ. One way of thinking of the limit set of Φ in this case is that the points from the limit set are in one-to-one correspondence with the points of the attractor of Φ, that is, the unique non-empty compact set F that satisfies $\phi_1(F) \cup \phi_2(F) = F$. In the case we are considering here, this attractor is a Cantor set. The coding map corresponds to zooming into a point in F through the basic intervals containing that point. The final basic ingredient is the *geometric potential function* $\phi : \{1,2\}^{\mathbb{N}} \to \mathbb{R}$, which is defined by setting

$$\phi((\omega_1, \omega_2, \omega_3, \ldots)) := \log \left|\phi'_{\omega_1}(\omega_2, \omega_3, \ldots)\right| = \log(r_{\omega_1}).$$

Note that "potential" is simply another word for a continuous function.

Given a Hölder continuous function $\psi : \{1,2\}^{\mathbb{N}} \to \mathbb{R}$, we say that the multifractal analysis of the system Φ with respect to ψ is the analysis of the *level sets*

$$\mathscr{F}_s := \pi_\Phi \left\{ \omega \in \{1,2\}^{\mathbb{N}} : \lim_{k\to\infty} \frac{S_k\psi(\omega)}{S_k\phi(\omega)} = s \right\}$$

in terms of the *Hausdorff dimension function* $f(s) := \dim_H(\mathscr{F}_s)$. Here, recall that $S_k f$ denotes the Birkhoff sum of f with respect to the shift map which is given by $S_k f(\omega) := \sum_{n=0}^{k-1} f(\sigma^n(\omega))$.

It will turn out that the multifractal spectrum as defined above is related to the *Legendre transformation* of a particular function. The Legendre transformation t^* of a function $t : \mathbb{R} \to \mathbb{R}$ is defined to be

$$t^*(s) := \inf\{vs + t(v) : v \in \mathbb{R}\}.$$

Note that in case the function t is convex and differentiable everywhere, the Legendre transformation of t has a nice geometric description in terms of tangents to the graph of t. If t is a convex function, then there exists a range of s, say $s \in (s_-, s_+)$, for which the graph of t has a tangent with slope $-s$. In this case, $t^*(s)$ is the intersection of this tangent with the vertical axis. This is illustrated in Figure 5.1, below.

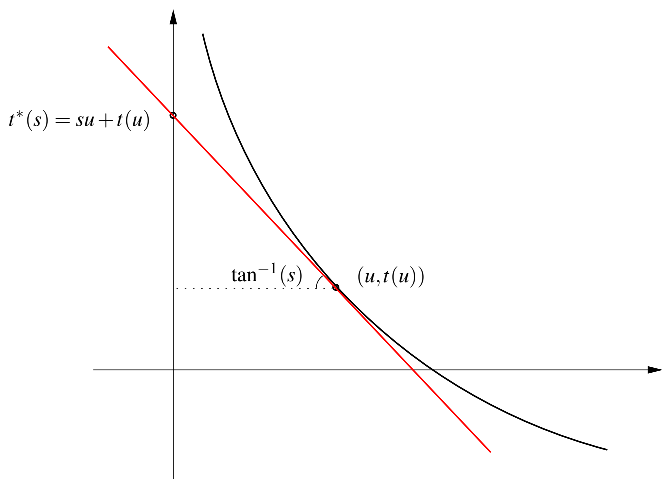


Figure 5.1: The Legendre transform $t^*(s)$ of the convex function t is the intersection of the tangent to t of slope $-s$ with the vertical axis. Here, $s = -t'(u)$.

Of course, the Legendre transformation of t is still well-defined when t is not differentiable everywhere. Consider a decreasing function t which has a point v_0 for which the right derivative t^+ of t at v_0 is equal to $-s_1$ and the left derivative t_- of t at v_0 is equal to $-s_2$. Then for all $s \in [s_1, s_2]$, we have that $t^*(s)$ can be explicitly determined by $t^*(s) = sv_0 + t(v_0)$. We will see an example like this in Section 5.4.

Returning now to the analysis at hand, notice that we have two canonical measures associated with the system Φ. These are the measure of maximal entropy μ_0, which is sometimes referred to as the arithmetic measure, and the δ-dimensional Hausdorff measure μ_δ, where δ denotes the Hausdorff dimension of F. That is, for each cylinder set $[\omega_1 \dots \omega_n]$, we have

$$\mu_0([\omega_1 \dots \omega_n]) = 2^{-n} \text{ and } \mu_\delta([\omega_1 \dots \omega_n]) = (r_{\omega_1} \cdots r_{\omega_n})^\delta.$$

Since we assumed that $r_1 \neq r_2$, these measures are not equivalent. Notice that

$$\mu_0([\omega_1 \dots \omega_n]) = e^{S_n \psi(\omega)} \text{ and } \mu_\delta([\omega_1 \dots \omega_n]) = e^{\delta S_n \phi(\omega)}.$$

It can be shown that these two measures in fact correspond to extreme points on a whole spectrum of measures $\{\mu_\beta : \beta \in I\}$ for some interval $I \subset \mathbb{R}$, with the property that

$$\mu_\beta([\omega_1 \dots \omega_n]) \asymp \exp(t(\beta) S_n \phi(\omega) + \beta S_n \psi(\omega)) = (r_{\omega_1} \cdots r_{\omega_n})^{\delta t(\beta)} (2^{-n})^\beta,$$

where the function t is defined implicitly by the *pressure equation* $P(t(\beta)\phi + \beta\psi) = 0$. Here $P(f)$ denotes the *topological pressure* of the system Φ with respect to the potential f, which is defined, in this situation, by

$$P(f) := \lim_{n \to \infty} \frac{1}{n} \log \sum_{\substack{(\omega_1, \dots, \omega_n) \\ \in \{1,2\}^n}} \exp\left(\sup_{\tau \in [\omega_1 \dots \omega_n]} S_n f(\tau) \right).$$

Note that this limit always exists. The pressure function here is convex and real-analytic; moreover, it always has a unique zero. Further, the function t defined by the pressure equation is also strictly convex. The measures μ_β are said to be *Gibbs measures*.

Now, for any given $\beta \in I$, the next step is to show that $\mu_\beta(\mathscr{F}_s) = 1$, where $s := -t'(\beta)$. From this it follows that for each $\omega \in \{1,2\}^{\mathbb{N}}$, we have

$$r^{t^*(s)+\varepsilon} \ll \mu_\beta(B(\omega, r)) \ll r^{t^*(s)-\varepsilon},$$

and so the mass distribution principle (see Proposition 2.2 in [23]), implies that

$$f(s) := \dim_H(\mathscr{F}_s) = t^*(s).$$

We remark that there are three major steps involved in rigorously proving the result outlined above. The first two are the existence of the function t and the existence of the family of Gibbs measures. The third is that the Gibbs measures are supported on the level sets.

5.2 General multifractal results of Jaerisch and Kesseböhmer

We come now to the general multifractal results of Jaerisch and Kesseböhmer, proved in [38]. In this paper, multifractal spectra are studied in the context of infinite conformal iterated function systems (cIFS). We have already seen many examples of such objects in this thesis but without explicitly describing them as such. An infinite iterated function system (IFS) is given by an at most countable family of injective contractions $\Phi := \{\phi_i : i \in I \subseteq \mathbb{N}\}$ on a compact connected subset of a Euclidean space $(\mathbb{R}^d, ||\cdot||)$, for $d \geq 1$. As the name suggests, a cIFS is an IFS consisting of conformal maps. The cIFSs with which we shall mainly be concerned in this chapter are the α-Lüroth systems, described as a family of linear maps $\Phi_\alpha := \{L_{\alpha,n} : \mathscr{U} \to A_n : n \in \mathbb{N}\}$ rather than as a piecewise linear map on $\mathscr{U}$. That is, the cIFS Φ_α is the family of inverse branches of L_α, which were defined in Definition 2.1.8. The precise definition of a cIFS is as follows.

Definition 5.2.1. An iterated function system $\Phi := \{\phi_i : i \in I\}$ is said to be *conformal* if the following conditions are satisfied.

(a) The *phase space* X is a compact connected subset of $\mathbb{R}^d$, $d \geq 1$, such that X is equal to the closure of its interior.

(b) *The open set condition.* For all $i, j \in I$ with $i \neq j$,

$$\phi_i(\mathrm{Int}(X)) \cap \phi_j(\mathrm{Int}(X)) = \emptyset.$$

(c) There exists an open connected set $W \supset X$ such that for every $i \in I$ the map ϕ_i extends to a C^1 conformal diffeomorphism of W into W.

(d) (*Cone property*) There exist $\gamma, l > 0$, $\gamma < \pi/2$, such that for every $x \in X$ there exists an open cone $\mathrm{Con}(x, \gamma, l) \subset \mathrm{Int}(X)$ with vertex x, central angle of measure γ and altitude l.

(e) There are two constants $L_\Phi \geq 1$ and $\alpha_\Phi > 0$ such that for every $i \in I$ and every pair of points $x, y \in X$,

$$||\phi_i'(y)| - |\phi_i'(x)|| \leq \frac{L_\Phi}{||(\phi_i')^{-1}||_X} ||y - x||^{\alpha_\Phi}.$$

Here, $||\phi_i'||_X := \sup_{x\in X} |\phi_i'(x)|$ with $|\phi_i'(x)|$ denoting the operator norm of the derivative.

Remark 5.2.2. For our situation of the IFS coming from the α-Lüroth system, Φ_α, these conditions are all clearly satisfied. In particular, $X = \mathscr{U}$, so (a) is obvious. The cone property does not apply in the one-dimensional case. The open set condition does not trouble us either. Since each map from Φ_α is linear, (c) is satisfied for any open interval containing $\mathscr{U}$. Finally, condition (e) is a sort of bounded distortion property, which is satisfied immediately for a linear map, since the derivative is simply a constant.

Just as in the finite case described previously, for each $\omega \in I^{\mathbb{N}}$ and for each $n \in \mathbb{N}$, let

$$\phi_{\omega|_n} := \phi_{\omega_1} \circ \cdots \circ \phi_{\omega_n} : X \to X.$$

Once again, it is a fact that the intersection $\bigcap_{n\in\mathbb{N}} \phi_{\omega|_n}(X)$ is a singleton, from which we again obtain the coding map $\pi_\Phi : I^{\mathbb{N}} \to X$ and the limit set $\pi_\Phi(I^{\mathbb{N}})$ of the iterated function system Φ. Given a Hölder continuous function $\psi : I^{\mathbb{N}} \to \mathbb{R}$, let us agree that the multifractal analysis of the system Φ with respect to the function ψ is the analysis of the level sets

$$\mathscr{F}_s := \pi_\Phi \left\{ \omega \in I^{\mathbb{N}} : \lim_{k\to\infty} \frac{S_k \psi(\omega)}{\log ||\phi'_{\omega|_k}||_X} = s \right\} \tag{5.1}$$

in terms of the Hausdorff dimension function $f(s) := \dim_H(\mathscr{F}_s)$.

Before going any further, we must return to the pressure function. In the setting of infinite iterated function systems, the pressure function is defined as follows.

Definition 5.2.3. For a Hölder continuous function $f : I^{\mathbb{N}} \to \mathbb{R}$, the *n-th partition function* $Z_n(f)$ is given by

$$Z_n(f) := \sum_{\omega\in I^{\mathbb{N}}} \exp \sup_{\tau\in[\omega]} (S_n f(\tau)).$$

It can be shown that this function is submultiplicative, so the following definition makes sense for each Hölder continuous function f. The *topological pressure* $\mathscr{P}(f)$ is defined to be

$$\mathscr{P}(f) := \lim_{n\to\infty} \frac{1}{n} \log Z_n(f) = \inf_{n\in\mathbb{N}} \frac{1}{n} \log Z_n(f).$$

At this point, exactly as in the finite case, let us define the geometric potential function $\phi : I^{\mathbb{N}} \to (-\infty, 0]$ by setting $\phi(\omega) := \log |\phi'_{\omega_1}(\pi(\sigma(\omega)))|$. It is well known that in the case of finite conformal iterated function systems (in particular for the linear case described in the preceding section), that the function f can be related to the Legendre transformation of the *free energy function* $t : \mathbb{R} \to \mathbb{R}$ which can be defined implicitly from the pressure equation

$$\mathscr{P}(t(\beta)\phi + \beta\psi) = 0. \tag{5.2}$$

Specifically, there exists a closed bounded interval $J \subset \mathbb{R}$ such that for all $s \in J$ we have

$$f(s) = t^*(s) := \inf_\beta \{t(\beta) + \beta s\},$$

and for $s \notin J$ we have that $\mathscr{F}_s = \emptyset$. (This can be found as Theorem 21.1 in [65], for example.) The main difficulty that arises when considering the case of infinite iterated function systems is that the pressure function $\mathscr{P}(f)$ may behave irregularly[1], so that we cannot find a solution to the equation in (5.2). For the case that there does exist a unique solution to this equation, the multifractal analysis has been discussed by Mauldin and Urbański in [58] and Roy and Urbański in [67]. For the very general results obtained by Jaerisch and Kesseböhmer, the first step is to generalise the definition of the free energy function to account for the fact that a unique solution to (5.2) may not always be found. Their definition is the following one.

[1]A complete description of the behaviour of the pressure function for infinite cIFSs can be found in [58].

Definition 5.2.4. Let Φ be a cIFS and $\psi : I^{\mathbb{N}} \to \mathbb{R}$ be a potential function. Then the *free energy function* $t : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ for the pair (Φ, ψ) is given by

$$t(\beta) := \inf\{t \in \mathbb{R} : \mathscr{P}(t\phi + \beta\psi) \leq 0\}.$$

We are now almost in a position to state the main result from [38]. The one remaining ingredient is to set

$$s_- := \inf\{-t^-(x) : x \in \mathrm{Int}(\mathrm{dom}(t))\}$$

and

$$s_+ := \sup\{-t^+(x) : x \in \mathrm{Int}(\mathrm{dom}(t))\},$$

where t^- denotes the derivative of t from the left, t^+ denotes the derivative of t from the right and $\mathrm{Int}(\mathrm{dom}(t))$ denotes the interior of the the effective domain $\mathrm{dom}(t) := \{x \in \mathbb{R} : t(x) < +\infty\}$ of t.

Theorem 5.2.5. The general multifractal result of Jaerisch and Kesseböhmer. *For all $s \in \mathbb{R}$ we have that $f(s) \leq \max\{t^*(s), 0\}$ and for $s \in (s_-, s_+)$ we have that $f(s) = t^*(s)$.*

The basic idea behind the proof is to exhaust the infinite system (Φ, ψ) with finite subsystems. They introduce the notion of regular convergence for families of cIFS not necessarily sharing the same index set, which guarantees the convergence of the multifractal spectra on the interior of their domain. In this way, results from finite systems can be carried over to infinite systems and the multifractal dimension spectrum can be established without such restrictive conditions as are usual for infinite alphabets.

We will now translate Theorem 5.2.5 into the situation of the α-Lüroth system. Recall that we are interested in the cIFS Φ_α given by $\Phi_\alpha := \{\phi_n = L_{\alpha,n} : x \mapsto t_n - a_n x \mid n \in \mathbb{N}\}$. The symbolic space is now $\mathbb{N}^{\mathbb{N}}$. Each ω in $\mathbb{N}^{\mathbb{N}}$ corresponds to some $x \in \mathscr{U}$, in that if $\omega = \ell_1\ell_2\ell_3\ldots \in \mathbb{N}^{\mathbb{N}}$, then

$$\pi_{\Phi_\alpha}(\ell_1\ell_2\ell_3\ldots) = [\ell_1, \ell_2, \ell_3, \ldots]_\alpha =: x \in \mathscr{U}.$$

Note that for each $n \in \mathbb{N}$ we have that $(L_{\alpha,n})'(x) = -a_n$ for all $x \in \mathscr{U}$. One immediately verifies that the geometric potential ϕ introduced above is in this case given by $\phi(\ell_1\ell_2\ell_3\ldots) = \log a_{\ell_1}$. Consider now the term $||\phi'_{\omega|_k}||_X$ from the definition of the level sets given in (5.1). By the chain rule, we have that

$$\phi'_{\omega|_k} = \phi'_{\omega_1}(\phi_{\omega_2} \circ \cdots \circ \phi_{\omega_k})\phi'_{\omega_2}(\phi_{\omega_3} \circ \cdots \circ \phi_{\omega_k}) \cdots \phi'_{\omega_{k-1}}(\phi_{\omega_k})\phi'_{\omega_k}.$$

It therefore follows that we obtain

$$\log(||\phi'_{\omega|_k}||_X) = \sum_{n=1}^{k} \log(\ell_n) = S_k\phi(\omega).$$

So, the level sets with which we are concerned can be written in terms of the two potential functions ψ and ϕ in the following way:

$$\mathscr{F}_s := \pi_{\Phi_\alpha}\left\{\omega \in I^{\mathbb{N}} : \lim_{k\to\infty} \frac{S_k\psi(\omega)}{S_k\phi(\omega)} = s\right\}.$$

Let $(z_n)_{n\in\mathbb{N}}$ be a sequence of negative real numbers and let the potential ψ be given by $\psi(x) = z_n$ for $x \in A_n$. We can then rewrite the level sets once again as

$$\mathscr{F}_s = \pi_{\Phi_\alpha}\left\{\omega \in I^{\mathbb{N}} : \lim_{k\to\infty} \frac{\sum_{n=1}^k z_{\ell_n}}{\sum_{n=1}^k \log(a_{\ell_n})} = s\right\}.$$

It remains to translate the free energy function. We must describe $\mathscr{P}(f)$ where $f(x) := t\log(a_n) + \beta z_n$, for $x \in A_n$. We have

$$\begin{aligned}
\mathscr{P}(f) &:= \lim_{n\to\infty}\frac{1}{n}\log Z_n(f)\\
&= \lim_{n\to\infty}\frac{1}{n}\log\sum_{\ell_1\ldots\ell_n\in\mathbb{N}^n}\exp\left(\log(a_{\ell_1\ldots\ell_n})^t + \beta\sum_{i=1}^n z_{\ell_i}\right)\\
&= \lim_{n\to\infty}\log\left(\sum_{k=1}^\infty a_k^t\exp(\beta z_k)\right)^{n\cdot\frac{1}{n}} = \log\sum_{k=1}^\infty a_k^t\exp(\beta z_k).
\end{aligned}$$

The next theorem gathers all of this information together and inserts it into Theorem 5.2.5 above. This is the form of the general result from [38] that we will need in the next section.

Theorem 5.2.6. *Let $\alpha := \{A_n : n\in\mathbb{N}\}$ be a given countable partition of $\mathscr{U}$ and consider the two potential functions $\phi,\psi : \mathscr{U} \to \mathbb{R}$ given for $x \in A_n$, $n\in\mathbb{N}$, by $\phi(x) := \log a_n$ and $\psi(x) := z_n$, for some fixed sequence $(z_n)_{n\in\mathbb{N}}$ of negative real numbers. For all $s\in\mathbb{R}$ we then have that*

$$\dim_H\left\{x\in\mathscr{U} : \lim_{n\to\infty}\left(\sum_{k=0}^{n-1}\psi(L_\alpha^k(x))\Big/\sum_{k=0}^{n-1}\phi(L_\alpha^k(x))\right) = s\right\} \le \max\{0, t^*(s)\}.$$

Here, the function $t : \mathbb{R} \to \mathbb{R}\cup\{\infty\}$ is given by

$$t(v) := \inf\left\{\sum_{n=1}^\infty a_n^u \exp(vz_n) \le 1 : u\in\mathbb{R}\right\}$$

and t^* is the Legendre transform of t, that is,

$$t^*(s) := \inf_{v\in\mathbb{R}}\{t(v) + vs\}.$$

Furthermore, there exist $r_-, r_+ \in\mathbb{R}$ such that for $s\in(r_-, r_+)$, we have

$$\dim_H\left\{x\in\mathscr{U} : \lim_{n\to\infty}\left(\sum_{k=0}^{n-1}\psi(L_\alpha^k(x))\Big/\sum_{k=0}^{n-1}\phi(L_\alpha^k(x))\right) = s\right\} = t^*(s).$$

In fact, the boundary points r_- and r_+ are determined explicitly by

$$r_- := \inf\left\{-t^-(v) : v\in\operatorname{Int}(\operatorname{dom}(t))\right\} \quad \text{and} \quad r_+ := \sup\left\{-t^+(v) : v\in\operatorname{Int}(\operatorname{dom}(t))\right\}.$$

Remark 5.2.7. Note that for $s \in \mathbb{R}$ we have

$$\left\{ x \in \mathscr{U} : \lim_{n\to\infty} \left(\sum_{k=0}^{n-1} \psi(L_\alpha^k(x)) / \sum_{k=0}^{n-1} \phi(L_\alpha^k(x)) \right) = s \right\} \neq \emptyset$$

if and only if $\inf\{z_n/\log a_n : n \in \mathbb{N}\} \leq s \leq \sup\{z_n/\log a_n : n \in \mathbb{N}\}$. It therefore follows that

$$r_- \geq \inf\{z_n/\log a_n : n \in \mathbb{N}\} \text{ and } r_+ \leq \sup\{z_n/\log a_n : n \in \mathbb{N}\}.$$

5.3 Main Theorems

In this section, we will give a description of the Lyapunov spectra arising from the α-Farey map and the α-Lüroth map. For this we use the general method obtained in [38] which was outlined in the previous section. Before stating and proving the main results, we first give a helpful lemma.

Lemma 5.3.1.

1. *Let $(a_n)_{n\in\mathbb{N}}$ be an eventually decreasing sequence of positive real numbers and for each $s \in \mathbb{R}$ let*
$$f(s) := \sum_{i=1}^{\infty} (a_i)^s \text{ and } f_n(s) := \sum_{i=n}^{\infty} (a_i)^s.$$
If $f(s_0)$ is finite for some $s_0 \in \mathbb{R}$ and if $f_n(s_0)$ is a slowly-varying function of n, then we have that s_0 is the abscissa of convergence of $f(s)$.

   In particular, if α is a partition of $\mathscr{U}$ that is expansive of exponent 0, so $t_n = \psi(n)$ for some slowly-varying function $\psi : \mathbb{N} \to \mathbb{R}^+$, we have that $t_\infty := \inf\{r > 0 : \sum_{k=1}^{\infty} a_n^r < \infty\} = 1$.

2. *Let α be a partition such that $\lim_{n\to\infty} t_n/t_{n+1} = \rho \geq 1$ and such that α is either expanding, or expansive of exponent $\theta > 0$ and eventually decreasing. Then there exists a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$, with $\lim_{n\to\infty} \varepsilon_n = 0$, such that for all $n \in \mathbb{N}$ and $x \in \bigcup_{k>n} A_k$ we have that*
$$\left| \frac{1}{n} \sum_{k=0}^{n-1} \log \left| F_\alpha'(F_\alpha^k(x)) \right| - \log \rho \right| < \varepsilon_n.$$

Proof. For the proof of the first part, let $(a_n)_{n\in\mathbb{N}}$ be as stated and for each $n \in \mathbb{N}$, define the set $C_n := \{k \in \mathbb{N} : e^{-n} \leq a_k < e^{-(n-1)}\}$. Further define $c_n := \#C_n$. Then for each $u > 0$ it follows that

$$\sum_{n=1}^{\infty} c_n e^{-nu} \leq \sum_{m=1}^{\infty} a_m^u. \tag{5.3}$$

Now let $\gamma_0 := \limsup_{n\to\infty} (\log c_n)/n$ and observe that (5.3) implies that $\gamma_0 \leq s_0$. Indeed, if on the contrary $\gamma_0 > s_0$, there would exist infinitely many $n \in \mathbb{N}$ such that $s_0 < (\log c_n)/n$, or in other words, $e^{ns_0} < c_n$. This would mean that the left-hand side of (5.3) for $u = s_0$ is infinite but, by assumption, the right-hand side is finite. This contradiction proves the claim.

We now aim to show that in fact $\gamma_0 = s_0$. To that end, suppose by way of contradiction that $\gamma_0 < s_0$ and let γ be arbitrary such that $\gamma_0 < \gamma < s_0$. Further, choose γ' such that $\gamma_0 < \gamma' < \gamma$. Then, by definition of γ_0, if $n \in \mathbb{N}$ is large enough we have that $c_n < e^{\gamma' n}$. Now, for each $n \in \mathbb{N}$, define $k_n := \sum_{i=1}^{n} c_i$. It follows, where $\kappa > 0$ is some constant and n is sufficiently large, that we have

$$k_n = \sum_{i=1}^{n} c_i \leq \kappa + \sum_{i=1}^{n} (e^{\gamma'})^i = \kappa + \frac{e^{\gamma'}}{e^{\gamma'} - 1}(e^{\gamma' n} - 1) < e^{\gamma n}.$$

Consequently,

$$n > \frac{\log(k_n)}{\gamma}$$

and, since $\gamma - s_0 < 0$, we have that

$$(\gamma - s_0)n < \log(k_n^{(\gamma - s_0)/\gamma}).$$

Recalling that the sequence $(a_n)_{n\in\mathbb{N}}$ is eventually decreasing, we infer that for large enough n the first k_n terms of (a_n) all lie in the union of the sets C_i for $1 \leq i \leq n$. Thus,

$$\begin{aligned} f_{k_n}(s_0) &= \sum_{m=k_n}^{\infty} a_m^{s_0} \leq \sum_{i=n}^{\infty} c_i e^{-s_0(i-1)} \\ &\leq e^{s_0} \sum_{i=n}^{\infty} \left(e^{(\gamma' - s_0)}\right)^i \leq e^{(\gamma - s_0)n} \leq k_n^{1 - s_0/\gamma}. \end{aligned}$$

Given that $\gamma < s_0$, and so $1 - s_0/\gamma < 0$, in light of Proposition 2.4.2 we have a contradiction to the fact that $f_n(s_0)$ is slowly varying. Therefore, $\gamma_0 := \limsup_{n\to\infty}(\log c_n)/n = s_0$.

To complete the proof of part 1, let $s < s_0$. Then there exist infinitely many $n \in \mathbb{N}$ such that $(\log c_n)/n \geq s$, from which we infer that $c_n \geq e^{ns}$ for infinitely many $n \in \mathbb{N}$. Therefore the sum in the left-hand side of the inequality in (5.3) is infinite, and consequently so is the one on the right. Hence, the sum $f(s)$ diverges for all $s < s_0$ and the abscissa of convergence of $f(s)$ is indeed equal to s_0.

For the proof of the second part, first recall that $F'_\alpha(x)$ is equal to the slope of the map F_α at the point x, so if $x \in A_n$, then $F'_\alpha(x) = \frac{a_{n-1}}{a_n}$. Thus, for fixed $n \in \mathbb{N}$ and $x \in A_{n+j}$ for some $j \in \mathbb{N}$, we have that

$$\frac{1}{n}\sum_{k=0}^{n-1} \log|F'_\alpha(F_\alpha^k(x))| = \frac{1}{n}\left(\log\left(\frac{a_{n+j-1}}{a_{n+j}} \cdot \frac{a_{n+j-2}}{a_{n+j-1}} \cdots \frac{a_j}{a_{j+1}}\right)\right) = \frac{1}{n}(\log a_j - \log a_{n+j}).$$

Therefore, for $x \in \bigcup_{i>n} A_i$, we obtain the following inequality:

$$\begin{aligned} \left| \frac{\sum_{k=0}^{n-1} \log\left|F'_\alpha(F_\alpha^k(x))\right|}{n} - \log\rho \right| &\leq \sup_{i\in\mathbb{N}} \left| \frac{\log a_i - \log a_{n+i}}{n} - \log\rho \right| \\ &= \sup_{i\in\mathbb{N}} \left| \frac{\log a_i}{i} \cdot \frac{i}{n} - \frac{\log a_{n+i}}{n+i} \cdot \frac{n+i}{n} - \log\rho \right| \\ &= \sup_{i\in\mathbb{N}} \left| \frac{i}{n}\left(\frac{\log a_i}{i} - \frac{\log a_{n+i}}{n+i}\right) - \frac{\log a_{n+i}}{n+i} - \log\rho \right| =: \varepsilon_n. \end{aligned}$$

Since by Lemma 2.4.8 (1) we have $\lim_{k\to\infty}(\log a_k)/k = -\log\rho$, it follows that $\lim_{k\to\infty}\varepsilon_k = 0$. □

In order to state the main results of this chapter, recall that the Lyapunov exponent of a differentiable map $S : \mathscr{U} \to \mathscr{U}$ at a point $x \in \mathscr{U}$ is defined, provided the limit exists, by

$$\Lambda(S,x) := \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \log|S'(S^k(x))|.$$

Our first main theorem gives a fractal-geometric description of the Lyapunov spectra associated with the map L_α. That is, we consider the Hausdorff dimension of the spectral sets $\{s \in \mathbb{R} : \{x \in \mathscr{U} : \Lambda(L_\alpha,x) = s\} \neq \emptyset\}$. This gives rise to the Hausdorff dimension function τ_α, which is given by

$$\tau_\alpha(s) := \dim_H(\{x \in \mathscr{U} : \Lambda(L_\alpha,x) = s\}).$$

In what follows, $p : \mathbb{R} \to \mathbb{R}\cup\{\infty\}$ denotes the *α-Lüroth pressure function*, which is defined by

$$p(u) := \log\sum_{n=1}^{\infty} a_n^u.$$

In addition, we say that L_α *exhibits no phase transition* if and only if the pressure function p is differentiable everywhere (that is, the right and left derivatives of p coincide everywhere, with the convention that $p'(u) = \infty$ if $p(u) = \infty$). For an interesting further discussion of the phenomenon of phase transition in the context of countable state Markov chains, we refer to Sarig [71].

