Universality of Lévy Matrices

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Abstract

Motivated by conjectures from physics, we study the eigenvalues and eigenvectors of Lévy matrices, which are symmetric random matrices whose upper triangular entries are independent, identically distributed α-stable distributions. For $\alpha < 2$, these distributions are heavy-tailed, with infinite second moment. For $\alpha \in (1, 2)$, we show that at all finite non-zero energies, Lévy matrices exhibit completely delocalized eigenvectors and local eigenvalue statistics that asymptotically match those of the Gaussian Orthogonal Ensemble. For almost all $\alpha \in (0, 2)$, we prove the same result for small energies, including zero. Additionally, for almost all $\alpha \in (0, 2)$, we analyze the statistics of eigenvector entries of Lévy matrices at small energies and show that the limiting distribution of any such entry is non-Gaussian. For entries of the eigenvector corresponding to the median eigenvalue, we identify this distribution explicitly. We also demonstrate the presence of non-trivial correlations between eigenvector entries corresponding to nearby eigenvalues. These findings contrast sharply with the known eigenvector behavior for other random matrix ensembles. Further, our results for both eigenvalues and eigenvectors generalize to a large class of heavy-tailed random matrices.

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Chapter 1

Introduction

1.1 Background

The spectral analysis of random matrices has been a topic of intense study since Wigner's pioneering investigations in the 1950s [99]. Wigner's central thesis asserts that the spectral statistics of random matrices are universal models for highly correlated systems. A concrete realization of his vision, the Wigner–Dyson–Mehta conjecture, states that the bulk local spectral statistics of an $N \times N$ real symmetric (or complex Hermitian) Wigner matrix should become independent of the laws of its entries as N tends to infinity [84, Conjecture 1.2.1]. This phenomenon is known as bulk universality. While Wigner was able to show the universality of the global spectral distribution, made precise by his famous semicircle law, rigorous results on local spectral statistics remained out of reach until only recently.

Over the past decade, a framework based on resolvent estimates and Dyson Brownian motion has been developed to establish the Wigner–Dyson–Mehta conjecture and extend its conclusion to a wide class of matrix models. These include Wigner matrices [41, 52, 57, 61, 62, 64, 70–72, 79, 95, 97], correlated random matrices [10, 48], random graph models [22, 54, 56, 69, 70], general Wigner-type matrices [11, 12], certain families of band matrices [39, 40, 43, 45, 100], and various other models. All these models require that the variance of each matrix entry is finite, an assumption already present in the original universality conjectures [84]. The moment assumption required for the bulk universality of Wigner matrices has been progressively improved, and universality is now known to hold for matrix entries with finite $(2+\varepsilon)$-th moments [4, 54].

While finite variance might seem to be the natural assumption for the Wigner–Dyson–Mehta conjecture, in 1994 the physicists Cizeau and Bouchaud [50] asked to what extent local eigenvalue statistics and related phenomena remain universal once the finite variance constraint is removed. Their work was motivated by heavy-tailed phenomena in physics [35, 90], including the study of spin glass models with power-law interactions [49], and applications to finance [36–38, 46, 66, 75, 76]. Recent work has also shown the appearance of heavy-tailed spectral statistics in neural networks [81–83]. Heavy-tailed random matrices may therefore be regarded as exemplars of a new universality class for highly correlated systems.

The authors of [50] proposed a family of symmetric random matrix models, called Lévy matrices, whose entries are random variables in the domain of attraction of an α-stable law.[1] Based on numerical simulations, they predicted that bulk universality should still hold in certain regimes when $\alpha < 2$. In particular, for $\alpha < 1$ they proposed that the local statistics of Lévy matrices should exhibit a sharp phase transition from GOE at small energies to Poisson at large energies.

Such a transition is called a mobility edge (also known as an Anderson transition or Mott transition, depending on the physical context) and is a principal concept in the pathbreaking work of the physicists Anderson and Mott on metal–insulator transitions in condensed matter physics [3, 16, 17, 87, 88]. It is widely believed to exist in the context of random Schrödinger operators, particularly in the Anderson tight-binding model [1, 2, 8, 9, 21], but rigorously establishing this statement has remained a fundamental open problem in mathematical physics for decades. While localization and Poisson statistics at large energies in the Anderson model have been known since the 1980s, even the existence of a delocalized phase

[1]When $\alpha < 2$, we recall that the densities of such laws decay asymptotically like $x^{-\alpha-1}\,dx$. In particular, they have infinite second moment.

with GOE local statistics has not been rigorously verified for any model exhibiting a conjectural mobility edge [7,51,65,67,80,85,92]. As explained below, Lévy matrices provide one of the few examples of a random matrix ensemble for which such a transition is also believed to appear. Consequently, the predictions of [50] have attracted significant attention among mathematicians and physicists over the past 25 years [18,19,24,25,25,30–34,47,91,98].

1.2 Delocalization and eigenvalue statistics

The work [50] further analyzed the large N limiting profile for the empirical spectral distribution of a Lévy matrix $\mathbf{H}$, defined by $\mu_{\mathbf{H}} = N^{-1}\sum_{j=1}^{N}\delta_{\lambda_j}$, where $\lambda_1, \lambda_2, \ldots, \lambda_N$ denote the eigenvalues of $\mathbf{H}$. They predicted that $\mu_{\mathbf{H}}$ should converge to a deterministic, explicit measure μ_α as N tends to infinity, which was later proven by Ben Arous and Guionnet [18]. This measure μ_α is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$ and therefore admits a density ϱ_α, which is symmetric and behaves as $\varrho_\alpha(x) \sim \frac{\alpha}{2x^{\alpha+1}}$ as x tends to infinity [18,24,32]. In particular, ϱ_α is supported on all of $\mathbb{R}$ and has an α-heavy tail. This contrasts with the limiting spectral density for Wigner matrices, given by the semicircle law,

$$\varrho_{\mathrm{sc}}(x) = (2\pi)^{-1}\mathbf{1}_{|x|<2}\sqrt{4-x^2}, \tag{1.2.1}$$

which is compactly supported on $[-2, 2]$.

Two other phenomena of interest are eigenvector delocalization and local spectral statistics. Associated with any eigenvalue λ_k of $\mathbf{H}$ is an eigenvector $\mathbf{u}_k = (u_{1k}, u_{2k}, \ldots, u_{Nk}) \in \mathbb{R}^N$, normalized such that $\|\mathbf{u}_k\|_2^2 = \sum_{i=1}^{N} u_{ik}^2 = 1$. If $\mathbf{H} = \mathbf{GOE}_N$ is instead taken from the Gaussian Orthogonal Ensemble[2] (GOE), then the law of $\mathbf{u}_k$ is uniform on the $(N-1)$-sphere, and so $\max_{1\le i\le N}\left|u_{ik}\right| \le N^{\delta-1/2}$ holds with high probability for any $\delta > 0$. This bound is referred to as complete eigenvector delocalization. The local spectral statistics of $\mathbf{H}$ concern

[2]This is defined to be the $N \times N$ real symmetric random matrix $\mathbf{GOE}_N = \{g_{ij}\}$, whose upper triangular entries g_{ij} are mutually independent Gaussian random variables with variances $2N^{-1}$ if $i = j$ and N^{-1} otherwise.

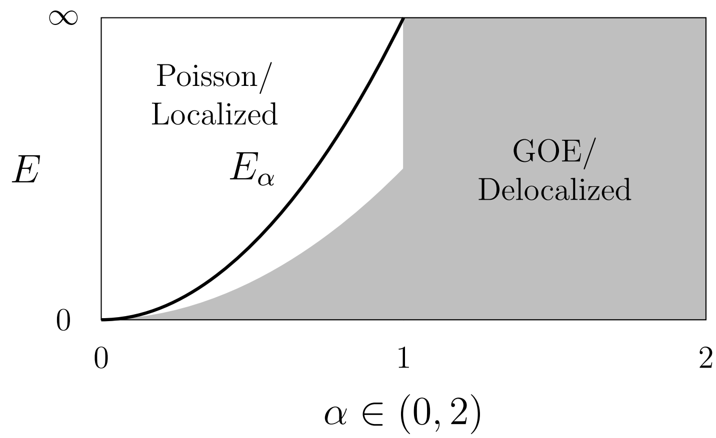

Figure 1.1: Phase diagram. The thick line indicates the location of the conjectural mobility edge, which separates the localized phase from the delocalized phase. The gray area indicates the scope of our results.

the behavior of its neighboring eigenvalues close to a fixed energy level $E \in \mathbb{R}$.

The main predictions of [50] were certain transitions in the eigenvector behavior and local spectral statistics of Lévy matrices. Their predictions are not fully consistent with the recent work [98] by Tarquini, Biroli, and Tarzia, based on the supersymmetric method. The latter predictions can be summarized as follows.

A $(1 \leq \alpha < 2)$ All eigenvectors of $\mathbf{H}$ corresponding to finite eigenvalues are completely delocalized. Further, for any $E \in \mathbb{R}$, the local statistics of $\mathbf{H}$ near E converge to those of the GOE as N tends to infinity.

B $(0 < \alpha < 1)$ There exists a mobility edge E_α such that (i) if $|E| < E_\alpha$ then the local statistics of $\mathbf{H}$ near E converge to those of the GOE and all eigenvectors in this region are completely delocalized; (ii) if $|E| > E_\alpha$, then the local statistics of $\mathbf{H}$ near E converge to those of a Poisson point process and all eigenvectors in this region are localized.

The earlier predictions of [50] are different: **A' $(1 \leq \alpha < 2)$:** There are two regions: (i) for sufficiently small energies, the eigenvectors are completely delocalized and the local statistics are GOE; (ii) for sufficiently large energies, the eigenvectors are weakly localized according to a power law decay, and the local statistics are given by certain non-universal laws that converge to Poisson statistics in the infinite energy limit; **B' ($0 < \alpha < 1$):** essentially the same as prediction **B** above except that in the delocalized region the eigenvectors were predicted to only be partially delocalized, in that a positive proportion of the mass is completely delocalized and a positive proportion of the mass is completely localized. In addition, [50] proposes an equation for the mobility edge E_α; a much simpler (but equivalent) version of this equation was predicted in [98].

The problem of rigorously establishing this mobility edge remains open. In fact, there have been no previous mathematical results on local statistics for Lévy matrices in any regime. However, partial results on eigenvector (de)localization were established by Bordenave and Guionnet in [33,34]. If $1 < \alpha < 2$, they showed that almost all eigenvectors $\mathbf{u}_k$ satisfy $\max_{1 \le i \le N} |u_{ik}| < N^{\delta - \rho}$ for any $\delta > 0$ with high probability, where $\rho = \frac{\alpha - 1}{\max\{2\alpha, 8 - 3\alpha\}}$ [33]. For almost all $\alpha \in (0, 2)$, they also proved the existence of some $c = c(\alpha)$ such if $\mathbf{u}_k$ is an eigenvector of $\mathbf{H}$ corresponding to an eigenvalue $\lambda_k \in [-c, c]$, then $\max_{1 \le i \le N} |u_{ik}| < N^{\delta - \alpha/(4+2\alpha)}$ for any $\delta > 0$ with high probability [34]. These estimates remain far from the complete delocalization bounds that have been established in the Wigner case. Furthermore, if $0 < \alpha < \frac{2}{3}$ and $\mathbf{G}(z) = \big\{ G_{ij}(z) \big\} = (\mathbf{H} - z)^{-1}$, then they showed that $\mathbb{E}\big[\big(\operatorname{Im} G_{11}(z) \big)^{\alpha/2} \big] = O(\eta^{\alpha/2 - \delta})$ for any $\delta > 0$ if $\operatorname{Re} z$ is sufficiently large and $\eta = \operatorname{Im} z \gg N^{-(2+\alpha)/(4\alpha+12)}$, which implies eigenvector localization in a certain weak sense at large energy [33].

In this dissertation, we establish complete delocalization and bulk universality for Lévy matrices for all energies in any fixed compact interval away from $E = 0$ if $1 < \alpha < 2$. In addition, for $0 < \alpha < 2$ outside a (deterministic) countable set, we prove that there exists $\widetilde{E}_\alpha$ such that complete delocalization and bulk universality hold for all energies in $[-\widetilde{E}_\alpha, \widetilde{E}_\alpha]$. These results establish the prediction **A** of [98] essentially completely for $1 < \alpha < 2$ and also the existence of the GOE regime for $0 < \alpha < 1$, with completely delocalized eigenvectors. Before describing these results in more detail, we recall the three-step strategy for establishing bulk universality of Wigner matrices developed in [52, 58, 60–62, 95] (see [26, 53] or the book

[52] for a survey).

The first step is to establish a local law for $\mathbf{H}$, meaning that the spectral density of $\mathbf{H}$ asymptotically follows that of its deterministic limit on small scales of order nearly N^{-1}, the typical inter-eigenvalue distance. The second step is to consider a Gaussian perturbation $\mathbf{H}+t^{1/2}\mathbf{GOE}_N$ of $\mathbf{H}$, for some small t, and then use the local law to show that the local statistics of the perturbed matrix are universal. The third step is to compare the local statistics of $\mathbf{H}$ and its perturbed variant $\mathbf{H}+t^{1/2}\mathbf{GOE}_N$, and show that they are asymptotically the same. The comparison of the local statistics can be most efficiently obtained by comparing the entries of the resolvents of the ensembles; this is often referred to as a Green's function (resolvent) comparison theorem [64].

There are two issues with adapting this framework to the heavy-tailed setting. First, we do not know of a direct way to establish a local law for the α-stable matrix $\mathbf{H}$ on the optimal scale of roughly N^{-1}. Second, justifying the removal of the Gaussian perturbation in the third step has intrinsic problems since the entries of $\mathbf{H}$ have divergent variances (and possibly divergent expectations).

