Applications of Equivariant Cohomology to Enumerative Geometry

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Abstract

We show how equivariant cohomology can be applied to enumerative geometry in three different settings: orbits of plane curves, strata of points on a line, and effective divisors on the moduli space of curves. We first give a brief introduction to equivariant cohomology. Then, we include three different applications that are essentially unchanged from their published versions and contain joint work with Mitchell Lee, Anand Patel, and Hunter Spink. We conclude with a short section with unpublished observations and conjectures stemming from a concrete connection between counting singularities and equivariant cohomology.

Contents

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Part 1. Introduction

1. Introduction

This thesis will cover three different applications of equivariant intersection theory, where appreciating the applications does not require knowledge of equivariant theory. It is our opinion that the tools of equivariant intersection theory are not widely used enough by algebraic geometers, even though a working knowledge can be obtained relatively quickly.

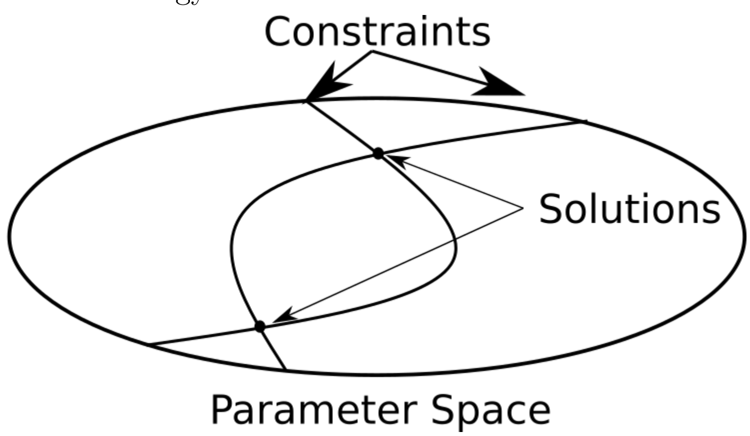
In some sense, the benefits of working equivariantly are psychological. Every equivariant construction can be restated in terms of a slightly more complicated analogue in usual intersection theory, but the benefits are twofold. First, working equivariantly highlights some approaches that wouldn't be obvious nonequivariantly. Secondly, and probably more important, there are results in the literature stated equivariantly that are harder to interpret without knowing the language.

The main example we have in mind is in the field of counting singularities, where we have Thom polynomials for singularities of maps and singularities of families of hypersurfaces. The main tool in this case is to take test families of maps or hypersurfaces parameterized by the classifying space of a torus (e.g. a product of infinite projective spaces). This can be clearly stated nonequivariantly but it would not seem like a natural approach, as it was not tried by algebraic geometers working independently on the same problem. It also seems like the researchers using the language of Thom polynomials have the stronger results.

We will concentrate on finding other applications of equivariant methods, though counting singularities will appear in Part 2 in the context of reducing our problem to counting quartic plane curves with prescribed singularities. For the remainder of the introduction, we will present a simplified view of enumerative geometry and equivariant intersection theory, and give some basics of equivariant intersection theory in Section 2.

1.1. **Enumerative Geometry.** Algebraic geometry is the study of varieties, geometric objects given as zero loci of polynomial equations, and intersection theory aims to understand the (singular) cohomology rings of varieties and cohomology classes of their subvarieties.

Enumerative geometry aims to count such objects subject to certain constraints. For example, one may ask: how many lines in 3-space intersect 4 generic lines? Instead of actually finding the lines, we can use the fact that the number of points in the intersection of two submanifolds can be computed using the cup product of their classes in singular cohomology.



In our example, the Grassmannian $\mathbb{G}(1,3)$ is a 4-dimensional variety parameterizing all complex lines in $\mathbb{C}P^3$. The lines in $\mathbb{C}P^3$ meeting a general line form a codimension 1 locus in $\mathbb{G}(1,3)$ whose class α in cohomology is well-understood. One can then compute that α^4 in the cohomology ring of $\mathbb{G}(1,3)$ is 2 times the class of a point, showing there are 2 lines in 3-space meeting 4 generic lines.[1]

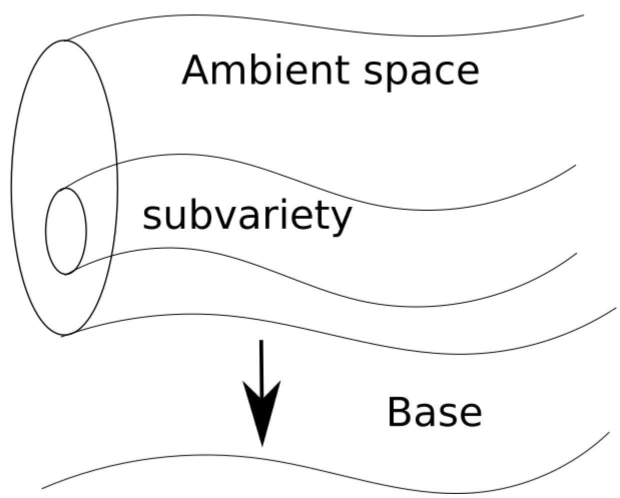
1.2. **Equivariant Cohomology.** Given an action of a topological group G on a space X, the equivariant cohomology ring $H^\bullet_G(X)$ is the ordinary cohomology ring of $(X \times EG)/G$. The idea is to replace X with a homotopy equivalent space with a free G-action, before taking the quotient by G.

[1] Those two lines might not be defined over the reals, which is why we work over the complex numbers. See [18, Section 1.2] for the expected number of real lines meeting four real lines and [95] for a count that works over arbitrary base fields.

Just like the singular cohomology ring is analogous to the Chow ring, there is an equivariant Chow ring corresponding to the equivariant cohomology. The foundations were given by Totaro, Edidin and Graham [96, 27, 29], where the idea is to approximate the infinite dimensional space EG with a sequence of finite dimensional varieties.

Perhaps the most compelling reason for the correctness of the definition of the equivariant Chow ring $A_G^\bullet(X)$ is that it is the integral Chow ring of the quotient stack $[X/G]$, defined functorially [27, Section 5.3]. This connection has been exploited to compute Chow rings of various moduli spaces, see for example [25, 24, 48, 88, 14, 72].

However, in this thesis, we will exploit a different connection between equivariant intersection theory and algebraic geometry. This connection is more naive and does not require knowledge of stacks or moduli spaces in algebraic geometry.



Given an algebraic group G acting on a variety X and a principal G-bundle $\mathcal{P} \to B$, we have the X-bundle $(X \times \mathcal{P})/G \to B$. Similarly, given a G-invariant subvariety $Z \subset X$, we have a subbundle $(Z \times \mathcal{P})/G \subset (X \times \mathcal{P})/G$, and it is natural to ask for its class.

In this context, the equivariant cohomology ring $H_G^\bullet(X)$ is the ordinary cohomology ring of $(X \times EG)/G$, where we let $\mathcal{P} \to B$ be the universal G-bundle $EG \to BG$.

The equivariant class $[Z] \in H_G^\bullet(X)$ is the class of $(Z \times EG)/G \subset (X \times EG)/G$. Therefore, the equivariant class $[Z] \in H_G^\bullet(X)$ contains the information of the class of the subbundle $(Z \times \mathcal{P})/G \subset (X \times \mathcal{P})/G$ for every principal G-bundle $\mathcal{P} \to B$.

Example 1.1. (Porteous formula) Given a map of two vector bundles $V \to W$ over a complex variety B, the Porteous formula computes the class of $\{b \in B \mid \operatorname{rank}(V_b \to W_b) \leq k\}$ for each k [32, Chapter 12].

If V and W are of ranks r_1 and r_2 respectively, then we can let X be the vector space $\operatorname{Hom}(\mathbb{C}^{r_1}, \mathbb{C}^{r_2})$, $G = GL_{r_1} \times GL_{r_2}$, and $Z \subset \operatorname{Hom}(\mathbb{C}^{r_1}, \mathbb{C}^{r_2})$ consist of matrices of rank at most k. The $GL_{r_1} \times GL_{r_2}$-equivariant class of Z is equivalent to Porteous formula, and allows for alternate derivations [43, Section 6]. Besides the psychological advantage of working with a G-invariant subvariety $Z \subset X$ instead of inside a non-trivial X-bundle, there are various methods for computing equivariant cohomology classes [44].

2. Background on Equivariant Cohomology in Algebraic Geometry

In this thesis, we will assume the basics of algebraic geometry [58, 100] and intersection theory [32]. However, since equivariant cohomology is not completely standard yet as a tool in intersection theory, we will give an abbreviated introduction that is general enough for our purposes. We will not strive for complete generality in this section, instead we will try to focus on the special cases that will suffice for most of the thesis, focusing on examples. The author originally learned equivariant cohomology from David Anderson's notes [10], and feels they are the best source for a quick and clear introduction to the topic. The reader can also see the original papers introducing equivariant methods in the algebraic setting [27, 29]. We will let G denote an algebraic group. When we appeal to singular cohomology, then it is assumed we are working over the complex numbers, but the constructions for equivariant intersection theory should hold regardless of the characteristic of the base field.

2.1. **Equivariant Chow ring of a point.** The equivariant cohomology ring $H^\bullet_G(\mathrm{pt})$ is $H^\bullet(BG)$. For example,

$$H^\bullet_{GL_n}(\mathrm{pt}) := H^\bullet(BGL_n) = H^\bullet(G(n,\infty)) = \mathbb{Z}[c_1,\dots,c_n].$$

Similarly, for the diagonal torus $T \subset GL_n$,

$$H^\bullet(BT) = H^\bullet((\mathbb{P}^\infty)^n) = \mathbb{Z}[t_1,\dots,t_n],$$

where $t_1,\dots,t_n$ are naturally identified with the standard characters of the torus T. The map $T \to GL_n$ induces a map

$$H^\bullet(BGL_n) \to H^\bullet(BT),$$

sending c_i to the i^{th} symmetric function in the characters $t_1,\dots,t_n$. This map is equivalently induced by $(\mathbb{P}^\infty)^n \dashrightarrow G(n,\infty)$ as the indeterminacy locus happens in infinite codimension.

Remark 2.1. The classes c_i in $H^\bullet(BGL)n)$ are the chern classes of complex vector bundles and the fact that $H^\bullet(BGL_n) \to H^\bullet(BT)$ is injective is equivalent to the splitting principle. Just as BGL_n classifies vector bundles, BT classifies totally split vector bundles with a choice of splitting.

In order to define the equivariant Chow rings $A^\bullet_{GL_n}(\mathrm{pt})$, we replace $G(n,\infty)$ and $\mathbb{P}^\infty$ with their finite dimensional approximations $G(n,N)$ and $\mathbb{P}^N$, as the groups $A^i(G(n,N))$ and $A^i(\mathbb{P}^N)$ stabilize for fixed i and $N >> 0$. To define $A^\bullet_G(\mathrm{pt})$ for an arbitrary algebraic group G, one needs to approximate BG. To do this, one takes a representation V of G on which G acts freely away from a set Z of large codimension and let $(V\backslash Z)/G$ be the approximation for BG. See [27, Section 2.2] for a reference.

2.2. **Equivariant Chow ring of a variety.** Let X be a smooth variety with a G action. The equivariant cohomology ring $H^\bullet_G(X)$ is the cohomology ring of $H^\bullet(X \times^G$

EG). Since $EG/G = BG$, this specializes to $H^\bullet_G(\mathrm{pt})$ if X is a point. However, in general, $X \times^G EG \to BG$ is an X-bundle.

Similarly to Section 2.1 above, the definition of $A^i_G(X)$ is given by finding a finite dimensional approximation $EG_N \to BG_N$ to the universal principal G-bundle $EG \to BG$. If $G = GL_n(\mathbb{C})$, then this is given by $A^i(X \times^G F(S))$ where $S \to G(n, N)$ is the tautological subbundle and $F(S) \to G(n, N)$ is the associated frame bundle for $N >> 0$.

2.3. **Examples of equivariant Chow rings.** Equivariant Chow rings can sometimes be computed directly from the definitions. In this thesis, the equivariant Chow rings are often very simple, with the following two basic examples sufficing for most applications.

2.3.1. *Affine space.* If we have a G-action on affine space $\mathbb{A}^n$ acting as a subgroup of GL_n, then $A^\bullet_G(\mathbb{A}^n) \cong A^\bullet_G(\mathrm{pt})$ as the Chow ring of a vector bundle is isomorphic to that of its base.

2.3.2. *Projective space.* Given a G-action on affine space $\mathbb{A}^n$ acting as a subgroup of GL_n, then

$$A^\bullet_G(\mathbb{P}^{n-1}) = A^\bullet_G(\mathrm{pt})[H]/(H^n + c_1^G H^{n-1} + \cdots + c_n^G).$$

Here, c_i^G are the equivariant chern classes of $\mathbb{A}^n$, viewed as an equivariant vector bundle over a point. Equivariant chern classes can be defined by using the usual chern classes and finite approximations of $EG \to BG$ as above. The presentation of the Chow ring follows from the projective bundle theorem, and can be viewed as an equivariant version of the projective bundle theorem.

2.3.3. *Extensions.* Similarly to the above, one can find the equivariant Chow ring of a product of projective spaces by applying the projective bundle theorem iteratively.

Also, the equivariant Chow ring of a Grassmannian can be deduced from the Chow ring of a Grassmannian bundle [32, Theorem 9.18].

2.4. **Equivariant class of a subvariety.** If X is a smooth variety with a G-action and $Z \subset X$ is a subvariety preserved by G, then one can define its equivariant cohomology class $[Z]_G \in H^\bullet_G(X)$ as the usual cohomology class

$$[Z \times^G EG] \in H^\bullet_G(X) = H^\bullet(X \times^G EG). \tag{2.1}$$

Algebraically, one can repeat the same construction, as long as we replace $EG \to BG$ with a finite dimensional approximation as above. The equivariant class $[Z]_G$ can be regarded as a universal formula for the class of the Z-bundle $Z \times^G P$ inside of the X-bundle $X \times^G P$ for every principal G-bundle $P \to B$. In equivariant cohomology, this follows from the construction. In equivariant Chow rings, this is less obvious but can also be shown quickly. This is given in the arXiv preprint [94, Section 2.2] with Hunter Spink, which can also be found in this thesis Section 12.2.

2.4.1. *Example: a linear subspace.* This thesis is essentially devoted to examples of equivariant classes of subvarieties and their applications. In terms of basic examples, the equivariant class of a linear subspace is equivalent to the formula for the projectivization of a subbundle [32, Proposition 9.13].

Instead of deducing it from [32, Proposition 9.13], we present an alternative way using equivariant intersection theory. Given a subbundle $W \subset V$, we want to compute the class $[\mathbb{P}(W)] \in A^\bullet(\mathbb{P}(V))$. By the splitting principle, it suffices to consider the case where W and V are totally split and W is a subset of the factors.

This is equivalent to computing the T-equivariant class of a torus invariant linear space $\Lambda \subset \mathbb{P}^n$, where $n = \operatorname{rank}(V) - 1$ and Λ is defined by the vanishing of the first $\dim(V) - \dim(W)$ coordinates. Let $\widetilde{\Lambda} \subset \mathbb{A}^{n+1}$ be the affine cone over Λ. Then, under the surjection

$$\mathbb{Z}[H, t_0, \ldots, t_n] \cong A^\bullet_{T\times\mathbb{G}_m}(\mathbb{A}^{n+1}) \to A^\bullet_T(\mathbb{P}^n) \cong \mathbb{Z}[H, t_0, \ldots, t_n]/((H + t_0)\cdots(H + t_n)),$$

$[\widetilde{\Lambda}]$ maps to $[\Lambda]$. Now, to compute $[\widetilde{\Lambda}]$, we consider $\mathbb{A}^{n+1}$ as the affine bundle $\mathbb{A}^{n+1} \to \mathbb{A}^{\dim(W)}$, where $\widetilde{\Lambda}$ is the zero section. Therefore, the class of $\widetilde{\Lambda}$ is the top chern class of this totally split vector bundle, which is $(H + t_0)\cdots(H + t_{\dim(V)-\dim(W)-1})$.

To interpret this in the original setting of computing $[\mathbb{P}(W)] \in A^\bullet(\mathbb{P}(V))$, we specialize H to the $\mathscr{O}(1)$ class and the $t_0, \ldots, t_{\dim(V)-\dim(W)-1}$ to the chern roots of V/W.

Part 2. Equivariant classes of plane curve orbits

This part of the thesis contains the arXiv preprint [74] joint with Mitchell Lee and Anand Patel. The key observation was that the GL_3-equivariant class of a general GL_3 orbit closure in $\mathrm{Sym}^4\mathbb{C}^3$ (e.g. the orbit of a general quartic plane curve) can be computed by a degeneration to a double conic. Surprisingly, this reduces the problem of computing the equivariant class to a problem in enumerating A_6 singularities in a family of curves, which is known [65]. We also explore classes of more special orbits, yielding connections with enumerating D_6 and E_6 singularities as well.

Abstract: In a series of papers, Aluffi and Faber computed the degree of the GL_3 orbit closure of an arbitrary plane curve. We attempt to generalize this to the equivariant setting by studying how orbits degenerate under some natural specializations, yielding a fairly complete picture in the case of plane quartics

3. Introduction

Let V be an $(r+1)$-dimensional vector space, and let $F \in \mathrm{Sym}^d V^\vee$ be a nonzero degree d homogeneous form on V. F naturally produces two varieties, $O_F \subset \mathrm{Sym}^d V^\vee$ and $\mathbb{P}O_F \subset \mathbb{P}\,\mathrm{Sym}^d V^\vee$, namely the GL_{r+1}-orbit closures of F and $[F]$ respectively. Basic questions about the relationship between the geometry of $\mathbb{P}O_F$ and the geometry of hypersurface $\{F = 0\}$ remain unanswered. Consider, for example, the enumerative problem of computing the degree of $\mathbb{P}O_F$. The analysis of the degrees of these orbit closures was carried out for the first two cases $r = 1, 2$ in a series of remarkable papers by Aluffi and Faber [3, 4, 6, 7, 5, 8, 9]. For instance, Aluffi and Faber's computation in the special case $r = 2, d = 4$ of quartic plane curves yields the enumerative consequence: in a general 6-dimensional linear system of quartic curves, a general genus 3 curve arises 14280 times. When $\{F = 0\}$ is a hyperplane arrangement, the degree of $\mathbb{P}O_F$ was studied in [98, 75].

One can interpret the calculation of the degree of $\mathbb{P}O_F$ as computing the fundamental class $[\mathbb{P}O_F] \in A^\bullet(\mathbb{P}\operatorname{Sym}^d V^\vee)$. Since $\mathbb{P}O_F$ is evidently preserved by the action of GL_{r+1}, one obtains a natural equivariant extension of the problem: to compute the equivariant fundamental class $[\mathbb{P}O_F]_{GL_{r+1}} \in A^\bullet_{GL_{r+1}}(\mathbb{P}\operatorname{Sym}^d V^\vee)$. In simple terms, beginning with a rank $r+1$ vector bundle $\mathcal{V}$, the class $[\mathbb{P}O_F]$ encodes the universal expressions in the chern classes $c_1, ..., c_{r+1}$ appearing in the fundamental class of the relative orbit closure cycle $(\mathbb{P}O_F)_{\mathcal{V}} \subset \mathbb{P}\operatorname{Sym}^d \mathcal{V}^\vee$. This larger equivariant setting encapsulates many more enumerative problems. For instance, by studying the particular case $r = 2, d = 4$ we will show: a general genus 3 curve appears 510720 times as a 2-plane slice of a fixed general quartic threefold.

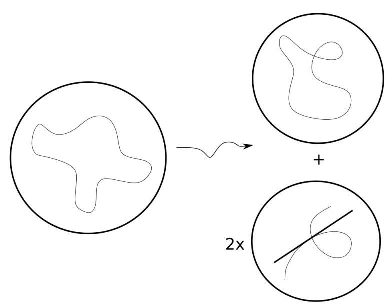
Very few equivariant classes $[\mathbb{P}O_F]_{GL_{r+1}}$ are known. When $d = 2$, the class $[\mathbb{P}O_F]_{GL_{r+1}}$ is determined by the rank of the quadric $F = 0$ and recovers the Porteous formula for symmetric maps [57]. The authors' work with H. Spink in [73] establishes the equivariant class when $F = 0$ defines a hyperplane arrangement. In this paper, we study the frontier case $r = 2$ of plane curves. As in [73], our strategy is to degenerate $[\mathbb{P}O_F]$ into a union of other orbits $[\mathbb{P}O_{F_i}]$ whose classes we can compute directly. To do this, we initiate a detailed study of how orbits of plane curves behave under particular specializations.

In the remainder of the introduction we summarize our results on degenerations of plane curve orbits. The particular case of quartic plane curves is especially beautiful – we deduce interesting relations among different orbit closures $[\mathbb{P}O_F]$ for F ranging over several types of quartic plane curves possessing special geometric properties. Since the computation of equivariant orbit classes does not have a strong presence in the literature, in the appendix we have included the cases of points on a line and cubic plane curves. The case of points on a line is done in two independent ways: one by specializing the results in [73] and the other by applying the Atiyah-Bott formula to the resolution of the orbit given in [3].

3.1. **Summary of Degenerations.** When degenerating orbit closures, it is often convenient to work with not the cycle of the orbit closure, but rather that cycle weighted by the number of linear automorphisms of the curve. In what follows, we will describe how these weighted orbit closures specialize.

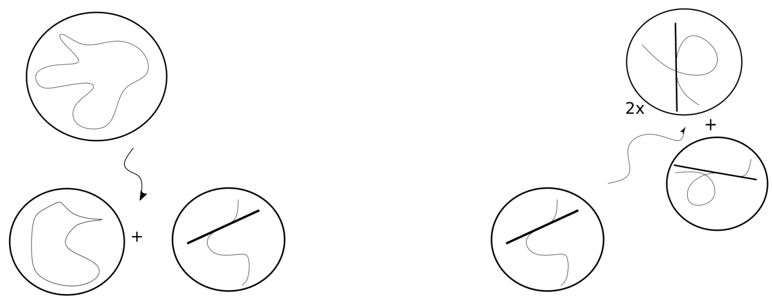
If a C_t is a family smooth curves specializing at $t = 0$ to a curve with nodes and cusps, this induces a specialization of (weighted) orbit closures. We obtain a description of which other orbits appear in the flat limit (see Theorem 8.1). To illustrate this theorem, we will describe what happens in the special case where the curve acquires a single node or a single cusp. The general case is simply a sum of the contributions for each node or cusp.

3.1.1. *Acquiring a node.* If C_t acquires a single node in the limit C_0, then as a limit of weighted orbits, one obtains the weighted orbit $\mathbb{P}C_0$ along with one other weighted orbit, $\mathbb{P}O_{C_{BN}}$, which occurs with multiplicity 2.



The curve C_{BN} is a nodal cubic union a $(d-3)$-fold line tangent to a branch of the node.

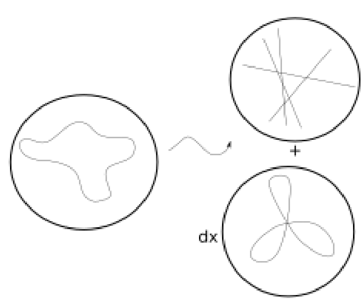
3.1.2. *Acquiring a cusp.* If C_t acquires a single cusp in the limit C_0, then as a limit of weighted orbits, one obtains the weighted orbit of the cuspidal curve $\mathbb{P}C_0$ along with another weighted orbit, $\mathbb{P}O_{C_{\text{flex}}}$.



The curve C_{flex} is a smooth cubic union a $(d-3)$-fold flex line. We can degenerate the weighted orbit of C_{flex} further to get the weighted orbit of C_{BN} with multiplicity 2 together with the weighted orbit of C_{AN}, where C_{AN} is a nodal cubic union a $(d-3)$-fold flex line (at a smooth point).

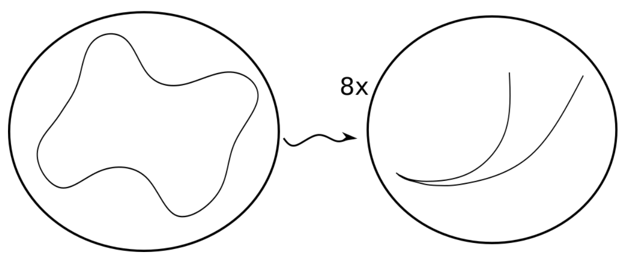
The subscripts of C_{AN} and C_{BN} are just to help us remember that the line meets "away from the node" or at a "branch of the node".

3.1.3. *Splitting off a line.* Since the equivariant class of the orbit closure of a union of lines can be deduced using the results of [73], it is natural to try to specialize a degree d plane curve to a union of lines. For example, we show that if C_t is a family of general curves and C_0 is a general union of lines, then in addition to the orbit class of $\mathbb{P}O_{C_0}$ we also get d times the weighted orbit class of a general irreducible plane curve with a multiplicity $d-1$ point.



More generally, it is also possible to specialize so that C_0 is a union of a general degree e curve union a $d-e$ general lines (see Proposition 7.2)

3.1.4. *Degeneration to the Double Conic.* Next suppose C_t is a family of general plane quartics specializing to a double conic. The orbit of the double conic has smaller dimension, so it will not appear as a component of the $t \to 0$ limit of orbit closures. We will show Theorem 10.5 that in this case, the weighted orbit of C_t specializes to 8 times the weighted orbit of a rational quartic curve with an A_6 singularity. Equations for quartic plane curves with A_6, D_6 and E_6 singularities are given in (5.1) in Section 5 below.



In addition, we can also let C_t be a family where the general member is a general curve with an A_n singularity where $3 \le n \le 6$ and find the limit is $(7-n)$ times the weighted orbit of a rational quartic curve with an A_6 singularity (see Theorem 10.7).

It is somewhat remarkable that the limit consists set-theoretically of the closure of the quartic plane curves with an A_6 singularity. This fact is related to the question of which planar quartics can yield a general hyperelliptic curve after semistable reduction. Furthermore, the multiplicity 8 we obtain corresponds to the 8 Weierstrass points of a genus 3 hyperelliptic curve. Tails arising from semistable reductions of singularities have been studied in [89, 59, 21, 38].

3.2. **Orbit classes of quartic curves.** In the specific setting of quartic curves, we find that the orbit class of an *arbitrary* smooth quartic can be deduced in a direct way from the orbit classes of special quartics with A_6 and E_6 singularities. We have already explained the relation with curves having an A_6 singularity above. By borrowing and adapting an idea of Aluffi and Faber [2, Theorem IV(2)], we specialize the GL_3-orbit closure of a general quartic plane curve to the GL_3-orbit of any particular smooth quartic plane curve (possibly having hyperflexes). In the flat limit, the GL_3-orbit closure of a rational quartic with an E_6 singularity appears (with multiplicity twice the number of hyperflexes of the limiting smooth quartic). In this way, we can express the orbit closure class of an arbitrary smooth quartic in terms of orbit classes of strata of rational curves with an A_6 singularity or with an E_6 singularity. From here, we conclude the analysis by invoking Kazarian's work [65] on counting A_6 and E_6 singularities in families of curves.

We can also compute the equivariant classes of orbit closure for many singular quartics using the degenerations in Section 3.1. In particular, the curves with a D_6 singularity arise when specializing to a node as in Section 3.1.1. We summarize the results in Theorem 3.1.

Theorem 3.1. *We can compute the equivariant classes of orbit closures of quartic plane curves for: an arbitrary smooth quartic, a general union of 4 lines, a general union of 2 lines and a conic, a general union of a cubic and a line, an irreducible quartic with δ ordinary nodes and κ ordinary cusps without hyperflexes, a general quartic with an A_n singularity for $n \leq 6$, a nodal cubic union a line tangent to a branch at the node, a cubic union a flex line, a smooth cubic union a flex line, a rational curve with an E_6 singularity. The formulas are given in Figure 1.*

For a plane curve $C \subset \mathbb{P}^2$ with an 8-dimensional PGL_3 orbit, the expressions p_C are defined to be the GL_3-equivariant classes $[O_C]_{GL_3}$ times the number of PGL_3-automorphisms of C. We note that the classes $[\mathbb{P}O_C]_{GL_3}$ are related to $[O_C]_{GL_3}$ by a simple substitution (see Proposition 4.4).

Remark 3.2. It is tempting to apply Kazarian's work on multisingularities [66] to the locus of curves with an A_5 and an A_1 singularity to find $p_{C_{AN}}$ in Figure 1. However, in addition to $O_{C_{AN}}$, there is another component of the locus of quartics with an A_5

and an A_1 singularity, namely two conics meeting at two points with multiplicities 3 and 1 respectively. Therefore, it was necessary to compute $O_{C_{AN}}$ independently (see Section 9).

3.2.1. *Sections of a Quartic Threefold.* Starting with a smooth quartic threefold $X \subset \mathbb{P}^4$, one obtains a rational map

$$\Phi : \mathbb{G}(2,4) \dashrightarrow \overline{M_3}$$

sending a general 2-plane $\Lambda \subset \mathbb{P}^4$ to the moduli of the plane curve $X \cap \Lambda$. Our calculation of the equivariant class of GL_3-orbit closure of the general quartic plane curve gives

Corollary 3.3. *If X is general, the map Φ has degree 510720.*

We note that the same computation as the proof of Corollary 3.3 also computes the number of times we see each curve in Theorem 3.1 with prescribed moduli (subject to transversality assumptions). This is given in Figure 2.

Example 3.4. The number of tricuspidal curves arising as a section of a quartic threefold is 27520 by applying Kazarian's theory of multisingularities. More precisely, the number can in principle be deduced from [66, Section 8], but the formula for 2-planes in $\mathbb{P}^4$ meeting a degree d hypersurface in a curve with three cusps can be found on Kazarian's website. From Figure 2, we get $510720 - 3 \cdot 2 \cdot 57600 = 6 \cdot 27520$, accounting for the 6 automorphisms of the tricuspidal quartic. This agrees with Kazarian's formula. However, for example, our Figure 2 also computes the number of 1-cuspidal and 2-cuspidal curves having prescribed moduli, which is not covered by the theory of multisingularities.

Remark 3.5. The formula for the general orbit in Theorem 3.1 and the answer 510720 has also been verified independently by the authors using the SAGE Chow ring package [93] to implement the resolution used by Aluffi and Faber [4] for smooth plane curves relatively. However, the computations were too cumbersome to verify by hand.

Proof of Corollary 3.3. We find the degree of Φ directly by choosing a general quartic plane curve $C \subset \mathbb{P}^2$ and counting the number of two planes Λ such that $X \cap \Lambda$ is isomorphic to C.

Let G be a quartic homogenous form cutting out a general quartic threefold $X \subset \mathbb{P}^4$, and let $\pi : \mathcal{S} \to \mathbb{G}(2,4)$ denote the rank 3 tautological subbundle over the Grassmannian. The form G defines a section of $\mathscr{O}_{\mathbb{P}(\mathcal{S})}(4)$ on $\mathbb{P}(\mathcal{S})$, which in turn induces a section $s : \mathbb{G}(2,4) \to \mathrm{Sym}^4(\mathcal{S}^\vee)$. Let $(O_C)_{\mathcal{S}} \subset \mathrm{Sym}^4(\mathcal{S}^\vee)$ be the relative orbit as in Definition 4.1. Since G is general, the section s will intersect $(O_C)_{\mathcal{S}}$ only in the interior of the relative orbit. Since C is general the intersection will consists of reduced points by generic reducedness in characteristic zero. In this paper, we will assume the characteristic is at least 7 (see Section 3.4), and there is a traversality argument that can be made to show that the intersection is also still reduced in this case. Therefore, s and $(O_C)_{\mathcal{S}}$ are smooth at the the scheme $s \cap (O_C)_{\mathcal{S}}$, and $\deg(\Phi) = \int_{\mathbb{G}(2,4)} s^*[(O_C)_{\mathcal{S}}]$. Expanding the formula for $p_C = [(O_C)_{\mathcal{S}}]$ in Theorem 3.1, we get

$$48384c_1(S)^6 + 88704c_1(\mathcal{S})^4c_2(\mathcal{S}) + 32256c_1(\mathcal{S})^2c_2(\mathcal{S})^2 - 34944c_1(\mathcal{S})^3c_3(\mathcal{S})$$
$$+2688c_1(\mathcal{S})c_2(\mathcal{S})c_3(\mathcal{S}) - 29568c_3(\mathcal{S})^2.$$

By evaluating on the Grassmannian, we conclude:

$$\deg \Phi = 48384 \cdot 5 + 88704 \cdot 3 + 32256 \cdot 2 - 34944 + 2688 - 29568 = 510720.$$

□

3.3. **Related Work.** This paper was heavily influenced and inspired by Aluffi and Faber's computation of degrees of orbit closures of plane curves of arbitrary degree. Zinger also computed the degree of the orbit closure of a general quartic as a special case of interpolating genus 3 plane curves with a fixed complex structure [102].

3.3.1. *Planar sections of a hypersurface of fixed moduli.* Counting linear sections of a hypersurface with fixed moduli has been considered in the case of line sections of a quintic curve [19] and generalized to line sections of hypersurfaces of degree $2r+1$ hypersurfaces in $\mathbb{P}^r$ [73] by extending the computation of orbits of points on a line [3] to the equivariant setting.

3.3.2. *Counting curves with prescribed singularities.* In addition to Kazarian's work [65], there have been independent efforts to count plane curve singularities at one point, including [15, 69, 91].[2] For us, Kazarian's work has the advantage that it can be directly applied to counting curve singularities in a family of surfaces. In fact, Kazarian's work applies to hypersurface singularities as well. We will not make essential use of Kazarian's generalization to multisingularities [66].

3.4. **Assumptions on the characteristic of the base field.** For our work on degenerations on orbits of plane curves, we will work over an algebraically closed field of arbitrary characteristic. For computations of equivariant classes, we work over an algebraically closed field of characteristic at least 7, because of our use of Kazarian's work on enumerating singularities. Kazarian works over the complex numbers, but it is possible to use equivariant intersection theory to show the existance of a universal formula algebraically. Then, the computation of his formulas using test classes [65, Section 2.5] can also be carried out algebraically. Also, one can give a transversality argument to show a general fiber of Φ in Corollary 3.3 is reduced in positive characteristic.

[2]For the reader's convenience, we note that numerical errors in [69] have been fixed in an updated arXiv version. Also, there are errors in the formulas for counting A_6 and A_7 singularities in [91].

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4. Definitions and Conventions

In this section, we define equivariant generalizations of predegrees of orbits of hypersurfaces as studied by Aluffi, Faber, and Tzigantchev [98, 3, 4, 5]. We will only deal with the case of points on a line, and plane cubics and quartics, but we give the general definition for clarity and to emphasize the potential for future work.

As a rule, the projectivization of a vector bundle parametrizes 1-dimensional subspaces, not quotients.

4.1. GL_{r+1}-equivariant Chow classes.

In this subsection, we will define the $GL(V)$-equivariant Chow class $[Z]_{GL_{r+1}}$ of $GL(V)$-invariant subvariety Z of $\operatorname{Sym}^d V^\vee$ and similarly for $\mathbb{P}(\operatorname{Sym}^d V^\vee)$. Our definitions are a special case of the definitions of equivariant intersection theory [27, 10], but we hope our setup will be self-contained and understandable without the general theory.

Definition 4.1. *Let V be an $r+1$-dimensional vector space and $Z \subset \operatorname{Sym}^d V^\vee$ be a $GL(V)$-invariant subvariety. Given a variety B and a rank $r+1$ vector bundle $\mathcal{V}$ on B, define the subvariety $Z_{\mathcal{V}}$ of the rank $\binom{d+r}{r}$ vector bundle $\operatorname{Sym}^d \mathcal{V}^\vee \to B$, to be the locus which restricts in every fiber to Z after choosing a basis.*

Although $Z_{\mathcal{V}}$ depends on B and $\mathcal{V}$, its class in $A^\bullet(B) \cong A^\bullet(\operatorname{Sym}^d \mathcal{V}^\vee)$ is a universal expression in chern classes of $\mathcal{V}$. By choosing $B = \mathbb{G}(r, N)$ for $N \gg 0$ and $\mathcal{V}$ to be the tautological subbundle, the construction of equivariant intersection theory [27] shows there is a single formula that works for all such choices of B and $\mathcal{V}$. Therefore, throughout this paper we will fix a base variety B and rank $r+1$ vector bundle $\mathcal{V}$ on B.

Definition 4.2. *Given Z as in Definition 4.1, let $[Z]_{GL_{r+1}}$ be the polynomial in $c_1, \ldots, c_{r+1}$ such that the class of $Z_{\mathcal{V}}$ is $[Z]_{GL_{r+1}}$ with the chern classes of $\mathcal{V}$ substituted for $c_1, \ldots, c_{r+1}$. Equivalently, $[Z]_{GL_{r+1}}$ is the $GL(V)$-equivariant class of Z in $A^{\bullet}_{GL_{r+1}}(\operatorname{Sym}^d V^{\vee}) \cong \mathbb{Z}[c_1, \ldots, c_{r+1}]$.*

Similarly to Definition 4.2, we can define an equivariant Chow class for a GL_{r+1}-invariant subvariety of $\mathbb{P}(\operatorname{Sym}^d V^{\vee})$. Following Definition 4.1, let $\mathbb{P}Z \subset \mathbb{P}(\operatorname{Sym}^d V^{\vee})$ be the projectivization of Z and $\mathbb{P}Z_{\mathcal{V}} \subset \mathbb{P}(\operatorname{Sym}^d \mathcal{V}^{\vee})$ be the projectivization of $Z_{\mathcal{V}}$. Then, as before, there is a single formula in the chern classes of $\mathcal{V}$ and $\mathscr{O}_{\mathbb{P}(\operatorname{Sym}^d \mathcal{V}^{\vee})}(1)$ that gives the class of $[\mathbb{P}Z_{\mathcal{V}}] \in A^{\bullet}(\mathbb{P}(\operatorname{Sym}^d \mathcal{V}^{\vee}))$ for every choice of $\mathcal{V} \to B$.

Definition 4.3. *Given Z as in Definition 4.1, let $[\mathbb{P}Z]_{GL_{r+1}}$ be the polynomial in $c_1, \ldots, c_{r+1}$ and H such that the class of $\mathbb{P}Z_{\mathcal{V}}$ is $[Z]_{GL_{r+1}}$ with the chern classes of $\mathcal{V}$ substituted for $c_1, \ldots, c_{r+1}$ and $\mathscr{O}_{\mathbb{P}(\operatorname{Sym}^d \mathcal{V}^{\vee})}(1)$ substituted for H. Equivalently, $[Z]_{GL_{r+1}}$ is the $GL(V)$-equivariant class of Z in*

$$A^{\bullet}_{GL_{r+1}}(\mathbb{P}(\operatorname{Sym}^d V^{\vee})) \cong \mathbb{Z}[c_1, \ldots, c_{r+1}][H]/(H^{r+1} + c_1 H^r + \cdots + c_{r+1}).$$

It seems like $[\mathbb{P}Z]_{GL_{r+1}}$ contains more information than $[Z]_{GL_{r+1}}$, but they are actually related by a simple substitution. Let $u_1, \ldots, u_{r+1}$ denote the formal chern roots of the vector bundle $\mathcal{V}$. More precisely, using the inclusion $\mathbb{Z}[c_1, \ldots, c_{r+1}] \hookrightarrow \mathbb{Z}[u_1, \ldots, u_{r+1}]$ where c_i maps to the ith elementary symmetric function, we can view $[Z]_{GL_{r+1}}$ as a symmetric polynomial in $u_1, \ldots, u_{r+1}$ and similarly $[\mathbb{P}Z]_{GL_{r+1}}$ as a polynomial in $u_1, \ldots, u_{r+1}$ and H symmetric in the u_i's.

Proposition 4.4 ([41, Theorem 6.1])**.** *We have:*

$$[Z]_{GL_{r+1}}(u_1, \ldots, u_{r+1}) = [\mathbb{P}Z]_{GL_{r+1}}(u_1, \ldots, u_{r+1}, 0)$$
$$[\mathbb{P}Z]_{GL_{r+1}}(u_1, \ldots, u_{r+1}, H) = [Z]_{GL_{r+1}}(u_1 - \frac{H}{d}, \ldots, u_{r+1} - \frac{H}{d}).$$

When $r = 1$ and 2, we will use (u, v) for (u_1, u_2) and (u, v, w) for (u_1, u_2, u_3), respectively.

4.2. **Weighting orbits by automorphism groups.**

Definition 4.5. *Given $X \subset \mathbb{P}(V)$ a degree d hypersurface, let $O_X \subset \operatorname{Sym}^d V^\vee$ be the $GL(V)$-orbit closure of any defining equation $F = 0$ of X.*

Definition 4.6. *Let $F \in \operatorname{Sym}^d V^\vee$ be a degree d homogenous form cutting out $X \subset \mathbb{P}(V)$. Then, define*

$$p_X := \begin{cases} \#\operatorname{Aut}(X)[O_X]_{GL_{r+1}} & \text{if } \#\operatorname{Aut}(X) < \infty \\ 0 & \text{if } \#\operatorname{Aut}(X) = \infty. \end{cases}$$

$$P_X := \begin{cases} \#\operatorname{Aut}(X)[\mathbb{P}O_X]_{GL_{r+1}} & \text{if } \#\operatorname{Aut}(X) < \infty \\ 0 & \text{if } \#\operatorname{Aut}(X) = \infty. \end{cases}$$

We include the factor of $\#\operatorname{Aut}(X)$ because it naturally arises when specializing orbits. The polynomials P_X are an equivariant generalization of predegrees as defined by Aluffi and Faber [4, Definition].

Definition 4.7. *The predegree of a hypersurface X having full dimensional orbit is $\#\operatorname{Aut}(X)$ times the degree of its orbit in the projective space $\mathbb{P}^{\binom{d+r}{r}-1}$. If the orbit of X is not full dimensional then we define its predegree to be zero.*

Remark 4.8. The predegree of a hypersurface X is the coefficient of $H^{\binom{d+r}{r}-(r+1)^2}$ in P_X. Thus, the equivariant classes contain much more enumerative data than the predegree. However, we will often critically use the knowledge of the pre-degree in equivariant arguments.

4.3. **Notation for GL_{r+1}-equivariant degeneration.** Our equivalences between GL_{r+1}-equivariant classes will be given by degeneration, so we will introduce notation to reflect this. Next we let R be a DVR with uniformizer t, and let $\Delta = \operatorname{Spec}(R)$. Let 0 and η denote the special and generic point of Δ, respectively. We will often denote by F_t a family of hypersurfaces parametrized by Δ.

Notation 4.9. Let Z_t be a flat family of $GL(V)$-invariant subvarieties of $\operatorname{Sym}^d V^\vee$. If the generic fiber Z_η specializes in the flat limit to a union of $GL(V)$-invariant subvarieties (with multiplicities) $Z_0 = \sum_i m_i Z_0^i$, then we write

$$Z_\eta \rightsquigarrow \sum_i m_i Z_0^i.$$

Remark 4.10. We do not expect it to be true that the flat limit of orbit closures is a union of orbit closures.

Notation 4.11. Consider the abelian group generated $\mathbb{Z}$-linearly by e_Z where Z varies over all $GL(V)$-invariant subvarieties of $\operatorname{Sym}^d V^\vee$ and with relations generated by all $e_{Z_\eta} - \sum_i m_i e_{Z_0^i}$ for all $Z_\eta \rightsquigarrow \sum_i m_i Z_0^i$. Then, we define $\sum_i m_i Z_i \sim \sum_j n_j Z'_j$ if $\sum_i m_i e_{Z_i} = \sum_j n_j e_{Z'_j}$ in the abelian group.

Note that

$$\begin{aligned}\sum_i m_i Z_i \sim \sum_j n_j Z'_j \Rightarrow \sum_i m_i [Z_i]_{GL_{r+1}} &= \sum_j n_j [Z'_j]_{GL_{r+1}} \\ \sum_i m_i [\mathbb{P}Z_i]_{GL_{r+1}} &= \sum_j n_j [\mathbb{P}Z'_j]_{GL_{r+1}}.\end{aligned}$$

5. Known classes of orbits of special quartics

In this section, we record Kazarian's formulas for counting curves with A_6, D_6 and E_6 singularities. It is known that in the space of quartic curves, the set of curves with such singularities form three respective 8-dimensional orbits (Proposition 5.2). Kazarian's formulas then directly yield the equivariant orbit classes of these three orbits (Corollary 5.4). We also record the computation for the equivariant class of unions of four lines.

We make essential use of a calculation of Kazarian [65, Theorem 1]:

Proposition 5.1. *Let $\mathcal{S} \to B$ be a smooth morphism of varieties whose fibers are smooth surfaces. Let L be a line bundle on $\mathcal{S}$ and σ be a section of L cutting out a family of curves $\mathcal{C} \subset \mathcal{S}$. The virtual classes $[Z_{A_6}]$ (respectively $[Z_{D_6}]$ and $[Z_{E_6}]$) supported on points $p \in \mathcal{S}$ where the fiber of $\mathcal{C} \to B$ has an A_6 (respectively D_6 and E_6) singularity at p is given by:*

$$\begin{aligned}
[Z_{A_6}] &= u(-c_1 + u)(c_2 - c_1 u + u^2)(720c_1^4 - 1248c_1^2 c_2 + 156c_2^2 - 1500c_1^3 u \\
&\qquad + 1514c_1 c_2 u + 1236c_1^2 u^2 - 485c_2 u^2 - 487c_1 u^3 + 79u^4) \\
[Z_{D_6}] &= 2u(-c_1 + u)(4c_2 - 2c_1 u + u^2)(c_2 - c_1 u + u^2)(12c_1^2 - 6c_2 - 13c_1 u + 4u^2) \\
[Z_{E_6}] &= 3u(-c_1 + u)(2c_1^2 + c_2 - 3c_1 u + u^2)(4c_2 - 2c_1 u + u^2)(c_2 - c_1 u + u^2)
\end{aligned}$$

where $c_i := c_i(T_{\mathcal{S}/B})$ and $u = c_1(L)$.

Proposition 5.2. *The set of irreducible quartic plane curves with an A_6 (respectively D_6 and E_6) singularity forms a single 8-dimensional orbit.*

Proof. The case of D_6 singularities is clear, since one of the branches of the singularity must be a line. Hence such a curve must be the union of a nodal cubic with a tangent branch line, constituting a single orbit.

The fact that irreducible plane quartics with an A_6 or E_6 singularity form an irreducible subvariety of codimension 6 in the projective space $\mathbb{P}^{14}$ of all quartics follows from their classification, for example [82, Section 3.4].

That an orbit of a general curve with such a singularity is 8-dimensional can be checked by the formulas for their pre-degrees as found in [5, Examples 5.2 and 5.4], which gives a nonzero result. This proves the proposition. $\square$

Definition 5.3. *Let C_{A_6} and C_{E_6} denote rational quartic curves with an A_6 and E_6 singularity respectively, whose PGL_3-orbits are 8-dimensional. By Proposition 5.2, this definition is well-defined up to projective equivalence.*

There are explicit equations for C_{A_6} and C_{E_6} (see for example [82, Section 3.4]):

$$C_{A_6}: \{(x^2+yz)^2+2yz^3=0\} \qquad C_{D_6}: \{Z(ZXY+X^3+Z^3)=0\} \tag{5.1}$$

$$C_{E_6}: \{y^3z+x^4+x^2y^2=0\}. \tag{5.2}$$

Corollary 5.4. *We have*

$$p_{C_{A_6}} = 3\cdot 112(9c_1^3+12c_1c_2-11c_3)(2c_1^3+c_1c_2+c_3)$$

$$p_{C_{D_6}} = 3\cdot 64(18c_1^6+33c_1^4c_2+12c_1^2c_2^2-85c_1^3c_3-11c_1c_2c_3-7c_3^2)$$

$$p_{C_{E_6}} = 2\cdot 48(2c_1^3+c_1c_2+c_3)(9c_1^3-6c_1c_2+7c_3),$$

where $\#\,\mathrm{Aut}(C_{A_6}) = \#\,\mathrm{Aut}(C_{D_6}) = 3$ *and* $\#\,\mathrm{Aut}(C_{E_6}) = 2$.

We will also verify the result for $p_{C_{D_6}}$ independently in Section 9.