Theorem 5.3.2. *For an arbitrary given partition α, the Hausdorff dimension function of the Lyapunov spectrum associated with L_α is given as follows. For $t_- := \inf\{-\log a_n : n \in \mathbb{N}\}$ we have that τ_α vanishes on $(-\infty, t_-)$, and for each $s \in (t_-,\infty)$ we have*

$$\tau_\alpha(s) = \inf_{u\in\mathbb{R}}\left(u + s^{-1}p(u)\right).$$

Moreover, $\tau_\alpha(s)$ tends to $t_\infty := \inf\{r > 0 : \sum_{k=1}^{\infty} a_n^r < \infty\} \leq 1$ for s tending to infinity. Concerning the possibility of phase transitions for L_α, the following hold:

- *If α is expanding, then L_α exhibits no phase transition and $t_\infty = 0$.*
- *If α is expansive of exponent $\theta > 0$ and eventually decreasing, then L_α exhibits no phase transition if and only if $\sum_{n=1}^{\infty} \psi(n)^{1/(1+\theta)}(\log n)/n$ diverges. Moreover, in this situation we have that $t_\infty = 1/(1+\theta)$.*
- *If α is expansive of exponent $\theta = 0$, then L_α exhibits no phase transition if and only if $\sum_{n=1}^{\infty} a_n \log(a_n)$ diverges. Moreover, in this situation we have that $t_\infty = 1$.*

Finally, for partitions which are either expanding or expansive of exponent $\theta > 0$ and eventually decreasing, t_∞ is also equal to the Hausdorff dimension of the Good-type set $G_\infty^{(\alpha)}$ associated to L_α, given by

$$G_\infty^{(\alpha)} := \{[\ell_1,\ell_2,\ldots]_\alpha : \lim_{n\to\infty}\ell_n = \infty\}.$$

Proof of Theorem 5.3.2. Directly from the definition of the Lyapunov exponent for the map L_α, provided that the limit exists, we obtain that

$$\Lambda(L_\alpha,x) := \lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\log\left|L'_\alpha\cdot L_\alpha^k(x)\right| = \lim_{n\to\infty} -\frac{1}{n}\sum_{k=1}^{n}\log(a_{\ell_k(x)}).$$

We aim to apply the general result by Jaerisch and Kesseböhmer, as stated above, to the special situation in which $z_n := -1$, for each $n\in\mathbb{N}$. In other words, the potential function ψ, instead of being a step function, is the constant function $\psi : x\mapsto -1$. Notice that in this case we have that

$$\begin{aligned}\{x\in\mathscr{U} : \Lambda(L_\alpha,x)=s\} &= \left\{x\in\mathscr{U} : \lim_{n\to\infty}\frac{\sum_{k=0}^{n-1}\phi\circ L_\alpha^k(x)}{\sum_{k=0}^{n-1}\psi\circ L_\alpha^k(x)}=s\right\}\\ &= \left\{x\in\mathscr{U} : \lim_{n\to\infty}\frac{\sum_{k=0}^{n-1}\psi\circ L_\alpha^k(x)}{\sum_{k=0}^{n-1}\phi\circ L_\alpha^k(x)}=\frac{1}{s}\right\}.\end{aligned}$$

We must now determine the free energy function t. In order to do this, consider the function $v : (t_\infty,\infty)\to\mathbb{R}$, which is given by $v(u) := \log\sum_{n=1}^\infty a_n^u$, where $t_\infty := \inf\{r>0 : \sum_{k=1}^\infty a_k^r<\infty\}$. (Note that this is the α-Lüroth pressure function on a restricted domain.) On the one hand, if $\lim_{t\to t_\infty} v(t)$ is infinite, then the function t appearing in the result of Jaerisch and Kesseböhmer is identically equal to the inverse v^{-1} of v. To see this, note that

$$\begin{aligned} t\circ v(u) &= \inf_{s\in\mathbb{R}}\left\{\sum_{n=1}^\infty a_n^s\exp(-v(u))\le 1\right\}\\ &= \inf_{s\in\mathbb{R}}\left\{\frac{\sum_{n=1}^\infty a_n^s}{\exp\left(\log\sum_{n=1}^\infty a_n^u\right)}\le 1\right\}=u.\end{aligned}$$

On the other hand, if $\lim_{t\to t_\infty} v(t)$ is finite, say equal to some real number c, then $t(s)=v^{-1}(s)$ for all $s\in(-\infty,c)$, whereas $t(s)=t_\infty$ for all $s\in[c,+\infty)$.

In order to determine the boundary points of the non-trivial part of the Lyapunov spectrum associated with the map L_α, in view of the general thermodynamical result stated above, we consider the asymptotic slopes r_- and r_+ of t. It is clear that in both of the above cases, we have $r_-=0$ and hence it follows that $t_+ := +\infty$. We also have that

$$r_+ = \lim_{v\to-\infty}\frac{-t(v)}{v} = \lim_{u\to\infty}\frac{-u}{v(u)}$$

and so,

$$t_- := \frac{1}{r+} = \lim_{u\to\infty}\frac{-v(u)}{u}.$$

For this, we let $a_{\max} := \max\{a_n : n \in \mathbb{N}\}$ and make the following calculation.

$$\begin{aligned}
\lim_{u\to\infty}\frac{-v(u)}{u} &= \lim_{u\to\infty}\frac{-\log\sum_{n=1}^{\infty}a_n^u}{u}\\
&= \lim_{u\to\infty}\frac{-\log\left(a_{\max}^u\sum_{n=1}^{\infty}\left(\frac{a_n}{a_{\max}}\right)^u\right)}{u}\\
&= -\log(a_{\max}) - \lim_{u\to\infty}\frac{-\log\sum_{n=1}^{\infty}\left(\frac{a_n}{a_{\max}}\right)^u}{u}\\
&= \inf\{-\log a_n : n\in\mathbb{N}\} - \lim_{u\to\infty}\frac{\log\left(1+\sum_{a_n\neq a_{\max}}\left(\frac{a_n}{a_{\max}}\right)^u\right)}{u}\\
&= \inf\{-\log a_n : n\in\mathbb{N}\}.
\end{aligned}$$

Here, the limit in the penultimate line of the calculation above is equal to 0, because each term $a_n/a_{\max}$ is strictly less than 1 and these terms are eventually decreasing in n. Thus, we obtain that $t_- = \inf\{-\log a_n : n \in \mathbb{N}\}$.

Therefore, for both the case that $v(t_\infty)$ is finite and the case that it is infinite, Theorem 5.2.6 shows that the Hausdorff dimension function associated with the Lyapunov spectrum of L_α vanishes for $s < t_-$ and is given, for $s \in (t_-, +\infty)$, by

$$\tau_\alpha(s) = t^*(1/s) = \inf_{v\in\mathbb{R}}\left(t(v)+s^{-1}v\right) = \inf_{u\in\mathbb{R}}\left(u+s^{-1}\log\sum_{n=1}^{\infty}a_n^u\right).$$

For the discussion of the phase transition phenomena for L_α, one immediately verifies that for the right derivative of the pressure function p of L_α, where the reader might like to recall that p is given by $p(u) := \log\sum_{n=1}^{\infty}a_n^u$, we have that

$$p^+(u) = \frac{\sum_{n=1}^{\infty}a_n^u\log a_n}{\sum_{n=1}^{\infty}a_n^u}.$$

Clearly, p is real-analytic on (t_∞,∞). Hence, we have that L_α exhibits no phase transition if and only if $\lim_{u\searrow t_\infty} -p^+(u) = +\infty$. First consider α expansive of exponent 0. In this case, we have proved in part 1 of Lemma 5.3.1 that $t_\infty = 1$ and so L_α exhibits no phase transition if and only if $-\sum_{n=1}^{\infty}a_n\log(a_n) = \infty$. We now distinguish the following further two cases.

If α is expanding, then there is no phase transition. This follows, since, by Lemma 2.4.8, we have that $\lim_{n\to\infty}(a_{n+1}/a_n)^u = 1/\rho^u < 1$ for all $u > 0$ and hence, by the ratio test for series convergence, $p(u) < \infty$, for all $u > 0$. In particular, $t_\infty = 0$ and $p(t_\infty) = \infty$.

If α is expansive of exponent $\theta > 0$, so that $t_n = \psi(n)n^{-\theta}$, then Proposition 2.4.7 implies that there exists ψ_0 such that $\psi_0(n) \sim \theta\psi(n)$ and $a_n = \psi_0(n)n^{-(1+\theta)}$. Consequently, one immediately verifies that $t_\infty = 1/(1+\theta)$. Hence, we now observe that

$$\lim_{u\searrow t_\infty} -p^+(u) = (1+\theta)\lim_{u\searrow t_\infty}\frac{\sum_{n=1}^{\infty}\left(n^{-(1+\theta)}\psi_0(n)\right)^u\log\left(n(\psi_0(n))^{-1/(1+\theta)}\right)}{\sum_{n=1}^{\infty}\left(n^{-(1+\theta)}\psi_0(n)\right)^u}.$$

We now split the discussion as follows. Firstly, if $\sum_{n=1}^{\infty} \psi(n)^{1/(1+\theta)}(\log n)/n$ converges, then, clearly, in the above expression the numerator and the denominator both converge, and hence, $\lim_{u\searrow t_\infty} -p^+(u)$ is finite, showing that in this case the system exhibits a phase transition. Secondly, if $\sum_{n=1}^{\infty} \psi(n)^{1/(1+\theta)}(\log n)/n$ diverges, then we have to consider the following two sub-cases. If $\sum_{n=1}^{\infty} n^{-1}\psi_0(n)^{1/(1+\theta)}$ converges, then the denominator in the expression for $-p^+(u)$ tends to a finite value, but the numerator clearly does not. So, $\lim_{u\searrow t_\infty} -p^+(u) = \infty$. On the other hand, if $\sum_{n=1}^{\infty} n^{-1}\psi_0(n)^{1/(1+\theta)}$ diverges, then for every $k \in \mathbb{N}$ we have that

$$\frac{(k^{-(1+\theta)}\psi_0(k))^u}{\sum_{n=1}^{\infty}(n^{-(1+\theta)}\psi_0(n))^u} \to 0 \text{ as } u \to \frac{1}{1+\theta}$$

and hence for every $\varepsilon > 0$ and for all $N \in \mathbb{N}$, there exists $u > 1/(1+\theta)$ such that

$$\frac{\sum_{k=1}^{N}(k^{-(1+\theta)}\psi_0(k))^u}{\sum_{n=1}^{\infty}(n^{-(1+\theta)}\psi_0(n))^u} < \varepsilon.$$

Thus, it follows that

$$\frac{\sum_{n=1}^{\infty}\left(n^{-(1+\theta)}\psi_0(n)\right)^u \log\left(n(\psi_0(n))^{-1/(1+\theta)}\right)}{\sum_{n=1}^{\infty}\left(n^{-(1+\theta)}\psi_0(n)\right)^u} \geq (1-\varepsilon)\log\left(N(\psi_0(N))^{-1/(1+\theta)}\right)$$

and hence we have that $\lim_{u\searrow t_\infty} -p^+(u) = \infty$. Therefore, in both of these sub-cases the system exhibits no phase transition.

Finally, for the interpretation of t_∞ in terms of the Hausdorff dimension of the Good-type set $G_\infty^{(\alpha)}$, as stated in the theorem, we have shown above that $t_\infty = 1/(1+\theta)$ for α expansive of exponent $\theta > 0$ and $t_\infty = 0$ for α expanding. By Theorem 2.5.5 and Proposition 2.5.8, this corresponds to the stated Hausdorff dimension. This finishes the proof of Theorem 5.3.2.

□

Remark 5.3.3. Note that the Lyapunov spectrum for the Gauss map and Farey map have been determined in [47]. Moreover, the Lyapunov spectrum for the classical (non-alternating) Lüroth map has been explicitly stated in [4], where the authors refer to the proof given in [47] for the Gauss map.

Our second main aim in the present section is to determine the Lyapunov exponent of the map F_α. In order to do this, the following proposition is essential. In this proposition, we consider the potential function $N : \mathscr{U} \to \mathbb{N} \cup \{\infty\}$, which is given by

$$N(x) := \begin{cases} n & \text{for } x \in A_n, \text{ for } n \in \mathbb{N}\,; \\ \infty & \text{for } x = 0. \end{cases}$$

Proposition 5.3.4. *Let α be a partition which is either expanding, or expansive of exponent θ and eventually decreasing. With $\Pi(L_\alpha, x) := \lim_{n\to\infty}\left(\sum_{k=0}^{n-1}\log\left|L_\alpha'(L_\alpha^k(x))\right| / \sum_{k=0}^{n-1} N(L_\alpha^k(x))\right)$, we then have for each $s \geq 0$ that the sets*

$$\{x \in \mathscr{U} : \Pi(L_\alpha, x) = s\} \text{ and } \{x \in \mathscr{U} : \Lambda(F_\alpha, x) = s\}$$

coincide up to a countable set of points.

Proof. Let us restrict the discussion to only the α-irrational numbers, that is, all those $x \in \mathscr{U}$ with infinite α-Lüroth expansion. Since the α-rational numbers are but a countable set, it is obviously of no consequence to do this. In this case, we then have for each x that $\sum_{k=0}^{n-1} N(L_\alpha^k(x)) = \sum_{k=1}^n \ell_k(x)$. For ease of exposition, set

$$\mathbf{L}_n(x) := \sum_{k=0}^{n-1} \log \left| L'_\alpha(L_\alpha^k(x)) \right| = -\sum_{k=1}^{n} \log(a_{\ell_k(x)}) \text{ and } \mathbf{F}_n(x) := \sum_{k=0}^{n-1} \log \left| F'_\alpha(F_\alpha^k(x)) \right|.$$

One immediately verifies that the sequence $(\mathbf{L}_n(x)/\sum_{k=1}^n \ell_k(x))_{n\in\mathbb{N}}$ is a subsequence of the sequence $(\mathbf{F}_n(x)/n)_{n\in\mathbb{N}}$. Thus, if $s \geq 0$ and if $\lim_{n\to\infty} \mathbf{F}_n(x)/n = s$, that is, if $\Lambda(F_\alpha, x) = s$, it follows directly that $\lim_{n\to\infty} \mathbf{L}_n(x)/\sum_{k=1}^n \ell_k(x) = s$. Therefore, we have for all $s \geq 0$ that

$$\{x \in \mathscr{U} : \Lambda(F_\alpha, x) = s\} \subset \{x \in \mathscr{U} : \Pi(L_\alpha, x) = s\}.$$

We will now consider the opposite inclusion. The aim is to show that if $\Pi(L_\alpha, x) = s$ then we also have that $\Lambda(F_\alpha, x) = s$. So, fix $s \geq 0$ and suppose that

$$\Pi(L_\alpha, x) = \lim_{n\to\infty} \frac{\mathbf{L}_n(x)}{\sum_{k=1}^n \ell_k(x)} = s.$$

Let us fix one further notation. Let

$$k(n) := \begin{cases} 0 & \text{for } 1 \leq n < \ell_1(x); \\ \sup\left\{k \in \mathbb{N} : \sum_{i=1}^k \ell_i(x) \leq n\right\} & \text{for } n \geq \ell_1(x). \end{cases}$$

That is, if $\ell_1(x) + \ldots + \ell_m(x) \leq n < \ell_1(x) + \ldots + \ell_m(x) + \ell_{m+1}(x)$, then $k(n) = m$. Let us also define $j(n) := n - \sum_{i=1}^{k(n)} \ell_i(x)$ and observe that since $L_\alpha^{k(n)}(x) \in A_{\ell_{k(n)+1}}$ and $j(n) < \ell_{k(n)+1}$, by Lemma 5.3.1 we have that

$$\begin{aligned} \frac{\mathbf{F}_n(x)}{n} &= \frac{1}{n} \sum_{k=0}^{n-1} \log |F'_\alpha(F_\alpha^k(x))| = \frac{1}{n}\left(\mathbf{L}_{k(n)}(x) + \mathbf{F}_{j(n)}(L_\alpha^{k(n)}(x))\right) \\ &= \frac{\mathbf{L}_{k(n)}(x)}{\sum_{i=1}^{k(n)} \ell_i(x) + j(n)} + \frac{j(n)}{\sum_{i=1}^{k(n)} \ell_i(x) + j(n)} \cdot \frac{1}{j(n)} \sum_{k=0}^{j(n)-1} \log \left| F'_\alpha(F_\alpha^k(L^{k(n)}(x))) \right| \\ &= \frac{\sum_{i=1}^{k(n)} \ell_i(x)}{\sum_{i=1}^{k(n)} \ell_i(x) + j(n)} \cdot \frac{\mathbf{L}_{k(n)}(x)}{\sum_{i=1}^{k(n)} \ell_i(x)} + \frac{j(n)}{\sum_{i=1}^{k(n)} \ell_i(x) + j(n)} (\log \rho \pm \varepsilon_{j(n)}). \end{aligned} \tag{5.4}$$

Here, the terms $\varepsilon_{j(n)}$ belong to the sequence $(\varepsilon_k)_{k\in\mathbb{N}}$ which was obtained in Lemma 5.3.1. We will split the remainder of the proof into two cases; namely, when $s = \log \rho$ and when $s \neq \log \rho$. Let us first consider the case where $s = \log \rho$. Let $\varepsilon > 0$. By assumption, we have that $\lim_{n\to\infty} \mathbf{L}_{k(n)}(x)/\sum_{i=1}^{k(n)} \ell_i(x) = \log \rho$. Thus, there exists $N_1 \in \mathbb{N}$ such that if $n \geq N_1$ we have that

$$\frac{\sum_{i=1}^{k(n)} \ell_i(x)}{\sum_{i=1}^{k(n)} \ell_i(x) + j(n)} \cdot (\log \rho - \varepsilon) + \frac{j(n)}{\sum_{i=1}^{k(n)} \ell_i(x) + j(n)} (\log \rho - \varepsilon_{j(n)}) \leq \frac{\mathbf{F}_n(x)}{n} \leq \cdots$$

$$\cdots \leq \frac{\sum_{i=1}^{k(n)} \ell_i(x)}{\sum_{i=1}^{k(n)} \ell_i(x) + j(n)} \cdot (\log \rho + \varepsilon) + \frac{j(n)}{\sum_{i=1}^{k(n)} \ell_i(x) + j(n)} (\log \rho + \varepsilon_{j(n)}).$$

If it happens that for $n \geq N_1$ we have that $\varepsilon_{j(n)} < \varepsilon$, since the above is a convex combination, we obtain that

$$\log \rho - \varepsilon \leq \frac{\mathbf{F}_n(x)}{n} \leq \log \rho + \varepsilon.$$

Otherwise, if $\varepsilon_{j(n)} \geq \varepsilon$, since $\lim_{n\to\infty} \varepsilon_n = 0$, it follows that $j(n) < M$, for some $M := M(\varepsilon) \in \mathbb{N}$. Thus there exists $N_2 \in \mathbb{N}$ such that if $n \geq N_2$, we have that

$$\frac{\varepsilon}{\log \rho - c} < \frac{j(n)}{\sum_{i=1}^{k(n)} \ell_i(x) + j(n)} < \frac{\varepsilon}{\log \rho + c},$$

where $c = \sup_{n\in\mathbb{N}} \varepsilon_n$. In this case we obtain that

$$j(n) < \frac{\sum_{i=1}^{k(n)} \ell_i(x)}{(\log \rho + c)/\varepsilon - 1} \quad \text{and so} \quad \frac{\sum_{i=1}^{k(n)} \ell_i(x)}{\sum_{i=1}^{k(n)} \ell_i(x) + j(n)} > 1 - \frac{\varepsilon}{\log \rho + c}.$$

Hence, for all $n \geq \max\{N_1, N_2\}$, we obtain

$$\left(1 - \frac{\varepsilon}{\log \rho + c}\right)(\log \rho - \varepsilon) + \frac{\varepsilon}{\log \rho - c}(\log \rho - c) \leq \frac{\mathbf{F}_n(x)}{n} \leq \log \rho + \varepsilon + \frac{\varepsilon}{\log \rho + c}(\log \rho + c)$$

and rewriting this yields that

$$\log \rho - \varepsilon < \log \rho - \varepsilon \left(\frac{\log \rho - \varepsilon}{\log \rho + c}\right) \leq \frac{\mathbf{F}_n(x)}{n} \leq \log \rho + 2\varepsilon$$

Since ε was arbitrary, we have that $\lim_{n\to\infty} \mathbf{F}_n(x)/n = \Lambda(F_\alpha, x) = \log \rho$.

It now remains only to consider the case where $s \neq \log \rho$. Continuing further from Equation (5.4), we obtain that

$$\frac{\mathbf{F}_n(x)}{n} = \frac{1}{1 + j(n)/\sum_{i=1}^{k(n)} \ell_i(x)} \cdot \frac{\mathbf{L}_{k(n)}(x)}{\sum_{i=1}^{k(n)} \ell_i(x)} + \frac{1}{1 + \sum_{i=1}^{k(n)} \ell_i(x)/j(n)} \cdot (\log \rho \pm \varepsilon_{j(n)}).$$

Thus, since we have that $\lim_{n\to\infty} \mathbf{L}_{k(n)}(x)/\sum_{i=1}^{k(n)} \ell_i(x) = s$ (by assumption), it suffices to show that $\lim_{n\to\infty} j(n)/\sum_{i=1}^{k(n)} \ell_i(x) = 0$.

The basic idea now is to examine the terms of the sequence $(\mathbf{L}_{k(n)}(x)/\sum_{i=1}^{k(n)} \ell_i(x))_{n\in\mathbb{N}}$ at time $k(n)$ and $k(n)+1$. Observe that

$$\begin{aligned} s &= \lim_{n\to\infty} \frac{\mathbf{L}_{k(n)+1}(x)}{\sum_{i=1}^{k(n)+1} \ell_i(x)} = \lim_{n\to\infty} \frac{\mathbf{L}_{k(n)}(x) - \log(a_{\ell_{k(n)+1}(x)})}{\sum_{i=1}^{k(n)} \ell_i(x) + \ell_{k(n)+1}(x)} \\ &= \lim_{n\to\infty} \frac{\mathbf{L}_{k(n)}(x)}{\sum_{i=1}^{k(n)} \ell_i(x)} \cdot \frac{\left(1 + \log(1/a_{\ell_{k(n)+1}(x)})/\mathbf{L}_{k(n)}(x)\right)}{\left(1 + \ell_{k(n)+1}(x)/\sum_{i=1}^{k(n)} \ell_i(x)\right)}. \end{aligned}$$

Recall the present aim: to prove that $\lim_{n\to\infty} j(n)/\sum_{i=1}^{k(n)} \ell_i(x) = 0$. So, by way of contradiction, suppose that $\lim_{n\to\infty} j(n)/\sum_{i=1}^{k(n)} \ell_i(x) \neq 0$. Then, since $j(n) \in \{0,1,\ldots,\ell_{k(n)+1}(x)\}$, it follows that

$$\lim_{n\to\infty} \frac{\ell_{k(n)+1}(x)}{\sum_{i=1}^{k(n)} \ell_i(x)} \neq 0. \tag{5.5}$$

As each of these terms is positive, it follows that $\limsup_{n\to\infty} \ell_{k(n)+1}(x)/\sum_{i=1}^{k(n)} \ell_i(x) > 0$ and so there exists a sequence $(n_i)_{i\in\mathbb{N}}$ such that

$$\lim_{i\to\infty} \frac{\ell_{k(n_i)+1}(x)}{\sum_{m=1}^{k(n_i)} \ell_m(x)} = c > 0, \tag{5.6}$$

where c could be either finite or infinite. From this, we infer that for large enough i, the quantity $\ell_{k(n_i)+1}(x)/\sum_{m=1}^{k(n_i)} \ell_m(x)$ is bounded away from zero. Let us mention one general analytic fact. If $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are sequences of positive real numbers satisfying the properties that $\lim_{n\to\infty}(1+a_n)/(1+b_n) = 1$ and the sequence (b_n) is bounded away from zero for all sufficiently large n, then also $\lim_{n\to\infty} a_n/b_n = 1$. (Note that this makes no claim about the existence of any limit of either of the sequences (a_n) or (b_n).) From this fact and from the discussion above, we deduce that

$$\lim_{i\to\infty} \frac{-\log(a_{\ell_{k(n_i)+1}(x)})/\mathbf{L}_{k(n_i)}(x)}{\ell_{k(n_i)+1}(x)/\sum_{m=1}^{k(n_i)} \ell_m(x)} = 1. \tag{5.7}$$

Also, since the sequence $\left(\sum_{m=1}^{k(n_i)} \ell_m(x)\right)_{i\in\mathbb{N}}$ is increasing, it follows from Equation (5.6) that along the subsequence $(n_i)_{i\in\mathbb{N}}$ we have that

$$\lim_{i\to\infty} \ell_{k(n_i)+1}(x) = \infty. \tag{5.8}$$

In view of (5.8), we can apply Lemma 2.4.8 (1) to infer that

$$\lim_{i\to\infty} \frac{-\log(a_{\ell_{k(n_i)+1}(x)})}{\ell_{k(n_i)+1}(x)} = \log\rho. \tag{5.9}$$

Finally, combining Equations (5.9) and (5.7), we obtain that

$$1 = \lim_{i\to\infty} \frac{-\log(a_{\ell_{k(n_i)+1}(x)})\ \sum_{m=1}^{k(n_i)} \ell_m(x)}{\ell_{k(n_i)+1}(x)\ \mathbf{L}_{k(n_i)}(x)} = \frac{\log\rho}{s} \neq 1,$$

which is a contradiction and hence finishes the proof in the case that $s \neq \log\rho$. As both cases are proved, this finishes the proof of the proposition.

□

We are now in a position to consider the Lyapunov spectra arising from the maps F_α. In other words, we consider the spectral sets $\{s \in \mathbb{R} : \sigma_\alpha(s) \neq \emptyset\}$, arising from the Hausdorff dimension-function $\sigma_\alpha(s)$, given by

$$\sigma_\alpha(s) := \dim_H(\{x \in \mathscr{U} : \Lambda(F_\alpha, x) = s\}).$$

Theorem 5.3.5. *Let α be a partition that is either expanding or expansive and eventually decreasing. The Hausdorff dimension function of the Lyapunov spectrum associated with F_α is then given as follows. For*

$$r_- := \inf\{-t^-(v) : v \in \mathrm{Int}(dom(t))\} \text{ and } r_+ := \sup\{-t^+(v) : v \in \mathrm{Int}(dom(t))\},$$

we have that $\sigma_\alpha(s)$ vanishes outside the interval $[1/r_+, 1/r_-]$ and for each $s \in (1/r_+, 1/r_-)$, we have

$$\sigma_\alpha(s) = \inf_{v \in \mathbb{R}} \left(s^{-1} \cdot v + t(v)\right).$$

Proof. We aim to use the general result of Jaerisch and Kessböhmer again, in the special situation that the potential ψ is defined by $\psi(x) := -n$, for $x \in A_n$. Note that by Proposition 5.3.4 we have that

$$\begin{aligned} \{x \in \mathscr{U} : \Lambda(F_\alpha, x) = s\} &= \{x \in \mathscr{U} : \Pi(L_\alpha, x) = s\} \\ &= \left\{x \in \mathscr{U} : \lim_{n\to\infty} \frac{\sum_{k=1}^n \log(a_{\ell_k(x)})}{\sum_{k=1}^n \ell_k(x)} = s\right\} \\ &= \left\{x \in \mathscr{U} : \lim_{n\to\infty} \frac{\sum_{k=0}^{n-1} \phi \circ L_\alpha^k(x)}{\sum_{k=0}^{n-1} \psi \circ L_\alpha^k(x)} = s\right\}. \end{aligned}$$

It therefore follows directly from Theorem 5.2.6 that for each $s \in (1/r_+, 1/r_-)$, we have that $\sigma_\alpha(s) = t^*(1/s) = \inf_{v\in\mathbb{R}} \left(s^{-1} \cdot v + t(v)\right)$.

□

Remark 5.3.6. It is also possible to phrase Theorem 5.3.5 in terms of the *α-Farey free energy function* $v : \mathbb{R} \to \mathbb{R}$, which is given by

$$v(u) := \inf\left\{r \in \mathbb{R} : \sum_{n=1}^{\infty} a_n^u \exp(-rn) \leq 1\right\}.$$

The boundary points of the F_α spectrum are given by the asymptotic slopes of the function v. These are given by $s_- := \inf\{-(\log a_n)/n : n \in \mathbb{N}\}$ and $s_+ := \sup\{-(\log a_n)/n : n \in \mathbb{N}\}$. This follows, since for each $\varepsilon > 0$ and for $u > 0$, resp. $u < 0$, we have

$$\sum_{n=1}^{\infty} \exp\left(nu\left(\frac{\log a_n}{n} + s_\mp \mp \varepsilon\right)\right) \begin{cases} \leq \sum_{n=1}^{\infty} \exp(\mp nu\varepsilon) \longrightarrow 0 & \text{for } u \longrightarrow \pm\infty \\ \geq \exp(\pm u\varepsilon) \longrightarrow +\infty & \text{for } u \longrightarrow \pm\infty. \end{cases}$$

Then, for any $s \in (s_-, s_+)$, we have that

$$\sigma_\alpha(s) = \inf_{u \in \mathbb{R}} \left(u + s^{-1} v(u)\right).$$

As in the case of the map L_α, the map F_α also in some cases displays phase transition behaviour. The definition is equivalent to that for the map L_α; we say that the map F_α exhibits no phase transition if and only if the α-Farey free energy function v is differentiable everywhere. It turns out that the following holds:

- If α is expanding, then F_α exhibits no phase transition. In particular, v is strictly decreasing and bijective.
- If α is expansive of exponent $\theta > 0$ and eventually decreasing, then F_α exhibits no phase transition if and only if α is of infinite type. In particular, v is non-negative and vanishes on $[1, \infty)$.