To explain the first problem, we introduce some notation. We recall the Stieltjes transform of the empirical spectral distribution $\mu_{\mathbf{H}}$ is defined by the function

$$m_N(z) = m_{N,\mathbf{H}}(z) = \frac{1}{N}\sum_{j=1}^{N}\frac{1}{\lambda_j - z} = \frac{1}{N}\operatorname{Tr}\big(\mathbf{H}-z\big)^{-1}, \tag{1.2.2}$$

for any $z \in \mathbb{H}$. Since $\mu_{\mathbf{H}}$ converges weakly to μ_α as N tends to infinity, one expects $m_N(z)$ to converge to $m_\alpha(z) = \int_{\mathbb{R}}(x-z)^{-1}\varrho_\alpha(x)\,dx$. The imaginary part of the Stieltjes transform represents the convolution of the empirical spectral distribution with an approximate identity, the Poisson kernel, at scale $\eta = \operatorname{Im} z$. Hence, control of the Stieltjes transform at scale η can be thought of as control over the eigenvalues averaged over windows of size approximately η.

A local law for $\mathbf{H}$ is an estimate on $\big|m_N(z) - m_\alpha(z)\big|$ when $\eta = \operatorname{Im} z$ scales like $N^{-1+\varepsilon}$.

The typical procedure [55, 56, 59, 60, 64, 79] for establishing a local law relies on a detailed understanding of the resolvent of $\mathbf{H}$, defined to be the $N \times N$ matrix $\mathbf{G}(z) = \big(\mathbf{H} - z\big)^{-1} = \big\{G_{ij}(z)\big\}$. Indeed, since $m_N(z) = N^{-1} \operatorname{Tr} \mathbf{G}(z)$, it suffices to estimate the diagonal entries of $\mathbf{G}$. In many of the known finite variance cases, (almost) all of the entries G_{ij} converge to a deterministic quantity in the large N limit.

This is no longer true in the heavy-tailed setting, where the limiting resolvent entries are instead random away from the real axis [32]. While the idea that the resolvent entries should satisfy a self-consistent equation (which has been a central concept in proving local laws for Wigner matrices [59]) is still applicable to the current setting [18, 24, 33, 34], the random nature of these resolvent entries poses many difficulties in analyzing the resulting self-consistent equation. This presents serious difficulties in applying previously developed methods to establish a local law for α-stable matrices on the optimal scale. While local laws on intermediate scales $\eta \gg N^{-1/2}$ were established for such matrices in [33, 34] if α is sufficiently close to two, the value of η allowed in these estimates deteriorates to 1 as α decreases to zero.

For the second problem, all existing methods of comparing two matrix ensembles $\mathbf{H}$ and $\widetilde{\mathbf{H}}$ [4,41,54,56,70,79,94,95,97] involve Taylor expanding the matrix entries of their resolvents to order at least three and then matching the expectations of the first and second order terms of this expansion, which correspond to the first and second moments of the matrix entries. If the entries of $\mathbf{H}$ and $\widetilde{\mathbf{H}}$ are heavy-tailed, then all second and higher moments of these matrix entries diverge, and this expansion is meaningless.

These two difficulties are in fact intricately related, and our solution to them consists of the following steps.

1. We first rewrite the matrix as $\mathbf{H} = \mathbf{X} + \mathbf{A}$, where $\mathbf{A}$ consists of the "small terms" of $\mathbf{H}$ that are bounded by $N^{-\nu}$ in magnitude for some constant $0 < \nu < \frac{1}{\alpha}$, and $\mathbf{X}$ consists of the remaining terms. We prove a comparison theorem for the resolvent entries of $\mathbf{H} = \mathbf{X} + \mathbf{A}$ and those of $\mathbf{V}_t = \mathbf{X} + t^{1/2}\mathbf{GOE}_N$, where $\mathbf{GOE}_N$ is independent from $\mathbf{X}$. The parameter

$t \sim N^{\nu(\alpha-2)}$ will be chosen so that the variances of matrix entries of $t^{1/2}\mathbf{GOE}_N$ and $\mathbf{A}$ match. By construction, $\mathbf{A}$ and $\mathbf{X}$ are symmetric, so the first and third moments of the matrix entries vanish. Hence in the comparison argument, the problem is reduced to considering the second and fourth order terms.

Notice that $\mathbf{A}$ and $\mathbf{X}$ are dependent, so the previous heuristic cannot be applied directly. To remove their correlation, we expand upon a procedure introduced in [4] to produce a three-level decomposition of $\mathbf{H}$. By conditioning on the decomposition into large and small field regions, $\mathbf{A}$ and $\mathbf{X}$ are independent and a version of the comparison theorem can be proven.

2. From the work of [78], the GOE component in $\mathbf{V}_t$ improves the regularity of the initial data $\mathbf{V}_0$, which is a manifestation of the parabolic regularization of the Dyson Brownian motion flow. Roughly speaking, if the spectral density of $\mathbf{V}_0$ is bounded above and below at a scale $\eta \leq N^{-\delta}t$, then the following three properties for $\mathbf{V}_t$ hold: (i) universality of local statistics, (ii) complete eigenvector delocalization, and (iii) a local law at all scales up to $\eta \geq N^{\delta-1}$ for any $\delta > 0$ [42, 63, 77, 78].

The fundamental input for this method is an intermediate local law for $\mathbf{X}$ on a scale $\eta \ll N^{\nu(\alpha-2)} \sim t$. The existing intermediate local laws for heavy tailed matrices established in [33, 34] are unfortunately only valid on scales larger than this critical scale when α is close to one. Our second main result is to improve these laws to scales below $N^{\nu(\alpha-2)}$. Our method uses self-consistent equations for the resolvent entries and special tools developed in [33] for Lévy matrices. Note that the resolvent entries of $\mathbf{X}$ are random and self-consistent equations for them are difficult to work with. Still, we are able to derive effective upper bounds on the diagonal resolvent entries of $\mathbf{X}$, which enable us to improve the intermediate local laws to scales below $N^{\nu(\alpha-2)}$.

3. Combining steps 1 and 2, we are able to transport the three properties for $\mathbf{V}_t$ to our original matrix $\mathbf{H}$. Recall that in the standard comparison theorem, resolvent bounds on the optimal scale are required on both ensembles. Since our initial estimates on the resolvent of

the original matrix $\mathbf{H}$ are far from on the optimal scale, a different approach is required. In particular, it is known that one can induct on the scale η to transfer resolvent estimates from one ensemble to another using the comparison method [74]. Although technical estimates must be supplied, the upshot of this step is that all three properties for $\mathbf{V}_t$ hold for the original matrix $\mathbf{H}$. The eigenvector delocalization and universality of local statistics constitute our main results. For the sake of brevity, we will not pursue the local law on the optimal scale of approximately N^{-1}, since it will not be needed to prove our main results.

We prove our results on delocalization and GOE statistics in Chapter 2. In Section 2.1 we explain our results in more detail. In Section 2.2 we state the comparison between $\mathbf{H}$ and $\mathbf{V}_t$, as well as the intermediate local laws for $\mathbf{X}$ in the $\alpha \in (1, 2)$ case and the small energy $\alpha \in (0, 2)$ case. Then, assuming these estimates, we establish our results (given by Theorem 2.1.4 and Theorem 2.1.5). In Section 2.3 we establish the comparison between $\mathbf{H}$ and $\mathbf{V}_t$. In Section 2.4 and Section 2.5 we establish the intermediate local law on $\mathbf{X}$ at all energies away from 0 when $\alpha \in (1, 2)$. In Section 2.6 and Section 2.7 we show a similar intermediate local law on $\mathbf{X}$, but at sufficiently small energies and for almost all $\alpha \in (0, 2)$.

1.3 Eigenvector statistics

Universality for the eigenvector entries of Wigner matrices was recently proven in [44] by Bourgade and Yau. There, they introduced the eigenvector moment flow, a system of stochastic differential equations which govern the evolution of the moments of eigenvector entries of a matrix under the addition of Gaussian noise. Through a careful analysis of these dynamics, they prove asymptotic normality for the eigenvector entries of Wigner matrices. Extensions of this method later enabled the analysis of eigenvector statistics for sparse and deformed Wigner matrices in [27, 42], and for other eigenvector observables in [28, 45].

While the results and predictions in the previous section address (de)localization of Lévy matrix eigenvectors, little is known about refined properties of their entry fluctuations. In

[25], Benaych-Georges and Guionnet showed that averages of $O(N^2)$ of these eigenvector entries converge to Gaussian processes, after scaling by $N^{-1/2}$. However, until now, we have not been aware of any results or predictions concerning fluctuations for individual entries of Lévy matrix eigenvectors.

In this dissertation we establish several such results, which in many respects contrast with their known counterparts for Wigner matrices (and all other random matrix models for which the eigenvector entry distributions have previously been identified). We establish, for almost all $\alpha < 2$, the following statements concerning the unit (in L^2) Lévy eigenvectors $\mathbf{u}_k = \big(u_k(1), u_k(2), \ldots, u_k(N)\big)$ whose associated eigenvalues $\lambda_k \approx E$ are sufficiently small.

1. An eigenvector entry $\sqrt{N} u_j(i)$ is not asymptotically normal: its square converges to $\mathcal{N}^2 \cdot \mathcal{U}_\star(E)$ as N tends to infinity, where $\mathcal{N}$ is a standard normal random variable and $\mathcal{U}_\star(E)$ is an independent (non-constant and typically non-explicit) random variable that depends on E.

2. Different entries of the same eigenvector are asymptotically independent.

3. Entries of different eigenvectors with the same index are not asymptotically independent: if $k_1, k_2, \ldots, k_n \in [1, N]$ and $i \in [1, N]$ are indices such that $\lambda_{k_n} \approx E$, then the vector $\big(N\mathbf{u}_{k_1}(i)^2, N\mathbf{u}_{k_2}(i)^2, \ldots, N\mathbf{u}_{k_n}(i)^2\big)$ converges to $\big(\mathcal{N}_1^2 \cdot \mathcal{U}_\star(E), \mathcal{N}_2^2 \cdot \mathcal{U}_\star(E), \ldots, \mathcal{N}_n^2 \cdot \mathcal{U}_\star(E)\big)$, where the $\mathcal{N}_j$ are i.i.d. standard Gaussians that are independent from $\mathcal{U}_\star(E)$.

4. The law of $\mathcal{U}_\star(0)$ is given explicitly as the inverse of a one-sided $\frac{\alpha}{2}$-stable law. In particular, all asymptotic moments of the median eigenvector are also explicit.

To contextualize our results, we recall the asymptotic normality statements for Wigner eigenvectors proved in [44]. First, an individual eigenvector entry converges to a standard normal random variable. Second, different entries of the same eigenvector are asymptotically independent. Third, the same is true for entries of different eigenvectors with the same index. Our results show that, although the second of these phenomena persists in the Lévy case, the first and third do not. We further note that while Lévy matrices exhibit GOE local statistics at small energy, which only differ through a rescaling factor corresponding to the eigenvalue density, the eigenvector statistics follow a family of distinct random variables that vary in the energy parameter $E \in \mathbb{R}$. This is again unlike the Wigner eigenvectors, which display the same Gaussian statistics throughout the spectrum.

Our work confronts one of the major differences between Lévy matrices and Wigner matrices: the non-concentration of the resolvent entries. In the Wigner case, these diagonal resolvent entries converge to a deterministic quantity (the Stieltjes transform of the semicircle law) as N tends to infinity. However, for Lévy matrices $\mathbf{H}$, the diagonal entries $G_{jj}(z)$ of the resolvent $\mathbf{G}(z) = (\mathbf{H} - z)^{-1}$ converge for fixed $z \in \mathbb{H}$ to nontrivial (complex) random variables $R_\star(z)$ as N tends to infinity [32]. This is the mechanism that generates the non-Gaussian scaling limit for the limiting eigenvector entries. Indeed, the random variable $\mathcal{U}_\star(E)$ mentioned above is defined as a (multiple of a) weak limit of $R_\star(E + \mathrm{i}\eta)$, as η tends to 0.

Our proof strategy is dynamical. First, we define a matrix $\mathbf{X}$, which is coupled to the Lévy matrix $\mathbf{H}$ and obtained by setting all sufficiently small (in absolute value) entries of $\mathbf{H}$ to zero. We also introduce the Gaussian perturbation $\mathbf{X}_s = \mathbf{X} + \sqrt{s}\mathbf{W}$, where $\mathbf{W}$ is a GOE and $s \ll 1$. Under a certain choice of $s = t$, we are able to show that the eigenvector statistics of $\mathbf{H}$ are asymptotically the same as those of $\mathbf{X}_t$. Second, we identify the moments of the eigenvector entries of $\mathbf{X}_t$ in terms of entries of the resolvent matrix $\mathbf{R}(t, z) = (\mathbf{X}_t - z)^{-1}$, where $z = E + \mathrm{i}\eta$ and η tends to 0 as N tends to infinity. Third, we compute the large N limit of these resolvent entries and deduce the above claims from the behavior of the resulting scaling limits. We now describe the steps of our argument, and their associated challenges, in more detail.

1. The first step is a comparison of eigenvector statistics, which has been achieved before for Wigner matrices with entries matching in the first four moments [73, 97]. However, these results do not apply to Lévy matrices, since the second moments of their entries are infinite. Instead, we use the comparison scheme introduced above that conditions on the locations of

the large entries of $\mathbf{H}$ and matches moments between the small entries of $\mathbf{H}$ and the Gaussian perturbation $\sqrt{t}\mathbf{W}$ in $\mathbf{X}_t = \mathbf{X} + \sqrt{t}\mathbf{W}$; these have all moments finite by construction. While we previously considered the comparison for Green's function elements, we apply it to general smooth functions of the matrix entries and write the eigenvector statistics as such functions using ideas from [42, 70].