Proof. We apply Proposition 5.1 to the case where $B = \mathbb{G}(2,N)$ for $N >> 0$ and $\mathcal{S} = \mathbb{P}(\mathcal{V})$ where $\mathcal{V}$ is the tautological subbundle. Let T be the relative tangent bundle of $\mathbb{P}(\mathcal{V}) \to B$. By the splitting principle and the relative Euler exact sequence for projective bundles, we get:

$$c_1(T) = c_1(\mathcal{V}) + 3c_1(\mathscr{O}_{\mathbb{P}(\mathcal{V})}(1))$$

$$c_2(T) = c_2(\mathcal{V}) + 2c_1(\mathcal{V})c_1(\mathscr{O}_{\mathbb{P}(\mathcal{V})}(1)) + 3c_1(\mathscr{O}_{\mathbb{P}(\mathcal{V})}(1))^2.$$

Now, we let $u = 4c_1(\mathscr{O}_{\mathbb{P}(\mathcal{V})}(1))$ in the formulas for $[Z_{A_6}]$, $[Z_{D_6}]$ and $[Z_{E_6}]$ Proposition 5.1 and apply push-forward along the projection $\mathbb{P}(\mathcal{V}) \xrightarrow{\pi} B$. This yields

$$\pi_*[Z_{A_6}] = 112(9c_1(\mathcal{V})^3 + 12c_1(\mathcal{V})c_2(\mathcal{V}) - 11c_3(\mathcal{V}))(2c_1(\mathcal{V})^3 + c_1(\mathcal{V})c_2(\mathcal{V}) + c_3(\mathcal{V}))$$

$$\pi_*[Z_{D_6}] = 64(18c_1(\mathcal{V})^6 + 33c_1(\mathcal{V})^4c_2(\mathcal{V}) + 12c_1(\mathcal{V})^2c_2(\mathcal{V})^2 - 85c_1(\mathcal{V})^3c_3(\mathcal{V}) - 11c_1(\mathcal{V})c_2(\mathcal{V})c_3(\mathcal{V}) - 7c_3(\mathcal{V})^2)$$

$$\pi_*[Z_{E_6}] = 48(2c_1(\mathcal{V})^3 + c_1(\mathcal{V})c_2(\mathcal{V}) + c_3(\mathcal{V}))(9c_1(\mathcal{V})^3 - 6c_1(\mathcal{V})c_2(\mathcal{V}) + 7c_3(\mathcal{V})).$$

Now, $\pi_*[Z_{A_6}]$, $\pi_*[Z_{D_6}]$, and $\pi_*[Z_{E_6}]$ respectively give the formulas for $[O_{C_{A_6}}]_{GL_3}$, $[O_{C_{D_6}}]_{GL_3}$, $[O_{C_{E_6}}]_{GL_3}$, as they are also the result of pulling back $(O_{C_{A_6}})_{\mathcal{S}}$, $(O_{C_{D_6}})_{\mathcal{S}}$, and $(O_{C_{E_6}})_{\mathcal{S}}$ under a generic section $\mathbb{G}(2,N) \to \operatorname{Sym}^4 \mathcal{S}^\vee$.

The statement on the automorphisms of C_{A_6} and C_{E_6} come from the equations. Alternatively, one could compare the predegrees of C_{A_6} and C_{E_6} with the projective versions of $[Z_{A_6}]$ and $[Z_{E_6}]$ using [5, Examples 5.2 and 5.4] and Proposition 4.4. □

In order to calculate the orbit class of a general quartic with a triple point, we will need to know p_C in the case where C is the union of four lines, with no three concurrent.

Proposition 5.5. *Let C be the union of four lines, no three concurrent. Then,*

$$p_C = 24 \cdot 16(18c_1^6 + 33c_1^4c_2 + 12c_1^2c_2^2 + 131c_1^3c_3 + 153c_1c_2c_3 - 147c_3^2).$$

Here, $\#\operatorname{Aut}(C) = 24$.

Proof. We will closely follow [42, Theorem 3.1]. Consider the map $\phi : \mathbb{P}(\mathcal{V}^\vee)^4 \to \mathbb{P}(\operatorname{Sym}^4 \mathcal{V}^\vee)$, which restricts to the multiplication map $(\mathbb{P}^2)^4 \to \mathbb{P}^{14}$ on each fiber. Then, ϕ maps 4! to 1 onto $\mathbb{P}O_C$ so $[\mathbb{P}O_C] = \frac{1}{24}\phi_*(1)$.

Let $H = \mathscr{O}_{\mathbb{P}(\mathrm{Sym}^4 \mathcal{V}^\vee)}(1)$ and

$$\alpha = H^{14} + c_1(\mathrm{Sym}^4 \mathcal{V}^\vee)H^{13} + \cdots + c_{14}(\mathrm{Sym}^4 \mathcal{V}^\vee).$$

Using the fact that $\alpha H + c_{15}(\mathrm{Sym}^4 \mathcal{V}^\vee) = 0$, we get that the integral

$$\int_{\mathbb{P}(\mathrm{Sym}^4 \mathcal{V}^\vee) \to S} \alpha\beta$$

is equal to the constant term (with respect to H) of β. By this, we mean that any class $\beta \in A^\bullet(\mathbb{P}(\mathrm{Sym}^4 \mathcal{V}^\vee)$ can be written as a polynomial in H and pullbacks of classes of $A^\bullet$ and integrating against α extracts the constant term.

To finish, we let $\beta = \frac{1}{24}\phi_*(1)$ and apply the projection formula to reduce our problem to the evaluation of

$$\frac{1}{24}\int_{\mathbb{P}(\mathcal{V}^\vee)^4 \to S} \phi^*(\alpha).$$

This equals the answer claimed in the proposition, after multiplying by 24. $\square$

6. Families of orbits

The purpose of this section is to gather the basic degeneration tools we will use repeatedly throughout the paper.

Given a degree d plane curve with an 8-dimensional orbit, we can consider the orbit map

$$\begin{tikzcd} PGL_3 \arrow[r] \arrow[d, hook] & \mathbb{P}^{\binom{d+2}{2}-1} \\ \mathbb{P}^8 \arrow[ur, dashed, "\phi"] & \end{tikzcd}$$

inducing a rational map $\mathbb{P}^8 \dashrightarrow \mathbb{P}^{\binom{d+2}{2}-1}$. Resolving this map and pushing forward the fundamental class yields $\#\,\mathrm{Aut}(C)$ times the class of the orbit closure of C, which is the definition of the predegree (see Definition 4.7).

Suppose we have a family $\gamma : \Delta \to \mathbb{P}^{\binom{d+2}{2}-1}$ of plane curves parameterized by a smooth (affine) curve or DVR Δ. Pulling back the universal curve yields $\mathcal{C} \to \Delta$. Let C_t be a general fiber of $\mathcal{C}_t$ and C_0 be the special fiber over $0 \in \Delta$. In all our applications, Δ is an open subset of $\mathbb{A}^1$. Then, by taking the orbit map fiberwise, we get

$$\begin{tikzcd} PGL_3 \times \Delta \arrow[r] \arrow[d, hook] & \mathbb{P}^{\binom{d+2}{2}-1} \\ \mathbb{P}^8 \times \Delta \arrow[ur, dashed, "\Phi"] & \end{tikzcd}$$

Resolving $\Phi : \mathbb{P}^8 \times \Delta \dashrightarrow \mathbb{P}^{\binom{d+2}{2}-1}$ yields a degeneration of the orbit closure of C_t (with multiplicity $\#\operatorname{Aut}(C)$) to a union of 8-dimensional cycles. Our goal will be to identify those cycles in the limit. To do so, we will frequently apply Principles 6.1 and 6.2 below.

Let $\Delta^\times = \Delta \backslash \{0\}$.

Principle 6.1. Let $\mu : PGL_3 \times \mathbb{P}^{\binom{d+2}{2}-1} \to \mathbb{P}^{\binom{d+2}{2}-1}$ be the action of PGL_3 on $\mathbb{P}^{\binom{d+2}{2}-1}$ by pullback. Suppose for $1 \le i \le n$ we have found maps $\gamma_i : \Delta^\times \to PGL_3$ such that

(1) The unique extension $\mu(\gamma_i, \gamma) : \Delta \to \mathbb{P}^{\binom{d+2}{2}-1}$ sends 0 to a plane curve C_i which has an 8-dimensional orbit closure.

(2) The images of $\gamma_i(0) \in \mathbb{P}^8$ are pairwise distinct.

Then, the equivariant class $p_{C_t} - \sum_{i=1}^n p_{C_i}$ can be represented by a nonnegative sum of equivariant classes of effective cycles. Suppose in addition the predegrees of $C_1, \ldots, C_n$ adds up to the predegree of C_t. Then, $O_{C_t} \rightsquigarrow \sum_{i=1}^n O_{C_i}$.

Proof. Given a rank 3 vector bundle $\mathcal{V} \to B$, the degeneration given by resolving Φ also relativizes to a degeneration of $\#\operatorname{Aut}(C_t)[(O_{C_t})_{\mathcal{V}}]$ into a union of relative cycles in $\operatorname{Sym}^d \mathcal{V}^\vee$ given an equality in $A^\bullet(\operatorname{Sym}^d \mathcal{V}^\vee) \cong A^\bullet(B)$. Therefore, to prove Principle 6.1, we can and will assume B is a point.

For each curve γ_i, we can multiply by PGL_3 to get a map

$$\begin{tikzcd} \Delta \times PGL_3 \arrow[r] \arrow[rr, bend left, "f_i"] & \Delta \times \mathbb{P}^8 \arrow[r, dashed, "\Phi"] & \mathbb{P}^{\binom{d+2}{2}-1} \end{tikzcd}$$

where $(f_i)_*(1)$ is $\# \operatorname{Aut}(C_i)[O_{C_i}]$. Let $\mathcal{X} \subset \Delta \times \mathbb{P}^8 \times \mathbb{P}^{\binom{d+2}{2}-1}$ be the closure of the graph of Φ. We see $\mathcal{X} \to \mathbb{P}^{\binom{d+2}{2}-1}$ resolves Φ. Each γ_i corresponds to an 8-dimensional component Y_i of the special fiber X_0 of $\mathcal{X} \to \Delta$ over $0 \in \Delta$. Each Y_i pushes forward to a positive multiple of $\# \operatorname{Aut}(C_i)[O_{C_i}]$ in $\mathbb{P}^{\binom{d+2}{2}-1}$.

Each Y_i lies over precisely the orbit closure of $PGL_3 \cdot \gamma_i(0)$ in $\mathbb{P}^8$ under the map $X_0 \to \mathbb{P}^8$ given by restricting the resolution $\mathcal{X} \to \Delta \times \mathbb{P}^8$ over $0 \in \Delta$. Since the assumption on the images of $\gamma_i(0)$ are equivalent to the orbits $PGL_3 \cdot \gamma_i(0)$ being distinct as we vary over $1 \leq i \leq n$, the Y_i's correspond to distinct components of X_0.

Therefore, we find the difference

$$\# \operatorname{Aut}(C_t)[O_{C_t}] - \sum_{i=1}^{n} \# \operatorname{Aut}(C_i)[O_{C_i}]$$

is a nonnegative combination of 8-dimensional cycles. If we assume the equality of predegrees, the degrees of those 8-dimensional cycles sum to zero. This means the difference is identically zero. $\square$

For many of our applications, Principle 6.1 will suffice. But in Section 8.1, we will have two different maps γ_1, γ_2 whose images $\gamma_i(0)$ are the same, but will both still contribute to p_{C_t}. We will show this by first blowing up $\Delta \times \mathbb{P}^8$. The proof method is the same as Principle 6.1, but with $\Delta \times \mathbb{P}^8$ replaced by X for $X \to \Delta \times \mathbb{P}^8$ a PGL_3-equivariant birational map, so we omit it.

Principle 6.2. Let $X \to \Delta \times \mathbb{P}^8$ be a PGL_3-equivariant birational map. Suppose we have found $\gamma_1, \ldots, \gamma_n$ maps $\gamma_i : \Delta^\times \to PGL_3$ such that

(1) The unique extension $\mu(\gamma_i, \gamma) : \Delta \to \mathbb{P}^{\binom{d+2}{2}-1}$ sends 0 to the curve C_i with an 8-dimensional orbit closure.

(2) The unique extensions $\gamma_i : \Delta \to X$ have the property that the points $\gamma_i(0) \in X$ are in different PGL_3-orbits of X.

Then, the equivariant class $p_{C_t} - \sum_{i=1}^n p_{C_i}$ can be represented by a nonnegative sum of effective cycles. Suppose in addition the predegrees of $C_1, \ldots, C_n$ adds up to the predegree of C_t. Then, $p_{C_t} = \sum_{i=1}^n p_{C_i}$.

7. Splitting off a line as a component

In this section, we analyze how the orbit changes as a degree d smooth curve degenerates to a general degree e smooth curve together with $d-e$ general lines.

Lemma 7.1. *Let $F(X,Y,Z)$ and $G(X,Y,Z)$ cut out plane curves of degrees $d-1$ and d respectively. If $\{F=0\}$ or $\{G=0\}$ does not contain $\{X=0\}$, then*

$$\lim_{t\to 0} t^{-1}((tX)F(tX,Y,Z)+tG(tX,Y,Z)) = XF(0,Y,Z)+G(0,Y,Z)$$

Proof. Consider $t^{-1}((tX)F(tX,Y,Z)+tG(tX,Y,Z)) = XF(tX,Y,Z)+G(tX,Y,Z)$. Setting $t=0$ yields $XF(0,Y,Z)+G(0,Y,Z)$. □

Proposition 7.2. *Let $d \geq 4$ and let C, C_{d-1}, D be a general curve of degree d, a general degree d curve with a point of multiplicity $d-1$, and a general degree $e \neq 1$ curve union $d-e$ lines respectively. Then,*

$$O_C \rightsquigarrow (d-e)O_{C_{d-1}} + \#\operatorname{Aut}(O_D)O_D.$$

Proof. This follows from Lemma 7.1 and Principle 6.1 provided we can show the equality of predegrees. To show the equality on predegrees, it suffices to consider the case $e=0$, where we want to see that the predegree of a general degree d curve is d times the predegree of a general degree d curve with a point of multiplicity $d-1$ plus the predegree of the union of d general lines. The result follows from plugging into the formulas in [5, Examples 3.1, 4.2] and [4]. □

8. Degeneration to nodes and cusps

In this section, we establish the effect of acquiring a node or cusp (with analytic equation $y^2 = x^3$) on the polynomial p_C for arbitrary plane curves d. In what follows, a node singularity p of a plane curve C is called *ordinary* if both tangent lines intersect C with multiplicity 3 at p. A similar definition for cusp singularities is not necessary as no line meets the cusp with multiplicity ≥ 4. Throughout, let Δ be $\operatorname{Spec}(R)$ where R is a DVR with uniformizer t, valuation v and residue field $\mathbb{C}$.

Our objective in this section is to prove:

Theorem 8.1. *Let $\mathscr{C} \to \Delta$ be a family of degree d plane curves whose generic fiber C_ν is a smooth curve with no hyperflexes and whose special fiber C_0 has exactly δ ordinary nodes and κ cusps. If the total space $\mathscr{C}$ is smooth and C_0 has no hyperflexes then:*

(8.1)
$$\#\operatorname{Aut}(C_\nu)O_{C_\nu} \sim \#\operatorname{Aut}(C_0)O_{C_0} + 2\delta(3O_{C_{BN}}) + \kappa(2O_{C_{\text{flex}}}) \Rightarrow p_{C_\nu} = p_{C_0} + 2\delta \cdot p_{C_{BN}} + \kappa \cdot p_{C_{\text{flex}}}$$

where C_{BN} is curve defined by $Z^{d-3}(XYZ + X^3 + Z^3)$ and C_{flex} is the curve defined by $Z^{d-3}(Y^2Z - X^3 - aXZ^2 - bZ^3)$, where $a, b \in K$ are general.

In words, C_{BN} is a nodal cubic union a multiplicity $d-3$ line tangent to one of the branches at the node and C_{flex} is a general smooth cubic union a flex line with multiplicity $d-3$. In Theorem 8.1, we note that C_{flex} still has moduli because we can vary the j-invariant. A study of how C_{flex} degenerates as we send the j-invariant to ∞ yields the following

Proposition 8.2. *Let C_{AN} is a nodal cubic union a multiplicity $d-3$ flex line. We have $O_{C_{\text{flex}}} \rightsquigarrow O_{C_{AN}} + 3O_{C_{BN}}$. In particular, $p_{C_{\text{flex}}} = p_{C_{AN}} + 2p_{C_{BN}}$.*

8.1. **Degeneration to a node.**

Lemma 8.3. *Let $F(X, Y, Z)$ be a homogenous degree d polynomial with coefficients in R cutting out $\mathscr{C} \subset \Delta \times \mathbb{P}^2$ such that the special fiber C_0 has an ordinary node at $[0:0:1]$ and branches tangent to $X = 0$ and $Y = 0$ and $\mathscr{C}$ is smooth at the node of C_0. Then,*

$$\lim_{t \to 0} t^{-1}(F(t^{\frac{1}{3}}X, t^{\frac{2}{3}}Y, Z))$$

is projectively equivalent to $Z^{d-3}(ZXY + X^3 + Z^3)$, where $t^{\frac{1}{3}}$ is a third root of t in R after an order three base change. In particular, the limit plane curve is a nodal cubic with a multiple line tangent to a branch of the node. Similarly,

$$\lim_{t \to 0} t^{-1}(F(t^{\frac{2}{3}}X, t^{\frac{1}{3}}Y, Z))$$

is projectively equivalent to $Z^{d-3}(ZXY + Y^3 + Z^3)$.

Proof. From our setup, the coefficient a_{ij} of each monomial $X^iY^jZ^{d-i-j}$ of $F(X, Y, Z)$ is an element of R. By the assumption on the tangents to the branches to the special fiber at $[0:0:1]$,

$$v(a_{0,0}), v(a_{1,0}), v(a_{0,1}), v(a_{2,0}), v(a_{0,2}) \geq 1.$$

Since the node singularity is assumed to be simple, $v(a_{3,0}) = v(a_{0,3}) = 0$. Since $\mathscr{C}$ is smooth at the node $v(a_{0,0}) = 1$. Now, a direct check shows $\frac{2}{3}i + \frac{1}{3}j - 1 + v(a_{i,j})$ is zero if $(i, j) \in \{(0, 3), (1, 1), (0, 0)\}$ and strictly positive otherwise. The proof of the second half is similar. $\square$

Remark 8.4. In the case $d = 4$, the orbit closure of a nodal cubic union a line tangent contains all curves possessing a D_6 singularity.

Definition 8.5. *Let C_{BN} be the curve defined by $Z^{d-3}(ZXY + X^3 + Z^3)$.*

Definition 8.5 depends on d, but it will be clear what d is from context.

8.2. **The degree of the orbit of C_{BN}.** In light of Proposition 8.16, we will now compute the degree of the orbit closure of C_{BN}. In principle, this can be deduced by applying the algorithm of Aluffi and Faber in [5]. We provide an independent calculation in this section.

Proposition 8.6. *Let $d \geq 4$. As a function of the degree d, the degree of the orbit of C_{BN} is the quadratic polyonomial $24 + 144 \cdot (d-3) + 140 \cdot (d-3)^2$. The predegree of the orbit of C_{BN} is $3(24 + 144 \cdot (d-3) + 140 \cdot (d-3)^2)$.*

We will prove Proposition 8.6 in pieces below. Given the calculation of the degree of the orbit, the assertion on the predegree follows from the fact that the curve C_{BN} has order 3 automorphism group.

Lemma 8.7. *Let $d \geq 4$. As a function of the degree d, the degree of the orbit of C_{BN} is a quadratic polyonomial $a + b \cdot (d-3) + c \cdot (d-3)^2$ with $a, b, c \geq 0$.*

Explicitly a, b, c are the answers to the following enumerative problems:

$$a = 2\#\{\textit{singular cubics through 8 points}\} = 24$$

$$b = \binom{8}{1}\#\{\textit{nodal cubics through 7 points with a nodal branch line containing a fixed 8th point}\}$$

$$c = \binom{8}{2}\#\{\textit{nodal cubics through 6 points with specified nodal branch line}\}$$

Proof. Let V denote the 8 dimensional smooth variety parametrizing triples (C, L, p) where C is a cubic curve singular at the point $p \in \mathbb{P}^2$ and L is a line containing p whose intersection multiplicity with C is greater than 2.

The variety V possesses a natural map to the projective space $\mathbb{P}^9$ of cubic curves in $\mathbb{P}^2$ by forgetting L and p – let H denote the divisor class on V induced by the $\mathcal{O}(1)$ on $\mathbb{P}^9$. Similarly, let h denote the divisor class induced by the forgetful map $V \mapsto \mathbb{P}^{2*}$ given by forgetting C and p.

Let $\nu : \mathbb{P}^9 \times \mathbb{P}^{2*} \to \mathbb{P}^{\binom{d+2}{2}-1}$ denote the map which sends a pair (C, L) to the degree d curve $C \cup (d-3)L$. Then the composite map $V \to \mathbb{P}^9 \times \mathbb{P}^{2*} \to \mathbb{P}^{\binom{d+2}{2}-1}$ is induced by the divisor class $H + (d-3)h$ on V, and the image of this map is precisely the orbit closure of the curve C_{BN}. Therefore, the degree of the orbit closure of C_{BN} is given by the intersection number $(H + (d-3)h)^8$ on V.

Since $h^3 = 0$ on V, this intersection number is equal to

$$H^8 + 8(d-3)H^7h + \binom{8}{2}(d-3)^2H^6h^2.$$

The numbers a, b, c in the lemma are the monomials H^8, H^7h, H^6h^2. By treating H^ih^j as i general point conditions on the cubic C and j general point conditions on the line L, we see that

$$H^8 = 2\#\{\text{singular cubics through 8 points}\},$$

where the coefficient of 2 arises because $V \to \mathbb{P}^9$ is 2 to 1 onto its image. Furthermore,

$$H^7h = \#\{\text{nodal cubics through 7 points with a nodal branch line containing a fixed 8th point}\}$$

$$H^6h^2 = \#\{\text{nodal cubics through 6 points with specified nodal branch line}\}.$$

This proves the lemma. The value of a comes from the fact that there are twelve nodal cubics in a pencil [101]. $\square$

Lemma 8.8. *The sum $a + b + c$ in Lemma 8.7 is* 308.

Proof. To compute $a + b + c$, we need to know the degree of the orbit of C_{D_6} in the $\mathbb{P}^{14}$ of quartic plane curves. To compute the degree, we apply Corollary 5.4 together with Proposition 4.4. Explicitly, we take

$$\frac{1}{\#\operatorname{Aut}(C_{D_6})} p_{C_{D_6}} = 64(18c_1^6 + 33c_1^4c_2 + 12c_1^2c_2^2 - 85c_1^3c_3 - 11c_1c_2c_3 - 7c_3^2)$$

from Corollary 5.4, make the substitution $c_1 \mapsto u+v+w$, $c_2 \mapsto uv+uw+vw$, $c_3 \mapsto uvw$ followed by $u \mapsto u-\frac{H}{4}$, $v \mapsto v-\frac{H}{4}$, $w \mapsto w-\frac{H}{4}$, and extract the coefficient of H^6. □

Lemma 8.9. *The coefficient c in Lemma 8.7 is* $5 \cdot \binom{8}{2}$.

Proof. By the proof of Lemma 8.7 it suffices to demonstrate the following enumerative statement: Fix 6 general points $p_1, ...p_6$ in $\mathbb{P}^2$ and fix a general line $L \subset \mathbb{P}^2$. Then there are 5 singular cubics containing the points p_i singular at a point on L and meeting L with multiplicity ≥ 3 at the singular point.

For this, we recast the problem as the degree of the degeneracy locus of a map between two rank 4 vector bundles $e : A \to B$ on the line L. The vector bundle A is simply the trivial vector bundle with fiber the vector space of cubic curves containing the 6 points $p_1, ..., p_6$. We now describe the second vector bundle B, used previously by the second author in [87, Section 5.2.2].

For each point $p \in L$, let $\mathcal{J}_p \subset \mathcal{O}_{\mathbb{P}^2}$ denote the ideal defining the divisor $3p$ in L, and let $\mathfrak{m}_p^2 \subset \mathcal{O}_{\mathbb{P}^2}$ denote the square of the maximal ideal. Let $W_p \subset Z_p$ denote the subschemes defined by $\mathcal{J}_p$ and $\mathcal{J}_p \cap \mathfrak{m}_p^2$ respectively. We define B' to the the rank 3 jet bundle on L whose fiber at a point p is

$$B'|_p = \{\text{degree 3 forms}\}/\{\text{degree 3 forms vanishing on } W_p\}$$

and we define B to be the rank 4 jet bundle on L whose fiber at a given point p is

$$B|_p = \{\text{degree 3 forms}\}/\{\text{degree 3 forms vanishing on } Z_p\}.$$

The quotient space $\mathcal{J}_p/\left(\mathcal{J}_p \cap \mathfrak{m}_p^2\right)$ can naturally be identified with the conormal space $(\mathcal{I}_L/\mathcal{I}_L^2)\,|_p$: In local affine coordates (x, y), if L is the line $x = 0$ and p is the origin, then $\mathcal{J}_p = (x, y^3)$, $\mathcal{J}_p \cap \mathfrak{m}_p^2 = (x^2, xy, y^3)$, and $\mathcal{J}_p/\mathcal{J}_p \cap \mathfrak{m}_p^2$ is generated by $\bar{x}$, the local generator for $(\mathcal{I}_L/\mathcal{I}_L^2)\,|_p$.

Putting these observations together, we obtain a short exact sequence of vector bundles:

$$0 \to \mathcal{I}_L/\mathcal{I}_L^2 \otimes \mathcal{O}_L(3) \to B \to B' \to 0. \tag{8.2}$$

Therefore, the degree of B (as vector bundle on L) is equal to the degree of B' plus the degree of the line bundle $\mathcal{I}_L/\mathcal{I}_L^2 \otimes \mathcal{O}_L(3)$. The latter clearly has degree 2. B' is the standard second order jet bundle for the line bundle $\mathcal{O}_L(3)$, which has degree 3. Therefore, the degree of B is 5.

The map $e : A \to B$ is the natural evaluation map. Since A is trivial, the number of points where e is degenerate is the degree of B, which is 5. The lemma follows.

□

Proof of Proposition 8.6. Since $a = 24$ and $c = 5 \cdot 28 = 140$ from Lemma 8.9, $b = 144$ from Lemma 8.8. □

8.3. **Degeneration to a cusp.** In what follows, a cusp singularity of a plane curve C is called *ordinary* if no line meets C with multiplicity 4 at p. Let Δ be the spectrum of a DVR with uniformizer t and residue field $\mathbb{C}$.

Lemma 8.10. *Let $F(X, Y, Z)$ be a homogenous degree d polynomial with coefficients in R cutting out $\mathscr{C} \subset \Delta \times \mathbb{P}^2$ such that the special fiber C_0 has an ordinary cusp at $[0 : 0 : 1]$ meeting the line $\{X = 0\}$ to order 3 and suppose $\mathscr{C}$ is smooth at the cusp of C_0. Then,*

$$\lim_{t \to 0} t^{-1}(F(t^{\frac{1}{3}}X, t^{\frac{1}{2}}Y, Z))$$

is the curve $Z^{d-3}(X^3 + Y^2Z + Z^3)$ up to rescaling the coordinates, where $t^{\frac{1}{6}}$ is a 6th root of t obtained after performing an order 6 base change on Δ.

Proof. The proof is identical to Lemma 8.3. From our setup, the coefficient a_{ij} of each monomial $X^iY^jZ^{d-i-j}$ of $F(X, Y, Z)$ is an element of R. By the assumption on the tangents to the branches to the special fiber at $[0:0:1]$,

$$v(a_{0,0}), v(a_{1,0}), v(a_{0,1}), v(a_{2,0}), v(a_{1,1}) \geq 1.$$

By the assumption that the cusp of C_0 is an A_2 singularity, $v(a_{3,0}) = 0$. By the assumption $\mathscr{C}$ is smooth at $[0:0:1]$ in the central fiber, $v(a_{0,0}) = 1$. One can check that $-1 + \frac{i}{3} + \frac{j}{2} + v(a_{i,j})$ is zero for $(i,j) \in \{(3,0),(0,2),(0,0)\}$ and strictly positive otherwise. $\square$

Definition 8.11. *Let C_{flex} to be the curve which is the union of a general cubic with $(d-3)$ times one of its flex lines. Let C_{AN} denote the curve which is the union of a nodal cubic with $(d-3)$ times one of the flex lines through a smooth point.*

As in the previous section, we have dropped the dependence on d.

8.4. **A degeneration of the orbit of C_{flex}.** In this subsection, we will study how the orbit of C_{flex} degenerates as we vary the j-invariant to ∞.

Proof of Proposition 8.2. Let W denote the smooth variety parametrizing triples (C, L, p) where C is a plane cubic, $p \in C$ is a point and L is a line containing p which meets C with multiplicity at least 3 at p. (W is similar to the variety V from the proof of Lemma 8.7, however we allow C to be an arbitrary smooth cubic curve. Therefore, V is a closed subvariety of W.)

The 9 dimensional variety W has a natural projection to the projective space $\mathbb{P}^9$ of cubic plane curves, and also has a projection to the projective space $\mathbb{P}^{2*}$ of lines in $\mathbb{P}^2$. We let H and h denote the divisor classes on W corresponding to the respective pullbacks of $\mathcal{O}(1)$ under these projections. Just as in the proof of Lemma 8.7, the divisor class $H + (d-3)h$ is the pullback of $\mathcal{O}(1)$ under the map

$$f : W \to \mathbb{P}^{\binom{d+2}{2}-1}$$

sending (C, L, p) to the curve $C + (d-3)L$.

For each $j \in \mathbb{P}^1$, define the 8 dimensional subvariety $W_j \subset W$ to be the closure of the locus of triples (C, L, p) where C is a cubic with j-invariant j. Let $W_{\infty,AN}, W_{\infty,BN} \subset W_\infty$ be the two components of W_∞, where $W_{\infty,AN}$ consists of triples (C, L, p) where C is singular at p and $W_{\infty,BN}$ is the closure of the triples (C, L, p) where C is smooth at p. By specializing j to ∞, we get

$$[W_j] \simeq A[W_{\infty,AN}] + B[W_{\infty,BN}] + C[Z],$$

with A, B, C positive integers and j general. Here, Z consists of (C, L, p) where C is the union of a conic and L and p is on L. Now, we intersect both sides of the equation with H^8, where H is the hyperplane class pulled back from the $\mathbb{P}^9$ of cubic plane curves.

Specifically, $[W_j]H^8 = 12 \cdot 9 = 108$, where 12 is the degree of the orbits closure of a cubic with a fixed j-invariant and 9 is the number of flexes on such a cubic. Similarly, we can compute $[W_{\infty,AN}]H^8 = 3 \cdot 12 = 36$ and $[W_{\infty,BN}]H^8 = 2 \cdot 12 = 24$, where we have 3 smooth flexes of a nodal cubic and 2 branches at a node, respectively. We know that the intersections are all multiplicity 1 by Bertini. Finally, $H^8 \cdot [Z] = 0$.

We conclude by noting that $A = 1$ and $B = 3$ are the only positive integer solutions to $108 = 36A + 24B$. $\square$

8.5. **The degree of the orbit of C_{flex}.** Next, we compute the degree of the orbit closure of the curve C_{flex} in the projective space $\mathbb{P}^{\binom{d+2}{2}-1}$. Again, although this can be computed in principle using the algorithm of Aluffi and Faber [5], we have decided to proceed independently.

Lemma 8.12. *As a function of d, the degree of the orbit closure of C_{flex} is a quadratic polynomial $a + b \cdot (d-3) + c \cdot (d-3)^2$, where the coefficients a, b, c are the answers* *to the following enumerative problems:*

$$a = 9 \cdot 12\#\{\textit{Cubics through 9 points}\} = 108$$

$$b = 12 \cdot \binom{8}{1}\#\{\textit{Cubics through 8 points with flex line containing a fixed 9th point}\}$$

$$c = 12 \cdot \binom{8}{2}\#\{\textit{Cubics through 7 points flexed at a specified line}\}$$

Proof. We wil reuse the notation of W and W_j in Proposition 8.2. Our objective is to calculate the degree of the image $f(W_j)$, as this is precisely the orbit closure of C_{flex}. Thus, we must compute $(H + (d-3)h)^8 \cdot [W_j]$ in the Chow ring of W. Since $h^3 = 0$, we get that the degree of the orbit closure of C_{flex} is:

$$H^8 \cdot [W_j] + 8(d-3)H^7h \cdot [W_j] + \binom{8}{2}(d-3)^2H^6h^2 \cdot [W_j]. \tag{8.3}$$

Next, we observe that the divisor W_j is linearly equivalent to $12 \cdot H$, since the degree of the divisorial locus in $\mathbb{P}^9$ consisting of the closure of plane cubics with given generic j-invariant is 12. Therefore, the degree of the orbit closure of C_{flex} is

$$12\left(H^9 + 8(d-3)H^8h + \binom{8}{2}(d-3)^2H^7h^2\right).$$

The lemma now follows by interpreting the three intersection numbers H^9, H^8h, H^7h^2 as the quantities appearing in the descriptions of a, b and c in the statement of the lemma. □

Lemma 8.13. *There are* 9 *cubics passing through eight general points and having a flex line containing a general fixed ninth point, i.e.* $H^8h = 9$.

Proof. Let Λ denote the Hesse pencil

$$s(X^3 + Y^3 + Z^3) + tXYZ = 0.$$

Recall that the 9 base points of the Hesse pencil consist of the 9 flexes of every smooth member of Λ. At each base point p of the pencil, the flex lines of the cubic curves in the pencil at p in turn sweep out a pencil of lines in $\mathbb{P}^2$. Therefore, a general point x in $\mathbb{P}^2$ is contained in exactly 9 flex lines of members of the Hesse pencil, one per basepoint.

Thus, if we use the Hesse pencil Λ to represent the curve class H^8 in W, then we get $H^8 h = 9$ as claimed. □

Lemma 8.14. *There are 3 cubic curves passing through 7 general points and possessing a particular line as flex line, i.e.* $H^7 h^2 = 3$.

Proof. Let L be a fixed line, and suppose $p_1, ..., p_7$ are general points. Then the net of cubic curves containing the points p_i restricts to a general net in the linear system $|\mathcal{O}_L(3)|$. A general such net maps L to a nodal cubic in $\mathbb{P}^2$, which has exactly 3 flex points. These three flexes, in turn, correspond to the solutions to the enumerative problem in the statement of the lemma. □

Corollary 8.15. *The degree of the orbit closure of* C_{flex} *is* $12\,(9 + 72(d-3) + 84(d-3)^2)$. *The predegree of* C_{flex} *is* $24\,(9 + 72(d-3) + 84(d-3)^2)$.

Proof. Combine Lemma 8.12, Lemma 8.13, Lemma 8.14. The second statement follows from the fact that the curve C_{flex} has an order 2 automorphism group, since the generic elliptic curve has an order 2 automorphism group. □

8.6. **Proof of Theorem 8.1.** We now have all ingredients for the proof of Theorem 8.1.

Proposition 8.16. *Let* $\mathscr{C} \to \Delta$ *be a family of degree* d *plane curves whose generic fiber* C_ν *is a smooth curve with no hyperflexes and whose special fiber* C_0 *has* δ *ordinary nodes and* κ *cusps. If the total space* $\mathscr{C}$ *is smooth at those nodes and cusps then* *the equivariant class*

$$p_{C_\nu} - p_{C_0} - 2\delta \cdot p_{C_{BN}} - \kappa \cdot p_{C_{\text{flex}}} \tag{8.4}$$

is a nonnegative sum of equivariant classes of effective cycles.

Proof. Let $p_1, \ldots, p_\delta$ be the simple nodes and $q_1, \ldots, q_\kappa$ be the simple cusps of $\{F = 0\}$. For each p_i, let $B_i \simeq \mathbb{P}^2 \subset \mathbb{P}^8$ denote the linear space corresponding to matrices with image equal to p_i. For each p_i, we have two 1-parameter families $\gamma_{i,1}$ and $\gamma_{i,2}$ given by Lemma 8.3. For each q_i we have the 1-parameter family γ_i' from Lemma 8.10.

We apply Principle 6.2 to X being the blowup of $\Delta \times \mathbb{P}^8$ along the subvarieties $\Delta \times B_i$ and the 1-parameter families $\gamma_{i,j}$ for $1 \le i \le \delta$ and $j \in \{1, 2\}$ and to the 1-parameter families γ_i' for $1 \le i \le \kappa$. The only thing one needs to check is that $\gamma_{i,1}$ and $\gamma_{i,2}$ limit to points in different PGL_3-orbits of X.

In coordinates, if $p_i = [0 : 0 : 1]$, then

$$B_i = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ * & * & * \end{pmatrix} \right\} \qquad E = \left\{ \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & 0 \end{pmatrix} \right\} \qquad \gamma_{i,1} = \begin{pmatrix} t & 0 & 0 \\ 0 & t^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \gamma_{i,2} = \begin{pmatrix} t^2 & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where E is the exceptional divisor of the blowup of $\mathbb{P}^8$ along B_i. The central fiber of $X \to \Delta$ can be identified with the blowup of $\mathbb{P}^8$ along B_i, and PGL_3 action on E is given by column operations. The curves $\gamma_{i,1}$ and $\gamma_{i,2}$ limit to

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

respectively on E which are not in the same PGL_3-orbit because they have different images. □

Proof of Theorem 8.1. From Proposition 8.16, $p_{C_\nu} - p_{C_0} - 2\delta \cdot p_{C_{BN}} - \kappa \cdot p_{C_{\text{flex}}}$ is effective. To see that this class is in fact zero, we need the corresponding linear combination of the predegrees to be zero. By [5, Example 4.1 and Example 5.2] the contributions of a node and cusp to the predegree are respectively

$$24(35d^2 - 174d + 213) = 2 \cdot 3(24 + 144(d-3) + 140(d-3)^2)$$

$$72(28d^2 - 144d + 183) = 24(9 + 72(d-3) + 84(d-3)^2)$$

which are precisely twice the predegree of $p_{C_{BN}}$ and the predegree of $p_{C_{\text{flex}}}$ respectively by Proposition 8.6 and Corollary 8.15. $\square$

9. Computation of $[O_{C_{AN}}]$ and $[O_{C_{BN}}]$

In this section we provide a method for computing the equivariant classes of $O_{C_{AN}}$ and $O_{C_{BN}}$ and apply it to the case $d = 4$. Recall that when $d = 4$, C_{BN} is a nodal cubic union a line tangent to a branch of the singularity and C_{AN} is a nodal cubic union a flex line at a smooth point.

Proposition 9.1. *When $d = 4$,*

$$[O_{C_{BN}}] = 64(18c_1^6 + 33c_1^4c2 + 12c_1^2c_2^2 - 85c_1^3c_3 - 11c_1c_2c_3 - 7c_3^2)$$

$$[O_{C_{AN}}] = 192(18c_1^6 + 33c_1^4c_2 + 12c_1^2c_2^2 + 19c_1^3c_3 - 7c_1c_2c_3 - 35c_3^2).$$

Proof. We will use the variety W parameterizing triples (C, L, p) given in the proof of Proposition 8.2 in Section 8.4. We can regard W as an iterated projective bundle, by first forgetting C, then forgetting p, and then forgetting L (to map to a point). Each of these projective bundles are given by the projectivization of a vector bundle over their associated base spaces, and so the Chow ring of W is determined by the chern classes of these vector bundles.

Furthermore, there is a generically finite map $c : W \to \mathbb{P}^9$ mapping (C, L, p) to C. Applying Riemann-Hurwitz to c yields the ramification divisor, which has two components:

(1) $W_{\infty,BN}$ consisting of (C, L, p) for which C is nodal at p and L meets C at p to multiplicity 3

(2) Z consisting of (C, L, p) for which C is the union of a conic and L.

Using the classical fact that a branch line of a node is the limit of three flexes, $W_{\infty,BN}$ appears with multiplicity 2 in the ramification divisor of c. We will also be able to deduce that Z appears with multiplicity 1 in the ramification divisor from the formula.

Since we can compute the class Z in W, this computes $W_{\infty,BN}$. To get $W_{\infty,AN}$, we pull back the discriminant locus $\Delta \subset \mathbb{P}^9$ under the map c and note that this pullback is $3 \cdot W_{\infty,BN} + W_{\infty,AN} + 2Z$. Subtracting off the contributions of $W_{\infty,BN}$ and Z yields $W_{\infty,AN}$.

Finally, we note that the whole construction above is compatible with the standard action of GL_3 on $W \subset \mathbb{P}^2 \times \mathbb{P}^{2\vee} \times \mathbb{P}^{9\vee}$ so given a vector bundle $\mathcal{V} \to B$ of rank 3, there is a relative version $W_{\mathcal{V}} \subset \mathbb{P}(\mathcal{V}) \times \mathbb{P}(\mathcal{V}^\vee) \times \mathbb{P}(\operatorname{Sym}^3 \mathcal{V}^\vee)$. The argument above yield the classes of $W_{\infty,AN}$ and $W_{\infty,BN}$ in $A^\bullet(W_{\mathcal{V}})$. Let H_{curve}, H_{point}, H_{line} be the $\mathscr{O}(1)$ classes of $\mathbb{P}(\operatorname{Sym}^3 \mathcal{V}^\vee)$, $\mathbb{P}(\mathcal{V})$ and $\mathbb{P}(\mathcal{V}^\vee)$ respectively. To finish, one applies the same integration trick as in the proof of Proposition 5.5 and given in [42, Theorem 3.1], where we pullback a particular class ϕ under the canonical map

$$W_{\mathcal{V}} \to \mathbb{P}(\operatorname{Sym}^4 \mathcal{V}^\vee)$$

and integrate to B to get a formula in terms of the chern classes of $\mathcal{V}$.

We now do the computation. By abuse of notation, we suppress all pullbacks. Let c_1, c_2, c_3 be the chern classes of $\mathcal{V}$. To perform the computation, let $S \to \mathbb{P}(\mathcal{V}^\vee)$ be the universal subbundle. Its total chern class is $\frac{c(\mathcal{V})}{c(\mathscr{O}_{\mathcal{P}(\mathcal{V})}(1))}$. Next, over $\mathbb{P}(S) \to \mathbb{P}(\mathcal{V}^\vee)$, we have a rank 7 vector bundle V_{flex}, which over each point of B restricts to the cubic curves meeting l at p to order 3. More precisely, on $\mathbb{P}(S)$, we have an exact sequence

$$0 \to V_{\text{flex}} \to \operatorname{Sym}^3 \mathcal{V}^\vee \to J^3_{\mathbb{P}(S)/\mathbb{P}(\mathcal{V}^\vee)}(\mathscr{O}_S(3)) \to 0,$$

where $\operatorname{Sym}^3 \mathcal{V}^\vee$ in the sequence is pulled back from B since we suppressed pullbacks in our notation. Finally, $W_{\mathcal{V}} = \mathbb{P}(V_{\text{flex}})$.

Using the structure of $W_{\mathcal{V}}$ as an iterated bundle over B, we compute the relative canonical of $W_{\mathcal{V}} \to B$ is $-7H_{\text{curve}} + H_{\text{line}} + H_{\text{point}} + 7u + 7v + 7w$. Using the fact that Z is the projectived subbundle of $\mathbb{P}(V_{\text{flex}})$ given as the kernel of $\operatorname{Sym}^3 \mathcal{V}^\vee \to J^4_{\mathbb{P}(S)/\mathbb{P}(\mathcal{V}^\vee)}(\mathscr{O}_S(3))$, the class of Z is

$$3H_{\text{point}} + 3K_{\mathbb{P}(S)/\mathbb{P}(\mathcal{V}^\vee)} + H_{\text{curve}} = H_{\text{curve}} - 3H_{\text{point}} + 3H_{\text{line}} - 3c_1.$$

Applying Riemann-Hurwitz, we find the ramification divisor is

$$K_{W_{\mathcal{V}}/\mathbb{P}(\mathcal{V}^\vee)} - K_{\mathbb{P}(\operatorname{Sym}^3 \mathcal{V}^\vee)} = 3H_{\text{curve}} + H_{\text{line}} + H_{\text{point}} - 3c_1$$

If we work nonequivariantly (set the $c_i = 0$), we find that the H_{curve} coefficient of Z and $W_{\infty,BN}$ is 1, meaning the multiplicity of Z in the ramification divisor must be 1. Solving for $[W_{\infty,BN}]$ yields $H_{\text{curve}} - H_{\text{line}} + 2H_{\text{point}}$.

The class of the relative discriminant divisor of $\mathbb{P}(\operatorname{Sym}^3 \mathcal{V}^{vee})$ is $12\mathscr{O}_{\mathbb{P}(\operatorname{Sym}^3 \mathcal{V}^{vee})}(1) - 12c_1$. Pulling back to $W_{\mathcal{V}}$ and subtracting off $3W_{\infty,BN}$ and $2Z$, we get

$$W_{\infty,AN} = 7H_{\text{curve}} - 3H_{\text{line}} - 6c_1.$$

Now, according to the proof of Proposition 5.5 and [42, Theorem 3.1], we want to pullback

$$\phi := H^{14} + c_1(\operatorname{Sym}^4 \mathcal{V}^\vee)H^{13} + \cdots + c_{14}(\operatorname{Sym}^4 \mathcal{V}^\vee)$$

under $W_{\mathcal{V}} \to \mathbb{P}(\mathrm{Sym}^4 \mathcal{V}^\vee)$, multiply by $W_{\infty,AN}$ and integrate to B using the projective bundle structure. This computes $[O_{C_{AN}}]$. Doing the same with $W_{\infty,BN}$ gives $[O_{C_{BN}}]$. $\square$

Remark 9.2. The only place $d = 4$ was used in the the proof of Proposition 9.1 was the definition of class ϕ and the pullback map $A^\bullet(\mathbb{P}(\mathrm{Sym}^4 \mathcal{V}^\vee)) \to A^\bullet(W_{\mathcal{V}})$, meaning that we have an algorithm to get the formulas for $[O_{C_{AN}}]$ and $[O_{C_{BN}}]$ for all d, but we have not tried to use the algorithm to find a closed expression.

10. Degenerations of Quartic Plane Curves

In this section, we record the degenerations that are are proven only for quartics, namely the degeneration to a double conic and acquiring a hyperflex. We think the specialization to a hyperflex can be done in arbitrary degree, but the algorithm in [5] was too complicated for us to apply with confidence.

10.1. **Degeneration to the double conic.** In this section, we study how p_C changes as a general smooth quartic C specializes to a double conic.

10.1.1. *Preliminary lemmas.*

Lemma 10.1. *Let Q be a smooth conic and $p \in Q$ a point. Let p_1 and p_2 (respectively p_1) be points of $\mathbb{P}^2$ so that p_1, p_2 and p are not collinear (respectively not lying on the tangent line to Q at p). Then, there exists a unique smooth conic Q' meeting Q at p to order 3 (respectively 4) and containing p_1 and p_2 (respectively p_1).*

Proof. Let Z be the curvilinear scheme of length 3 (respectively 4) in a neighborhood of $p \in Q$. By counting conditions, we see that there is a conic Q' containing Z, $p_1, \ldots, p_{5-n}$. If $n = 3$, then Q' cannot be a double line since Z, p_1, p_2 are not set-theoretically contained in a line, and the conic cannot be the union of two distinct lines since Z is not contained in a line. Therefore the conic is smooth.

If $n = 4$, then Q' cannot be a double line since the underlying line must be tangent to Q at p, but that line does not pass through p_1 by assumption. We also cannot have Q' be the union of two distinct lines or else Q' can only meet Q at p to order 3. Therefore Q' is smooth.

In both cases, Q' is unique because the space of all such conics is a linear system and any nontrivial linear system of conics contains singular conics. □

Lemma 10.2. *Let $3 \leq n \leq 7$ and let C be a general quartic curve with an A_n singularity. Then, there is a smooth conic meeting C at its singular point to order $n + 1$ and meeting C transversely at $7 - n$ other points.*

Proof. We will do this case by case. Let $p \in C$ be the singular point, For the case $n = 3$, the conic needs to pass through p with a specified tangent direction and otherwise intersect C transversely. There is 3-dimensional linear system of conics passing through p with a specified tangent direction. In that 3-dimensional linear system, the conics that intersect C at 4 other distinct points form a nonempty open set, as it contains the union of the unique line passing through p in the specified tangent direction with a line intersecting C transversely. Since the space of smooth conics in that 3-dimensional linear system is also nonempty, there exists a smooth conics passing through p in the specified tangent direction and C at four other points.

For the case $n = 4$, we need to resort to equations. The space of conics meeting C at p to order 5 is the same as the space of conics containing a specified length 3 curvilinear scheme Z, and we can assume $p = [0 : 0 : 1]$ and Z is given by the length 3 neighborhood of $X^2 + YZ$ around p. We can specialize C while preserving p and Z, and it suffices to prove the result for the specialized curve. Consider the rational quartic curve C_0 given by

$$(X^2 + YZ)^2 + X^3Y = 0,$$

which has a rhamphoid cusp at $[0:0:1]$ and an ordinary cusp at $[0:1:0]$.

Consider the conic given by $X^2+YZ+aXY+bY^2=0$. If we restrict C_0 to the conic, then we get $(aXY+bY^2)^2+X^3Y=(X^3+a^2X^2Y+2abXY^2+b^2Y^3)Y$. Therefore, the restriction of C_0 to the conic is also given by the union of 4 lines through $p=[0:0:1]$. The line given by $Y=0$ is tangent to the conic at the point, to it suffices to check the remaining three lines are distinct. This can be shown by noting that the discriminant of the cubic polynomial $X^3+a^2X^2Y+2abXY^2+b^2Y^3$ does not vanish identically (indeed it is not even homogenous).