For a detailed discussion of these phase transition phenomena for the α-Farey map and the boundary points of the spectrum, the reader is referred to the paper [42].

5.4 Examples

In this section, we give various examples which demonstrate the diversity of different behaviours of the spectra given by Theorem 5.3.2 and Theorem 5.3.5 in dependence on the chosen partition α. Each partition α under consideration here is eventually decreasing and either expanding or expansive of exponent $\theta > 0$.

Our first example is that of the alternating Lüroth map L_{α_H} and the α_H-Farey map. Recall that the partition α_H is expansive of exponent 1 and is strictly decreasing. In this case we can explicitly calculate that the starting point of the α_H-Lüroth spectrum is given by $\log 2$ and t_∞ is equal to $1/2$. The α_H-Farey spectrum starts at 0 (this is easily verified for any expansive partition) and ends at the point $(\log 6)/2$. We see that the spectra overlap in the interval $(\log 2, (\log 6)/2)$.

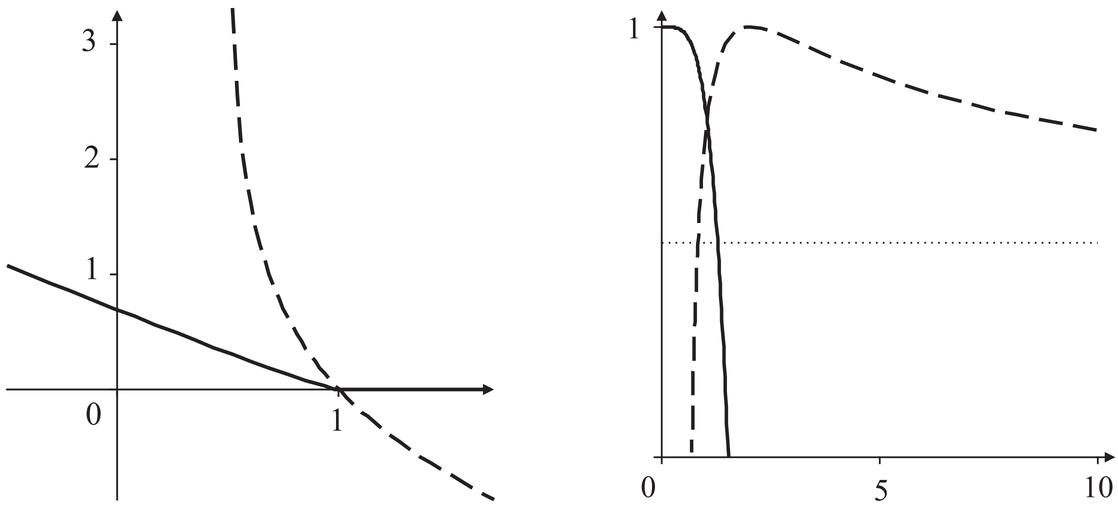
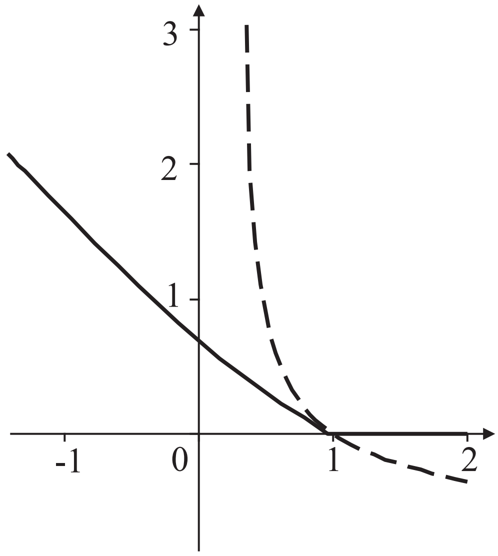


Figure 5.2: **The α_H-Farey free energy function and the α_H-Lüroth pressure function.** The figure shows the α_H-Farey free energy v (solid line), the α_H-Lüroth pressure function p (dashed line), and the associated dimension graphs σ_α and τ_α of the alternating Lüroth system.

Let us comment on the value of s for which $\tau_\alpha(s) = 1$. In order to calculate this, we have to find the maximum value of the Legendre transformation of the pressure function p. By convexity, this maximum must be at the point $u = 1$. Therefore, $\tau_\alpha(s) = 1$ when $s = -p'(1) = \sum_{n=1}^{\infty} a_n \log(a_n)$. Recall from Remark 3.2.4 that in case α is expansive of exponent $\theta > 0$ or expanding, this value is also equal to the measure-theoretic entropy of the map L_α with respect to the Lebesgue measure. It therefore follows from considerations of t_∞ that for partitions α that are expansive of exponent $\theta > 0$ or expanding, the measure-theoretic entropy of the map L_α is finite.

Figure 5.3 below shows the α-Farey free energy v, the α-Lüroth pressure function p and the associated dimension graphs for the partition α defined by $a_n := \zeta(3)^{-1} n^{-3}$, where ζ denotes the Riemann zeta function. The partition α is of finite type, due to Proposition 2.4.6. Therefore, by Remark 5.3.6, F_α exhibits a phase transition. However, since it is clear that $\sum_{n=1}^{\infty} a_n^{t_\infty}$ diverges, where $t_\infty = 1/3$, we have that $p(t_\infty)$ is infinite and so L_α exhibits no phase transition.



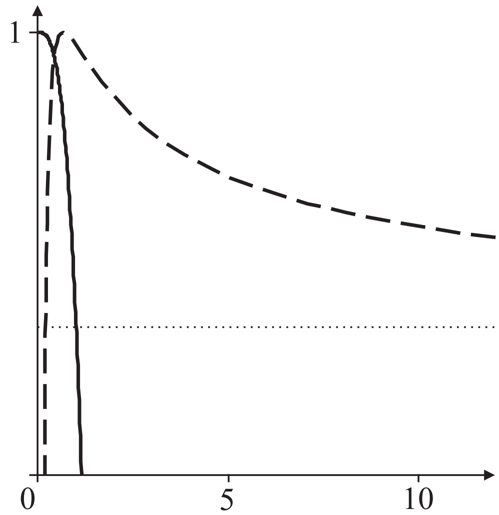


Figure 5.3: **Phase transition for the α-Farey free energy function, no phase transition for the the α-Lüroth pressure function with α expansive of exponent** 2. The α-Farey free energy v (solid line), the α-Lüroth pressure function p (dashed line), and the associated dimension graphs for $a_n := \zeta(3)^{-1} n^{-3}$. Here, F_α has a phase transition, namely, p is not differentiable at 1, whereas L_α exhibits no phase transition and $p(t_\infty) = \infty$.

Figure 5.4 shows the α-Lüroth pressure function p, and the associated dimension graph for the α-Lüroth system defined by the partition α which is given by $a_n := n^{-2} \cdot (\log(n+5))^{-4}/C$, where $C := \sum_{n\geq 1} n^{-2} \cdot (\log(n+5))^{-4}$. Since the partition α is expansive of exponent 1, we immediately obtain that $t_\infty = 1/2$. We have that $p(t_\infty)$ is finite, since

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{n(\log(n+5))^2} &\leq \left(\frac{1}{\log 6}\right)^2 + \sum_{n=2}^{\infty} \frac{1}{n(\log(n))^2} \\ &\leq \int_2^{\infty} \frac{1}{x(\log x)^2}\, dx = \int_{\log 2}^{\infty} \frac{du}{u^2} \\ &= \frac{1}{\log 2}.\end{aligned}$$

However, we have in this case that $\lim_{t\to t_\infty} -p^+(t) = +\infty$, where the reader might like to recall that $p^+(t) = \left(\sum_{n=1}^\infty a_n^t \cdot \log(a_n)\right) / \left(\sum_{n=1}^\infty a_n^t\right)$. Indeed, this follows directly from the fact that

$$\begin{aligned}\sum_{n=1}^{\infty} a_n^t \log(1/a_n) &= \sum_{n=1}^{\infty} (n^{-2}(\log(n+5))^{-4})^t \log(n^2(\log(n+5))^4) \\ &\leq -a_1^t \log(a_1) + \sum_{n=2}^{\infty} \frac{\log(n+5)}{(n^2(\log(n+5))^4)^t}.\end{aligned}$$

This means that the map L_α exhibits no phase transition. Alternatively, this can be proved via the fact that $\sum_{n=1}^\infty \psi(n)^{1/(1+\theta)}(\log n)/n$ converges (as in the condition stated in Theorem 5.3.2 for the existence of phase transitions).

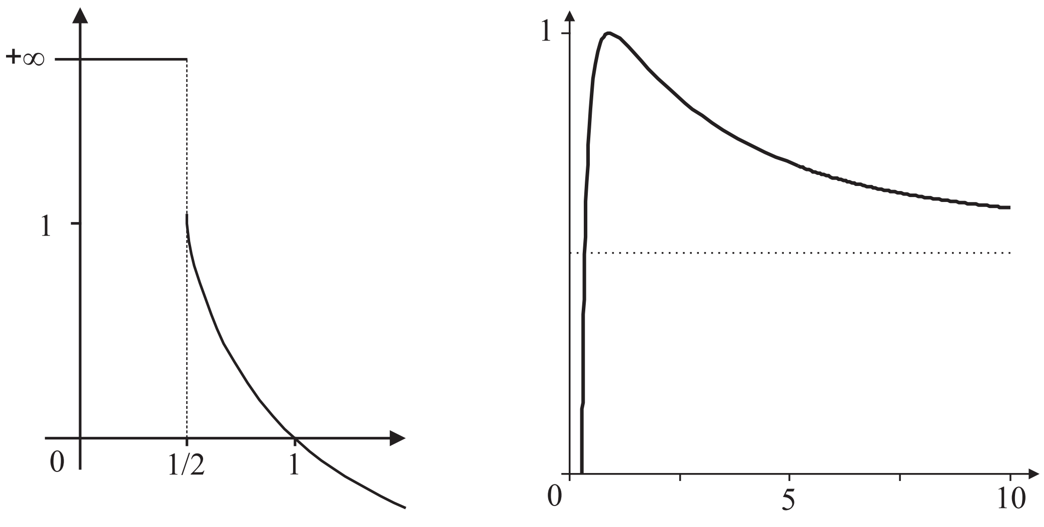


Figure 5.4: **Finite critical value $p(t_\infty) < \infty$ and no phase transition for the α-Lüroth pressure function and α expansive.** The α-Lüroth pressure function p, and the associated dimension graph for the α-Lüroth system with $a_n := n^{-2} \cdot (\log(n+5))^{-4}/C$, where $C := \sum_{n\geq 1} n^{-2} \cdot (\log(n+5))^{-4}$. In this case $t_\infty = 1/2$ and $p(1/2) < \infty$, but L_α exhibits no phase transition.

Our next example shows the α-Lüroth spectrum for a partition α that is expansive with exponent 1 and exhibits a phase transition. Firstly, it is clear that $p(t_\infty) = p(1/2) = \log \sum_{n=1}^{\infty} a_n^{1/2} < \infty$, by the comparison test for series convergence, where the series we are comparing with is that given by the partition in the example directly above. In this case, in contrast to the preceding example, we also have that $\sum_{n=1}^{\infty} \phi(n)^{1/2}(\log(n))/n$ converges, where $\phi(n) := 1/C \cdot (\log(n+5))^{1/2}$. To show this, we make the following calculation:

$$\sum_{n=2}^{\infty} \left(\frac{1}{(\log(n+5))^6}\right)^{1/2} \frac{\log(n)}{n} = \sum_{n=2}^{\infty} \frac{\log(n)}{n(\log(n+5))^6} \leq \sum_{n=2}^{\infty} \frac{1}{n(\log(n))^5} < \infty.$$

Then, in light of Theorem 5.3.2, we have that L_α exhibits a phase transition.

For this example, we can also calculate that the value of $t_- := \inf\{-\log a_n : n \in \mathbb{N}\}$, which is achieved for $n = 1$, is equal to $\log(C(\log 6)^{12})$. Recall that for each $s \in (t_-, \infty)$ we have that $\tau_\alpha(s) = \inf_{u \in \mathbb{R}}\{u + s^{-1}p(u)\}$. The dashed line for the latter part of the spectrum pictured in the right-hand side graph in Figure 5.5 indicates that part of the spectrum which does not come from the real-analytic part of the pressure function, i.e., if $t_0 := \lim_{t \to t_\infty} -p^+(t)$, which we have just shown to be finite, the dashed line denotes that part of the spectrum for values of s that lie in the interval (t_0, ∞). In this part of the spectrum, we can explicitly calculate that $\tau_\alpha(s) = s^{-1}p(t_\infty) + t_\infty = s^{-1} \cdot \log\left(\sum_{n=1}^{\infty} a_n^{t_\infty}\right) + t_\infty$.

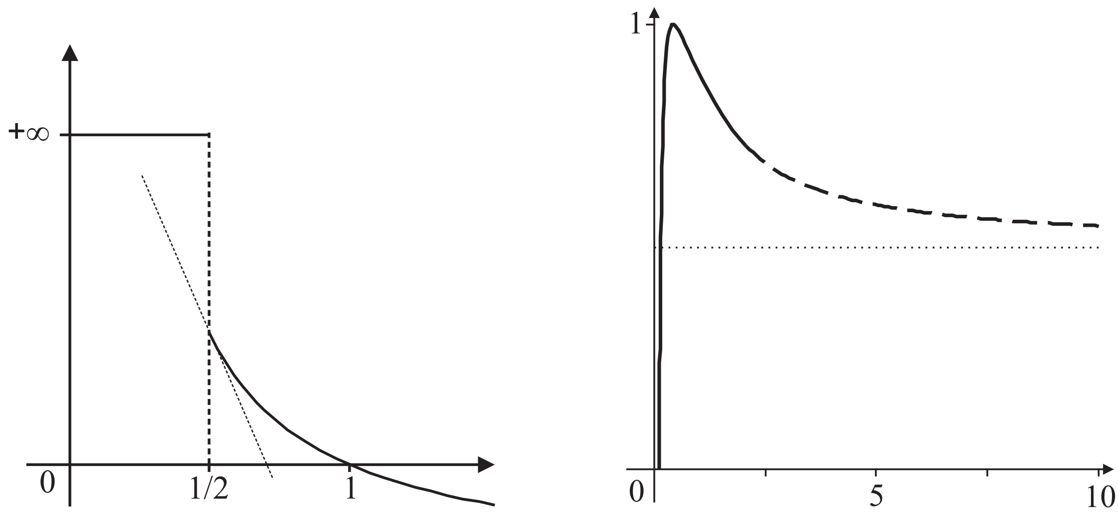


Figure 5.5: **Finite critical value $p(t_\infty) < \infty$ with phase transition for the α-Lüroth pressure function and α expansive of exponent $\theta = 1$.** The α-Lüroth pressure function p, and the associated dimension graph for the α-Lüroth system with $a_n := n^{-2} \cdot (\log(n+5))^{-12}/C$, where $C := \sum_{n \geq 1} n^{-2} \cdot (\log(n+5))^{-12}$. In this case $t_\infty = 1/2$ and $p(1/2) < \infty$ and L_α has a phase transition, namely, p is not differentiable at $1/2$. The dotted line in the left-hand side graph has slope equal to the right derivative of p at the point t_∞.

In Figure 5.6 below, an example of a partition for which the spectral set of the α-Lüroth map and that of the α-Farey map intersect in a single point is shown. Let α be defined by the condition $a_n := \zeta(5/4)^{-1} n^{-5/4}$ for each $n \in \mathbb{N}$. Then the the partition α is expansive of exponent $\theta = 1/4$. In particular, by Proposition 2.4.6, we have that α is of infinite type and thus, according to Remark 5.3.6, it follows that F_α exhibits no phase transition. It is clear that for $t_\infty = 4/5$ we have that $p(t_\infty)$ is infinite and so L_α exhibits no phase transition either. One immediately verifies that $t_- = s_+ = \log(\zeta(5/4))$.

As has been shown in [47], the spectral sets of the Farey map and Gauss map intersect at the single point $2\log((\sqrt{5}+1)/2)$. As the examples above have already demonstrated, for the linear systems considered here this is by no means canonical.

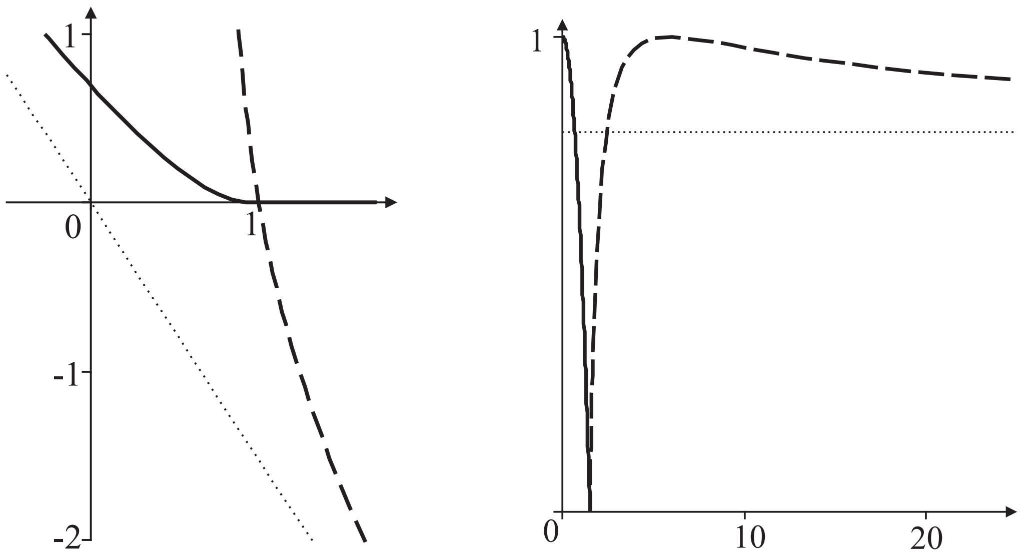


Figure 5.6: **The α-Farey spectrum and the α-Lüroth spectrum intersect in a single point, for α expansive.** The α-Farey free energy v (solid line), the α-Lüroth pressure function p (dashed line), and the associated dimension graphs for $a_n := \zeta(5/4)^{-1} n^{-5/4}$.

The next example, illustrated in Figure 5.7 below, is of a situation that is in some sense as far from the preceding one as it is possible to get. For the expanding partition α defined by $a_n := 2 \cdot 3^{-n}$ for all $n \in \mathbb{N}$, we have that the spectral set of F_α is completely contained in the spectral set of L_α. It is easy to show that $t_- := \inf\{-\log(a_n) : n \in \mathbb{N}$ is in this case given by $\log(3/2)$. From Remark 2.3.6 and Example 2.3.5, we can calculate that $s_- = \log(3/2)$ and $s_+ = \log(3)$.

Let us make a few further remarks on the starting point of the spectrum σ_α. If α is a partition which is expanding and eventually decreasing, as in the example shown in Figure 5.7, it follows that $s_- > 0$, whereas $\sigma_\alpha(s_-)$ can be either zero or strictly positive. To see that $s_- > 0$, it is sufficient to recall that for any $\varepsilon > 0$, if n is large enough, we have that

$$a_n \leq \left(\frac{1}{\rho - \varepsilon}\right)^n, \tag{5.10}$$

where $1 < \rho := \lim_{n\to\infty} t_n/t_{n+1} = \lim_{n\to\infty} a_n/a_{n+1}$. Fix $\varepsilon > 0$ small enough that $\rho - \varepsilon > 1$ and let $N \in \mathbb{N}$ be such that (5.10) is satisfied for all $n \geq N$. It follows that

$$s_- := \inf\left\{\frac{-\log a_n}{n} : n \in \mathbb{N}\right\} \geq \min\left\{\frac{-\log a_1}{1}, \frac{-\log a_2}{2}, \ldots, \frac{-\log a_N}{N}, \log(\rho - \varepsilon)\right\} > 0.$$

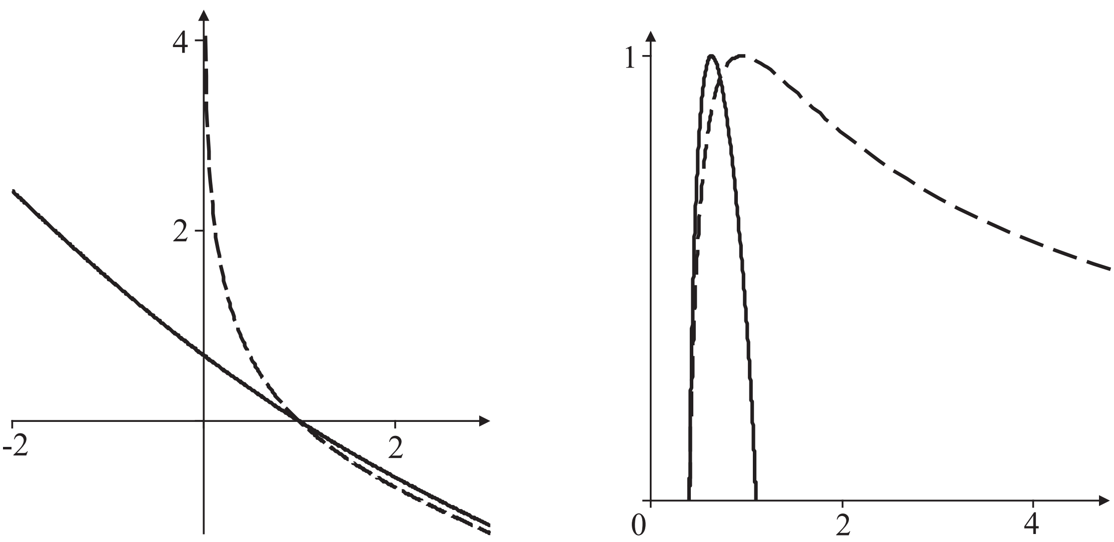


Figure 5.7: **The set of points for which the α-Farey spectrum is non-zero is completely contained in the set of points for which the α-Lüroth spectrum is non-zero, for α expanding.** The α-Farey free energy v (solid line), the α-Lüroth pressure function p (dashed line), and the associated dimension graphs for the α-Farey and α-Lüroth systems with $a_n := 2 \cdot 3^{-n}$, $n \in \mathbb{N}$. The α-Farey system is given in this situation by the skewed tent map with slopes 3 and $-3/2$.

On the other hand, if α is expansive of exponent $\theta > 0$ we always have that $s_- = 0$ and $\sigma_\alpha(0) = 1$. In order to see that $\sigma_\alpha(0) = 1$ is in fact true for any such partition α, one argues as follows. If α is a partition of finite type (recall that this means that $\sum_{n=1}^\infty t_n < \infty$), then the proof follows along the lines of the proof of [30, Proposition 10]. However, if α is of infinite type, then this follows from the fact that $\Lambda(F_\alpha, x) = 0$, for λ-almost all $x \in \mathscr{U}$. To see why this last statement is true, we turn to Hopf's Ratio Ergodic Theorem [37] (also see [86], for a very neat proof). This theorem states that if T is an ergodic measure-preserving transformation on the σ-finite measure space $(X, \mathscr{A}, \mu)$ and if $f, g \in L^1(\mu)$ with $g \geq 0$ and $\int_X g\, d\mu > 0$, then

$$\lim_{n\to\infty} \frac{S_n f(x)}{S_n g(x)} = \frac{\int_X f\, d\mu}{\int_X g\, d\mu}, \quad \text{for } \mu\text{-a.e. } x \in X.$$

Therefore, letting $\mu = \lambda$, $T = L_\alpha$, $f = \phi$ and $g = N$, as in Proposition 5.3.4, we obtain, for λ-a.e. $x \in \mathscr{U}$, that

$$\lim_{n\to\infty} \frac{S_n \phi(x)}{S_n N(x)} = \frac{\sum_{n=1}^\infty a_n \log a_n}{\sum_{n=1}^\infty n \cdot a_n} = \frac{\sum_{n=1}^\infty a_n \log a_n}{\sum_{n=1}^\infty t_n}.$$

Since for any α that is expansive of exponent $\theta > 0$ we always have that $\sum_{n=1}^{\infty} a_n \log a_n$ is finite (see the discussion after Figure 5.2, above), if α is of infinite type, that is, if α is such that $\sum_{n=1}^{\infty} t_n$ diverges, it is clear that the limit above is equal to 0 for λ-a.e. $x \in \mathscr{U}$. Thus, by Proposition 5.3.4, $\lambda(\{x \in \mathscr{U} : \Lambda(F_\alpha, x) = 0\}) = 1$ and so $\dim_H(\{x \in \mathscr{U} : \Lambda(F_\alpha, x) = 0\}) = 1$. This finishes the proof of the claim.

So far, each example that we have shown for the Lyapunov spectrum of the α-Lüroth map has $\tau_\alpha(t_-) = 0$ and each example shown for the α-Farey map has $\sigma_\alpha(s_+) = 0$. However, this is not necessarily always the case. For instance, if we let α be a partition with $a_1 = a_2$ and with all other partition elements having smaller measure, it follows immediately that the Hausdorff dimension of the set $\{x \in \mathscr{U} : \Lambda(L_\alpha, x) = -\log a_1\}$ is at least equal to $\log 2/(-\log a_1)$, since it contains a Cantor set of this dimension.

Also, let α be a partition such that $a_1 = \sqrt{a_2}$ and all other elements have smaller measure. In this case, we have that

$$s_+ := \sup\{\frac{-\log a_1}{1}, \frac{-\log a_2}{2}, \frac{-\log a_3}{3}, \ldots\} = \sup\{-\log a_1, \frac{-\log a_3}{3}, \ldots\} = -\log a_1.$$

One immediately verifies that for any $x = [\ell_1(x), \ell_2(x), \ldots]_\alpha \in \mathscr{U}$ with all entries satisfying $\ell_i(x) \in \{1, 2\}$, we have that $\Lambda(F_\alpha, x) = -\log a_1$. Therefore, the set $\{x \in \mathscr{U} : \Lambda(F_\alpha, x) = s_+\}$ at least contains a so-called *asymmetric Cantor set*. The Hausdorff dimension of this Cantor set, say C, is given by Hutchinson's formula, namely,

$$\dim_H(C) = s, \text{ where } a_1^s + a_2^s = a_1^s + a_1^{2s} = 1.$$

It is straightforward to verify that the value of s satisfying Hutchinson's formula for the set C is given by $s = (\log((1+\sqrt{5})/2))/(-\log a_1)$. Finally, we then have that $\sigma_\alpha(s_+) \geq \dim_H(C) > 0$.

It is worth pointing out here that for the partition α_D, which is defined by setting $a_n := 2^{-n}$, we have that $s_- = s_+ = \log 2$. Therefore, in this case the α_D-Farey spectrum is trivial. However, we still obtain a proper spectrum for the α_D-Lüroth map.

Chapter 6

Good sets and strict Jarník sets for geometrically finite Fuchsian groups

In this chapter we will study certain subsets of the limit set of a non-elementary Fuchsian group. Throughout we assume that the group in question is geometrically finite and has one parabolic element. The sets we are interested in are analogous to those considered in Theorem 2.5.4 and Theorem 2.5.10 for the α-Lüroth system. The chapter will be organised as follows. In Section 6.1, we give the basic facts necessary from hyperbolic geometry to describe the set-up in the subsequent sections. In Section 6.2, we describe a certain collection of subsets of $\mathbb{S}^1$ that will be used to calculate the Hausdorff dimension results we obtain in subsequent sections. In Section 6.3, we define τ-Good sets for a group G and study the Hausdorff dimension of these sets as τ tends to infinity. In Section 6.4, we define sets analogous to those given in Section 2.5.2 and use them to obtain the Hausdorff dimension of the strict-Jarník limit set for G. Finally, in Section 6.5, we apply the results of Section 6.4 to derive a weak multifractal spectrum for the Patterson measure.

6.1 Hyperbolic Geometry preliminaries

In this first section, we gather together all the background material from Hyperbolic Geometry necessary to understand the subsequent sections. The treatment here is fairly brief; further details may be found in the books of A. Beardon [7] and P. Nicholls [62].

6.1.1 The Poincaré disc and upper half-plane models

Let us begin by defining the space and the metric comprising the *Poincaré disc model* $(\mathbb{D}^2, d_h)$ of 2-dimensional hyperbolic space.

Definition 6.1.1.