2. The second step uses the eigenvector moment flow to show that moments of the eigenvector entries of $\mathbf{X}_t$ approximate moments of the $\operatorname{Im} \mathbf{R}(t, z)$ entries. As in [42, 44], a primary idea here is to apply the maximum principle to show under these dynamics that eigenvector moment observables equilibriate after a short period of time to a polynomial in the entries of $\operatorname{Im} \mathbf{R}(t, z)$. However, unlike in those previous works, the entries of $\operatorname{Im} \mathbf{R}(s, z)$ do not concentrate and therefore might be unstable under the dynamics. To address this, we condition on the initial data $\mathbf{X}_0$, which one might hope renders the $R_{jj}(s, z)$ essentially constant under the flow $\mathbf{X}_s$ for $s \ll 1$. Unfortunately, we cannot show this directly, and in fact it appears that these resolvent entries can be unstable in the beginning of the dynamics even after this conditioning. Therefore, we run the flow for a short time τ before beginning the main analysis. This has a regularizing effect and ensures the stability of the resolvent entries of $\mathbf{X}_s$ for $s \in [\tau, t]$. Our analysis then proceeds by running the dynamics for a further amount of time $t - \tau \gg N^{-1/2}$ to prove convergence to equilibrium, given this initial regularity.

3. The third step asymptotically equates moments of $\operatorname{Im} R_{ii}(t, z)$ with those of $\operatorname{Im} R_\star(z)$, as $\eta = \operatorname{Im} z$ tends to 0 and N tends to infinity simultaneously. To analyze the former, we first use the Schur complement formula and a certain integral identity to express arbitrary moments of $\operatorname{Im} R_{ii}(t, z)$ through the $\frac{\alpha}{2}$-th moments of (real and imaginary parts of) $R_{ii}(t, z)$, as in [33, 34]. Next, using a local law from the previous section, we approximate these $\frac{\alpha}{2}$-th moments by corresponding ones for $R_\star(z)$. To analyze the moments of $\operatorname{Im} R_\star(z)$, we use the same integral identity and a recursive distributional equation for $R_\star(z)$ from [32] to express them through $\frac{\alpha}{2}$-moments of (real and imaginary parts of) $R_\star(z)$. We then observe the two

expressions are equal as N tends to infinity.

We prove our results on eigenvector statistics in Chapter 3. In Section 3.1 we state our results in detail. In Section 3.2 we give a full proof outline and establish our main results, assuming several preliminary claims which are shown in the remainder of the chapter. Section 3.3 recalls results on Lévy matrices from previous works that are required for the argument. Section 3.4 details the comparison part of the argument. Section 3.5 analyzes the eigenvector moment flow. Section 3.6 computes the scaling limits of the resolvent entries mentioned above. Section 3.7 provides some preliminary results needed in the previous sections, and Section 3.8 addresses convergence in distribution. In Section 3.9, we discuss quantum unique ergodicity (QUE) for eigenvectors of Lévy matrices, whose analogue for Wigner matrices was established in [44].

1.4 About this dissertation

This dissertation is a compilation of two papers. The first, [6], appears here as Chapter 2. This is joint work with Amol Aggarwal and Horng-Tzer Yau. The second, [5], constitutes Chapter 3. This is joint work with Amol Aggarwal and Jake Marcinek. In each case, I made minor typographical changes from the versions currently on arXiv to make the sections stylistically coherent, but did not alter the mathematical content. Further, instead of reproducing the introduction to each paper individually, I combined them to form the introduction to this work.

The work [6] will be published in *Journal of the European Mathematical Society*, while [5] is an unpublished preprint currently under peer review. Therefore, it is likely that the published version of the latter will incorporate changes made during peer review and differ from the one here. The latest version of each paper will always be available on the arXiv. The copyright to [6] is held by JEMS and the work is reproduced here for non-commercial purposes, in accordance with the journal's copyright policy. I and the other authors of [5]

retain the copyright to that work.

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Chapter 2

Eigenvalue Statistics of Lévy Matrices

2.1 Results

We fix parameters $\alpha \in (0,2)$ and $\sigma > 0$. A random variable Z is a $(0,\sigma)$ *α-stable law* if

$$\mathbb{E}\big[e^{\mathrm{i}tZ}\big] = \exp\big(-\sigma^\alpha |t|^\alpha\big), \quad \text{for all } t \in \mathbb{R}. \tag{2.1.1}$$

While many previous works have considered only matrices whose entires are distributed as α-stable laws, the methods of this work apply to a fairly broad range of symmetric power-law distributions. We now define the entry distributions we consider in this paper; the end of this section discusses an extension to more general ones. For simplicity, the reader may consider the concrete case of an α-stable distribution. The proof for this case contains all essential features of the general one.

Definition 2.1.1. Let Z be a $(0,\sigma)$ α-stable law with

$$\sigma = \left(\frac{\pi}{2\sin\left(\frac{\pi\alpha}{2}\right)\Gamma(\alpha)} \right)^{1/\alpha} > 0. \tag{2.1.2}$$

Let J be a symmetric[1] random variable (not necessarily independent from Z) such that

[1]By *symmetric*, we mean that J has the same distribution as $-J$.

$\mathbb{E}[J^2] < \infty$, $Z + J$ is symmetric, and

$$\frac{C_1}{\big(|t|+1\big)^{\alpha}} \le \mathbb{P}\big[|Z+J| \ge t\big] \le \frac{C_2}{\big(|t|+1\big)^{\alpha}}, \quad \text{for each } t \ge 0 \text{ and some constants } C_1, C_2 > 0. \tag{2.1.3}$$

Denoting $\mathfrak{z} = Z + J$, the symmetry of J and the condition $\mathbb{E}[J^2] < \infty$ are equivalent to imposing a coupling between $\mathfrak{z}$ and Z such that $\mathfrak{z} - Z$ is symmetric and has finite variance, respectively.

For each positive integer N, let $\{H_{ij}\}_{1 \le i \le j \le N}$ be mutually independent random variables that each have the same law as $N^{-1/\alpha}(Z + J) = N^{-1/\alpha}\mathfrak{z}$. Set $H_{ij} = H_{ji}$ for each i, j, and define the $N \times N$ random matrix $\mathbf{H} = \mathbf{H}_N = \{H_{ij}\} = \{H_{i,j}^{(N)}\}$, which we call an *$\alpha$-Lévy matrix.*

The $N^{-1/\alpha}$ scaling in the H_{ij} is different from the more standard $N^{-1/2}$ scaling that occurs in the entries of Wigner matrices. This is done in order to make the typical row sum of $\mathbf{H}$ of order one. Furthermore, the explicit constant σ (3.1.2) was chosen to make our notation consistent with that of previous works, such as [18, 33, 34], but can be altered by rescaling $\mathbf{H}$.

It was shown as Theorem 1.1 of [18] that, as N tends to ∞, the empirical spectral distribution of $\mathbf{H}$ converges to a deterministic measure μ_α. This is the (unique) probability distribution μ on $\mathbb{R}$ whose Stieltjes transform $\int_{\mathbb{R}} (x - z)^{-1}\, d\mu(x)$ is equal to the function $m_\alpha(z)$, which can be explicitly described as follows. Denote the upper half plane by $\mathbb{H} = \{z \in \mathbb{C}\colon \operatorname{Im} z > 0\}$ and its image under multiplication by $-\mathrm{i}$ by $\mathbb{K} = \{z \in \mathbb{C}\colon \operatorname{Re} z > 0\}$. For any $z \in \mathbb{H}$, define the functions $\varphi = \varphi_{\alpha,z}\colon \mathbb{K} \to \mathbb{C}$ and $\psi = \psi_{\alpha,z}\colon \mathbb{K} \to \mathbb{C}$ by

$$\varphi_{\alpha,z}(x) = \frac{1}{\Gamma(\alpha/2)} \int_0^\infty t^{\alpha/2-1} e^{\mathrm{i}tz} e^{-\Gamma(1-\alpha/2) t^{\alpha/2} x}\, dt, \quad \psi_{\alpha,z}(x) = \int_0^\infty e^{\mathrm{i}tz} e^{-\Gamma(1-\alpha/2) t^{\alpha/2} x}\, dt, \tag{2.1.4}$$

for any $x \in \mathbb{K}$. For each $z \in \mathbb{H}$ there exists a unique solution $y(z) \in \mathbb{K}$ to the equation

$y(z) = \varphi_{\alpha,z}\big(y(z)\big)$, so let us define

$$m_\alpha(z) = \mathrm{i}\psi_{\alpha,z}\big(y(z)\big). \tag{2.1.5}$$

The probability density function of the measure μ_α is given by ϱ_α, which is defined by setting

$$\varrho_\alpha(E) = \frac{1}{\pi}\lim_{\eta\to 0} \operatorname{Im} m_\alpha(E + \mathrm{i}\eta), \quad \text{for each } E \in \mathbb{R}.$$

The term *bulk universality* refers to the phenomenon that, in the bulk of the spectrum, the correlation functions of an $N \times N$ random matrix should converge to those of an $N \times N$ GOE matrix in the large N limit. This is explained more precisely through the following definitions.

Definition 2.1.2. Let N be a positive integer and $\mathbf{H}$ be an $N \times N$ real symmetric random matrix. Denote by $p_{\mathbf{H}}^{(N)}(\lambda_1, \lambda_2, \ldots, \lambda_N)$ the density of the symmetrized joint eigenvalue distribution of $\mathbf{H}$.[2] For each integer $k \in [1, N]$, define the *k-th correlation function* of $\mathbf{H}$ by

$$p_{\mathbf{H}}^{(k)}(x_1, x_2, \ldots, x_k) = \int_{\mathbb{R}^{N-k}} p_{\mathbf{H}}^{(N)}(x_1, x_2, \ldots, x_k, y_{k+1}, y_{k+2}, \ldots, y_N) \prod_{j=k+1}^{N} dy_j.$$

Definition 2.1.3. Let $\{\mathbf{H} = \mathbf{H}_N\}_{N\in\mathbb{Z}_{\geq 1}}$ be a set of matrices, $\{\varrho = \varrho_N\}_{N\in\mathbb{Z}_{\geq 1}}$ be a set of a probability density functions, and $E \in \mathbb{R}$ be a fixed real number. We say that *the correlation functions of* $\mathbf{H}$ *are universal at energy level* E *with respect to* ϱ if, for any positive integer k and compactly supported smooth function $F \in \mathcal{C}_0^\infty(\mathbb{R}^k)$, we have that

$$\lim_{N\to\infty} \left| \int_{\mathbb{R}^k} F(\mathbf{a}) \left(p_{\mathbf{H}_N}^{(k)}\left(E + \frac{\mathbf{a}}{N\varrho_N(E)}\right) - p_{\mathbf{GOE}_N}^{(k)}\left(\frac{\mathbf{a}}{N\varrho_{\mathrm{sc}}(0)}\right)\right) d\mathbf{a} \right| = 0, \tag{2.1.6}$$

where $d\mathbf{a}$ denotes the Lebesgue measure on $\mathbb{R}^k$ and we recall that ϱ_{sc} was defined by (1.2.1).

[2]In particular, with respect to the symmetrized density, $\lambda_1, \lambda_2, \ldots, \lambda_N$ are exchangeable random variables. Such a density exists because each entry distribution of the random matrix has a density.

Now we can state our main results. In what follows, we set $\|\mathbf{v}\|_\infty = \max_{j \in [1,d]} |v_j|$ for any $\mathbf{v} = (v_1, v_2, \ldots, v_d) \in \mathbb{R}^d$.

Theorem 2.1.4. *Let* $\mathbf{H}$ *denote an* $N \times N$ α*-stable matrix, as in Definition 3.1.1. Suppose that* $\alpha \in (1, 2)$*, and fix some compact interval* $K \subset \mathbb{R} \setminus \{0\}$*.*

1. *For any* $\delta > 0$ *and* $D > 0$*, there exists a constant* $C = C(\alpha, \delta, D, K) > 0$ *such that*

$$\mathbb{P}\left[\max\left\{\|\mathbf{u}\|_\infty : \mathbf{H}\mathbf{u} = \lambda \mathbf{u}, \|\mathbf{u}\|_2 = 1, \lambda \in K\right\} > N^{\delta - 1/2}\right] < CN^{-D}.$$

2. *Fix some* $E \in K$*. Then the correlation functions of* $\mathbf{H}$ *are universal at energy level* E *with respect to* ϱ_α*, as in Definition 2.1.3.*

Theorem 2.1.5. *Let* $\mathbf{H}$ *denote an* α*-stable matrix, as in Definition 3.1.1. There exists a countable set* $\mathcal{A} \subset (0, 2)$ *with no accumulation points in* $(0, 2)$ *such that for any* $\alpha \in (0, 2) \setminus \mathcal{A}$*, there exists a constant* $c = c(\alpha) > 0$ *such that the following holds.*

1. *For any* $\delta > 0$ *and* $D > 0$*, there exists a constant* $C = C(\alpha, \delta, D) > 0$ *such that*

$$\mathbb{P}\left[\max\left\{\|\mathbf{u}\|_\infty : \mathbf{H}\mathbf{u} = \lambda \mathbf{u}, \|\mathbf{u}\|_2 = 1, \lambda \in [-c, c]\right\} > N^{\delta - 1/2}\right] < CN^{-D}.$$

2. *Fix* $E \in [-c, c]$*. Then, the correlation functions of* $\mathbf{H}$ *are universal at energy level* E *with respect to* ϱ_α*, as in Definition 2.1.3.*

The above two theorems comprise the first complete eigenvector delocalization and bulk universality results for a random matrix model whose entries have infinite variance. For $\alpha \in (1, 2)$, Theorem 2.1.4 completely settles the bulk universality and complete eigenvector delocalization for all energies (except if $\alpha \in \mathcal{A}$ and $E = 0$), consistent with prediction **A** in **??**. When $\alpha < 1$, Theorem 2.1.5 can be viewed as establishing a lower bound on the mobility edge in prediction **B**.