For the cases $n=5,6$, we use Lemma 10.1. In both cases, we have a curvilinear scheme Z of length $n-2$ contained in a conic, and we want to find a smooth conic containing Z and passing through $7-n$ distinct other points of C. If $n=5$, then it suffices to pick the remaining 2 points p_1,p_2 of C so that p, p_1, and p_2 do not all lie on a line. If $n=6$, it suffices to pick the remaining point p to not be contained in the tangent line to Z. □

10.2. **Sibling orbit with A_6 singularity.**

Lemma 10.3. *Let $F(X,Y,Z)$ cut out a quartic plane curve and let $Q(X,Y,Z)=X^2+YZ$. Suppose F and Q meet transversely at $[0:0:1]$. Then,*

$$\lim_{t\to 0} t^{-4}(t^3F(t^2X,tY,Z)+Q(t^2X,tY,Z))$$

is projectively equivalent to C_{A_6}.

Proof. Note $Q(t^2X,tY,Z)^2=t^4Q(X,Y,Z)$. Also, the only coefficients of $t^3F(t^2X,tY,Z)$ whose vanishing order with respect to t is at most 4 are the coefficients of Z^4 and Z^3X. Since F vanishes at $p=[0:0:1]$ by assumption, the coefficient of Z^4 is zero.

The tangent line to $\{Q = 0\}$ at p is given by $Y = 0$. Since $\{F = 0\}$ is transverse to $\{Q = 0\}$ at p, the coefficient of Z^3X is nonzero. Therefore,

$$\lim_{t\to 0} t^3F(t^2X, tY, Z) + Q(t^2X, tY, Z)^2 = (X^2 + YZ)^2 + aZ^3X$$

for $a \neq 0$, which is the unique, up to projective equivalence, rational curve with an A_6 singularity with a full dimensional orbit given in [5, Example 5.4]. $\square$

Corollary 10.4. *For a general quartic plane curve C,*

$$O_C \rightsquigarrow 8(3O_{C_{A_6}}) \Rightarrow p_C = 8p_{C_{A_6}},$$

where C_{A_6} is a general quartic curve with an A_6 singularity.

Proof. Let $F(X,Y,Z)$ cut out C. Pick a conic intersecting C transversely in 8 points and let $Q(X,Y,Z)$ cut out the conic. Then, consider the family of curves over $\mathbb{A}^1$ given by

$$t^3F(X,Y,Z) + Q(X,Y,Z)^2.$$

Applying Lemma 10.3 gives 8 choices $\gamma_i : \mathbb{A}^1 \to PGL_3$, where $1 \leq i \leq 8$, to use in Principle 6.1. To conclude, we use either [5, Example 5.4] or Corollary 5.4 to see the predegree of a general rational quartic C_{A_6} with an A_6 singularity is 1785, and $1785 \cdot 8 = 14280$, which is the predegree the orbit of a general quartic curve [4]. To finish, we note $\#\operatorname{Aut}(C_{A_6}) = 3$ by the equation in [5, Example 5.4]. $\square$

Theorem 10.5. *For a smooth quartic plane curve C with no hyperflexes,*

$$\#\operatorname{Aut}(C)O_C \rightsquigarrow 8(3O_{C_{A_6}}) \Rightarrow p_C = 8p_{C_{A_6}},$$

where C_{A_6} is a general quartic curve with an A_6 singularity.

Proof. Let F cut out a general plane quartic D and G cut out C. Consider the family of curves given by $tF + G$ and apply Principle 6.1 in the special case where $n = 1$ and $\gamma_i : \mathbb{A}^1 \backslash \{0\} \to PGL_3$ is the identity. Then, the fact that the predegree of C is the same as the the predegree of a general plane quartic D [4] means $p_C = p_D$. We conclude by Corollary 10.4. $\square$

Remark 10.6. We remark that our usage of the predegree computation of Aluffi and Faber [4] can in principle be replaced by the explicit description of the semistable reduction of an A_n singularity given in [21].

Theorem 10.7. *Let C_{A_n} be a general curve with an A_n singularity, where $3 \leq n \leq 6$. Then,*

$$\# \operatorname{Aut}(C_{A_n}) O_{C_{A_n}} \rightsquigarrow (7-n)(3 O_{C_{A_6}}) \Rightarrow p_{C_{A_n}} = (7-n) p_{C_{A_6}}.$$

Proof. By Lemma 10.2 we can find a smooth conic that meets C_{A_n} at its singular point to order $n+1$ and meets C transversely at $7-n$ other points $p_1, \ldots, p_{7-n}$. Let $F(X, Y, Z)$ cut out C_{A_n} and $Q(X, Y, Z)$ cut out the conic.

Consider the family of quartic curves given by

$$t^3 F(X, Y, Z) + Q(X, Y, Z)^2.$$

Note in particular that for general fixed t, we get a curve with an A_n singularity. From Lemma 10.3 gives $7-n$ choices for $\gamma_i : \mathbb{A}^1 \backslash \{0\} \to PGL_3$ to use in Principle 6.1. Applying [5, Example 5.4], we find the predegree of C_{A_n} is $(7-n)$ times the predegree of C_{A_6} if $n \geq 3$. $\square$

Remark 10.8. The argument in Theorem 10.7 still works for $n = 1, 2$, except the predegrees don't add up. This suggests there are more orbits to identify. For the cases $n = 1, 2$, we choose to instead use the degeneration in Section 8.

10.3. **Quartic acquiring hyperflexes.** Aluffi and Faber already considered the case of a smooth plane curve with no hyperflexes degenerating to a smooth curve with a hyperflex [2, Theorem IV(2)]. However, in order to run their argument, we need to take a pencil of curves, where each member is tangent to the hyperflex of the special curves. Since a smooth quartic can have up to twelve hyperflexes [71, Section 4], some adjustment has to be made. Instead of using equations as in [2] and the rest of our degenerations, we use ideas of limit linear series.

Lemma 10.9. *Let $C \subset \mathbb{P}^2$ be a rational quartic with an E_6 singularity and two simples flexes. Then, C has an 8-dimensional orbit, so in particular is projectively equivalent to C_{E_6}.*

Proof. We will show $\mathrm{Aut}(C)$ is finite by showing that only a finite subgroup of PGL_3 preserves the flexes and the tangent vector to the singularity. Let G be the component of $\mathrm{Aut}(C)$ containing the identity.

Without loss of generality, we can assume the E_6 singularity is at $[0:0:1]$ and the two flexes are at $[0:1:0]$ and $[1:0:0]$. The group G fixes these three points, so G is a subgroup of $\begin{pmatrix} a & & \\ & b & \\ & & c \end{pmatrix}$.

In addition, G fixes the tangent vector to the singularity. The line L tangent to the singularity meets the curve to order 4 at $[0:0:1]$, so it cannot intersect $[0:1:0]$ or $[1:0:0]$. Since G must preserve L, $a=b$.

Let L_1 be the tangent to C at $[0:1:0]$ and L_2 be the tangent to C at $[1:0:0]$. By Bezout we know neither line passes through $[0:0:1]$. We also cannot have $L_1 = L_2$ by Bezout's theorem. Therefore, L_1 does not pass through $[0:1:0]$ or L_2 does not pass through $[1:0:0]$. In the first case, we find $a=c$ and in the second case we find $b=c$. $\square$

Lemma 10.10. *Let Δ be a smooth (affine) curve, $0 \in \Delta$ be a closed point, and t be a uniformizer at 0. Let $\mathcal{C} \subset \mathbb{P}^2 \times \Delta$ be a family of smooth quartic curves where the general member C_t has general flex behavior and C_0 has a hyperflex at $p \in C_0$.*

Base changing and restricting Δ to an open neighborhood of zero if necessary, there is a family of matrices $\gamma : \Delta \backslash \{0\} \to PGL_3$ such that, in the $\mathbb{P}^8$ of matrices modulo scalars, $\lim_{t \to 0} \gamma(t)$ has image exactly the point p, and, in the $\mathbb{P}^{14}$ of quartics,

$$\lim_{t \to 0} C_t(\gamma(t) \cdot (X, Y, Z))$$

is projectively equivalent to C_{E_6}.

Proof. We will use ideas from limit linear series (see [30] for a reference), but it is not necessary to know the theory to understand the argument.

After base change, we can assume we have two sections $\sigma_1, \sigma_2 : \Delta \to \mathcal{C}$, where σ_1 and σ_2 trace out two flexes in the family limiting to the hyperflex $p_1 := p \in C_0$.

We blow up C_0 at p_1 in order to try to separate σ_1 and σ_2. Since $\mathcal{C}$ is smooth, the exceptional divisor is a rational curve D_1 attached to C_0 at p_1.

The family of curves $\mathcal{C}$ carries a line bundle $\mathcal{L}$ giving the map $\mathcal{C} \to \mathbb{P}^2$. Shrinking Δ to an open neighborhood around 0 if necessary, pick sections s_0, s_1, s_2 of $\mathcal{L}$ such that, when restricted to C_0, we have s_0, s_1, s_2 vanish to orders 4, 1, and 0 respectively.

Let $\pi_1 : \operatorname{Bl}_{p_1} \mathcal{C} \to \mathcal{C}$ be the blowup map. Considered as meromorphic sections of the line bundle $\mathcal{L}(-4D_1)$, $\pi_1^* s_0$, $\pi_1^* s_1$, $\pi_1^* s_2$ have poles of orders 0, 3, 4 respectively. Therefore, to make them regular sections, we have to multiply them by t^0, t^3, and t^4, respectively. Then, $\pi_1^* s_0, t^3 \pi_1^* s_1, t^4 \pi_1^* s_2$ vanish to orders 0, 3 and 4 respectively on C_0, so they also vanish to orders 0, 3, and 4 respectively at $C_0 \cap D_1$ when restricted to D_1.

To summarize, $\pi_1^* s_0, t^3 \pi_1^* s_1, t^4 \pi_1^* s_2$ are regular sections of $\pi_1^* \mathcal{L}(-4D_1)$. When restricted to C_0 the sections correspond to a constant map $C_0 \to \mathbb{P}^2$. When restricted to D_1, the sections map $D_1 \cong \mathbb{P}^1$ into $\mathbb{P}^2$ such that the image is an irreducible quartic plane curve. It cannot map multiple to 1 onto its image because $t^3\pi_1^*s_1$ vanishes to order 3 at p_1, which is relatively prime to 4. Furthermore, $p_1 \in D_1$ maps to a unibranch triple point singularity. No other point in D_1 cannot map to the image of p_1 since $t^4\pi_1^*s_2$ already vanishes to order 4 and p. Therefore the image of D_1 in $\mathbb{P}^2$ is a rational quartic with an E_6 singularity.

Consider the proper transforms $\widetilde{\sigma_1}$ and $\widetilde{\sigma_2}$ of σ_1 and σ_2. They cannot pass through $C_0 \cap D_1$ because σ_1 and σ_2 intersect C_0 with multiplicity 1 at p. If $\widetilde{\sigma_1}$ and $\widetilde{\sigma_2}$ intersect D_1 at two distinct points, then the image of D_1 in $\mathbb{P}^2$ also has two simple flexes. Applying Lemma 10.9 shows that this is projectively equivalent to C_{E_6}, and so has an 8-dimensional orbit under PGL_3. To find the family of matrices $\gamma : \Delta\backslash\{0\} \to PGL_3$, we note that the construction above yields a family of matrices parameterized by Δ that sends $\pi_1^*s_0, t^3\pi_1^*s_1, t^4\pi_1^*s_2$ to $\pi_1^*s_0, t^{-3}\pi_1^*s_1, t^{-4}\pi_1^*s_2$ respectively, which is equivalent to $t^4\pi_1^*s_0, t\pi_1^*s_1, \pi_1^*s_2$ in PGL_3. This is our $\gamma : \Delta\backslash\{0\} \to PGL_3$. We see $\gamma(0)$ is precisely the point p.

If $\widetilde{\sigma_1}$ and $\widetilde{\sigma_2}$ intersect D_1 at the same point p_2, then the image of D_1 in $\mathbb{P}^2$ has a hyperflex and its orbit under PGL_3 is smaller than 8-dimensional. Let them intersect D_1 at p_2. Let π_2 be the blowup map at p_2 and D_2 be the exceptional divisor. Then, we pullback $\pi_1^*\mathcal{L}(-4D_1)$ and $\pi_1^*s_0, t^3\pi_1^*s_1, t^4\pi_1^*s_2$ under π_2.

As before, pick a basis s_0^1, s_1^1, s_2^1 for the vector space spanned by $\pi_1^*s_0, t^3\pi_1^*s_1, t^4\pi_1^*s_2$ such that s_0^1, s_1^1, s_2^1 vanish to orders 4, 1, and 0 at p_2 when restricted to D_1. Then, as above, we twist $\pi_2^*\pi_1^*\mathcal{L}(-4D_1)$ down by $-D_2$ and replace s_0^1, s_1^1, s_2^1 with s_0^1, $t^3s_1^1$, $t^4s_2^1$. If the proper transforms of $\widetilde{\sigma_1}$ and $\widetilde{\sigma_2}$ intersect D_2 at distinct points, we are done by the same argument as above. The image of the family of matrices $\gamma(t)$ will now be the point p_2, but p_2 maps to the same point as p_1 in $\mathbb{P}^2$.

If the proper transforms of $\widetilde{\sigma_1}$ and $\widetilde{\sigma_2}$ intersect D_2 at the same point, we let $p_3 \in D_2$ be the common point of intersection, blow up at p_3 and repeat.

In summary, we found a family of matrices $\gamma : \Delta\backslash\{0\} \to PGL_3$ that takes the original map $\mathcal{C} \to \mathbb{P}^2$ and creates a rational map $\mathcal{C} \dashrightarrow \mathbb{P}^2$ under *pullback* under family of linear maps $\mathbb{P}^2 \times \Delta \dashrightarrow \mathbb{P}^2$ given by γ. Furthermore $\lim_{t\to 0}\gamma(t)$ in the $\mathbb{P}^8$ of 3×3 matrices up to scalar has image precisely the point p. To resolve $\mathcal{C} \dashrightarrow \mathbb{P}^2$, we blow up the special fiber of $\mathcal{C} \to \Delta$ repeatedly to get a chain of rational curve $D_m \cup D_{m-1} \cup \cdots \cup D_1 \cup C_0$ in the special fiber. Here, D_i is attached to D_{i-1} and D_1 is attached to C_0 at p. The resolved map collapses $D_{m-1} \cup \cdots \cup D_1 \cup C_0$ to the same point as p and maps D_m onto a curve that is projectively equivalent to C_{E_6}. □

Theorem 10.11. *Let C be a smooth quartic plane curve with n hyperflexes. Then,*

$$\#\operatorname{Aut}(C)O_C \sim 8(3O_{C_{A_6}}) - n(2O_{C_{E_6}}) \Rightarrow p_C = 8p_{C_{A_6}} - np_{C_{E_6}}.$$

Proof. We consider a family of smooth quartic curves, where the general member C' has no hyperflexes, where C is the special fiber.

From applying Lemma 10.10 and Principle 6.1, we see that

$$p_{C'} - p_C - np_{C_{E_6}}$$

represents a sum of effective cycles, so it suffices to check the predegree of C' is the predegree of C plus n times the predegree of C_{E_6}. The predegree of C is $294n$ less than the predegree of C' [4, Section 3.6]. Also, the predegree of C_{E_6} is 294 from [5, bottom of page 36] or Corollary 5.4 (noting $\#\operatorname{Aut}(C_{E_6}) = 2$).

Finally, we use $p_{C'} = 8p_{C_{A_6}}$ from Theorem 10.5. □

A. Points on $\mathbb{P}^1$

In this section, we compute p_X in the case X is a hypersurface in $\mathbb{P}^1$. We will let u, v be the chern roots of c_1 and c_2. In the case $X \subset \mathbb{P}^1$ is supported on at most three points, these are strata of coincident root loci, which were first computed in [42] and generalized to PGL_2-equivariant cohomology in [94]. Therefore, we only have to deal with the case where X is supported on at least four points, and we give two separate proofs. Note that since Definition 4.1 involves taking the dual, our sign convention differs from the usual in the case of points, and we will always be computing $p_X(-u,-v)$ instead of $p_X(u,v)$.

Theorem A.1. *Let $X \subset \mathbb{P}^1$ be a subscheme of length d supported on points $p_1,\dots,p_n$ with multiplicities $m_1,\dots,m_n$ with $n \geq 3$. Then,*

$$p_X(-u,-v) = \frac{\prod_{i=0}^{d}(iu+(d-i)v)}{(u-v)^2}\left(\frac{n-2}{duv} + \sum_{i=1}^{n}\frac{2m_i-d}{(m_iv+(d-m_i)u)(m_iu+(d-m_i)v)}\right)$$

We give a proof of Theorem A.1 using the resolution given by Aluffi and Faber [3] together with the Atiyah-Bott formula [29] in Section B. This proof is self-contained and direct. The second proof we give is from the machinery developed in [73] that apply to arbitrary hyperplane arrangements. A computation is required to specialize the results from the case of ordered points on $\mathbb{P}^1$ to unordered points on $\mathbb{P}^1$, we do this now.

Proof using [73]. We use the same argument in [73, Theorem 12.5], so we only describe the computation, and refer the motivation and proof of correctness to [73]. Because our sign convention is opposite that of [73], we will actually compute $p_X(-u,-v)$. Let $d = \sum_{i=1}^{n} m_i$ and $G(z) = \prod_{i=0}^{d}(H+iu+(d-i)v) \in \mathbb{Z}[u,v][z]$. Let $L(z) = \frac{G(z)-G(0)}{z}$. Let $\overline{L}(H_1,\dots,H_n)$ be the result of reducing $L(m_1H_1+\cdots+m_nH_n)$ modulo $(H_i+u)(H_i+v)$ for each i. Now, we carry out the three steps in the proof of [73, Theorem 12.5].

Step 1 By Lagrange interpolation,

$$L(m_1H_1+\cdots+m_nH_n) = \frac{G(m_1H_1+\cdots+m_nH_n)-L(0)}{m_1H_1+\cdots+m_nH_n}$$

$$\overline{L}(H_1,\dots,H_n) = \sum_{T\subset\{1,\dots,n\}}\frac{-G(0)}{-\sum_{i\in T}m_iv-\sum_{i\notin T}m_iu}\left(\prod_{i\in T}\frac{H_i+u}{-v+u}\right)\left(\prod_{i\notin T}\frac{H_i+v}{-u+v}\right).$$

Step 2 Substituting z for each H_i yields

(A.1)

$$\overline{L}(z,\dots,z) = G(0)\sum_{T\subset\{1,\dots,n\}} \frac{1}{\sum_{i\in T} m_i v + \sum_{i\notin T} m_i u}\frac{(z+u)^{\#T}(z+v)^{d-\#T}}{\prod_{i\in T}(-v+u)\prod_{i\notin T}(-u+v)}.$$

Step 3 Let $F(z)=(z+u)(z+v)$. All terms of (A.1) are divisible by $F(z)^2$ unless $\#T\in\{0,1,n-1,n\}$. Thus, $[z^1][F(z)^1]\overline{L}(z,\dots,z)$ is

$$\frac{G(0)}{(u-v)^n}[z^1][F(z)^1](\frac{(-1)^n(z+v)^n}{du}+\frac{(z+u)^n}{dv}+\sum_{i=1}^{n}\frac{(-1)^{n-1}F(z)(z+v)^{n-2}}{m_i v+(d-m_i)u}$$
$$+\sum_{i=1}^{n}\frac{(-1)F(z)(z+u)^{n-2}}{m_i u+(d-m_i)v}).$$

As in the proof of [73, Theorem 12.5],

$$[z^1][F(z)^1]F(z)(z+u)^k = (u-v)^{k-1} \qquad [z^1][F(z)^1](z+u)^k = (k-2)(u-v)^{k-3}$$
$$[z^1][F(z)^1]F(z)(z+v)^k = (v-u)^{k-1} \qquad [z^1][F(z)^1](z+v)^k = (k-2)(v-u)^{k-3},$$

so $[z^1][F(z)^1]\overline{L}(z,\ldots,z)$ simplifies to

$$\frac{G(0)}{(u-v)^n}(\frac{(-1)^n(n-2)(v-u)^{n-3}}{du}+\frac{(n-2)(u-v)^{n-3}}{dv}+\sum_{i=1}^{n}\frac{(-1)^{n-1}(v-u)^{n-3}}{m_iv+(d-m_i)u}$$

$$+\sum_{i=1}^{n}\frac{(-1)(u-v)^{n-3}}{m_iu+(d-m_i)v})$$

$$\frac{G(0)}{(u-v)^n}(\frac{(-1)(n-2)(u-v)^{n-3}}{du}+\frac{(n-2)(u-v)^{n-3}}{dv}+\sum_{i=1}^{n}\frac{(u-v)^{n-3}}{m_iv+(d-m_i)u}$$

$$+\frac{(-1)(u-v)^{n-3}}{m_iu+(d-m_i)v})$$

$$\frac{G(0)}{(u-v)^n}(\frac{(n-2)(u-v)^{n-2}}{duv}+\sum_{i=1}^{n}\frac{(2m_i-d)(u-v)^{n-2}}{(m_iv+(d-m_i)u)(m_iu+(d-m_i)v)})$$

$$\frac{G(0)}{(u-v)^2}(\frac{n-2}{duv}+\sum_{i=1}^{n}\frac{2m_i-d}{(m_iv+(d-m_i)u)(m_iu+(d-m_i)v)})$$

□

In the case all the multiplicities are all one, the formula in Theorem A.1 simplifies. We will also give a direct proof by slow projection.

Corollary A.2. *In the setting of Theorem A.1 if each $m_i = 1$, then*

$$p_X(-u,-v) = n(n-1)(n-2)\prod_{j=2}^{n-2}(H+(ju+(n-j)v)).$$

Proof using Theorem A.1. Applying Theorem A.1, we find $p_X(-u,-v)$ is

$$\frac{1}{(u-v)^2}\prod_{i=0}^{n}(iu+(n-i)v)\left(\frac{n-2}{nuv}-\frac{(-2+n)n}{((n-1)u+v)((n-1)v+u)}\right)=$$

$$\frac{n(n-2)}{(u-v)^2}\prod_{i=0}^{n}(iu+(n-i)v)\left(\frac{1}{(nu)(nv)}-\frac{1}{((n-1)u+v)((n-1)v+u)}\right)=$$

$$\frac{n(n-2)}{(u-v)^2}\prod_{i=0}^{n}(iu+(n-i)v)\left(\frac{n-1}{(nu)((n-1)u+v)((n-1)v+u)(nv)}\right).$$

Applying [41, Theorem 6.1] yields the answer. □

Proof by slow projection. Let V be a 2-dimensional vector space, $v_1, \ldots, v_n$ pairwise linearly independent vectors of V, $X \subset \mathbb{P}^1$ the corresponding point configuration supported on $p_1, \ldots, p_n$, and $Z \subset \mathbb{P}(\operatorname{Sym}^n V)$ the orbit closure. The key fact we will use is

Claim A.3. *Every point in the boundary of Z corresponds to a point configuration in $\mathbb{P}^1$ supported on two points with multiplicities $n-1$ and 1 or one point with multiplicity n.*

Proof of Claim A.3. Let $A(t)$ be a 1-parameter family of matrices, or more precisely a map from the spectrum of a discrete valuation ring to $\operatorname{End}(V)$ where the generic point maps to an element of $GL(V)$. We want to show that the multiset $S = \{\lim_{t\to 0} A(t)p_i \mid 1 \leq i \leq n\}$ does not have two copies each of two distinct points. First, we can assume the rank of $A(0) = 1$. If the rank of $A(0)$ is 2, then S consists of distinct points. If $A(0) = 0$, we can divide out by a power of the uniformizing parameter so that $A(0) \neq 0$. Then, $\{\lim_{t\to 0} A(t)p_i \mid 1 \leq i \leq n\}$ is the point in $\mathbb{P}(V)$ corresponding to the 1-dimensional image of $A(0)$ if v_i is not in the kernel of $A(0)$. Otherwise, there is at most one v_i in the kernel of $A(0)$ and $\{\lim_{t\to 0} A(t)p_i \mid 1 \leq i \leq n\}$ is otherwise unrestricted. $\square$

Let x, y be a basis for V. Then, a basis of $\operatorname{Sym}^n V$ is $x^n, x^{n-1}y, \ldots, y^n$. Let $T \subset GL(V)$ be the maximal torus corresponding to the basis x, y. Since $A^\bullet_{GL(V)}(\mathbb{P}(\operatorname{Sym}^d V)) \to A^\bullet_T(\mathbb{P}(\operatorname{Sym}^d V))$ is injective, we can use a T-equivariant degeneration and compute the T-equivariant class. Our T-equivariant degeneration will be to scale the coordinates corresponding to $x^{n-2}y^2, \ldots, x^2y^{n-2}$ to zero.

By Claim A.3, Z is disjoint from the source of this "slow projection," so the T-equivariant class of Z is a multiple of the class of the 3-plane in $\mathbb{P}(\operatorname{Sym}^d V)$ given by the vanishing of the coordinates corresponding to $x^{n-2}y^2, \ldots, x^2y^{n-2}$. The class of that 3-plane is $\prod_{i=2}^{n-2} (iu + (n-i)v)$. The multiple we need is the degree of Z as a projective variety, with is $n(n-1)(n-2)$ by the combinatorial argument given in [3, Introduction]. $\square$

Remark A.4. Corollary A.2 can be generalized in a different direction. Suppose we fix n general points $p_1, \ldots, p_n \in \mathbb{P}^r$ and consider all configurations of n points given by mapping $p_1, \ldots, p_n$ via a linear rational map $\mathbb{P}^r \to \mathbb{P}^1$. Let the closure of these configurations in $\operatorname{Sym}^n \mathbb{P}^1$ be $Z_{r,n}$. The same proof of Claim A.3 using slow projection shows the equivariant class of $Z_{r,n}$ in $A^\bullet(\operatorname{Sym}^n \mathbb{P}^1) = \mathbb{Z}[u,v][H]/(\prod_{i=0}^n H + iu + (n-i)v)$ has constant term

$$\prod_{i=0}^{2r+1} (n-i) \prod_{i=r+1}^{n-r-1} (iu + (n-i)r),$$

and the full class is given by substituting $u \to u + \frac{H}{n}, v \to v + \frac{H}{n}$ into the constant term [41, Theorem 6.1]. These are examples of *generalized matrix orbits* defined in [73]. Also see [97, Example 1.3].

B. Points on $\mathbb{P}^1$ via Atiyah-Bott

The method in Section A was closer to the theme of equivariant degeneration explored in this paper. We note that there is self-contained proof given by the Atiyah-Bott formula, or equivalently resolution and integral via localization [44, Section 4]. The authors attempted to perform the same method for smooth plane curves using the resolution given by [4], but the computation of the normal bundles quickly became intractable.

B.1. **General setup.** Let V be a 2-dimensional vector space with $T = (\mathbb{C}^\times)^2$ acting by scaling. Then, we have T-action on $\mathbb{P}^3 \cong \mathbb{P}\mathrm{Hom}(V, \mathbb{C}^2)$. Given a point configuration of d-points in $\mathbb{P}^1$ (a central hyperplane configuration in $\mathbb{C}^2$), we have a rational map

$$\mathbb{P}\mathrm{Hom}(V, \mathbb{C}^2) \dashrightarrow \mathbb{P}(\operatorname{Sym}^n V)$$

The base locus is n-disjoint lines, where n is the number of distinct points, given by the matrices with image contained in each p_i for $1 \le i \le d$. Picking a basis, we find $\mathbb{P}\mathrm{Hom}(V, \mathbb{C}^2)$ is given by 2 by 2 matrices $\begin{pmatrix} * & * \\ * & * \end{pmatrix}$ up to scaling. The T action is by scaling the columns, and each of the base loci (after base change via a left GL_2-action) looks like $\begin{pmatrix} * & * \\ 0 & 0 \end{pmatrix}$.

Let X be the blow up of $\mathbb{P}^3$ along these base loci $R_1, \dots, R_n$. This resolves the rational map above [4, Proposition 1.2].

B.2. **Normal bundle to a proper transform.**

Lemma B.1. *Let $Z \subset Y$ be an inclusion of smooth varieties. Let $W \subset Y$ be a smooth subvariety and $\widetilde{W} \subset \mathrm{Bl}_Z Y$ be the proper transform of W. If $\pi : \mathrm{Bl}_Z Y \to Y$ is the blowup map, then we have the short exact sequence*

$$0 \to \operatorname{coker}(\pi^* N^\vee_{W/Y} \to N^\vee_{\widetilde{W}/\mathrm{Bl}_Z Y}) \to \Omega_{\mathrm{Bl}_Z Y/Y}|_{\widetilde{W}} \to \Omega_{\widetilde{W}/W} \to 0.$$

Proof. Consider the following diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & \pi^* N^\vee_{W/Y} \arrow[r] \arrow[d] & N^\vee_{\widetilde{W}/\mathrm{Bl}_Z Y} \arrow[d] & & \\
0 \arrow[r] & \pi^*\Omega_Y|_{\widetilde{W}} \arrow[r] \arrow[d] & \Omega_{\mathrm{Bl}_Z Y}|_{\widetilde{W}} \arrow[r] \arrow[d] & \Omega_{\mathrm{Bl}_Z Y/Y}|_{\widetilde{W}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \pi^*\Omega_W \arrow[r] \arrow[d] & \Omega_{\widetilde{W}/W} \arrow[r] \arrow[d] & \Omega_{\widetilde{W}/W} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

The bottom two rows are exact by the relative cotangent sequence for a generically separable morphism of integral smooth varieties [77, Remark 4.17]. The lemma follows from the nine lemma. ∎

B.3. **Setup for Atiyah-Bott integration.** In order to apply Atiyah-Bott integration to X, we need to identify the fixed loci, their normal bundles, and how classes restrict from X to the fixed point loci.

B.4. **Fixed point loci.** First, we note that the fixed-point loci of $\mathbb{P}^3$ under the action of T consists of two disjoint $\mathbb{P}^1$'s which we will call C_1, C_2.

$$\begin{pmatrix} * & 0 \\ * & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & * \\ 0 & * \end{pmatrix}$$

The fixed-point loci of X under the action of T must lie over the fixed-point loci of $\mathbb{P}^3$ under T. We claim that there are $2n+2$ fixed-point loci:

(1) 2 fixed point loci corresponding to $\mathbb{P}^1$'s that are the proper transforms $\widetilde{C_1}$ and $\widetilde{C_2}$ of C_1 and C_2. If we suppose C_1 is the $\mathbb{P}^1$ consisting of the matrices $\begin{pmatrix} * & * \\ 0 & 0 \end{pmatrix}$, then the point of the proper transform lying above $C_1 \cap R_1$ is given by the limiting point of

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + t \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

as $t \to 0$.

(2) $2n$ isolated points that lie over the $2n$ pairwise intersections of C_1, C_2 with $R_1, \ldots, R_n$. If we suppose C_1 is the $\mathbb{P}^1$ consisting of the matrices $\begin{pmatrix} * & * \\ 0 & 0 \end{pmatrix}$, then the point lying above $C_1 \cap R_1$ is given by the limiting point of

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + t \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

as $t \to 0$.

B.4.1. *Normal bundles and restriction of proper transforms.* Let H be the $c_1(\mathscr{O}_{\mathbb{P}^3}(1))$ on $\mathbb{P}^3$ pulled back to X and E an exceptional divisor of $X \to \mathbb{P}^3$. We have $\widetilde{C}_1$ is $\mathbb{P}^1$ with a trivial T-actions. Therefore, $A^\bullet(\widetilde{C}_1) \cong \mathbb{Z}[z][u,v]/(z^2)$. We have the following restrictions:

$$H \mapsto H = z - u$$

$$E \mapsto z = H + u.$$

Here, E is any of the m exceptional divisors. (We are thinking of $\widetilde{C}_1$ as the $\mathbb{P}^1$ embedded as the first column of 2 by 2 matrices $\mathbb{P}^3$ up scaling. Therefore, it's actually natural to think of it as the projectivization of a vector bundle with a nontrivial T-action, so it is a projective bundle over a point that is trivial, but $\mathscr{O}(1) = H$ is twisted. The Leray relation in this case is $(H+u)^2 = z^2$.)

We need to compute the normal bundle to the proper transform of C_1. The normal bundle of C_1 in $\mathbb{P}^3$ is

$$\frac{c(\mathbb{P}^3)}{c(C_1)} = \frac{(1+u+H)^2(1+v+H)^2}{(1+u+H)^2} = (1+v+H)^2.$$

Note that this also makes sense as C_1 is a complete intersection cut out by $(v+H)^2$. Applying Lemma B.1 yields

$$0 \to \pi^* N^\vee_{C_1/\mathbb{P}^3} \to N^\vee_{\widetilde{C}_1/X} \to \Omega_{X/\mathbb{P}^3}|_{\widetilde{C}_1} \to 0.$$

The term on the right is a skyscraper sheaf supported on the intersection of $\widetilde{C}_1 \cong \mathbb{P}^1$ with the exceptional locus. We need to find the torus action on the bundle $T_{X/\mathbb{P}^3}|_{\widetilde{C}_1}$ supported on E at the intersection $E \cap \widetilde{C}_1$. There is an affine neighborhood of $E \cap \widetilde{C}_1$ in X of the form

$$\begin{pmatrix} 1 & \frac{a_{01}}{a_{00}} \\ \frac{a_{10}}{a_{00}} & \frac{a_{11}}{a_{00}} \end{pmatrix} + t \begin{pmatrix} 0 & 0 \\ 1 & \frac{A_{11}}{A_{10}} \end{pmatrix}$$

with coordinates given by $\frac{a_{01}}{a_{00}}, \frac{a_{10}}{a_{00}}, \frac{A_{11}}{A_{10}}$ as $\frac{A_{11}}{A_{10}}a_{10} = a_{11}$. We have the short exact sequence

$$0 \to \widetilde{C}_1(-z) \otimes \mathbb{C}_{v-u} \to \widetilde{C}_1 \otimes \mathbb{C}_{v-u} \to \mathbb{C}_{v-u}|_{\widetilde{C}} \cap \pi^{-1}(R_i) \to 0,$$

where $\mathbb{C}_{v-u}$ is the nonequivariantly trivial line bundle with an action of T by the character $v-u$. The torus action has character $v-u$ on the coordinate $\frac{A_{11}}{A_{10}}$, so the term on the right has chern class $\frac{1+v-u}{1-z+v-u}$. We apply this for each i to find

$$\begin{aligned} c(N_{\widetilde{C}_1/X}) &= (1+H+v)^2 \frac{(1-z+v-u)^m}{(1+v-u)^n} \\ &= (1+v-u)^2(1+\frac{z}{1+v-u})^2(1-\frac{z}{1+v-u})^n \\ &= (1+v-u)^2(1+\frac{(2-n)z}{1+v-u}) \\ &= (1+z+v-u)(1+(1-n)z+v-u). \end{aligned}$$

B.4.2. *Restriction to isolated points.* Suppose we are considering the isolated fixed point p given by the limit as $t \to 0$ of

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + t \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Then, we have the restrictions

$$H \mapsto -u$$

$$E \mapsto v-u.$$

Here, E is the exceptional divisor containing p. The first one is by restricting the tautological line bundle and considering the torus action. To see the restriction of E, we note that the restriction of E to itself is $\mathscr{O}_{\mathbb{P}(N_{R_1/\mathbb{P}^3})}(-1)$. Then, we take the local chart around $\pi(p)$ consisting of

$$\begin{pmatrix} 1 & \frac{a_{01}}{a_{00}} \\ \frac{a_{10}}{a_{00}} & \frac{a_{11}}{a_{00}} \end{pmatrix}$$

and find the action on the coordinate $\frac{a_{11}}{a_{00}}$ is $v-u$. Also the normal bundle to p in X has chern class

$$(1+v-u)^2(1+u-v).$$

To see this, consider the local chart around p

$$\begin{pmatrix} 1 & \frac{a_{01}}{a_{00}} \\ \frac{a_{10}}{a_{00}} & \frac{a_{11}}{a_{00}} \end{pmatrix} + t \begin{pmatrix} 0 & 0 \\ \frac{A_{10}}{A_{11}} & 1 \end{pmatrix}$$

which has local coordinates $\frac{a_{01}}{a_{00}}, \frac{a_{11}}{a_{00}}, \frac{A_{10}}{A_{11}}$ on which T acts by characters $v-u$, $v-u$ and $u-v$ respectively.

B.5. **Application of Atiyah-Bott.**

Proof of Theorem A.1. As before, we compute $p_X(-u,-v)$ due to our sign conventions. Let

$$\phi(H) = \frac{(H+du)(H+(d-1)u+v)\cdots(H+dv)-(du)\cdots(dv)}{H}.$$

We want to pull $\phi(H)$ back to X and integrate using Atiyah-Bott. We first integrate over $\widetilde{C_1}$. Since H pulls back to

$$dH - \sum_{i=1}^{n} m_i E_i = d(z-u) - dz = -du,$$

this is

$$[z]\frac{1}{(z+v-u)((1-n)z+v-u)}\phi(-du) =$$

$$[z]\frac{\frac{1}{(v-u)^2}}{(1+\frac{z}{v-u})(1+\frac{(1-n)z}{v-u})}\frac{-\prod_{i=0}^{d}(iu+(d-i)v)}{-du} =$$

$$\frac{(n-2)\prod_{i=1}^{d}(iu+(d-i)v)}{(v-u)^3}.$$

Adding this to the contribution of $\widetilde{C_2}$ yields

$$(n-2)\prod_{i=0}^{d}(iu+(d-i)v)\frac{1}{(v-u)^3}\left(\frac{1}{du}-\frac{1}{dv}\right) =$$

$$(n-2)\prod_{i=0}^{d}(iu+(d-i)v)\frac{1}{(v-u)^2}\frac{1}{duv}$$

For each $1 \leq i \leq n$, we have a point in the configuration of multiplicity m_i. We have two isolated fixed points corresponding to i lying above $R_i \cap C_1$ and $R_i \cap C_2$. For the point lying above $R_i \cap C_1$, H pulls back to $dH - m_iE$, where E is the exceptional divisor lying above R_i. This restricts to

$$-du - n(v-u) = (-d+m_i)u - m_iv$$

at the fixed point. The contribution to Atiyah Bott is

$$\frac{1}{(u-v)^3}\phi((-d+m_i)u - m_iv) =$$

$$\frac{1}{(u-v)^3}\frac{-\prod_{j=0}^{d}(ju+(d-j)v)}{(-d+m_i)u-m_iv}.$$

Adding this to the contribution of the fixed point lying above $R_i \cap C_2$, we get

$$\frac{1}{(u-v)^3}\prod_{j=0}^{d}(ju+(d-j)v)\left(\frac{1}{(d-m_i)u+m_iv}-\frac{1}{(d-m_i)v+m_iu}\right)=$$

$$-\frac{1}{(u-v)^2}\prod_{j=0}^{d}(ju+(d-j)v)\frac{d-2m_i}{((d-m_i)u+m_iv)((d-m_i)v+m_iu)}.$$

Adding the contributions up yields the result. □

C. Cubic plane curves

Although the computations of p_C for cubic plane curves C are elementary, we provide them here for the sake of completeness.

The following table provides a complete list of polynomials p_C:

Cubic Curve C	$p_C(c_1, c_2, c_3)$	# Aut
Triple Line	$-(72c_1^3c_2^2 + 36c_1c_2^3 + 36c_1^4c_3 - 162c_1^2c_2c_3 + 243c_1c_3^2)$	∞
Double Line plus Line	$-(72c_1^3c_2 + 36c_1c_2^2 - 108c_1^2c_3)$	∞
Three concurrent lines	$12c_1^4 + 6c_1^2c_2 + 27c_1c_3$	∞
Conic plus tangent line	$-36c_1^3 - 18c_1c_2$	∞
Triangle	$-(12c_1^3 + 6c_1c_2 + 27c_3)$	∞
Conic plus line	$18c_1^2 + 9c_2$	∞
Cuspidal cubic	$24c_1^2$	∞
Irreducible nodal cubic	$(-12c_1)6$	6
Smooth cubic ($j \neq 0, 1728$)	$(-12c_1)18$	18
Smooth cubic with $j = 1728$	$(-6c_1)36$	36
Smooth cubic with $j = 0$	$(-4c_1)54$	54

The formulas for a triple line, double line plus line, conic plus line, and triangle can all be obtained via presentation and integration along the lines of [42, Theorem 3.1] and Proposition 5.5. This is the method of resolution and integration [44, Section 3].

The formula for three concurrent lines, conic plus tangent line and cuspidal cubic can be gotten by applying Kazarian's formula [65, Theorem 1] for counting D_4, A_3, and A_2 singularities respectively. This was carried out for the case of quartic plane curves for A_6, D_6, and E_6 in the proof of Corollary 5.4. The formula for smooth and nodal cubics and be obtained by their predegree formulas [4, Section 3.6] and Proposition 4.4.

Part 3. PGL_2-equivariant strata of point configurations in $\mathbb{P}^1$

This part of the thesis contains the arXiv preprint [94] joint with Hunter Spink. The idea was to extend the GL_2-equivariant computation of Fehér, Némethi, and Rimányi [42] of strata of point configurations on $\mathbb{P}^1$ and determine the relations between them. Here, the main obstacle is that PGL_2 is harder to work with than GL_2 as it is nonspecial, so its Chow ring contains torsion.

Abstract: We compute the integral Chow ring of the quotient stack $[(\mathbb{P}^1)^n/PGL_2]$, which contains $\mathcal{M}_{0,n}$ as a dense open, and determine a natural $\mathbb{Z}$-basis for the Chow ring in terms of certain ordered incidence strata. We further show that all $\mathbb{Z}$-linear relations between the classes of ordered incidence strata arise from an analogue of the WDVV relations in $A^\bullet(\overline{\mathcal{M}}_{0,n})$.

Next we compute the classes of unordered incidence strata in the integral Chow ring of the quotient stack $[\mathrm{Sym}^n\mathbb{P}^1/PGL_2]$ and classify all $\mathbb{Z}$-linear relations between the strata via these analogues of WDVV relations.

Finally, we compute the rational Chow rings of the complement of a union of unordered incidence strata.

11. Introduction

We consider PGL_2-equivariant Chow classes of incidence strata corresponding to point configurations in $\mathbb{P}^1$. Our results concern both ordered point configurations, parametrized by $(\mathbb{P}^1)^n$, and unordered point configurations, parametrized by $\mathrm{Sym}^n\mathbb{P}^1 \cong \mathbb{P}^n$. The equivariant Chow rings $A^\bullet_{PGL_2}((\mathbb{P}^1)^n)$ and $A^\bullet_{PGL_2}(\mathrm{Sym}^n\mathbb{P}^1)$ can be defined as the integral Chow rings of the quotient stacks $[(\mathbb{P}^1)^n/PGL_2]$ and $[\mathrm{Sym}^n\mathbb{P}^1/PGL_2]$ respectively, so the PGL_2-equivariant Chow classes of incidence strata specialize to relative classes in $A^\bullet(\mathcal{P}^n)$ and $A^\bullet(\mathrm{Sym}^n\mathcal{P})$ respectively for any $\mathbb{P}^1$-bundle $\mathcal{P} \to B$.

Our computations of the PGL_2-equivariant classes of unordered strata generalizes the GL_2-equivariant computation of Fehér, Némethi, and Rimányi [42]. From our results we are able to deduce all integral relations between these PGL_2-equivariant classes. Surprisingly, despite the presence of nontrivial 2-torsion, every integral relation between GL_2-equivariant classes also holds PGL_2-equivariantly. Even though classes of incidence strata do not generate $A^\bullet_{PGL_2}(\mathrm{Sym}^n\mathbb{P}^1)$ integrally, we find a $\mathbb{Q}$-linear basis of $A^\bullet_{PGL_2}(\mathrm{Sym}^n\mathbb{P}^1)$ in terms of certain incidence strata.

Ordered strata of point configurations are products of diagonal classes in $(\mathbb{P}^1)^n$. The stack $[(\mathbb{P}^1)^n/PGL_2]$ contains $\mathcal{M}_{0,n}$ as a dense open, and we show that its Chow ring behaves much like the Chow ring of $\overline{\mathcal{M}}_{0,n}$ [68], with incidence strata in place of boundary divisors. We also find a $\mathbb{Z}$-linear basis for $A^\bullet_{PGL_2}((\mathbb{P}^1)^n)$ in terms of certain incidence strata.

For any $\mathbb{P}^1$-bundle $\mathcal{P} \to B$ and PGL_2-invariant subvarieties $X \subset (\mathbb{P}^1)^n$ and $Y \subset \mathrm{Sym}^n\mathbb{P}^1$, we have relative versions $\mathcal{X} \subset \mathcal{P}^n$ and $\mathcal{Y} \subset \mathrm{Sym}^n\mathcal{P}$ restricting to X and Y in every fiber of $\mathcal{P}^n \to B$ and $\mathrm{Sym}^n\mathcal{P}$ respectively. Our additive bases express the non-equivariant integral Chow class of $\mathcal{X}$ (resp. rational Chow class of $\mathcal{Y}$) as integral (resp. rational) combinations of certain incidence strata.

Finally, we remark two advantages of working PGL_2-equivariantly rather than GL_2-equivariantly (where restriction to the torus-fixed points is an injection). First, our results specialize to arbitrary $\mathbb{P}^1$-bundles $\mathcal{P} \to B$ instead of only projectivizations of rank 2 vector bundles. Second, in the corresponding GL_2-invariant Chow ring, there are additional classes not generated by GL_2-invariant cycles. Thus it does not seem clear how to recover results such as the ones in the previous paragraph directly from the GL_2-equivariant theory.

The reader may refer to Section 12.1 and Section 12.2 for an exposition on how equivariant Chow classes yield universal relations between relative Chow classes in bundles, and Example 11.11 for example consequences.

11.1. **Ordered strata in** $[(\mathbb{P}^1)^n/PGL_2]$**.** The moduli space $\mathcal{M}_{0,n}$ of n distinct points on $\mathbb{P}^1$ is the quotient of $(\mathbb{P}^1)^n \setminus \bigcup_{i<j} \Delta_{i,j}$ by the free action of PGL_2, where $\Delta_{i,j}$ is the locus in $(\mathbb{P}^1)^n$ where the ith and jth coordinates are equal. This is classically compactified by the variety $\overline{\mathcal{M}}_{0,n}$ of stable genus zero n-pointed curves.

We study the integral Chow ring $A^\bullet_{PGL_2}((\mathbb{P}^1)^n)$ (as defined in [27, Section 5]) of the quotient stack $[(\mathbb{P}^1)^n/PGL_2]$ containing $\mathcal{M}_{0,n}$ as a dense open. This stack is stratified by certain incidence strata $\Delta_P \subset (\mathbb{P}^1)^n$ for P a partition of $[n] := \{1, \ldots, n\}$, the loci where the ith and jth coordinates are equal if i and j are in the same part of P.

We compute a ring presentation in Theorem 11.1 for $A^\bullet_{PGL_2}((\mathbb{P}^1)^n)$ similar to that of $A^\bullet(\overline{\mathcal{M}}_{0,n})$ computed by Keel [68]. The incidence strata Δ_P play a fundamental role in the equivariant Chow ring: in Theorem 11.3 we compute a $\mathbb{Z}$-basis for $A^\bullet_{PGL_2}((\mathbb{P}^1)^n)$, which consists in degree $\leq n-2$ of certain incidence strata, and in Theorem 11.5 we show all relations between incidence strata arise from an analogue of the WDVV relation in $A^\bullet(\overline{\mathcal{M}}_{0,4})$ (see Section 11.1.1).

Theorem 11.1. *The following are true.*

(1) *(Theorem 15.16) For $n \geq 3$, the ring $A^\bullet_{PGL_2}((\mathbb{P}^1)^n) = \frac{\mathbb{Z}[\Delta_{i,j}]_{1 \leq i < j \leq n}}{relations}$, where the relations are (notating $\Delta_{j,i} := \Delta_{i,j}$ for $j > i$)*

  (a) *$\Delta_{i,j} + \Delta_{k,l} = \Delta_{i,k} + \Delta_{j,l}$ for distinct i, j, k, l (square relations)*

  (b) *$\Delta_{i,j}\Delta_{i,k} = \Delta_{i,j}\Delta_{j,k}$ for distinct i, j, k. (diagonal relations)*

(2) *(Lemma 15.4) For $n \geq 1$, the group $A^k_{PGL_2}((\mathbb{P}^1)^n)$ is a free $\mathbb{Z}$-module of rank*

$$\sum_{\substack{i \leq k \\ i \equiv k \text{ mod } 2}} \binom{n}{i}.$$

(3) *(Theorem 13.3) For $n \geq 1$, the natural map from $A^\bullet_{PGL_2}((\mathbb{P}^1)^n)$ to*

$$A^\bullet_{GL_2}((\mathbb{P}^1)^n) \cong \mathbb{Z}[u, v]^{S_2}[H_1, \ldots, H_n]/(F(H_1), \ldots, F(H_n)),$$

is injective, where $u + v$ and uv are the first and second chern classes of the standard representation of GL_2, $F(z) = (z + u)(z + v)$, and H_i is $c_1(\mathscr{O}(1)) \in A^\bullet_{PGL_2}((\mathbb{P}^1))$ pulled back via projection to the ith factor.