1. Let $\mathbb{D}^2$ denote the interior of the unit ball in the complex plane, that is, let

$$\mathbb{D}^2 := \{z \in \mathbb{C} : |z| < 1\}.$$

Denote by $\mathbb{S}^1$ the boundary of this disc, that is,

$$\mathbb{S}^1 := \{z \in \mathbb{C} : |z| = 1\}.$$

2. Let $\lambda : \mathbb{D}^2 \to \mathbb{R}$ be given by

$$\lambda(z) := \frac{2}{1-|z|^2},$$

for all $z \in \mathbb{D}^2$. Then the metric $d_h : \mathbb{D}^2 \times \mathbb{D}^2 \to \mathbb{R}^+$ is given by

$$d_h(u,v) := \inf\left\{ \int_\gamma \lambda(z)|dz| : \gamma \text{ is a smooth curve joining u and v} \right\}.$$

Remark 6.1.2. It is immediately apparent from the definition that the function d_h is positive and that the triangle inequality $d_h(z,w) \leq d_h(z,u) + d_h(u,w)$ holds for all $z,w,u \in \mathbb{D}^2$. It is also clear that if $x = y$ then $d_h(x,y) = 0$. To show that this is a metric, it only remains to show that the converse of this last statement also holds. For this, we refer to [7].

Recall that a map $g : \mathbb{D}^2 \to \mathbb{D}^2$ is said to be a *conformal automorphism* if and only if it is bijective, differentiable and preserves angles (magnitude and orientation) between smooth curves in $\mathbb{D}^2$. The set of all conformal automorphisms of $\mathbb{D}^2$ forms a group under composition of mappings. This group will be denoted by

$$Con(1) := \left\{g : g \text{ is a conformal automorphism of } \mathbb{D}^2\right\}.$$

The elements of $Con(1)$ are a certain type of Möbius transformation[1]. It can be shown that if $g \in Con(1)$, then there exist complex numbers a and c with the property that $|a|^2 - |c|^2 = 1$ and

$$g(z) = \frac{az+\bar{c}}{\bar{c}z+a}.$$

Lemma 6.1.3. *For each $g \in Con(1)$ we have that*

1. $|g'(z)| = \frac{1-|g(z)|^2}{1-|z|^2}$ *for all $z \in \mathbb{D}^2$. In particular,* $|g'(0)| = 1 - |g(0)|^2$.

2. $\frac{|g(x)-g(y)|^2}{|x-y|^2} = |g'(x)||g'(y)|$

Proof. By direct calculation, letting $g(z) = \frac{az+\bar{c}}{\bar{c}z+a}$. □

In the sequel, we shall be interested in subgroups of the group of *isometries* of hyperbolic space with respect to the hyperbolic metric. Recall that an isometry is a transformation of a metric space that preserves distances between points. We now show that the elements of $Con(1)$ are isometries of the metric space $(\mathbb{D}^2, d_h)$.

[1] For more information regarding general Möbius transformations, the reader is referred to the book Visual Complex Analysis, by Needham [61].

Proposition 6.1.4. *For each $g \in Con(1)$ we have that*

$$d_h(z,w) = d_h(g(z), g(w)) \text{ for all } z, w \in \mathbb{D}^2.$$

That is, g is an isometry of $(\mathbb{D}^2, d_h)$.

Proof. Let γ be a smooth curve between z and w and let $g \in Con(1)$. Then, using the substitution $u = g(v)$, Lemma 6.1.3 yields

$$\int_{g(\gamma)} \frac{2|du|}{1-|u|^2} = \int_\gamma \frac{2|g'(v)||dv|}{1-|g(v)|^2} = \int_\gamma \frac{2|dv|}{1-|v|^2}.$$

Therefore, since elements of $Con(1)$ map smooth curves onto smooth curves, taking the infimum on both sides gives the desired result.

□

We now give one explicit formulation of the hyperbolic distance between points of the unit disc. There are, of course, a great many other such formulae. For more details we refer to [7].

Lemma 6.1.5. *For all $z \in \mathbb{D}^2$, we have that*

$$d_h(0,z) = \log\left(\frac{1+|z|}{1-|z|}\right).$$

Proof. This result can be found in Section 7.2 of [7].

□

Corollary 6.1.6. *Hyperbolic geodesics in $\mathbb{D}^2$ are given by Euclidean straight lines through the origin, or circles orthogonal to $\mathbb{S}^1$.*

Proof. The fact that a hyperbolic geodesic through the origin is a Euclidean straight line is obtained in the proof of Lemma 6.1.5. The second half of the statement follows from the fact that each $g \in Con(1)$ consists of a composition of translations, rotations and reflections.

□

In addition to the Poincaré disc model, we will also have occasion to use the *upper half-plane* model of hyperbolic space, which we now define.

Definition 6.1.7.

1. Let $\mathbb{H}$ denote the upper half of the complex plane $\mathbb{C}$, so

$$\mathbb{H} := \{z = x + iy \in \mathbb{C} : y > 0\}.$$

   The boundary of $\mathbb{H}$ is then the set $\mathbb{R} \cup \{\infty\}$.

2. The metric in the upper half-plane is given by the map $d_{\mathbb{H}} : \mathbb{H} \times \mathbb{H} \to \mathbb{R}^+$, which is defined for all z, w in $\mathbb{H}$ by

$$d_{\mathbb{H}}(z,w) := \inf\left\{\int_\gamma \frac{|dz|}{y} : \gamma \text{ is a smooth curve between } z \text{ and } w\right\}.$$

The geodesics in $\mathbb{H}$ are either vertical Euclidean straight lines or semicircles orthogonal to the real axis. The following lemma is an easy consequence of this fact.

Lemma 6.1.8. *For all $z,w \in \mathbb{H}$ with $\mathrm{Re}(z) = \mathrm{Re}(w)$, we have that*

$$d_{\mathbb{H}}(z,w) = \left|\log \frac{\mathrm{Im}(z)}{\mathrm{Im}(w)}\right|.$$

The reason for having more than one model of hyperbolic space is purely practical - some results are easier to phrase in terms of one model than another. In order for this to make sense, though, the models must be equivalent in some way. The equivalence we require is *conformal equivalence*, which means that there exists a conformal map from one model to the other. We will now define a conformal map from $\mathbb{H}$ to $\mathbb{D}^2$. Consider the following three maps:

- Let ρ_1 be reflection at the line $\{z = x+iy \in \mathbb{C} : y = 0\}$, so

  $$\rho_1(z) := \bar{z},$$

  where $\bar{z}$ denotes the complex conjugate of z.

- Let ρ_2 be the reflection at the circle centred at i with radius $\sqrt{2}$, so

  $$\rho_2(z) := i + \left(\frac{\sqrt{2}}{|z-i|}\right)^2 (z-i).$$

- Let ρ_3 be the map given by clockwise rotation around 0 by $\frac{\pi}{2}$, so

  $$\rho_3(z) := -iz.$$

Note that each of these three maps is obviously conformal. Now let $\Phi := \rho_3 \circ \rho_2 \circ \rho_1$. It is easily verifiable that $\Phi(\mathbb{H}) = \mathbb{D}^2$, $\Phi(\mathbb{R}) = \mathbb{S}^1 \setminus \{1\}$ and $\Phi(\{\infty\}) = 1$. Also, it is easy to check that, for each $z \in \mathbb{H}$,

$$\Phi(z) = \frac{z-i}{z+i} \text{ and } \Phi^{-1}(z) = -i\frac{z+1}{z-1}.$$

Definition 6.1.9. The map $\Phi : \mathbb{H} \to \mathbb{D}^2$ is called the *Cayley transformation*.

It can be directly calculated that $d_{\mathbb{H}}(z,w) := d_h(\Phi(z), \Phi(w))$ for each $z,w \in \mathbb{H}$. Also, the group of isometries of $(\mathbb{H}, d_{\mathbb{H}})$ can be obtained by conjugating with $Con(1)$, that is,

$$Isom(\mathbb{H}) = \Phi^{-1} Con(1) \Phi.$$

Furthermore, the group of isometries of $(\mathbb{H}, d_{\mathbb{H}})$ is isomorphic to the group $PSL_2(\mathbb{R})$, where

$$PSL_2(\mathbb{R}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,b,c,d, \in \mathbb{R} \text{ and } ad - bc = 1 \right\} / \{\pm I\}.$$

The quotient here indicates that we identify the elements $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}$.

The group $PSL_2(\mathbb{R})$ acts on $\mathbb{H}$ via linear fractional transformations. That is, the action is described by the function $\psi_{\mathbb{H}}$ which is given by

$$\psi_{\mathbb{H}} : PSL_2(\mathbb{R}) \times \mathbb{H} \to \mathbb{H},$$

where for each $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL_2(\mathbb{R})$ and $z \in \mathbb{H}$, we have

$$\psi_{\mathbb{H}}(g,z) := g(z) = \frac{az+b}{cz+d}.$$

6.1.2 The cross-ratio and some hyperbolic distance estimates

Let us begin by recalling the definition of the cross-ratio.

Definition 6.1.10. The *cross-ratio* of four points x,y,z,t in $\mathbb{H} \cup \mathbb{R} \cup \{\infty\}$ is given by

$$[x,y,z,t] := \frac{(x-y)(z-t)}{(y-z)(t-x)}.$$

A straightforward calculation shows that this quantity is always equal to a real number.

Let us now consider the following situation. Let x and y be two distinct points in the upper-half plane. Suppose that either $\mathrm{Re}(x) < \mathrm{Re}(y)$, or, if $\mathrm{Re}(x) = \mathrm{Re}(y)$, suppose that $\mathrm{Im}(x) < \mathrm{Im}(y)$. Let ξ and η denote the start and end points of the oriented geodesic that joins x to y (see Figure 6.1).

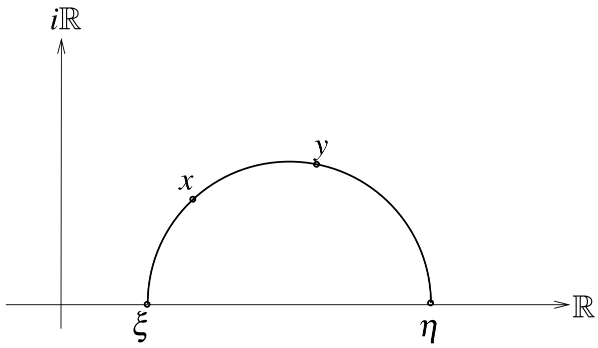


Figure 6.1: The oriented geodesic through x and y with startpoint ξ and endpoint η.

In this situation we have an extremely useful hyperbolic distance formula, which is given in the following proposition.

Proposition 6.1.11. *Let x, y, ξ and η be as described above. Then*

$$d_{\mathbb{H}}(x,y) = \log([y,\xi,x,\eta]).$$

Proof. Let $g(z) := (az+b)/(cz+d) \in PSL_2(\mathbb{R})$. Then, using the fact that $g'(z) = 1/(cz+d)^2$, it can be shown that the cross-ratio is g-invariant. In other words, for all $g \in PSL_2(\mathbb{R})$ and all distinct points $x,y,z,t \in \mathbb{H} \cup \mathbb{R} \cup \{\infty\}$, we have $[g(x),g(y),g(z),g(t)] = [x,y,z,t]$. Now define the map $g \in PSL_2(\mathbb{R})$ by setting

$$g(z) := \frac{z-\xi}{z-\eta}.$$

This map sends ξ to zero and η to $\{\infty\}$, therefore it maps the points x and y to two points on the imaginary axis, say ia and ib, respectively. Then, by Lemma 6.1.8, we have that

$$\begin{aligned} d_{\mathbb{H}}(x,y) &= d_{\mathbb{H}}(g(x),g(y)) = \log(b/a) \\ &= \log \frac{0-ib}{0-ia} = \log([ib,0,ia,\infty]) \\ &= \log([g(y),g(\xi),g(x),g(\eta)]) = \log([y,\xi,x,\eta]). \end{aligned}$$

□

6.1.3 Triangles and circles

Another feature peculiar to hyperbolic geometry is the the *double triangle inequality*. This comes from the hyperbolic cosine rule, which we now state.

Proposition 6.1.12. *With a,b and c referring to the hyperbolic lengths of the three sides of an arbitrary hyperbolic triangle and θ referring to the angle opposite the side of length c, we have*

$$\cosh c = \cosh a \cosh b - \sinh a \sinh b \cos\theta.$$

Proof. This result can be found in Section 7.12 of [7].

□

If the triangle in question is right-angled, Proposition 6.1.12 reduces to what is often known as the *hyperbolic Pythagoras' Theorem.*

Corollary 6.1.13. *With a,b and c referring to the hyperbolic lengths of the three sides of a right-angled hyperbolic triangle, where c is the length of the side opposite the right angle, we have*

$$\cosh c = \cosh a \cosh b$$

The double triangle inequality is given in the following corollary. Note that it is only valid if the angle θ is bounded away from zero.

Corollary 6.1.14. *With a,b,c and θ as above, there exists a positive constant $K := K(\theta)$ such that*

$$a+b-K \leq c \leq a+b.$$

Proof. The right-hand side of this inequality is simply the ordinary triangle inequality. For the left-hand side, from Proposition 6.1.12 and the definitions of the functions cosh and sinh, we obtain that

$$\begin{aligned} 2(e^c+e^{-c}) &= (e^a+e^{-a})(e^b+e^{-b})-(e^a-e^{-a})(e^b-e^{-b})\cos\theta \\ &\geq e^a e^b - e^a e^b \cos\theta = e^a e^b(1-\cos\theta). \end{aligned}$$

Hence, $4e^c \geq e^a e^b(1-\cos\theta)$ and so we infer that

$$c \geq a+b+\log\left(\frac{1-\cos\theta}{4}\right).$$

Therefore, by setting $K := \log\left(\frac{4}{1-\cos\theta}\right)$, the proof is finished.

□

Finally in this section, we discuss hyperbolic circles. Directly from Lemma 6.1.5, it follows that if $C_e(R)$ denotes the Euclidean circle around the origin of radius R and $C_h(r)$ denotes the hyperbolic circle around the origin of hyperbolic radius r, then $C_e(R) = C_h(r)$ if and only if $r = \log((1+R)/(1-R))$. From this, we deduce the following lemma.

Lemma 6.1.15. *With $\mathscr{L}(C_h(r))$ referring to the hyperbolic length of the hyperbolic circle $C_h(r)$, we have for every $r \in \mathbb{R}$,*

$$\mathscr{L}(C_h(r)) = 2\pi\sinh(r).$$

Proof. This is a straightforward calculation using the integral definition of hyperbolic length.

□

Corollary 6.1.16. *The statement of the previous lemma holds for any arbitrary hyperbolic circle, that is, for hyperbolic circles centred at any point $z \in \mathbb{D}^2$.*

Proof. This is proved by first noting that any hyperbolic circle is the image under some hyperbolic isometry $g \in Con(1)$ of a circle centred at the origin. It is then not difficult to show that $\mathscr{L}(C_h(r)) = \mathscr{L}(g(C_h(r)))$.

□

Corollary 6.1.17. *For all sufficiently large r,*

$$\mathscr{L}(C_h(r)) \asymp e^r.$$

This last statement should be contrasted with the corresponding statement for Euclidean circles, which is that the Euclidean length of a circle of Euclidean radius R is equal to $2\pi R$, so the length grows linearly.

6.1.4 Classification of isometries

In this section, we give a classification of hyperbolic isometries in terms of fixed points and geometric actions. For convenience, we will work in the upper half-space model, but it is to be understood that all results here are also valid in the disc model of hyperbolic space (or, for that matter, any other model).

Let $g \in PSL_2(\mathbb{R})$. Then g is of the form $g(z) = \frac{az+b}{cz+d}$ where a, b, c and d are real numbers. It is clear, on setting $g(z) = z$, that the fixed points of g are the roots of a quadratic equation with real coefficients. These will either be two points in $\mathbb{R} \cup \{\infty\}$, one point in $\mathbb{R} \cup \{\infty\}$ or complex conjugate roots, giving one fixed point inside the upper half plane. We make the following definition.

Definition 6.1.18. Each element g of $PSL(2, \mathbb{R})$ is of exactly one of the following three forms:

1. g is said to be *hyperbolic* if g has exactly two fixed points and these lie on the boundary of hyperbolic space.

2. g is said to be *parabolic* if g has exactly one fixed point that lies on the boundary of hyperbolic space.

3. g is said to be *elliptic* if g has exactly one fixed point that lies in the interior of hyperbolic space.

It is a fact that the isometries of hyperbolic space are *triply transitive*. That is, if (z_1, z_2, z_3) is any distinct triple of points in hyperbolic space or its boundary, and (z_1', z_2', z_3') is any other such triple, then there exists a unique element of $PSL_2(\mathbb{R})$ which maps z_i to z_i' for $i = 1, 2, 3$. (This is true of general Möbius transformations and can be proved via the cross-ratio.) In particular, we can map the fixed points of a hyperbolic transformation to the points $\{0, \infty\}$, the fixed point of a parabolic transformation to $\{\infty\}$ and the fixed point of an elliptic transformation to $\{i\}$. The isometry is then said to be in *standard form.*

In the sequel, we will be mostly interested in parabolic points. We have the following proposition.

Proposition 6.1.19. *Let $g \in PSL_2(\mathbb{R})$. Then g is parabolic if and only if the standard form of g is given by a translation $z \mapsto z + \beta$, for non-zero $\beta \in \mathbb{R}$. So the standard form of g maps every horizontal Euclidean straight line in $\mathbb{H}$ into itself. More generally, if g is parabolic then there exists a Euclidean circle tangent to $\mathbb{R}$ or a horizontal Euclidean straight line in $\mathbb{H}$ left invariant by g.*

Proof. Denote the standard form of g by $\hat{g} : z \mapsto (az+b)/(cz+d)$ and suppose that it is parabolic and fixes $\{\infty\}$. By considering the fixed point equation, we immediately obtain that $c = 0$. So, $\hat{g}(z) = (az+b)/d$. If $a/d \neq 1$, the map $\hat{g}$ would also fix the point $z = -b/(a-d)$. In that case, the map $\hat{g}$ would be hyperbolic, not parabolic. So, $a/d = 1$. It is also required that $ad = 1$, which implies that $a = d = \pm 1$. It follows that $\hat{g}(z) = z + b$ or $\hat{g}(z) = z - b$, for some $b > 0$. This proves the first assertion of the proposition. From this, the second statement follows.

Finally, note that if $h \in \mathrm{PSL}_2(\mathbb{R})$ is such that h does not fix the point at infinity, then the map $h\hat{g}h^{-1}$ is parabolic with fixed point $h(\{\infty\})$. One immediately verifies that the image of a Euclidean horizontal straight line under such a map h is a Euclidean circle which touches $\mathbb{R}$ at the point $h(\{\infty\})$. Consequently, all of these circles are left invariant by the map $h\hat{g}h^{-1}$.

□

These circles and straight lines are called *horoballs*. In the Poincaré disc model of hyperbolic space, the horoballs are Euclidean circles internally tangent to $\mathbb{S}^1$.

6.1.5 Fuchsian Groups

We can equip the group $PSL_2(\mathbb{R})$ with a topology inherited from $\mathbb{R}^4$ by identifying the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with the vector $(a,b,c,d) \in \mathbb{R}^4$, then defining the norm on $PSL_2(\mathbb{R})$ to be the Euclidean norm on $\mathbb{R}^4$. This norm then induces a metric, which in turn induces the metric topology. Recall that a set E in a topological space (X,τ) is *discrete* if for each $e \in E$ there exists an open subset $G \in \tau$ such that $E \cap G = \{e\}$. We make the following definition.

Definition 6.1.20. Let G be a subgroup of $PSL_2(\mathbb{R})$. Then G is said to be a *Fuchsian group* if and only if G is a discrete subset of the topological space $PSL_2(\mathbb{R})$.

Another way to describe a Fuchsian group G is in terms of properly discontinuous group actions. We say that a group G acts *properly discontinuously* on a metric space X if and only if the orbit $G(x) := \{g(x) : g \in G\}$ is locally finite for all $x \in X$. That is, given an orbit $G(x)$, every compact subset $K \subset X$ contains at most finitely many points of $G(x)$. Note that the statement that a group acts properly discontinuously is equivalent to the statement that each orbit of G is a discrete set of points.

Proposition 6.1.21. *Let G be a subset of $Con(1)$. Then G is Fuchsian group if and only if G acts properly discontinuously on $\mathbb{D}^2$.*

Proof. See Theorem 5.3.2 in [7].

□

Definition 6.1.22. Let G be a Fuchsian group. A *fundamental domain F* for G is an open subset of $\mathbb{D}^2$ such that the following conditions are satisfied.

1. $\bigcup_{g \in G} g(\overline{F}) = \mathbb{D}^2$,

2. $g(F) \cap h(F) = \emptyset$, for all $g,h \in G$ with $g \neq h$.

Thus, each fundamental domain for a Fuchsian group G gives rise to a *tessellation* of hyperbolic space. Let us now describe a particular type of fundamental domain. Let G be a Fuchsian group acting on $\mathbb{D}^2$ and let $z_0 \in \mathbb{D}^2$ be a point that is not fixed be any elliptic element of the group G. Then the *Dirichlet fundamental domain $D_{z_0}(G)$* of G at the point z_0 is given by

$$D_{z_0}(G) := \{z \in \mathbb{D}^2 : d_h(z,z_0) < d_h(z,g(z_0)) \ \forall g \in G/\{id\}\}.$$

Alternatively, for each $g \in G/\{id\}$ consider the perpendicular bisector of the geodesic segment joining z_0 and $g(z_0)$. This divides $\mathbb{D}^2$ into two half-spaces. With H_g referring to the half-space containing z_0, we have that

$$D_{z_0}(G) = \bigcap_{g \in G/\{id\}} H_g.$$

That D_{z_0} really is a fundamental domain requires proof, but we leave this to the reader. As usual, this definition could equally well have been written in terms of $\mathbb{H}$.

Example 6.1.23. Consider the group

$$PSL_2(\mathbb{Z}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,b,c,d, \in \mathbb{Z} \text{ and } ad-bc=1 \right\} / \{\pm I\}.$$

This group is referred to as the *modular group*. The modular group is generated by one parabolic element P and one elliptic element Q, where

$$P(z) = z+1 \quad \text{and} \quad Q(z) = -\frac{1}{z}.$$

It is clear that this is a discrete subgroup of $PSL_2(\mathbb{R})$. It can be shown that the region bounded by the lines $\mathrm{Re}(z) = 1/2$, $\mathrm{Re}(z) = -1/2$ and the unit circle is the Dirichlet fundamental domain at the point $z_0 = 2i$ for the modular group. (A proof of this fact is contained in Appendix B.)

We can use the notion of a Dirichlet fundamental domain to obtain an important theorem for Fuchsian groups. In the following discussion, let $D_{z_0}(G)$ refer to a Dirichlet fundamental domain for a Fuchsian group G constructed at the base point z_0. The region $D_{z_0}(G)$ is a hyperbolic polygon in $\mathbb{D}^2 \cup \mathbb{S}^1$ (in a wider sense than is usual, since we allow vertices and edges on $\mathbb{S}^1$ and allow the possibility that there be infinitely many edges) . Let us consider the edges which bound the polygon $D_{z_0}(G)$, where we let s_g denote the edge that is part of the perpendicular bisector of the segment joining z_0 to $g(z_0)$. Observe that

$$z \in s_g \;\Leftrightarrow\; d_h(z,z_0) = d_h(z,g(z_0)) \;\Leftrightarrow\; d_h(g^{-1}(z),g^{-1}(z_0)) = d_h(g^{-1}(z),z_0) \;\Leftrightarrow\; g^{-1}(z) \in s_{g^{-1}}.$$

In other words, we have that for the edges s_g which bound the polygon $D_{z_0}(G)$ we have that

$$g^{-1}(s_g) = s_{g^{-1}} \quad \text{and} \quad g(s_{g^{-1}}) = s_g.$$

We refer to these identifications of edges under elements of G as the *side-pairing transformations* of G. Note that if g is a parabolic element that is also a side-pairing transformation, then the sides that are paired by g will meet at the fixed point of g.

Theorem 6.1.24. *The side-pairing transformations of a Fuchsian group G for a Dirichlet fundamental domain $D_{z_0}(G)$ are generators of the group G.*

Proof. See Theorem 9.3.3 in [7].

□

Definition 6.1.25. A Fuchsian group G is said to be *geometrically finite* if there exists a fundamental domain for G with only finitely many edges.

From the example given above of a fundamental region for the modular group, it is apparent that the modular group is geometrically finite.

Remark 6.1.26. A Fuchsian group G is geometrically finite if and only if G is finitely generated. This can be deduced immediately from Theorem 6.1.24. (The equivalence no longer holds for discrete groups of isometries of higher dimensional hyperbolic spaces.)

Recall that a *Riemann surface* is a connected, analytic, complex 1-dimensional manifold. A Riemann surface S is called *simply connected* if every closed curve on S can be continuously deformed into a single point (so the surface of the 2-sphere is simply connected, whereas the torus is not). It is a very deep theorem in the theory of complex functions - the Riemann Mapping Theorem, sometimes called the First Uniformization Theorem - that every simply connected Riemann surface is conformally equivalent to one of $\mathbb{C}$, $\mathbb{C}\cup\{\infty\}$ or $\mathbb{D}^2$. Further, the Second Uniformization Theorem states that every Riemann surface S is conformally equivalent to a quotient $\tilde{S}/G$ for some simply connected Riemann surface $\tilde{S}$ and for some group G of conformal automorphisms which acts properly discontinuously on $\tilde{S}$. The quotient $\tilde{S}/G$ comprises equivalence classes of points in $\tilde{S}$, where two points are equivalent if and only if they belong to the same G-orbit. If we are in the case where $\tilde{S}$ is conformally equivalent to $\mathbb{D}^2$, then every properly discontinuous group G is a Fuchsian group. So, here we always have that a Riemann surface conformally equivalent to $\mathbb{D}^2/G$ is represented by a fundamental domain for the action of G. We can also think of this the other way around - that every Fuchsian group G has an associated Riemann surface, obtained by "gluing" the edges of a fundamental domain F for G.

In the figures below, we illustrate various types of surfaces obtainable as the Riemann surface associated to a Fuchsian group. Figure 6.2 shows a compact surface, which occurs when the fundamental domain of the group G does not have any vertices on $\mathbb{S}^1$ and also shows an example of a surface with *funnels*. This happens when the fundamental domain has edges contained in $\mathbb{S}^1$. Finally, going back to the example of the modular group, in Figure 6.3 we see the *modular surface*. This surface has what is known as a *cusp*, which happens when the group G contains a parabolic element. There is then a parabolic fixed point as a vertex of the fundamental domain.

6.1.6 The Limit Set of a Fuchsian Group

Definition 6.1.27. Let $w \in \mathbb{D}^2$ (or $\mathbb{H}$) be given. Then the limit set $L(G)$ of the Fuchsian group G is the set

$$L(G) := \{\xi \in \mathbb{C}\cup\{\infty\} : \xi \text{ is an accumulation point of the orbit } G(w)\}.$$

In fact, the limit set is independent of the choice of w in this definition. This definition implies that the limit set of a Fuchsian group is always a closed set. We also have that the limit set is G-invariant, meaning that $g(L(G)) = L(G)$ for each g in G. It is a consequence of the discontinuous

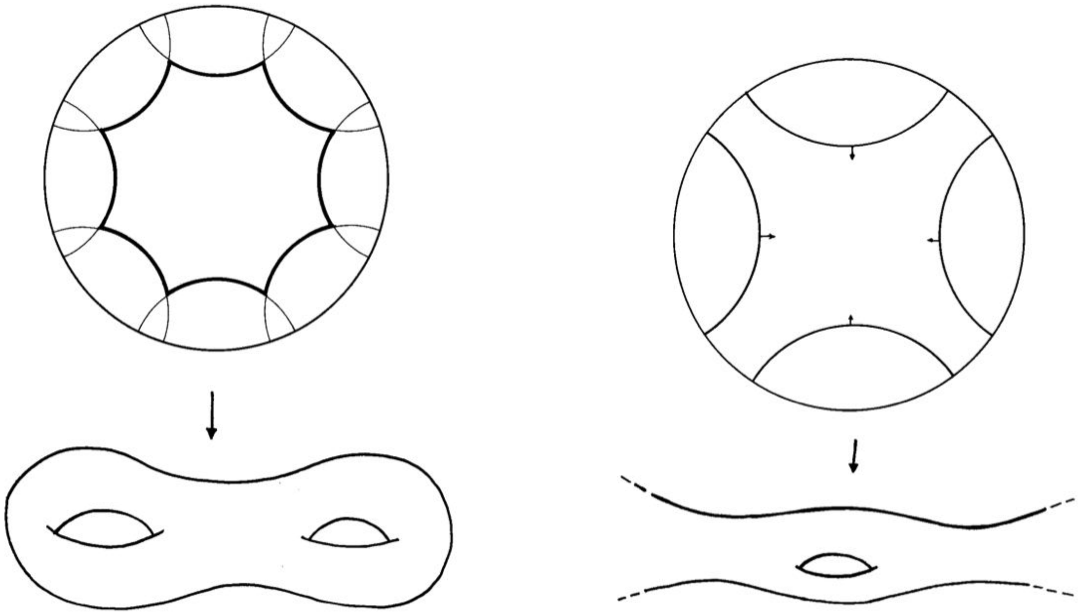


Figure 6.2: On the left is an example of a Fuchsian group whose associated Riemann surface is compact. On the right is an example of a Fuchsian group whose fundamental domain has edges at infinity, which leads to a Riemann surface with funnels.