Let us make four additional comments about the results above. First, although they are only stated for real symmetric matrices, they also apply to complex Hermitian random matrices. In order to simplify notation later in the paper, we only pursue the real case here.

Second, the exceptional set $\mathcal{A}$ of α to which Theorem 2.1.5 does not apply should be empty. Its presence is due to the fact that we use results of [34] stating that certain deterministic, α-dependent fixed point equations can be inverted when $\alpha \notin \mathcal{A}$.

Third, our conditions in Definition 3.1.1 allow for the entries of $\mathbf{H}$ to be not exactly α-stable, but they are not optimal. Although our methods currently seem to require the symmetry of J and $Z + J$, they can likely be extended to establish our main results under weaker moment constraints on J. In particular, they should apply assuming only this symmetry, (3.1.3), and that $\mathbb{E}[|J|^{\beta}] < \infty$, for some fixed $\beta > \alpha$. Pursuing this improvement would require altering the statements and proofs of (2.5.13), Lemma 2.5.8, and Lemma 2.9.1 below (with the primary effort being in the former).[3] However, for the sake of clarity and brevity, we do not develop this further here.

Fourth, local statistics of a random matrix $\mathbf{H}$ are also quantified through *gap statistics*. For some fixed (possibly N-dependent) integer i and uniformly bounded integer $m \geq 0$, these statistics provide the joint distribution of the gaps between the (nearly) neighboring eigenvalues $\big\{N(\lambda_j - \lambda_k)\big\}_{|j-i|,|k-i|\leq m}$. Our methods can be extended to establish universality of gap statistics of Lévy matrices, by replacing the use of Proposition 2.2.11 below with Theorem 2.5 of [78], but we do not pursue this here.

2.2 Proofs of delocalization and bulk universality

In this section we establish the theorems stated in Section 2.1 assuming some results that will be proven in later parts of this paper. For the remainder of this paper, all matrices under consideration will be real and symmetric.

[3] For the improvement of Theorem 2.1.5, which considers $\alpha \in (0, 2) \setminus \mathcal{A}$ and small energies, it suffices to modify only the statements and proofs of Lemma 2.5.8, and Lemma 2.9.1.

Throughout this section, we fix a compact interval $K \subset \mathbb{R}$ and parameters $\alpha, b, \nu, \rho > 0$ satisfying

$$\alpha \in (0,2), \quad \nu = \frac{1}{\alpha} - b > 0, \quad 0 < \rho < \nu < \frac{1}{2}, \quad \frac{1}{4-\alpha} < \nu < \frac{1}{4-2\alpha}, \quad \alpha\rho < (2-\alpha)\nu. \tag{2.2.1}$$

Viewing $\alpha \in (0,2)$ as fixed, one can verify that it is possible to select the remaining parameters $b, \nu, \rho > 0$ such that the conditions (2.2.1) all hold. The reason for these constraints will be explained in Section 2.3.2. The proofs of Theorem 2.1.4 and Theorem 2.1.5 will proceed through the following three steps.

1. First we define a matrix $\mathbf{X}$ obtained by setting all entries of $\mathbf{H}$ less than $N^{-\nu}$ to zero, and we establish an intermediate local law for $\mathbf{X}$ on a certain scale $\eta = N^{-\varpi}$ with $\varpi > \nu(2-\alpha)$.

2. Next we study $\mathbf{V} = \mathbf{V}_t = \mathbf{X} + t^{1/2}\mathbf{W}$, for a GOE matrix $\mathbf{W}$ and $t \sim N^{\nu(\alpha-2)}$. The results of [42,77] imply that if the Stieltjes transform and diagonal resolvent entries of $\mathbf{X}$ are bounded on some scale $\eta_0 \ll t$, then all resolvent entries of $\mathbf{V}$ are bounded by N^{δ} on the scale $\eta \sim N^{\delta-1}$ for any $\delta > 0$, and bulk universality holds for $\mathbf{V}$. In particular, this does not require that the resolvent entries of $\mathbf{X}$ approximate a deterministic quantity. Thus, the previously mentioned local law for $\mathbf{X}$ (which takes place on scale $N^{-\varpi}$, which is less than $t \sim N^{(\alpha-2)\nu}$) implies that the resolvent entries of $\mathbf{V}$ are bounded by N^{δ} when $\eta = N^{\delta-1}$, and that the local statistics of $\mathbf{V}$ are universal.

3. Then we establish a comparison theorem between the resolvent entries of $\mathbf{H}$ and $\mathbf{V}$. Combining this with the estimates on the resolvent entries of $\mathbf{V}$ from the previous step, this allows us to conclude that that the resolvent entries of $\mathbf{H}$ are bounded by N^{δ} on the scale $\eta = N^{\delta-1}$, implying complete eigenvector delocalization for $\mathbf{H}$. Further combining this comparison with bulk universality for $\mathbf{V}$ will also imply bulk universality for $\mathbf{H}$.

We will implement the first, second, and third steps outline above in Section 2.2.1, Section 2.2.2, and Section 2.2.3, respectively.

Remark 2.2.1. In the above outline we use [77] to prove the strongest form of convergence of local statistics, which is given by (2.1.6). However, if one is content to establish this convergence after averaging the eigenvalues over a small interval of size $N^{\delta-1}$ (known as *averaged energy universality*), then one can instead use Theorem 2.4 of the shorter work [78]. Moreover, if one is only interested in proving complete delocalization for the eigenvectors of $\mathbf{H}$, then it suffices to instead use Theorem 2.1 and Proposition 2.2 of [42].

2.2.1 The intermediate local laws

In this section we introduce a *removal* variant, denoted by $\mathbf{X}$, of our α-stable matrix $\mathbf{H}$, given by Definition 2.2.2 and Definition 3.2.3 below. Then, we state two intermediate local laws for $\mathbf{X}$, depending on whether $\alpha \in (1,2)$ or $\alpha \in (0,2)$. These are given by Theorem 2.2.4 and Theorem 2.2.5, respectively.

Definition 2.2.2. Fix constants α and b satisfying (2.2.1), and let Z, J, and $\mathfrak{z} = Z + J$ be as in Definition 3.1.1. Let $Y = \mathfrak{z}\mathbf{1}_{|\mathfrak{z}| \le N^b}$, and let $X = \mathfrak{z} - Y$. We call X the *b-removal of a deformed* $(0, \sigma)$ *α-stable law.*

Definition 2.2.3. Let $\{X_{ij}\}_{1 \le i \le j \le N}$ be mutually independent random variables that each have the same law as $N^{-1/\alpha}X$, where X is given by Definition 2.2.2. Set $X_{ij} = X_{ji}$ for each $1 \le j < i \le N$, and define the $N \times N$ matrix $\mathbf{X} = \{X_{ij}\}$. We call $\mathbf{X}$ a *b-removed α-Lévy matrix.* For any complex number $z \in \mathbb{H}$, define the resolvent $\mathbf{R} = \mathbf{R}(z) = \{R_{ij}\}_{1 \le i,j \le N} = (\mathbf{X} - z)^{-1}$. Further denote $m = m_N = m_N(z) = N^{-1} \operatorname{Tr} \mathbf{R}$, and also set $z = E + \mathrm{i}\eta$ with $E, \eta \in \mathbb{R}$ and $\eta > 0$.

Now, let $\{Z_{ij}\}_{1 \le i \le j \le N}$ and $\{J_{ij}\}_{1 \le i \le j \le N}$ mutually independent random variables that have the same laws as $N^{-1/\alpha}Z$ and $N^{-1/\alpha}J$, respectively, where Z and J are as in Definition 3.1.1. Let $\{H_{ij}\}_{1 \le i \le j \le N}$ be mutually independent random variables such that $H_{ij} =$

$Z_{ij} + J_{ij}$. Couple each H_{ij} with X_{ij} so that $X_{ij} = H_{ij} - H_{ij}\mathbf{1}_{N^{1/\alpha}|H_{ij}|\le N^b}$. Set $H_{ij} = H_{ji}$ for each $1 \le j < i \le N$, and define the $N \times N$ matrix $\mathbf{H} = \{H_{ij}\}$. The matrix $\mathbf{H}$ is an α-Lévy matrix that is coupled with $\mathbf{X}$, and we refer to this coupling as the *removal coupling*. For any $z \in \mathbb{H}$, let $\mathbf{G}(z) = \big\{G_{ij}(z)\big\} = (\mathbf{H} - z)^{-1}$.

Now let us state intermediate local laws for the removal matrix $\mathbf{X}$ at all energies away from 0 when $\alpha \in (1,2)$ (given by Theorem 2.2.4 below) and at sufficiently small energies for almost all $\alpha \in (0,2)$ (given by Theorem 2.2.5 below). The scale at which the former local law will be stated is $\eta = N^{-\varpi}$ for some $\varpi \in \big((2-\alpha)\nu, \nu\big)$, and the scale at which the latter will be is $\eta = N^{\delta-1/2}$ for any $\delta > 0$. These should not be optimal and do not match that at which local laws were proven in finite variance cases, which is $\eta = N^{\delta-1}$ [4,10,12,23,48,52,53,55,56,59,60,64,69], but they will suffice for our purposes. In fact, one can establish a local law on this optimal scale by combining Theorem 2.2.15 and Theorem 2.2.16 with Theorem 2.2.4 and Theorem 2.2.5, but we will not pursue this here.

The below result will be established in Section 2.4.1.

Theorem 2.2.4. *Fix $\alpha, b, \nu > 0$ satisfying (2.2.1). Assume that $\alpha \in (1,2)$ and $K \subset \mathbb{R}\setminus\{0\}$. Let ϖ be such that*

$$(2-\alpha)\nu < \varpi < \nu,$$

and define the domain

$$\mathcal{D}_{K,\varpi,C} = \big\{z = E + \mathrm{i}\eta \in \mathbb{H} : \quad E \in K, \quad N^{-\varpi} \le \eta \le C\big\}, \tag{2.2.2}$$

There exists a small constant $\varkappa = \varkappa(\alpha, b, \nu, \varpi, K) > 0$ and large constants $\mathfrak{B} = \mathfrak{B}(\alpha) > 0$

and $C = C(\alpha, b, \nu, \varpi, K) > 0$ *such that*

$$\begin{aligned}\mathbb{P}\left[\sup_{z\in\mathcal{D}_{K,\varpi,\mathfrak{B}}}\big|m_N(z) - m_\alpha(z)\big| > \frac{C}{N^{\varkappa}}\right] &\leq C\exp\left(-\frac{(\log N)^2}{C}\right),\\ \mathbb{P}\left[\sup_{z\in\mathcal{D}_{K,\varpi,\mathfrak{B}}}\max_{1\leq j\leq N}\big|R_{jj}(z)\big| > C(\log N)^{30/(\alpha-1)}\right] &\leq C\exp\left(-\frac{(\log N)^2}{C}\right).\end{aligned} \tag{2.2.3}$$

Theorem 2.2.4 is similar to Theorem 3.5 of [33], but there are several differences. For appropriate choices of constants satisfying constraints (2.2.1), we control the Stieltjes transform for $\eta \gg N^{-1/2}$, which essentially equals the scale achieved for $\alpha \in \left(\frac{8}{5}, 2\right)$ in [33] and improves the scale $\eta \gg N^{-\alpha/(8-3\alpha)}$ achieved for $\alpha \in \left(1, \frac{8}{5}\right)$ in [33]. The latter improvement is important for our work because it permits us to access the the critical scale $t \sim N^{(\alpha-2)\nu}$ for all $\alpha \in (1, 2)$. This would not have been possible for α near 1 using the scales achieved in [33]. Theorem 2.2.4 also asserts estimates on the diagonal resolvent entries $R_{jj}(z)$, which are crucial for our main results but were not estimated in [33] for any α. Finally, in Theorem 3.5 of [33], a finite, non-explicit set of energies must be excluded, while we need only exclude the energy 0.

Next let us state the intermediate local law for $\mathbf{X}$ at sufficiently small energies when $\alpha \in (0, 2) \setminus \mathcal{A}$, which is a consequence of Theorem 2.6.6 (and Remark 2.6.7), stated in Section 2.6.1 below.

Theorem 2.2.5. *There exists a countable set* $\mathcal{A} \subset (0, 2)$*, with no accumulation points in* $(0, 2)$*, that satisfies the following property. Fix* α *and* b *satisfying* (2.2.1)*, set* $\theta = \frac{(b-1/\alpha)(2-\alpha)}{20}$*, and let* $\delta \in (0, \theta)$*. Define the domain*

$$\mathcal{D}_{C,\delta} = \left\{z = E + \mathrm{i}\eta \in \mathbb{H}: \quad E \leq \frac{1}{C}, \quad N^{\delta-1/2} \leq \eta \leq \frac{1}{C}\right\}. \tag{2.2.4}$$

Then there exists a large constant $C = C(\alpha, b, \delta) > 0$ such that

$$\mathbb{P}\left[\sup_{z \in \mathcal{D}_{C,\delta}} \left| m_N(z) - m_\alpha(z) \right| > \frac{1}{N^{\alpha\delta/8}}\right] < C \exp\left(-\frac{(\log N)^2}{C}\right), \tag{2.2.5}$$

and

$$\mathbb{P}\left[\sup_{z \in \mathcal{D}_{C,\delta}} \max_{1 \le j \le N} \left| R_{jj}(z) \right| > (\log N)^C\right] < C \exp\left(-\frac{(\log N)^2}{C}\right). \tag{2.2.6}$$

Theorem 2.2.5 is similar to Proposition 3.2 of [34], except that it also bounds the diagonal resolvent entries $R_{jj}(z)$. Furthermore, Theorem 2.2.5 estimates the Stieltjes transform $m_N(z)$ for smaller values of $\eta = \operatorname{Im} z \gg N^{-1/2}$ than in Proposition 3.2 of [34], which requires $\eta \gg N^{-\alpha/(2+\alpha)}$. This improvement is again important for us to access the the critical scale $t \sim N^{(\alpha-2)\nu}$ for all $\alpha \in (0, 2)$.