This identifies $A^\bullet_{PGL_2}((\mathbb{P}^1)^n)$ with the subring of $A^\bullet_{GL_2}((\mathbb{P}^1)^n)$ generated by $H_i + H_j + u + v$ for distinct i, j and $2H_i + u + v$ for all i, and this maps

$$\Delta_{i,j} \mapsto H_i + H_j + u + v.$$

(4) *(Remark 13.5) If the base field is $\mathbb{C}$, then for all $n \geq 1$ the map $A^\bullet_{PGL_2}((\mathbb{P}^1)^n) \to H^\bullet_{PGL_2}((\mathbb{P}^1)^n)$ to equivariant cohomology is an isomorphism.*

The square relations $\Delta_{i,j} + \Delta_{k,l} = \Delta_{i,k} + \Delta_{j,l}$ for distinct i, j, k, l are analogous to the WDVV relations on $A^\bullet(\overline{\mathcal{M}}_{0,n})$ pulled back from $A^\bullet(\overline{\mathcal{M}}_{0,4}) \cong A^\bullet(\mathbb{P}^1)$ (see Section 11.1.1).

The diagonal relations $\Delta_{i,j}\Delta_{i,k} = \Delta_{i,j}\Delta_{j,k}$ are geometrically obvious as $\Delta_{i,j} \cap \Delta_{i,k}$ and $\Delta_{i,j} \cap \Delta_{j,k}$ both give the locus where the ith, jth, and kth coordinates are all equal. In particular, repeated intersections in this fashion allow us to reconstruct all Δ_P.

We will in fact show that the classes of the Δ_P for P a partition of $\{1, \ldots, n\}$ into $d \geq 2$ parts generate $A^{n-d}_{PGL_2}((\mathbb{P}^1)^n)$ $\mathbb{Z}$-linearly. Surprisingly, we can produce a $\mathbb{Z}$-basis for $A^{\bullet}_{PGL_2}((\mathbb{P}^1)^n)$ represented by certain Δ_P (at least in degrees $\leq n-2$).

Definition 11.2. *Call a partition P of $\{1, \ldots, n\}$* good *if it can be written as $P = \{A_1, \ldots, A_d\}$ with $A_1 \sqcup A_2$ an initial segment of $\{1, \ldots, n\}$, and $A_3, \ldots, A_d$ intervals.*

Theorem 11.3 (Theorem 15.16)**.** *For $n \geq 3$, the additive group $A^k_{PGL_2}((\mathbb{P}^1)^n)$ has a $\mathbb{Z}$-basis consisting of the following.*

1. *If $k \leq n-2$, the classes Δ_P for P a good partition into $n-k$ parts.*
2. *If $k > n-2$, the classes $\Delta^{k-n+2}_{i_P, j_P} \Delta_P$ for P a partition of $\{1, \ldots, n\}$ into two parts and $\Delta^{k-n+1}_{i_{\{[n]\}}, j_{\{[n]\}}} \Delta_{\{[n]\}}$, where for each P the pair i_P, j_P are chosen to lie in the same part of P.*

In Section 15.1 we describe a simple algorithm to write arbitrary classes in this $\mathbb{Z}$-basis, along with a worked example.

In addition, we show that all relations between the Δ_P are generated by pushforwards of square relations. The method of proof will in fact supply an algorithm to write every Δ_Q as a $\mathbb{Z}$-linear combination of Δ_P for P a good partition using only these relations.

Definition 11.4. *Denote by* $\mathrm{Part}(d, n)$ *the set of partitions of $[n]$ into d parts. Let* $\mathrm{Sq}(d, n)$ *be the subgroup of the free abelian group $\mathbb{Z}^{\mathrm{Part}(d,n)}$ generated by formal square relations $P_{i,j} - P_{j,k} + P_{k,l} - P_{l,i}$ for $P \in \mathrm{Part}(d+1, n)$ and $i, j, k, l \in \{1, \ldots, n\}$ indices in different parts of P, where $P_{x,y}$ denotes the partition formed by merging the parts of P containing x and y.*

Theorem 11.5 (Corollary 15.14)**.** *For $d \geq 2$, the map*

$$\mathbb{Z}^{\mathrm{Part}(d,n)} / \operatorname{Sq}(d, n) \to A^{n-d}_{PGL_2}((\mathbb{P}^1)^n)$$

sending $P \mapsto \Delta_P$ is an isomorphism.

In particular, since every square relation between the Δ_P classes comes from an explicit PGL_2-invariant degeneration in $(\mathbb{P}^1)^n$ (see Section 11.1.1), Theorem 11.5 implies that all linear relations between the Δ_P classes can be realized by a sequence of PGL_2-invariant degenerations in $(\mathbb{P}^1)^n$.

Non-equivariantly, there are relations between the classes $\Delta_P \in A^\bullet((\mathbb{P}^1)^n)$ not generated by these square relations. For example, if $n = 4$ we have

$$\begin{aligned}\Delta_{\{\{1,2,3\},\{4\}\}} &+ \Delta_{\{\{1,2,4\},\{3\}\}} + \Delta_{\{\{1,3,4\},\{2\}\}} + \Delta_{\{\{2,3,4\},\{1\}\}} \\ &= \Delta_{\{\{1,2\},\{3,4\}\}} + \Delta_{\{\{1,3\},\{2,4\}\}} + \Delta_{\{\{1,4\},\{2,3\}\}}\end{aligned}$$

in $A^2((\mathbb{P}^1)^4)$.

Remark 11.6. All of our theorems can be extended to $n = 1, 2$ if we include the classes $\psi_i = \pi_i^* c_1(T^\vee\mathbb{P}^1) \in A^\bullet_{PGL_2}((\mathbb{P}^1)^n)$ pulled back from the ith projection π_i, which for $n \geq 3$ can be written in terms of the $\Delta_{j,k}$-classes via $\psi_i = \Delta_{j,k} - \Delta_{i,j} - \Delta_{i,k}$ for any $j, k \neq i$. They correspond to $-(2H_i + u + v)$ under the map from item (3) of Theorem 11.1 (see Proposition 14.4) and their definition is analogous to the ψ-classes on $\overline{M}_{0,n}$ [86, Section 2].

11.1.1. *Relation of the square relation to the $WDVV$ relation.* The $WDVV$ relation in $A^\bullet(\overline{\mathcal{M}}_{0,4})$ says two points in $\overline{\mathcal{M}}_{0,4} \cong \mathbb{P}^1$ corresponding to reducible curves have the same class [78, Section 0.1]. It was shown by Keel [68] that $A^\bullet(\overline{\mathcal{M}}_{0,n})$ is generated as a ring by its boundary divisors, and the only nontrivial relations come from pulling back the $WDVV$ relation under forgetful maps $\overline{\mathcal{M}}_{0,n} \to \overline{\mathcal{M}}_{0,4}$. The square relations relate to the $WDVV$ relations as follows. Consider the diagram

$$\begin{array}{ccc} \overline{\mathcal{M}}_{0,4}(\mathbb{P}^1, 1) & \xrightarrow{\text{ev}} & (\mathbb{P}^1)^4 \\ \big\downarrow{\scriptstyle\pi} & & \\ \overline{\mathcal{M}}_{0,4} & & \end{array}$$

where ev is the (PGL_2-equivariant) total evaluation map from the Kontsevich mapping space [51, Section 1] and π remembers only the source of the stable map and stabilizes. The square relation is $\mathrm{ev}_* \pi^*$ applied to the $WDVV$ relation.

Equivalently, for any closed point $a \in \mathbb{P}^1 \cong \overline{\mathcal{M}}_{0,4}$, we can consider the locus $A_a \subset (\mathbb{P}^1)^n$ consisting of the quadruples of points with cross ratio a. The square relation comes from equating the classes of A_0 and A_∞.

11.1.2. *Relation to other moduli spaces.* If we pick a linearization of the PGL_2-action on $(\mathbb{P}^1)^n$ and there are no strictly semistable points, then excising the unstable locus and applying [27, Theorem 3] gives the rational Chow ring of the GIT quotient. In this case, the ideal given by excision is generated by the classes of the excised strata. See [47] for an approach via quiver representations. These GIT quotients are Hassett spaces with total weight $2+\epsilon$ [60, Section 8] and receive maps from $\overline{\mathcal{M}}_{0,n}$ via reduction morphisms [60, Theorem 4.1], as induced maps between GIT quotients [61, Theorem 3.4], or by viewing $\overline{\mathcal{M}}_{0,n}$ as a Chow quotient [63].

11.2. **Unordered strata in** $[\mathrm{Sym}^n\mathbb{P}^1/PGL_2]$**.** The PGL_2-action on $\mathbb{P}^1$ induces an action on the symmetric power $\mathrm{Sym}^n\mathbb{P}^1 \cong \mathbb{P}^n$, which parameterizes degree n divisors on $\mathbb{P}^1$. For each partition $\lambda = \{\lambda_1, \dots, \lambda_d\}$ of n, we have the PGL_2-invariant subvariety $Z_\lambda \subset \mathbb{P}^n$ consisting of divisors that can be written in the form $\sum_{i=1}^d \lambda_i p_i$ where $p_i \in \mathbb{P}^1$. For convenience we often write $\lambda = a_1^{e_1} \dots a_k^{e_k}$ to be the partition of n where a_i appears e_i times.

11.2.1. *Integral classes of strata.* We compute the class of $[Z_\lambda]$ in $A^\bullet_{PGL_2}(\mathbb{P}^n)$. The class of $[Z_\lambda]$ in $A^\bullet_{GL_2}(\mathbb{P}^n)$ was given in [42], and we will give a quick independent proof and more compact form in Theorem 14.5. If n is odd, the map $A^\bullet_{PGL_2}(\mathbb{P}^n) \to A^\bullet_{GL_2}(\mathbb{P}^n)$ induced by the projection $GL_2 \to PGL_2$ is injective (see Proposition 13.7). Therefore, all of the difficulty lies in computing $[Z_\lambda]$ in $A^\bullet_{PGL_2}(\mathbb{P}^n)$ for n even. It turns out (see Section 17) that it suffices to compute the class in $A^\bullet_{PGL_2}(\mathbb{P}^n) \otimes \mathbb{Z}/2\mathbb{Z}$, which takes on a particularly simple form.

Theorem 11.7. *Let n be even and $\lambda = a_1^{e_1} \dots a_k^{e_k}$ be a partition of n into $d = e_1 + \ldots + e_k$ parts. The class of $[Z_\lambda] \in A^\bullet_{PGL_2}(\mathbb{P}^n) \otimes \mathbb{Z}/2\mathbb{Z} \cong \mathbb{F}_2[c_2, c_3, H]/(q_n(H))$ where*

$$q_n(t) = \begin{cases} t^{(n+4)/4}(t^3 + c_2 t + c_3)^{n/4} & n \equiv 0 \; mod \; 4, \; and \\ t^{(n-2)/2}(t^3 + c_2 t + c_3)^{(n+2)/4} & n \equiv 2 \; mod \; 4 \end{cases}$$

is non-zero precisely when all a_i and $\frac{d!}{e_1! \ldots e_k!}$ are odd and all e_i are even, in which case it is equal to $(\frac{q_n}{q_d})(H)$.

11.2.2. *Relations between strata.* If $\lambda = \{\lambda_1, \ldots, \lambda_d\} = a_1^{e_1} \dots a_k^{e_k}$ is a partition of n, then taking $\Phi : (\mathbb{P}^1)^n \to \mathrm{Sym}^n \mathbb{P}^1$ to be the multiplication map, if $P = \{A_1, \ldots, A_d\}$ is any partition of $[n]$ with $|A_i| = \lambda_i$, we have

$$\Phi_* \Delta_P = (\prod e_i!)[Z_\lambda].$$

In particular, every square relation between the classes of ordered strata induces a relation between $[Z_\lambda]$ classes by pushing forward along Φ.

Theorem 11.8. *(Section 17) Fix n and choose $a_\lambda \in \mathbb{Z}$ for each partition of n. The following are equivalent:*

(1) $\sum a_\lambda [Z_\lambda] = 0$ in $A^\bullet_{PGL_2}(\mathbb{P}^n)$
(2) $\sum a_\lambda [Z_\lambda] = 0$ in $A^\bullet_{GL_2}(\mathbb{P}^n)$
(3) $\sum a_\lambda [Z_\lambda]$ is formally a rational linear combination of pushforwards of square relations from $A^\bullet_{PGL_2}((\mathbb{P}^1)^n)$
(4) The following identity holds in $\mathbb{Q}[z]$:

$$\sum_{\lambda = a_1^{e_1} \dots a_k^{e_k}} \frac{a_\lambda}{\prod_{i=1}^k e_i!} \prod_{i=1}^k (z^{a_i} - 1)^{e_i} = 0.$$

Corollary 11.9. *Every $\mathbb{Z}$-linear relation that holds between Chow classes of relative Z_λ-cycles in $A^\bullet(\mathrm{Sym}^n\mathbb{P}(V))$ for every rank 2 vector bundle $V \to B$ and base B holds in $A^\bullet(\mathrm{Sym}^n\mathcal{P})$ for every $\mathbb{P}^1$-bundle $\mathcal{P} \to B$ and base B.*

We remark that there is 2-torsion in $A^\bullet_{PGL_2}(\mathbb{P}^n)$ for n even, but Theorem 11.8 implies that if each a_λ is even and $\sum a_\lambda[Z_\lambda]$ is zero in $A^\bullet_{PGL_2}(\mathbb{P}^n)$, then in fact the same is true for $\sum \frac{a_\lambda}{2}[Z_\lambda]$.

Rather than search for linear relations between $[Z_\lambda]$ classes using Theorem 11.8 (4), the following corollary identifies certain partitions whose corresponding strata are a $\mathbb{Q}$-linear basis for $A^{\leq n-2}_{PGL_2}(\mathbb{P}^n) \otimes \mathbb{Q}$, and gives an explicit formula for writing every such class in this basis. Every part of Corollary 11.10 can be deduced from Theorem 11.8 except that the strata that we choose span $A^{\leq n-2}_{PGL_2}(\mathbb{P}^n) \otimes \mathbb{Q}$.

For $\lambda = a_1^{e_1} \dots a_k^{e_k}$, denote by $[\lambda]$ the normalization

$$[\lambda] = (\prod e_i!)[Z_\lambda].$$

Corollary 11.10 (Theorem 16.4 and Corollary 16.2)**.** *For fixed $d \geq 2$, the classes $[\{a, b, 1^{d-2}\}]$ form a $\mathbb{Q}$-basis for $A^{n-d}_{PGL_2}(\mathbb{P}^n) \otimes \mathbb{Q} \subset A^{n-d}_{GL_2}(\mathbb{P}^n) \otimes \mathbb{Q}$. Writing the polynomial*

$$-\frac{1}{(z-1)^{d-2}}\prod_{i=1}^{d}(z^{a_i}-1) = \sum_{\substack{0 \leq k_1 \leq k_2 \\ k_1+k_2=n-d+2}} \alpha_{k_1}(z^{k_1}+z^{k_2}),$$

we have $\alpha_i \in \mathbb{Z}$ and

$$[\{a_1, \dots, a_d\}] = \sum_{\substack{1 \leq k_1 \leq k_2 \\ k_1+k_2=n-d+2}} \alpha_{k_1}[\{k_1, k_2, 1^{d-2}\}].$$

Each relation between classes $[Z_\lambda]$ in the equivariant Chow ring $A^\bullet_{PGL_2}(\mathbb{P}^n)$ gives relations between enumerative problems.

Example 11.11. Suppose $n = 6$, then Corollary 11.10 implies

$$[Z_{\{4,1,1\}}] + 3[Z_{\{2,2,2\}}] = [Z_{\{3,2,1\}}].$$

Consider the following two instances:

(1) Let $C_t \subset \mathbb{P}^2$ be a general pencil of degree 6 plane curves. Then, as we vary C_t over $t \in \mathbb{P}^1$, the number of hyperflex lines plus thrice the number of tritangent lines is equal to the number of lines that are both flex and bitangent.

(2) Let $X \subset \mathbb{P}^3$ be a general degree 6 surface. Then in $\mathbb{G}(1,3)$, the class of the curve of lines that meet X to order 4 at a point plus three times the class of the curve of tritangent lines to X is equal to the class of the curve of lines that meet X at three points with multiplicities $1, 2, 3$.

Note that in both examples, in the absence of a transversality argument, the equalities need to be taken with appropriate multiplicities.

Remark 11.12. Lines with prescribed orders of contact with a hypersurface were also studied in [99, Section 5]. Counts of these lines are also related to counting line sections of a hypersurface with fixed moduli [20, 73]. For the surface $X \subset \mathbb{P}^3$ in Example 11.11, the points $p \in X$ for which a line meets X at p to order 4 is the *flecnode curve*, which is always of expected dimension 1 if X is not ruled by lines by the Cayley-Salmon theorem [64, Theorem 6], which is a primary tool for bounding the number of lines on a smooth surface in $\mathbb{P}^3$ (see [92] and [17, Appendix]).

Also, there is no reason not to consider a general variety $X \subset \mathbb{P}^N$ other than the difficulty of finding a projective variety of higher codimension that has at least a 3-dimensional family of 6-secant lines.

11.3. **Excision.** As an application of our results, we compute the rational equivariant Chow ring of the complement of a union of unordered strata $A^\bullet_{PGL_2}(\mathbb{P}^n \setminus \cup_\lambda Z_\lambda) \otimes \mathbb{Q} = (A^\bullet_{PGL_2}(\mathbb{P}^n) \otimes \mathbb{Q})/(\sum_\lambda I_\lambda \otimes \mathbb{Q})$, where I_λ is the ideal of excision for Z_λ.

We show that $I_\lambda \otimes \mathbb{Q}$ is generated by the classes of strata contained in Z_λ.

Theorem 11.13 (Lemma 18.8). *Given a partition λ of n, $I_\lambda \otimes \mathbb{Q}$ is generated by $[Z_{\lambda'}]$ for all λ' that can be obtained from λ by merging parts.*

Remark 11.14. Theorem 11.13 is false if we replace $I_\lambda \otimes \mathbb{Q}$ with I_λ. This already fails nonequivariantly in the case $n = 4$ and $\lambda = \{2, 1, 1\}$. Indeed, $\Phi : \mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^4$ maps birationally onto Z_λ. Let H_1 and H_2 be the hyperplane classes in the factors of $\mathbb{P}^1 \times \mathbb{P}^2$ and H be the hyperplane class of $\mathbb{P}^4$. Then $\Phi_* H_1 = H^2$, while $[Z_{\{2,2\}}] = 8H^2$, $[Z_{\{3,1\}}] = 6H^2$, and $[Z_{\{2,1,1\}}] = 6H$.

We typically don't need to use every merged partition λ' for dimension reasons by Corollary 11.10. When $\lambda = \{a, 1^{n-a}\}$ is a partition with only one part of size greater than 1, we in fact show that $I_\lambda \otimes \mathbb{Q}$ is generated by just two generators.

Theorem 11.15 (Theorem 18.2). *Given the partition $\lambda = \{a, 1^{n-a}\}$ of n, $I_\lambda \otimes \mathbb{Q}$ is generated by $[Z_\lambda]$ and $[Z_{\lambda'}]$, where*

$$\lambda' = \begin{cases} \{a+1, 1^{n-a-1}\} & \text{if } a \neq \frac{n}{2} \\ \{a, 2, 1^{n-a-2}\} & \text{if } a = \frac{n}{2}. \end{cases}$$

In fact we will also show the analogous results with $A^\bullet_{GL_2}(\mathbb{P}^n \setminus \cup_\lambda Z_\lambda) \otimes \mathbb{Q}$, and if we further replace $\mathbb{P}^n \setminus \cup_\lambda Z_\lambda$ with its affine cone $\mathbb{A}^{n+1} \setminus \cup_\lambda \widetilde{Z_\lambda}$ and consider $A^\bullet_{GL_2}(\mathbb{A}^{n+1} \setminus \cup_\lambda \widetilde{Z_\lambda})$ (see Theorem 19.2).

In the special case $\lambda = \{2, 1^{n-2}\}$, computing I_λ is the technical heart of the computation of Edidan and Fulghesu of the Chow ring of the stack of hyperelliptic curves of even genus [26].

For n odd and Z_λ the unstable locus, i.e with $\lambda = \{\frac{n+1}{2}, 1^{\frac{n-1}{2}}\}$, the rational Chow ring $A^\bullet_{GL_2}(\mathbb{P}^n \setminus Z_\lambda) \otimes \mathbb{Q}$ equals $A^\bullet(\mathbb{P}^n // GL_2) \otimes \mathbb{Q}$, the rational Chow ring of the GIT quotient [27, Theorem 3]. For all n and $Z_\lambda \subset \mathbb{P}^n$ the locus of unstable and strictly semistable points, Fehér, Némethi, and Rimányi computed $A^\bullet_{GL_2}(\mathbb{P}^n \backslash Z_\lambda) \otimes \mathbb{Q}$ using a spectral sequence and used the result to compute the rational Chow ring of the GIT quotient [42, Theorems 4.3 and 4.10]. They actually work with the affine space $\mathrm{Sym}^n K^2$ instead of $\mathbb{P}^n$, but the two settings are essentially the same (see Lemma 19.5).

Remark 11.16. The affine analogue of Theorem 11.15 as given in Theorem 19.2 in the special case $a = \lceil \frac{n}{2} \rceil$ recovers the GL_2-equivariant Chow rings of the stable locus computed in [42, Theorems 4.3 and 4.10] as described above. The Chow ring of the semistable locus required a separate argument.

11.3.1. *Multiplicative relations of affine analogues.* We conclude in Section D by describing a combinatorial branching rule for multiplying the affine analogue of the class of a strata $[\widetilde{Z_\lambda}] \in A^\bullet_{GL_2}(\mathrm{Sym}^n K^2) \cong \mathbb{Z}[u, v]^{S_2}$ by a generator $u + v$ or uv. This generalizes [42, Remark 3.9 (1)].

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12. Background and conventions

Conventions:

(1) The base field K is algebraically closed of arbitrary characteristic
(2) GL_2 acts linearly on $\mathbb{P}^1$ and hence on all products $(\mathbb{P}^1)^n$, symmetric powers $\mathrm{Sym^n}\mathbb{P}^1 \cong \mathbb{P}^n$, and their duals
(3) $T \subset GL_2$ is the standard maximal torus with standard characters u and v
(4) $[n]$ denotes the set $\{1, \ldots, n\}$
(5) $\Phi : (\mathbb{P}^1)^n \to \mathrm{Sym^n}\mathbb{P}^1 \cong \mathbb{P}^n$ denotes the multiplication map, where n will be clear from context.

12.1. **Universal relations and equivariant intersection theory.** Equivariant intersection theory was formalized in [27] and will be used to help us analyze the following situation. See also [11] for an exposition.

Suppose we have a group G (typically $G = T, GL_2, PGL_2$) acting on a variety X (typically $(\mathbb{P}^1)^n, \mathrm{Sym}^n\mathbb{P}^1 = \mathbb{P}^n$), and G-invariant subvarieties Y_i (typically incidence strata in $(\mathbb{P}^1)^n$ or $\mathbb{P}^n$). Given a principal G-bundle $\mathcal{P} \to B$, we have the X-bundle $X_{\mathcal{P}} \to B$, where $X_{\mathcal{P}} := \mathcal{P} \times^G X$. Inside $X_{\mathcal{P}}$, we have the cycles

$$(Y_i)_{X_{\mathcal{P}}} := (Y_i)_{\mathcal{P}} \subset X_{\mathcal{P}}$$

restricting to Y_i in each fiber X, inducing classes $[Y_i]_{X_{\mathcal{P}}} \in A_\bullet(X_{\mathcal{P}})$. We would like to understand what "universal" linear relations exist between these classes (i.e. which don't depend on B or $\mathcal{P}$).

For example, if we take $G = PGL_2$, then we are seeking universal relations between classes $[Z_P]_{\mathcal{F}^n}$ and between classes $[Z_\lambda]_{\mathrm{Sym}^n\mathcal{F}}$ for $\mathcal{F} \to B$ a $\mathbb{P}^1$-bundle. If we use $G = GL_2$ instead the relations hold a priori only for $\mathcal{F}$ the projectivization of a rank 2 vector bundle on B.

As we will see in Section 12.2, there is a universal group $A^G_\bullet(X)$ approximated by certain $A_\bullet(X_{\mathcal{P}'})$ which is equipped with maps $A^G_\bullet(X) \to A_\bullet(X_{\mathcal{P}})$ for all $\mathcal{P}$ and there are classes $[Y_i] \in A^G_\bullet(X)$ such that $[Y_i] \mapsto [Y_i]_{\mathcal{P}}$, so any relations in $A^G_\bullet(X)$ between the $[Y_i]$ descend to relations between the $[Y_i]_{\mathcal{P}}$. Conversely, we will see by construction that any relation between the $[Y_i]_{\mathcal{P}}$ for all $\mathcal{P}$ induces a relation between the $[Y_i]$.

12.2. **Equivariant intersection theory.** The equivariant Chow group $A^G_\bullet(X)$ is defined as follows. Suppose G acts linearly on a vector space V with an open subset U of codimension c on which it acts freely. Then for any $k < c$, we define $A^G_{\dim(X)-k}(X) := A_{\dim(X \times^G U)-k}(X \times^G U)$. Note that $X \times^G U = X_{\mathcal{P}}$ where $\mathcal{P}$ is the principal G-bundle $U \to U/G$. This does not depend on the choice of V [27, Definition-Proposition 1].

For $\mathcal{P} \to B$ a principal G-bundle over an equidimensional base B, we have a map

$$A_\bullet^G(X) \to A_{\dim(B)+\bullet}(P \times^G X)$$

via the composition

$$\begin{aligned} A^G_{>\dim(X)-c}(X) &\cong A_{>\dim(X\times^G U)-c}(X \times^G U) \\ &\to A_{>\dim((P\times X)\times^G U)-c}((P \times X) \times^G U) \\ &\cong A_{>\dim((P\times X)\times^G U)-c}((P \times X) \times^G V) \\ &\cong A_{>\dim(P\times^G X)-c}(P \times^G X) \end{aligned}$$

where the second map is induced by flat pullback from the projection, the third map follows from excising $(P \times X) \times^G (V \setminus U)$, and the last map follows from the Chow groups of a vector bundle [50, Theorem 3.3(a)].

Now, we define $A^\bullet_G(X)$ to be the ring of operational G-equivariant Chow classes on X, i.e. $A^i_G(X)$ is all assignments

$$(Y \to X) \mapsto (A^G_\bullet(Y) \to A^G_{\bullet-i}(Y))$$

for every G-equivariant map $Y \to X$, compatible with the standard operations on Chow groups [27, Section 2.6]. In our case X is always smooth, and we have the Poincaré duality isomorphism $A^\bullet_G(X) = A^G_{\dim(X)-\bullet}(X)$ [27, Proposition 4], and the identification

$$A^\bullet([X/G]) \cong A^\bullet_G(X),$$

where $[X/G]$ is the quotient stack [27, Section 5.3].

12.3. GL_2 **and** T**-equivariant Chow rings of** $(\mathbb{P}^1)^n$ **and** $\mathbb{P}^n$**.** We will postpone discussing PGL_2-equivariant intersection rings to Section 13. The equivariant Chow rings $A_T^\bullet((\mathbb{P}^1)^n), A_T^\bullet(\mathbb{P}^n)$, (respectively $A_{GL_2}^\bullet((\mathbb{P}^1)^n), A_{GL_2}^\bullet(\mathbb{P}^n)$) can be approximated by the ordinary Chow rings of $(\mathbb{P}^1)^n$ and $\mathbb{P}^n$ bundles over $\mathbb{P}^N \times \mathbb{P}^N$ (respectively the Grassmannian of lines $\mathbb{G}(1, N)$) for $N >> 0$.

Let u, v be the standard characters of T. If pt is a point with trivial GL_2 action, then

$$A_T^\bullet(\mathrm{pt}) = \mathbb{Z}[u, v], \qquad A_{GL_2}^\bullet(\mathrm{pt}) = \mathbb{Z}[u, v]^{S_2}$$

where S_2 acts on $\mathbb{Z}[u, v]$ by swapping u, v. By the Chow ring of a vector bundle [50, Theorem 3.3(a)], the T (respectively GL_2) equivariant Chow ring of an affine space is isomorphic to the equivariant Chow ring of a point. By the projective bundle theorem [32, Theorem 9.6], we have

$$A_T^\bullet((\mathbb{P}^1)^n) = \mathbb{Z}[u, v][H_1, \dots, H_n]/(F(H_i)), \qquad A_T^\bullet(\mathbb{P}^n) = \mathbb{Z}[u, v][H]/(G(H)),$$
$$A_{GL_2}^\bullet((\mathbb{P}^1)^n) = \mathbb{Z}[u, v]^{S_2}[H_1, \dots, H_n]/(F(H_i)), \qquad A_{GL_2}^\bullet(\mathbb{P}^n) = \mathbb{Z}[u, v]^{S_2}[H]/(G(H))$$

where H_i is $c_1(\mathscr{O}_{\mathbb{P}^1}(1))$ pulled back to $(\mathbb{P}^1)^n$ under the ith projection and H is $c_1(\mathscr{O}_{\mathbb{P}^n}(1))$, and we define

$$F(z) = (z + u)(z + v), \qquad G(z) = \prod_{k=0}^{n}(z + ku + (n - k)v)$$

for the rest of the document. Even though one might want to use GL_2-equivariant Chow rings for applications, GL_2-equivariant Chow rings inject into T-equivariant Chow rings, so it suffices to only consider T-equivariant Chow rings.

The formula for the class of the projectivization of a subbundle [32, Proposition 9.13] shows the ith coordinate hyperplane in $\mathbb{P}^n$ has class $H + iu + (n - i)v$. This gives the formula for any torus fixed linear space (for example the torus-fixed points) in $(\mathbb{P}^1)^n$ or $\mathbb{P}^n$ by multiplying a subset of these classes.

12.4. **Ordered and unordered strata of n points on $\mathbb{P}^1$.**

Definition 12.1. *Given a collection $P = \{A_1, \dots, A_d\}$ of disjoint subsets of $[n]$, let $\Delta_P \subset (\mathbb{P}^1)^n$ denote the d-dimensional locus of points $(p_1, \dots, p_n)$ where $p_i = p_j$ whenever i, j are in the same set A_k of P.*

Example 12.2. If $P = \{\{1, 2, 4\}, \{3, 6\}\}$ and $A = [6]$, then $Z_P \subset (\mathbb{P}^1)^6$ consists of points $(p_1, \dots, p_6)$ such that $p_1 = p_2 = p_4$ and $p_3 = p_6$.

Definition 12.3. *Given a partition $\lambda = \{\lambda_1, \dots, \lambda_d\}$ of a positive integer n, we define the d-dimensional subvariety $Z_\lambda \subset \mathrm{Sym}^n\mathbb{P}^1 \cong \mathbb{P}^n$ to be the image of Δ_P under the multiplication map $\Phi : (\mathbb{P}^1)^n \to \mathbb{P}^n$, where $P = \{A_1, \dots, A_d\}$ is any partition of $[n]$ with $|A_i| = \lambda_i$.*

Remark 12.4. If we view $\mathrm{Sym}^n\mathbb{P}^1 \cong \mathbb{P}^n$ as binary degree n forms on the dual of $\mathbb{P}^1$, then Z_λ is the closure of the degree n forms with multiplicity sequence given by λ, whose equivariant Chow classes were studied by Fehér, Némethi, and Rimányi [42].

In order to compactify notation, we make the following definitions.

Definition 12.5. *Given P a partition of $[n]$ and λ a partition of n, we let*

$$\Delta_P := [\Delta_P] \in H^\bullet_G((\mathbb{P}^1)^n)$$
$$[\lambda] := (\prod_{i=1}^n e_i^\lambda!)[Z_\lambda] \in H^\bullet_G(\mathbb{P}^n),$$

where G is T, GL_2 or PGL_2, depending on the context and $e_i^\lambda = \#\{j \mid \lambda_j = i\}$. For $\lambda = \{a_1, \dots, a_d\}$, we will often write $[a_1, \dots, a_d]$ or $[1^{e_1^\lambda}, \dots, n^{e_n^\lambda}]$ for $[\lambda]$.

Remark 12.6. For any such partition P and λ as in Definition 12.3, then Φ maps Δ_P onto Z_λ with degree $\prod_{i=1}^n e_i^\lambda!$, so $\Phi_*\Delta_P = [\lambda]$.

12.5. **Affine and projective Thom polynomials.**

Definition 12.7. *Given a T-invariant subvariety $V \subset \mathbb{P}^n$, let $\widetilde{V} \subset \mathbb{A}(\mathrm{Sym}^n K^2)$ denote the cone of $V \subset \mathbb{P}^n$ in $\mathbb{A}(\mathrm{Sym}^n K^2) \cong \mathbb{A}^{n+1}$.*

Given a T-invariant subvariety $V \subset \mathbb{P}^n$, its class $[V] \in A_T^\bullet(\mathbb{P}^n)$ is a polynomial $p(H, u, v)$ of degree at most n. The degree 0 term in H, $p_0(u, v)$, is $[\widetilde{V}] \in A_T^\bullet(\mathbb{A}^{n+1}) \cong \mathbb{Z}[u, v]$. This is seen by considering the diagram

$$A_T^\bullet(\mathbb{P}^n) \xleftarrow{\sim} A_{T\times\mathbb{G}_m}^\bullet(\mathbb{A}^{n+1}\backslash\{0\}) \to A_T^\bullet(\mathbb{A}^{n+1}\backslash\{0\})$$

and noting that $A_T^k(\mathbb{A}^{n+1}\backslash\{0\}) \cong A_T^k(\mathbb{A}^{n+1})$ for $k \leq n$.

It turns out $p_0(u, v)$ determines all of p.

Lemma 12.8 ([41, Theorem 6.1])**.** *We have $p(u, v) = p_0(u + \frac{H}{d}, v + \frac{H}{d})$.*

Proof sketch. As $(\mathbb{A}^{n+1}\backslash\{0\})/\mathbb{G}_m \cong \mathbb{P}^n$, p can be computed from $[\widetilde{V}] \in A_{T\times\mathbb{G}_m}^\bullet(\mathbb{A}^{n+1})$ by mapping to $A_T^\bullet(\mathbb{P}^n)$ via

$$A_{T\times\mathbb{G}_m}^\bullet(\mathbb{A}^{n+1}) \to A_{T\times\mathbb{G}_m}^\bullet(\mathbb{A}^{n+1}\backslash\{0\}) \cong A_T^\bullet(\mathbb{P}^n).$$

However, the diagonal action of $\mathbb{G}_m$ on $\mathbb{A}^{n+1}$ actually factors through the action of T on $\mathbb{A}^{n+1}$, so $A_{T\times\mathbb{G}_m}^\bullet(\mathbb{A}^{n+1})$ contains no more information than $A_T^\bullet(\mathbb{A}^{n+1})$. Taking the class p_0 and following it from $A_T^\bullet(\mathbb{A}^{n+1})$ to $A_{T\times\mathbb{G}_m}^\bullet(\mathbb{A}^{n+1})$ and finally to $A_T^\bullet(\mathbb{P}^n)$ yields Lemma 12.8. This argument is written down precisely and in its natural generality in [41, Theorem 6.1]. $\square$

13. PGL_2 and GL_2-equivariant Chow rings

In this section we compare certain PGL_2-equivariant Chow rings to their GL_2-equivariant counterparts, which are easier to work with because GL_2 is special, so restricting to the maximal torus is an injection on equivariant Chow rings [27, Proposition 6].

In particular, we show in Theorem 13.3 that $A^\bullet_{PGL_2}((\mathbb{P}^1)^n) \to A^\bullet_{GL_2}((\mathbb{P}^1)^n)$ is injective and identify its image. For the unordered case, we show in Proposition 13.7 that $A^\bullet_{PGL_2}(\mathbb{P}^n) \to A^\bullet_{GL_2}(\mathbb{P}^n)$ is injective for n odd and injective up to 2-torsion when n is even.

To start, we recall a lemma.

Lemma 13.1 ([80, Lemma 2.1])**.** *Given a linear algebraic group G acting on a smooth variety X, let H be a normal subgroup of G that acts freely on X with quotient X/H. Then, there is a canonical isomorphism of graded rings*

$$A^\bullet_G(X) \cong A^\bullet_{G/H}(X/H).$$

Remark 13.2. Lemma 13.1 was proven in [80, Lemma 2.1] directly from the definitions, but it can also be seen as a consequence of the fact that the ring $A^\bullet_G(X)$ depends only on the quotient stack $[X/G]$ [27, Proposition 16] and $[[X/H]/(G/H)] \cong [X/G]$ (see [90, Remark 2.4] or [14, Lemma 4.3]).

Theorem 13.3. *For $n \geq 1$, the ring homomorphism*

$$A^\bullet_{PGL_2}((\mathbb{P}^1)^n) \to A^\bullet_{GL_2}((\mathbb{P}^1)^n)$$

induced by the quotient map $GL_2 \to PGL_2$ is an injection, and the image is generated by the classes $-(2H_i + u + v)$ and $\Delta_{i,j} = H_i + H_j + u + v$.

Remark 13.4. We will show in Proposition 14.4 that $\psi_i := \pi_i^* c_1(T^\vee \mathbb{P}^1) = -(2H_i + u + v)$, as mentioned in Remark 11.6. For $n \geq 3$ this class is redundant as

$$-(2H_i + u + v) = \Delta_{j,k} - \Delta_{i,j} - \Delta_{i,k}.$$

Proof. We show the injectivity of $A^\bullet_{PGL_2}((\mathbb{P}^1)^n) \xrightarrow{\iota} A^\bullet_{GL_2}((\mathbb{P}^1)^n)$ using the commutativity of the diagram

$$\begin{array}{ccc}
A^\bullet_{PGL_2}((\mathbb{P}^1)^n) & \xrightarrow[q_1]{\sim} & A^\bullet_{GL_2}((\mathbb{A}^2\backslash 0) \times (\mathbb{P}^1)^{n-1}) \\
\downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle f} \\
A^\bullet_{GL_2}((\mathbb{P}^1)^n) & \xrightarrow[q_2]{\sim} & A^\bullet_{GL_2\times\mathbb{G}_m}((\mathbb{A}^2\backslash\{0\}) \times (\mathbb{P}^1)^{n-1})
\end{array}$$

with f induced by the multiplication map $GL_2 \times \mathbb{G}_m \to GL_2$.

We have the isomorphisms q_1 and q_2 by Lemma 13.1.

To prove commutativity of the diagram, we can identify each of the rings $A^\bullet_G(X)$ with $A^\bullet([X/G])$ as in Section 12.2, so it suffices to show the following diagram of stacks is commutative.

$$\begin{array}{ccc}
[(\mathbb{P}^1)^n/PGL_2] & \xleftarrow{\sim} & [(\mathbb{A}^2\backslash\{0\}) \times (\mathbb{P}^1)^{n-1}/GL_2] \\
\uparrow & & \uparrow \\
[(\mathbb{P}^1)^n/GL_2] & \xleftarrow{\sim} & [(\mathbb{A}^2\backslash\{0\}) \times (\mathbb{P}^1)^{n-1}/(GL_2 \times \mathbb{G}_m)]
\end{array}$$

Suppose we start with a principal $GL_2 \times \mathbb{G}_m$-bundle $P \to S$ together with a $GL_2 \times \mathbb{G}_m$-equivariant map $P \to (\mathbb{A}^2\backslash\{0\}) \times (\mathbb{P}^1)^{n-1}$, giving a map $S \to [(\mathbb{A}^2\backslash\{0\}) \times (\mathbb{P}^1)^{n-1}/(GL_2 \times \mathbb{G}_m)]$. Following the diagram around clockwise or counterclockwise, we get a map $S \to [(\mathbb{P}^1)^n/PGL_2]$ given by a PGL_2-equivariant morphism

$$P \times^{GL_2\times\mathbb{G}_m} GL_2 \times^{GL_2} PGL_2 \cong P \times^{GL_2\times\mathbb{G}_m} PGL_2 \to (\mathbb{P}^1)^n.$$

When going counterclockwise, the product $P \times^{GL_2\times\mathbb{G}_m} GL_2$ is taken with respect to the multiplication map $GL_2 \times \mathbb{G}_m \to GL_2$, while when going clockwise, the product is taken with respect to the projection map $GL_2 \times \mathbb{G}_m \to GL_2$. However, the resulting principal PGL_2-bundle is the same as the compositions with the quotient $GL_2 \to PGL_2$ are identical.

Now, we will find the induced map

$$A^\bullet_{GL_2}((\mathbb{A}^2\backslash 0) \times (\mathbb{P}^1)^{n-1}) \to A^\bullet_{GL_2}((\mathbb{P}^1)^n)$$

in terms of generators and show it is injective. Consider the diagram

$$\begin{tikzcd}
A^\bullet_{GL_2}((\mathbb{A}^2\backslash 0) \times (\mathbb{P}^1)^{n-1}) \arrow[r, "f"] \arrow[d, "\sim"] & A^\bullet_{GL_2\times\mathbb{G}_m}((\mathbb{A}^2\backslash\{0\}) \times (\mathbb{P}^1)^{n-1}) \arrow[r, "q_2", "\sim"'] \arrow[d, "\sim"] & A^\bullet_{GL_2}((\mathbb{P}^1)^n) \arrow[d, "\sim"] \\
A^\bullet_{GL_2\times(\mathbb{G}_m)^{n-1}}((\mathbb{A}^2\backslash 0)^n) \arrow[r, "f'"] & A^\bullet_{GL_2\times(\mathbb{G}_m)^n}((\mathbb{A}^2\backslash\{0\})^n) \arrow[r, "q_2'", "\sim"'] & A^\bullet_{GL_2\times(\mathbb{G}_m)^n}((\mathbb{A}^2\backslash\{0\})^n)
\end{tikzcd}$$

where GL_2 acts in the standard way in all cases. In the middle term of the top row, $\mathbb{G}_m$ acts by scaling $\mathbb{A}^2 \setminus \{0\}$. In the last term of the second row, $(\mathbb{G}_m)^n$ acts by having the ith copy of $\mathbb{G}_m$ scale the ith copy of $\mathbb{A}^2 \setminus \{0\}$. In the middle term of the second row, $(\mathbb{G}_m)^n$ acts by having the first copy of $\mathbb{G}_m$ act by scaling all copies of $\mathbb{A}^2 \setminus \{0\}$ and the ith copy of $\mathbb{G}_m$ with $2 \leq i \leq n$ acting by scaling the ith copy of $\mathbb{A}^2 \setminus \{0\}$. In the first term of the second row, the ith copy of $\mathbb{G}_m^{n-1}$ scales the $i+1$st copy of $\mathbb{A}^2 \setminus \{0\}$.

To compute f', we let H_1 be the standard character on the first factor of $\mathbb{G}_m$ in $GL_2 \times (\mathbb{G}_m)^n$ and let $H_2, \ldots, H_n$ be the standard characters on the remaining $n-1$ factors and the $n-1$ factors of $\mathbb{G}_m$ in $GL_2 \times (\mathbb{G}_m)^{n-1}$. The induced map $T \times (\mathbb{G}_m)^n \to T \times (\mathbb{G}_m)^{n-1}$ of tori induces $u \mapsto u + H_1$ and $v \mapsto v + H_1$. Therefore,

$$f' : \frac{\mathbb{Z}[u,v]^{S_2}[H_2, \ldots, H_n]}{(uv, F(H_2), \ldots, F(H_n))} \to \frac{\mathbb{Z}[u,v]^{S_2}[H_1][H_2, \ldots, H_n]}{(uv, F(H_2 + H_1), \ldots, F(H_n + H_1))},$$

where $u \mapsto u + H_1$, $v \mapsto v + H_1$, and $H_i \mapsto H_i$.

For q_2', the induced map $T \times (\mathbb{G}_m)^n \to T \times (\mathbb{G}_m)^n$ of tori induces $H_1 \mapsto H_1$, $H_i \mapsto H_i - H_1$ for $2 \leq i \leq n$ and $u \mapsto u$, $v \mapsto v$, and gives the map

$$q_2' : \frac{\mathbb{Z}[u,v]^{S_2}[H_1][H_2, \ldots, H_n]}{(uv, F(H_2 + H_1), \ldots, F(H_n + H_1))} \to \frac{\mathbb{Z}[u,v]^{S_2}[H_1, \ldots, H_n]}{(F(H_1), \ldots, F(H_n))}.$$

The composite

$$q_2' \circ f' : \frac{\mathbb{Z}[u,v]^{S_2}[H_2, \ldots, H_n]}{(uv, F(H_2), \ldots, F(H_n))} \to \frac{\mathbb{Z}[u,v]^{S_2}[H_1, \ldots, H_n]}{(F(H_1), \ldots, F(H_n))}$$

is given by $u \mapsto u + H_1$, $v \mapsto v + H_1$, $H_i \mapsto H_i - H_1$ for $2 \leq i \leq n$. The image is therefore generated by $2H_1 + u + v$ and $H_i - H_1$ for $2 \leq i \leq n$. If $n \geq 3$, then this is generated by the collection

$$\{H_i + H_j + u + v \mid 1 \le i < j \le n\} = \{\Delta_{i,j} \mid 1 \le i < j \le n\}$$

(see Proposition 14.4). $\square$

Remark 13.5. Suppose our base field is $\mathbb{C}$. We have a commutative diagram

$$\begin{tikzcd} A^\bullet_{PGL_2}((\mathbb{P}^1)^n) \arrow[r, hook, "q_1"] \arrow[d] & A^\bullet_{GL_2}((\mathbb{P}^1)^n) \arrow[d] \\ H^\bullet_{PGL_2}((\mathbb{P}^1)^n) \arrow[r, "q_1^H"'] & H^\bullet_{GL_2}((\mathbb{P}^1)^n) \end{tikzcd}$$

The map $A^\bullet_{GL_2}((\mathbb{P}^1)^n) \to H^\bullet_{GL_2}((\mathbb{P}^1)^n)$ is an isomorphism by the Leray-Hirsch theorem applied to $\mathbb{P}^1_{\mathbb{C}}$-bundles. Running the proof of Theorem 13.3 for the map q_1^H shows q_1^H is injective. Here we replace the projective bundle theorem in algebraic geometry by the Leray-Hirsch theorem applied to $\mathbb{P}^1_{\mathbb{C}}$-bundles and the application of Lemma 13.1 with the fact that if G acts on X and H is a normal subgroup which acts freely, then $(X \times EG)/G \cong ((X \times EG)/H)/(G/H)$, and $(X \times EG)/H$ is homotopy equivalent to X/H and has a free action by G/H.

This implies $A^\bullet_{PGL_2}((\mathbb{P}^1)^n) \to H^\bullet_{PGL_2}((\mathbb{P}^1)^n)$ is an isomorphism.

By [85, Theorem 1], the injection $SO(3) \to GL_3$ induces a surjection $A^\bullet_{GL_3}(\mathrm{pt}) \to A^\bullet_{SO(3)}(\mathrm{pt})$ expressing $A^\bullet_{SO(3)}(\mathrm{pt}) \cong \mathbb{Z}[c_1, c_2, c_3]/(c_1, 2c_3)$, where c_1, c_2, c_3 are the generators of $A^\bullet_{GL_3}(\mathrm{pt})$. Lemma 13.6 expresses the map $A^\bullet_{PGL_2}(\mathrm{pt}) \to A^\bullet_{GL_2}(\mathrm{pt})$ in terms of this presentation.

Lemma 13.6. *Under the composition,*

$$A^\bullet_{GL_3}(\mathrm{pt}) \to A^\bullet_{SO(3)}(\mathrm{pt}) \cong A^\bullet_{PGL_2}(\mathrm{pt}) \to A^\bullet_{GL_2}(\mathrm{pt}) \to A^\bullet_T(\mathrm{pt}),$$

we have $c_1 \mapsto 0$, $c_2 \mapsto -(u-v)^2$, $c_3 \mapsto 0$.

Proof. Lemma 13.6 amounts to finding the map

$$T \to GL_2 \to PGL_2 \cong SO(3) \to GL_3$$

inducing the maps of rings.

To describe the isomorphism $SO(3) \cong PGL_2$, recall that GL_2 acts on the space $K^{2\times 2}$ of 2 by 2 matrices by conjugation. There is a pairing $\langle \bullet, \bullet \rangle$ on $K^{2\times 2}$ given by $\langle A, B\rangle = \mathrm{Tr}(AB)$ that restricts to a nondegenerate form on the three-dimensional vector space of trace zero matrices $V \subset K^{2\times 2}$. Since the action of GL_2 preserves $\langle \bullet, \bullet \rangle$ and the scalar matrices inside GL_2 act trivially, we have an injection $PGL_2 \to SO(3)$, which is an isomorphism for dimension reasons.