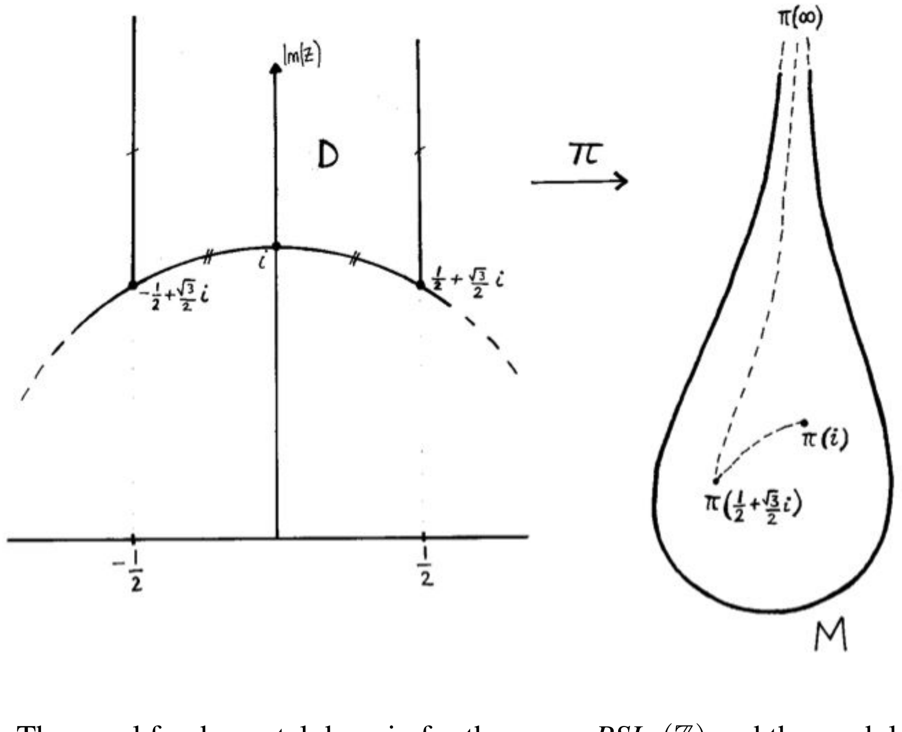


Figure 6.3: The usual fundamental domain for the group $PSL_2(\mathbb{Z})$ and the modular surface.

action of a Fuchsian group that $L(G) \subseteq \mathbb{S}^1$. It is a well-known fact that if $L(G)$ has more than two points, then $L(G)$ has uncountably many points. We say that G is *elementary* if $L(G)$ is either empty (so G generated only by elliptic elements), or consists of only one or two points (so G generated by either a single parabolic element or a single hyperbolic element). Otherwise, G is said to be *non-elementary*. From this point on, we always assume that G is non-elementary.

Let us now define certain subsets of the limit set $L(G)$. First we fix some notation. Let s_ξ denote the hyperbolic ray from the origin to the point $\xi \in \mathbb{S}^1$ and, for $t \in \mathbb{R}$, let ξ_t be the point on s_ξ such that $d_h(0, \xi_t) = t$. Also, for a Fuchsian group G and $t > 0$, define $\Delta(\xi_t)$ by setting $\Delta(\xi_t) := d_h(\xi_t, G(0))$. In other words, $\Delta(\xi_t)$ is the smallest hyperbolic distance from the point ξ_t to an orbit point of 0.

Definition 6.1.28. Let G be a Fuchsian group. A point $\xi \in L(G)$ is said to be a *radial limit point* if there exists a positive constant c such that

$$\liminf_{t\to\infty} \Delta(\xi_t) < c.$$

Denote the set of radial limit points by $L_r(G)$. A point $\eta \in L(G)$ is said to be a *uniformly radial limit point* if there exists a positive constant c such that

$$\limsup_{t\to\infty} \Delta(\xi_t) < c.$$

Denote the set of uniformly radial limit points by $L_{ur}(G)$. Finally, let $L_p(G)$ denote the set of *parabolic limit points*, where a point p is parabolic if it is the fixed point of some parabolic map in G.

Geometrically, a point $\xi \in L(G)$ is a radial limit point if the ray s_ξ intersects infinitely many balls of radius c around orbit points of 0 and ξ is a uniformly radial limit point if the ray s_ξ is covered by such balls. Note that a parabolic fixed point cannot also be a radial limit point.

It is sometimes helpful to imagine each limit point ξ of G as represented by the geodesic ray s_ξ from 0 to ξ. So, if $L(G)$ is the whole of $\mathbb{S}^1$, every geodesic direction from 0 represents a limit point. If $L(G)$ is a proper subset of $\mathbb{S}^1$, certain directions do not represent limit points. On the Riemann surface $M(G)$ associated to G, the limit points are represented by all those geodesics which do not escape out of a funnel. The parabolic limit points are represented by geodesics which end up in a cusp.

The following result is due to A.F. Beardon and B. Maskit [8].

Theorem 6.1.29. *Let G be a Fuchsian group. Then G is geometrically finite if and only if*

$$L(G) = L_r(G) \cup L_p.$$

Definition 6.1.30.

1. For any Fuchsian group G, the *Poincaré series* is defined for $s \in \mathbb{R}$ and $x, y \in \mathbb{D}^2$ to be

$$\sum\nolimits_s(x,y) := \sum_{g \in G} e^{-s d_h(x, g(y))}.$$

2. The *exponent of convergence* $\delta(G)$ of a group G is defined to be the infimum of all those s for which the Poincaré series converges. That is,

$$\delta(G) := \delta := \inf\{s \in \mathbb{R}^+ : \sum_s(x,y) < \infty\}.$$

More explicitly,

$$\sum_s(x,y) = \begin{cases} \infty & \text{for } s < \delta; \\ < \infty & \text{for } s > \delta. \end{cases}$$

From the triangle inequalities

$$d_h(x,g(y)) \leq d_h(x,y) + d_h(y,g(y)) \text{ and } d_h(x,g(y)) \geq d_h(y,g(y)) - d_h(x,y),$$

we can see that

$$e^{-sd_h(x,y)} \sum_s(y,y) \leq \sum_s(x,y) \leq e^{sd_h(x,y)} \sum_s(y,y),$$

so the convergence depends only upon G and not upon the points x and y and we are justified in writing simply $\delta(G)$. Note that without some more information we do not know if the Poincaré series diverges or converges at δ. So we define a group G to be of *divergence type* if the Poincaré series $\sum_s(x,y)$ diverges at the critical exponent $s = \delta$ and say a group G is of *convergence type* otherwise. It was proved by Sullivan [79] that if G is a geometrically finite Fuchsian group, then G is of divergence type. The proof is decidedly non-trivial and we do not reproduce it here. This divergence at the critical exponent is important for the definition of the Patterson measure (see Appendix A).

It is a result of Bishop and Jones [12] (see also the paper of Stratmann [77]), that for *any* non-elementary Fuchsian group G,

$$\dim_H(L_{ur}(G)) = \dim_H(L_r(G)) = \delta(G).$$

Combining this result with Theorem 6.1.29 above implies that if G is a geometrically finite Fuchsian group, then $\dim_H(L(G)) = \delta(G)$. Finally, let us mention another result proved by Beardon [6].

Theorem 6.1.31. *Let G be a non-elementary geometrically finite Fuchsian group with at least one parabolic element. Then, we have that $\delta(G) > \frac{1}{2}$.*

As a direct corollary of Theorem 6.1.31 and the discussion above, for each geometrically finite Fuchsian group G with at least one parabolic element, it follows that

$$\dim_H(L(G)) > \frac{1}{2}. \tag{6.1}$$

6.2 Standard horoballs

From this point on, let G be a non-elementary, geometrically finite Fuchsian group which has one parabolic element, say γ, in its generating set. Let p be the fixed point of the isometry γ, so that $\gamma(p)=p$. Let F_G be a fundamental domain for G, fixed so that F_G contains the origin of $\mathbb{D}^2$. Denote by $C(L(G))$ the intersection of all convex sets containing the set of geodesics in $\mathbb{D}^2$ with both endpoints belonging to the set $L(G)\times L(G)\setminus\{(x,x):x\in L(G)\}$.

Let us now describe a certain set of horoballs associated to the orbit of the parabolic fixed point p under the group G, called a *standard set* of horoballs. This was first introduced, in a more general situation, by Stratmann and Velani [78]. Assign a horoball H_γ to the point p, and let H_g be the image of H_γ under the map $g\in G$. Note that if the map g belongs to the stabiliser G_p of p, which is given by $G_p:=\{g\in G:g(p)=p\}$, then the horoball H_g is equal to H_γ. It is well known that the set $\{H_g:g\in G/G_p\}$ can be chosen in such a way that it is a pairwise disjoint collection of horoballs. This set is a standard set of horoballs for G. From now on, let $\{H_g:g\in G/G_p\}$ denote a fixed standard set of horoballs for G.

Let s_ξ denote the hyperbolic half-ray between the origin and the point ξ on $\mathbb{S}^1$. We will think of this ray as having an orientation, so that we travel from 0 towards $\mathbb{S}^1$. Define the *top* of the standard horoball H_g to be the first point on the boundary of H_g reached whilst traveling along $s_{g(p)}$, that is,

$$\tau_g := s_{g(p)}\cap\partial H_g\cap\mathbb{D}^2.$$

It was shown in [78] that the point τ_g lies a bounded distance away from the orbit of the origin under G. For completeness, we include the proof here in the Fuchsian groups setting.

Lemma 6.2.1. *There exists a positive constant ρ, depending only on G, such that for each $g(p)\in G(p)$, there exists $f\in G$ such that*

$$d_h(\tau_g,f(0))\le\rho.$$

Proof. First, note that the each of the sets representing $(C(L(G))\cap\partial H_\gamma)/G_p$ is compact. Then, from this fact and the fact that τ_γ lies in $C(L(G))$, it follows that there exists a compact arc $K(p)$ of $\partial H_\gamma\setminus\{p\}$ containing τ_γ, as shown in Figure 6.4. Denote by d_K the diameter of $K(p)$ with respect to the hyperbolic metric and let t_p denote the distance from the origin to τ_γ, that is, $t_p:=d_h(0,\tau_\gamma)$. Fix $g\in G$. Then $g(K(p))$ is a compact subset of ∂H_g containing the point $g(\tau_\gamma)$, see Figure 6.4. The point τ_g is contained in $C(L(G))$ and, since $C(L(G))$ is a G-invariant set, the point $g(\tau_\gamma)$ is also contained in $C(L(G))$. Thus, there exists some map h in the stabiliser $G_{g(p)}=gG_pg^{-1}$ such that both τ_g and $h\circ g(\tau_\gamma)$ lie in $h\circ g(K(p))$. Note that the diameter of $h\circ g(K(p))$ is equal to d_K. It follows that

$$\begin{aligned} d_h(\tau_g,h\circ g(0)) &\le d_h(\tau_g,h\circ g(\tau_\gamma))+d_h(h\circ g(\tau_\gamma),h\circ g(0))\\ &= d_h(\tau_g,h\circ g(\tau_\gamma))+d_h(\tau_\gamma,0)\\ &\le d_K+t_p. \end{aligned}$$

Setting $f:=h\circ g$ and $\rho:=d_K+t_p$, the proof is finished.

□

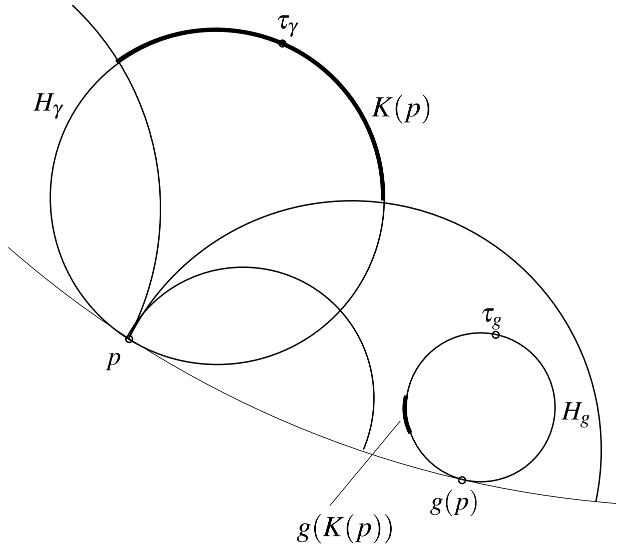


Figure 6.4: Illustration of the set $K(p)$ mapped under g to the horoball H_g.

Examining the proof of Lemma 6.2.1, we see that it is possible to choose the map g in such a way that $g(K(p))$ and the intersection of ∂H_g with the image of F_G containing τ_g are one and the same thing. We can now choose a set $\mathfrak{T}$ of coset representatives of G/G_p in a geometric way, namely, let g be in $\mathfrak{T}$ if the orbit point $g(0)$ lies in a ρ-neighbourhood of τ_g, the top of the horoball H_g, where ρ comes from Lemma 6.2.1. That is

$$g \in \mathfrak{T} \Rightarrow d_h(\tau_g, g(0)) \leq \rho.$$

We refer to this as the *top representation* and from here on we will write $\{H_g : g \in \mathfrak{T}\}$ for a fixed standard set of horoballs for G with top representation.

Definition 6.2.2. The *shadow map* $\Pi : \mathscr{P}(\mathbb{D}^2) \to \mathscr{P}(\mathbb{S}^1)$ is defined by

$$\Pi(A) := \{\xi \in \mathbb{S}^1 : s_\xi \cap A \neq \emptyset\}.$$

Here, $\mathscr{P}(X)$ denotes the set of all subsets of X.

We now give an estimate of the size of the shadow of the standard horoball H_g. Recall that two functions $f, g : \mathbb{R} \to \mathbb{R}$ are said to be *comparable*, denoted $f \asymp g$, if there exists a universal constant $c \geq 1$ such that for every $x \in \mathbb{R}$ we have $c^{-1}g(x) \leq f(x) \leq cg(x)$. If we want to refer only to one side of this inequality, we write either $f \ll g$ or $g \ll f$.

Proposition 6.2.3. *For every standard horoball with top representation from the set* $\{H_g : g \in \mathfrak{T}\}$ *we have, where* $|\Pi(H_g)|$ *refers to the length of* $\Pi(H_g) \subset \mathbb{S}^1$*, that*

$$|\Pi(H_g)| \asymp e^{-d_h(0,\tau_g)}.$$

Proof. Firstly, notice that the Euclidean radius r_g of H_g is given by $2r_g = 1 - |\tau_g|$. Then, from Lemma 6.1.5 we obtain that

$$r_g \asymp e^{-d_h(0,\tau_g)}.$$

Secondly, if we draw the hyperbolic circle $C_h(1-r_g)$, centred at zero and with radius $1-r_g$, from Corollary 6.1.17 we obtain for the length of this circle that

$$\mathscr{L}(C_h(1-r_g)) \asymp e^{1-r_g}.$$

This circle can be covered by approximately $e^{1-r_g}/2r_g$ balls of radius r_g, so the corresponding shadow of each of these balls must be comparable to $2\pi r_g e^{-(1-r_g)}$. Since $e^{-1} < e^{-(1-r_g)} < 1$, we finally obtain the required comparability, namely,

$$|\Pi(H_g)| \asymp r_g \asymp e^{-d_h(0,\tau_g)}.$$

□

Remark 6.2.4. Notice that combining Lemma 6.2.1 and the double triangle inequality with the above proposition yields the corollary that

$$|\Pi(H_g)| \asymp e^{-d_h(0,g(0))}.$$

Recall the stabiliser G_p of the parabolic point p, which is given by $G_p := \{g \in G : g(p) = p\}$. For example, in the case of the map $z \mapsto z+1$ which fixes the point at infinity, the stabiliser is given by $G_{\{\infty\}} = \{g(z) := z+n : n \in \mathbb{Z}\}$. We began this section by letting F_G be a fundamental region for G with the property that one vertex of F_G is equal to p. In the tessellation of hyperbolic space given by the region F_G, each map in the stabiliser of p sends F_G to a region that also has one vertex equal to p. This structure is illustrated for the map $P : z \mapsto z+1$ in the upper half-plane and also for the Poincaré disc equivalent in Figure 6.5 below. We will refer to the countably many copies of F_G that cluster down to each point in $G(p)/G_p$ as *petals*. In somewhat of an abuse of notation, we will use this word interchangeably to mean the arc of $\mathbb{S}^1$ enclosed by the two edges of the petal.

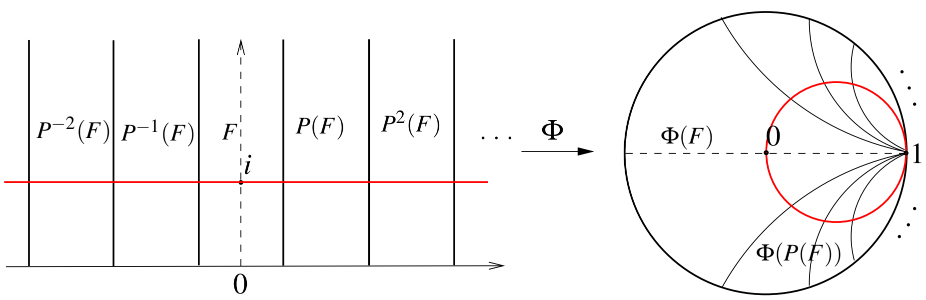


Figure 6.5: Petals around the parabolic point at infinity on the boundary of $\mathbb{H}$ and the point 1 on the boundary of $\mathbb{D}^2$.

Our next aim is to obtain an estimate of the size of the petals around the point $g(p)$ for all $g \in G$. This will be achieved with the help of the cross-ratio formula for distances given in Proposition 6.1.11. Let us first consider the petals around the point p itself. First of all, without loss of generality, suppose that the top of the horoball H_γ is actually at 0; this will not alter any of the estimates by any more than a constant amount, due to Lemma 6.2.1. Then, let r denote the rotation around 0 that moves p to 1. This rotates the entire horoball H_γ. Now send this rotated picture into the upper half-plane, by way of the inverse of the Cayley transformation. This procedure sends the map γ to a parabolic element of $PSL_2(\mathbb{R})$ in standard form, that is, in the form $z \mapsto z+\beta$, for $\beta \neq 0$. Again without loss of generality, suppose that $\beta = 1$. We will now make a particular hyperbolic distance estimate, which is illustrated in Figure 6.6. The notation"$a \asymp_+ b$" means that there exists a constant $K > 0$ such that $a - K \leq b \leq a + K$.

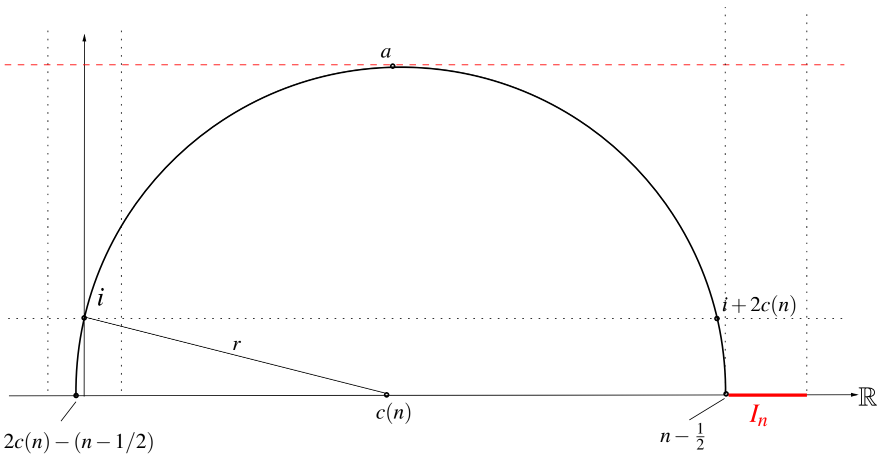


Figure 6.6: Illustration of the hyperbolic geodesic joining i to $i+2c$.

Lemma 6.2.5. *As in Figure 6.6, with $c(n)$ referring to the centre of the circle whose top half forms the geodesic in $\mathbb{H}$ joining i to $n-1/2$, we have that*

$$d_h(i, i+2c(n)) \asymp_+ 2\log n.$$

Proof. To shorten the notation, let $c := c(n)$. By Proposition 6.1.11, we have that

$$\begin{aligned} d_h(i,i+2c) &= \log([i+2c, 2c-(n-1/2), i, n-1/2]) \\ &= \log\left(\frac{(i+2c-2c+(n-1/2))(i-(n-1/2))}{(2c-(n-1/2)-i)(n-1/2-(i+2c))}\right) \\ &= \log\left(\frac{1+(n-1/2)^2}{(1+(2c-(n-1/2))^2)}\right). \end{aligned}$$

Let r denote the radius of the semi-circle forming the geodesic joining i to $n-1/2$. Then, on the one hand, we have that $r^2 = 1 + c^2$, but on the other hand, $r = n - 1/2 - c$. So, after some elementary algebra, we obtain that $c = (4n^2 - 4n - 3)/(4(2n-1))$. It follows that

$$2c - (n - 1/2) = \frac{-2}{2n-1} \text{ and so, } 1 \le 1 + (2c - (n-1/2))^2 \le 5. \tag{6.2}$$

Also, $1 + (n-1/2)^2 = n^2 - n + 5/4$, which implies that

$$\frac{n^2}{2} \le 1 + (n-1/2)^2 \le n^2, \tag{6.3}$$

provided that $n \ge 2$. Combining (6.2) and (6.3) yields that

$$\log(n^2) - \log 10 \le d_h(i, i+2c) \le \log(n^2).$$

This finishes the proof.

□

Note that where a refers to the point at the top of the semi-circle forming the geodesic between i and $n - 1/2$, as in Figure 6.6 above, it immediately follows that

$$d_h(i,a) \asymp_+ \log n. \tag{6.4}$$

Remark 6.2.6. The cross-ratio can be used in a similar way to find numerous other estimates of this type. For instance, we can also show that $d_h(i, i+n) \asymp_+ 2\log n$.

If we now return to the situation we were in before Lemma 6.2.5, that is, if we return to the picture of the petals around the parabolic point $p \in \mathbb{S}^1$, we are now in a position to estimate the size of these petals. Let us denote the petal containing 0 by I_0 and then call the petals $(I_n)_{n\in\mathbb{N}}$ in sequence as they move around $\mathbb{S}^1$ to the point p. We are being a little vague here, because there are actually two sequences of petals clustering down to p, but since they are completely symmetric there is no real problem. We have for the hyperbolic distance between 0 and the point $r^{-1} \circ \Phi(i+2c)$, where r is the rotation bringing p to 1 and c is as in Lemma 6.2.5, that

$$d_h(0, r^{-1} \circ \Phi(i+2c)) \asymp_+ \log(n^2).$$

We also have, where a is as in (6.4), that

$$d_h(0, r^{-1} \circ \Phi(a)) \asymp_+ \log n.$$

Note that $r^{-1} \circ \Phi(a)$ lies on a horoball at p whose shadow contains all the petals I_k for $k \ge n$. We can calculate that the hyperbolic distance from 0 to the top of this horoball is also comparable to $\log n$, either using the double triangle inequality or directly via Lemma 6.1.8. (To make this clearer, see Figure 6.7, below.) From this and from Proposition 6.2.3 it immediately follows that the size of the shadow of this horoball is comparable to $1/n$. Given that this holds for every $n \in \mathbb{N}$, we finally obtain that

$$|I_n| \asymp \frac{1}{n^2}. \tag{6.5}$$

Let us now consider an arbitrary horoball H_g from the standard set with top representation $\{H_g : g \in \mathfrak{T}\}$. Then $H_g = g(H_\gamma)$ and we will assume, again without loss of generality (in light of Lemma 6.2.1), that $\tau_g = g(0)$. As already mentioned, each image of the parabolic point p has the same petal structure. It is then straightforward to calculate in a similar way to that above that if $I_n^{(g)}$ denotes the n-th petal around $g(p)$, we have that

$$\left| I_n^{(g)} \right| \asymp e^{-d_h(0,g(0))} \cdot \frac{1}{n^2}. \tag{6.6}$$

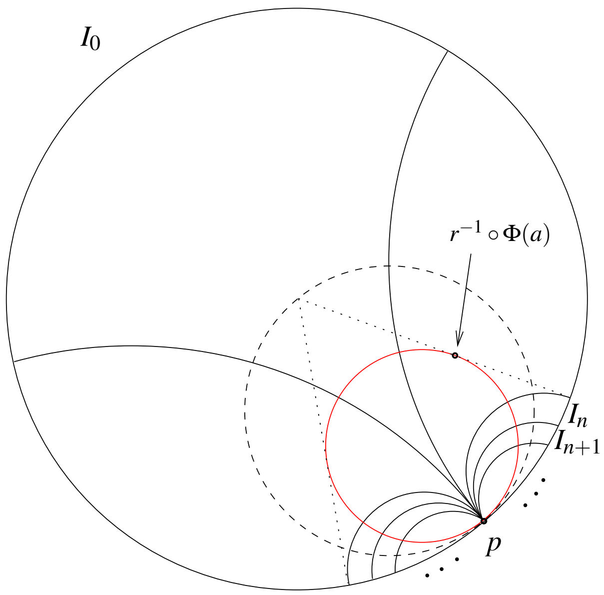


Figure 6.7: Illustration of the petals around p and the horoball used to obtain the estimate of the size of I_n.

6.3 Good sets

In this section, we give the first of our results concerning the Hausdorff dimension of certain subsets of the limit set of a non-elementary, geometrically finite Fuchsian group. The first definition describes the sets we are interested in.

Definition 6.3.1.

1. Let $\mathscr{L}(G)$ denote the set of all those $\xi \in L(G)$ with the property that the ray s_ξ intersects infinitely many standard horoballs $H_{g_1}(\xi), H_{g_2}(\xi), H_{g_3}(\xi), \ldots$, which we always assume to be ordered according to their appearance when traveling from 0 to ξ.

2. We call the distance traveled by a ray s_ξ inside a standard horoball a *cusp excursion*. For each $\xi \in \mathscr{L}(G)$, let $d_n(\xi)$ denote the depth of the n-th cusp excursion, that is,

$$d_n(\xi) := \max\{d_h(\eta, \partial H_{g_n}(\xi)) : \eta \in s_\xi \cap \mathrm{Int}(H_{g_n}(\xi))\}.$$

3. For $\kappa > 0$, let $\mathscr{B}_\kappa$ denote the set of all those $\xi \in \mathscr{L}(G)$ with the property that the distance traveled between each cusp excursion is bounded by κ. In other words, $\mathscr{B}_\kappa$ is defined to be

$$\mathscr{B}_\kappa(G) := \{\xi \in \mathscr{L}(G) : d_h(H_{g_n}(\xi), H_{g_{n+1}}(\xi)) < \kappa, \text{ for all } n \in \mathbb{N}\}.$$

4. Now, for $\kappa, \tau > 0$, define the (τ, κ)-*Good set* by

$$\mathscr{C}_{\tau,\kappa}(G) := \{\xi \in \mathscr{B}_\kappa(G) : d_n(\xi) > \log\tau, \text{ for all } n \in \mathbb{N}\}.$$

   Then, finally, let the τ-*Good set* be given by

$$\mathscr{C}_\tau(G) := \bigcup_{\kappa>0} \mathscr{C}_{\tau,\kappa}(G).$$

We now come to the main result of this section, which is to give an estimate of the Hausdorff dimension of the τ-Good set $\mathscr{C}_\tau(G)$. We have the following theorem.