2.2.2 Estimates for V

In this section we implement the second step of our outline, in which we define a matrix $\mathbf{V} = \mathbf{X} + t^{1/2}\mathbf{W}$, establish that its resolvent entries are bounded by N^δ on scale $N^{\delta-1}$, and show that its local statistics are universal.

Recall that $\alpha, b, \nu, \rho > 0$ are parameters satisfying (2.2.1), and define $t = t(\rho, \nu)$ by the conditional expectation

$$t = N\mathbb{E}\Big[H_{11}^2 \mathbf{1}_{|H_{11}| < N^{-\nu}} \Big| |H_{11}| < N^{-\rho}\Big] = \frac{N\mathbb{E}\big[H_{11}^2 \mathbf{1}_{|H_{11}| < N^{-\nu}}\big]}{\mathbb{P}\big[|H_{11}| < N^{-\rho}\big]}. \tag{2.2.7}$$

We require the following lemma that provides large N asymptotics for t; with the definitions of (2.2.1), it in particular implies $t = o(1)$. Its proof will be given in Section 2.3.1 below.

Lemma 2.2.6. *There exist a small constant $c = c(\alpha, \nu, \rho) > 0$ and a large constant $C =$*

$C(\alpha, \nu, \rho) > 0$ *such that*

$$cN^{(\alpha-2)\nu} \le t \le CN^{(\alpha-2)\nu}. \tag{2.2.8}$$

Now let us define a matrix $\mathbf{V}$ that we will compare to $\mathbf{H}$.

Definition 2.2.7. Define the $N \times N$ random matrix $\mathbf{V} = \{v_{ij}\} = \mathbf{X} + t^{1/2}\mathbf{W}$, where t is given by (3.2.10); $\mathbf{X}$ is the removal matrix from Definition 3.2.3; and $\mathbf{W} = \{w_{ij}\}$ is an $N \times N$ GOE matrix independent from $\mathbf{X}$. For any $z \in \mathbb{H}$, let $\mathbf{T} = \mathbf{T}(z) = \{T_{ij}(z)\} = (\mathbf{V} - z)^{-1}$.

Now we would like to bound the entries of $\mathbf{T}$ and show that bulk universality holds for $\mathbf{V}$. To do this, we first require the following definition from [78], which defines a class of initial data on which Dyson Brownian motion is well-behaved.

Definition 2.2.8 ([78, Definition 2.1])**.** Let N be a positive integer, let $\mathbf{H}_0$ be an $N \times N$ matrix, and set $m_0(z) = N^{-1} \operatorname{Tr} \big(\mathbf{H}_0 - z\big)^{-1}$. Fix $E_0 \in \mathbb{R}$, $\delta \in (0,1)$, and $\gamma > 0$ independently of N. Let η_0 and r be two (N-dependent) parameters satisfying $N^{\delta-1} \le \eta_0$ and $N^{2\delta}\eta_0 < r \le 1$. Define

$$\mathcal{D}(E_0, r, \eta_0, \gamma) = \big\{ z = E + \mathrm{i}\eta \in \mathbb{H} : \quad E \in \big[E_0 - r, E_0 + r\big], \quad \eta \in [\eta_0, \gamma] \big\}. \tag{2.2.9}$$

Although $\mathcal{D}(E_0, r, \eta_0, \gamma)$ in the above definition depends on δ through the choice of η_0, we omit this from the notation.

We say that $\mathbf{H}_0$ is (η_0, γ, r)-*regular with respect to* E_0 if there exists a constant $A > 1$ (independent of N) such that

$$\|\mathbf{H}_0\| \le N^A, \qquad \frac{1}{A} < \sup_{z \in \mathcal{D}(E_0, r, \eta_0, \gamma)} \operatorname{Im} m_0(z) \le A. \tag{2.2.10}$$

Now let N be a positive integer and $\mathbf{H}_0$ denote an $N \times N$ matrix. Recall that $\mathbf{W} = \{w_{ij}\}$ is an $N \times N$ GOE matrix (which we assume to be independent from $\mathbf{H}_0$), and define $\mathbf{H}_s = \mathbf{H}_0 + s^{1/2}\mathbf{W}$ for each $s > 0$. For each $z \in \mathbb{H}$, let $\mathbf{G}_s = \mathbf{G}_s(z) = \big\{G_{ij}(s, z)\big\} = \big(\mathbf{H}_s - z\big)^{-1}$.

If $\mathbf{H}_0$ is (η_0, γ, r)-regular and s is between η_0 and r, then the following proposition estimates the entries of $\mathbf{G}_s(E + \mathrm{i}\eta)$, when η can be nearly of order N^{-1}, in terms of estimates on the diagonal entries of $\mathbf{G}_0(E + \mathrm{i}\eta_0)$. Its proof will appear in Section 2.8 and is based on results of [42, 78].

Proposition 2.2.9. *Adopt the notation of Definition 2.2.8, and let* $B \in \big(1, \frac{1}{\eta_0}\big)$ *be an* N*-dependent parameter. Assume that* $\mathbf{H}_0$ *is* (η_0, γ, r)*-regular with respect to* E_0 *and that* $\max_{1 \le j \le N} \big| G_{jj}(0, z) \big| \le B$ *for all* $z \in \mathcal{D}(E_0, r, \eta_0, \gamma)$*. Let* $s \in \big(N^{\delta} \eta_0, N^{-\delta} r\big)$*. Then, for any* $D > 1$ *there exists a large constant* $C = C(\delta, D) > 0$ *such that*

$$\mathbb{P}\left[\sup_{z \in \mathfrak{D}} \max_{1 \le i,j \le N} \big| G_{ij}(s, z) \big| > N^{\delta} B \right] \le C N^{-D},$$

where we have abbreviated $\mathfrak{D} = \mathcal{D}(E_0, \frac{r}{2}, N^{\delta - 1}, \gamma - \frac{r}{2})$*.*

Now we can bound the entries of $\mathbf{T}$.

Corollary 2.2.10. *Let* $\alpha, b, \nu, \rho > 0$ *satisfy* (2.2.1)*. For given* $E_0 \in \mathbb{R}$ *and* $\delta, \gamma, r > 0$*, we abbreviate* $\mathfrak{D} = \mathcal{D}\big(E_0, \frac{r}{2}, N^{\delta - 1}, \gamma - \frac{r}{2}\big)$ *(as in* (2.2.9)*).*

1. *If* $\alpha \in (1, 2)$ *and* $K \subset \mathbb{R} \backslash \{0\}$ *is a compact interval, let* γ *denote the constant* $\mathfrak{B} = \mathfrak{B}(\alpha)$ *from Theorem 2.2.4. Let* $E_0 \in K$ *and* $\delta, r > 0$ *be constants (independent of* N*) such that* $[E_0 - r, E_0 + r] \subset K$ *and* $r < \gamma$*. Then, for any* $D > 0$ *there exists a large constant* $C = C(\alpha, \nu, \rho, \delta, D, K) > 0$ *such that*

$$\mathbb{P}\left[\sup_{z \in \mathfrak{D}} \max_{1 \le i,j \le N} \big| T_{ij}(z) \big| > N^{\delta} \right] \le C N^{-D}. \tag{2.2.11}$$

2. *If* $\mathcal{A} \subset (0, 2)$ *is as in Theorem 2.2.5 and* $\alpha \in (0, 2) \setminus \mathcal{A}$*, then let* $\gamma = \frac{1}{2C}$*, where the constant* C *is from Theorem 2.2.5. Further let* $E_0 \in \mathbb{R}$ *and* $r \in (0, \gamma)$ *be constants (independent of* N*) such that* $[E_0 - r, E_0 + r] \subset \big[- 2\gamma, 2\gamma \big]$*. Then, for any* $\delta, D > 0$*, there exists a large constant* $C = C(\alpha, \nu, \rho, \delta, D) > 0$ *such that* (2.2.11) *holds.*

Proof. We assume $\alpha \in (1,2)$, since the case $\alpha \in (0,2) \setminus \mathcal{A}$ is entirely analogous. By Theorem 2.2.4, there exist large constants $\mathfrak{B} = \mathfrak{B}(\alpha) > 0$ and $C = C(\alpha, b, \varpi, \delta, D, K) > 0$ such that

$$\mathbb{P}\left[\sup_{z \in \mathcal{D}_{K,\varpi,\mathfrak{B}}} \max_{1 \le j \le N} \big|R_{jj}(z)\big| > N^{\delta/4}\right] < C \exp\left(-\frac{(\log N)^2}{C}\right), \tag{2.2.12}$$

for any $(2-\alpha)\nu < \varpi < \nu$, where we recall the definition of $\mathcal{D}_{K,\varpi,\mathfrak{B}}$ from (2.2.2). Furthermore, observe (after increasing C if necessary) that $\mathbb{P}\big[\|\mathbf{X}\| > N^{(2D+3)/\alpha}\big] \le CN^{-2D}$, since $\alpha < 2$ and the probability that the magnitude of a given entry of $\mathbf{H}$ is larger than $N^{(2D+1)/\alpha}$ is at most CN^{-2D-2}.

Therefore, we may apply Proposition 2.2.9 with that $\mathbf{H}_0$ equal to our $\mathbf{X}$; that η_0 equal to our $N^{-\varpi}$; that t equal to our t, which is defined by (3.2.10) and satisfies $t \sim N^{(\alpha-2)\nu}$ by Lemma 3.2.6; that δ to be sufficiently small, so that it is less than our $\frac{\delta}{4}$ and $\frac{1}{4}\big(\varpi - (2-\alpha)\nu\big)$ (if α were in $(0,2)$, then we would require that δ be less than $\frac{1}{4}\big(\frac{1}{2} - (2-\alpha)\nu\big)$ instead); that E_0 equal to the E_0 here; that γ equal to our $\mathfrak{B}$; that r equal to the min $\big\{r, \frac{\mathfrak{B}}{4}\big\}$ here; and that A sufficiently large. Under this choice of parameters, $\mathbf{G}_t = \mathbf{T}$, so Proposition 2.2.9 implies (2.2.11). $\square$

We will next show that the local statistics of $\mathbf{V}$ are universal, which will follow from the results of [63, 77, 78] together with the intermediate local laws Theorem 2.2.4 and Theorem 2.2.5. Specifically, the results of [63, 77, 78] state that, if we start with a (η_0, γ, r)-regular matrix (recall Definition 2.2.8) and then add an independent small Gaussian component of order greater than η_0 but less than r, then the local statistics of the result will asymptotically coincide with those of the GOE. To state this more precisely, we must introduce the free convolution [29] of a probability distribution with the semicircle law.

Fix $N \in \mathbb{Z}_{>0}$ and an $N \times N$ matrix $\mathbf{A}$. For each $s \ge 0$, define $\mathbf{A}^{(s)} = \mathbf{A} + s^{1/2}\mathbf{W}$, where $\mathbf{W}$ is an $N \times N$ GOE matrix. For any $z \in \mathbb{H}$, also define $m^{(s)}(z) = N^{-1} \operatorname{Tr}\big(\mathbf{A}^{(s)} - z\big)^{-1}$ to be the Stieltjes transform of the (N-dependent) empirical spectral density of $\mathbf{A}^{(s)}$, which we denote by $\rho^{(s)}(x) = \pi^{-1} \lim_{\eta \to 0} \operatorname{Im} m^{(s)}(E + \mathrm{i}\eta)$.

The following proposition establishes the universality of correlation functions of the random matrix $\mathbf{M}^{(s)}$, assuming that $\mathbf{M}$ is regular in the sense of Definition 2.2.8.

Proposition 2.2.11 ([77, Theorem 2.2])**.** *Fix some $\delta \in (0,1)$ and $\gamma > 0$, let N be a positive integer, and let $r \in (0, N^{-\delta})$ and $\eta_0 \in (N^{\delta-1}, 1)$ be N-dependent parameters satisfying $\eta_0 < N^{-2\delta} r$. Let $\mathbf{M}$ be an $N \times N$ matrix, and assume that $\mathbf{M}$ is (η_0, γ, r)-regular with respect to some fixed $E \in K$. Then, for any $s \in \big(N^{\delta}\eta_0, N^{-\delta} r\big)$, the correlation functions of $\mathbf{M}^{(s)}$ are universal at energy level E with respect to $\rho^{(s)}$, as in Definition 2.1.3.*

Using Proposition 2.2.11, one can deduce the following result. In what follows, we recall the matrices $\mathbf{X}$ and $\mathbf{V} = \mathbf{X}^{(t)}$ from Definition 3.2.3 and Definition 2.2.7, respectively (where t was given by (3.2.10)).

Proposition 2.2.12. *Assume $\alpha \in (1,2)$ and $K \subset \mathbb{R} \setminus \{0\}$, and let $E \in K$. Then the correlation functions of $\mathbf{V}$ are universal at energy level E with respect to ϱ_α, as in Definition 2.1.3. Moreover, the same statement holds if $\mathcal{A}$ and C are as in Theorem 2.2.5, $\alpha \in (0,2) \setminus \mathcal{A}$, and $E \subset \big[-\frac{1}{2C}, \frac{1}{2C} \big]$.*

To establish this proposition, one conditions on $\mathbf{X}$ and uses its intermediate local law (Theorem 2.2.4 or Theorem 2.2.5) and Lemma 3.2.6 to verify the assumptions of Proposition 2.2.11. Then, the latter proposition implies that the correlation functions of $\mathbf{V}$ are universal at E with with respect to $\rho^{(s)}$. The remaining difference between universality with respect to $\rho^{(s)}(E)$ and the desired result is in the scaling in (2.1.6). Specifically, one must approximate the factors of $\rho^{(s)}(E)$ by $\varrho_\alpha(E)$ in Definition 2.1.3. This approximation can be justified using the intermediate local law (Theorem 2.2.4 or Theorem 2.2.5) for $\mathbf{X}$ through a very similar way to what was explained in Lemma 3.3 and Lemma 3.4 of [70]. Thus, we omit further details.