Under this isomorphism, $\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \in T$ maps into diagonal matrices in GL_3 and acts on V with characters $u - v$, $v - u$ and 0 (written additively). Therefore,

$$c_1 \mapsto (u-v) + (v-u) = 0$$

$$c_2 \mapsto (u-v)(v-u) = -(u-v)^2$$

$$c_3 \mapsto 0(u-v)(v-u) = 0.$$

$\square$

Proposition 13.7. *We have*

$$A^\bullet_{PGL_2}(\mathbb{P}^n) \cong \begin{cases} \mathbb{Z}[u,v]^{S_2}/(\prod_{i=0}^n((\frac{n+1}{2} - i)u + (\frac{-n+1}{2} + i)v)) & \textit{if } n \textit{ is odd} \\ \mathbb{Z}[c_2, c_3, H]/(2c_3, p_n(H)) & \textit{if } n \textit{ is even} \end{cases}$$

where $p_n(t) \in A^\bullet_{PGL_2}(\mathrm{pt})[t]$ *is defined as*

$$p_n(t) = \begin{cases} t\prod_{k=1}^{\frac{n}{2}}(t^2 + k^2c_2) + t^{\frac{n}{4}+1}\sum_{k=1}^{\frac{n}{4}}\binom{\frac{n}{4}}{k}(t^3 + c_2t)^{\frac{n}{4}-k}c_3^k & n \equiv 0 \pmod 4 \\ t\prod_{k=1}^{\frac{n}{2}}(t^2 + k^2c_2) + t^{\frac{n-2}{4}}\prod_{k=1}^{\frac{n+2}{4}}\binom{\frac{n+2}{4}}{k}(t^3 + c_2t)^{\frac{n+2}{4}-k}c_3^k & n \equiv 2 \pmod 4. \end{cases}$$

The map

$$A^\bullet_{PGL_2}(\mathbb{P}^n) \to A^\bullet_{GL_2}(\mathbb{P}^n)$$

induced by $GL_2 \to PGL_2$ is given by

$$\begin{aligned} &u \mapsto H + \frac{n+1}{2}u + \frac{n-1}{2}v \quad v \mapsto H + \frac{n-1}{2}u + \frac{n+1}{2}v && \text{if } n \text{ is odd, and} \\ &c_2 \mapsto -(u-v)^2 \quad c_3 \mapsto 0 \quad H \mapsto H + \frac{n}{2}(u+v) && \text{if } n \text{ is even.} \end{aligned}$$

Finally, $A^\bullet_{PGL_2}(\mathbb{P}^n) \to A^\bullet_{GL_2}(\mathbb{P}^n)$ is injective for n odd and injective up to 2-torsion when n is even.

Proof. The injectivity statements immediately follow from the explicit descriptions of all of the rings maps in the statement of Proposition 13.7, we omit the verification.

We do the cases n is odd and even separately. First suppose n is odd. Consider the commutative diagram

$$\begin{tikzcd} & A^\bullet_{PGL_2}(\mathbb{P}^n) \arrow[r] \arrow[d, "\sim"] & A^\bullet_{GL_2}(\mathbb{P}^n) \arrow[d, "\sim"] \\ A^\bullet_{GL_2}(\mathbb{A}^{n+1} \setminus \{0\}) \arrow[r, "\sim", "\phi_1"'] & A^\bullet_{GL_2/\mu_n}(\mathbb{A}^{n+1} \setminus \{0\}) \arrow[r, "\phi_2"] & A^\bullet_{GL_2 \times \mathbb{G}_m}(\mathbb{A}^{n+1} \setminus \{0\}) \end{tikzcd}$$

The map ϕ_1 is induced by the isomorphism $GL_2/\mu_n \to GL_2$ given by $A \mapsto (\det A)^{\frac{n-1}{2}} A$ [13, Proposition 4.4]. To determine $A^\bullet_{GL_2}(\mathbb{A}^{n+1} \setminus \{0\})$ it suffices to check how the maximal torus $T \subset GL_2$ acts on $\mathbb{A}^{n+1}$. Since the inverse of $GL_2 \to GL_2/\mu_n$ is given by $A \mapsto (\det A)^{\frac{1-n}{2n}} A$, $\begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix}$ maps to $\begin{pmatrix} \lambda_1^{\frac{n+1}{2n}} \lambda_2^{\frac{1-n}{2n}} & \\ & \lambda_1^{\frac{1-n}{2n}} \lambda_2^{\frac{n+1}{2n}} \end{pmatrix}$ in GL_2/μ_n and acts on $\mathbb{A}^{n+1}$ with characters $\{(\frac{n+1}{2} - i)u + (\frac{-n+1}{2} + i)v \mid 0 \leq i \leq n\}$. This shows

$$A^\bullet_{PGL_2}(\mathbb{P}^n) = \mathbb{Z}[u,v]^{S_2}/(\prod_{i=0}^{n}((\frac{n+1}{2} - i)u + (\frac{-n+1}{2} + i)v))$$

in this case.

To find the map $A^\bullet_{GL_2}(\mathbb{A}^{n+1} \setminus \{0\}) \to A^\bullet_{GL_2 \times \mathbb{G}_m}(\mathbb{A}^{n+1} \setminus \{0\})$, we consider the map $GL_2 \times \mathbb{G}_m \to GL_2$ and find it maps the pair $\begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix}, \lambda$ to $\lambda^{\frac{1}{n}} \begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix}$ in GL_2/μ_n and $\begin{pmatrix} \lambda\lambda_1^{\frac{n+1}{2}}\lambda_2^{\frac{n-1}{2}} & \\ & \lambda\lambda_1^{\frac{n-1}{2}}\lambda_2^{\frac{n+1}{2}} \end{pmatrix}$ in GL_2. This shows the map

$$\mathbb{Z}[u,v]^{S_2}/(\prod_{i=0}^{n}((\frac{n+1}{2} - i)u + (\frac{-n+1}{2} + i)v)) \to \mathbb{Z}[u,v]^{S_2}[H]/(\prod_{i=0}^{n}(H + iu + (n-i)v))$$

giving $A^\bullet_{GL_2}(\mathbb{A}^{n+1} \setminus \{0\}) \to A^\bullet_{GL_2 \times \mathbb{G}_m}(\mathbb{A}^{n+1} \setminus \{0\})$ is given by

$$u \mapsto H + \frac{n+1}{2}u + \frac{n-1}{2}v \quad v \mapsto H + \frac{n-1}{2}u + \frac{n+1}{2}v.$$

Now, we do the case n is even. Let $V \cong K^2$ be a 2-dimensional vector space with the standard representation of GL_2. Let $D \cong K$ be a 1-dimensional vector space where GL_2 acts by multiplication by the determinant. Then, $(\mathrm{Sym}^n V) \otimes (D^\vee)^{\otimes n}$ is a GL_2 representation that descends to a PGL_2 representation.

To determine

$$A^\bullet_{PGL_2}(\mathbb{P}^n) \cong A^\bullet_{PGL_2}(\mathbb{P}((\mathrm{Sym}^n V) \otimes (D^\vee)^{\otimes \frac{n}{2}}))$$

it suffices to find the chern classes of the PGL_2 representation $(\mathrm{Sym}^n V) \otimes (D^\vee)^{\otimes \frac{n}{2}}$ regarded as a PGL_2-equivariant vector bundle over a point. These chern classes are given in [49, Corollary 6.3]. The reader should also note that [49] contains mistakes elsewhere in the document (see [24, Introduction]). As a result, we have $A^\bullet_{PGL_2}(\mathbb{P}^n)$ is $\mathbb{Z}[c_2, c_3, H]/(2c_3, p_n(H))$, where $p_n(t) \in A_{PGL_2}(\mathrm{pt})[t]$ is given as in the statement of the proposition.

Therefore, we have

$$A^\bullet_{PGL_2}(\mathbb{P}((\mathrm{Sym}^n V) \otimes (D^\vee)^{\otimes \frac{n}{2}})) \to A^\bullet_{GL_2}(\mathbb{P}((\mathrm{Sym}^n V) \otimes (D^\vee)^{\otimes \frac{n}{2}}))$$

given by $c_2 \mapsto -(u-v)^2$ and $c_3 \mapsto 0$ by Lemma 13.6. Also, the $\mathscr{O}_{\mathbb{P}((\mathrm{Sym}^n V)\otimes (D^\vee)^{\otimes \frac{n}{2}})}(1)$ class in $A^\bullet_{PGL_2}(\mathbb{P}((\mathrm{Sym}^n V)\otimes (D^\vee)^{\otimes \frac{n}{2}})$ maps to the $\mathscr{O}_{\mathbb{P}((\mathrm{Sym}^n V)\otimes (D^\vee)^{\otimes n})}(1)$ class in $A^\bullet_{GL_2}(\mathbb{P}((\mathrm{Sym}^n V)\otimes (D^\vee)^{\otimes \frac{n}{2}})$ by the projective bundle formula.

Finally, since $(\mathrm{Sym}^n V)\otimes (D^\vee)^{\otimes \frac{n}{2}}$ is a twist of $\mathrm{Sym}^n V$ by a GL_2-equivariant line bundle, the $\mathscr{O}_{\mathbb{P}((\mathrm{Sym}^n V)\otimes (D^\vee)^{\otimes \frac{n}{2}})}(1)$ class in $A^\bullet_{GL_2}(\mathbb{P}((\mathrm{Sym}^n V)\otimes (D^\vee)^{\otimes \frac{n}{2}})$ maps to $\mathscr{O}_{\mathbb{P}(\mathrm{Sym}^n V)}(1) + c_1^{GL_2}(D^{\otimes \frac{n}{2}})$ in $A^\bullet_{GL_2}(\mathbb{P}(\mathrm{Sym}^n V)^{\otimes \frac{n}{2}})$. Since $c_1^{GL_2}(D^{\otimes n/2}) = \frac{n}{2}(u+v)$, we find the composite map

$$A^\bullet_{PGL_2}(\mathbb{P}((\mathrm{Sym}^n V)\otimes (D^\vee)^{\otimes \frac{n}{2}}) \to A^\bullet_{GL_2}(\mathbb{P}(\mathrm{Sym}^n V))$$

is given by

$$c_2 \mapsto -(u-v)^2 \quad c_3 \mapsto 0 \quad H \mapsto H + \frac{n}{2}(u+v).$$

□

14. Formulas and initial reductions

In this section we express the Δ_P and $[\lambda]$ classes in terms of our equivariant Chow ring presentations.

After this, we compute formulas for $\Delta_P \in A^\bullet_T((\mathbb{P}^1)^n)$ and give a quick, alternative computation of the classes $[Z_\lambda] \in A^\bullet_T(\mathbb{P}^n)$ given in [42, Theorem 3.4]. The simple presentation for the class of the diagonal in $(\mathbb{P}^1)^n$ works especially well with the formula for the pushforward $\Phi_* : A^\bullet_T((\mathbb{P}^1)^n) \to A^\bullet_T(\mathbb{P}^n)$ via the classes of torus fixed points, and appears not to have been previously exploited in this fashion.

14.1. **Class of the diagonal in $(\mathbb{P}^1)^n$.** We now compute the T-equivariant class of the diagonal $\Delta_{\{[n]\}} \subset (\mathbb{P}^1)^n$. This formula would also follow from localization to the torus fixed points, but the derivation below is simpler.

Proposition 14.1. *The class of $\Delta_{\{[n]\}}$ in $A^\bullet_T((\mathbb{P}^1)^n)$ is given by*

$$\Delta_{\{[n]\}} = \frac{1}{u-v}\left(\prod_{i=1}^n (H_i+u) - \prod_{i=1}^n (H_i+v)\right).$$

Proof. This is a result of the fact that $\Delta_{\{[n]\}}$ intersected with $\{[0:1]\} \times (\mathbb{P}^1)^{n-1}$ and $\{[1:0]\} \times (\mathbb{P}^1)^{n-1}$ are the torus-fixed points $[0:1]^n$ and $[1:0]^n$ respectively, so

$$((H_1+u)-(H_1+v))\Delta_{\{[n]\}} = \prod_{i=1}^{n}(H_i+u) - \prod_{i=1}^{n}(H_i+v).$$

□

14.2. **Formula for Δ_P.** When two strata Δ_P and $\Delta_{P'}$ intersect transversely in $(\mathbb{P}^1)^n$, it is easy to describe their intersection as another stratum.

Proposition 14.2. *The class $\Delta_P \in A^{n-d}_{PGL_2}((\mathbb{P}^1)^n)$ for P a partition of $[n]$ into d parts is given by the product $\prod_{\{i,j\}\in Edge(T)} \Delta_{i,j}$, where T is any forest with vertex set $[n]$ consisting of one spanning tree for each part of P. In particular,*

(1) If i, j are in distinct parts of P, then if P_{ij} is the partition merging the parts containing i and j, we have $\Delta_{i,j}\Delta_P = \Delta_{P_{ij}}$.
(2) If i, j, i', j' are in the same part of P, we have $\Delta_{i,j}\Delta_P = \Delta_{i',j'}\Delta_P$.

Proof. Item (1) follows from the transversality of the intersection $\Delta_{i,j} \cap \Delta_P$, from which we can deduce the first part of the proposition, and item (2) then follows from the first part and repeated applications of the diagonal relation $\Delta_{i,j}\Delta_{i,k} = \Delta_{i,j}\Delta_{j,k}$. □

Proposition 14.3. *Let $P = \{V_1, \ldots, V_d\}$ be a partition of $[n]$, then*

$$\Delta_P = \frac{1}{(u-v)^d}\prod_{i=1}^{d}\left(\prod_{j\in V_i}(H_j+u) - \prod_{j\in V_i}(H_j+v)\right).$$

Proof. From Proposition 14.2, $\Delta_P = \prod_{i=1}^{d} \Delta_{\{V_i\}}$. Now apply Proposition 14.1. □

14.3. **The ψ_i and $\Delta_{i,j}$ classes.** At this point, we can prove the formula for $\Delta_{i,j}$ in item (3) of Theorem 11.1 and for ψ_i as mentioned in Remark 11.6.

Proposition 14.4. *We have*

$$\Delta_{i,j} = H_i + H_j + u + v$$

$$\psi_i = -(2H_i + u + v).$$

Proof. The formula for $\Delta_{i,j}$ is an immediate consequence of Proposition 14.3.

To compute ψ_i, it suffices to show that $c_1(T_{\mathbb{P}^1}) \in A^\bullet_{GL_2}(\mathbb{P}^1)$ is $2H + u + v$, where $H = c_1(\mathscr{O}(1))$. We note that $c_{\text{top}}(T_X)$ for any smooth X is the pullback of the diagonal under the diagonal map $X \to X \times X$. The pullback $A^\bullet_{GL_2}((\mathbb{P}^1)^2) \to A^\bullet_{GL_2}(\mathbb{P}^1)$ under the inclusion $\mathbb{P}^1 \cong \Delta_{1,2} \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ is given by $H_1, H_2 \mapsto H$. Under this map, $\Delta_{1,2}$ pulls back to $2H + u + v$ as desired. $\square$

14.4. **Pullback and Pushforward under Φ.** The pullback map $\Phi^* : A^\bullet_T(\mathbb{P}^n) \to A^\bullet_T((\mathbb{P}^1)^n)$ is induced by

$$\Phi^*(H) = \sum_{i=1}^n H_i.$$

We now consider Φ_*. By considering the classes of the torus-fixed loci, we have for any $A \subset [n]$,

$$\Phi_*\left(\prod_A (H_i + u) \prod_{[n]\setminus A} (H_j + v)\right) = \prod_{k \in [n]\setminus\{|A|\}} (H + kv + (n-k)u).$$

This in fact uniquely characterizes Φ_*, which can be seen either from localization [29, Theorem 2] or because

$$\frac{\prod_A (H_i + u) \prod_{[n]\setminus A} (H_j + v)}{\prod_A (-v + u) \prod_{[n]\setminus A} (-u + v)}$$

is a Lagrange interpolation basis for polynomials in $H_1, \ldots, H_n$ modulo $F(H_i)$ for each i.

14.5. **Formula for** $[\lambda]$**.** Fehér, Némethi, and Rimányi computed the class of $[\lambda]$ for λ a partition of n [42, Theorem 3.4]. We can give a quick self-contained computation from Section 14.4 and Proposition 14.1 as follows.

Theorem 14.5 ([42, Theorem 3.4])**.** *The class* $[a_1, \ldots, a_d]$ *is the result of first expanding the polynomial*

$$\prod_{i=1}^{d}(z^{a_i} - 1) = \sum_{k \geq 0} c_k z^k \qquad (c_k \in \mathbb{Z}),$$

and then replacing each monomial

$$z^k \mapsto \frac{1}{(u-v)^d} \prod_{j \in [n] \setminus \{k\}} (H + jv + (n-j)u).$$

Proof. Let $P = \{V_1, \ldots, V_d\}$ be a partition of $[n]$ with $|V_i| = a_i$. We expand the formula from Proposition 14.3

$$\Delta_P = \frac{1}{(u-v)^d} \prod_{i=1}^{d} \left(\prod_{j \in V_i} (H_i + u) - \prod_{j \in V_i} (H_i + v) \right)$$

to a sum of terms of the form $\prod_{i \in A}(H_i + u) \prod_{j \in [n] \setminus A}(H_j + v)$. Then, Section 14.4 implies that each such term pushes forward to $\prod_{j \in [n] \setminus \{|A|\}}(H + jv + (n-j)u)$. The result follows immediately. $\square$

15. Strata in $[(\mathbb{P}^1)^n/PGL_2]$

In this section we prove all of our results on ordered point configurations in $\mathbb{P}^1$. Up to Section 15.1, the only result that we use is Theorem 13.3, and in particular the identification of $\Delta_{i,j}$ in $A^{\bullet}_{GL_2}((\mathbb{P}^1)^n)$ as $H_i + H_j + u + v$.

Remark 15.1. Whenever we write $\Delta_{i,j}$ in any context, we will always treat $\{i, j\}$ as an unordered tuple, so that implicitly

$$\Delta_{i,j} := \Delta_{j,i}$$

for $i > j$.

Recall from Theorem 13.3 and Section 12.3, we have the inclusions

$$A^\bullet_{PGL_2}((\mathbb{P}^1)^n) \subset A^\bullet_{GL_2}((\mathbb{P}^1)^n) \subset A^\bullet_T((\mathbb{P}^1)^n).$$

We first consider the square relation in $(\mathbb{P}^1)^4$.

Proposition 15.2. *In $A^\bullet_{PGL_2}((\mathbb{P}^1)^4)$, we have the square relation*

$$\Delta_{1,2} + \Delta_{3,4} = \Delta_{2,3} + \Delta_{4,1}.$$

Proof. Both sides are equal to $H_1+H_2+H_3+H_4+2(u+v)$ by Proposition 14.4. This can also be shown using the fact that the diagonal $\Delta \subset \mathbb{P}^1\times\mathbb{P}^1$ has a torus-equivariant deformation to $\{0\} \times \mathbb{P}^1 \cup \mathbb{P}^1 \times \{\infty\}$. It also holds by Section 11.1.1. □

Definition 15.3. *Let $R(n)$ be the ring*

$$R(n) = \mathbb{Z}[\{\Delta_{i,j} \mid 1 \le i < j \le n\}]/\mathit{relations},$$

generated by the symbols $\Delta_{i,j}$ together with the relations

(1) $\Delta_{i,j} + \Delta_{k,l} = \Delta_{i,k} + \Delta_{j,l}$ for distinct i, j, k, l (square relations)
(2) $\Delta_{i,j}\Delta_{i,k} = \Delta_{i,j}\Delta_{j,k}$ for distinct i, j, k. (diagonal relations)

given in Theorem 11.1 (1). If n is clear from context or irrelevant, we will let $R := R(n)$. If we let each $\Delta_{i,j}$ have degree 1, then the ideal of relations is homogenous, so R is a graded ring, and we will denote by R_k the kth graded part of R.

By Theorem 13.3, we can identify $A^\bullet_{PGL_2}((\mathbb{P}^1)^n)$ as a subring of $A^\bullet_{GL_2}((\mathbb{P}^1)^n)$, where the image

$$A^\bullet_{PGL_2}((\mathbb{P}^1)^n) \hookrightarrow A^\bullet_{GL_2}((\mathbb{P}^1)^n) = \mathbb{Z}[u,v]^{S_2}[H_1,\dots,H_n]/(F(H_1),\dots,F(H_n))$$

is generated by $\Delta_{i,j} = H_i + H_j + u + v$ for $n \geq 3$. If $n \leq 2$, we also have to add the classes $\psi_i = -(2H_i + u + v)$ (see Proposition 14.4). Therefore for $n \geq 3$ by Proposition 15.2, we have a surjective map

$$R \twoheadrightarrow A^\bullet_{PGL_2}((\mathbb{P}^1)^n), \tag{15.1}$$

sending each symbol $\Delta_{i,j} \in R$ to $\Delta_{i,j} \in A^\bullet_{PGL_2}((\mathbb{P}^1)^n)$. To show Theorem 11.1 (1), we need to show this surjection is an isomorphism for $n \geq 3$.

As $A^\bullet_{GL_2}((\mathbb{P}^1)^n)$ is free as an abelian group, $A^k_{PGL_2}((\mathbb{P}^1)^n)$ is a free abelian group for each k. We first compute the rank of these groups for varying k.

Lemma 15.4. *For every $n \geq 1$, the free abelian group $A^k_{PGL_2}((\mathbb{P}^1)^n)$ has rank*

$$\sum_{\substack{i \leq k \\ i \equiv k \ mod\ 2}} \binom{n}{i}.$$

Proof. We compute the rank of $A^k_{PGL_2}((\mathbb{P}^1)^n)$ by working instead with the rational subring

$$A^\bullet_{PGL_2}((\mathbb{P}^1)^n) \otimes \mathbb{Q} \subset A^\bullet_{GL_2}((\mathbb{P}^1)^n)) \otimes \mathbb{Q},$$

which is generated by the elements $H'_i := H_i + \frac{1}{2}(u+v)$ by Theorem 13.3. Noting that $H'^2_i = \frac{1}{4}(u-v)^2$, we see the $\mathbb{Q}$-vector space $A^k_{PGL_2}((\mathbb{P}^1)^n) \otimes \mathbb{Q}$ is spanned by the elements

$$\mathcal{B} = \{\left(\frac{u-v}{2}\right)^{n-d-|B|} \prod_{i \in B} H'_i \mid B \subset [n],\ |B| \leq n-d,\ |B| \equiv n-d \pmod 2\},$$

which has size

$$|\mathcal{B}| = \sum_{\substack{i \leq k \\ i \equiv k \text{ mod } 2}} \binom{n}{i}.$$

To finish, it suffices to show that the elements of $\mathcal{B}$ are linearly independent. Indeed, the elements of $\mathcal{B}$ become distinct monomials in the H'_i after setting $u = 1$ and $v = -1$ (after which the defining relations $F(H_i) = 0$ become $H'^2_i = 1$ for each i). $\square$

Definition 15.5. *Let* $\operatorname{Part}(d,n)$ *denote the set of partitions of* $[n]$ *into* d *parts. For* $P \in \operatorname{Part}(d,n)$*, for any forest* T *with vertex set* $[n]$ *consisting of one spanning tree for each part of* P*, we define*

$$\Delta_P = \prod_{\{i,j\} \in Edge(T)} \Delta_{i,j} \in R$$

Note that by the diagonal relations this is independent of the choice of T*, and* $\Delta_P \mapsto \Delta_P$ *under the map* $R \to A^{\bullet}_{PGL_2}((\mathbb{P}^1)^n)$ *by Proposition 14.2.*

Remark 15.6. The two items (1), (2) in Proposition 14.2 are also true for the elements $\Delta_P \in R$ as the proof only uses the diagonal relations in $A^{\bullet}_{PGL_2}((\mathbb{P}^1)^n)$.

Lemma 15.7. *For* $k \leq n-2$*,* R_k *is generated by* $\{\Delta_P \mid P \in \operatorname{Part}(n-k,n)\}$.

Proof. Given a product $\prod_{\ell=1}^{k} \Delta_{i_\ell, j_\ell}$, we will produce an algorithm for rewriting this product in terms of Δ_P with P a partition of $[n]$ into $n-k$ parts.

By induction, we can write $\prod_{\ell=1}^{k-1} \Delta_{i_\ell, j_\ell}$ as $\sum_{P' \in \operatorname{Part}(n-k+1,n)} a_{P'} \Delta_{P'}$, so it suffices to show that $\Delta_{i_k,j_k}\Delta_{P'}$ for $P' \in \operatorname{Part}(n-k+1,n)$ can be written as a $\mathbb{Z}$-linear combination $\sum_{P \in \operatorname{Part}(n-k,n)} a_P \Delta_P$.

If i_k, j_k are in different parts of P', then $\Delta_{i_k,j_k}\Delta'_P = \Delta_P$ where P merges the parts containing i_k and j_k, and we are done. Otherwise, if they are in the same part A_1, let A_2, A_3 be two parts of P' distinct from A_1 (which exist as $n-k+1 \geq 3$), with elements $x_2 \in A_2$ and $x_3 \in A_3$. By applying a square relation, we have

$$\Delta_{i_k,j_k}\Delta_{P'} = (\Delta_{i_k,x_2} - \Delta_{x_2,x_3} + \Delta_{x_3,j_k})\Delta_{P'},$$

and each of the three terms on the right is some Δ_P with $P \in \operatorname{Part}(n-k,n)$. $\square$

Definition 15.8. *Given a partition* P *of* $[n]$ *and* $i,j \in [n]$ *in distinct parts of* P*, let* $P_{i,j}$ *be the partition of* $[n]$ *obtained by merging the parts in* P *containing* i *and* j*.*

From Remark 15.6, the following relations hold in $R(n)$ (and hence also in $A^{\bullet}_{PGL_2}((\mathbb{P}^1)^n)$).

Definition 15.9. *For i_1, i_2, i_3, i_4 in distinct parts of a partition P of $[n]$, define the* square relation *for P associated to i_1, i_2, i_3, i_4 to be the relation*

$$\Delta_{P_{i_1,i_2}} - \Delta_{P_{i_2,i_3}} + \Delta_{P_{i_3,i_4}} - \Delta_{P_{i_4,i_1}} = 0.$$

Definition 15.10. *Inside the free abelian group $\mathbb{Z}^{\mathrm{Part}(d,n)}$, denote by $\mathrm{Sq}(d,n)$ the subgroup generated by formal square relations $P_{i,j} - P_{j,k} + P_{k,l} - P_{l,i}$ for $P \in \mathrm{Part}(d+1, n)$ and i, j, k, l distinct. Then we define*

$$\mathcal{A}(d,n) := \mathbb{Z}^{\mathrm{Part}(d,n)} / \mathrm{Sq}(d,n).$$

Lemma 15.7 shows for $d \geq 2$ we have a surjection

$$\mathcal{A}(d,n) \twoheadrightarrow R_{n-d}$$

that sends $P \mapsto \Delta_P$. We will in fact show this is an isomorphism.

Definition 15.11. *Say a partition $P \in \mathrm{Part}(d,n)$ for $d \geq 2$ is* good *if P can be written as $P = \{A_1, \ldots, A_n\}$ with $A_1 \sqcup A_2$ a partition of an initial segment of $[n]$, and $A_3, \ldots, A_n$ all contiguous intervals. Denote*

$$\mathrm{Good}(d,n) := \{P \in \mathrm{Part}(d,n) \mid P \text{ good}\}.$$

Lemma 15.12. *For $d \leq n-2$, $\mathcal{A}(d,n)$ is generated by the set of $P \in \mathrm{Good}(d,n)$.*

Proof. We use induction on n and d. For $d = 2$ every partition is good, and for $n = 2$ the result is trivial. Suppose now we have $n, d > 2$. Take $Q \in \mathrm{Part}(d,n)$.

If $n-1$ and n are in the same part, then $Q' := Q \setminus n \in \mathrm{Part}(d, n-1)$, and by the induction hypothesis applied to $\mathcal{A}(d, n-1)$ we can write $Q' = \sum_{P' \in \mathrm{Good}(d,n-1)} a_{P'} P'$. There is a map

$$\mathcal{A}(d, n-1) \to \mathcal{A}(d,n)$$

mapping each P' for $P' \in \text{Part}(d, n-1)$ to P, where P is obtained by adding n to the same part as $n-1$ in P'. Furthermore, under this map $P \in \text{Good}(d, n)$ if $P' \in \text{Good}(d, n-1)$, so we get Q as a $\mathbb{Z}$-linear combination of P for $P \in \text{Good}(d, n)$.

If n is isolated in Q, then let $Q' = Q \setminus n \in \text{Part}(d-1, n-1)$. By the induction hypothesis applied to $\mathcal{A}(d-1, n-1)$, we can write $Q = \sum_{P' \in \text{Good}(d-1,n-1)} a_{P'} P'$. There is a map

$$\mathcal{A}(d-1, n-1) \to \mathcal{A}(d, n)$$

mapping each P' for $P' \in \text{Part}(d-1, n-1)$ to P, where P is obtained by adding n as an isolated part. Furthermore, under this map $P \in \text{Good}(d, n)$ if $P' \in \text{Good}(d-1, n-1)$, so we get Q as a $\mathbb{Z}$-linear combination of P for $P \in \text{Good}(d, n)$.

If neither of the above two cases hold, then $n-1$ and n are not in the same part and n is not isolated in Q. Let $x \in [n]$ be another element in the same part as n, and let $y \in [n]$ be in a different part as $n-1$ and n (which exists as $d > 2$). Then if we let $\widetilde{Q} \in \text{Part}(d+1, n)$ be the result of taking Q and isolating n into its own part, the square relation for $\widetilde{Q}$ associated to $n-1, n, x, y$ yields Q as a combination of 3 terms, each of which either has n isolated or $n-1, n$ in the same group. □

Lemma 15.13. *For $d \geq 2$,*

$$\#\,\text{Good}(d, n) = \sum_{\substack{i \leq n-d \\ i \equiv n-d \bmod 2}} \binom{n}{i}.$$

Proof. From the definition of Good(d, n),

$$\#\,\text{Good}(d, n) = \sum_{k=1}^{n-d+2} (2^{k-1} - 1) \binom{n-k-1}{n-k-d+2}.$$

Let

$$G_{d,n} = \sum_{k=1}^{n-d+2} (2^{k-1} - 1)\binom{n-k-1}{n-k-d+2}$$
$$G'_{d,n} = \sum_{\substack{i \le n-d \\ i \equiv n-d \bmod 2}} \binom{n}{i}.$$

We will show $G_{d,n} = G'_{d,n}$ for all $n \geq 2$ and $d \geq 2$ by induction on n. For the base case if $n = 2$ and $d \geq 2$ arbitrary, we have two cases: if $d = 2$, $|\mathcal{G}(2,2)| = G_{2,2} = 1$ and if $d > 2$, $|\mathcal{G}(d,2)| = G_{d,2} = 0$. If $d = 2$ and $n \geq 2$ arbitrary, then $G'_{d,n} = 2^{n-1} - 1$ by the binomial theorem, and $G_{d,n} = 2^{n-1} - 1$ because only the $k = n$ term $(2^{n-1}-1)\binom{-1}{0}$ contributes.

Now, assume we know $G_{d,n} = G'_{d,n}$ for some n and all $d \geq 2$. For the induction step,

$$\begin{aligned} G_{d,n} + G_{d+1,n} &= \sum_{k=1}^{n-d+2} (2^{k-1} - 1)\left(\binom{n-k-1}{n-k-d+2} + \binom{n-k-1}{n-k-d+1}\right) \\ &= \sum_{k=1}^{n-d+2} (2^{k-1} - 1)\binom{n-k}{n-k-d+2} = G_{d+1,n+1}, \end{aligned}$$

and similarly applying Pascal's identity, $G'_{d,n} + G'_{d+1,n} = G'_{d+1,n+1}$. □

Corollary 15.14. *For $d \geq 2$, and $n \geq 3$ we have the isomorphisms*

$$\mathbb{Z}^{\mathrm{Good}(d,n)} \xrightarrow{\sim} \mathcal{A}(d,n) \xrightarrow{\sim} R_{n-d} \xrightarrow{\sim} A^{n-d}_{PGL_2}((\mathbb{P}^1)^d).$$

Proof. By Lemmas 15.7 and 15.12 and (15.1), we have

$$\mathbb{Z}^{\mathrm{Good}(d,n)} \twoheadrightarrow \mathcal{A}(d,n) \twoheadrightarrow R^{n-d} \twoheadrightarrow A^{n-d}_{PGL_2}((\mathbb{P}^1)^d).$$

Since $A^{n-d}_{PGL_2}((\mathbb{P}^1)^d)$ is a finitely generated, free $\mathbb{Z}$-module of rank equal to the rank of $\mathbb{Z}^{\mathrm{Good}(d,n)}$ by Lemmas 15.4 and 15.13, the composite $\mathbb{Z}^{\mathrm{Good}(d,n)} \to A^{n-d}_{PGL_2}((\mathbb{P}^1)^d)$ is an isomorphism. □

We now find an explicit basis for R_k for $k > n-2$ of size 2^{n-1}.

Lemma 15.15. *For each partition $P \in \mathrm{Part}(d,n)$ for $d \leq 2$, arbitrarily choose i_P, j_P that lie in the same part. Then for $k > n-2$, R_k is generated by the 2^{n-1} elements*

$$S_k := \{\Delta_{\{[n]\}}\Delta_{i_{\{[n]\}},j_{\{[n]\}}}^{k-n+1}\} \cup \{\Delta_P \Delta_{i_P,j_P}^{k-n+2} \mid P \in \mathrm{Part}(2,n)\}.$$

Proof. Let $P = \{A, B\} \in \mathrm{Part}(2,n)$. By Lemma 15.7, it suffices to show $\Delta_P \prod_{a=1}^{k-n+2} \Delta_{i_a,j_a}$ is generated by S_k for any choices of $i_a \neq j_a$. We proceed by induction on $k > n-2$. For the base case $k = n-1$, it suffices to show $\Delta_{i,j}\Delta_P$ is generated by S_k for any $i \neq j$. If $k > n-1$, then by the induction hypothesis, it suffices to show $\Delta_{i,j}\Delta_P\Delta_{i_P,j_P}^{k-n+1}$ and $\Delta_{i,j}\Delta_{\{[n]\}}\Delta_{i_{\{[n]\}},j_{\{[n]\}}}^{k-n}$ are generated by S_k. Both the base case and the induction step will work in the same way.

First, $\Delta_{i,j}\Delta_{\{[n]\}}\Delta_{i_{\{[n]\}},j_{\{[n]\}}}^{k-n} = \Delta_{\{[n]\}}\Delta_{i_{\{[n]\}},j_{\{[n]\}}}^{k-n+1}$ by Remark 15.6 (2). To deal with $\Delta_{i,j}\Delta_P\Delta_{i_P,j_P}^{k-n+1}$, we have two cases.

(1) If $\{i,j\}$ is not contained in A or B, then $\Delta_{i,j}\Delta_P$ is the diagonal $\Delta_{\{[n]\}}$ by Remark 15.6 (1). Then, by Remark 15.6 (2), $\Delta_{i,j}\Delta_P\Delta_{i_P,j_P}^{k-n+1} = \Delta_{i_{\{[n]\}},j_{\{[n]\}}}^{k-n+1}\Delta_{\{[n]\}}$.

(2) Suppose now each $\{i,j\}$ is in A or B, and that without loss of generality, $i_P, j_P \in A$. If $i,j \in A$, then using Remark 15.6 (2) we may replace $\Delta_{i,j}$ with Δ_{i_P,j_P}. If $i,j \in B$, we can use a square relation to replace it with $\Delta_{i,i_P} - \Delta_{i_P,j_P} + \Delta_{j_P,i}$. We then have $\Delta_{i,i_P}\Delta_P = \Delta_{\{[n]\}} = \Delta_{j_P,i}\Delta_P$, so

$$\Delta_{i,j}\Delta_P\Delta_{i_P,j_P}^{k-n+1} = 2\Delta_{i_{\{[n]\}},j_{\{[n]\}}}^{k-n+1}\Delta_{\{[n]\}} - \Delta_P\Delta_{i_P,j_P}^{k-n+2}$$

by Remark 15.6 (2).

$\square$

Theorem 15.16. *For $n \geq 3$, the natural surjection $R \twoheadrightarrow A^{\bullet}_{PGL_2}((\mathbb{P}^1)^n)$ is an isomorphism. Furthermore, R_k has $\mathbb{Z}$-basis given by*

(1) $\{\Delta_P \mid P \in \mathrm{Good}(n-k,n)\}$ *for* $k \leq n-2$

(2) $S_k = \{\Delta_{\{[n]\}}\Delta^{k-n+1}_{i_{\{[n]\}},j_{\{[n]\}}}\} \cup \{\Delta_P\Delta^{k-n+2}_{i_P,j_P} \mid P \in \mathrm{Part}(2,n)\}$, *where for each partition* $P \in \mathrm{Part}(d,n)$ *for* $d \leq 2$, *arbitrarily choose* i_P, j_P *that lie in the same part.*

Proof. If $k \leq n-2$, we have $R_k \twoheadrightarrow A^k_{PGL_2}((\mathbb{P}^1)^n)$ is an isomorphism with $\mathbb{Z}$-basis given by $\{\Delta_P \mid P \in \mathrm{Good}(n-k,n)\}$ by Corollary 15.14. Now, we consider the case $k > n-2$.

The S_k span R_k by Lemma 15.15, so applying (15.1) yields

$$\mathbb{Z}^{S_k} \twoheadrightarrow R_k \twoheadrightarrow A^k_{PGL_2}((\mathbb{P}^1)^n),$$

whose composite is a surjection of free $\mathbb{Z}$-modules of the same rank 2^{n-1} by Lemmas 15.4 and 15.15, so it is an isomorphism. This proves $R_k \twoheadrightarrow A^k_{PGL_2}((\mathbb{P}^1)^n)$ is an isomorphism and identifies S_k as a basis. □

15.1. **Algorithm and Example.** We can describe an algorithm for writing arbitrary classes in $A^\bullet_{PGL_2}((\mathbb{P}^1)^d)$ in terms of our $\mathbb{Z}$-basis. The key fact is that if $\mathrm{pr}^n : (\mathbb{P}^1)^n \to (\mathbb{P}^1)^{n-1}$ is projection by forgetting the last factor, then by definition of the pushforward of a cycle

$$\mathrm{pr}^n_* \Delta_P = \begin{cases} \Delta_{P\setminus n} & \text{if } n \text{ is not isolated, and} \\ 0 & \text{if } n \text{ is isolated.} \end{cases}$$

At the level of formulae, if we write our class as a polynomial in the H_i, u, v with each H_i appearing to degree at most 1, then pr^n_* extracts the H_n-coefficient. Also, if we have a Δ_P and we know that either n is isolated or $n-1, n$ are in the same part, then as $(H_n - H_{n-1}) \cap \Delta_{n-1,n} = 0$ we also have

$$\mathrm{pr}^n_*(\Delta_P \cap (H_n - H_{n-1})) = \begin{cases} 0 & \text{if } n-1, n \text{ are in the same group, and} \\ \Delta_{P\setminus n} & \text{if } n \text{ is isolated.} \end{cases}$$

Suppose we have a class

$$\alpha = \sum_{P \in \mathrm{Good}(d,n)} a_P \Delta_P = \sum_{\substack{P \in \mathrm{Good}(d,n) \\ n \text{ isolated}}} a_P \Delta_P + \sum_{\substack{P \in \mathrm{Good}(d,n) \\ n-1,n \text{ together}}} a_P \Delta_P$$

and we want to find the coefficients a_P.

We first show how to reduce down to the case $d = 2$. By the above, we have

$$\mathrm{pr}^n_* \alpha = \sum_{\substack{P \in \mathrm{Good}(d,n) \\ n-1,n \text{ together}}} a_P \Delta_{P \setminus n}, \qquad \mathrm{pr}^n_*(\alpha \cap (H_n - H_{n-1})) = \sum_{\substack{P \in \mathrm{Good}(d,n) \\ n \text{ isolated}}} a_P \Delta_{P \setminus n}.$$

In the first case each $P \setminus n \in \mathrm{Good}(d-1, n)$, and in the second case each $P \setminus n \in \mathrm{Good}(d-1, n-1)$ so we can apply induction to determine all of these coefficients.

Once we have reduced down to the case $d = 2$, we can now identify each a_P separately for $P = \{A, B\}$ a partition of $[n]$ into two parts by evaluating at $H_i = -u$ for $i \in A$ and $H_i = -v$ for $i \in B$ (which is localization at a torus-fixed point). By Proposition 14.3, this evaluates to $a_{\{A,B\}}(u-v)^{n-2}(-1)^{|A|-1}$.

The same method for $d = 2$ works for elements $\alpha \in A^k((\mathbb{P}^1)^n)$ with $k > n - 2$. Applying the same substitution to

$$\alpha = \sum a_P \Delta^{k-n+2}_{i_P, j_P} \Delta_P + a_{\{[n]\}} \Delta^{n-k+1}_{i_{\{[n]\}}, j_{\{[n]\}}} \Delta_{\{[n]\}}$$

extracts the a_P-coefficient for $P = \{A, B\}$ a partition of $[n]$ into two parts as this is the only term that does not vanish under this substitution. Then, we subtract off all of these terms to recover $a_{[n]}$.

Example 15.17. As a simple example, consider the PGL_2-orbit closure of a generic point in $(\mathbb{P}^1)^5$. The formula computed in [73, Corollary 4.8] shows that the class of this orbit is

$$\alpha = e_2(H_1, H_2, H_3, H_4, H_5) + 2(u+v)(H_1 + H_2 + H_3 + H_4 + H_5) + (3u^2 + 4uv + 3v^2),$$

where e_2 is the second elementary symmetric polynomial. We have

$$
\begin{tikzcd}[column sep=tiny]
 & & & \alpha \arrow[dl, dash] \arrow[dr, dash] & \\
 & & \mathrm{pr}^5_*(\alpha) \arrow[dl, dash] \arrow[dr, dash] & & \begin{array}{c} \mathrm{pr}^5_*(\alpha \cap (H_5 - H_4)) \\ = \Delta_{\{\{1,2,3\},\{4\}\}} \end{array} \\
 & \mathrm{pr}^4_*(\mathrm{pr}^*_5(\alpha)) \arrow[dl, dash] \arrow[dr, dash] & & \begin{array}{c} \mathrm{pr}^4_*(\mathrm{pr}^5_*(\alpha) \cap (H_4 - H_3)) \\ = \Delta_{\{\{1,2\},\{3\},\{4\}\}} \end{array} & \\
 \begin{array}{c} \mathrm{pr}^3_*(\mathrm{pr}^4_*(\mathrm{pr}^5_*(\alpha))) \\ = 0 \end{array} & & \begin{array}{c} \mathrm{pr}^3_*(\mathrm{pr}^4_*(\mathrm{pr}^*_5(\alpha)) \cap (H_3 - H_2)) \\ = \Delta_{\{\{1\},\{2\}\}} \end{array} & &
\end{tikzcd}
$$

$$
\begin{aligned}
&\mathrm{pr}^5_* \alpha = (H_1 + H_2 + H_3 + H_4) + 2(u + v) \\
&\mathrm{pr}^5_*(\alpha \cap (H_5 - H_4)) = e_2(H_1, H_2, H_3) + (u + v)(H_1 + H_2 + H_3) + (u^2 + uv + v^2) \\
&\mathrm{pr}^4_*(\mathrm{pr}^5_* \alpha) = 1 \\
&\mathrm{pr}^4_*(\mathrm{pr}^5_* \alpha \cap (H_4 - H_3)) = H_1 + H_2 + u + v \\
&\mathrm{pr}^3_*(\mathrm{pr}^4_*(\mathrm{pr}^5_* \alpha)) = 0 \\
&\mathrm{pr}^3_*(\mathrm{pr}^4_*(\mathrm{pr}^5_* \alpha) \cap (H_3 - H_2)) = 1.
\end{aligned}
$$

The only non-trivial identification was $\mathrm{pr}^5_*(\alpha \cap (H_5 - H_4)) = \Delta_{\{\{1,2,3\},\{4\}\}}$, which we can identify as follows. Substitute $-u$'s and $-v$'s for the H_i corresponding to all nontrivial partitions $\{A, B\}$ of $[4]$ into two parts. We find the only choice that gives a nonzero result is $A = \{1, 2, 3\}, B = \{4\}$, yielding $(u - v)^2$, which is the same as for $\Delta_{\{\{1,2,3\},\{4\}\}}$ by Proposition 14.3. Putting this together yields

$$\alpha = \Delta_{\{\{1\},\{2\},\{3,4,5\}\}} + \Delta_{\{\{1,2\},\{3\},\{4,5\}\}} + \Delta_{\{\{1,2,3\},\{4\},\{5\}\}}.$$

We remark that the PGL_2-orbit closure $X_n \subset (\mathbb{P}^1)^n$ of a general point in $(\mathbb{P}^1)^n$ decomposes into good incidence strata as

$$[X_n] = \sum_{a=1}^{n-2} \Delta_{\{\{1,\ldots,a\},a+1,\{a+2,\ldots,n\}\}} \tag{15.2}$$

which can be geometrically explained as follows. Consider the diagram

$$\begin{array}{ccc} \overline{\mathcal{M}}_{0,n}(\mathbb{P}^1,1) & \xrightarrow{\mathrm{ev}} & (\mathbb{P}^1)^n \\ \downarrow{\scriptstyle \pi} & & \\ \overline{\mathcal{M}}_{0,n} & & \end{array}$$

(see Section 11.1.1 for notation). The left and right hand side of (15.2) can both be described as $\mathrm{ev}_*\pi^*(\mathrm{pt})$ for $\mathrm{pt} \in \overline{\mathcal{M}}_{0,n}$ being a general point and the point corresponding to a chain of $n-2$ rational curves (respectively), and the result follows from the flatness of π. See [73, Section 4] for a generalization of this degeneration to PGL_{r+1} orbits closures of general points in $(\mathbb{P}^r)^n$.

16. GL_2-equivariant classes of strata in $\mathrm{Sym}^n\mathbb{P}^1$

Recall from Definition 12.5 that $[\lambda] \in A^\bullet_{GL_2}(\mathbb{P}^n)$ for λ a partition of n is the push-forward of $\Delta_P \in A^\bullet_{GL_2}((\mathbb{P}^1)^n)$ under the multiplication map $(\mathbb{P}^1)^n \to \mathbb{P}^n$ for P a partition of $[n]$ into subsets with cardinalities given by λ. Up to a constant factor given in Definition 12.5, this is the class of the closure Z_λ given in Definition 12.3 of degree n forms on $(\mathbb{P}^1)^\vee$ whose roots have multiplicities given by λ as studied by Fehér, Némethi, and Rimányi [42].

Definition 16.1. *Denote by* $[a, b, 1^c] := [\{a, b, 1, 1, \ldots, 1\}]$ *where there are* c *1's.*

From writing the expressions for $[\lambda]$ in Theorem 14.5 using generating functions, we find the following new Corollary.

Corollary 16.2. *For $d \geq 2$, consider the polynomial*

$$-\frac{1}{(z-1)^{d-2}}\prod_{i=1}^{d}(z^{a_i}-1) = \sum_{\substack{0\leq k_1\leq k_2\\ k_1+k_2=n-d+2}} \alpha_{k_1}(z^{k_1}+z^{k_2}).$$

Then $\alpha_i \in \mathbb{Z}$ and

$$[a_1,\ldots,a_d] = \sum_{\substack{1\leq k_1\leq k_2\\ k_1+k_2=n-d+2}} \alpha_{k_1}[k_1,k_2,1^{d-2}]$$

Proof. Clearly all $\alpha_i \in \mathbb{Z}$ except possibly $\alpha_{\frac{n-d+2}{2}}$, which a priori only lies in $\mathbb{Z}[\frac{1}{2}]$. But plugging in $z=1$ to both sides shows the integrality.

By Theorem 14.5, it suffices to show

$$\prod_{i=1}^{d}(z^{a_i}-1) = \sum_{\substack{1\leq k_1\leq k_2\\ k_1+k_2=n-d+2}} \alpha_{k_1}(z^{k_1}-1)(z^{k_2}-1)(z-1)^{d-2}.$$

or equivalently

$$\frac{1}{(z-1)^{d-2}}\prod_{i=1}^{d}(z^{a_i}-1) = \sum_{\substack{k_1\leq k_2\\ k_1+k_2=n-d+2}} \alpha_{k_1}(z^{k_1}-1)(z^{k_2}-1).$$

By definition of α_k, the coefficients of both sides agree except possibly the z^0 and $z^{n-(d-2)}$-coefficient. Also, the coefficients of z^0 and $z^{n-(d-2)}$ are equal to each other on the left hand side, and the same is true on the right side. To see they agree between the left and right sides, we note both sides are 0 after substituting $z=1$. $\square$

Lemma 16.3. *The rational GL_2-equivariant classes in $\mathbb{P}^n$ of the torus fixed points*

$$\prod_{j\in[n]\setminus\{k\}} (H+jv+(n-j)u) \in A_T^\bullet(\mathbb{P}^n)\otimes\mathbb{Q}$$

are linearly independent.