Theorem 6.3.2.

$$\lim_{\tau\to\infty} \dim_H(\mathscr{C}_\tau(G)) = \frac{1}{2}.$$

Proof. First let $\kappa > 0$ be given and let us consider $\dim_H(\mathscr{C}_{\tau,\kappa}(G))$. We begin with the upper bound. Notice that the set $\mathscr{C}_{\tau,\kappa}(G)$ can be covered by any of the families

$$\{\Pi(H_{g_1}(\xi)) : \xi \in \mathscr{C}_{\tau,\kappa}(G)\},$$

$$\vdots$$

$$\left\{\Pi(H_{g_k}(\xi)) : \xi \in \mathscr{C}_{\tau,\kappa}(G)\right\}$$

$$\vdots$$

Eventually, if k is chosen sufficiently large, the cover $\{\Pi(H_{g_k}(\xi)) : \xi \in \mathscr{C}_{\tau,\kappa}(G)\}$ will consist of sets of diameter less than any fixed positive δ. Now, let $s = \frac{1}{2}(1+\varepsilon_\tau)$, where ε_τ is chosen such that $\varepsilon_\tau < 1$ and $\varepsilon_\tau(\lfloor\tau\rfloor - 1)^{\varepsilon_\tau} > 1$. (This is certainly possible. As an example, for any $\tau \in [3,5)$, let $\varepsilon_\tau = 3/4$ and for any $\tau \in [n^n+1, (n+1)^{(n+1)}+1)$, let $\varepsilon_\tau = 1/n$, for $n \geq 2$.) Then, note that for each $\xi \in \mathscr{B}_\kappa(G)$, the shadows of the standard horoballs intersected by the ray s_ξ form a nested sequence of intervals of $\mathbb{S}^1$ which cluster down to the point ξ. We can associate to the sequence of horoballs a sequence of positive integers $a_1(\xi), a_2(\xi), \ldots$ with the property that

$$\log(a_n(\xi)) \leq d_n(\xi) < \log(a_n(\xi)+1).$$

Consequently, for any $\xi \in \mathscr{B}_\kappa(G)$ and any $n \in \mathbb{N}$, by Proposition 6.2.3 above, we have the following estimate.

$$\frac{1}{e^{(n+1)\kappa}((a_1(\xi)+1)\dots(a_n(\xi)+1))^2} \ll |\Pi(H_{g_n}(\xi))| \ll \frac{1}{(a_1(\xi)\dots a_n(\xi))^2}. \tag{6.7}$$

It then follows, by the choice of ε_τ, that

$$\begin{aligned}
\mathscr{H}^s_\delta(\mathscr{C}_{\tau,\kappa}(G)) &\leq \sum_{\xi\in\mathscr{C}_{\tau,\kappa}(G)} |\Pi(H_{g_k}(\xi))|^s \\
&\ll \sum_{a_1\geq\lfloor\tau\rfloor} \frac{1}{a_1^{2s}} \left(\sum_{a_2\geq\lfloor\tau\rfloor} \frac{1}{a_2^{2s}} \cdots \left(\sum_{a_k\geq\lfloor\tau\rfloor} \frac{1}{a_k^{2s}} \right) \cdots \right) \\
&\ll \left(\int_{\lfloor\tau\rfloor}^{\infty} \frac{1}{x^{2s}}\, dx \right)^k \\
&= \left(\frac{1}{\varepsilon_\tau(\lfloor\tau\rfloor-1)^{\varepsilon_\tau}} \right)^k < 1.
\end{aligned}$$

As this is true for any arbitrary $\delta > 0$, it follows that $\mathscr{H}^s(\mathscr{C}_{\tau,\kappa}(G))$ is finite and consequently that $\dim_H(\mathscr{C}_{\tau,\kappa}(G)) \leq s = \frac{1}{2}(1+\varepsilon_\tau)$. If for each τ we then choose ε_τ such that $\lim_{\tau\to\infty}\varepsilon_\tau = 0$, we obtain the desired upper bound, that is,

$$\lim_{\tau\to\infty} \dim_H(\mathscr{C}_{\tau,\kappa}(G)) \leq \frac{1}{2}.$$

For the lower bound, again fix $\tau \geq 3$ and $\kappa > 0$. We first describe a subset of the set in question and then employ Frostman's Lemma to estimate from below the dimension of this subset. So, to that end, choose τ' to satisfy the equation

$$\sum_{i=\lfloor\tau\rfloor}^{\lfloor\tau'+1\rfloor} \frac{1}{i+1} > e^{\kappa/2}.$$

Denote this sum by S. Let $\mathscr{C}_{\tau',\kappa}(G)$ be the set

$$\mathscr{C}_{\tau',\kappa}(G) := \{\xi \in \mathscr{B}_\kappa : \log\tau < d_n(\xi) \leq \log\tau' \text{ for all } n \in \mathbb{N}\}.$$

Let ν be a measure supported on the limit set $L(G)$ with the property that

$$\nu(\Pi(H_{g_k}(\xi))) = \frac{1}{S^k} \cdot \frac{1}{(a_1(\xi)+1)\dots(a_k(\xi)+1)} \leq \frac{c_1\, e^{((k+1)\kappa)/2}}{S^k} |\Pi(H_{g_k}(\xi))|^{\frac{1}{2}},$$

where c_1 is a constant. Note that by the choice of τ', the term $c_1 \left(e^{\kappa/2}/S\right)^k e^{\kappa/2}$ is simply another constant, say c_2. Now, let $\xi \in \mathscr{C}_{\tau',\kappa}$ and let $r > 0$. Then choose the first k such that the shadow of the $(k+1)$th level horoball $H_{g_{k+1}}(\xi)$ is at most equal to r, that is, choose k such that

$$|\Pi(H_{g_{k+1}}(\xi))| \leq r < |\Pi(H_{g_k}(\xi))|.$$

Note that for each $k \in \mathbb{N}$ there can only be a fixed finite number of petals around any point $g_k(p)$ in which it is possible for a point in the set $\mathscr{C}_{\tau',\kappa}(G)$ to end up. Therefore there can only be a fixed finite number of horoballs that $B(\xi,r)$ could possibly intersect at each level k and furthermore, each of these shadows has comparable ν-measure, so, without loss of generality we suppose that

$$\Pi(H_{g_{k+1}}(\xi)) \subset B(\xi,r) \subset \Pi(H_{g_k}(\xi)).$$

We now need to compare the sizes of the above shadows. Directly from Proposition 6.2.3, we obtain that

$$\frac{|\Pi(H_{g_k}(\xi))|}{|\Pi(H_{g_{k+1}}(\xi))|} \asymp e^{d_h(0,\tau_{g_{k+1}}(\xi))-d_h(0,\tau_{g_k}(\xi))}.$$

It can be shown, via the cross-ratio distance formula again, that if z_{g_k} denotes the point that the geodesic segment joining 0 and $\tau_{g_{k+1}}$ first intersects the horoball $H_{g_k}(\xi)$, then $d_h(0,\tau_{g_k}) \asymp d_h(0,z_{g_k})$. It follows that

$$d_h(0,\tau_{g_{k+1}}) \asymp d_h(0,\tau_{g_k}) + 2d_k(\xi) + \kappa.$$

Therefore, keeping in mind that $d_k(\xi) \leq \log \tau'$, we obtain that

$$d_h(0,\tau_{g_{k+1}}(\xi)) - d_h(0,\tau_{g_k}(\xi)) \ll \log \tau' + \kappa.$$

Finally, then, since τ' is fixed for each τ, there exists another constant c_3 such that $|\Pi(H_{g_k}(\xi))| \leq (c_3)^2|\Pi(H_{g_{k+1}}(\xi))|$. Then,

$$\begin{aligned}
\nu(B(\xi,r) &\leq \nu(\Pi(H_{g_k}(\xi))) \leq c_2|\Pi(H_{g_k}(\xi))|^{\frac{1}{2}} \\
&\leq c_2c_3|\Pi(H_{g_{k+1}}(\xi))|^{\frac{1}{2}} \\
&\leq c_4 \cdot r^{\frac{1}{2}},
\end{aligned}$$

where $2c_4 := c_2c_3$. Thus, by Frostman's Lemma, we have that $\dim_H(\mathscr{C}_{\tau',\kappa}(G)) \geq 1/2$ and therefore $\dim_H(\mathscr{C}_{\tau,\kappa}(G) \geq 1/2$, too. Combining this and the upper bound obtained previously, we have, for every $\kappa > 0$, that

$$\lim_{\tau\to\infty} \dim_H(\mathscr{C}_{\tau,\kappa}(G)) = \frac{1}{2}.$$

Therefore, as this does not depend on the choice of κ we obtain that

$$\lim_{\tau\to\infty} \dim_H(\mathscr{C}_{\tau}(G)) = \frac{1}{2}.$$

□

Remark 6.3.3.

1. If the group G is chosen to be $PSL_2(\mathbb{Z})$, this result provides the corollary that if we define the set $F_N := \{x = [a_1(x),a_2(x),\ldots] : a_n(x) \geq N \text{ for all } n \in \mathbb{N}\}$, we have

$$\lim_{N\to\infty} \dim_H(F_N) = \frac{1}{2}.$$

This result can also be obtained from Theorem 2 in the 1941 paper of I.J. Good [31], where he gives upper and lower bounds for the Hausdorff dimension of each set F_N. For the details of exactly how the group $PSL_2(\mathbb{Z})$ relates to the continued fraction expansion, the reader is referred either to Appendix B or to Series [74].

2. Since a pairwise disjoint set of standard horoballs with top representation can be associated to any geometrically finite Fuchsian group with any finite number of parabolic elements, the restriction to only one is not really necessary, although it does simplify the notation significantly. This comment also applies to the next section.

6.4 Strict Jarník sets

In this section we will derive the Hausdorff dimension of another family of subsets of the limit set of a non-elementary geometrically finite Fuchsian group G with one parabolic element. The sets we are interested in here can be described by prescribing a particular geometric behaviour. Before we arrive at this description, we begin with a simple, purely analytical lemma.

Lemma 6.4.1. *Suppose that $(\sigma_n)_{n\in\mathbb{N}}$ is a sequence with each $0<\sigma_n<1$ and $\lim_{n\to\infty}\sigma_n=1$. Then, if $(a_n)_{n\in\mathbb{N}}$ is a sequence of positive real numbers,*

$$\limsup_{n\to\infty} a_n\sigma_n = \limsup_{n\to\infty} a_n.$$

Proof. Choose a subsequence $(a_{n_j})_{j\in\mathbb{N}}$ such that $\lim_{j\to\infty} a_{n_j}=\alpha:=\limsup_{n\to\infty} a_n$. Then,

$$\lim_{j\to\infty} a_{n_j}\sigma_{n_j} = \lim_{j\to\infty} a_{n_j}\cdot\lim_{j\to\infty}\sigma_{n_j}=\alpha.$$

Therefore, for every $\varepsilon>0$ and for all sufficiently large j we have that

$$\alpha(1-\varepsilon)\le a_{n_j}\sigma_{n_j}.$$

In other words, $\limsup_{n\to\infty} a_n\sigma_n\ge\alpha$. In addition, given that $a_n\sigma_n<a_n$ for every $n\in\mathbb{N}$, we obtain the opposite inequality, that is, $\limsup_{n\to\infty} a_n\sigma_n\le\alpha$.

□

In preparation for our first main result in this section, fix $\omega\in\mathbb{R}$ and let $s:=(s_n)_{n\in\mathbb{N}}$ be a sequence of positive integers such that $\lim_{n\to\infty}s_n=\infty$ and

$$\limsup_{n\to\infty}\frac{\log(s_n)}{2\log(s_1\dots s_{n-1})}=\omega.$$

Recall the definition of $d_n(\xi)$ from Definition 6.3.1. Then, with a constant $\kappa>0$ and positive integer $N>3$, define the set $F_{s,N,\kappa}(G)$ to be

$$F_{s,N,\kappa}(G):=\{\xi\in\mathscr{B}_\kappa:\log s_n\le d_n(\xi)<\log Ns_n\ \text{ for all } n\in\mathbb{N}\}.$$

Further, define

$$F_s(G) := \bigcup_{\substack{N>3 \\ \kappa>0}} F_{s,N,\kappa}.$$

We then have the following result.

Lemma 6.4.2.

$$\dim_H(F_s(G)) = \frac{1}{2(1+\omega)}.$$

Before starting the proof of the lemma, let us gather a few useful facts that will be required in the proof. The first of these has already been used in the proof of Theorem 2.5.10, but we find it helpful to restate it here.

- Since $\lim_{n\to\infty} s_n = \infty$, it follows that $\lim_{n\to\infty} \log s_n = \infty$ and thus that

$$\lim_{n\to\infty} \frac{\log(s_1 \dots s_n)}{n} = \infty. \tag{6.8}$$

- Define

$$\rho := \liminf_{n\to\infty} \frac{\log(s_1 \dots s_n)}{\log((s_1 \dots s_n)^2 s_{n+1})} = \frac{1}{2(1+\omega)}.$$

  Then, for all $K > 0$, we have that

$$\liminf_{n\to\infty} \frac{\log(s_1 \dots s_n)}{\log((K^n s_1 \dots s_n)^2 s_{n+1})} = \rho. \tag{6.9}$$

  Indeed, if we write

$$\frac{\log(s_1 \dots s_n)}{\log((K^n s_1 \dots s_n)^2 s_{n+1})} = \frac{\log(s_1 \dots s_n)}{\log((s_1 \dots s_n)^2 s_{n+1})} \cdot \frac{1}{1 + \frac{2n\log K}{\log((s_1 \dots s_n)^2 s_{n+1})}},$$

  then the statement in (6.9) follows immediately from Lemma 6.4.1 and (6.8).

- For all $K > 0$, all $\rho' < \rho$ and sufficiently large $n \in \mathbb{N}$, we have that

$$\frac{1}{s_1 \dots s_n} \leq \left(\frac{1}{K^{2n}(s_1 \dots s_n)^2 s_{n+1}} \right)^{\rho'}. \tag{6.10}$$

  This follows directly from (6.9) and the definition of the lower limit.

Proof of Lemma 6.4.2. Again, we first establish the upper bound, then the lower bound. To begin, just as in the proof of Theorem 6.3.2, we can associate to each point $\xi \in F_{\omega,\kappa,N}(G)$ a sequence of positive integers $(a_n(\xi))_{n\geq 1}$, where $a_n(\xi)$ is determined by

$$\log(a_n(\xi)) \leq d_n(\xi) < \log(a_n(\xi) + 1).$$

Notice that this implies that the point ξ lies, up to some constant of comparability, in the $a_n(\xi)$-th petal around the point $g_n(p)$, for each $n \in \mathbb{N}$. We therefore have, from Proposition 6.2.3 and the extra information given by the sequence $(s_n)_{n\geq 1}$, that

$$\frac{1}{e^{(n+1)\kappa}(N^n s_1 \dots s_n)^2} \ll |\Pi(H_{g_{n+1}}(\xi))| \ll \frac{1}{(a_1(\xi)\dots a_n(\xi))^2} \leq \frac{1}{(s_1 \dots s_n)^2}.$$

For each positive integer n, define the "shrunken" horoball $\widetilde{H}_{g_n}(\xi)$ to be the horoball with base point $g_n(p)$ and top $\tilde{\tau}_{g_n}$ given by

$$d_h(0, \tilde{\tau}_{g_n}) = d_h(0, \tau_{g_n}) + \log s_n.$$

It follows immediately that

$$\frac{1}{e^{(n+1)\kappa}(N^n s_1 \dots s_n)^2 s_{n+1}} \ll |\Pi(\widetilde{H}_{g_{n+1}}(\xi))| \ll \frac{1}{(s_1 \dots s_n)^2 s_{n+1}}. \tag{6.11}$$

We will now provide the upper bound. This is based on covers of arbitrarily small diameter for the set $F_{\omega,\kappa,N}(G)$, which are given by the shrunken horoballs defined above. First, let us make the observation that if $\xi \in F_{\omega,\kappa,N}(G)$, it follows that $\log s_n \leq d_n(\xi) < \log(N s_n)$ and thus that ξ could lie in any of the petals around $g_n(p)$ from the s_n-th up to the $N s_n$-th. So, there are $c(N-1)s_n$ shrunken horoballs in the n-th layer that the point ξ could lie in the shadow of, where c is the fixed constant number of these horoballs that have their base point in any given petal.

Now, by the definition of ρ given above, it follows that if we let $\rho' \in (\rho, 3\rho)$, we have for all sufficiently large n that

$$\frac{\rho' - \rho}{2} \leq \frac{\log(s_1 \dots s_n)}{\log((s_1 \dots s_n)^2 s_{n+1})}.$$

Consequently, from the identity $(1/b)^{\log a/\log b} = 1/a$, we obtain the inequality

$$\left(\frac{1}{(s_1 \dots s_n)^2 s_{n+1}}\right)^{(\rho'-\rho)/2} < \left(\frac{1}{(s_1 \dots s_n)^2 s_{n+1}}\right)^{\log(s_1 \dots s_n)/\log((s_1 \dots s_n)^2 s_{n+1})} = \frac{1}{s_1 \dots s_n}.$$

In other words,

$$s_1 \dots s_n \leq ((s_1 \dots s_n)^2 s_{n+1})^{(\rho'-\rho)/2}.$$

Then we observe by equation (6.8) that for all sufficiently large n we have that $\log(N-1) < \log(s_1 \dots s_n)/n$ (since the left-hand side is simply a constant depending only on N). Therefore, for sufficiently large n,

$$(N-1)^n < ((s_1 \dots s_n)^2 s_{n+1})^{(\rho'-\rho)/2}. \tag{6.12}$$

Also directly from the definition of ρ, for any $\rho' > \rho$, there exists a sequence $(n_k)_{k\in\mathbb{N}}$ with the property that

$$\frac{\log(s_1 \dots s_{n_k})}{\log((s_1 \dots s_{n_k})^2 s_{n_k+1})} \leq \frac{\rho' + \rho}{2}, \quad \text{for all } k \geq 1.$$

Hence, on rearranging the above expression, we obtain that

$$s_1 \ldots s_{n_k} \leq ((s_1 \ldots s_{n_k})^2 s_{n_k+1})^{(\rho'+\rho)/2}. \tag{6.13}$$

Consequently, if we neglect any terms of the sequence $(n_k)_{k\in\mathbb{N}}$ that are too small and rename the sequence accordingly, we have on combining (6.12) and (6.13) that for all $k \geq 1$ and any $\rho' > \rho$,

$$(N-1)^{n_k} s_1 \ldots s_{n_k} < ((s_1 \ldots s_{n_k})^2 s_{n_k+1})^{\rho'}.$$

Thus, from (6.11) and the above inequality, we infer that

$$\begin{aligned} \mathscr{H}^{\rho'}(F_{s,\kappa,N}(G)) &\leq \liminf_{k\to\infty} \sum_{\substack{s_m \leq a_m(\xi) < N s_m \\ 1\leq m\leq n_k}} |\Pi(\widetilde{H}_{g_{n_k+1}}(\xi))|^{\rho'} \\ &\ll ((N-1)^{n_k} s_1 \ldots s_{n_k}) \left((s_1 \ldots s_{n_k})^2 s_{n_k+1}\right)^{-\rho'} \leq 1. \end{aligned}$$

Hence, as this is true for all $\rho' \in (\rho, 3\rho)$, it follows that

$$\dim_H(F_{s,\kappa,N}(G)) \leq \rho = 1/(2(1+\omega)).$$

For the lower bound, we will again use Frostman's Lemma. So, to that end, in a similar way to that in Theorem 2.5.10, define a finite Borel measure m on the limit set $L(G)$ with the property that

$$m(\Pi(H_{g_{k+1}}(x)) = \frac{1}{a_1(x) \ldots a_k(x)}.$$

Let $\xi \in F_{s,\kappa,N}(G)$. Then for each small enough $r > 0$, we can find a unique k such that

$$|\Pi(\widetilde{H}_{g_{k+1}}(\xi))| \leq r < |\Pi(\widetilde{H}_{g_k}(\xi))|.$$

The difference in the sizes of these sets is too large to proceed directly from here. All we could hope to obtain using only this inequality is a lower bound of $1/2$. So we consider two further possibilities. Either,

$$|\Pi(\widetilde{H}_{g_{k+1}}(\xi))| \leq r < |\Pi(H_{g_{k+1}}(\xi))|, \tag{6.14}$$

or,

$$|\Pi(H_{g_{k+1}}(\xi))| \leq r < |\Pi(\widetilde{H}_{g_k}(\xi))|. \tag{6.15}$$

First note that by inequality (6.10), if we let $\rho' < \rho$, then there exists k_0 such that for all $k \geq k_0$, we have that

$$\frac{1}{s_1 \ldots s_k} \leq \left(\frac{1}{(e^\kappa)^{k+1}(N^k s_1 \ldots s_k)^2 s_{k+1}}\right)^{\rho'}. \tag{6.16}$$

Now, suppose that we are in the situation of (6.14). Choose r so that $k+1 \geq k_0$. In order to estimate the m-measure of the ball $B(\xi,r)$, we must first identify the number of shadows of standard horoballs in the $(k+1)$-th layer that said ball can intersect. Since there are a fixed number in each petal, it suffices to calculate the number of petals around $g_k(p)$ that the ball $B(\xi,r)$ can intersect. First of all, note that for large enough k, $B(\xi,r)$ cannot extend further than the $(a_k(\xi)-1)$-th petal, because the petals are decreasing in size. To finish the proof that $B(\xi,r)$ can only intersect a finite number of petals around $g_k(p)$, we must show that these petals are not shrinking too fast. It suffices to show that where c comes from the comparability given in (6.6), there exists a $k_0 \in \mathbb{N}$ such that if $k > k_0$ there exists M such that

$$\sum_{i=1}^{M} \frac{1}{(a_k(\xi)+i)^2} \geq \frac{c^2}{(a_k(\xi))^2}. \tag{6.17}$$

This follows immediately from the fact that the sequence $(a_k(\xi))_{k\in\mathbb{N}}$ tends to infinity and that $n-1 < \sum_{i=1}^{\infty} n^2/(n+i)^2 < n$ for all $n \in \mathbb{N}$. Consequently, $B(\xi,r)$ can only intersect a fixed finite number of petals around $g_k(p)$ for each $k \in \mathbb{N}$ and can thus only intersect a fixed finite number of shadows of standard horoballs in each layer. It follows, via (6.16), (6.11) and (6.14), that for any $\rho' < \rho$ we have

$$\begin{aligned} m(B(\xi,r)) &\ll m\left(\Pi(H_{g_{k+1}}(\xi))\right) \\ &= \frac{1}{a_1(\xi)\ldots a_k(\xi)} \leq \frac{1}{s_1\ldots s_k} \\ &\leq \left(\frac{1}{e^{\kappa(k+1)}(N^k s_1\ldots s_k)^2 s_{k+1}}\right)^{\rho'} \\ &\ll r^{\rho'}. \end{aligned}$$

If we are in the situation of (6.15), it is clear, by similar reasoning to that above, that $B(\xi,r)$ cannot intersect more than two petals in the k-th layer, which means that there is again only a fixed finite number of shadows of shrunken horoballs that $B(\xi,r)$ can intersect. In addition to this, a maximum of $2r(\kappa N^2)^k(s_1\ldots s_k)^2$ of the $(k+1)$-th layer shadows of standard horoballs are intersected by $B(\xi,r)$. So, denoting by $\Pi(\widetilde{H}_{g_k})$ and $\Pi(H_{g_{k+1}})$ the largest possible shadow in layer k and $k+1$ respectively, we have that

$$\begin{aligned} m(B(\xi,r)) &\ll \min\{2m(\Pi(\widetilde{H}_{g_k})), 2r(\kappa N^2)^k(s_1\ldots s_k)^2 m(\Pi(H_{g_{k+1}}))\} \\ &\leq \frac{2}{s_1\ldots s_k}\min\{1,(\kappa N^2)^{k+1}(s_1\ldots s_k)^2 s_{k+1} r\} \end{aligned}$$

and, using equation (6.16) and the fact that $\min\{a,b\} \leq a^{1-s}b^s$ for any $0<s<1$, it follows that for $\rho' < \rho$ we have

$$\begin{aligned} m(B(\xi,r)) &\leq 2\left(\frac{1}{(\kappa N^2)^k(s_1\ldots s_k)^2 s_{k+1}}\right)^{\rho'} \cdot \left(\left(\kappa N^2\right)^{k+1}(s_1\ldots s_k)^2 s_{k+1} r\right)^{\rho'} \\ &= 2\kappa N^2 r^{\rho'}. \end{aligned}$$

Thus, in each case, on applying Frostman's Lemma and letting ρ' tend to ρ,the proof is finished. □

Remark 6.4.3.

1. Note that the result in Lemma 6.4.2 does not depend on the particular sequence $(s_n)_{n\in\mathbb{N}}$. By this, what is meant is that in the definition of F_s we could instead use any other sequence $(\hat{s}_n)_{n\in\mathbb{N}}$ with the property that $\limsup_{n\to\infty} \frac{\log(\hat{s}_n)}{2\log(\hat{s}_1\ldots\hat{s}_{n+1})} = \omega$ and the resulting set $F_{\hat{s}}$ would have precisely the same Hausdorff dimension.

2. Similarly to the corollary to Theorem 6.3.2, we can also derive a continued fractions result from Lemma 6.4.2. With $a_n(x)$ referring to the n-th partial quotient of x, the sequence $(s_n)_{n\geq 1}$ defined as above and $N \geq 2$, we have that

$$\dim_H\{x \in [0,1) : s_n \leq a_n(x) < Ns_n \ \forall n \in \mathbb{N}\} = \liminf_{n\to\infty} \frac{\log(s_1\ldots s_n)}{\log((s_1\ldots s_n)^2 s_n + 1)}.$$

   This result can be found in the paper by Fan *et al.* [24].

We are now ready to state and prove the main result of this section. First, define $t_n(\xi) := d_h(0, z_{g_n}) + d_n(\xi)$, where z_{g_n} is the point the ray from 0 to ξ enters the n-th horoball (i.e., the point just before the n-th cusp excursion begins). Then, for $\kappa > 0$ and $\theta \in [0,1]$, define the *strict* (θ,κ)*-Jarník limit set* $\mathscr{J}^*_{\theta,\kappa}(G)$ by setting

$$\mathscr{J}^*_{\theta,\kappa}(G) := \left\{ \xi \in \mathscr{B}_\kappa : \lim_{n\to\infty} d_n(\xi) = \infty \text{ and } \limsup_{n\to\infty} \frac{d_n(\xi)}{t_n(\xi)} = \theta \right\}.$$

Define the *strict* θ*-Jarník limit set* to be

$$\mathscr{J}^*_\theta(G) := \bigcup_{\kappa>0} \mathscr{J}^*_{\theta,\kappa}(G).$$

We have the following theorem.

Theorem 6.4.4. *For the strict* θ*-Jarník limit set* $\mathscr{J}^*_\theta(G)$*, we have that*

$$\dim_H(\mathscr{J}^*_\theta(G)) = \frac{1}{2}(1-\theta).$$

Proof. Fix $\kappa > 0$. The first step of the proof is to show that the condition $\limsup_{n\to\infty} \frac{d_n(\xi)}{t_n(\xi)} = \theta$ is equivalent to the condition that

$$\limsup_{n\to\infty} \frac{d_n(\xi)}{2(d_1(\xi)+\cdots+d_{n-1}(\xi))} = \frac{\theta}{1-\theta}.$$

In order to do this, we begin by claiming that

$$\limsup_{n\to\infty} \frac{d_n(\xi)}{t_n(\xi)} = \theta \Leftrightarrow \limsup_{n\to\infty} \frac{d_n(\xi)}{d_h(0,z_{g_n})} = \frac{\theta}{1-\theta}. \tag{6.18}$$

Indeed, if $\theta > 0$, we have that

$$\theta = \limsup_{n\to\infty} \frac{d_n(\xi)}{t_n(\xi)} = \limsup_{n\to\infty} \frac{d_n(\xi)}{d_h(0,z_{g_n}) + d_n(\xi)} = \frac{1}{1 + \liminf_{n\to\infty} \frac{d_h(0,z_{g_n})}{d_n(\xi)}}. \tag{6.19}$$

Therefore,

$$\limsup_{n\to\infty} \frac{d_n(\xi)}{d_h(0,z_{g_n})} = \frac{1}{\frac{1}{\theta} - 1} = \frac{\theta}{1-\theta}.$$

On the other hand, if $\theta = 0$, we have from (6.19) that

$$\liminf_{n\to\infty} \frac{d_h(0,z_{g_n})}{d_n(\xi)} = \infty \;\Rightarrow\; \limsup_{n\to\infty} \frac{d_n(\xi)}{d_h(0,z_{g_n})} = 0.$$

Thus, since these arguments work equally well backwards, the claim in (6.18) is proved. Next, notice that

$$\limsup_{n\to\infty} \frac{d_n(\xi)}{d_h(0,z_{g_n})} = \frac{\theta}{1-\theta} \Leftrightarrow \limsup_{n\to\infty} \frac{d_n(\xi)}{2(d_1(\xi) + \cdots + d_{n-1}(\xi))} = \frac{\theta}{1-\theta}. \tag{6.20}$$

The reason for this is that we have

$$2(d_1(\xi) + \cdots + d_{n-1}(\xi)) \le d_h(0,z_{g_n}) \le n\kappa + 2(d_1(\xi) + \cdots + d_{n-1}(\xi)),$$

so

$$\frac{d_n(\xi)}{n\kappa + 2(d_1(\xi) + \cdots + d_{n-1}(\xi))} \le \frac{d_n(\xi)}{d_h(0,z_{g_n})} \le \frac{d_n(\xi)}{2(d_1(\xi) + \cdots + d_{n-1}(\xi))}. \tag{6.21}$$

Then, from the second inequality in (6.21), it is immediate that

$$\frac{\theta}{1-\theta} \le \limsup_{n\to\infty} \frac{d_n(\xi)}{2(d_1(\xi) + \cdots + d_{n-1}(\xi))}.$$

Rewriting the first inequality from (6.21), we obtain that

$$\frac{d_n(\xi)}{2(d_1(\xi) + \cdots + d_{n-1}(\xi))} \cdot \frac{1}{1 + \frac{n\kappa}{2(d_1(\xi) + \cdots + d_{n-1}(\xi))}} \le \frac{d_n(\xi)}{d_h(0,z_{g_n})}.$$

Consequently, by Lemma 6.4.1 and from the fact that $\lim_{n\to\infty} d_n(\xi) = \infty$, we also obtain the opposite inequality, namely,

$$\limsup_{n\to\infty} \frac{d_n(\xi)}{2(d_1(\xi) + \cdots + d_{n-1}(\xi))} \le \frac{\theta}{1-\theta}.$$

Combining (6.18) and (6.20) establishes the first step of the proof.