2.2.3 Proofs of Theorem 2.1.4 and Theorem 2.1.5

In this section we establish Theorem 2.1.4 and Theorem 2.1.5. This will proceed through a comparison between the resolvent entries of $\mathbf{H}$ and $\mathbf{V}$ (from Definition 2.2.7). In Section 2.2.3, we state this comparison; we will provide a heuristic for its proof in Section 2.3.2, and the result will be established in detail in Section 2.3. We will then in Section 2.2.3 use the comparison to deduce eigenvector delocalization and bulk universality for $\mathbf{H}$ from the corresponding results for $\mathbf{V}$ established in Section 2.2.2.

The comparison theorem

To formulate our specific comparison statement, we require a certain way of decomposing the matrix $\mathbf{H}$ so that the elements of this decomposition remain largely independent. A less general version of this procedure was described in [4] under different notation to establish bulk universality for Wigner matrices whose entries have finite $(2+\varepsilon)$-th moment. This is done through the following two definitions.

Definition 2.2.13. Let ψ and χ be independent Bernoulli $0-1$ random variables defined by

$$\mathbb{P}\big[\psi = 1\big] = \mathbb{P}\big[|H_{ij}| \geq N^{-\rho}\big], \qquad \mathbb{P}\big[\chi = 1\big] = \frac{\mathbb{P}\big[|H_{ij}| \in [N^{-\nu}, N^{-\rho})\big]}{\mathbb{P}\big[|H_{ij}| < N^{-\rho}\big]}.$$

In particular, ψ has the same law as the indicator of the event that $|H_{ij}| \geq N^{-\rho}$. Similarly, χ has the same law as the indicator of the event that $|H_{ij}| \geq N^{-\nu}$, conditional on $|H_{ij}| < N^{-\rho}$.

Additionally, let a, b, and c be random variables such that

$$\mathbb{P}[a_{ij} \in I] = \frac{\mathbb{P}\Big[H_{ij} \in (-N^{-\nu}, N^{-\nu}) \cap I\Big]}{\mathbb{P}\big[|H_{ij}| < N^{-\nu})\big]},$$

$$\mathbb{P}[b_{ij} \in I] = \frac{\mathbb{P}\Big[H_{ij} \in \big((-N^{-\rho}, -N^{-\nu}] \cup [N^{-\nu}, N^{-\rho})\big) \cap I\Big]}{\mathbb{P}\big[|H_{ij}| \in [N^{-\nu}, N^{-\rho})\big]},$$

$$\mathbb{P}[c_{ij} \in I] = \frac{\mathbb{P}\Big[H_{ij} \in \big((-\infty, -N^{-\rho}] \cup [N^{-\rho}, \infty)\big) \cap I\Big]}{\mathbb{P}\big[|H_{ij}| \geq N^{-\rho}\big]},$$

for any interval $I \subset \mathbb{R}$. Again, a has the same law as H_{ij} conditional on $|H_{ij}| < N^{-\nu}$; similar statements hold for b and c.

Observe that if a, b, c, ψ, and χ are mutually independent, then H_{ij} has the same law as $(1-\psi)(1-\chi)a + (1-\psi)\chi b + \psi c$ and X_{ij} has the same law as $(1-\psi)\chi b + \psi c$. Thus, although the random variables $H_{ij}\mathbf{1}_{|H_{ij}|\geq N^{-\rho}}$, $H_{ij}\mathbf{1}_{N^{-\nu}\leq|H_{ij}|<N^{-\rho}}$, and $H_{ij}\mathbf{1}_{|H_{ij}|<N^{-\nu}}$ are correlated, this decomposition expresses their dependence through the Bernoulli random variables ψ and χ.

Definition 2.2.14. For each $1 \leq i \leq j \leq N$, let a_{ij}, b_{ij}, c_{ij}, ψ_{ij}, and χ_{ij} be mutually independent random variables whose laws are given by those of a, b, c, ψ, and χ from Definition 3.2.4 respectively. For each $1 \leq j < i \leq N$, define $a_{ij} = a_{ji}$ by symmetry, and similarly for each b_{ij}, c_{ij}, ψ_{ij}, and χ_{ij}. Let $\mathbb{P}$ and $\mathbb{E}$ denote the probability measure and expectation with respect to the joint law of these random variables, respectively.

Now for each $1 \leq i, j \leq N$, set

$$A_{ij} = (1-\psi_{ij})(1-\chi_{ij})a_{ij}, \quad B_{ij} = (1-\psi_{ij})\chi_{ij}b_{ij}, \quad C_{ij} = \psi_{ij}c_{ij}, \tag{2.2.13}$$

and define the four $N \times N$ matrices $\mathbf{A} = \{A_{ij}\}$, $\mathbf{B} = \{B_{ij}\}$, $\mathbf{C} = \{C_{ij}\}$, and $\Psi = \{\psi_{ij}\}$.

Sample $\mathbf{H}$ and $\mathbf{X}$ by setting $\mathbf{H} = \mathbf{A} + \mathbf{B} + \mathbf{C}$ and $\mathbf{X} = \mathbf{B} + \mathbf{C}$. We will commonly refer to Ψ as the *label* of $\mathbf{H}$ (or of $\mathbf{X}$). Defining $\mathbf{H}$ and $\mathbf{X}$ in this way ensures that they have the same laws as in Definition 3.1.1 and Definition 3.2.3, respectively. Furthermore, this sampling induces a coupling between $\mathbf{H}$ and $\mathbf{X}$, which coincides with the removal coupling of Definition 3.2.3.

To state our comparison results, we require some additional notation. Define $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{H}$, and $\mathbf{X}$ as in Definition 3.2.5, and let $\mathbf{W} = \{w_{ij}\}$ be an independent $N \times N$ GOE matrix. Recalling the parameter t from (3.2.10), define for each $\gamma \in [0,1]$ the $N \times N$ random matrices

$$\mathbf{H}^{\gamma} = \{H_{ij}^{\gamma}\} = \gamma \mathbf{A} + \mathbf{X} + (1-\gamma^2)^{1/2} t^{1/2} \mathbf{W}, \qquad \mathbf{G}^{\gamma} = \{G_{ij}^{\gamma}\} = \big(\mathbf{H}^{\gamma} - z\big)^{-1}.$$

Observe in particular that $\mathbf{H}^0 = \mathbf{V}$, $\mathbf{G}^0 = \mathbf{T}$, $\mathbf{H}^1 = \mathbf{H}$, and $\mathbf{G}^1 = \mathbf{G}$, where we recall the matrices $\mathbf{V}$ and $\mathbf{T}$ from Definition 2.2.7. Our comparison result will approximate the entries of $\mathbf{G}^{\gamma}$ by those of $\mathbf{G}^0$ for any $\gamma \in [0,1]$, after conditioning on Ψ and assuming it to be in an event with high probability with respect to $\mathbb{P}$.

So, it will be useful to consider the laws of $\mathbf{H}$ and $\mathbf{X}$ conditional on their label Ψ. This amounts to conditioning on which entries of $\mathbf{H}$ are at least $N^{-\rho}$. For any $N \times N$ symmetric $0-1$ matrix Ψ, let $\mathbb{P}_{\Psi}$ and $\mathbb{E}_{\Psi}$ denote the probability measure and expectation with respect to the joint law of the random variables $\{a_{ij}, b_{ij}, c_{ij}, \psi_{ij}, \chi_{ij}\}$ from Definition 3.2.5 conditional on the event that $\{\psi_{ij}\}$ is equal to Ψ. This induces a probability measure and expectation on the $\mathbf{H}^{\gamma}$ and $\mathbf{G}^{\gamma}$, denoted by $\mathbb{P}_{\Psi}$ and $\mathbb{E}_{\Psi}$, respectively.

It will also useful for us to further condition on a single χ_{ij}. Thus, for any $\chi \in \{0,1\}$ and $1 \le p, q \le N$, let $\mathbb{P}_{\Psi}\big[\,\cdot\,\big|\chi_{pq}\big] = \mathbb{P}_{\Psi}\big[\,\cdot\,\big|\chi_{pq} = \chi\big]$ denote the probability measure $\mathbb{P}_{\Psi}$ after additionally conditioning on the event that $\chi_{pq} = \chi$, and let $\mathbb{E}_{\Psi}\big[\,\cdot\,\big|\chi_{pq}\big] = \mathbb{E}_{\Psi}\big[\,\cdot\,\big|\chi_{pq} = \chi\big]$ denote the associated expectation. Observe in particular that $\mathbb{E}^{\chi}\big[\mathbb{E}_{\Psi}[\cdot|\chi_{pq}]\big] = \mathbb{E}_{\Psi}\big[\,\cdot\,\big]$, where $\mathbb{E}^{\chi}$ denotes the expectation with respect to the Bernoulli $0-1$ random variable χ from Definition 3.2.4.

The following theorem, which will be a consequence of Proposition 2.3.4 stated in Section 2.3.4 below, provides a way to compare conditional expectations of $\mathbf{G}^0$ to those of $\mathbf{G}^{\gamma}$ for any $\gamma \in [0,1]$. After conditioning on the label Ψ to not have too many entries equal to

1, it roughly states that one compare expectations of smooth functions of these resolvent entries, assuming a bound on the probability they are large.

Theorem 2.2.15. *Let α, b, ρ, ν satisfy (2.2.1), and fix a positive integer m. Then, there exist (sufficiently small) constants $\varepsilon = \varepsilon(\alpha, \nu, \rho, m) > 0$ and $\omega = \omega(\alpha, \nu, \rho, m) > 0$ such that the following holds. Let N be a positive integer. For each integer $j \in [1, m]$, fix real numbers $E_j \in \mathbb{R}$ and $\eta_j > N^{-2}$, and denote $z_j = E_j + \mathrm{i}\eta_j$ for each $j \in [1, m]$. Furthermore, let $F : \mathbb{R}^m \to \mathbb{R}$ be a function such that*

$$\sup_{\substack{0 \le |\mu| \le d \\ |x_j| \le 2N^{\varepsilon}}} \left| F^{(\mu)}(x_1, \ldots, x_m) \right| \le N^{C_0 \varepsilon}, \qquad \sup_{\substack{0 \le |\mu| \le d \\ |x_j| \le 2N^{2}}} \left| F^{(\mu)}(x_1, \ldots, x_m) \right| \le N^{C_0}, \tag{2.2.14}$$

for some real numbers $C_0, d > 0$. Here $\mu = (\mu_1, \mu_2, \ldots, \mu_m)$ is an m-tuple of nonnegative integers, $|\mu| = \sum_{j=1}^m \mu_j$, and $F^{(\mu)} = \prod_{j=1}^m \left(\frac{\partial}{\partial x_j} \right)^{\mu_j} F$. Assume that $d > d_0(\alpha, \nu, \rho, m, C_0)$ is sufficiently large. For any symmetric $0 - 1$ matrix Ψ and complex number z, define the quantities $\mathfrak{J} = \mathfrak{J}(\Psi)$ and $Q_0 = Q_0(\varepsilon, z_1, z_2, \ldots, z_m, \Psi)$ and the event $\Omega_0 = \Omega_0(\varepsilon, z)$ by

$$\mathfrak{J} = \max_{0 \le |\mu| \le d} \sup_{\substack{1 \le i_s, j_s \le N \\ 0 \le \gamma \le 1}} \mathbb{E}_{\Psi} \left[\left| F^{(\mu)} \big(\operatorname{Im} G^{\gamma}_{i_1 j_1}, \ldots, \operatorname{Im} G^{\gamma}_{i_m j_m} \big) \right| \right], \tag{2.2.15}$$

and

$$\Omega_0 = \left\{ \sup_{\substack{1 \le i, j \le N \\ 0 \le \gamma \le 1}} \left| G^{\gamma}_{ij}(z) \right| \le N^{\varepsilon} \right\}, \qquad Q_0 = 1 - \sum_{j=1}^m \mathbb{P}_{\Psi} \big[\Omega_0(z_j) \big]. \tag{2.2.16}$$

Now let Ψ be a symmetric $0 - 1$ random matrix with at most $N^{1 + \alpha\rho + \varepsilon}$ entries equal to 1. Then, there exists a large constant $C = C(\alpha, \nu, \rho, m) > 0$ such that

$$\begin{aligned} \sup_{0 \le \gamma \le 1} \bigg| \mathbb{E}_{\Psi} \Big[F \big(\operatorname{Im} G^{\gamma}_{a_1 b_1}, \ldots, \operatorname{Im} G^{\gamma}_{a_m b_m} \big) \Big] - \mathbb{E}_{\Psi} \Big[F \big(\operatorname{Im} G^{0}_{a_1 b_1}, \ldots, \operatorname{Im} G^{0}_{a_m b_m} \big) \Big] \bigg| & \\ < C N^{-\omega} (\mathfrak{J} + 1) + C Q_0 N^{C + C_0}, & \end{aligned} \tag{2.2.17}$$

for any indices $1 \leq a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_m \leq N$. *The same estimate* (2.2.17) *holds if some of the* $\operatorname{Im} G^0_{a_j b_j}$ *and* $\operatorname{Im} G^\gamma_{a_j b_j}$ *are replaced by* $\operatorname{Re} G^0_{a_j b_j}$ *and* $\operatorname{Re} G^\gamma_{a_j b_j}$, *respectively.*

Although the conditioning on the label Ψ might notationally obscure the statement of Theorem 2.2.15, we will see in Section 2.3.3 that this particular statement of the result will be useful for the proof of Proposition 2.2.17 below. Additionally, we note the constants ε, ω, and d_0 from Theorem 2.2.15 are explicit; see (2.3.25) and (2.3.26) for their values in the case $m = 1$.