Proof. For fixed k, $H \mapsto -ku-(n-k)v$ maps $\prod_{j\in[n]\setminus\{k'\}}(H+jv+(n-j)u)$ to 0 if and only if $k' \neq k$ $\square$

Theorem 16.4. *For fixed $c \geq 0$, the classes $[a,b,1^c]$ with $a+b=n-c$ and $a \geq b$ form a $\mathbb{Q}$-basis for $A^{n-c-2}_{PGL_2}(\mathbb{P}^n)\otimes\mathbb{Q} \subset A^{n-c-2}_{GL_2}(\mathbb{P}^n)\otimes\mathbb{Q}$.*

Proof. To show the linear independence, first note that $\prod_{j\in[n]\setminus\{k\}}(H+jv+(n-j)u)$ are linearly independent in $A^\bullet_T(\mathbb{P}^n)\otimes\mathbb{Q}$ by Lemma 16.3. Therefore, it suffices to show for fixed c that the polynomials $(z^a-1)(z^b-1)(z-1)^c$ with $a \geq b$ and $a+b=n-c$ are linearly independent. Indeed, dividing out by $(z-1)^c$, we note that $(z^a-1)(z^b-1)$ is the only such polynomial which contains either of the monomials z^a or z^b.

To see that the $\mathbb{Q}$-linear span of the classes $[a,b,1^c]$ is precisely $A^{n-c-2}_{PGL_2}(\mathbb{P}^n)\otimes\mathbb{Q}$, we note that we have just shown that the dimension of the $\mathbb{Q}$-linear span of the $[a,b,1^c]$ is precisely $\lfloor\frac{n-c}{2}\rfloor$ by linear independence, which we can check is the same as the dimension of $A^{n-c-2}_{PGL_2}(\mathbb{P}^n)\otimes\mathbb{Q}$ by Proposition 13.7.

$\square$

17. Integral classes of unordered strata in $[\mathrm{Sym}^n\mathbb{P}^1/PGL_2]$

In this section, we compute the integral classes of $[Z_\lambda] \in A^\bullet_{PGL_2}(\mathbb{P}^n)$. By Proposition 13.7, if n is odd, then $A^\bullet_{PGL_2}(\mathbb{P}^n) \to A^\bullet_{GL_2}(\mathbb{P}^n)$ is injective and we know the image of the $[Z_\lambda]$ in $A^\bullet_{GL_2}(\mathbb{P}^n)$ by Theorem 14.5, so it suffices to consider the case n is even, which we assume for the remainder of this section.

Recall the polynomials $p_n(t) \in A^\bullet_{PGL_2}(\mathrm{pt})[t]$ defined in Proposition 13.7 for even n and let q_n be the image of p_n in $A^\bullet_{PGL_2}(\mathrm{pt})/(2)[t] \cong \mathbb{F}_2[c_2,c_3,t]$. It is easy to see by the binomial theorem or directly from [49, Lemma 6.1] that

$$q_n(t) = \begin{cases} t^{(n+4)/4}(t^3+c_2t+c_3)^{n/4} & \text{if } n \equiv 0 \bmod 4, \text{ and} \\ t^{(n-2)/4}(t^3+c_2t+c_3)^{(n+2)/4} & \text{if } n \equiv 2 \bmod 4, \end{cases}$$

and $q_n(t) \mid q_{n+k}(t)$ for $k=0$ or $k \geq 4$ for any n.

By Proposition 13.7, for n even,

$$A^\bullet_{PGL_2}(\mathbb{P}^n) \cong \mathbb{Z}[c_2, c_3, H]/(2c_3, p_n(H)),$$

which is isomorphic to

$$\left(\bigoplus_{i=0}^{n} \mathbb{Z}[c_2]H^i\right) \oplus \left(\bigoplus_{i=0}^{n} c_3\mathbb{F}_2[c_2, c_3]H^i\right)$$

as abelian groups. So to determine the class $[Z_\lambda] \in A^\bullet_{PGL_2}(\mathbb{P}^n)$, it suffices to find its image in $\bigoplus_{i=0}^n \mathbb{Z}[c_2]H^i$ and $\bigoplus_{i=0}^n c_3\mathbb{F}_2[c_2, c_3]H^i$. Equivalently, if we write the class of $[Z_\lambda]$ as a polynomial in c_2, c_3, and H with degree at most n in H, then it suffices to consider the terms not containing c_3 and the terms containing c_3 separately. Under the map $A^\bullet_{PGL_2}(\mathbb{P}^n) \to A^\bullet_{GL_2}(\mathbb{P}^n)$, Proposition 13.7 shows that the first factor maps injectively and the second factor maps to zero.

We can determine the image of $[Z_\lambda]$ in the first factor using Theorem 14.5, so it suffices to determine the image of $[Z_\lambda]$ in the second factor to identify its class. To do this, we will work modulo 2 and determine $[Z_\lambda] \in A^\bullet_{PGL_2}(\mathbb{P}^n) \otimes \mathbb{Z}/2\mathbb{Z}$. Discarding those monomials not containing c_3 then yields the image of $[Z_\lambda]$ in the second factor.

Definition 17.1. *We say a partition $\lambda = a_1^{e_1} \ldots a_k^{e_k}$ of n into $d = \sum_{i=1}^k e_i$ parts is* special *if all a_i and $\frac{d!}{e_1! \cdots e_k!}$ are odd, and all e_i are even.*

Theorem 17.2. *Let d and n be integers with n even. The class of $[Z_\lambda] \in A^\bullet_{PGL_2}(\mathbb{P}^n) \otimes \mathbb{Z}/2\mathbb{Z}$ for λ a partition of n into d parts is given by*

$$\begin{cases} \frac{q_n}{q_d}(H) & \text{if } \lambda \text{ is special, and} \\ 0 & \text{otherwise.} \end{cases}$$

Remark 17.3. If $[Z_\lambda] \in A^\bullet_{PGL_2}(\mathbb{P}^n) \otimes \mathbb{Z}/2\mathbb{Z}$ is zero, then the component in $\bigoplus_{i=0}^n \mathbb{Z}[c_2]H^i$ is a multiple of 2, and the component in $\bigoplus_{i=0}^n c_3\mathbb{F}_2[c_2, c_3]H^i$ is zero.

Furthermore, given the statement of the theorem, if $[Z_\lambda] \in A^\bullet_{PGL_2}(\mathbb{P}^n) \otimes \mathbb{Z}/2\mathbb{Z}$ is non-zero, then the component in $\bigoplus_{i=0}^n c_3\mathbb{F}_2[c_2, c_3]H^i$ is non-zero and is given by discarding anything with a c_3^0-coefficient in $\frac{q_n}{q_d}(H)$.

Lemma 17.4. *Given a ring $R[H]/(P(H))$ for P a monic polynomial of degree $n+1$, define the R-linear map $\int : R[H]/(P(H)) \to R$ given by taking a polynomial $f(H)$, and outputting the H^n-coefficient of the reduction $\widetilde{f}(H)$ of $f(H)$ $\pmod{P(H)}$ to a polynomial of degree $\leq n$. Then letting t be an indeterminate, we have*

$$\int \frac{P(H) - P(t)}{H - t} f(H) = \widetilde{f}(t).$$

Proof. We have

$$\begin{aligned}
&\int \frac{P(H) - P(t)}{H - t} f(H) \\
=&\int \frac{P(H) - P(t)}{H - t} \widetilde{f}(H) \\
=&\int P(H) \frac{\widetilde{f}(H) - \widetilde{f}(t)}{H - t} - \int P(t) \frac{\widetilde{f}(H) - \widetilde{f}(t)}{H - t} + \int \frac{P(H) - P(t)}{H - t} \widetilde{f}(t) \\
=&0 + 0 + \widetilde{f}(t) = \widetilde{f}(t).
\end{aligned}$$

Where in the second last equality, the first term is zero because the integrand is a multiple of $P(H)$, the second term is zero because $\frac{\widetilde{f}(H) - \widetilde{f}(t)}{H-t}$ is a polynomial of degree at most $n-1$, and the last term is $\widetilde{f}(t)$ because $\frac{P(H)-P(t)}{H-t}$ is monic of degree n. $\square$

Remark 17.5. Let G be a linear algebraic group and V be a representation. Then,

$$A^\bullet_G(\mathbb{P}(V)) \cong A^\bullet_G(\mathrm{pt})[H]/(P(H))$$

$$A^\bullet_G(\mathbb{P}(V) \times \mathbb{P}(V)) \cong A^\bullet_G(\mathrm{pt})[H_1, H_2]/(P(H_1), P(H_2)),$$

where $P \in A^\bullet_G[T]$ is $T^{\dim(V)} + c_1^G(V)T^{\dim(V)-1} + \cdots + c^G_{\dim(V)}(V)$ by the projective bundle theorem and the class of the diagonal in $\mathbb{P}(V) \times \mathbb{P}(V)$ is $(P(H_1) - P(H_2))/(H_1 -$

H_2), giving a geometric interpretation of Lemma 17.4. This can be proven, for example, by first noting that it suffices to consider the case $G = GL(V)$. Then, we can restrict to a maximal torus [27, Proposition 6] and use the fact that the diagonal in $\mathbb{P}(V) \times \mathbb{P}(V)$ admits a torus-equivariant deformation into a union of products of coordinate linear spaces [16, Theorem 3.1.2].

Proof of Theorem 17.2. Note that when all a_i are odd and all e_i are even then $n = \sum a_i e_i$ is either equal to $\sum e_i$, or exceeds it by at least 4, so $q_{e_1+\ldots+e_k} \mid q_n$ and the claimed expression for $[Z_\lambda]$ is well-defined.

We resolve Z_λ birationally with the map

$$\Psi : \prod_{i=1}^{k} \mathbb{P}^{e_i} \to \mathbb{P}^n$$

taking $(D_1, \ldots, D_k) \mapsto a_1 D_1 + \ldots + a_k D_k$ (treating $P^r = \mathrm{Sym}^r \mathbb{P}^1$ for all r).

If at least one e_i is odd, then we claim $c_3[Z_\lambda] = 0$. Indeed,

$$c_3[Z_\lambda] = \Psi_* c_3,$$

and $c_3 \in A^\bullet_{PGL_2}(\mathrm{pt})$ maps to 0 in $A^\bullet_{PGL_2}(\prod_{i=1}^k \mathbb{P}^{e_i})$ as the projection $\prod_{i=1}^k \mathbb{P}^{e_i} \to \mathrm{pt}$ can be factored as the composite $\prod_{i=1}^k \mathbb{P}^{e_i} \to \mathbb{P}^{e_i} \to \mathrm{pt}$, and if e_i is odd then c_3 pulls back to zero in $A^\bullet_{PGL_2}(\mathbb{P}^{e_i})$ by Proposition 13.7.

Hence, as $c_3[Z_\lambda] = 0$, we must have $[Z_\lambda]$ is zero in $A^\bullet_{PGL_2}(\mathbb{P}^n) \otimes \mathbb{Z}/2\mathbb{Z}$.

Now, suppose that all e_i are even. This means each $\mathbb{P}^{e_i}$ is the projectivization of a PGL_2-representation with Chern classes given as the coefficients of $p_{e_i}(t)$, so we have the Chow ring

$$A^\bullet_{PGL_2}(\prod_{i=1}^{k} \mathbb{P}^{e_i}) \cong A^\bullet_{PGL_2}(\mathrm{pt})[H_1, \ldots, H_k]/(p_{e_1}(H_1), \ldots, p_{e_k}(H_k))$$

by repeatedly applying the projective bundle formula.

For the remainder of the proof all integrals are in Chow rings after tensoring with $\mathbb{Z}/2\mathbb{Z}$, so each $p_r(t)$ gets replaced with $q_r(t)$. By Lemma 17.4, it suffices to show

$$\int_{\mathbb{P}^n} \frac{q_n(t) - q_n(H)}{t - H} \cap \Psi_* 1 = \begin{cases} \frac{q_n}{q_d}(t) & \text{if all } a_i \text{ and } \frac{d!}{e_1! \dots e_k!} \text{ are odd and} \\ 0 & \text{otherwise.} \end{cases}$$

By the projection formula applied to Ψ, we have

$$\int_{\mathbb{P}^n} \frac{q_n(t) - q_n(H)}{t - H} \cap \Psi_* 1 = \int_{\prod_{i=1}^k \mathbb{P}^{e_i}} \frac{q_n(t) - q_n(\sum a_i H_i)}{t - \sum a_i H_i}.$$

Now, if any a_i is even, then as we are working modulo 2, $\frac{q_n(t) - q_n(\sum a_i H_i)}{t - \sum a_i H_i}$ will not contain H_i, so the integral is clearly zero. Hence we may assume from now on that all a_i are odd, so that $\sum a_i H_i = \sum H_i \bmod 2$.

We claim that $q_d(\sum H_i) = 0$ and that

$$\int_{\prod_{i=1}^k \mathbb{P}^{e_i}} \frac{q_d(t) - q_d(\sum H_i)}{t - \sum H_i} = \frac{d!}{e_1! \cdots e_k!}.$$

The first of these follows from pulling back $q_d(H)$ under the multiplication map $\prod_{i=1}^k \mathbb{P}^{e_i} \to \mathbb{P}^d$, and the second of these follows from applying Lemma 17.4 to $1 \in A^\bullet_{PGL_2}(\mathbb{P}^d)$ together with the projection formula as the multiplication map has degree $\frac{d!}{e_1! \cdots e_k!}$.

From the vanishing of $q_d(\sum H_i)$, we have

$$\begin{aligned} \frac{q_n(t) - q_n(\sum H_i)}{t - \sum H_i} &= \frac{q_n}{q_d}(t) \frac{q_d(t) - q_d(\sum H_i)}{t - \sum H_i} + q_d\left(\sum H_i\right) \frac{\frac{q_n(t)}{q_d(t)} - \frac{q_n(\sum H_i)}{q_d(\sum H_i)}}{t - \sum H_i} \\ &= \frac{q_n}{q_d}(t) \frac{q_d(t) - q_d(\sum H_i)}{t - \sum H_i}, \end{aligned}$$

and the result now follows from the second claim after applying $\int_{\prod_{i=1}^k \mathbb{P}^{e_i}}$ to both sides. $\square$

We now prove surprisingly that despite the presence of occasional 2-torsion, integral relations between $[Z_\lambda]$ classes in $A^\bullet_{GL_2}(\mathbb{P}^n)$ are equivalent to integral relations between $[Z_\lambda]$-classes in $A^\bullet_{PGL_2}(\mathbb{P}^n)$.

Theorem 17.6. *Let n, d be integers. A linear combination $\sum a_\lambda[Z_\lambda]$ with $a_\lambda \in \mathbb{Z}$ and each λ a partition of n into d parts is zero in $A^\bullet_{PGL_2}(\mathbb{P}^n)$ if and only if it is zero in $A^\bullet_{GL_2}(\mathbb{P}^n)$. In particular, $\sum a_\lambda[Z_\lambda] = 0$ if and only if*

$$\sum_{\lambda=a_1^{e_1}\dots a_n^{e_k}} a_\lambda \prod_{i=1}^{k} \frac{(z^{a_i}-1)^{e_i}}{e_i!} = 0.$$

Proof. One direction is trivial, as we have the map $A^\bullet_{PGL_2}(\mathbb{P}^n) \to A^\bullet_{GL_2}(\mathbb{P}^n)$ induced by $GL_2 \to PGL_2$, so if a linear relation holds in $A^\bullet_{PGL_2}(\mathbb{P}^n)$, then it also holds in $A^\bullet_{GL_2}(\mathbb{P}^n)$. Conversely, suppose that we have $\sum a_\lambda[Z_\lambda] = 0$ in $A^\bullet_{GL_2}(\mathbb{P}^n)$. We only have to care about the case that n is even, because when n is odd, $A^\bullet_{PGL_2}(\mathbb{P}^n) \hookrightarrow A^\bullet_{GL_2}(\mathbb{P}^n)$ is an injection by Proposition 13.7.

For n even, suppose we have a sum $\sum a_\lambda[Z_\lambda]$, which is 0 in $A^\bullet_{GL_2}(\mathbb{P}^n)$. Then since the kernel of $A^\bullet_{PGL_2}(\mathbb{P}^n) \to A^\bullet_{GL_2}(\mathbb{P}^n)$ is 2-torsion by Proposition 13.7, we know $\sum a_\lambda[Z_\lambda]$ is 2-torsion in $A^\bullet_{PGL_2}(\mathbb{P}^n)$. By Theorem 17.2, the class $[Z_\lambda]$ in $A^\bullet_{PGL_2}(\mathbb{P}^n) \otimes \mathbb{Z}/2\mathbb{Z}$ is either 0 or $\frac{q_n}{q_d}(H)$, and the second possibility occurs precisely when λ is special. Hence to prove Theorem 17.6, by Theorem 14.5 and Lemma 16.3 it suffices to show that if

$$\sum_{\lambda=a_1^{e_1}\dots a_n^{e_k}} a_\lambda \prod_{i=1}^{k} \frac{(z^{a_i}-1)^{e_i}}{e_i!} = 0, \tag{17.1}$$

then

$$\sum_{\lambda \text{ special}} a_\lambda \equiv 0 \pmod 2.$$

Note first that if no special λ appears we are done, so we may assume that at least one special λ appears. As $d = \sum_{i=1}^{k} e_i$ for any partition $\lambda = a_1^{e_1} \dots a_n^{e_k}$ appearing, we must have d is even if a special λ appears. Multiplying (17.1) by $\frac{d!}{(z-1)^d}$ and plugging in $z = 1$, we have

$$\sum_{\lambda = a_1^{e_1} \ldots a_n^{e_n}} a_\lambda \frac{d!}{e_1! \cdots e_k!} \prod_{i=1}^{k} a_i{}^{e_i} = 0.$$

Now we claim that $\frac{d!}{e_1! \cdots e_k!}$ is even if any e_i is odd. Indeed, as d is even, if not all e_i are even, then at least two of the e_i are odd. If e_i, e_j are both odd, then replacing $e_i!e_j!$ in $\frac{d!}{e_1! \cdots e_k!}$ with $(e_i - 1)!(e_j + 1)!$ yields an integer with a smaller power of 2 dividing it.

Hence, $\frac{d!}{e_1! \cdots e_k!} \prod_{i=1}^{k} a_i{}^{e_i}$ is odd precisely when λ is special. Taking the equality (mod 2) then yields the desired result. $\square$

We complete the proof of Theorem 11.8.

Proof of Theorem 11.8. We have (1), (2) and (4) are equivalent by Theorem 17.6. Also (3) implies (2) is clear as $A^\bullet_{GL_2}(\mathbb{P}^n)$ is free as an abelian group, so $A^\bullet_{GL_2}(\mathbb{P}^n) \hookrightarrow A^\bullet_{GL_2}(\mathbb{P}^n) \otimes \mathbb{Q}$.

To finish, it suffices to show (2) implies (3). Let $\lambda = (\lambda_1, \ldots, \lambda_d)$ for $\lambda_1 \geq \cdots \geq \lambda_d$.

Claim. Suppose $\lambda_3 > 1$. Then using pushforwards of square relations in $A^\bullet_{PGL_2}((\mathbb{P}^1)^n)$, we can express $[\lambda] \in A^\bullet_{PGL_2}(\mathbb{P}^n)$ in terms of classes $[\lambda']$ where $\lambda' = (\lambda'_1, \ldots, \lambda'_d)$ where $\lambda_1 + \lambda_2 > \lambda'_1 + \lambda'_2$.

Proof of Claim. Pick a partition $P = \{A_1, \ldots, A_d\}$ of $[n]$ with $|A_i| = \lambda_i$. Since $|A_3| > 1$, we can partition it as $A_3 = A'_3 \sqcup A''_3$ into nonempty parts. Now, applying the square relation associated to $P' = \{A_1, A_2, A'_3, A''_3, \ldots, A_d\}$ of $[n]$ into $d+1$ parts and the parts A_1, A_2, A_3, A''_3 shows

$$[\lambda] = [\lambda_1] + [\lambda_2] - [\lambda_3],$$

where $\lambda_3' = |A_3'|$ and $\lambda_3'' = |A_3''|$ and

$$\lambda_1 = \{\lambda_1 + \lambda_3', \lambda_2, \lambda_3'', \ldots, \lambda_d\}$$

$$\lambda_2 = \{\lambda_1, \lambda_2 + \lambda_3'', \lambda_3', \ldots, \lambda_d\}$$

$$\lambda_3 = \{\lambda_1 + \lambda_2, \lambda_3', \lambda_3'', \ldots, \lambda_d\}.$$

□

Returning to the proof of Theorem 11.8, iterating the claim shows that the push-forward of square relations allow us to rewrite any $[\lambda]$ in terms of the $\mathbb{Q}$-basis found in Theorem 16.4, which shows (2) implies (3). □

18. Excision of unordered strata in $[\mathrm{Sym}^n\mathbb{P}^1/PGL_2]$

As an application of our results in the ordered case, we will prove the following result on the PGL_2-equivariant Chow ring of $\mathbb{P}^n$ with strata excised, which we will adapt in the next section to the case of GL_2-equivariant Chow rings with strata in both $\mathbb{P}^n$ and in $\mathbb{A}^{n+1}$.

Theorem 18.1. *Given a partition $\lambda = \{\lambda_1, \ldots, \lambda_d\}$ of n,*

$$A^\bullet_{PGL_2}(\mathbb{P}^n \setminus Z_\lambda) = A^\bullet_{PGL_2}(\mathbb{P}^n)/I,$$

where the ideal $I \otimes \mathbb{Q} \subset A^\bullet_{PGL_2}(\mathbb{P}^n) \otimes \mathbb{Q}$ is generated by all $[\lambda']$ for λ' a partition formed by merging some of the parts of λ.

Even though Theorem 18.1 requires many generators for I, in some cases fewer generators suffice.

Theorem 18.2. *Given the partition* $\lambda = \{a, 1^{n-a}\}$ *of* n*, the ideal* $I \otimes \mathbb{Q}$ *in Theorem 18.1 is generated by* $[\lambda]$ *and* $[\lambda']$*, where*

$$\lambda' = \begin{cases} \{a+1, 1^{n-a-1}\} & \text{if } a \neq \frac{n}{2} \\ \{a, 2, 1^{n-a-2}\} & \text{if } a = \frac{n}{2}. \end{cases}$$

See Remark 19.3 for the connection to similar results proved in [42].

By the excision exact sequence [50, Proposition 1.8], the ideal I is the same as the pushforward ideal I_λ which we define in Definition 18.3.

Definition 18.3. *Given a partition* λ *of* n *and for* $G = PGL_2$ *or* GL_2*, let* I_λ^G *be the ideal of* $A_G^\bullet(\mathbb{P}^n)$ *given by the pushforward via the inclusion* $\iota_\lambda : Z_\lambda \hookrightarrow \mathbb{P}^n$

$$I_\lambda^G = (\iota_\lambda)_* A_\bullet^G(Z_\lambda) \subset A_\bullet^G(\mathbb{P}^n)$$

and the identification $A_\bullet^G(\mathbb{P}^n) \cong A_G^{n-\bullet}(\mathbb{P}^n)$ *via Poincaré duality* [27, Proposition 4]. *When* G *is clear from context we will simply write* I_λ*.*

Since Z_λ is possibly singular, we will want to instead work with a desingularization (as was done in [42]).

Definition 18.4. *Given a partition* $\lambda = \{\lambda_1, \ldots, \lambda_d\}$ *of* n*, let* $e_i^\lambda = \#\{j \mid \lambda_j = i\}$ *and* $Y_\lambda = \prod_{i=1}^n \mathbb{P}^{e_i^\lambda}$*. We have a map*

$$\hat{\iota}_\lambda : Y_\lambda \to \mathbb{P}^n$$

that is birational onto its image Z_λ *given by the composition*

$$Y_\lambda \hookrightarrow \prod_{i=1}^n \mathbb{P}^{ie_i^\lambda} \to \mathbb{P}^n$$

of the ith power map on each factor $\mathbb{P}^{e_i}$ together with the multiplication map. Equivalently, if we view projective space $\mathbb{P}^n$ as parameterizing degree n divisors on $\mathbb{P}^1$, then the map is given by $(D_1, \ldots, D_n) \mapsto \sum_{i=1}^n iD_i$.

In particular, I_λ is also given by the image of $(\hat{\iota}_\lambda)_*$. Since we are working rationally, we can take a finite cover of Y_λ.

Definition 18.5. *Given a partition $\lambda = \{\lambda_1, \ldots, \lambda_d\}$ of n, define the finite map $\Phi_\lambda : (\mathbb{P}^1)^d \to Y_\lambda$ to be*

$$\Phi_\lambda : (\mathbb{P}^1)^d = \prod_{i=1}^n (\mathbb{P}^1)^{e_i^\lambda} \to \prod_{i=1}^n \mathbb{P}^{e_i^\lambda} = Y_\lambda$$

given by the multiplication map $(\mathbb{P}^1)^{e_i^\lambda} \to \mathbb{P}^{e_i^\lambda}$ on each factor.

Since Φ_λ is finite,

$$(\Phi_\lambda)_* : A^\bullet_{PGL_2}((\mathbb{P}^1)^d) \otimes \mathbb{Q} \to A^\bullet_{PGL_2}(Y_\lambda) \otimes \mathbb{Q}$$

is surjective, so $I_\lambda \otimes \mathbb{Q}$ is the image of

$$(\hat{\iota}_\lambda \circ \Phi_\lambda)_* : A^\bullet_{PGL_2}((\mathbb{P}^1)^d) \otimes \mathbb{Q} \to A^\bullet_{PGL_2}((\mathbb{P}^1)^n) \otimes \mathbb{Q}.$$

The map Φ_λ has the nice property that given a partition P of $[d]$, the pushforward of the strata $(\hat{\iota}_\lambda \circ \Phi_\lambda)_* \Delta_P$ is $[\lambda']$, where λ' is the partition of n given by merging the parts of λ according to the partition P. From this, we will be able to deduce certain symmetrized strata generate $I_\lambda \otimes \mathbb{Q}$ based on the generation properties of strata in $(\mathbb{P}^1)^d$.

Definition 18.6. *Given a set of partitions $\mathcal{P}$ of $[d]$ and $G = PGL_2$ or GL_2, let $\Lambda^G_{\mathcal{P}} \subset A^\bullet_G((\mathbb{P}^1)^d) \otimes \mathbb{Q}$ be the submodule over $A^\bullet_G(\mathbb{P}^n) \otimes \mathbb{Q}$ generated by the classes Δ_P. Explicitly,*

$$\Lambda^G_{\mathcal{P}} = \sum_{P \in \mathcal{P}} \Delta_P \cap \Phi^*_\lambda \hat{\iota}^*_\lambda (A^\bullet_G(\mathbb{P}^n) \otimes \mathbb{Q}).$$

When G is clear from context we will notate $\Lambda_{\mathcal{P}}^{G}$ simply by $\Lambda_{\mathcal{P}}$.

Lemma 18.7. *Let $\lambda = \{\lambda_1, \ldots, \lambda_d\}$ be a partition of n, and let $G = PGL_2$ or GL_2. Suppose we have a collection of partitions $\mathcal{P}$ of $[d]$ such that in $A_G^\bullet((\mathbb{P}^1)^d) \otimes \mathbb{Q}$*

$$A_G^\bullet((\mathbb{P}^1)^d)^{\prod_{i=1}^n S_{e_i^\lambda}} \otimes \mathbb{Q} \subset \Lambda_{\mathcal{P}}^G.$$

Then $\{(\hat{\iota}_\lambda \circ \Phi_\lambda)_* \Delta_P \mid P \in \mathcal{P}\}$ generates $I_\lambda^G \otimes \mathbb{Q} \subset A_G^\bullet(\mathbb{P}^n) \otimes \mathbb{Q}$.

Proof. Since

$$\Phi_\lambda^*(A_G^\bullet(Y_\lambda) \otimes \mathbb{Q}) \subset A_G^\bullet((\mathbb{P}^1)^d)^{\prod_{i=1}^n S_{e_i^\lambda}} \otimes \mathbb{Q} \subset \Lambda_{\mathcal{P}}^G,$$

we have

$$(\Phi_\lambda)_* \Lambda_{\mathcal{P}}^G \supset (\Phi_\lambda)_*(\Phi_\lambda^*(A_G^\bullet(Y_\lambda)) \otimes \mathbb{Q}) = A_G^\bullet(Y_\lambda) \otimes \mathbb{Q}$$

and by the projection formula, $(\Phi_\lambda)_* \Lambda_{\mathcal{P}}^G$ is

$$(\Phi_\lambda)_* \sum_{P \in \mathcal{P}} \Delta_P \cap \Phi_\lambda^* \hat{\iota}_\lambda^*(A_G^\bullet(\mathbb{P}^n) \otimes \mathbb{Q}) = \sum_{P \in \mathcal{P}} (\Phi_\lambda)_* \Delta_P \cap \hat{\iota}_\lambda^*(A_G^\bullet(\mathbb{P}^n) \otimes \mathbb{Q}).$$

By the projection formula again, we thus have

$$I_\lambda^G \otimes \mathbb{Q} = (\hat{\iota}_\lambda)_*(A_G^\bullet(Y_\lambda) \otimes \mathbb{Q}) = \sum_{P \in \mathcal{P}} (\hat{\iota}_\lambda \circ \Phi_\lambda)_* \Delta_P \cap A_G^\bullet(\mathbb{P}^n) \otimes \mathbb{Q}$$

as desired. $\square$

Lemma 18.8. *Let $\lambda = \{\lambda_1, \ldots, \lambda_d\}$ be a partition of $[n]$ and $\mathcal{P}$ be all partitions of $[d]$. Then*

$$\Lambda_{\mathcal{P}}^{PGL_2} = \begin{cases} A_{PGL_2}^\bullet((\mathbb{P}^1)^2)^{S_2} \otimes \mathbb{Q} & \text{if } d = 2 \text{ and } \lambda_1 = \lambda_2, \text{ and} \\ A_{PGL_2}^\bullet((\mathbb{P}^1)^d) \otimes \mathbb{Q} & \text{otherwise.} \end{cases}$$

In particular, given a partition $\lambda = \{\lambda_1, \ldots, \lambda_d\}$ of n, $I_\lambda^{PGL_2} \otimes \mathbb{Q}$ is generated by all $[\lambda']$ with λ' formed by merging parts of λ.

Proof. Given the description of $\Lambda_{\mathcal{P}}^{PGL_2}$, the result about $I_\lambda^{PGL_2} \otimes \mathbb{Q}$ follows directly from Lemma 18.7. We will now show the description of $\Lambda_{\mathcal{P}}^{PGL_2}$.

We may identify $A^\bullet_{PGL_2}(\mathbb{P}^n) \otimes \mathbb{Q} \subset A^\bullet_{GL_2}(\mathbb{P}^n) \otimes \mathbb{Q}$ as the subring generated by $H + \frac{n}{2}(u+v)$ and $(u-v)^2$ by Proposition 13.7. Define

$$H_i' = H_i + \frac{1}{2}(u+v) \quad \text{and} \quad H' = H + \frac{n}{2}(u+v).$$

Note that with these definitions, we have

$$\Phi_\lambda^* \hat{\iota}_\lambda^*(H') = \sum \lambda_i H_i', \qquad H_i'^2 = \frac{1}{4}(u-v)^2.$$

We have the $\mathbb{Q}$-linear span

$$\Lambda_{\mathcal{P}} = \mathrm{Span}_{\mathbb{Q}}(\{\Delta_P (u-v)^{2k} (\sum \lambda_i H_i')^\ell \mid k, l \geq 0, P \in \mathcal{P}\}).$$

The trivial partition is in $\mathcal{P}$, so 1 is automatically in $\Lambda_{\mathcal{P}}$.

Recall by Proposition 14.4 that

$$\Delta_{i,j} = H_i' + H_j',$$

and that $A^\bullet_{PGL_2}((\mathbb{P}^1)^d) \otimes \mathbb{Q}$ is generated by the H_i' and $(u-v)^2$. As $H_i'^2 = \frac{1}{4}(u-v)^2$, to show $\Lambda_{\mathcal{P}} = A^\bullet_{PGL_2}((\mathbb{P}^1)^d) \otimes \mathbb{Q}$ it suffices to show that every monomial $\prod_{i \in C} H_i'$ is in $\Lambda_{\mathcal{P}}$ for $C \subset [n]$.

For $d = 1$, $\lambda = \{[n]\}$, we are done as $H_1' = \frac{1}{\lambda_1} \Phi_\lambda^* \iota_\lambda^* H'$.

For $d = 2$ and $\lambda_1 \neq \lambda_2$,

$$\begin{aligned} H_1' &= \frac{1}{\lambda_1 - \lambda_2}(\Phi_\lambda^* \hat{\iota}_\lambda^*(H') - \lambda_2 \Delta_{1,2}) \\ H_2' &= \frac{1}{\lambda_2 - \lambda_1}(\Phi_\lambda^* \hat{\iota}_\lambda^*(H') - \lambda_1 \Delta_{1,2}) \\ H_1' H_2' &= \frac{1}{2\lambda_1 \lambda_2}(\Phi_\lambda^* \hat{\iota}_\lambda^*(H')^2 - \frac{1}{4}(\lambda_1^2 + \lambda_2^2)(u-v)^2). \end{aligned}$$

For $d = 2$ and $\lambda_1 = \lambda_2 = a$, we have to show $\Lambda_{\mathcal{P}} = A^{\bullet}_{PGL_2}((\mathbb{P}^1)^2)^{S_2} \otimes \mathbb{Q}$. As $H_i'^2 = \frac{1}{4}(u-v)^2$, it suffices to show $1, H_1' + H_2'$ and $H_1'H_2'$ are in $\Lambda_{\mathcal{P}}$. We already know that $1 \in \Lambda_{\mathcal{P}}$, and

$$\begin{aligned} H_1' + H_2' &= \frac{1}{a}\Phi_\lambda^* \hat{\iota}_\lambda^* H', \\ H_1'H_2' &= \frac{1}{2a^2}(\Phi_\lambda^* \hat{\iota}_\lambda^* (H')^2 - \frac{1}{2}a^2(u-v)^2). \end{aligned}$$

We will now show that $\Lambda_{\mathcal{P}} = A^{\bullet}_{PGL_2}((\mathbb{P}^1)^d) \otimes \mathbb{Q}$ when $d \geq 3$.

Up to degree $d-2$, we can take $k, \ell = 0$ as the classes Δ_P for $P \in \mathcal{P}$ generate $A^{\leq d-2}_{PGL_2}((\mathbb{P}^1)^d)$ by Lemma 15.12. Hence to conclude the proof of Lemma 18.8, it suffices to show that $\prod_{k\neq i} H_k'$ for all i and $\prod H_k'$ are in $\Lambda_{\mathcal{P}}$.

For $\prod_{k\neq i} H_k$, without loss of generality suppose $i = 1$. We have each of

$$\begin{aligned} \frac{1}{a_1a_2}(\prod_{k\neq 1,2} H_k') \cap \Phi_\lambda^* \hat{\iota}_\lambda^* H' &= \frac{1}{a_1}\prod_{k\neq 1} H_k' + \frac{1}{a_2}\prod_{k\neq 2} H_k' + \frac{1}{4a_1a_2}(u-v)^2 \sum_{j\neq 1,2} a_j \prod_{k\neq 1,2,j} H_k' \\ \frac{1}{a_1a_3}(\prod_{k\neq 1,3} H_k') \cap \Phi_\lambda^* \hat{\iota}_\lambda^* H &= \frac{1}{a_1}\prod_{k\neq 1} H_k + \frac{1}{a_3}\prod_{k\neq 3} H_k + \frac{1}{4a_2a_3}(u-v)^2 \sum_{j\neq 2,3} a_j \prod_{k\neq 2,3,j} H_k' \\ \frac{1}{a_2a_3}(\prod_{k\neq 2,3} H_k') \cap \Phi_\lambda^* \hat{\iota}_\lambda^* H &= \frac{1}{a_2}\prod_{k\neq 2} H_k + \frac{1}{a_3}\prod_{k\neq 3} H_k + \frac{1}{4a_1a_3}(u-v)^2 \sum_{j\neq 1,3} a_j \prod_{k\neq 1,3,j} H_k' \end{aligned}$$

lie in $\Lambda_{\mathcal{P}}$ as we have already shown each $\prod_{k\neq i,j} H_k$ lies in Λ. Also, the last term on each right hand side lies in $\Lambda_{\mathcal{P}}$ as the number of terms in the H_k' monomial is $d-3$. Hence taking a linear combination we get $\prod_{k\neq 1} H_k' \in \Lambda_{\mathcal{P}}$.

To show $\prod_{i=1}^n H_i' \in \Lambda_{\mathcal{P}}$, we can proceed similarly to above, or expand

$$\frac{1}{a_1 \ldots a_n}\Phi_\lambda^* \hat{\iota}_\lambda^* (H')^d = \prod H_i' + (u-v)^2 \,(\text{lower order terms in the } H_i')\,,$$

using $H_i'^2 = \frac{1}{4}(u-v)^2$. $\square$

Proof of Theorem 18.1. This follows from the excision exact sequence [50, Proposition 1.8] and Lemma 18.8. $\square$

Lemma 18.9. *Let $\lambda = \{a, 1^b\}$ be a partition of n. Define $\mathcal{P}_\lambda$ to be the set of partitions*

$$\mathcal{P}_\lambda = \{T\} \sqcup \begin{cases} \{T_{1,i}\}_{i \geq 2} & a \neq b \\ \{T_{i,j}\}_{2 \leq i < j \leq n} & a{=}b, \end{cases}$$

where T is the trivial partition and $T_{i,j}$ is the partition with $n-1$ parts and i, j in the same part. Then

$$\Lambda_{\mathcal{P}_\lambda}^{PGL_2} = A^\bullet_{PGL_2}((\mathbb{P}^1)^{b+1})^{S_1 \times S_b} \otimes \mathbb{Q}.$$

Proof. Define

$$H' = H + \frac{n}{2}(u+v) \quad \text{and} \quad H_i' = H_i + \frac{1}{2}(u+v).$$

Then in particular,

$$\Delta_{i,j} = H_i' + H_j'$$

$$\Phi_\lambda^* \hat{\iota}_\lambda^*(H') = aH_1' + H_2' + \ldots + H_{b+1}',$$

so $\Lambda_{\mathcal{P}_\lambda}$ is the $\mathbb{Q}$-linear span

$$\Lambda_{\mathcal{P}_\lambda} = \mathrm{Span}_{\mathbb{Q}}\{\Delta_P (u-v)^{2k}(aH_1' + H_2' + \ldots + H_{b+1}')^\ell \mid k, \ell \geq 0,\ P \in \mathcal{P}_\lambda\}.$$

We first show that $H_1' \in \Lambda_{\mathcal{P}_\lambda}$. Consider the case $b \neq a$. Then

$$H_1' = \frac{1}{a-b}\left(\Phi_\lambda^* \hat{\iota}_\lambda^*(H') - \sum_{i \geq 2} \Delta_{1,i}\right) \in \Lambda_{\mathcal{P}_\lambda}.$$

Now consider the case $b = a$. Then

$$H_1' = \frac{1}{a}\left(\Phi_\lambda^* \hat{\iota}_\lambda^*(H') - \frac{1}{a-1}\sum_{2 \leq i < j \leq a+1} \Delta_{i,j}\right) \in \Lambda_{\mathcal{P}_\lambda}.$$

Now that we have shown that $H_1' \in \Lambda$, it therefore suffices to show that the invariant subring $A^{\bullet}_{PGL_2}((\mathbb{P}^1)^{b+1})^{S_1\times S_b}$ is given by

$$\mathrm{Span}_{\mathbb{Q}}\{(u-v)^{2k}(aH_1'+H_2'+\ldots+H_{b+1}')^{\ell},\ H_1'(u-v)^{2k}(aH_1'+H_2'+\ldots+H_{b+1}')^{\ell} \mid k,\ell\geq 0\}$$

Note that by using the relation $H_1'^2 = \frac{1}{4}(u-v)^2$, we see this is the same as

$$\begin{aligned}
&\mathrm{Span}_{\mathbb{Q}}\{H_1'^k(aH_1'+H_2'+\ldots+H_{b+1}')^{\ell} \mid k,\ell\geq 0\}\\
=&\mathrm{Span}_{\mathbb{Q}}\{H_1'^k(H_2'+\ldots+H_{b+1}')^{\ell} \mid k,\ell\geq 0\}\\
=&\mathrm{Span}_{\mathbb{Q}}\{(u-v)^{2k}(H_1'+H_2'+\ldots+H_{b+1}')^{\ell},\\
&H_1'(u-v)^{2k}(H_1'+H_2'+\ldots+H_{b+1}')^{\ell} \mid k,\ell\geq 0\}.
\end{aligned}$$

By using the relations $H_i'^2 = \frac{1}{4}(u-v)^2$ whenever possible, we see that an element of the invariant subring is a sum of terms of the form $(u-v)^{2k}e_j(H_2',\ldots,H_{b+1}')$ and $(u-v)^{2k}H_1'e_j(H_2',\ldots,H_{b+1}')$ where e_j is the jth elementary symmetric polynomial, hence it suffices to show that

$$\mathrm{Span}_{\mathbb{Q}}\{(u-v)^{2k}e_j(H_2',\ldots,H_{b+1}') \mid j,k\geq 0\} \subset \mathrm{Span}_{\mathbb{Q}}\{(u-v)^{2k}(H_2'+\ldots+H_{b+1}')^{\ell} \mid k,\ell\geq 0\}.$$

This follows by induction on j and the relation

$$\begin{aligned}
&e_j(H_2',\ldots,H_{b+1}')(H_2'+\ldots+H_{b+1}')\\
&=(j+1)e_{j+1}(H_2'+\ldots+H_{b+1}')+\frac{1}{4}(u-v)^2(n-j+1)e_{j-1}(H_2',\ldots,H_{b+1}').
\end{aligned}$$

□

Proof of Theorem 18.2. This follows from the excision exact sequence [50][Proposition 1.8], Lemma 18.7, and Lemma 18.9. □

19. Excision of unordered strata in $[\mathrm{Sym}^n\mathbb{P}^1/GL_2]$ and $[\mathrm{Sym}^n K^2/GL_2]$

In this section, we show how our results about excision of unordered strata in $[\mathrm{Sym}^n\mathbb{P}^1/PGL_2]$ imply similar results in $[\mathrm{Sym}^n\mathbb{P}^1/GL_2]$ and $[\mathrm{Sym}^n K^2/GL_2]$, recovering and extending some results of [42] (see Remark 19.3).

Definition 19.1. *Given a partition λ of n, let $\widetilde{I}_\lambda$ be the ideal of $A^\bullet_{GL_2}(\mathbb{A}^{n+1})$ given by the image of the pushforward $A^{GL_2}_\bullet(\widetilde{Z_\lambda}) \hookrightarrow A^{GL_2}_\bullet(\mathbb{A}^{n+1})$ and the identification $A^{GL_2}_\bullet(\mathbb{A}^{n+1}) \cong A^{n+1-\bullet}_{GL_2}(\mathbb{A}^{n+1})$ via Poincaré duality* [27, Proposition 4].

Theorem 19.2. *$I^{GL_2}_\lambda \otimes \mathbb{Q}$ (respectively $\widetilde{I}_\lambda \otimes \mathbb{Q}$) is generated by all $[Z_{\lambda'}]$ (respectively $[\widetilde{Z}_{\lambda'}]$) with λ' formed by merging parts of λ. For $\lambda = \{a, 1^{n-a}\}$ only two generators are required, namely $[Z_\lambda]$ (respectively $[\widetilde{Z}_\lambda]$) and $[Z_{\lambda'}]$ (respectively $[\widetilde{Z}_{\lambda'}]$) where*

$$\lambda' = \begin{cases} \{a+1, 1^{n-a-1}\} & \text{if } a \neq \frac{n}{2} \\ \{a, 2, 1^{n-a-2}\} & \text{if } a = \frac{n}{2}. \end{cases}$$

Remark 19.3. In the affine case, when n is odd and $a = \lceil \frac{n}{2} \rceil$ this recovers [42, Theorem 4.3], and when n is even and $a = \frac{n}{2}$ this recovers the rational Chow ring of the stable locus in [42, Theorem 4.10].

Lemma 19.4. *We have*

$$\mathbb{Q}[u,v]^{S_2}\left(A^\bullet_{PGL_2}((\mathbb{P}^1)^d) \otimes \mathbb{Q}\right)^{S_{e_1}\times\ldots\times S_{e_k}} = \left(A^\bullet_{GL_2}((\mathbb{P}^1)^d) \otimes \mathbb{Q}\right)^{S_{e_1}\times\ldots\times S_{e_k}} \text{ and}$$

$$\mathbb{Q}[u,v]^{S_2}\left(A^\bullet_{PGL_2}(\mathbb{P}^n) \otimes \mathbb{Q}\right) = A^\bullet_{GL_2}(\mathbb{P}^n) \otimes \mathbb{Q}.$$

In particular, if a set of partitions $\mathcal{P}$ satisfies the hypotheses of Lemma 18.7 for $G = PGL_2$, then they also satisfy the hypotheses of Lemma 18.7 for $G = GL_2$.

Proof. We identify $A^\bullet_{PGL_2}((\mathbb{P}^1)^d) \otimes \mathbb{Q}$ as the subring of $A^\bullet_{GL_2}((\mathbb{P}^1)^d) \otimes \mathbb{Q}$ via Proposition 13.7 generated by $H' := H + \frac{n}{2}(u+v)$ and $(u-v)^2$. Since $A^\bullet_{GL_2}((\mathbb{P}^1)^d) \otimes$

$\mathbb{Q}$ is generated by H' over $\mathbb{Q}[u,v]^{S_2}$, and $(u-v)^2$ and $u+v$ generate $\mathbb{Q}[u,v]^{S_2}$, $\mathbb{Q}[u,v]^{S_2}\left(A^\bullet_{PGL_2}(\mathbb{P}^n)\otimes\mathbb{Q}\right) = A^\bullet_{GL_2}(\mathbb{P}^n)\otimes\mathbb{Q}$.

For the other equality, we use Theorem 13.3 to identify $A^\bullet_{PGL_2}(\mathbb{P}^n)\otimes\mathbb{Q}$ as the subring of $A^\bullet_{GL_2}(\mathbb{P}^n)\otimes\mathbb{Q}$ generated by $H_i' := H_i+\frac{u+v}{2}$. Then, $\left(A^\bullet_{PGL_2}((\mathbb{P}^1)^d)\otimes\mathbb{Q}\right)^{S_{e_1}\times\ldots\times S_{e_k}}$ is generated $\mathbb{Z}$-linearly by all $p(H_1',\ldots,H_n')$, where p is a polynomial invariant under the action of $S_{e_1}\times\cdots\times S_{e_k}$. Similarly, $\left(A^\bullet_{GL_2}((\mathbb{P}^1)^d)\otimes\mathbb{Q}\right)^{S_{e_1}\times\ldots\times S_{e_k}}$ is generated by all such $p(H_1',\ldots,H_n')$, together with $u+v$ and uv. Therefore,

$$\mathbb{Q}[u,v]^{S_2}\left(A^\bullet_{PGL_2}((\mathbb{P}^1)^d)\otimes\mathbb{Q}\right)^{S_{e_1}\times\ldots\times S_{e_k}} = \left(A^\bullet_{GL_2}((\mathbb{P}^1)^d)\otimes\mathbb{Q}\right)^{S_{e_1}\times\ldots\times S_{e_k}}.$$

□

As we will now see, the cones over generators of $I_\lambda^{GL_2}\otimes\mathbb{Q}$ also generate $\widetilde{I}_\lambda\otimes\mathbb{Q}$. We will use a certain property about the classes of unordered strata to prove this, which as we will see is that Z_λ contains a cycle whose class divides the class of the origin in $A^\bullet_{GL_2}(\mathbb{A}^{n+1})\otimes\mathbb{Q}$.

Lemma 19.5. *Given a partition λ of n and a set of generators S of $I_\lambda^{GL_2}\otimes\mathbb{Q}$ of degree at most n, $\widetilde{I}_\lambda\otimes\mathbb{Q}$ is generated by*

$$\{\alpha_0 \mid \alpha\in S\},$$

where α_0 is the constant term of $\alpha\in A^\bullet_{GL_2}(\mathbb{P}^n)\otimes\mathbb{Q}$, after writing α as a polynomial in H,u,v that is degree at most n in H using the relation $G(H)=0$ (see Section 12.3).