We now aim to use this equivalent definition to find an upper bound for the sought-after Hausdorff dimension. So, suppose that $\xi \in \mathscr{J}^*_{\theta,\kappa}(G)$, that is, suppose that $\xi \in \mathscr{B}_\kappa$ is such that $\lim_{n\to\infty} d_n(\xi) = \infty$ and

$$\limsup_{n\to\infty} \frac{d_n(\xi)}{2(d_1(\xi)+\cdots+d_{n-1}(\xi))} = \frac{\theta}{1-\theta}.$$

Then, for each $n \in \mathbb{N}$ pick an integer $\hat{s}_n$ such that

$$\log(\hat{s}_n) \le d_n(\xi) < \log(\hat{s}_n + 1).$$

Since $\lim_{n\to\infty} d_n(\xi) = \infty$, we can immediately infer that $\lim_{n\to\infty} \hat{s}_n = \infty$. It is clear that we can write, say, $\log \hat{s}_n \le d_n(\xi) < \log 3\hat{s}_n$. So then, since $\xi \in F_{\hat{s},3,\kappa}$, it suffices to show that $\limsup\limits_{n\to\infty} \frac{\log \hat{s}_n}{2\log(\hat{s}_1\ldots\hat{s}_{n-1})} = \frac{\theta}{1-\theta}$. But,

$$\frac{d_n(\xi)}{2(d_1(\xi)+\cdots+d_{n-1}(\xi))} \le \frac{\log(3\hat{s}_n)}{2\log(\hat{s}_1\ldots\hat{s}_{n-1})}$$

and

$$\frac{d_n(\xi)}{2(d_1(\xi)+\cdots+d_{n-1}(\xi))} \ge \frac{\log(\hat{s}_n)}{2\log(\hat{s}_1\ldots\hat{s}_{n-1}) + 2n\log 3}.$$

From these two inequalities, we see that this reduces to basically the same argument again, involving another application of Lemma 6.4.1. Thus we obtain that

$$\dim_H(\mathscr{J}^*_{\theta,\kappa}(G)) \le \dim_H(F_{\hat{s},3,\kappa}) = \frac{1}{2(1+\frac{\theta}{1-\theta})} = \frac{1}{2}(1-\theta).$$

Finally, suppose now that $N > 2$ and $\xi \in F_{s,N,\kappa}(G)$, where the sequence $(s_n)_{n\in\mathbb{N}}$ (which is not necessarily the same as the sequence $(\hat{s}_n)_{n\in\mathbb{N}}$), satisfies

$$\lim_{n\to\infty} s_n = \infty,\ \limsup_{n\to\infty} \frac{\log s_n}{\log(s_1\ldots s_{n-1})} = \frac{\theta}{1-\theta} \ \text{ and } \ \log s_n \le d_n(\xi) < \log N s_n \ \forall n \in \mathbb{N}.$$

By similar reasoning to that above, it is clear that $\xi \in \mathscr{J}^*_{\theta,\kappa}(G)$. Consequently, we have that

$$\frac{1}{2}(1-\theta) = \dim_H(F_{s,N,\kappa}(G)) \le \dim_H(\mathscr{J}^*_{\theta,\kappa}(G)).$$

Thus, combining this with the opposite inequality achieved above finishes the proof.

□

6.5 Weak multifractal spectra for the Patterson measure.

In this section we again assume that G is a non-elementary geometrically finite Fuchsian group with one parabolic element. We remind the reader that the definition of $t_n(\xi)$ was given in Section 6.4. We assume that the reader is familiar with the construction and basic properties of the Patterson measure; if not, a very short introduction can be found in Appendix A. The Global Measure Formula, which is one important component of the proof of Theorem 6.5.1 below, is stated there as Theorem A.2.13.

Let $b(\xi_t, e^{-t})$ denote the shadow of the geodesic which intersects the ray s_ξ orthogonally at the point ξ_t, where ξ_t is defined to be the point on the ray s_ξ at a distance t from the origin. Then we define the β-strict-Jarník level sets for the Patterson measure μ to be

$$\mathscr{F}_\beta^* := \left\{ \xi \in L(G) : \limsup_{n\to\infty} \frac{\log \mu(b(\xi, e^{-t_n(\xi)}))}{-t_n(\xi)} = \beta \right\}.$$

Further, let $\mathscr{B}(G) := \bigcup_{\kappa>0} \mathscr{B}_\kappa(G)$. We obtain the following theorem.

Theorem 6.5.1. *For each $\beta \in [2\delta-1, \delta]$, we have that*

$$\dim_H \left(\mathscr{F}_\beta^* \cap \mathscr{B}(G) \right) = \frac{1}{2} \cdot f_p(\beta),$$

where $f_p(\beta) := (\beta - (2\delta-1))/(1-\delta)$.

Proof. The Global Measure Formula for μ gives the existence of a constant $c > 1$ (depending only on G), such that for each $\xi \in L(G)$ and every $t > 0$ we have that

$$\frac{1}{c} e^{-t\delta} e^{-(\delta - k(\xi_t))\Delta(\xi_t)} \le \mu(b(\xi_t, e^{-t})) \le c e^{-t\delta} e^{-(\delta - k(\xi_t))\Delta(\xi_t)}.$$

Consequently, where $c_1 := \log c > 0$, we have that

$$\delta + (\delta - k(\xi_t))\frac{\Delta(\xi_t)}{t} - \frac{c_1}{t} \le \frac{\log \mu(b(\xi_t, e^{-t})}{\log e^{-t}} \le \delta + (\delta - k(\xi_t))\frac{\Delta(\xi_t)}{t} + \frac{c_1}{t}.$$

Thus, if we let $t = t_n(\xi)$, which implies that $\Delta(\xi_t) = d_n(\xi)$ and $k(\xi_t) = 1$, we immediately deduce that

$$\limsup_{n\to\infty} \frac{\log \mu \left(b(\xi_{t_n(\xi)}, e^{-t_n(\xi)}) \right)}{-t_n(\xi)} = \delta + (\delta - 1) \limsup_{n\to\infty} \frac{d_n(\xi)}{t_n(\xi)}.$$

It therefore follows that $\xi \in \mathscr{J}_\theta^*(G)$ if and only if $\xi \in \mathscr{B}(G)$ and

$$\limsup_{n\to\infty} \frac{\log \mu \left(b(\xi_{t_n(\xi)}, e^{-t_n(\xi)}) \right)}{-t_n(\xi)} = \delta - (1-\delta)\theta.$$

Consequently, if $\beta := \delta - (1-\delta)\theta$, or in other words, if $\theta = (\delta - \beta)/(1-\delta)$, we have by an application of Theorem 6.4.4 that

$$\dim_H \left(\mathscr{F}_\beta^* \cap \mathscr{B}(G)\right) = \dim_H \left(\mathscr{J}^*_{\frac{\delta-\beta}{1-\delta}}\right) = \frac{1}{2}\left(1 - \frac{\delta-\beta}{1-\delta}\right) = \frac{1}{2} \cdot \frac{1-2\delta+\beta}{1-\delta}.$$

□

Remark 6.5.2. Note that in [75] a "weak multifractal analysis" of the Patterson measure was given. The analysis there was based on investigations of the Hausdorff dimension of the associated *θ-Jarník limit set*

$$\mathscr{J}_\theta(G) := \left\{\xi \in L(G) : \limsup_{t\to\infty} \frac{\Delta(\xi_t)}{t} \geq \theta\right\}.$$

In Stratmann [75] (see also [76] and [36]) the result was obtained that

$$\dim_H(\mathscr{J}_\theta(G)) = (1-\theta)\delta, \text{ for each } \theta \in [0,1].$$

In Stratmann [76] it was then shown how to use this result in order to derive the following "weak multifractal spectrum" of the Patterson measure:

$$\dim_H\left(\mathscr{F}_\beta\right) = \begin{cases} 0 & \text{for} \quad 0 < \beta \leq 2\delta - 1 \\ \delta \cdot f_p(\beta) & \text{for} \quad 2\delta - 1 < \beta \leq \delta \\ \delta & \text{for} \quad \beta > \delta. \end{cases},$$

where f_p is given, as before, by $f_p(\beta) = (\beta - (2\delta - 1))/(1-\delta)$ and where $\mathscr{F}_\beta(G)$ is defined by

$$\mathscr{F}_\beta(G) := \left\{\xi \in L(G) : \liminf_{n\to\infty} \frac{\log \mu(b(\xi, e^{-t_n(\xi)}))}{-t_n(\xi)} \leq \beta\right\}.$$

The outcome here should be compared with the result in Theorem 6.5.1. The two spectra are illustrated in Figure 6.8, below.

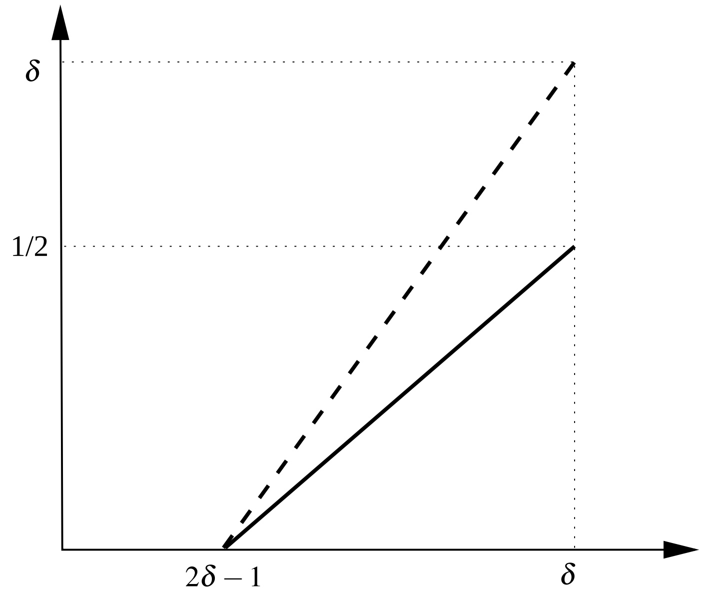


Figure 6.8: Weak multifractal spectra for the Patterson measure. The upper (dashed) line is the graph of the spectrum $\dim_H(\mathscr{F}_\beta(G))$ and the lower (solid) line is the graph of the spectrum $\dim_H(\mathscr{F}^*_\beta(G))$.

Appendix A

The Patterson Measure

This appendix describes the construction and basic properties of the Patterson measure. We begin with a section on the geometric properties of the Poisson kernel, then move on to the Patterson itself in Section A.2.

A.1 The Poisson kernel

We begin by investigating the geometry of the *Poisson kernel*. Recall from Chapter 6 the definition of horoballs. We can also define horoballs in terms of the Poisson kernel $P(z,\xi)$, which is given for $z \in \mathbb{D}^2$ and $\xi \in \mathbb{S}^1$ by

$$P(z,\xi) := \frac{1-|z|^2}{|z-\xi|^2}.$$

We have the following two lemmas.

Lemma A.1.1. *For $\xi \in \mathbb{S}^1$, $x \in \mathbb{D}^2$ and $0 < k < 1$, we have that x lies on the horoball of radius k with base point ξ if and only if*

$$P(x,\xi) = \frac{1-k}{k}.$$

In particular, this implies that $P(x,\xi) = P(y,\xi)$ for any two $x,y \in \mathbb{D}^2$ lying on the same horoball based at $\xi \in \mathbb{S}^1$.

Proof. First, suppose that $P(x,\xi) = (1-k)/k$. Then,

$$k(1-|x|^2) \quad = \quad (1-k)(|x|^2 - 2(x\overline{\xi} + \overline{x}\xi) + 1)$$

and so, on rearranging this expression, we obtain

$$2(x\overline{\xi} + \overline{x}\xi) = \frac{|x|^2 + 1 - 2k}{1-k}.$$

Consider now the distance between x and the point $(1-k)\xi$, which is the the centre of the horoball of radius k with base point ξ. We have that

$$\begin{aligned}|x-(1-k)\xi|^2 &= |x|^2 - 2(1-k)(x\overline{\xi}+\overline{x}\xi)+(1-k)^2 \\ &= |x|^2-(1-k)\cdot\frac{|x|^2+1-2k}{1-k}+(1-k)^2 = k^2.\end{aligned}$$

Therefore, x lies on the horoball of radius k with base point ξ (see Figure A.1, below). Since this argument works equally well backwards, we are done.

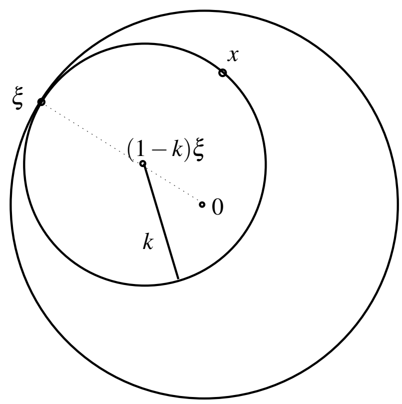


Figure A.1: Pictured is the unit disc with a horoball of radius k at the point $\xi \in \mathbb{S}^1$.

□

Lemma A.1.2. *Given $x, y \in \mathbb{D}^2$ and $\xi \in \mathbb{S}^1$, we have that*

$$\lim_{w\to\xi}\frac{e^{d_h(x,w)}}{e^{d_h(y,w)}} = \frac{P(y,\xi)}{P(x,\xi)}.$$

Proof. The lemma follows immediately from the formula

$$\sinh^2(d_h(x,w)/2) = \frac{|x-w|^2}{(1-|x|^2)(1-|w|^2)},$$

which can be found in section 7.2 of [7], and the fact that $\sinh^2(x/2) \asymp e^x$, as x tends to infinity. □

Remark A.1.3. It follows from this last lemma that

$$\lim_{w\to\xi}\frac{e^{d_h(0,w)}}{e^{d_h(x,w)}} = \frac{P(x,\xi)}{P(0,\xi)} = P(x,\xi).$$

The geometric interpretation of this fact is that the Poisson kernel $P(x,\xi)$ is the "signed distance" between the horoballs H_0, with base point ξ through 0 and H_x, with base point ξ through x.

In other words, if the distance between the horoballs H_0 and H_x is denoted by D_x, as $w \to \xi$, the quantity $d_h(0,w) - d_h(x,w)$ approaches D_x when x is inside H_0 and $-(d_h(0,w) - d_h(x,w))$ approaches this same distance when x is outside H_0. So, if x lies inside H_0, the Poisson kernel $P(x,\xi)$ is given by e^{D_x}, whereas if x lies outside H_0 we have that the Poisson kernel $P(x,\xi)$ is given by e^{-D_x}.

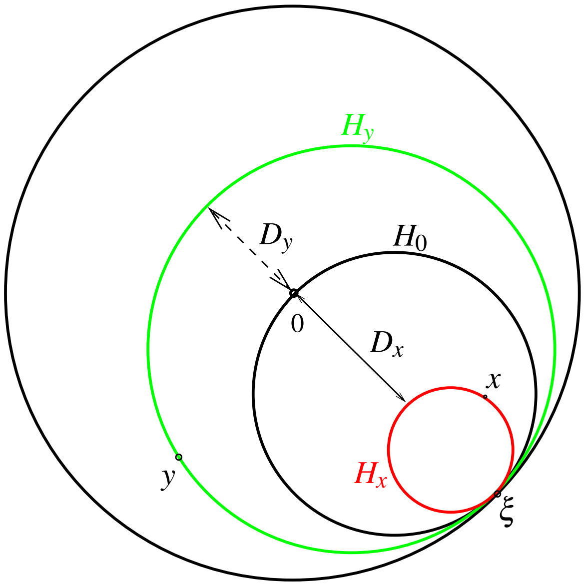


Figure A.2: We have that $P(x,\xi) = e^{D_x}$ and $P(y,\xi) = e^{-D_y}$, where D_x denotes the hyperbolic distance between the horoballs H_0 and H_x and D_y denotes the hyperbolic distance between the horoballs H_0 and H_y.

Let us now provide a formula for the Poisson kernel in $\mathbb{H}$. The reason for giving this version here is that it appears incorrectly in various sources in the literature.

Proposition A.1.4. *The Poisson kernel $P_{\mathbb{H}} : \mathbb{H} \times \mathbb{R} \cup \{\infty\} \to \mathbb{R}$ is given by*

$$P_{\mathbb{H}}(z,r) := \begin{cases} \mathrm{Im}(z) & \text{for } r = \infty; \\ \frac{\mathrm{Im}(z)(r^2+1)}{|z-r|^2} & \text{for } r \in \mathbb{R}. \end{cases}$$

Proof. Suppose first that $r = \infty$. We have that $P_{\mathbb{H}}(z,\infty) = P(\phi(z),\phi(r)) = P(\phi(z),1)$, where ϕ here denotes the Cayley transform introduced above. Therefore,

$$P_{\mathbb{H}}(z,\infty) = \frac{1 - \left|\frac{z-i}{z+i}\right|^2}{\left|\frac{z-i}{z+i} - 1\right|^2} = \frac{|z+i|^2 - |z-i|^2}{|-2i|^2} = \frac{4 \cdot \mathrm{Im}(z)}{4} = \mathrm{Im}(z).$$

For the second case, when r is a finite real number, we have

$$\begin{aligned} P_{\mathbb{H}}(z,r) &= \frac{1-\left|\frac{z-i}{z+i}\right|^2}{\left|\frac{z-i}{z+i}-\frac{r-i}{r+i}\right|^2} = \frac{|z+i|^2-|z-i|^2}{\frac{1}{|r+i|^2}(|(z-i)(r+i)-(r-i)(z+i)|^2)} \\ &= \frac{4\cdot \mathrm{Im}(z)(r^2+1)}{|2i(z-r)|^2} = \frac{\mathrm{Im}(z)(r^2+1)}{|z-r|^2}. \end{aligned}$$

$\square$

Remark A.1.5. We can also see that the geometric intuition behind the Poisson kernel is valid in the upper half-plane model of hyperbolic space. Since i is mapped to 0 under the Cayley transformation, in this case we are considering the distance between a horoball H_i through i with base point r and another horoball H_z with base point r through any other point z. For $r=\infty$, it is immediately clear from Lemma 6.1.8 that the Poisson kernel is given by $P_{\mathbb{H}}(z,\infty)=\mathrm{Im}(z)$. If $r\in\mathbb{R}$ and $z=x+iy\in\mathbb{H}$ lies inside H_i, referring to Figure A.3, basic Euclidean geometry gives that the height of the horoball H_z is given by $h_1=((r-x)^2+y^2)/y$ and the height of the horoball H_i is given by $h_2=r^2+1$. The distance between these two horoballs is then, by Lemma 6.1.8 again, given by $D_z=\log(h_2/h_1)$. So we obtain that

$$P(z,r)=e^{D_z}=\frac{y(r^2+1)}{(r-x)^2+y^2}.$$

This coincides with the formula given in Proposition A.1.4. The case where z lies outside the horoball H_i proceeds similarly, except that there we have that $D_z=\log(h_1/h_2)$.

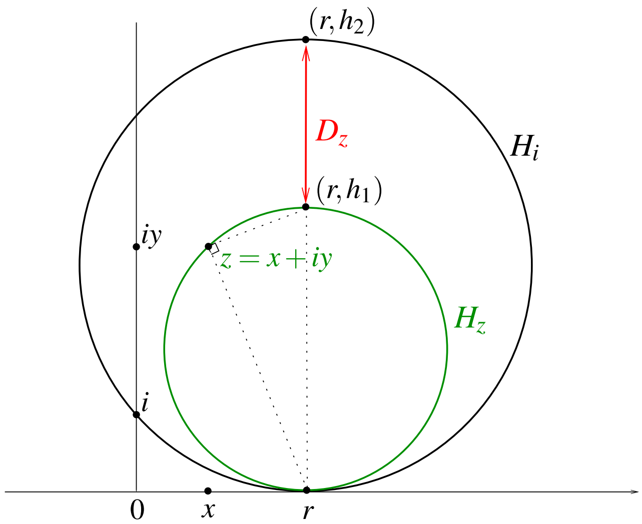


Figure A.3: We have that $P_{\mathbb{H}}(z,r)=e^{D_z}$.

A.2 The Patterson measure

The Patterson measure was constructed by S.J. Patterson in 1976 [64]. His work was motivated by a number theoretical problem in the theory of Diophantine approximation. The Patterson measure is a very effective tool for examining the limit set of a Fuchsian group. For further details about the Patterson measure, in addition to [64], the reader is referred to the book [62].

Let us now begin the construction. For each $s > \delta$, we start by defining the measure

$$\mu_{x,s}(A) := \frac{\sum_{g \in G} e^{-sd(x,g(0))} \delta_{g(0)}(A)}{\sum_{g \in G} e^{-sd(0,g(0))}} = \frac{\sum_{g \in G} e^{-sd(x,g(0))} \delta_{g(0)}(A)}{\Sigma_s(0,0)}.$$

Here, $\delta_{g(0)}$ is a Dirac point-mass at the point $g(0)$, that is to say,

$$\delta_{g(0)}(A) = \begin{cases} 1 & \text{if } g(0) \in A; \\ 0 & \text{otherwise.} \end{cases}$$

Let us now investigate what happens when we fix some $s > \delta$ and consider the measures $\mu_{x,s}$ as the base point x is allowed to vary. Recall that $P(z, \xi)$ denotes the Poisson kernel in $\mathbb{D}^2$.

Theorem A.2.1. *Let G be a Fuchsian group with exponent of convergence δ. Suppose that $s > \delta$ and choose $x, y \in \mathbb{D}^2$ and $\xi \in \mathbb{S}^1$. Let A be a Borel subset of $\mathbb{D}^2 \cup \mathbb{S}^1$ and for $t > 0$ let $A(\xi, t)$ denote that part of A within Euclidean distance t of ξ. Then for every $\varepsilon > 0$ there exists $t(\varepsilon)$ such that if $t < t(\varepsilon)$,*

$$\left[\left(\frac{P(x,\xi)}{P(y,\xi)} \right)^s - \varepsilon \right] \mu_{y,s}(A(\xi,t)) \le \mu_{x,s}(A(\xi,t)) \le \left[\left(\frac{P(x,\xi)}{P(y,\xi)} \right)^s + \varepsilon \right] \mu_{y,s}(A(\xi,t)).$$

Proof. We have that

$$\begin{aligned} \mu_{x,s}(A(\xi,t)) &= \frac{1}{\Sigma_s(0,0)} \sum_{g \in G} e^{-sd_h(x,g(0))} \delta_{g(0)}(A(\xi,t)) \\ &= \frac{1}{\Sigma_s(0,0)} \sum_{g \in G} e^{-sd_h(x,g(0))} \cdot \frac{e^{-sd_h(y,g(0))}}{e^{-sd_h(y,g(0))}} \delta_{g(0)}(A(\xi,t)). \end{aligned}$$

From Lemma A.1.2 it follows that there exists a $t(\varepsilon)$ such that if $t < t(\varepsilon)$ and $g(0) \in A(\xi,t)$, then

$$\left| \frac{e^{-sd_h(x,g(0))}}{e^{-sd_h(y,g(0))}} - \left(\frac{P(y,\xi)}{P(x,\xi)} \right)^{-s} \right| \le \varepsilon.$$

The theorem follows immediately on combining these two observations.

□

Corollary A.2.2. *With x, y and ξ as in Theorem A.2.1,*

$$\lim_{t\to 0} \frac{\mu_{x,s}(A(\xi,t))}{\mu_{y,s}(A(\xi,t))} = \left(\frac{P(x,\xi)}{P(y,\xi)}\right)^s.$$

In particular, if $y = 0$, we obtain

$$\lim_{t\to 0} \frac{\mu_{x,s}(A(\xi,t))}{\mu_{0,s}(A(\xi,t))} = (P(x,\xi))^s.$$

We are now almost ready to define the Patterson measure. First, we recall the notion of the *weak limit* of a sequence of measures.

Definition A.2.3. Let $C(X)$ denote the set of real-valued continuous functions on a measurable space X. If $(\nu_n)_{n\geq 1}$ and ν are measures on the measure space $(X,\mathscr{B})$ satisfying

$$\lim_{n\to\infty} \int_X f\, d\nu_n = \int_X f\, d\nu \text{ for all } f \in C(X),$$

we say that the sequence $(\nu_n)_{n\geq 1}$ *converges weakly* to the measure ν.

Remark A.2.4. Recall that if X is a set and X_α a topological space, then the *weak topology* induced on X by a collection of functions $\{f_\alpha : X \to X_\alpha : \alpha \in A\}$ is the smallest topology on X such that each f_α is continuous. Evidently, the sets $f_\alpha^{-1}(U_\alpha)$, for U_α open in X_α, constitute a subbase for the weak topology. Weak convergence of measures is weak convergence in a weak topology induced by $C(X)$ on the dual space

$$\big(C(X)\big)^* := C^*(X) = \{F : C(X) \to \mathbb{R} : F \text{ is continuous and linear}\}.$$

This can be slightly confusing on first sight, because in functional analysis this topology is usually referred to as the *weak*-topology* on $C(X)$, but it is a weak topology nevertheless.

The most important result for us concerning weak convergence of measures is the following. Note that in here the closure is with respect to the norm (or strong) topology on $C^*(X)$.

Theorem A.2.5. Alaoglu's Theorem. *The closed unit ball in the dual space X^* of a Banach space X is compact in the weak*-topology on X^*. Further, every closed bounded subset of X^* is compact in the weak*-topology on X^*.*

Proof. See Dunford and Schwartz [19], Theorem V.4.2.

□

Observe that the set of all measures on a space $(X,\mathscr{B})$ is contained in $C^*(X)$, since if ν is a measure on $(X,\mathscr{B})$, then ν can be thought of as a function $\nu : C(X) \to \mathbb{R}$ defined by

$$\nu(f) := \int f\, d\nu.$$

Consider the set $M(X)$ consisting of all probability measures on $(X,\mathscr{B})$. This set is evidently closed with respect to the norm topology on $C^*(X)$. It is also bounded, since if μ is a probability measure, then

$$||\mu|| = \sup_{\substack{f\in C(X)\\ ||f||_\infty\leq 1}} \left\{ \left| \int_X f\, d\mu \right| \right\} \leq \left| \int_X \mathbb{1}\, d\mu \right| = 1.$$

Provided that X is a compact metric space, the space $C(X)$ is a Banach space. We can then infer from Alaoglu's Theorem that the set $M(X)$ is compact in the weak*-topology and hence every sequence of measures in $M(X)$ has a weakly convergent subsequence.

Returning now to the particular situation of the Patterson measure, from the triangle inequality we obtain that

$$d_h(0,g(0)) - d_h(x,0) \leq d_h(x,g(0)) \leq d_h(0,g(0)) + d_h(x,0)$$

and consequently, for any $s > \delta$ that

$$e^{-sd_h(x,0)} \cdot e^{-sd_h(0,g(0))} \leq e^{-sd_h(x,g(0))} \leq e^{sd_h(x,0)} \cdot e^{-sd_h(0,g(0))}.$$

Summing over all $g \in G$ yields that

$$\frac{e^{-sd_h(x,0)}}{\Sigma_s(x,0)} \leq \frac{1}{\Sigma_s(0,0)} \leq \frac{sd_h(x,0)}{\Sigma_s(x,0)},$$

which in turn implies that

$$e^{-sd_h(x,0)} \leq \mu_{x,s}(\mathbb{D}^2 \cup \mathbb{S}^1) \leq e^{sd_h(x,0)}. \tag{A.1}$$

Let $(s_j)_{j\in\mathbb{N}}$ be a sequence of real numbers from $(\delta, 2\delta)$, monotonically decreasing to δ. By (A.1), for each $j \in \mathbb{N}$ there exists a real number $\alpha_j \in [e^{-sd_h(x,0)}, e^{sd_h(x,0)}]$ with the property that $\nu_{x,s_j} := {\alpha_j}^{-1}\mu_{x,s_j}$ is a probability measure. Therefore, by Alaoglu's Theorem, we have that along a subsequence (which we rename (s_j) again),

$$\lim_{j\to\infty} \nu_{x,s_j} = \nu_x.$$

Note that ν_x is also a probability measure. The sequence $(\alpha_j)_{j\in\mathbb{N}}$ is uniformly bounded, so there exists a subsequence $(\alpha_{j_k})_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty} \alpha_k = \alpha$. Thus,

$$\lim_{k\to\infty} \mu_{x,s_{j_k}} := \lim_{k\to\infty} \alpha_{j_k} \nu_{x,s_{j_k}} = \alpha\nu_x$$

and we have shown that for each sequence $(s_j)_{j\in\mathbb{N}}$ of real numbers monotonically decreasing to δ, there exists a weak limit measure.

Some more work is required to show that such a measure is unique, indeed, in some cases it is not. However, in the situation where G is a geometrically finite Fuchsian group, the uniqueness was proved by Sullivan [79]. This proof is decidedly non-trivial and we will not reproduce it here. We make the following definition.

Definition A.2.6. Let G be a geometrically finite Fuchsian group with exponent of convergence δ and let $(s_j)_{j\geq 1}$ be a sequence of real numbers monotonically decreasing to δ. Then the *Patterson measure with base point x* is defined by setting

$$\mu_x(A) := \lim_{s_j\to\delta} \mu_{x,s_j}(A),$$

for each Borel set $A \subseteq \mathbb{D}^2 \cup \mathbb{S}^1$.

If the group G is of divergence type, this limit measure will be supported on the limit set $L(G)$, which we prove in the next proposition. However, if G is of convergence type, we will simply get another measure supported on the disc $\mathbb{D}^2$ with point masses on the orbit of 0. In order to get around this problem, Patterson introduced an ingenious multiplicative factor $h(e^{d_h(x,g(0))})$ which does not alter the exponent of convergence, but ensures that the Poincaré series at $\delta(G)$ diverges. However, as we are only interested here in geometrically finite groups and recalling from Section 6.1.6 that geometrically finite groups are of divergence type, in all that follows the factor h will be set equal to 1.