Eigenvector delocalization and bulk universality for *H*

In this section we establish Theorem 2.1.4 and Theorem 2.1.5. We first show that the resolvent entries of $\mathbf{H}$ are bounded by N^δ on the nearly optimal scale $\eta = N^{\delta - 1}$ for arbitrarily small $\delta > 0$.

Theorem 2.2.16. *In both regimes* (1) *and* (2) *in Corollary 2.2.10, we have for sufficiently large* $C = C(\alpha, \nu, \rho, \delta, D, K) > 0$ *that*

$$\mathbb{P}\left[\sup_{0 \leq \gamma \leq 1} \sup_{z \in \mathfrak{D}} \max_{1 \leq i,j \leq N} \left| G^\gamma_{ij}(z) \right| > N^\delta \right] \leq C N^{-D}. \tag{2.2.18}$$

Theorem 2.2.16 is a consequence of Corollary 2.2.10 and the following comparison result, which allows one to deduce bounds on the entries of $\mathbf{G}^\gamma$ from bounds on those of $\mathbf{T}$; the latter result will be established using Theorem 2.2.15 in Section 2.3.3 below.

Proposition 2.2.17. *Assume that* $\alpha, b, \nu, \rho > 0$ *satisfy* (2.2.1), *and recall that* $K \subset \mathbb{R}$ *is a compact interval. Fix* $\varsigma \geq 0$, *and suppose that for each* $\delta > 0$ *and* $D > 0$ *there exists a constant* $C = C(\alpha, \rho, \nu, \delta, D, K)$ *such that*

$$\mathbb{P}\left[\sup_{\eta \geq N^{\varsigma - 1}} \sup_{E \in K} \max_{1 \leq i,j \leq N} \left| T_{ij}(E + \mathrm{i}\eta) \right| \geq N^\delta \right] \leq C N^{-D}. \tag{2.2.19}$$

Then, for each $\delta > 0$ and $D > 0$ there exists a large constant $A = A(\alpha, \rho, \nu, \delta, D, K)$ such that

$$\mathbb{P}\left[\sup_{0 \leq \gamma \leq 1} \sup_{\eta \geq N^{\varsigma - 1}} \sup_{E \in K} \max_{1 \leq i,j \leq N} \left|G_{ij}^{\gamma}(E + \mathrm{i}\eta)\right| \geq N^{\delta}\right] \leq AN^{-D}. \tag{2.2.20}$$

Now we can establish Theorem 2.1.4 and Theorem 2.1.5.

Proof of Theorem 2.1.4 and Theorem 2.1.5. It is known from Corollary 3.2 of [64] that complete eigenvector delocalization of the form given by the first parts of Theorem 2.1.4 and Theorem 2.1.5 follows from bounds on the resolvent entries $\left|G_{ij}(z)\right| = \left|G_{ij}^{1}(z)\right|$ of the form (2.2.18). Thus, the first parts of Theorem 2.1.4 and Theorem 2.1.5 follow from Theorem 2.2.16.

To establish the second parts of these two theorems, fix a positive integer m, and let $z_1, z_2, \ldots, z_m \in \mathbb{C}$ be such that $\operatorname{Im} z_j \geq \frac{1}{N^2}$ for each $j \in [1, N]$. Furthermore, if we are in the setting of Theorem 2.1.4 then we additionally impose that each $\operatorname{Re} z_j \in K$: if we are in the setting of Theorem 2.1.5, then we require that each $\left|\operatorname{Re} z_j\right| < \frac{1}{2C}$, where C is from Theorem 2.2.5. We now apply Theorem 2.2.15 with $F(x_1, x_2, \ldots, x_m) = \prod_{i=1}^{m} x_i$.

Then, Theorem 2.2.16 implies that the quantity Q_0 from Theorem 2.2.15 is bounded above by N^{-D} for any $D > 0$ if N is sufficiently large. Furthermore, that theorem and the deterministic bounds $|T_{ij}|, |G_{ij}| \leq N^2$ (due to (2.3.2) below) imply that for each $\delta > 0$ there exists a constant $C = C(\delta)$ such that the quantity $\mathfrak{J}(\Psi)$ from (2.2.15) is bounded by CN^{δ}. Also observe from (3.1.3) and the Chernoff bound that there exists a large constant $C > 0$ such that

$$\mathbb{P}\left[\left|\left\{(i,j) \colon |H_{ij}| \in [N^{-\rho}, \infty)\right\}\right| \notin \left[\frac{N^{1+\alpha\rho}}{C}, CN^{1+\alpha\rho}\right]\right] < Ce^{-N/C}. \tag{2.2.21}$$

Thus, the probability that the matrix Ψ from Theorem 2.2.15 has more than $N^{1+\alpha\rho+\varepsilon}$ entries equal to one is bounded by $c^{-1}e^{-cN}$ for some constant $c > 0$. On this event, we apply the deterministic bounds $|T_{ij}|, |G_{ij}| \leq N^2$. Off of this event, we apply (2.2.17) (averaged over all $(a_1, a_2, \ldots, a_m) = (b_1, b_2, \ldots, b_m)$ in $[1, N]$) and then average over Ψ conditional on the event that Ψ has at most $N^{1+\alpha\rho+\varepsilon}$ entries equal to one. Combining these estimates implies

$$\left| \mathbb{E}\left[N^{-m} \prod_{j=1}^{m} \operatorname{Im} \operatorname{Tr} \mathbf{G}(z_j) - N^{-m} \prod_{j=1}^{m} \operatorname{Im} \operatorname{Tr} \mathbf{T}(z_j) \right] \right| \leq C N^{-c}, \tag{2.2.22}$$

after increasing C and decreasing c if necessary. It is known from Theorem 6.4 of [64] that a comparison of this form implies that the correlation functions of $\mathbf{G}$ and $\mathbf{T}$ asymptotically coincide. Now the universality of the correlation functions for $\mathbf{H}$ at energy level E follows from the corresponding statement for $\mathbf{V}$, given by Proposition 2.2.12. □

2.3 Comparison results

In this section we establish Theorem 2.2.15. After recalling several identities and estimates in Section 2.3.1, we provide a heuristic for the proof of Theorem 2.2.15 in Section 2.3.2. Next, assuming Theorem 2.2.15, we use it to establish Proposition 2.2.17 in Section 2.3.3. We then outline the proof of Theorem 2.2.15 in Section 2.3.4 and implement this outline in the remaining sections: Section 2.3.5, Section 2.3.6, and Section 2.3.7.

2.3.1 Estimates and identities

In this section we state several identities and estimates that will be used throughout this article. We first recall that, for any square matrices $\mathbf{M}$ and $\mathbf{K}$ of the same dimension, we have the resolvent identity

$$\mathbf{K}^{-1} - \mathbf{M}^{-1} = \mathbf{K}^{-1}\big(\mathbf{M} - \mathbf{K}\big)\mathbf{M}^{-1}. \tag{2.3.1}$$

Furthermore, for any symmetric matrix $\mathbf{M}$ and $z = E + \mathrm{i}\eta \in \mathbb{H}$ with $E, \eta \in \mathbb{R}$, we have the deterministic estimate (see equation (8.34) of [52])

$$\left|K_{ij}\right| \leq \frac{1}{\eta}, \quad \text{where } \mathbf{K} = \{K_{ij}\} = (\mathbf{M} - z)^{-1}. \tag{2.3.2}$$

Moreover, observe from (3.1.3) and the fact that H_{ij} has the same law as $N^{-1/\alpha}(Z + J)$ that

$$\frac{C_1}{Nt^\alpha + 1} \leq \mathbb{P}\big[|H_{ij}| \geq t\big] \leq \frac{C_2}{Nt^\alpha + 1}, \quad \text{for any } t > 0. \tag{2.3.3}$$

Using (2.3.3), we can establish the following lemma, which bounds moments of truncations of H_{ij}. As a consequence, we deduce Lemma 3.2.6.

Lemma 2.3.1. *Fix $R \geq N^{-1/\alpha}$ and let $s_{ij} = H_{ij}\mathbf{1}_{|H_{ij}|<R}$. For any positive real number $p > \alpha$, we have that $cN^{-1}R^{p-\alpha} \leq \mathbb{E}\big[|s_{ij}|^p\big] \leq CN^{-1}R^{p-\alpha}$, for a small constant $c = c(\alpha, p, C_2) > 0$ and a large constant $C = C(\alpha, p, C_1) > 0$.*

Proof. From (2.3.3), we have that

$$\mathbb{E}\big[|s_{ij}|^p\big] = p\int_0^R s^{p-1}\mathbb{P}\big[|H_{ij}| \geq s\big]ds \leq \frac{C_1 p}{N}\int_0^R s^{p-1-\alpha}ds = \frac{C_1 p R^{p-\alpha}}{N(p-\alpha)},$$

which establishes the upper bound in the lemma. To establish the lower bound, observe from (2.3.3) and the bound $R \geq N^{-1/\alpha}$ that

$$\begin{aligned}\mathbb{E}\big[|s_{ij}|^p\big] = p\int_0^R s^{p-1}\mathbb{P}\big[|H_{ij}| \geq s\big]ds &\geq \frac{C_1 p}{N}\int_{R/2}^R \frac{ds}{s^{\alpha+1-p} + N^{-1}s^{1-p}} \\ &\geq \frac{C_1 p}{5N}\int_{R/2}^R s^{p-\alpha-1}ds = \frac{C_1 p(1 - 2^{\alpha-p})R^{p-\alpha}}{5N(p-\alpha)}.\end{aligned}$$

□

Proof of Lemma 3.2.6. From Lemma 2.3.1 applied with $R = N^{-\nu}$ and $p = 2$, we deduce the existence of constants $C = C(\alpha, C_1) > 0$ and $c = c(\alpha, C_2) > 0$ such that $cN^{(\alpha-2)\nu-1} \leq \mathbb{E}\big[H_{11}^2\mathbf{1}_{|H_{11}|<N^{-\nu}}\big] \leq CN^{(\alpha-2)\nu-1}$. Combining this with the fact that $\mathbb{P}\big[|H_{11}| < N^{-\rho}\big] \geq \frac{1}{2}$ for sufficiently large N (due to (2.3.3)), we deduce the lemma. $\square$

We close this section with the following lemma, which bounds the conditional moments of the random variables A_{ij} and B_{ij} from Definition 3.2.5.

Lemma 2.3.2. *Let $p > \alpha$. There exists a large constant $C = C(\nu, \rho, p)$ such that, for any indices $1 \leq i, j \leq N$, we have that*

$$\mathbb{E}_{\Psi}\big[|A_{ij}|^p \big| \chi_{ij}\big] \leq C N^{\nu(\alpha - p) - 1}, \qquad \mathbb{E}_{\Psi}\big[|B_{ij}|^p\big] \leq C N^{\rho(\alpha - p) - 1}. \tag{2.3.4}$$

Proof. Let us first establish the bound on $\mathbb{E}_{\Psi}\big[|B_{ij}|^p\big]$. There are two cases to consider, depending on the entry $\psi \in \{0, 1\}$. If $\psi_{ij} = 1$, then $B_{ij} = 0$ and thus (2.3.4) holds. If $\psi_{ij} = 0$, then there exists a constant $C = C(\rho, p) > 0$ such that

$$\mathbb{E}_{\Psi}\big[|B_{ij}|^p\big] = \frac{\mathbb{E}\big[|B_{ij}|^p\big]}{\mathbb{P}\big[\psi_{ij} = 0\big]} \leq \frac{\mathbb{E}\big[|H_{ij}|^p \mathbf{1}_{|H_{ij}| \leq N^{-\rho}}\big]}{\mathbb{P}\big[|H_{ij}| \leq N^{-\rho}\big]} \leq C N^{\rho(\alpha - p) - 1},$$

where to deduce the last estimate above we used Lemma 2.3.1 and the fact that $\mathbb{P}\big[|H_{ij}| > N^{-\rho}\big] > \frac{1}{2}$ for sufficiently large N (due to (2.3.3)). This yields the second estimate in (2.3.4).

Through a very similar procedure, we deduce after increasing C if necessary that $\mathbb{E}\big[|a_{ij}|^p\big] \leq C N^{\nu(\alpha - p) - 1}$, where a_{ij} has the same law as the random variable a given in Definition 3.2.4. Now the first estimate in (2.3.4) follows from the deterministic bound $|A_{ij}| \leq |a_{ij}|$. $\square$

2.3.2 A heuristic for the comparison

Here we provide a heuristic for the estimate (2.2.17) if $a = i = b$ for some $i \in [1, N]$. Conditioning on Ψ (and abbreviating $\mathbb{E}_{\Psi}$ as $\mathbb{E}$ here for brevity), we obtain

$$\partial_{\gamma} \mathbb{E}\big[G_{ii}^{\gamma}\big] = \sum_{1 \leq j, k \leq N} \mathbb{E}\left[G_{ij}^{\gamma} \left(A_{jk} - \frac{\gamma t^{1/2}}{(1 - \gamma^2)^{1/2}} w_{jk}\right) G_{ki}^{\gamma}\right],$$

where we used (2.3.1) to compute the derivative on the left side.