Proof. Let $\widetilde{I}_\lambda'\subset A^\bullet_{GL_2}(\mathbb{A}^{n+1})\otimes\mathbb{Q}$ be the ideal generated by $\{\alpha_0\mid\alpha\in S\}$, so we want to show $\widetilde{I}_\lambda'=\widetilde{I}_\lambda\otimes\mathbb{Q}$. Consider the diagram of rational Chow rings (we omit $\otimes\mathbb{Q}$ for brevity)

$$\begin{tikzcd} A^\bullet_{GL_2}(\mathbb{P}^n) \arrow[d, two heads, "\pi_1"] & A^\bullet_{GL_2\times\mathbb{G}_m}(\mathbb{A}^{n+1}\backslash\{0\}) \arrow[l, "\sim"'] \arrow[r, two heads] \arrow[d, two heads, "\pi_2"] & A^\bullet_{GL_2}(\mathbb{A}^{n+1}\backslash\{0\}) \arrow[d, two heads, "\pi_3"] & A^\bullet_{GL_2}(\mathbb{A}^{n+1}) \arrow[l, two heads] \arrow[d, two heads, "\pi_4"] \\ A^\bullet_{GL_2}(\mathbb{P}^n\backslash Z_\lambda) & A^\bullet_{GL_2\times\mathbb{G}_m}(\mathbb{A}^{n+1}\backslash\widetilde{Z_\lambda}) \arrow[l, "\sim"'] \arrow[r, two heads] & A^\bullet_{GL_2}(\mathbb{A}^{n+1}\backslash\widetilde{Z_\lambda}) & A^\bullet_{GL_2}(\mathbb{A}^{n+1}\backslash\widetilde{Z_\lambda}) \arrow[l, "\sim"'] \end{tikzcd}$$

where $\mathbb{G}_m$ acts by scaling on $\mathbb{A}^{n+1}$.

We know $I_\lambda \otimes \mathbb{Q}$ is the kernel of π_1, so it maps surjectively to the kernel of π_3 in $A^\bullet_{GL_2}(\mathbb{A}^{n+1}\backslash\{0\})$. Each generator $\alpha \in S$ maps to the image of α_0 in $A^\bullet_{GL_2}(\mathbb{A}^{n+1}\backslash\{0\})\otimes\mathbb{Q}$. Since the kernel of $A^\bullet_{GL_2}(\mathbb{A}^{n+1}) \otimes \mathbb{Q} \to A^\bullet_{GL_2}(\mathbb{A}^{n+1}\backslash\{0\}) \otimes \mathbb{Q}$ is generated by $\prod_{i=0}^n (iu + (n-i)v)$, we have $\widetilde{I}'_\lambda + \langle \prod_{i=0}^n (iu + (n-i)v)\rangle = \widetilde{I}_\lambda \otimes \mathbb{Q}$. To finish, it suffices to see $\prod_{i=0}^n (iu + (n-i)v) \in \widetilde{I}'_\lambda$.

As $Z_{\{n\}}$ is a cycle in Z_λ, $[\{n\}]$ can be expressed as an $A^\bullet_{GL_2}(\mathbb{P}^n)\otimes\mathbb{Q}$-linear combination of the elements of S, and taking the constant terms yields

$$[\{n\}]_0 = n\prod_{i=1}^{n-1}(iu + (n-i)v) \in \widetilde{I}'_\lambda$$

by Proposition 14.1 and Section 14.4, which divides $\prod_{i=0}^n (iu + (n-i)v)$. □

Proof of Theorem 19.2. Apply Lemma 19.4 to Lemmas 18.8 and 18.9 to get the statements on $I_\lambda^{GL_2} \otimes \mathbb{Q}$. Then, apply Lemma 19.5 to get the statements on $\widetilde{I}_\lambda$. □

D. Multiplicative relations between symmetrized strata

In this section, we investigate certain multiplicative relations between the classes $[\widetilde{Z_\lambda}] \in A^\bullet_{GL_2}(\mathrm{Sym}^n K^2)$. These are equivalent to certain relations between the degree 0 terms of the expressions for $[\lambda] \in A^\bullet_{GL_2}(\mathbb{P}^n)$ by Section 12.5. For this, it suffices to restrict ourselves to the $\mathbb{Q}$-basis given by the $[a, b, 1^c]$-classes from Theorem 16.4.

Definition D.1. *Denote by $[a, b, 1^c]_0 \in \mathbb{Z}[u, v]^{S_2}$ be the term of $[a, b, 1^c] \in H^\bullet_{GL_2}(\mathbb{P}^n)$ that is degree zero in H.*

We show how to write $(u+v)[a, b, 1^c]_0$ and $uv[a, b, 1^c]_0$ as a $\mathbb{Q}$-linear combination of strata. A few of these multiplicative relations have been explicitly written down [42, Remark 3.9] and shown to exist abstractly [42, Theorems 4.3 and 4.10] using the degeneration of a spectral sequence of a filtered CW-complex. We give a combinatorial method to do this in general in Theorems D.2 and D.4.

Theorem D.2. *For $c \geq 1$ and $a+b+c=n$,*

$$\begin{aligned} n(u+v)[a,b,1^c]_0 = {} & (c+a-b)[a+1,b,1^{c-1}]_0 \\ & +(b+c-a)[a,b+1,1^{c-1}]_0 \\ & +(a+b-c)[a+b,1,1^{c-1}]_0. \end{aligned}$$

Proof. We will prove Theorem D.2 by pulling back to $(\mathbb{P}^1)^n$. By Lemma 12.8, we want to show

$$\begin{aligned} (2H+nu+nv)[a,b,1^c] = {} & (c+a-b)[a+1,b,1^{c-1}] \\ & +(b+c-a)[a,b+1,1^{c-1}] \\ & +(a+b-c)[a+b,1,1^{c-1}]. \end{aligned}$$

Let $A = \{1,\ldots,a\}$, $B = \{b+1,\ldots,a+b\}$. By the projection formula, the right hand side is

$$\Phi_*(\Delta_{\{A,B\}} \cap \Phi^*(2H+nu+nv)).$$

The pullback of $2H+nu+nv$ along Φ is

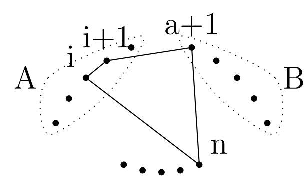
$$\begin{aligned} (H_1+H_2+u+v)+(H_2+H_3+u+v)+\ldots+(H_n+H_1+u+v) \\ = \Delta_{1,2}+\Delta_{2,3}+\ldots+\Delta_{n,1} \end{aligned}$$

by Proposition 14.4. In this way, we now only have to intersect strata using Proposition 14.2 and the square relation as in Proposition 15.2.

There are 6 cases: $1 \leq i \leq a-1$, $i = a$, $a+1 \leq i \leq a+b-1$, $i = a+b$, $a+b+1 \leq i \leq n-1$, and $i = n$. We will deal with each of these cases in the same way outlined above.

To calculate $\Phi_*(\Delta_{i,i+1}\Delta_{\{A,B\}})$ for $1 \leq i \leq a-1$, we use the square relation to replace $\Delta_{i,i+1}$ with $\Delta_{i,n} - \Delta_{n,a+1} + \Delta_{a+1,i+1}$. Using Proposition 14.2, each of the products is itself a strata, and the pushforward is

$$[a+1, b, 1^{c-1}] - [a, b+1, 1^{c-1}] + [a+b, 1, 1^{c-1}].$$



For $i = a$, Proposition 14.2 implies $\Delta_{a,a+1}\Delta_{A,B} = \Delta_{\{A \sqcup B\}}$, which pushes forward to

$$[a+b, 1, 1^{c-1}].$$

Similarly to before, for $a+1 \leq i \leq a+b-1$, the pushforward is

$$[a, b+1, 1^{c-1}] - [a+1, b, 1^{c-1}] + [a+b, 1, 1^{c-1}].$$

For $i = a+b$, Proposition 14.2 implies $\Delta_{a+b,a+b+1}\Delta_{\{A,B\}} = \Delta_{\{A,B \sqcup \{a+b+1\}\}}$, which pushes forward to

$$[a, b+1, 1^{c-1}].$$

For $a+b+1 \leq i \leq n-1$, replace $\Delta_{i,i+1}$ with $\Delta_{i,a} - \Delta_{a,a+1} + \Delta_{a+1,i+1}$, and similarly to before we get the pushforward is

$$[a+1, b, 1^{c-1}] - [a+b, 1, 1^{c-1}] + [a, b+1, 1^{c-1}].$$

Finally, for $i = n$, using Proposition 14.2, $\Delta_{n,1}\Delta_{\{A,B\}} = \Delta_{\{A\sqcup\{n\},B\}}$, so this will pushforward to

$$[a+1, b, 1^{c-1}].$$

Combining these yields the desired result. $\square$

Remark D.3. Given a partition λ of n with at least three nontrivial parts, the argument of Theorem D.2 is a combinatorial algorithm that can non-canonically express $n(u+v)[\lambda]$ in terms of other classes $[\lambda']$ with one fewer part. The number of square relations can be drastically reduced in practice by an appropriate choice of the partition pushing forward to $[a_1, \ldots, a_d]$.

Theorem D.4. *For $c \geq 2$, and $a+b+c = n$*

$$\begin{aligned}
n^2uv[a,b,1^c]_0 = (2ab+ac+bc+c(c-1))[a+1,b+1,1^{c-2}]_0 \\
+(-ab-bc)[a+2,b,1^{c-2}]_0 \\
+(-ab-ac)[a,b+2,1^{c-2}]_0 \\
+(-ac-bc-c(c-1))[a+b+1,1,1^{c-2}]_0 \\
+(ac+bc)[a+b,2,1^{c-2}]_0.
\end{aligned}$$

Proof. As in the previous theorem letting $A = \{1, \ldots, a\}$, $B = \{a+1, \ldots, a+b\}$ the statement is equivalent to

$$\begin{aligned}
&\Phi_*((\sum H_i + nu)(\sum H_i + nv)\Delta_{\{A,B\}}) \\
&\qquad = (2ab + ac + bc + c(c-1))[a+1, b+1, 1^{c-2}] \\
&\qquad\qquad + (-ab - bc)[a+2, b, 1^{c-2}] \\
&\qquad\qquad + (-ab - ac)[a, b+2, 1^{c-2}] \\
&\qquad\qquad + (-ac - bc - c(c-1))[a+b+1, 1, 1^{c-2}] \\
&\qquad\qquad + (ac + bc)[a+b, 2, 1^{c-2}].
\end{aligned}$$

We have $(H_i + u)(H_i + v) = 0$, so

$$\begin{aligned}
(\sum H_i + nu)(\sum H_i + nv) &= \sum_{1 \le i < j \le n} (H_i + u)(H_j + v) + (H_j + u)(H_i + v) \\
&= \sum_{1 \le i < j \le n} -(H_i - H_j)^2 \\
&= \sum_{1 \le i < j \le n} -(\Delta_{i,k_{i,j}} - \Delta_{j,k_{i,j}})^2
\end{aligned}$$

where $k_{i,j} \in [n] \setminus \{i, j\}$ is arbitrary. There are 6 cases depending on which of $A, B, [n] \setminus \{A, B\}$ each of i, j lie in, and for each of these cases an appropriate choice of $k_{i,j}$ can be made so that the strata combine via Proposition 14.2 as in the proof of Theorem D.2 and push forward to $[a', b', 1^{c-2}]$-classes. $\square$

Remark D.5. Similarly to Theorem D.2, the argument of Theorem D.4 is a combinatorial algorithm that can express $n^2uv[\lambda]$ in terms of other classes $[\lambda']$ with two fewer parts for any partition λ of n with at least four parts.

Part 4. Divisors on the moduli space of curves from divisorial conditions on hypersurfaces

In this part, we consider an application of equivariant intersection theory to the moduli space of curves. The original connection was given by Farkas and Rimányi [37]. The contribution of this work is to vastly simplify the part of the paper involving equivariant intersection to a basic lemma. Since the lemma now applies more generally, we obtain more examples of effective (virtual) divisors and used Mathematica to compute their slopes.

In the paper, we cannot show that these virtual divisors are actual divisors, and we can only compute the coefficients of λ and δ_0. I suspect that showing that these virtual divisors are actual divisors is hard. I have tried to see if I could compute the coefficients of δ_i for $i > 0$, but I am still unsure how to do so.

Abstract: In this note, we extend work of Farkas and Rimányi on applying quadric rank loci to finding divisors of small slope on the moduli space of curves by instead considering all divisorial conditions on the hypersurfaces of a fixed degree containing a projective curve. This gives rise to a large family of virtual divisors on $\overline{\mathcal{M}_g}$. We determine explicitly which of these divisors are candidate counterexamples to the Slope Conjecture. The potential counterexamples exist on $\overline{\mathcal{M}_g}$, where the set of possible values of $g \in \{1, \dots, N\}$ has density $\Omega(\log(N)^{-0.087})$ for $N >> 0$. Furthermore, no divisorial condition defined using hypersurfaces of degree greater than 2 give counterexamples to the Slope Conjecture, and every divisor in our family has slope at least $6 + \frac{8}{g+1}$.

20. Introduction

There has been much interest in bounding the effective cone of the moduli space of curves $\overline{\mathcal{M}_g}$. In the study of the effective cone, a fundamental invariant of $\overline{\mathcal{M}_g}$ is the slope $s(\overline{\mathcal{M}_g})$. Much work has been done in bounding $s(\overline{\mathcal{M}_g})$ from above by exhibiting special effective divisors on $\overline{\mathcal{M}_g}$ to show general typeness [56, 53, 31, 34, 76, 62] and to find counterexamples to the Slope Conjecture [54, Conjecture 0.1] stated by Harris and Morrison [36, 33, 35, 37, 70]. Despite this, many basic questions are still open. For example, we do not understand how $s(\overline{\mathcal{M}_g})$ behaves asymptotically [22, Problem 0.1].

In this note, we focus on extending the methods of Farkas and Rimányi [37]. The authors fixed g, r, d so the Brill Noether number $\rho = g-(r+1)(r-d+g)$ is 0 and asked for nondegenerate curves $C \to \mathbb{P}^r$ of degree d and genus g to either lie on a quadric of low rank or be contained in a degenerate pencil of quadrics. When either of these two conditions is a divisorial condition on the space of quadrics containing C, one gets a (virtual) divisor on $\overline{\mathcal{M}}_g$. The authors exhibited infinitely many examples of potential counterexamples to the Slope Conjecture and verified the potential counterexamples were actual divisors in small cases using *Macaulay* [37, Section 7].

Our contribution is twofold. First, we show their argument can both be easily simplified and generalized to apply to *any* divisorial condition on the hypersurfaces of degree $m \geq 2$ containing a curve (see Section 21). Second, we use the formulas to deduce three results (see Theorem 20.1):

(1) We show the slopes of all our divisors are all bounded below by $6 + \frac{8}{g+1}$. This gives evidence that $s(\overline{\mathcal{M}_g})$ approaches 6 as $g \to \infty$ in the context of [22, Problem 0.1].
(2) Only divisors defined using quadrics (instead of hypersurfaces of higher degree) can give counterexamples to the Slope Conjecture.
(3) We give virtual divisors that are candidate counterexamples to the Slope Conjecture on $\overline{\mathcal{M}_g}$ for all $g = (r+1)s$ with $\frac{r^2+1}{3r-1} < s \leq \frac{r}{2}$.

20.1. **Statement of results.**

20.1.1. *Definition of slope.* We recall for $g \geq 3$ [12, Theorem 1],

$$A^\bullet(\overline{\mathcal{M}_g}) \otimes \mathbb{Q} = \mathbb{Q}\lambda \oplus \bigoplus_{i=0}^{\lfloor \frac{g}{2} \rfloor} \mathbb{Q}\delta_i.$$

Given an effective divisor $D = a\lambda - \sum_{i=0}^{\lfloor \frac{g}{2} \rfloor} b_i\delta_i$ on $\overline{\mathcal{M}_g}$ with $a, b_i > 0$, define the slope

$$s(D) = \frac{a}{\min\{b_i : 0 \leq i \leq \lfloor \frac{g}{2} \rfloor\}}.$$

If a, b_i are not all positive, then we define $s(D) = \infty$. Define $s(\overline{\mathcal{M}_g})$ to be the infimum of $s(D)$ as D varies over all effective divisors.

Even though this is not standard, we will similarly define

$$s_0(D) = \begin{cases} \frac{a}{b_0} & \text{if } a, b_0 > 0 \\ \infty & \text{otherwise} \end{cases}$$

$$s_0(\overline{\mathcal{M}_g}) = \inf\{s_0(D) : D \text{ is effective}\}.$$

Clearly, $s_0(D) \leq s(D)$ and $s_0(\overline{\mathcal{M}_g}) \leq s(\overline{\mathcal{M}_g})$. Conjecturally, $s_0(\overline{\mathcal{M}_g}) = s(\overline{\mathcal{M}_g})$ [36, Conjecture 1.5], and this has been verified for $g \leq 23$ [36, Theorem 1.4]. For technical reasons (see Section 22.1), we will work with $s_0(D)$ instead of $s(D)$, which means our candidate counterexamples to the Slope Conjecture have only been checked with the coefficients of λ and δ_0. However, lower bounds for $s_0(D)$ clearly give lower bounds for $s(D)$.

20.1.2. *Definition of the divisors.* We will work with $\overline{\mathcal{M}_g}$ as a Deligne-Mumford stack instead of a coarse moduli space, but the distinction does not matter for the statement of Theorem 20.1. We will work over $\mathbb{C}$, but see Section 20.4 for more on characteristic assumptions on the base field. Fix r, g, d such that $\rho := g - (r+1)(g-d+r) = 0$. Equivalently, we have $s \geq 1, r \geq 1$ such that $g = (r+1)s$, $d = (s+1)r$. Since we are interested in the hypersurfaces containing a curve $C \to \mathbb{P}^r$, we also assume

$r \geq 3$. Given an integer $m \geq 2$ such that $\binom{r+m}{m} \geq md - g + 1$, fix a divisor $D \subset \mathrm{Hom}(\mathrm{Sym}^m \mathbb{C}^{r+1}, \mathbb{C}^{md-g+1})$ invariant under the action of $GL(\mathbb{C}^{r+1}) \times GL(\mathbb{C}^{md-g+1})$.

Let $\mathcal{M}_g^{\mathrm{irr}} \subset \overline{\mathcal{M}_g}$ denote the open substack parameterizing irreducible curves of genus g. In $\mathcal{M}_g^{\mathrm{irr}}$, consider the locus $Z_g^{m,r,s}$ consisting of curves C for which there exists a line bundle L of degree d mapping $C \to \mathbb{P}^r$ such that the induced map

$$H^0(\mathbb{P}^r, \mathscr{O}_{\mathbb{P}^r}(m)) \to H^0(C, L^{\otimes m})$$

is given by a map in D after choosing bases for $H^0(L)$ and $H^0(C, L^{\otimes m})$. Since D is invariant under $GL(\mathbb{C}^{r+1}) \times GL(\mathbb{C}^{md-g+1})$, this definition is independent of choice of bases.

Now, take the closure of $Z_{m,r,s}$ in $\overline{\mathcal{M}}_g$ to get $D_{m,r,s} \subset \overline{\mathcal{M}}_g$. If $Z_{m,r,s}$ is not dense, then $D_{m,r,s}$ is a divisor and we can compute its slope. Otherwise, we only know its slope as a virtual divisor, whose class can be defined using intersection theory. In either case, we can define $s(D_{m,r,s})$ and $s_0(D_{m,r,s})$ as above. Now, we state our main theorem

Theorem 20.1. *The slope $s_0(D_{m,r,s})$ is independent of the choice of the $GL(\mathbb{C}^{r+1}) \times GL(\mathbb{C}^{md-g+1})$-invariant divisor $D \subset \mathrm{Hom}(\mathrm{Sym}^m \mathbb{C}^{r+1}, \mathbb{C}^{md-g+1})$ given $m \geq 2, r \geq 3, s \geq 1$. Furthermore*

(1) If $m \geq 3$, $s_0(D_{m,r,s}) \geq 6 + \frac{12}{g+1}$, so considering hypersurfaces other than quadrics will not yield counterexamples to the Slope Conjecture. Equality holds if and only if $(m, r, s) = (3, 3, 2)$.

(2) We have $s_0(D_{2,r,s}) > 6 + \frac{8}{g+1}$.

(3) We have $s_0(D_{2,r,s}) < 6 + \frac{12}{g+1}$ if and only if $\frac{r^2+1}{3r-1} < s \leq \frac{r}{2}$.

The density of the potential counterexamples in Theorem 20.1 is $\Theta(\frac{1}{\log(g)^{\delta} \log(\log(x))^{\frac{3}{2}}})$ by [46, Corollary 2], where $\delta = 1 - \frac{1+\log(\log(2))}{\log(2)} \approx .086071$.

One can always choose D by taking the closure of a suitably large family of orbits as follows. Let $e = md - g + 1$. If $\binom{r+m}{m} = e$, we choose D to be the linear maps not of full rank. If $\binom{r+m}{m} = e + 1$, we can choose D to be linear maps whose kernel defines a singular hypersurface in $\mathbb{P}^r$. If $\binom{r+m}{m} \geq e + 2$, then one can check $\dim(\mathrm{Hom}(\mathrm{Sym}^m \mathbb{C}^{r+1}, \mathbb{C}^e)) \geq \dim(GL(\mathbb{C}^{r+1})) + \dim(GL(\mathbb{C}^e))$, so a general orbit in $\mathrm{Hom}(\mathrm{Sym}^m \mathbb{C}^{r+1}, \mathbb{C}^e)$ has codimension at least 1.

The proof of Theorem 20.1 follows from Lemma 21.2, a generalization of [37, Theorems 1.1 and 1.2] that can be proved easily using standard methods of equivariant intersection theory, together with straightforward, but tedious, formula manipulation using Mathematica.

20.2. **Example cases and comparison to literature.**

Example 20.2. If $\binom{r+m}{m} = md - g + 1$, then the unique choice of invariant divisor $D \subset \mathrm{Hom}(\mathrm{Sym}^m \mathbb{C}^{r+1}, \mathbb{C}^{md-g+1})$ consists of linear maps that are not of full rank. This is the locus of curves contained in a degree m hypersurface. In the case $(r, g, d, m) = (4, 10, 12, 2)$ this is the first known counterexample to the Slope Conjecture [36, Theorem 1.7(4)] and this was considered in general by Khosla [70, Section 3-B]. For the case of $m = 2$, it has been checked that the coefficient of δ_i for $i > 0$ do not contribute to the slope [35, Theorem 1.4].

Example 20.3. The case $(r, g, d, m) = (5, 12, 15, 2)$ was considered in [37, Section 8]. Given a general genus 12 curve C together with one of its finitely many degree 15 embeddings $C \subset \mathbb{P}^5$, there is a pencil of quadrics containing it. Pulling back the discriminant hypersurface of singular quadrics yields 6 points (possible non-distinct) on $\mathbb{P}^1$. To illustrate the independence of the slope on the choice of divisor $D \subset \mathrm{Hom}(\mathrm{Sym}^2 \mathbb{C}^6, \mathbb{C}^{19})$ in the statement of Theorem 20.1, the following divisorial conditions on those 6 points yield virtual divisors on $\overline{\mathcal{M}}_{12}$ with the same slopes, each contradicting the Slope Conjecture. To bound the coefficients of δ_i for $i > 0$, one can use [36, Corollary 1.2].

(1) 6 points on $\mathbb{P}^1$, where at least two points coincide. This was considered in [37, Section 8] and shown to be an actual divisor using *Macaulay*.
(2) 6 points on $\mathbb{P}^1$ with an involution.
(3) 6 points on $\mathbb{P}^1$ such that 4 of them have a fixed choice of moduli.
(4) 6 points on $\mathbb{P}^1$ that arise as the image of 6 points on $\mathbb{P}^2$ under a linear map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^1$. It is not necessary for the 6 points to be general, for example it suffices for 5 of them to be in general linear position.

Example 20.4. Let $m = 2$. If $r = 9\ell - 2$ and $s = 4\ell - 1$, this recovers [37, Theorem 7.1], and similarly if $r = 8\ell + 3$ and $s = 3\ell + 1$, this recovers [37, Theorem 7.2]. The authors state the result in terms of $s(D_{2,9\ell-2,4\ell-1})$ and $s(D_{2,8\ell+3,3\ell+1})$, but they also only computed $s_0(D_{2,9\ell-2,4\ell-1})$ and $s_0(D_{2,8\ell+3,3\ell+1})$.

Example 20.5. The smallest case of Theorem 20.1 that is new to our knowledge is when (g, r, d) is $(27, 8, 32)$. Given a line bundle L of degree 32 mapping a genus 27 curve $C \to \mathbb{P}^8$, we expect $\dim(\operatorname{Sym}^2 H^0(C, L)) = 45$ and $H^0(C, L^{\otimes 2}) = 38$, so we expect a $\mathbb{P}^6$ of quadrics containing C, and there to be $\frac{\binom{9}{3}\binom{10}{2}\binom{11}{1}}{\binom{1}{0}\binom{3}{1}\binom{5}{2}} = 1386$ quadrics of corank at least 3 [57, Proposition 12(b)]. We can choose D to be the divisor where at least two of these points coincide.

Example 20.6. If $g = 10 + 6j$ for $j \geq 0$, then [33, Theorem A] gives a virtual counterexample to the Slope Conjecture, where the coefficients of δ_i for $i > 0$ are also checked. There are cases where Theorem 20.1 and [33, Theorem A] overlap, and computing the slopes with Mathematica in small cases suggests that the divisor computed in [33, Theorem A] will always have smaller slope unless $r = 12\ell$, $s = 6\ell$, and $\binom{r+2}{2} = 2d - g + 1$. In this case, D corresponds to curves lying on a quadric surface. This has been tested for all values of $r < 1000$.

Example 20.7. In the equality case of Part 1 of Theorem 20.1, we are looking at genus 8 curves with a degree 9 map $C \to \mathbb{P}^3$ contained in a cubic surface. This set-theortically contains the Brill-Noether divisor of curves with a $\mathfrak{g}^2_7$. Suppose we have $f : C \to \mathbb{P}^2$ whose image is a septic plane curve with 7 nodes. The canonical divisor on the image is $4L$, where L is the class of a line in $\mathbb{P}^2$. The canonical divisor of C is then $4f^*L - \sum_{i=1}^{7}(p_i + q_i)$, where p_i, q_i are the preimages of the 7 nodes. Pick one of the nodes, for example the node corresponding to p_7, q_7. The lines through that node give a $\mathfrak{g}^1_5$ on C. Subtracting this $\mathfrak{g}^1_5$ from the canonical on C gives $3f^*L - \sum_{i=1}^{6}(p_i + q_i)$, which is the cubics in $\mathbb{P}^2$ passing through the other 6 nodes of the image of C. This gives $C \to \mathbb{P}^2 \dashrightarrow \mathbb{P}^3$ which yields a degree 9 embedding of C into $\mathbb{P}^3$ contained in a cubic surface. The class of C on the cubic surface is $7L - 2(E_1 + \cdots + E_6)$. It is not immediately clear to us, for example, whether curves corresponding to $9L - 3(E_1 + \cdots + E_6)$ or $11L - 4(E_1 + \cdots + E_6)$ could also contribute additional components to $D_{3,3,2}$.

20.3. **Classification of the divisors.** Given vector spaces V and W, it is natural to ask for a classification of divisors $D \subset \mathrm{Hom}(\mathrm{Sym}^m(V), W)$ invariant under the action of $GL(V) \times GL(W)$. There is a correspondence between such invariant divisors and divisors on the GIT quotient $G(\dim(W), \mathrm{Sym}^m(V))//SL(V)$ of the Grassmannian of quotients of $\mathrm{Sym}^m(V)$ under the action of $SL(V)$.

Thinking of $\mathrm{Hom}(\mathrm{Sym}^m(V), W)$ as an affine space, any divisor D must be cut out by a single polynomial f. If D is $GL(W)$ invariant, then it must act on f by a character of $GL(W)$, which is the k^{th} power of the determinant for some positive k. If D is also $SL(V)$ invariant, then D is also $GL(V)$-invariant.

Therefore, we have the trivial vector bundle $\mathbb{A}^1 \times \mathrm{Hom}(\mathrm{Sym}^m(V), W)$ together with an action of $SL(V) \times GL(W)$ lifting the action on the base $\mathrm{Hom}(\mathrm{Sym}^m(V), W)$. This action is unique up to taking a power. Taking the GIT quotient by $GL(W)$ shows all $GL(W)$-invariant divisors are given by pullbacks from the Grassmannian

$G(\dim(W), \mathrm{Sym}^m(V))$ parameterizing $\dim(W)$ *quotients* of $\mathrm{Sym}^m(V)$. Taking the GIT quotient by $SL(V)$ shows $SL(V) \times GL(W)$ divisors on $\mathrm{Hom}(\mathrm{Sym}^m(V), W)$ divisors on $G(\dim(W), \mathrm{Sym}^m(V))//SL(V)$.

The intersection of all the $SL(V) \times GL(W)$ invariant divisors on $\mathrm{Hom}(\mathrm{Sym}^m(V), W)$ is by definition the unstable points. The unstable points consist of the maps in $\mathrm{Hom}(\mathrm{Sym}^m(V), W)$ that are not of full rank and the pullback of the $SL(V)$-unstable points under $\mathrm{Hom}(\mathrm{Sym}^m(V), W) \dashrightarrow G(\dim(W), \mathrm{Sym}^m(V))$.

The semistable points of $G(\dim(W), \mathrm{Sym}^m(V))$ under the action of $SL(V)$ has appeared in the study of associated forms, for example [1, 39, 40].

20.4. **A note on characteristic assumptions.** We will work over $\mathbb{C}$ for notational convenience, but our proofs are algebraic, so everything automatically extends to when our base field is an algebraically closed field of characteristic zero.

Section 21 holds independent of characteristic. To extend Theorem 20.1 to positive characteristic, one would need to check that the setup in [70] or [35, Section 2] to pushforward classes from the stack parameterizing curves with a linear series to the moduli space of curves can be adapted to positive characteristic. The Picard group $\mathrm{Pic}(\overline{\mathcal{M}_{g,n}}) \otimes \mathbb{Q}$ is unchanged in positive characteristic [81]. More seriously, when applying limit linear series arguments in positive characteristic, we want to restrict ourselves to cases where ramification is imposed at at most two points on each component [83, 84]. For example, since we only compute the coefficients of λ and δ_0, [70, Lemma 4.5] suffices for our use, but Khosla degenerates further to a comb of elliptic curves with a rational backbone in the proof.

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21. Divisors from hypersurfaces

The goal of this section is to prove the following two lemmas that generalize [37, Theorems 1.1 and 1.2].

Lemma 21.1. *Let $D \subset \mathrm{Hom}(\mathbb{C}^e, \mathrm{Sym}^m \mathbb{C}^f)$ be a divisor, preserved under the natural actions of $GL(\mathbb{C}^e)$ and $GL(\mathbb{C}^f)$. Given vector bundles $\mathcal{E}$ and $\mathcal{F}$ of ranks e and f respectively over a scheme X together with a map $\phi : \mathcal{E} \to \mathrm{Sym}^m \mathcal{F}$, the class of the virtual divisor supported on points of X over which ϕ fiberwise restricts to maps in D is a positive multiple of*

$$mec_1(\mathcal{F}) - fc_1(\mathcal{E}).$$

Lemma 21.2. *Let $D \subset \mathrm{Hom}(\mathrm{Sym}^m \mathbb{C}^e, \mathbb{C}^f)$ be a divisor, preserved under the natural actions of $GL(\mathbb{C}^e)$ and $GL(\mathbb{C}^f)$. Given vector bundles $\mathcal{E}$ and $\mathcal{F}$ of ranks e and f respectively over a scheme X together with a map $\phi : \mathrm{Sym}^m \mathcal{E} \to \mathcal{F}$, the class of the virtual divisor supported on points of X over which ϕ fiberwise restricts to maps in D is a positive multiple of*

$$ec_1(\mathcal{F}) - mfc_1(\mathcal{E}).$$

Lemmas 21.1 and 21.2 are stated in a form that is easier to apply, but they are easier to prove in the language of equivariant intersection theory. Lemma 21.1 follows from Lemma 21.4 and Lemma 21.2 follows from Lemma 21.5 in Section 21.1 below. Finally, we note that we will be applying Lemma 21.2 in the case where X is the Deligne-Mumford stack of the moduli space of curves. To do so, one either pulls back to enough test curves or notes that the equivariant class computed in Lemma 21.5 below implies Lemma 21.2 in the necessary generality (for example the argument in [94, Section 2.2]).

21.1. **Proof of Lemmas 21.1 and 21.2.**

Lemma 21.3. *Let T be a torus acting on an affine space $\mathbb{A}^N$. Then, the equivariant Chow ring $A_T^\bullet(\mathbb{A}^N) \cong \mathbb{Z}[t_1, \dots, t_n]$, where $t_1, \dots, t_n$ $\mathbb{Z}$-linearly span the character lattice of T.*

If $D \subset \mathbb{A}^N$ is a T-invariant divisor, then it is defined by a polynomial $F(x_1, \dots, x_N)$ whose monomials have the same weight χ under the action of T. The equivariant class $[D] \in A_T^\bullet(\mathbb{A}^N)$ is χ.

Proof. The statement on $A_T^\bullet(\mathbb{A}^N) \cong \mathbb{Z}[t_1, \dots, t_n]$ is standard [28, Section 3.1]. The statement on the class of $[D]$ is used in [37, Theorem 5.1] and can be proven for example by scaling the coordinates of $\mathbb{A}^N$ to degenerate to the case where D is defined by a monomial. Then, we reduce to the case where F is simply a coordinate function of $\mathbb{A}^N$. □

Lemma 21.4. *If $D \subset \mathrm{Hom}(\mathbb{C}^e, \mathrm{Sym}^m \mathbb{C}^f)$ is a divisor, preserved under the natural actions of $GL(\mathbb{C}^e)$ and $GL(\mathbb{C}^f)$, then the equivariant class $[D]$ in*

$$A^1_{GL(\mathbb{C}^e) \times GL(\mathbb{C}^f)}(\mathrm{Hom}(\mathbb{C}^e, \mathrm{Sym}^m \mathbb{C}^f))$$

is a positive multiple of

$$me \sum \beta_i - f \sum \alpha_i,$$

where $\{\alpha_i\}$ and $\{\beta_i\}$ are the standard characters of the standard maximal tori of $GL(\mathbb{C}^e)$ and $GL(\mathbb{C}^f)$ respectively.

Lemma 21.5. *If $D \subset \mathrm{Hom}(\mathrm{Sym}^m \mathbb{C}^e, \mathbb{C}^f)$ is a divisor, preserved under the natural actions of $GL(\mathbb{C}^e)$ and $GL(\mathbb{C}^f)$, then the equivariant class $[D]$ in*

$$A^1_{GL(\mathbb{C}^e) \times GL(\mathbb{C}^f)}(\mathrm{Hom}(\mathrm{Sym}^m \mathbb{C}^e, \mathbb{C}^f))$$

is a positive multiple of

$$e\sum \beta_i - mf\sum \alpha_i,$$

where $\{\alpha_i\}$ *and* $\{\beta_i\}$ *are the standard characters of the standard maximal tori of* $GL(\mathbb{C}^e)$ *and* $GL(\mathbb{C}^f)$ *respectively.*

The proofs of Lemmas 21.4 and 21.5 follow easily from Lemma 21.3. For example, we prove Lemma 21.5.

Proof of Lemma 21.5. Let T_e and T_f be the standard maximal tori of $GL(\mathbb{C}^e)$ and $GL(\mathbb{C}^f)$ respectively. The restriction map

$$A^1_{GL(\mathbb{C}^e)\times GL(\mathbb{C}^f)}(\mathrm{Hom}(\mathrm{Sym}^m \mathbb{C}^e, \mathbb{C}^f)) \to A^1_{T_e\times T_f}(\mathrm{Hom}(\mathrm{Sym}^m \mathbb{C}^e, \mathbb{C}^f))$$

is injective [28, Proposition 6].

To determine $[D]$ we apply Lemma 21.3. Let $\alpha_1, \ldots, \alpha_e$ be the standard characters of T_e and let $\beta_1, \ldots, \beta_f$ be the standard characters of T_f. Viewing $\mathrm{Hom}(\mathrm{Sym}^m \mathbb{C}^e, \mathbb{C}^f)$ as the space of $\binom{e+1}{m} \times f$ matrices, T_e and T_f act by the characters $\{\beta_i - \sum_{j\in S}\alpha_j\}$ on the entries, where i ranges from 1 to f and S ranges over multisets of $\{1, \ldots, e\}$ with size m. Each monomial term of the hypersurface F defining D in $\mathrm{Hom}(\mathrm{Sym}^m \mathbb{C}^e, \mathbb{C}^f)$ has a certain weight χ.

Now, we use the fact that χ has to be invariant under permutation of the characters α_i and the characters β_i, which means that it must be

$$e\sum \beta_i - mf\sum \alpha_i$$

up to a power. $\square$

22. Application to Slopes of $\overline{\mathcal{M}}_g$

22.1. **Setup.** In addition to Lemma 21.2, we will need to pushforward classes from the moduli stack parameterizing a genus g curve together with a $\mathfrak{g}^r_d$. The key ingredients were first written in [70] and [35, Section 2]. The details of the setup will not be used, and the same setup as already been utilized for computations in [37, 70, 35, 23]. We will follow [23, Section 5.1].

As a first approximation, we want a stack $\widetilde{\mathcal{G}^r_d}$ parameterizing curves with a choice of $\mathfrak{g}^r_d$ together with a proper map $\widetilde{\mathcal{G}^r_d} \to \overline{\mathcal{M}_g}$. In order to be able to define the universal line bundle and vector bundle corresponding to choice of sections over $\widetilde{\mathcal{G}^r_d}$, we will work instead with $\overline{\mathcal{M}_{g,1}}$. (This is not strictly necessary, also see the second page of [35, Section 2].)

Recall for $g \geq 3$ [12, Theorem 2],

$$A^\bullet(\overline{\mathcal{M}_{g,1}}) \otimes \mathbb{Q} = \mathbb{Q}\lambda \oplus \bigoplus_{i=0}^{g-1} \mathbb{Q}\delta_i \oplus \mathbb{Q}\psi,$$

where δ_0 is the class of the irreducible nodal curves $\Delta_0 \subset \overline{\mathcal{M}_{g,1}}$, and δ_i for $i \geq 1$ is the class of the closure of the reducible nodal curves $\Delta_i \subset \overline{\mathcal{M}_{g,1}}$ where the component containing the marked point is genus i. Also, λ is the first chern class of the Hodge bundle and ψ is the relative dualizing sheaf of $\overline{\mathcal{M}_{g,1}} \to \overline{\mathcal{M}_g}$.

We restrict to an open substack $\widetilde{\mathcal{M}_{g,1}} \subset \overline{\mathcal{M}_{g,1}}$ whose compliment is codimension 2, so this step does not affect divisor calculations. Specifically, we first let $\widetilde{\mathcal{M}_{g,1}}$ be the complement of the closure of the locus of two smooth curves intersecting transversely at two points.

There is a Deligne-Mumford stack $\mathcal{G}^r_d \to \widetilde{\mathcal{M}_{g,1}}$ parameterizing the choice of a curve C, a rank 1 torsion free sheaf L, and an $r+1$-dimensional subspace of the global sections of the sheaf. The torsion free sheaf L is restricted to have degree d on the component of C containing the marked point and zero on the unmarked components.

Let $\pi : \mathcal{C}_d^r \to \mathcal{G}_d^r$ be the universal (quasi-stable) curve. Equivalently, $\mathcal{C}_d^r \to \mathcal{G}_d^r$ is the pullback of the universal curve over $\widetilde{\mathcal{M}_g}$ under $\mathcal{G}_d^r \to \widetilde{\mathcal{M}_{g,1}} \to \widetilde{\mathcal{M}_g}$.

On $\mathcal{C}_d^r$, there is a universal sheaf $\mathcal{L}$ whose restriction to each fiber of π is a torsion-free sheaf with degree d on the component with the marked point and degree zero on the other components. Furthermore, $\mathcal{L}$ is normalized to be trivial along the marked section of π. In addition, there is a subbundle $\mathcal{V} \to \pi_*\mathcal{L}$ that restricts to the marked aspect of the (limit) linear series in each fiber.

We want to apply Lemma 21.2 in the case where $\mathcal{E} = \mathcal{V}$ and $\mathcal{F} = \pi_*\mathcal{L}^{\otimes m}$. To do, we need $c_1(\pi_*\mathcal{L}^{\otimes m})$ and we need to know $\pi_*\mathcal{L}^{\otimes m}$ is locally free away from a set of codimension 2.

Unfortunately, $\pi_*\mathcal{L}^{\otimes m}$ jumps in rank over Δ_i for $i > 0$. Therefore, we restrict $\mathcal{G}_d^r \to \widetilde{\mathcal{M}_{g,1}}$ to $\mathcal{G}_d^{r,\mathrm{irr}} \to \mathcal{M}_{g,1}^{\mathrm{irr}}$, where $\mathcal{M}_{g,1}^{\mathrm{irr}} \subset \widetilde{\mathcal{M}_{g,1}}$ is the complement of Δ_i for $i > 0$ and $\mathcal{G}_d^{r,\mathrm{irr}}$ is the inverse image of $\mathcal{M}_{g,1}^{\mathrm{irr}}$ in $\mathcal{G}_d^r$.

Then, $A^\bullet(\mathcal{M}_{g,1}^{\mathrm{irr}}) \otimes \mathbb{Q} = \mathbb{Q}\lambda \oplus \mathbb{Q}\delta_0 \oplus \mathbb{Q}\psi$, which means we cannot compute the coefficients of δ_i for $i > 0$. Conjecturally this does not matter for computing the slope of $\overline{\mathcal{M}_g}$ [36, Conjecture 1.5].

22.2. **Computation.** By an abuse of notation, let us also refer to the restriction $\mathcal{C}_d^{r,\mathrm{irr}} \to \mathcal{G}_d^{r,\mathrm{irr}}$ of $\mathcal{C}_d^r \to \mathcal{G}_d^r$ as π and let ω be the dualizing sheaf of π. Then, following [70, 35], we define

$$\alpha = \pi_*(c_1(\mathcal{L})^2) \qquad \beta = \pi_*(c_1(\mathcal{L}) \cap c_1(\omega)) \qquad \gamma = c_1(\mathcal{V}),$$

where $\mathcal{L}$ and $\mathcal{V}$ are restricted to $\mathcal{C}_d^{r,\mathrm{irr}}$ and $\mathcal{G}_d^{r,\mathrm{irr}}$ respectively. Let η be the map $\mathcal{G}_d^{r,\mathrm{irr}} \to \mathcal{M}_{g,1}^{\mathrm{irr}}$.

In order to have $\rho = g - (r+1)(g-d+r) = 0$, g needs to be $s(r+1)$ for some $s > 1$. Solving for d, we have $d = r(s+1)$. Finally, for $(C, L) \in \mathcal{G}_d^{r,\mathrm{irr}}$ general, we need $\dim(\mathrm{Sym}^m H^0(L)) \geq \dim(H^0(L^{\otimes m}))$. If C is general, then the Geiseker-Petri theorem implies $h^1(L^{\otimes 2}) = 0$, so we must require

$$(22.1) \qquad \binom{r+m}{m} \geq md - g + 1.$$

The following lemma is already contained in [70, Section 3A], but we include it for completeness and to correct a typo in the proof.

Lemma 22.1. *We have $\pi_*\mathcal{L}^{\otimes m}$ is a vector bundle away from a set of codimension at least 2 and $c_1(\pi_*\mathcal{L}^{\otimes m}) = \frac{m^2}{2}\alpha - \frac{m}{2}\beta + \eta^*(\lambda)$.*

Proof. We first claim that for $(C, L) \in \mathcal{G}_d^{r,\mathrm{irr}}$, then $h^1(L^{\otimes m}) = 0$ for degree reasons away from a set of codimension at least 2. This implies $R^1\pi_*\mathcal{L}^{\otimes m} = 0$ and $\pi_*\mathcal{L}^{\otimes m}$ is a vector bundle away from a set of codimension at least 2 by Grauert's theorem. First, suppose C is smooth. If $m = 2$, then $2d - 2g + 2 = 2(r - s + 1)$. This is greater than zero as $s \leq \frac{r}{2}$ (which is equivalent to (22.1) when $m = 2$). If $m \geq 3$, we note

$$md - 2g - 2 \geq 3rs + 3r - 2rs - 2s + 2 = rs + 3r - 2s + 2 = (r-2)(s+3) + 8 \geq 0.$$

Now, if C is a general irreducible nodal curve, then [35, Proposition 2.3] says that L is locally free, and we can repeat the same argument above to see $h^1(L^{\otimes m}) = 0$.

To apply Grothendieck Riemann-Roch, we need the Todd class of π. This is pulled back from the Todd class of $\overline{\mathcal{M}_{g,1}} \to \overline{\mathcal{M}_g}$, which is computed in [55, page 158]. Applying Grothendieck Riemann-Roch yields

$$\begin{aligned} c_1(\pi_*\mathcal{L}^{\otimes m}) &= \pi_*\left[(1 + mc_1(\mathcal{L}) + \frac{m^2}{2}c_1(\mathcal{L})^2)(1 - \frac{1}{2}c_1(\omega) + \frac{c_1(\omega)^2 + [Z]}{12}\right]_2 \\ &= \frac{m^2}{2}\alpha - \frac{m}{2}\beta + \eta^*\frac{\pi_*c_1(\omega)^2 + \delta}{12} \end{aligned}$$

where $Z \subset \mathcal{C}_d^{r,\mathrm{irr}}$ is the singular locus of $\pi : \mathcal{C}_d^{r,\mathrm{irr}} \to \mathcal{G}_d^{r,\mathrm{irr}}$. At this point, we use the fact that the universal curve $\pi : \mathcal{C}_d^{r,\mathrm{irr}} \to \mathcal{G}_d^{r,\mathrm{irr}}$ is pulled back from $\mathcal{G}_d^r \to \widetilde{\mathcal{M}_{g,1}} \to \widetilde{\mathcal{M}_g}$. This means $\pi_*c_1(\omega)^2$ is the pullback of the κ divisor class on $\widetilde{\mathcal{M}_g}$ under $\mathcal{G}_d^r \to \widetilde{\mathcal{M}_{g,1}} \to \widetilde{\mathcal{M}_g}$.

Using the relation $\frac{\kappa+\delta}{12} = \lambda$ [55, page 158], we have $\eta^* \frac{\pi_* c_1(\omega)^2+\delta}{12} = \eta^*\lambda$, resulting in the claimed formula in Lemma 22.1. □

Theorem 22.2 ([70, Theorem 2.11])**.** *Choose $g, r, d \geq 1$ integers such that $\rho = g-(r+1)(g-d+r) = 0$. Then, pushing forward under $\eta : \mathcal{G}_d^{r,\mathrm{irr}} \to \mathcal{M}_{g,1}^{\mathrm{irr}}$, we have*

$$\begin{aligned}
\frac{6(g-1)(g-2)}{dN}\eta_*\alpha =& 6(gd-2g^2+8d-8g+4)\lambda + (2g^2-gd+3g-4d-2)\delta_0 \\
&- 6d(g-2)\psi \\
\frac{2(g-1)}{Nd}\eta_*\beta =& 12\lambda - \delta_0 - 2(g-1)\psi \\
\frac{2(g-1)(g-2)}{N}\eta_*\gamma =& ((-(g+3)\xi + 5r(r+2))\lambda - d(r+1)(g-2)\psi + \\
&\frac{1}{6}((g+1)\xi - 3r(r+2))\delta_0),
\end{aligned}$$

where

$$\begin{aligned}
N &= \frac{g!\prod_{i=1}^r i!}{\prod_{i=0}^r (g-d+r+i)} (= \deg(\eta)) \\
\xi &= 3(g-1) + \frac{(r-1)(g+r+1)(3g-2d+r-3)}{g-d+2r+1}.
\end{aligned}$$

Proof of Theorem 20.1. Following the notation of Section 22.1, apply Lemma 21.2 in the case where $\mathcal{E} = \mathcal{V}$ and $\mathcal{F} = \pi_*\mathcal{L}^{\otimes k}$ and $\pi : \mathcal{C}_d^{r,\mathrm{irr}} \to \mathcal{G}_d^{r,\mathrm{irr}}$. This yields a positive multiple of

$$(r+1)(\frac{m^2}{2}\alpha - \frac{m}{2}\beta + \sigma^*\lambda) - m(md-g+1)\gamma \tag{22.2}$$

on $\mathcal{G}_d^{r,\mathrm{irr}}$. Since we only care about the slope, we can scale by a constant factor and work with (22.2). We push forward (22.2) via $\eta : \mathcal{G}_d^{r,\mathrm{irr}} \to \mathcal{M}_{g,1}^{\mathrm{irr}}$ using Theorem 22.2 to get a class $a\lambda + b_0\delta_0 + c\psi$. This yields $c = 0$ (as expected) and rather complicated formulas for a and b_0. Checking these formulas using Mathematica yields the three statements of Theorem 20.1. For more details, the interested reader can refer to Section E. $\square$

E. Mathematica computation

Proof of Theorem 20.1 continued. Continuing the proof of Theorem 20.1, we find

$$a = \frac{N}{2(r+s+1)(rs+s-2)(rs+s-1)}(m^2r^5s^4 - m^2r^5s^2 + 3m^2r^4s^4 + 5m^2r^4s^3 + m^2r^4s^2 - m^2r^4s + m^2r^3s^4 + 12m^2r^3s^3 + 13m^2r^3s^2 - 2m^2r^3s - 4m^2r^3 - 3m^2r^2s^4 - 5m^2r^2s^3 + m^2r^2s^2 + 3m^2r^2s - 2m^2rs^4 - 12m^2rs^3 - 14m^2rs^2 + 4m^2r - mr^5s^4 + mr^5s^2 - 5mr^4s^4 - 5mr^4s^3 - mr^4s^2 + mr^4s - 9mr^3s^4 - 26mr^3s^3 - 13mr^3s^2 + 22mr^3s + 4mr^3 - 7mr^2s^4 - 37mr^2s^3 - 5mr^2s^2 + 57mr^2s - 2mrs^4 - 16mrs^3 + 6mrs^2 + 40mrs - 4mr + 2r^4s^2 + 2r^3s^3 + 8r^3s^2 - 6r^3s + 6r^2s^3 + 6r^2s^2 - 18r^2s + 4r^2 + 6rs^3 - 4rs^2 - 14rs + 8r + 2s^3 - 4s^2 - 2s + 4)$$

$$b_0 = -\frac{N}{12(r+s+1)(rs+s-2)(rs+s-1)}mr(r+1)(s+1)(mr^3s^3 - mr^3s^2 + 2mr^2s^3 + mr^2s^2 - mrs^3 + 5mrs^2 + mrs - 2mr - 2ms^3 - 5ms^2 - ms + 2m - r^3s^3 + r^3s^2 - 4r^2s^3 + r^2s^2 - 5rs^3 - 7rs^2 + 7rs + 2r - 2s^3 - 7s^2 + 17s - 2)$$

This yields $-\frac{a}{b_0}$ as a complicated rational function $F(m,r,s)$ for the slope $s_0(D_{m,r,s})$. We now prove each case individually. Recall in each case $g = (r+1)s$ and $d = r(s+1)$.