Proposition A.2.7. *Let G be a geometrically finite Fuchsian group. Then the support of the Patterson measure μ_x is the limit set, $L(G)$.*

Proof. Let δ denote the exponent of convergence of G. First suppose that A is a set fully contained in $\mathbb{D}^2$. Then

$$\mu_x(A) = \lim_{s_j\to\delta} \mu_{x,s_j}(A) = \lim_{s_j\to\delta} \frac{1}{\sum_{s_j}(0,0)} \sum_{g\in G} e^{-s_j d(x,g(0))} \delta_{g(0)}(A) = 0,$$

since the numerator for each s_j is necessarily finite, due to the discontinuous action of the group G, and the denominator tends to infinity. If $L(G)$ is the whole of $\mathbb{S}^1$, we are done. If not, since $L(G)$ is a closed set, around every point $\eta \in \mathbb{S}^1 \setminus L(G)$ there exists an open neighbourhood U of η so that U is fully contained in $\mathbb{S}^1 \setminus L(G)$. Then, let $\varepsilon > 0$ be not more than the diameter of the set U and define

$$U(\varepsilon) := \{x \in \mathbb{D}^2 \cup \mathbb{S}^1 : |x-\eta| < \varepsilon\}.$$

Then, as there are no limit points in this set, there are only finitely many orbit points of 0 in $U(\varepsilon)$, so, for the same reason as above, $\mu_x(U(\varepsilon)) = 0$. Thus, μ_x is supported on $L(G)$.

□

Remark A.2.8. Obviously, if the group G is elementary, the Patterson measure consists entirely of atoms. From here on, as usual, we will always assume that every Fuchsian group is non-elementary.

The Patterson measure has the following invariance property.

Lemma A.2.9. *Let G be a geometrically finite Fuchsian group. For each $g \in G$,*

$$\mu_0(g(A)) = \mu_{g^{-1}(0)}(A).$$

Proof. Fix an arbitrary $s > \delta$ and let $g \in G$. Then, by definition, we have that

$$\mu_{0,s}(g(A)) \quad = \quad \frac{1}{\Sigma_s(0,0)} \sum_{h \in G} e^{-sd_h(0,h(0))} \delta_{h(0)}(g(A)).$$

Set $f = g^{-1} \circ h$, so that $h(0)$ is an element of $g(A)$ precisely when $f(0)$ is an element of A. Then, as h runs over G, so too does f and we obtain that

$$\mu_{0,s}(g(A)) \quad = \quad \frac{1}{\Sigma_s(0,0)} \sum_{f \in G} e^{-sd_h(g^{-1}(0),f(0))} \delta_{f(0)}(A) = \mu_{g^{-1}(0),s}(A).$$

Since s was arbitrary, the proof is finished.

□

It was shown in [64] that the Patterson measure is a δ-harmonic measure. We state the result here only for the measure μ_0, but it is valid in more generality, see Theorem 3.4.1 of [62].

Lemma A.2.10. *For every Borel set $E \subset \mathbb{S}^1$ and every $g \in G$,*

$$\mu_0(g(E)) = \int_E P(g^{-1}(0),\xi)^\delta \, d\mu_0(\xi).$$

Proof. Recall that for each Borel subset A of $\mathbb{D}^2 \cup \mathbb{S}^1$ and for all $t > 0$, we define $A(\xi,t)$ to be that part of A within Euclidean distance t of ξ. By Corollary A.2.2, for all $s > \delta$, all positive ε and for all sufficiently small t, we have that

$$\left(\left(P(g^{-1}(0),\xi)\right)^s - \varepsilon\right) \mu_{0,s}(A(\xi,t)) \leq \mu_{g^{-1}(0),s}(A(\xi,t)) \leq \left(\left(P(g^{-1}(0),\xi)\right)^s + \varepsilon\right) \mu_{0,s}(A(\xi,t)).$$

If $(s_j)_{j\geq 1}$ is a sequence of real numbers monotonically decreasing to δ so that $\mu_0 = \lim_{s_j \to \delta} \mu_{0,s_j}$, then there exists a subsequence $(s_{j_k})_{k\geq 1}$ with the property that $\mu_{g^{-1}(0)} = \lim_{s_{j_k}\to\delta} \mu_{g^{-1}(0),s_{j_k}}$. Therefore,

$$\left(\left(P(g^{-1}(0),\xi)\right)^\delta - \varepsilon\right) \mu_0(A(\xi,t)) \leq \mu_{g^{-1}(0)}(A(\xi,t)) \leq \left(\left(P(g^{-1}(0),\xi)\right)^\delta + \varepsilon\right) \mu_0(A(\xi,t)).$$

Thus, the measures μ_0 and $\mu_{g^{-1}(0)}$ are absolutely continuous to each other and, moreover, the Radon-Nikodym derivative is given by

$$\frac{d\mu_{g^{-1}(0)}}{d\mu_0}(\xi) = \left(P(g^{-1}(0),\xi)\right)^\delta.$$

(For details on the differentiation of measures, the reader is referred either to Section 2 of Mattila [57] or to Chapter 7 of Rudin [68].) This finishes the proof.

□

Corollary A.2.11. *The Patterson measure μ_0 is a δ-conformal measure. That is,*

$$\mu_0(g(E)) = \int_E |g'(\xi)|^\delta \, d\mu_0(\xi).$$

Proof. From Lemma 6.1.3 (2), we have, for each $g \in G$ and $\xi \in \mathbb{S}^1$, that

$$|g'(g^{-1}(0))||g'(\xi)| = \frac{|0-g(\xi)|^2}{|g^{-1}(0)-\xi|^2} = \frac{1}{|g^{-1}(0)-\xi|^2}.$$

It follows from the chain rule that

$$g'(g^{-1}(0)) = ((g^{-1})'(0))^{-1}$$

and so, by Lemma 6.1.3 (1), we obtain that

$$|g'(\xi)| = \frac{|(g^{-1})'(0)|}{|g^{-1}(0)-\xi|^2} = \frac{1-|g^{-1}(0)|^2}{|g^{-1}(0)-\xi|^2} = P(g^{-1}(0),\xi).$$

□

Sullivan was the first to give a geometric interpretation of the Patterson measure (for this reason, it is sometimes referred to as the Patterson-Sullivan measure). An example of this geometric insight is the interpretation of δ-harmonicity to yield what is called the Sullivan Shadow Lemma ([79],[80], see also [62]).

Lemma A.2.12. (Sullivan's Shadow Lemma). *Let G be a Fuchsian group. Then, for all sufficiently large Δ and for every $g \in G$,*

$$\mu_0(\Pi(B(g(0),\Delta)) \asymp e^{\delta d_h(0,g(0))}$$

Sketch of proof. The first ingredient of the proof is the estimate (which depends on Δ),

$$|\Pi(B(g(0),\Delta))| \asymp 1-|g(0)|. \tag{A.2}$$

This is established in almost exactly the same way as the proof of Proposition 6.2.3. Secondly, noting that for $\xi \in \Pi(B(g(0),\Delta))$,

$$P(g(0),\xi) = \frac{1-|g(0)|^2}{|g(0)-\xi|^2} \asymp \frac{1-|g(0)|}{|g(0)-\xi|^2} \text{ and } |g(0)-\xi| \asymp 1-|g(0)|,$$

we infer that

$$P(g(0),\xi) \asymp (1-|g(0)|)^{-1}. \tag{A.3}$$

Finally, putting (A.2) and (A.3) together with δ-harmonicity and the fact that for all but finitely many $g \in G$, we have that

$$\mu_0(g^{-1}(\Pi(B(g(0),\Delta))) \asymp 1,$$

we obtain that

$$\begin{aligned} 1 &\asymp \mu_0(g^{-1}(\Pi(B(g(0),\Delta))) = \int_{\Pi(B(g(0),\Delta)} P(g(0),\xi)^\delta \, d\mu_0(\xi) \\ &\asymp \int_{\Pi(B(g(0),\Delta)} \left(\frac{1}{1-|g(0)|}\right)^\delta d\mu_0(\xi) = (1-|g(0)|)^{-\delta}\mu_0(\Pi(B(g(0),\Delta)). \end{aligned}$$

□

The Shadow Lemma gives us a way of estimating the measure of shadows of balls around orbit points of zero. It can also be phrased in terms of $\Delta(\xi_t)$, the distance of the point ξ_t from the orbit of zero. For $\xi \in L(G)$, and positive t, let $b(\xi_t, e^{-t})$ denote the shadow of the geodesic which intersects the ray s_ξ orthogonally at the point ξ_t. One immediately verifies that $b(\xi_t, e^{-t})$ is an arc of $\mathbb{S}^1$ centred at the point ξ with radius comparable to e^{-t}. As long as $\Delta(\xi_t)$ is bounded, which is to say that as long as we are traveling towards a radial limit point, we can use the Shadow Lemma to estimate the measure of $b(\xi_t, e^{-t})$.

The following estimate, called the *Global Measure Formula* by B. Stratmann and S. Velani [78], gives a uniform estimate for the measure of balls in $\mathbb{S}^1$ around any limit point of G. (Note that this estimate was first given by Sullivan [80].) In order to state the formula, we require the following notation. Define $k(\xi_t)$ to be equal to 1 if ξ_t is inside some standard horoball H_g and let $k(\xi_t)$ be equal to δ otherwise. (We have stated this only for the case of a Fuchsian group; in the paper [78], the authors are concerned with Kleinian groups, that is, discrete groups of isometries of three or more dimensional hyperbolic space.)

Theorem A.2.13. Global Measure Formula. *Let G be a non-elementary geometrically finite Fuchsian group with parabolic elements. If $\xi \in L(G)$ and t is positive, then*

$$\mu_0(b(\xi_t, e^{-t})) \asymp e^{-t\delta} e^{-(\delta - k(\xi_t))\Delta(\xi_t)}.$$

Proof. See Theorem 2 of [78].

□

Appendix B

Continued fractions and the modular group

In this second appendix, we outline a beautiful result which links the geometry of the modular group (defined in Example 6.1.23) to the regular continued fraction expansion of real numbers. This link can originally be traced back to a paper by Artin [3] in 1924, but is probably better known now from the expository works on the subject published in the 1980s by Series, see [73] and [74]. The first step is to describe a certain tessellation of the upper-half plane, known as the Farey tessellation. That this is connected to the Farey map gives the first clue that there is a connection between hyperbolic geometry and continued fractions.

Recall the definition of a fundamental domain from Chapter 6. One particular construction for the fundamental domain of a Fuchsian group can be very useful. This is the Dirichlet fundamental domain, which we now define. Strictly speaking, that this genuinely defines a fundamental domain requires proof, but this can be found in any book on hyperbolic geometry.

Definition B.0.14. Let G be a Fuchsian group acting on $\mathbb{D}^2$ and suppose that a point $z_0 \in \mathbb{D}^2$ is not the fixed point of any elliptic transformation belonging to G. Then the *Dirichlet fundamental domain* for G at the point z_0 is given by

$$D_{z_0}(G) := \{z \in \mathbb{D}^2 : d_h(z, z_0) < d(z, g(z_0)) \text{ for all } g \in G \setminus \{id\}\}.$$

In other words, for each $g \in G \setminus \{id\}$, consider the perpendicular bisector of the geodesic segment between z_0 and $g(z_0)$. This divides $\mathbb{D}^2$ into two half-spaces. Let S_g refer to the half-space containing z_0. Then,

$$D_{z_0}(G) = \bigcap_{g \in G \setminus \{id\}} S_g.$$

Recall the modular group:

$$PSL_2(\mathbb{Z}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d, \in \mathbb{Z} \text{ and } ad - bc = 1 \right\} / \{\pm I\}.$$

Also recall that the modular group is generated by one parabolic element P and one hyperbolic element Q, where

$$P(z) = z+1 \text{ and } Q(z) = -\frac{1}{z}.$$

We can calculate a Dirichlet fundamental region for the modular group at the point $2i$. First note that $2i$ is not fixed by any element of $PSL_2(\mathbb{Z})$. Then, let $D := S_P \cap S_{P^{-1}} \cap S_Q$, where

$$S_P = \{z \in \mathbb{H} : \mathrm{Re}(z) < 1/2\}, \quad S_{P^{-1}} = \{z \in \mathbb{H} : \mathrm{Re}(z) > -1/2\}$$

and

$$S_Q = \{z \in \mathbb{H} : |z| > 1\}.$$

It is readily verified that these are the half-spaces defined by the elements P, P^{-1} and Q exactly in the way described above. We want to show that $D_{2i}(G) = D$. It is clear that $D_{2i}(G) \subseteq D$. For the other direction, suppose by way of contradiction that $D_{2i}(G)$ is a proper subset of D. This would mean that there exists some $z \in D$ and $g \in PSL_2(\mathbb{Z})$ such that $g(z)$ also belongs to D. Suppose that this g is given by $g(z) = \frac{az+b}{cz+d}$, where $a,b,c,d \in \mathbb{Z}$ and $ad-bc = 1$. First, notice that

$$\begin{aligned} |cz+d|^2 &= c^2|z|^2 + 2cd\mathrm{Re}(z) + d^2 > c^2 - |2cd\mathrm{Re}(z)| + d^2 \\ &> c^2 - |cd| + d^2 = (|c|-|d|)^2 + |cd| \geq 1, \end{aligned}$$

since c and d cannot both equal zero simultaneously. Using the easily verified fact that $\mathrm{Im}(g(z)) = \mathrm{Im}(z)/|cz+d|^2$, we then deduce that

$$\mathrm{Im}(g(z)) = \frac{\mathrm{Im}(z)}{|cz+d|^2} < \mathrm{Im}(z).$$

On the other hand, we can argue in exactly the same way with z replaced by $g(z)$ and g replaced by g^{-1}. In this way, we obtain

$$\mathrm{Im}(z) = \mathrm{Im}(g^{-1}(g(z)) < \mathrm{Im}(g(z)).$$

This is the desired contradiction and hence $D \subseteq D_{2i}(G)$.

We begin by creating a new fundamental domain for the modular group, by cutting the Dirichlet fundamental domain D in half along the imaginary axis and shifting the left half by applying the map $P : z \mapsto z+1$. This gives the region F, a quadrilateral with vertices $\{i, i+1, 1/2(1+\sqrt{3}i), \{\infty\}\}$ (see Figure B.1 below). Now, let $S \in PSL_2(\mathbb{Z})$ be given by

$$S(z) := -\frac{1}{z-1} = Q \circ P(z).$$

The images of F under S and S^2 are also shown in Figure B.1.

The union $\Delta := F \cup S(F) \cup S^2(F)$ is the ideal triangle[1] with vertices $\{0, 1, \{\infty\}\}$. Finally, denote by $\mathbb{F}$ the tessellation of the upper half-plane obtained from the images $\{g(\Delta) : g \in PSL_2(\mathbb{Z})\}$. This is what is known as the Farey tessellation. It is illustrated in Figure B.2.

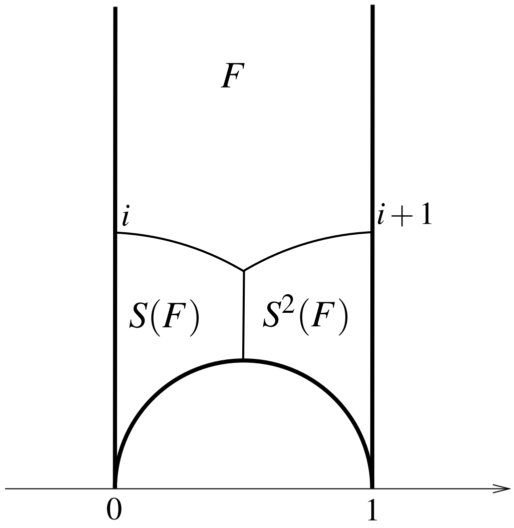


Figure B.1: The ideal triangle with vertices at 0, 1 and $\{\infty\}$.

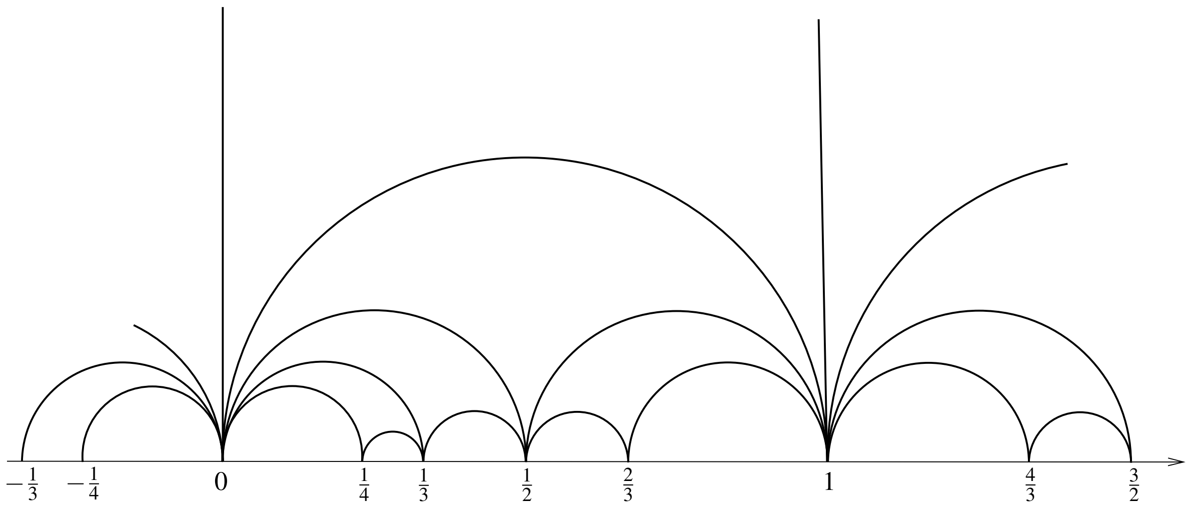


Figure B.2: The Farey tesselation.

Notice that the boundary lines of $\mathbb{F}$ can most easily be described by

$$\partial\mathbb{F} := \{g(\mathrm{Im}^+(z)) : g \in PSL_2(\mathbb{Z})\},$$

where $\mathrm{Im}^+(z)$ denotes the upper half of the imaginary axis. Also note that the vertices of the Farey tessellation are exactly the images under $PSL_2(\mathbb{Z})$ of the point at infinity; in other words, the vertices of $\mathbb{F}$ correspond on the modular surface M to the cusp associated with the parabolic element P. These images are precisely the set of rational numbers and the point at infinity, $\mathbb{Q} \cup \{\infty\}$. Moreover, if p/q and p'/q' are rational numbers such that $pq' - p'q = 1$, then they are linked by a line of $\partial\mathbb{F}$. In fact, the lines of the Farey tessellation are built up by successively joining each neighbouring pair of vertices to their mediant, as can be seen in Figure B.2 above. Since we can start with 0 and 1 and translate using the element P, the vertices appear as the Stern-Brocot series (see Remark 1.2.12 above). We are interested in geodesics travelling through the upper half-plane and the way that they cut through the lines of the Farey tessellation. Let us now describe the *cutting sequence* of a geodesic. An oriented geodesic ℓ is divided into segments as it cuts across the triangles which compose $\mathbb{F}$. Travelling along ℓ in the positive direction, each of these segments intersects a triangle so that there is a single vertex on either the left or the right of ℓ. If the single vertex is on the right-hand side, we label the segment R; if it is on the left, we label the segment L. This is illustrated in Figure B.3 below. We say that ℓ changes type at a point where we find two neighbouring segments with different labels. If it happens that our geodesic ℓ either starts or ends at a rational number, we only have finitely many segments to label, and for the final one, we could choose either R or L. In order to be consistent, we will label the final (or initial) segment with whichever label the one immediately before (or after) has. Therefore, there is no type change point directly before termination into a vertex of $\mathbb{F}$. For the special cases of geodesics joining -1 to 1 and 1 to -1, we label these as RyL and LyR, where y indicates a type change point at the imaginary axis. So, each geodesic has a cutting sequence given by $\ldots R^{n_{-2}}L^{n_{-1}}yR^{n_1}L^{n_2}\ldots$ or $\ldots L^{n_{-2}}R^{n_{-1}}yL^{n_1}R^{n_2}\ldots$, with respect to some type change point y.

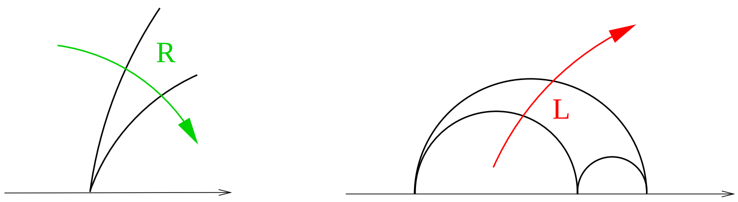


Figure B.3: The method of labelling an oriented geodesic as it travels through the triangles of the Farey tessellation.

Let $\mathscr{A}$ denote the set of all geodesics ℓ in $\mathbb{H}$ with ℓ_- and ℓ_+ satisfying $|\ell_+| \geq 1, 1 < |\ell_-| \leq 1$ and $\ell_-\ell_+ < 0$. Any such geodesic has a type change point on the imaginary axis in a point we will denote by y_ℓ. If $\ell_+ \geq 1$ then ℓ has cutting sequence $\ldots L^{n_{-2}}R^{n_{-1}}y_\ell L^{n_1}R^{n_2}\ldots$ and if $\ell_- \leq -1$, then ℓ has cutting sequence $\ldots R^{n_{-2}}L^{n_{-1}}yR^{n_1}L^{n_2}\ldots$. Note that any geodesic in $\mathbb{H}$ can can be sent

[1]A hyperbolic triangle is said to be ideal if all of its angles are equal to zero.

by an element of the modular group to a geodesic in the set $\mathscr{A}$, so the set $\mathscr{A}$ projects to the set of all geodesics on the modular surface M.

Now, since each of the boundary lines of $\mathbb{F}$ are the image of the upper half of the imaginary axis, it follows that they all project to the same line on the modular surface M. We will denote this line by I. Recall that the unit tangent bundle T_1M is the collection of all unit tangent vectors with base points on M. We consider a subset $X \subseteq T_1M$ consisting of all those unit tangent vectors with base point x in the line I which point along a geodesic that changes type at the point x. Let us now define a function $\phi : \mathscr{A} \to X$. The function ϕ maps the type change point y_ℓ on the imaginary axis to the corresponding unit tangent vector with base point $\pi(y_\ell)$ on $I \subset M$ which points in the direction of the geodesic ℓ. Note here that there is a slight ambiguity in the definition of the function ϕ as it is given in [74]. If we consider some $0 < a < 1$ and the geodesic ℓ_a with left endpoint at $-a$ and right endpoint at $1+a$ it is clear that there are two unit tangent vectors at the point $\pi(y_{\ell_a}) \in I$ which both point along the geodesic ℓ_a, but in different places when lifted back to $\mathbb{H}$. It is not immediately clear which vector we should take to be the image of y_{ℓ_a}. So, we really have to consider the tangent vectors in a small neighbourhood of the point y_ℓ for each geodesic $\ell \in A$ to ensure we always take the correct direction at a given base point. This observation was pointed out to the author by Anna Zielicz, then a student at St Andrews University, see [85].

The main result in [74] is the following.

Theorem B.0.15. *The map $\phi : \mathscr{A} \to X$given by $\phi(\ell) = \pi(u_{y_\ell})$ is surjective, continuous and open. It is injective except that the oppositely oriented geodesics joining $+1$ and -1 have the same image. Moreover, if u_x defines a geodesic in $\mathscr{A}$ with cutting sequence*

$$\dots L^{n_{-2}} R^{n_{-1}} x L^{n_1} R^{n_2} \dots$$

then $\ell = \phi^{-1}(u_x)$ has endpoints given by $\ell_+ = [n_1; n_2, n_3, \dots]$ and $\ell_- = -[n_{-1}, n_{-2}, n_{-3}, \dots]$. Alternatively, if the cutting sequence is

$$\dots R^{n_{-2}} L^{n_{-1}} x R^{n_1} L^{n_2} \dots$$

then $\ell = \phi^{-1}(u_x)$ has endpoints given by $\ell_+ = -[n_1; n_2, n_3, \dots]$ and $\ell_- = [n_{-1}, n_{-2}, n_{-3}, \dots]$.

Proof. First, to see that ϕ is surjective, if $u_x \in X$ defines a geodesic γ on M, then there is a lift of γ, call it ℓ, that has a type change point at some line S in $\partial\mathbb{F}$. Say this type change point occurs at the point η_ℓ and further suppose, without loss of generality, that the labeling of ℓ changes from L to R at η_ℓ. Choose $g \in PSL_2(\mathbb{Z})$ such that $g(\eta_\ell)$ lies on $\mathrm{Im}^+(z)$. Then, since labeling is invariant under the modular group, $g(\ell)$ also changes type from L to R at the imaginary axis. Bearing in mind that only geodesics in $\mathscr{A}$ change type at the imaginary axis, it follows, as $\phi(g(\ell) = u_x$, that the map ϕ is indeed surjective. For injectivity, suppose that there are two geodesics $\ell, \ell_1 \in \mathscr{A}$ with $\phi(\ell) = \phi(\ell_1) = u_y$. Both ℓ and ℓ_1, after crossing the imaginary axis at the points y_ℓ and y_{ℓ_1} respectively, can only travel into one of the regions $F, P^{-1}(F), Q(F)$ or $S(F) = Q \circ P^{-1}(F)$, as shown below in Figure B.4. Moreover, by considering the orientation as well as the crossing point, if ℓ first enters F, the only option for ℓ_1 (unless it actually equals ℓ), is to first go into the region $Q(F)$. The regions $P^{-1}(F)$ and $Q \circ P^{-1}(F)$ are paired up similarly.

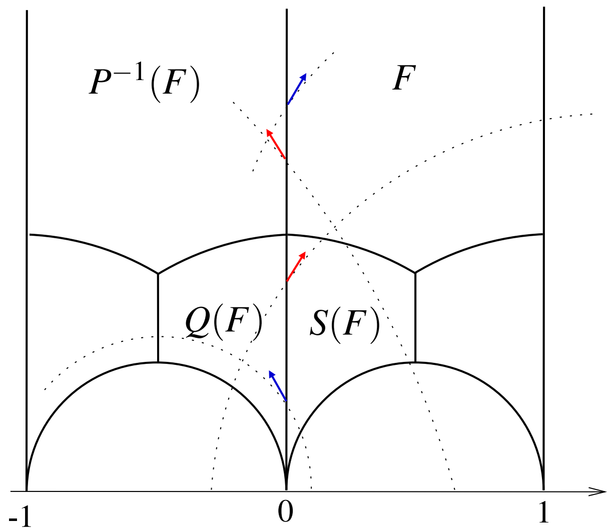


Figure B.4: The regions F and and $S(F)$ are paired up, as are the regions $P^{-1}(F)$ and $Q \circ P^{-1}(F)$.

Consequently, if $\phi(\ell) = \phi(\ell_1)$, it follows that $\ell_1 = Q(\ell)$. The only two distinct geodesics in $\mathscr{A}$ with this property are the two oppositely oriented geodesics joining 1 and -1.

For continuity and openness, we refer to [74]. It only remains to demonstrate the statements given concerning the endpoints of the geodesics in $\mathscr{A}$. First note that for the oppositely oriented geodesics joining 1 and -1, using the coding outlined above satisfies the conclusion here. So, suppose that $u_y \in X$ defines a geodesic on M that is different from either of the geodesics from -1 to 1. Then there exists a unique geodesic $\ell = \phi^{-1}(u_y)$ on $\mathbb{H}$ with cutting sequence given by either $\ldots L^{n_{-2}}R^{n_{-1}}yL^{n_1}R^{n_2}\ldots$ or $\ldots R^{n_{-2}}L^{n_{-1}}yR^{n_1}L^{n_2}\ldots$, where these can be infinite or finite. Suppose we are in the former case. (The second case can be proved by the same argument as the one that follows, but starting from the second step.) We have that $\ell_+ \geq 1$ and so the continued fraction expansion of ℓ_+ is given by $\ell_+ = [a_1; a_2, a_3, \ldots]$. It is geometrically obvious that $a_1 = n_1$.

If $\ell_+ = n_1$ we are done. If not, apply the map

$$Q \circ P^{-n_1} := \rho_1 : z \mapsto \frac{-1}{z - n_1}$$

to the geodesic ℓ. This reverses the orientation of ℓ, sending ℓ_- to the interval $(0,1]$ and ℓ_+ to the interval $(-\infty, -1)$. More precisely, $\rho_1(\ell_+) = [a_2; a_3, a_4, \ldots]$. With η_ℓ referring to the type change point of ℓ immediately after that at y_ℓ, we also have that $\rho_1(\eta_\ell) = y_{\rho_1(\ell)}$.

Now, it is clear just as before that since $\rho_1(\ell)$ changes type from R to L at the line $\mathrm{Re}(z) = -a_2 \in \partial\mathbb{F}$, that $a_2 = n_2$. If $\ell_+ = n_1 + 1/n_2$, we are done, if not, we continue in the same way by applying the map

$$Q \circ P^{n_2} := \rho_2 : z \mapsto \frac{-1}{z + n_2}.$$

Continuing in this manner until we either come to a stop with a finite continued fraction or indefinitely otherwise, we have shown that $\ell_+ = [n_1; n_2, n_3, \ldots]$. For the left endpoint, consider the geodesic $Q(\ell)$. First, we must reverse the orientation of this geodesic, otherwise it does not belong to the set $\mathscr{A}$. If ℓ has cutting sequence $\ldots L^{n_{-2}} R^{n_{-1}} y_\ell L^{n_1} R^{n_2} \ldots$ then $[Q(\ell)]^{-1}$, which travels from $Q(\ell_+)$ to $Q(\ell_-)$, has cutting sequence $\ldots R^{n_2} L^{n_1} Q(y_\ell) R^{n_{-1}} L^{n_{-2}} \ldots$. Hence, by the argument above, $Q(\ell_-) = -1/\ell_- = [n_{-1}; n_{-2}, n_{-3}, \ldots]$, as required.

$\square$

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