Proof of Part 1 of Theorem 20.1. Consider $F(m,r,s) - (6 + \frac{12}{g+1})$. This again is a complicated rational function $G(m,r,s)$ in m, r, s. To see $G(m,r,s) \geq 0$ if $m \geq 3$, $r \geq 3$, $s \geq 1$ subject to the constraint $\binom{r+m}{m} - (dm - g + 1) \geq 0$, we first note that

$$G(m+4, r+4, s+1) = (6(6+r+s)(3+r+5s+rs)(4+r+5s+rs)(156 + 120m + 24m^2 + 110r + 78mr + 14m^2r + 18r^2 + 12mr^2 + 2m^2r^2 + 2s + 36ms + 12m^2s + 24rs + 29mrs + 7m^2rs + 6r^2s + 5mr^2s + m^2r^2s))/((4+m)(4+r)(5+r)(2+s)(6+r+5s+rs)(162 + 54m + 81r + 27mr + 9r^2 + 3mr^2 + 513s + 207ms + 330rs + 120mrs + 58r^2s + 20mr^2s + 3r^3s + mr^3s + 543s^2 + 237ms^2 +$$

$410rs^2 + 154mrs^2 + 89r^2s^2 + 31mr^2s^2 + 6r^3s^2 + 2mr^3s^2 + 210s^3 + 90ms^3 + 167rs^3 + 63mrs^3 + 40r^2s^3 + 14mr^2s^3 + 3r^3s^3 + mr^3s^3))$

is clearly positive. To deal with the edge cases when $m = 3$ or $r = 3$, we first find

$$G(3, r+3, s+1) = (2(5+r+s)(2+r+4s+rs)(3+r+4s+rs)(11+15r+4r^2 - 11s - rs + r^2s))/((3+r)(4+r)(2+s)(5+r+4s+rs)(30+21r+3r^2+66s + 70rs + 16r^2s + r^3s + 52s^2 + 76rs^2 + 23r^2s^2 + 2r^3s^2 + 20s^3 + 29rs^3 + 10r^2s^3 + r^3s^3)).$$

The only factor of $G(3, r+3, s+1)$ that can be negative is $(11+15r+4r^2-11s-rs+r^2s)$. Now, we use the constraint $\binom{r+m}{m} - (dm - g + 1) \geq 0$. Substituting $m \to 3$ yields $\frac{r^3}{6} + r^2 - 2rs - \frac{7r}{6} + s \geq 0$, so $s \leq \frac{r^3+6r^2-7r}{6(2r-1)}$. Plugging in $s = \frac{(r+3)^3+6(r+3)^2-7(r+3)}{6(2(r+3)-1)} - 1$ into $(11 + 15r + 4r^2 - 11s - rs + r^2s)$ yields $\frac{r(r+1)(r+4)\left(r^2+9r+17\right)}{6(2r+5)}$, which is nonnegative. Furthermore, this is zero only when $r = 0$. Therefore, we have $G(3, r, s) \geq 0$ for $r \geq 3, s \geq 1$ and equality can hold only if $r = 3$. In this case, $s \leq \frac{r^3+6r^2-7r}{6(2r-1)} = 2$. Plugging in $s = 1, 2$ yields $G(3, 3, 1) > 0$ and $G(3, 3, 2) = 0$.

Now, we are left with the case $r = 3$, $m \geq 4$ and $s \geq 1$. Note

$$G(m+5, 3, s+1) = ((5+s)(1+2s)(3+4s)(65+39m+6m^2+s+12ms + 3m^2s))/((5+m)(2+s)(5+4s)(60+15m+172s+53ms+164s^2+56ms^2 + 60s^3 + 20ms^3))$$

is clearly positive, so we are left with the case $r = 3$, $m = 4$ and $s \geq 1$. Since

$$\binom{3+4}{4} - (4d - g + 1) \geq 0 \Leftrightarrow 22 - 8s \geq 0,$$

so our remaining candidates are $(m, r, s) = (4, 3, 1)$ or $(4, 3, 2)$. We evaluate

$$G(4, 3, s+1) = -\frac{2(s-5)(s+4)(2s-1)(4s-1)}{(s+1)(4s+1)\left(40s^3 - 12s^2 + 23s - 6\right)}$$

and note that it is positive for $s = 1, 2$. Tracing through the cases, we find $G(m, r, s) \geq 0$ for $m \geq 3, r \geq 3, s \geq 1$ subject to the constraint $\binom{r+m}{m} - (dm - g + 1) \geq 0$, and equality holds when $(m, r, s) = (3, 3, 2)$. $\square$

Proof of Part 2 of Theorem 20.1. Define $G(m, r, s) = F(m, r, s) - (6 + \frac{8}{g+1})$. We want to see $G(m, r, s) > 0$ if $m \geq 2$, $r \geq 3$, $s \geq 1$ subject to the constraint $\binom{r+m}{m} - (dm - g + 1) \geq 0$. First note

$$\begin{aligned}
&G(m+2, r+5, s+1) = (2(30240 + 51240m + 26880m^2 + 27168r + 48018mr + \\
&25176m^2r + 9774r^2 + 17994mr^2 + 9390m^2r^2 + 1770r^3 + 3378mr^3 + 1746m^2r^3 + \\
&162r^4 + 318mr^4 + 162m^2r^4 + 6r^5 + 12mr^5 + 6m^2r^5 + 76896s + 179100ms + \\
&102960m^2s + 79264rs + 171950mrs + 94152m^2rs + 31166r^2s + 64661mr^2s + \\
&34067m^2r^2s + 5928r^3s + 11947mr^3s + 6101m^2r^3s + 550r^4s + 1087mr^4s + 541m^2r^4s + \\
&20r^5s + 39mr^5s + 19m^2r^5s + 61560s^2 + 213960ms^2 + 132360m^2s^2 + 79312rs^2 + \\
&211992mrs^2 + 119522m^2rs^2 + 35270r^2s^2 + 81050mr^2s^2 + 42560m^2r^2s^2 + 7224r^3s^2 + \\
&15042mr^3s^2 + 7470m^2r^3s^2 + 700r^4s^2 + 1360mr^4s^2 + 646m^2r^4s^2 + 26r^5s^2 + 48mr^5s^2 + \\
&22m^2r^5s^2 + 21048s^3 + 109500ms^3 + 68280m^2s^3 + 36976rs^3 + 112180mrs^3 + 61426m^2rs^3 + \\
&18634r^2s^3 + 43891mr^2s^3 + 21769m^2r^2s^3 + 4096r^3s^3 + 8282mr^3s^3 + 3798m^2r^3s^3 + \\
&416r^4s^3 + 758mr^4s^3 + 326m^2r^4s^3 + 16r^5s^3 + 27mr^5s^3 + 11m^2r^5s^3 + 7056s^4 + \\
&24120ms^4 + 12240m^2s^4 + 10200rs^4 + 24564mrs^4 + 11028m^2rs^4 + 4828r^2s^4 + \\
&9598mr^2s^4 + 3916m^2r^2s^4 + 1034r^3s^4 + 1815mr^3s^4 + 685m^2r^3s^4 + 104r^4s^4 + 167mr^4s^4 + \\
&59m^2r^4s^4 + 4r^5s^4 + 6mr^5s^4 + 2m^2r^5s^4))/((2+m)(5+r)(6+r)(2+s)(7+r+ \\
&6s + rs)(84 + 84m + 33r + 33mr + 3r^2 + 3mr^2 + 208s + 348ms + 129rs + \\
&163mrs + 21r^2s + 23mr^2s + r^3s + mr^3s + 200s^2 + 424ms^2 + 162rs^2 + 222mrs^2 + \\
&33r^2s^2 + 37mr^2s^2 + 2r^3s^2 + 2mr^3s^2 + 84s^3 + 168ms^3 + 68rs^3 + 94mrs^3 + 15r^2s^3 + \\
&17mr^2s^3 + r^3s^3 + mr^3s^3)),
\end{aligned}$$

which is clearly positive. This leaves the cases when $r = 3$ and $r = 4$. To deal with the case $r = 4$, we evaluate

$G(m+2,4,s+2) = (7560+31584m+20832m^2+7561s+55078ms+37970m^2s+2083s^2+35878ms^2+25162m^2s^2+335s^3+10610ms^3+7190m^2s^3+125s^4+1250ms^4+750m^2s^4)/(5(2+m)(3+s)(11+5s)(84+196m+109s+317ms+53s^2+169ms^2+10s^3+30ms^3))$

$G(m+2,4,1) = \dfrac{2\,(11m^2+21m+13)}{15(m+1)(m+2)}.$

To deal with the case $r=3$, we evaluate

$G(m+3,3,s+1) = (885+825m+210m^2+2362s+2895ms+847m^2s+2007s^2+3410ms^2+1119m^2s^2+642s^3+1640ms^3+582m^2s^3+152s^4+320ms^4+104m^2s^4)/((3+m)(2+s)(5+4s)(30+15m+66s+53ms+52s^2+56ms^2+20s^3+20ms^3)),$

which reduces us to the case $r=3$, $m=2$. Now, we use the bound $\binom{3+2}{2}-(2d-g+1)\geq 0 \Leftrightarrow 3-2s\geq 0$. Plugging in $G(2,3,1)>0$ finishes this case. $\square$

Proof of Part 3 of Theorem 20.1. Define $G(m,r,s) = F(m,r,s)-(6+\frac{12}{g+1})$. We want to see when $G(m,r,s)<0$ if $m=2$, $r\geq 3$, $s\geq 1$ subject to the constraint $\binom{r+m}{m}-(dm-g+1)\geq 0$. First, note

$$\binom{r+m}{m}-(dm-g+1)\geq 0 \Leftrightarrow s\leq \frac{r}{2},$$

which is one of the constraints claimed in Part 3 of Theorem 20.1. Next, we evaluate

$G(2,r,s) = (6(1+r+s)(1+r^2+s-3rs)(-2+s+rs)(-1+s+rs))/(r(1+r)(1+s)(1+s+rs)(2-2r+15s+9rs-17s^2+3rs^2+3r^2s^2-r^3s^2-6s^3-7rs^3+r^3s^3))$

$G(2,3,s+1) = \dfrac{(s+5)(2s+1)(4s-1)(4s+3)}{(s+2)(4s+5)\,(4s^2-13s-15)}$

$G(2,r+4,s+1) = (6(6+r+s)(6+5r+r^2-11s-3rs)(3+r+5s+rs)(4+r+5s+rs))/((4+r)(5+r)(2+s)(6+r+5s+rs)(54+27r+3r^2+99s+90rs+18r^2s+r^3s+69s^2+102rs^2+27r^2s^2+2r^3s^2+30s^3+41rs^3+12r^2s^3+r^3s^3)).$

Therefore, if $r \geq 4$, then $G(2, r, s) < 0$ if and only if

$$1 + r^2 + s - 3rs < 0 \Leftrightarrow s > \frac{r^2 + 1}{3r - 1}.$$

If $r = 3$, then $s \leq \frac{3}{2}$, so $s = 1$ and $G(2, 3, 1) > 0$. □

□

Part 5. **Unpublished work and open questions**

In this section, I include some observations that are not yet available on arXiv and some open questions I find interesting. This way, this information will not disappear forever should I ever forget.

23. Generalized Matrix Orbits

Given a matrix $M \in \mathbb{A}^{d \times n}$ of full row rank, we can consider the locus $\mathcal{O}_M$, which is the closure of $\mathbb{A}^{r \times d} \cdot M \cdot T$, where we multiply M by the left by all $\mathbb{A}^{r \times d}$ matrices and on the right by all diagonal matrices.

We are interested in the equivariant class

$$[\mathcal{O}_M] \in A^{\bullet}_{GL_r \times T}(\mathbb{A}^{r \times n}) \cong \mathbb{Z}[u_1, \ldots, u_r]^{S_r}[t_1, \ldots, t_n].$$

Assuming $\mathcal{O}_M$ is of expected dimension, this class was computed in joint work with Mitchell Lee, Hunter Spink, and Anand Patel [73], by drawing an analogy of integrating against the classes with the quantum product of projective space and carefully looking at the degenerations. The approach in [73] tells us more, but the goal of this section is to give a quick proof of the formula. The method should be able to be generalized to slightly more general orbits, for example replacing $\mathbb{A}^{r \times d}$ with upper triangular matrices with a fixed block structure.

Theorem 23.1 ([73])**.** *Let M be a $d \times n$ matrix with nonzero columns $x_1, \ldots, x_n$. If* $\mathrm{rk}(M) < d$*, then $\mathcal{O}_M = 0$. Otherwise, for each permutation $\sigma \in S_n$, let $B(\sigma)$*

be the lexicographically first d-element subset $\{i_1, \ldots, i_d\} \subset \{1, \ldots, n\}$ *with respect to the ordering* $\sigma(1) \prec \cdots \prec \sigma(n)$ *such that* $x_{i_1}, \ldots, x_{i_d}$ *is a basis of* K^d. *Then for indeterminates* $z_1, \ldots, z_n$, *the expression*

$$\sum_{\sigma \in S_n} \left(\prod_{i \in \{1,\ldots,n\} \setminus B(\sigma)} \prod_{j=1}^{r} (t_i + u_j) \right) \frac{1}{(z_{\sigma(2)} - z_{\sigma(1)}) \ldots (z_{\sigma(n)} - z_{\sigma(n-1)})}$$

is a polynomial of degree at most r *in each* z_i *with coefficients in* $A^\bullet_{GL_r}(\mathrm{pt})$, *and the equivariant Chow class* $[\mathcal{O}_M]$ *is given by evaluating this polynomial at* $t_1, \ldots, t_n$.

Proof. The plan is to resolve $\mathcal{O}_M$ with a vector bundle over a compact base (which will be a permutohedral toric variety), and then applying the localization formula [45, Proposition 5.1].

If K is our base field, we can regard $\mathbb{A}^{r \times n}$ as $\mathrm{Hom}(K^r, K^n)$. Let $X \subset G(d, n)$ be the T-orbit of the row span of M.

We resolve the locus $\mathcal{O}_M$ by remembering the row span of M after acting by a torus action. Namely,

$$\{(\Lambda, N) | \text{ row span of } N \text{ is contained in } \Lambda\} \subset X \times \mathbb{A}^{r \times n}$$

maps onto $\mathcal{O}_M$. Furthermore, the fiber over any point of X is a vector space, given by $\Lambda \otimes K^r \subset \mathbb{A}^{r \times n}$.

Thus, if S is the tautological subbundle of $G(d, n)$, then $\mathcal{O}_M$ is the image under

$$(S \otimes K^r)|_X \hookrightarrow \mathbb{A}^{r \times n} \times X \to \mathbb{A}^{r \times n}.$$

Finally, to finish, we resolve X with the permutohedral toric variety $\widetilde{X} \to X$. To compute the pushforward of the class of $(S \otimes K^r)|_{\widetilde{X}}$ under the projection $(S \otimes K^r)|_{\widetilde{X}} \subset \mathbb{A}^{r \times n} \times \widetilde{X} \to \mathbb{A}^{r \times n}$, we apply [45, Proposition 5.1], gives the answer as a sum over all torus-fixed points. Here, we abuse notation by letting $(S \otimes K^r)|_{\widetilde{X}}$ denote the pullback of $S \otimes K^r$ to $\widetilde{X}$.

The torus-fixed point of $\widetilde{X}$ are indexed by permutations. To see which torus-fixed points of $\widetilde{X}$ map to which points in $G(d,n)$, we first note that moment polytope of X is the matroid polytope [63, Proposition 1.2.4], and X is a normal toric variety. Let $\mathcal{B}$ be the d-element subsets of $\{1,\ldots,n\}$ that index linearly independent columns of M. Recall the matroid polytope P_M of M is the convex hull of

$$\{\sum_{i\in B} e_i \mid \ B \in \mathcal{B}\},$$

and each vertex $\sum_{i\in B} e_i$ of P_M corresponds to the torus invariant linear space $\Lambda \subset \mathbb{P}^{n-1}$ given by setting the coordinates not in B to zero.

The fan corresponding to X is the normal fan to P_M. To see how the normal fan of the permutohedron refines the normal fan of P_M, let the permutohedron P be the convex hull of $(1,\ldots,n)$ together with all its permutations. Given a small real number ϵ, $\epsilon P + P_M$ is a deformation of P whose face structure is the same as the permutohedron P.

Given a torus-fixed point p of $\widetilde{X}$, p corresponds to a vertex of P, which corresponds to a vertex of $\epsilon P + P_M$. Letting $\epsilon \to 0$, that vertex limits to a vertex of P_M. If p corresponds to the vertex $(1,\ldots,n)$ of P, then $\epsilon p + \sum_{i\in B} e_i$ is a vertex of $\epsilon P + P_M$ exactly when B is minimal among all elements of $\mathcal{B}$ in lexicographic ordering. For general p corresponding to a permutation, the corresponding vertex of P_M corresponds to the element of $\mathcal{B}$ that is minimal in lexicographic ordering after we reorder the set $\{1,\ldots,n\}$ according to the permutation.

Finally, the formula in Theorem 23.1 follows from [45, Proposition 5.1]. The summation is a summation over the torus-fixed points of $\widetilde{X}$. The denominator is corresponds to the tangent space at each of the torus fixed points together with the torus action. The numerator corresponds to the equivariant class of the linear subspace $(S \otimes K^r)|_p \subset K^n \otimes K^r$ for for every torus fixed point $p \in \widetilde{X}$. $\square$

24. A_n SINGULARITIES ARE THE MOST COMMON

As a starting point, we recall the following powerful result counting isolated hypersurface singularities in families. Let $\pi\colon W \to B$ be a smooth morphism of complex varieties and $\mathcal{H} \subset W$ be a divisor. As we vary over points $b \in B$, $\mathcal{H}_b \subset W_b$ is a family of hypersurfaces. Let Ω be an isolated hypersurface singularity class.

Theorem 24.1 ([66, Theorem 1],[79, Section 11])**.** *The cohomology class of*

$$\{p \in \mathcal{H} \mid p \in \pi^{-1}(\pi(p)) \text{ is a singularity of type } \Omega\} \subset W$$

is a polynomial P_Ω in the class of $\mathcal{H}$ and the chern classes of the relative tangent bundle $T_{W/B}$. Furthermore, P_Ω is known when Ω is the A_n, D_n or E_n singularity class for $n \leq 8$.[3]

Letting $W = \mathbb{C}P^n \times S$ for S a smooth complex surface recovers counts of curves with an A_n, D_n or E_n singularity in a linear series. Kazarian also has a generalization of Theorem 24.1 to count hypersurfaces with multiple singularities [67]. Our goal is to build upon Theorem 24.1.

Let S be a smooth surface, L a line bundle, and $V \subset H^0(L)$ a general linear series with $\dim(V) = n+1$. From Theorem 24.1, one can deduce

Corollary 24.2. *Given a singularity type Ω that occurs in codimension n, there exist constants $a(\Omega), b(\Omega), c(\Omega), d(\Omega)$ such that the number of curves in V with a singularity of type Ω is $a(\Omega)L^2 + b(\Omega)c_1(T_S)L + c(\Omega)c_1(T_S)^2 + d(\Omega)c_2(T_S)$.*

24.0.1. *Counting curve singularities asymptotically.* The problem of counting curves with a prescribed singularity has been well-studied [66, 15, 91, 69], but the constants $a(\Omega), b(\Omega), c(\Omega), d(\Omega)$ have only been determined for Ω equal to A_n, D_n or E_n singularities with $n \leq 8$. Our interest in this comes from Proposition 24.3 below. The

[3] Kazarian's result in [66, Theorem 1] and [79, Section 11] actually gives a formula that works for all dimensions. So for example, there is one formula that counts A_8 singularities for hypersurfaces of all dimensions.

results in this section should generalize to the case of hypersurface, but we will focus on the case of plane curves.

To set up Proposition 24.3, let G be the group of automorphisms on $\mathbb{C}[x,y]/(x,y)^{N+1}$ given by change of variables $x \to f(x,y)$ and $y \to g(x,y)$. Suppose we are given a closed subvariety $Z \subset \mathbb{C}[x,y]/(x,y)^{N+1}$ invariant under the action of G. Given a family $W \to B$ of complex surfaces and a relative curve $\mathcal{H} \subset W$, it makes since to ask for the locus

$$\{p \in \mathcal{H} \mid \mathcal{H}|_{\pi^{-1}(\pi(p))} \subset W|_{\pi^{-1}(\pi(p))} \text{ has equation in } Z \text{ in local analytic coordinates}\}.$$

In words, given a point $b \in B$ and $p \in \mathcal{H}_b$, the local equation for $\mathcal{H}_b$ in W_b lies in $\mathbb{C}[[x,y]]$. It makes sense to ask whether its image in $\mathbb{C}[x,y]/(x,y)^{N+1}$ is in Z. This is independent on the choice of local coordinates since Z is G-invariant. Let Ω_Z denote the singularity type associated to Z. [4]

Proposition 24.3. *We have* $a(\Omega_Z) = \binom{n+2}{2}\deg(\overline{Z})$*, where* $\overline{Z}$ *is the image of* Z *in* $\mathbb{P}(\mathbb{C}[x,y]/(x,y)^{N+1})$.

Proof. To illustrate the method, we start with a family $W \to B$ of complex surfaces and a relative curve $\mathcal{H} \subset W$, and then specialize to the case of linear series on a surface. In the process, we will also show the existance of the polynomial P_{Ω_Z} in Theorem 24.1. On W, there is a relative sheaf of principal parts $P^N(L)$, where L is the line bundle given by the Cartier divisor $\mathcal{H}$.

There is a section σ of $P^N(L)$ induced by $\mathcal{H}$ and we are interested in the locus $p \in \mathcal{H}$ where

$$\sigma_p \in P^N(L)|_p \cong \mathbb{C}[x,y]/(x,y)^{N+1}$$

[4] We are defining singularity type as having an analytic expansion in Z. One could alternatively define it to be analytically equivalent or topological equivalent to a given curve at the origin, and our definition encompasses both of these.

is in the locus Z. Equivalently, we have a locus $\mathcal{Z} \subset P^N(L)$ that restricts to Z in each fiber. (Since Z is invariant under the group G consisting of change of coordinates, we can construct $\mathcal{Z}$ by taking a trivialization, and the result will be independent of the trivialization.) The class we are interested in

$$[\mathcal{Z}] \in A^\bullet(P^N(L)) \cong A^\bullet(W).$$

At this point, one way to approach the problem and is to note that this class is a specialization of the $\mathbb{G}_m \times G$-equivariant cohomology class of $Z \subset \mathbb{C}[x,y]/(x,y)^{N+1}$ and G is homotopy equivalent to GL_2. Instead of exploiting this, we will instead note that one can specialize $P^N(L)$ to $L \oplus L \otimes \Omega_{W/B} \oplus \cdots \oplus L \otimes \operatorname{Sym}^N \Omega_{W/B}$ and $\mathcal{Z}$ will specialize to the locus $\mathcal{Z}_0 \subset L \oplus L \otimes \Omega_{W/B} \oplus \cdots \oplus L \otimes \operatorname{Sym}^N \Omega_{W/B}$ that is Z in each fiber.

Now $\mathcal{Z}_0$ can be computed as the $\mathbb{G}_m \times GL_2$-equivariant cohomology of $Z \subset \mathbb{C}[x,y]/(x,y)^{N+1}$, where GL_2 acts linearly on x and y and $\mathbb{G}_m$ acts as scaling. This shows that the class of $\mathcal{Z}_0$ (and hence the class of $\mathcal{Z}$) can be expressed in $A^\bullet(W)$ as a polynomial P_{Ω_Z} in $\mathcal{H} = c_1(L)$ and $c_1(\Omega_{W/B}$ and $c_2(\Omega_{W/B})$.

Furthermore, the degree $\deg(\overline{Z})$ of $\overline{Z} \subset \mathbb{P}(\mathbb{C}[x,y]/(x,y)^{N+1})$ is the coefficient of P_{Ω_Z} when we set $c_1(\Omega_{W/B}$ and $c_2(\Omega_{W/B})$ to be zero, as this is the same as taking the $\mathbb{G}_m$-equivariant class of Z.

Finally, we specialize to the case $W = S \times B$ for S a smooth surface and $B = \mathbb{P}(V)$ for $V \subset H^0(L)$. Therefore, we are after the number

$$\int_{S \times \mathbb{P}(V)} P_{\Omega_Z}(c_1(L) + \mathscr{O}_{\mathbb{P}(V)}(1), c_1(S), c_2(S)).$$

The coefficient of $c_1(L)^2$ of $P_{\Omega_Z}(c_1(L) + \mathscr{O}_{\mathbb{P}(V)}(1), c_1(S), c_2(S))$ after pushing forward under $S \times \mathbb{P}(V) \to S$ is $\binom{n+2}{2}$ times the coefficient of $c_1(L)^{n+2}$, which is $\binom{n+2}{2} \deg(\overline{Z})$.

$\square$

24.0.2. *The squaring locus and A_n singularities.* The locus $Z \subset \mathbb{C}[x,y]/(x,y)^{N+1}$ associated to A_n singularities is easy to describe explicitly.

Proposition 24.4. *The locus $Z_n \subset \mathbb{C}[x,y]/(x,y)^{n+1}$ given as the squares*

$$\overline{\{f^2 \mid f \in \mathbb{C}[x,y], f(0,0) = 0\}}$$

corresponds to A_n singularities.

Proof. A function $f \in \mathbb{C}[[x,y]]$ with an A_n singularity is n-determined [52, Corollary 2.24], meaning $f \equiv g \pmod{(x,y)^{n+1}}$ implies f and g are analytically equivalent. Therefore, A_n singularities correspond to some locus $Z'_n \subset \mathbb{C}[x,y]/(x,y)^{n+1}$.

The cases $n = 1, 2$ are trivial, so we assume $n \geq 3$. Then, x^2 must be in Z'_n since a double line is a degeneration of an A_n singularity for all n. However, we know Z'_n must be G-invariant, so Z'_n contains the orbit of x^2, which is Z_n. Finally, by codimension reasons, Z_n and Z'_n coincide. $\square$

Given the explicit form of the locus Z_n and Proposition 24.3, it makes sense to ask

Question 24.5. *What is the degree of $\overline{Z}_n \subset \mathbb{P}(\mathbb{C}[x,y]/(x,y)^{n+1})$?*

The answer to Question 24.5 for $n = 1, \ldots, 7$ is $1, 2, 5, 12, 30, 79, 217$, which returns no hits when searched in the online encyclopedia of integer sequences.

Perhaps more interesting is the following question:

Question 24.6. *For $N >> 0$, among all codimension $n+2$ subvarieties of $\mathbb{P}(\mathbb{C}[x,y]/(x,y)^{N+1})$ that are preserved under the action of G, is the subvariety of maximal dimension the closure of the inverse image of $\overline{Z_n}$ under*

$$\mathbb{P}(\mathbb{C}[x,y]/(x,y)^{N+1}) \dashrightarrow \mathbb{P}(\mathbb{C}[x,y]/(x,y)^{n+1})?$$

Informally, if we invoke Proposition 24.3, this is asking if A_n singularities are the most common curve singularity occuring in codimension n we see with sufficient positivity. Given the formulas in [79, Section 11], one should suspect that not much positivity is actually required.

References

[1] J. Alper and A. Isaev. Associated forms and hypersurface singularities: the binary case. *J. Reine Angew. Math.*, 745:83–104, 2018.

[2] P. Aluffi and C. Faber. Linear orbits of smooth plane curves. *MPI Preprint*, 25, 1991.

[3] P. Aluffi and C. Faber. Linear orbits of d-tuples of points in $\mathbf{P}^1$. *J. Reine Angew. Math.*, 445:205–220, 1993.

[4] P. Aluffi and C. Faber. Linear orbits of smooth plane curves. *J. Algebraic Geom.*, 2(1):155–184, 1993.

[5] P. Aluffi and C. Faber. Linear orbits of arbitrary plane curves. *Michigan Math. J.*, 48:1–37, 2000. Dedicated to William Fulton on the occasion of his 60th birthday.

[6] P. Aluffi and C. Faber. Plane curves with small linear orbits. I. *Ann. Inst. Fourier (Grenoble)*, 50(1):151–196, 2000.

[7] P. Aluffi and C. Faber. Plane curves with small linear orbits. II. *Internat. J. Math.*, 11(5):591–608, 2000.

[8] P. Aluffi and C. Faber. Limits of PGL(3)-translates of plane curves. I. *J. Pure Appl. Algebra*, 214(5):526–547, 2010.

[9] P. Aluffi and C. Faber. Limits of PGL(3)-translates of plane curves. II. *J. Pure Appl. Algebra*, 214(5):548–564, 2010.

[10] D. Anderson. Introduction to equivariant cohomology in algebraic geometry. In *Contributions to algebraic geometry*, EMS Ser. Congr. Rep., pages 71–92. Eur. Math. Soc., Zürich, 2012.

[11] D. Anderson. Introduction to equivariant cohomology in algebraic geometry. In *Contributions to algebraic geometry*, EMS Ser. Congr. Rep., pages 71–92. Eur. Math. Soc., Zürich, 2012.

[12] E. Arbarello and M. Cornalba. The Picard groups of the moduli spaces of curves. *Topology*, 26(2):153–171, 1987.

[13] A. Arsie and A. Vistoli. Stacks of cyclic covers of projective spaces. *Compos. Math.*, 140(3):647–666, 2004.

[14] S. Asgarli and G. Inchiostro. The picard group of the moduli of smooth complete intersections of two quadrics. *preprint*, 2017. arXiv:1710.10113.

[15] S. Basu and R. Mukherjee. Enumeration of curves with one singular point. *J. Geom. Phys.*, 104:175–203, 2016.

[16] M. Brion. Lectures on the geometry of flag varieties. In *Topics in cohomological studies of algebraic varieties*, Trends Math., pages 33–85. Birkhäuser, Basel, 2005.

[17] T. D. Browning and D. R. Heath-Brown. The density of rational points on non-singular hypersurfaces. II. *Proc. London Math. Soc. (3)*, 93(2):273–303, 2006. With an appendix by J. M. Starr.

[18] P. Burgisser and A. Lerario. Probabilistic schubert calculus. *Journal für die reine und angewandte Mathematik (Crelles Journal)*, forthcoming.

[19] C. Cadman and R. Laza. Counting the hyperplane sections with fixed invariants of a plane quintic—three approaches to a classical enumerative problem. *Adv. Geom.*, 8(4):531–549, 2008.

[20] C. Cadman and R. Laza. Counting the hyperplane sections with fixed invariants of a plane quintic—three approaches to a classical enumerative problem. *Adv. Geom.*, 8(4):531–549, 2008.

[21] S. Casalaina-Martin and R. Laza. Simultaneous semi-stable reduction for curves with ADE singularities. *Trans. Amer. Math. Soc.*, 365(5):2271–2295, 2013.

[22] D. Chen, G. Farkas, and I. Morrison. Effective divisors on moduli spaces of curves and abelian varieties. In *A celebration of algebraic geometry*, volume 18 of *Clay Math. Proc.*, pages 131–169. Amer. Math. Soc., Providence, RI, 2013.

[23] E. Cotterill. Effective divisors on $\overline{\mathscr{M}}_g$ associated to curves with exceptional secant planes. *Manuscripta Math.*, 138(1-2):171–202, 2012.

[24] A. Di Lorenzo. The Chow Ring of the Stack of Hyperelliptic Curves of Odd Genus. *International Mathematics Research Notices*, 07 2019. rnz101.

[25] D. Edidin and D. Fulghesu. The integral Chow ring of the stack of hyperelliptic curves of even genus. *Math. Res. Lett.*, 16(1):27–40, 2009.

[26] D. Edidin and D. Fulghesu. The integral Chow ring of the stack of hyperelliptic curves of even genus. *Math. Res. Lett.*, 16(1):27–40, 2009.

[27] D. Edidin and W. Graham. Equivariant intersection theory. *Invent. Math.*, 131(3):595–634, 1998.

[28] D. Edidin and W. Graham. Equivariant intersection theory. *Invent. Math.*, 131(3):595–634, 1998.

[29] D. Edidin and W. Graham. Localization in equivariant intersection theory and the Bott residue formula. *Amer. J. Math.*, 120(3):619–636, 1998.

[30] D. Eisenbud and J. Harris. Limit linear series: basic theory. *Invent. Math.*, 85(2):337–371, 1986.

[31] D. Eisenbud and J. Harris. The Kodaira dimension of the moduli space of curves of genus ≥ 23. *Invent. Math.*, 90(2):359–387, 1987.

[32] D. Eisenbud and J. Harris. *3264 and all that—a second course in algebraic geometry*. Cambridge University Press, Cambridge, 2016.

[33] G. Farkas. Syzygies of curves and the effective cone of $\overline{\mathscr{M}}_g$. *Duke Math. J.*, 135(1):53–98, 2006.

[34] G. Farkas. Birational aspects of the geometry of $\overline{\mathscr{M}}_g$. In *Surveys in differential geometry. Vol. XIV. Geometry of Riemann surfaces and their moduli spaces*, volume 14 of *Surv. Differ. Geom.*, pages 57–110. Int. Press, Somerville, MA, 2009.

[35] G. Farkas. Koszul divisors on moduli spaces of curves. *Amer. J. Math.*, 131(3):819–867, 2009.

[36] G. Farkas and M. Popa. Effective divisors on $\overline{\mathscr{M}}_g$, curves on $K3$ surfaces, and the slope conjecture. *J. Algebraic Geom.*, 14(2):241–267, 2005.

[37] G. Farkas and R. Rimányi. Quadric rank loci on moduli of curves and k3 surfaces. *preprint*, 2018. arXiv:1707.00756.

[38] M. Fedorchuk. Moduli spaces of hyperelliptic curves with A and D singularities. *Math. Z.*, 276(1-2):299–328, 2014.

[39] M. Fedorchuk. GIT semistability of Hilbert points of Milnor algebras. *Math. Ann.*, 367(1-2):441–460, 2017.

[40] M. Fedorchuk and A. Isaev. Stability of associated forms. *J. Algebraic Geom.*, 28(4):699–720, 2019.

[41] L. M. Fehér, A. Némethi, and R. Rimányi. Degeneracy of 2-forms and 3-forms. *Canad. Math. Bull.*, 48(4):547–560, 2005.

[42] L. M. Fehér, A. Némethi, and R. Rimányi. Coincident root loci of binary forms. *Michigan Math. J.*, 54(2):375–392, 2006.

[43] L. M. Fehér and R. Rimányi. Calculation of Thom polynomials and other cohomological obstructions for group actions. In *Real and complex singularities. Proceedings of the 7th international workshop, Universidade de São Paulo, São Carlos, Brazil, July 29–August 2, 2002*, pages 69–93. Providence, RI: American Mathematical Society (AMS), 2004.

[44] L. M. Fehér and R. Rimányi. Thom polynomial computing strategies. A survey. In *Singularity theory and its applications*, volume 43 of *Adv. Stud. Pure Math.*, pages 45–53. Math. Soc. Japan, Tokyo, 2006.

[45] L. M. Fehér and R. Rimányi. Thom series of contact singularities. *Ann. of Math. (2)*, 176(3):1381–1426, 2012.

[46] K. Ford. The distribution of integers with a divisor in a given interval. *Ann. of Math. (2)*, 168(2):367–433, 2008.

[47] H. Franzen and M. Reineke. Cohomology rings of moduli of point configurations on the projective line. *Proc. Amer. Math. Soc.*, 146(6):2327–2341, 2018.

[48] D. Fulghesu and A. Vistoli. The Chow ring of the stack of smooth plane cubics. *Michigan Math. J.*, 67(1):3–29, 2018.

[49] D. Fulghesu and F. Viviani. The Chow ring of the stack of cyclic covers of the projective line. *Ann. Inst. Fourier (Grenoble)*, 61(6):2249–2275 (2012), 2011.

[50] W. Fulton. *Intersection theory*, volume 2 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, second edition, 1998.

[51] W. Fulton and R. Pandharipande. Notes on stable maps and quantum cohomology. In *Algebraic geometry—Santa Cruz 1995*, volume 62 of *Proc. Sympos. Pure Math.*, pages 45–96. Amer. Math. Soc., Providence, RI, 1997.

[52] G.-M. Greuel, C. Lossen, and E. Shustin. *Introduction to singularities and deformations*. Springer Monographs in Mathematics. Springer, Berlin, 2007.

[53] J. Harris. On the Kodaira dimension of the moduli space of curves. II. The even-genus case. *Invent. Math.*, 75(3):437–466, 1984.

[54] J. Harris and I. Morrison. Slopes of effective divisors on the moduli space of stable curves. *Invent. Math.*, 99(2):321–355, 1990.

[55] J. Harris and I. Morrison. *Moduli of curves*, volume 187 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1998.

[56] J. Harris and D. Mumford. On the Kodaira dimension of the moduli space of curves. *Invent. Math.*, 67(1):23–88, 1982. With an appendix by William Fulton.

[57] J. Harris and L. W. Tu. On symmetric and skew-symmetric determinantal varieties. *Topology*, 23(1):71–84, 1984.

[58] R. Hartshorne. *Algebraic geometry*. Springer-Verlag, New York-Heidelberg, 1977. Graduate Texts in Mathematics, No. 52.

[59] B. Hassett. Local stable reduction of plane curve singularities. *J. Reine Angew. Math.*, 520:169–194, 2000.

[60] B. Hassett. Moduli spaces of weighted pointed stable curves. *Adv. Math.*, 173(2):316–352, 2003.

[61] Y. Hu and S. Keel. Mori dream spaces and GIT. *Michigan Math. J.*, 48:331–348, 2000. Dedicated to William Fulton on the occasion of his 60th birthday.

[62] D. Jensen and S. Payne. On the strong maximal rank conjecture in genus 22 and 23. *preprint*, 2018. arXiv:1808.01285.

[63] M. M. Kapranov. Chow quotients of Grassmannians. I. In *I. M. Gel'fand Seminar*, volume 16 of *Adv. Soviet Math.*, pages 29–110. Amer. Math. Soc., Providence, RI, 1993.

[64] N. H. Katz. The flecnode polynomial: a central object in incidence geometry. In *Proceedings of the International Congress of Mathematicians—Seoul 2014. Vol. III*, pages 303–314. Kyung Moon Sa, Seoul, 2014.

[65] M. Kazarian. Thom polynomials for Lagrange, Legendre, and critical point function singularities. *Proc. London Math. Soc. (3)*, 86(3):707–734, 2003.

[66] M. E. Kazaryan. Multisingularities, cobordisms, and enumerative geometry. *Uspekhi Mat. Nauk*, 58(4(352)):29–88, 2003.

[67] M. E. Kazaryan. Multisingularities, cobordisms, and enumerative geometry. *Uspekhi Mat. Nauk*, 58(4(352)):29–88, 2003.

[68] S. Keel. Intersection theory of moduli space of stable n-pointed curves of genus zero. *Trans. Amer. Math. Soc.*, 330(2):545–574, 1992.

[69] D. Kerner. Enumeration of singular algebraic curves. *Israel J. Math.*, 155:1–56, 2006.

[70] D. Khosla. Tautological classes on moduli spaces of curves with linear series and a push-forward formula when $\rho = 0$. *preprint*, 2007. arXiv:0704.1340.

[71] A. Kuribayashi and K. Komiya. On Weierstrass points of non-hyperelliptic compact Riemann surfaces of genus three. *Hiroshima Math. J.*, 7(3):743–768, 1977.

[72] E. Larson. The integral chow ring of of $\overline{M_2}$. *preprint*, 2019. arXiv:1904.08081.

[73] M. Lee, A. Patel, H. Spink, and D. Tseng. Orbits in $(\mathbb{P}^r)^n$ and equivariant quantum cohomology. *preprint*, 2018. arXiv:1805.08181.

[74] M. Lee, A. Patel, and D. Tseng. Equivariant degenerations of plane curve orbits. *preprint*, 2019. arXiv:1903.10069.

[75] B. Li. Images of rational maps of projective spaces. *Int. Math. Res. Not. IMRN*, (13):4190–4228, 2018.

[76] F. Liu, B. Osserman, M. T. i Bigas, and N. Zhang. The strong maximal rank conjecture and moduli spaces of curves. *preprint*, 2018. arXiv:1808.01290.

[77] Q. Liu. *Algebraic geometry and arithmetic curves*, volume 6 of *Oxford Graduate Texts in Mathematics*. Oxford University Press, Oxford, 2002. Translated from the French by Reinie Erné, Oxford Science Publications.

[78] X. Liu and R. Pandharipande. New topological recursion relations. *J. Algebraic Geom.*, 20(3):479–494, 2011.

[79] M. g. Mikosz, P. Pragacz, and A. Weber. Positivity of Legendrian Thom polynomials. *J. Differential Geom.*, 89(1):111–132, 2011.

[80] L. A. Molina Rojas and A. Vistoli. On the Chow rings of classifying spaces for classical groups. *Rend. Sem. Mat. Univ. Padova*, 116:271–298, 2006.

[81] A. Moriwaki. The $\mathbb{Q}$-Picard group of the moduli space of curves in positive characteristic. *Internat. J. Math.*, 12(5):519–534, 2001.

[82] A. N. Nejad and A. Simis. The Aluffi algebra. *J. Singul.*, 3:20–47, 2011.

[83] B. Osserman. A simple characteristic-free proof of the Brill-Noether theorem. *Bull. Braz. Math. Soc. (N.S.)*, 45(4):807–818, 2014.

[84] B. Osserman. Connectedness of brill–noether loci via degenerations. *International Mathematics Research Notices*, page rnx325, 2018.

[85] R. Pandharipande. Equivariant Chow rings of $\mathrm{O}(k)$, $\mathrm{SO}(2k+1)$, and $\mathrm{SO}(4)$. *J. Reine Angew. Math.*, 496:131–148, 1998.

[86] R. Pandharipande. Three questions in Gromov-Witten theory. In *Proceedings of the International Congress of Mathematicians, Vol. II (Beijing, 2002)*, pages 503–512. Higher Ed. Press, Beijing, 2002.

[87] A. Patel and R. Vakil. On the chow ring of the hurwitz space of degree three covers of $\mathbb{P}^1$. *preprint*, 2016. arXiv:1505.04323.

[88] N. Penev and R. Vakil. The Chow ring of the moduli space of curves of genus six. *Algebr. Geom.*, 2(1):123–136, 2015.

[89] H. C. Pinkham. *Deformations of algebraic varieties with G_m action*. Société Mathématique de France, Paris, 1974. Astérisque, No. 20.

[90] M. Romagny. Group actions on stacks and applications. *Michigan Math. J.*, 53(1):209–236, 2005.

[91] H. Russell. Counting singular plane curves via Hilbert schemes. *Adv. Math.*, 179(1):38–58, 2003.

[92] B. Segre. The maximum number of lines lying on a quartic surface. *Quart. J. Math., Oxford Ser.*, 14:86–96, 1943.

[93] C. Sorger and M. Lehn. Chow - a sage library for computations in intersection theory, 2015.

[94] H. Spink and D. Tseng. PGL_2-equivariant strata of point configurations in $\mathbb{P}^1$. *preprint*, 2018. arXiv:1808.05719.

[95] P. Srinivasan and K. Wickelgren. An arithmetic count of the lines meeting four lines in $\mathbb{P}^3$. 2018.

[96] B. Totaro. The Chow ring of a classifying space. In *Algebraic K-theory (Seattle, WA, 1997)*, volume 67 of *Proc. Sympos. Pure Math.*, pages 249–281. Amer. Math. Soc., Providence, RI, 1999.

[97] D. Tseng. Divisors on the moduli space of curves from divisorial conditions on hypersurfaces. *preprint*, 2019. arXiv:1901.11154.

[98] D. Tzigantchev. Predegree polynomials of plane configurations in projective space. *Serdica Math. J.*, 34(3):563–596, 2008.

[99] I. Vainsencher. Counting divisors with prescribed singularities. *Trans. Amer. Math. Soc.*, 267(2):399–422, 1981.

[100] R. Vakil. MATH 216: Foundations of Algebraic Geometry.

[101] J. E. Wright. Nodal Cubics Through Eight Given Points. *Proc. London Math. Soc. (2)*, 6:52–57, 1908.

[102] A. Zinger. Enumeration of genus-three plane curves with a fixed complex structure. *J. Algebraic Geom.*, 14(1):35–81, 2005.

Quartic Plane Curve C	$p_C(c_1, c_2, c_3)$
$C_{A_6} : (x^2 + yz)^2 + 2yz^3 = 0$	$3 \cdot 112(9c_1^3 + 12c_1c_2 - 11c_3)(2c_1^3 + c_1c_2 + c_3)$
$C_{D_6} : z(xyz + x^3 + z^3)$	$3 \cdot 64(18c_1^6 + 33c_1^4c_2 + 12c_1^2c_2^2 - 85c_1^3c_3 - 11c_1c_2c_3 - 7c_3^2)$
$C_{E_6} : y^3z + x^4 + x^2y^2 = 0$	$2 \cdot 48(2c_1^3 + c_1c_2 + c_3)(9c_1^3 - 6c_1c_2 + 7c_3)$
C_{AN}: Nodal cubic union flex line	$2 \cdot 192(18c_1^6 + 33c_1^4c_2 + 12c_1^2c_2^2 + 19c_1^3c_3 - 7c_1c_2c_3 - 35c_3^2)$
C_{flex}: smooth cubic union flex line	$p_{C_{AN}} + 2p_{D_6}$
Q: Quadrilateral	$24 \cdot 16(18c_1^6 + 33c_1^4c_2 + 12c_1^2c_2^2 + 131c_1^3c_3 + 153c_1c_2c_3 - 147c_3^2)$
C_{D_4}: a general curve with D_4 singularity	$\frac{1}{4}(8p_{C_{A_6}} - p_Q)$
Two lines plus conic	$p_Q + 2p_{C_{D_4}}$
A line plus a general cubic	$p_Q + 3p_{C_{D_4}}$
Quartic with δ nodes and κ cusps and no hyperflexes	$8p_{C_{A_6}} - 2\delta p_{C_{D_6}} - \kappa p_{C_{\text{flex}}}$
A smooth quartic with n hyperflexes	$8p_{C_{A_6}} - np_{C_{E_6}}$
General smooth quartic	$8p_{C_{A_6}}$

FIGURE 1. Equivariant classes of orbits of quartic plane curves

Quartic Plane Curve C	(# Aut(C) $\cdot$ # planar sections of quartic threefold)
$C_{A_6} : (x^2 + yz)^2 + 2yz^3 = 0$	$3 \cdot 21280$
$C_{D_6} : z(xyz + x^3 + z^3)$	$3 \cdot 7040$
$C_{E_6} : y^3z + x^4 + x^2y^2 = 0$	$2 \cdot 4800$
C_{AN}: Nodal cubic union flex line	$2 \cdot 36480$
C_{flex}: smooth cubic union flex line	$2 \cdot 57600$
Q: Quadrilateral	$24 \cdot 5600$
C_{D_4}: a general curve with D_4 singularity	94080
Two lines plus conic	322560
A line plus a general cubic	416640
Quartic with δ nodes and κ cusps and no hyperflexes	$510720 - 2\delta(3 \cdot 7040) - \kappa(2 \cdot 57600)$
A smooth quartic with n hyperflexes	$510720 - n(2 \cdot 4800)$
General smooth quartic	510720

FIGURE 2. Number of times we see a particular curve as a planar section of a quartic threefold with *specified moduli*.