Now let us consider two cases. The first is the "large field case," meaning that $\psi_{jk} = 1$ (which implies that $A_{jk} = 0 = B_{jk}$ and $|H_{jk}| \geq N^{-\rho}$). Recall the formula for Gaussian integration by parts (see, for example, Appendix A.4 of [93]): for a differentiable function $F\colon \mathbb{R} \to \mathbb{R}$ subject to a mild growth condition, and a centered Gaussian g, $\mathbb{E}\left[gF(g)\right] = \mathbb{E}\left[g^2\right]\mathbb{E}\left[F'(g)\right]$. We integrate by parts with respect to the Gaussian random variable $x = N^{1/2}w_{jk}$, which is centered and has variance one. This yields

$$\frac{\gamma}{(1-\gamma^2)^{1/2}}\left(\frac{t}{N}\right)^{1/2}\mathbb{E}\big[G_{ij}^{\gamma} x G_{ki}^{\gamma}\big] = \frac{\gamma t}{N}\mathbb{E}\big[G_{ij}^{\gamma} G_{kk}^{\gamma} G_{ki}^{\gamma} + \cdots\big],$$

where the additional terms are degree three monomials in the G_{ij}^{γ} (and we again used (2.3.1) to compute the derivatives of the resolvent entries). Assuming that each $\left|G_{ij}^{\gamma}\right|$ is bounded, and using Lemma 3.2.6 and the fact that the number of pairs (j,k) for which $\psi_{jk} = 1$ is essentially bounded by $N^{\alpha\rho+1}$, we can bound the total contribution of these terms by a multiple of

$$tN^{-1}N^{\alpha\rho+1} \sim N^{\nu(\alpha-2)+\alpha\rho}.$$

The second is the "small field case," meaning that $\psi_{jk} = 0$ (so $|H_{jk}| < N^{-\rho}$). Recall that $A_{jk} = a_{jk}(1-\chi_{jk})$ and $B_{jk} = b_{jk}\chi_{jk}$, and abbreviate $a_{jk} = a$, $b_{jk} = b$, $\chi_{jk} = \chi$, and $w_{jk} = w$. Letting $\mathbf{U}^{\gamma} = \left\{U_{jk}^{\gamma}\right\}$ denote the resolvent of $\mathbf{H}$ whose (j,k) and (k,j) entries are set to zero, we can expand $\mathbf{G}^\gamma$ around $\mathbf{U}^\gamma$ using (2.3.1) to obtain

$$\begin{aligned}
&\mathbb{E}\left[G^\gamma_{ik}\left((1-\chi)a - \frac{\gamma t^{1/2}w}{(1-\gamma^2)^{1/2}}\right)G^\gamma_{ji}\right] \\
&\quad = \mathbb{E}\left[\left((1-\chi)a + \frac{\gamma t^{1/2}w}{(1-\gamma^2)^{1/2}}\right)\left(\gamma(1-\chi)a + \chi b + (1-\gamma^2)^{1/2}t^{1/2}w\right)\left(U^\gamma_{ij}U^\gamma_{kk}U^\gamma_{ji} + \cdots\right)\right] \\
&\quad + \mathbb{E}\left[\left((1-\chi)a + \frac{\gamma t^{1/2}w}{(1-\gamma^2)^{1/2}}\right)\left(\gamma(1-\chi)a + \chi b + (1-\gamma^2)^{1/2}t^{1/2}w\right)^3\left(U^\gamma \cdots U^\gamma + \cdots\right)\right] \\
&\quad = \gamma\mathbb{E}\left[(1-\chi)a^2 - tw^2\right]\mathbb{E}[U^\gamma_{ij}U^\gamma_{kk}U^\gamma_{ji} + \cdots] \\
&\quad + \gamma\mathbb{E}\left[\gamma^2(1-\chi)a^4 + 3\gamma^2(1-\chi)tw^2a^2 + 3\chi tw^2b^2 + (1-\gamma^2)t^2w^4\right]\mathbb{E}[U^\gamma \cdots U^\gamma + \cdots],
\end{aligned}$$

where the additional terms refer to polynomials in the entries of $\mathbf{U}$. To deduce the first equality, we used the fact that terms not involving a factor of $\gamma(1-\chi)a+\chi b+(1-\gamma^2)^{1/2}t^{1/2}w$ (first order terms) and those involving $\big(\gamma a(1-\chi)+b\chi+(1-\gamma^2)^{1/2}t^{1/2}w\big)^2$ (third order terms) vanish because a, b, and w are symmetric and $\mathbf{U}$, a, b, w, and χ are mutually independent.

From the choice of t, we have that

$$\gamma\mathbb{E}\big[(1-\chi)a^2 - tw^2\big] = 0.$$

Hence the second order terms vanish if $\psi_{jk} = 0$. Assuming that the entries of $\mathbf{U}^\gamma$ are bounded, we can also estimate the sum of all fourth order terms by a multiple of

$$\begin{aligned}
N^2\mathbb{E}\left[(1-\chi)a^4 + (1-\chi)tw^2a^2 + \chi tw^2b^2 + t^2w^4\right] &\leq N^{\nu(\alpha-4)+1} + N^{(\rho+\nu)(\alpha-2)} + N^{2\nu(\alpha-2)} \\
&\leq N^{\nu(\alpha-4)+1} + N^{-r},
\end{aligned}$$

for some $r > 0$. Here, we used (3.1.3), Lemma 3.2.6, and the facts that $(1-\chi)a = H_{ij}\mathbf{1}_{|H_{ij}|<N^{-\nu}}$ and $\chi b = H_{ij}\mathbf{1}_{N^{-\nu}\leq|H_{ij}|<N^{-\rho}}$ to deduce that $\mathbb{E}\big[(1-\chi)a^4\big] \sim N^{\nu(\alpha-4)-1}$, $\mathbb{E}\big[(1-\chi)a^2\big] \sim N^{(\alpha-2)\nu-1}$, and $\mathbb{E}[\chi b^2] \sim N^{\rho(\alpha-2)-1}$, as shown in Lemma 2.3.2.

Hence the total contribution from the second and fourth order terms is bounded by a multiple of

$$N^{\nu(\alpha-2)+\alpha\rho} + N^{\nu(\alpha-4)+1} + N^{-r}.$$

For this to tend to 0, we require

$$\nu > \frac{1}{4-\alpha}, \qquad \alpha\rho < (2-\alpha)\nu, \qquad 0 < \rho < \nu,$$

where the last restriction is by definition. This recovers a number of the constraints imposed by (2.2.1). To motivate the others, recall that the local law for $\mathbf{X}$ was proved for any scale $N^{-\varpi}$ with $(2-\alpha)\nu < \varpi < \nu < \frac{1}{2}$ for $\alpha \in (1,2)$ (Theorem 2.2.4) and at the scale $N^{\delta-1/2}$ for almost all $\alpha \in (0,2)$ in the small energy regime (Theorem 2.2.5). In order to apply the results on Dyson Brownian motion from Section 2.2.2, we need the scale of these local laws to be smaller than $t \sim N^{\nu(\alpha-2)}$. For $\alpha \in (1,2)$, this condition is guaranteed. For $\alpha \le 1$, this requires $\nu < \frac{1}{4-2\alpha}$, which is the remaining condition in (2.2.1).

2.3.3 Improving the scale

In this section we establish Proposition 2.2.17, assuming Theorem 2.2.15 holds, using an induction on the scale η.

To that end, recall the definitions of the matrices $\mathbf{G}^\gamma(z)$ for any $\gamma \in [0,1]$, and define

$$\mathfrak{P}(\delta,\eta) = \mathbb{P}\left[\max_{\substack{1\le i,j\le N\\ 0\le\gamma\le 1}} \left|G_{ij}^\gamma(E+\mathrm{i}\eta)\right| > N^\delta\right], \tag{2.3.5}$$

for any $E \in \mathbb{R}$, $\eta \ge N^{\varsigma-1}$, and $\delta > 0$. Moreover, fix ε and ω as in Theorem 2.2.15, choosing $k = 1$ in that theorem, and let $\sigma = \frac{\varepsilon}{4}$. We omit the dependence of $\alpha, b, \rho, \nu, \varepsilon, \omega$, and k in the notation for the constants appearing in the following lemma and view them as fixed parameters.

We begin with the following lemma.

Lemma 2.3.3. *Adopt the notation and assumptions of Proposition 2.2.17. For any $\delta > 0$ and integer $D > 0$, there exists a large constant $C = C(\delta, D)$ such that $\mathfrak{P}(\delta, \eta) \leq CN^{C}\mathfrak{P}\big(\frac{\varepsilon}{2}, N^{\sigma}\eta\big) + CN^{-D}$ for all $\eta \geq N^{\varsigma - 1}$.*

Proof. Let $p = \left\lceil \frac{D+30}{\delta} \right\rceil$, and define $F_p(x) = |x|^{2p} + 1$. Observe that there exists a constant C_p, only dependent on p (and therefore only dependent on δ and D) such that

$$\big|F_p^{(a)}(x)\big| \leq C_p F_p(x), \quad \text{for all } x \in \mathbb{R} \text{ and } a \in \mathbb{Z}_{\geq 0}. \tag{2.3.6}$$

Now we apply Theorem 2.2.15 with $F(x) = F_p(x)$. Observe that the C_0 from that theorem can be taken to be $4p$ and that d can also be taken to be bounded by constant multiple of p (where the implicit constants depend ν, ρ, and ε, although in the future we will not mention the dependence on these parameters, since they are already fixed). In view of (2.2.17) and (2.3.6), there exists a large constant B_p (only dependent p) such that

$$\mathbb{E}_{\Psi}\Big[F_p\big(\operatorname{Im} G_{ab}^{\gamma}(z)\big)\Big] \leq \mathbb{E}_{\Psi}\Big[F_p\big(\operatorname{Im} T_{ab}(z)\big)\Big] + B_p\big(N^{-\omega}\mathfrak{I}_p(\Psi) + Q_0(\varepsilon, \Psi)N^{B_p} + 1\big), \tag{2.3.7}$$

for any $0-1$ symmetric $N \times N$ matrix Ψ with at most $N^{1+\alpha\rho+\varepsilon}$ entries equal to 1, where

$$\mathfrak{I}_p\big(\Psi\big) = \sup_{\substack{1 \leq i,j \leq N \\ 0 \leq \gamma \leq 1}} \mathbb{E}_{\Psi}\big[F_p(\operatorname{Im} G_{ij}^{\gamma})\big], \quad Q_0\big(\varepsilon, z, \Psi\big) = \mathbb{P}_{\Psi}\Bigg[\max_{\substack{1 \leq i,j \leq N \\ 0 \leq \gamma \leq 1}} \big|G_{ij}^{\gamma}(z)\big| > N^{\varepsilon}\Bigg].$$

Now observe that taking the supremum over all $1 \leq a, b \leq N$ and $0 \leq \gamma \leq 1$ on the left side of (2.3.7) yields $\mathfrak{I}_p\big(\Psi\big)$. Therefore,

$$(1 - B_p N^{-\omega})\mathfrak{I}_p\big(\Psi\big) \leq \max_{1 \leq a,b \leq N} \mathbb{E}_{\Psi}\Big[F_p\big(\operatorname{Im} T_{ab}(z)\big)\Big] + B_p\left(Q_0\big(\varepsilon, z, \Psi\big)N^{B_p} + 1\right). \tag{2.3.8}$$

We now take the expectation of (2.3.8) over Ψ. On the event when there are at most $N^{1+\alpha\rho+\varepsilon}$ entries equal to C in Ψ, we apply (2.3.8). The complementary event has probability bounded by $c^{-1}e^{-cN}$, for some constant $c > 0$, due to (2.2.21); on this event, we apply the deterministic bounds $F_p\big(\operatorname{Im} G_{ab}^{\gamma}\big) \leq N^{5p}$ and $F_p(\operatorname{Im} T_{ab}) \leq N^{5p}$. Combining these estimates and fact that $B_p N^{-\omega} < \frac{1}{2}$ for sufficiently large N yields that

$$\mathfrak{J}_p \leq N^2 \max_{1 \leq a,b \leq N} \mathbb{E}\Big[F_p\big(\operatorname{Im} T_{ab}(z)\big)\Big] + B_p \left(N^{B_p} \mathfrak{P}(\varepsilon, \eta) + 1\right) + B_p N^{B_p} e^{-cN}, \tag{2.3.9}$$

where

$$\mathfrak{J}_p = \max_{\substack{1 \leq i,j \leq N \\ 0 \leq \gamma \leq 1}} \mathbb{E}\big[F_p(\operatorname{Im} G_{ij}^{\gamma})\big].$$

Here we increased B_p and used

$$\mathfrak{J} \leq \mathbb{E}\left[\mathfrak{J}_p(\Psi)\right]; \qquad \mathbb{E}\left[\max_{1 \leq a,b \leq N} \mathbb{E}_{\Psi}\Big[F_p\big(\operatorname{Im} T_{ab}(z)\big)\Big]\right] \leq N^2 \max_{1 \leq a,b \leq N} \mathbb{E}\Big[F_p\big(\operatorname{Im} T_{ab}(z)\big)\Big].$$

After increasing B_p again if necessary, we find from (2.2.19) and the trivial bound (2.3.2) that $\mathbb{E}\big[F_p\big(\operatorname{Im} T_{ab}(z)\big)\big] \leq B_p N$ for any $1 \leq a, b \leq N$. Inserting this into (2.3.9) yields

$$\mathfrak{J}_p \leq B_p N^3 + B_p N^{B_p} \mathfrak{P}(\varepsilon, \eta) \leq B_p N^3 + B_p N^{B_p} \mathfrak{P}(\varepsilon - \sigma, N^{\sigma} \eta), \tag{2.3.10}$$

where in the second estimate above we have used the fact that $\mathfrak{P}(\varepsilon, \eta) \leq \mathfrak{P}(\varepsilon - \sigma, N^{\sigma}\eta)$, which follows from the bound

$$\max\left\{\max_{1 \leq i,j \leq N} \big|G_{ij}^{\gamma}(E + \mathrm{i}\eta)\big|, 1\right\} \leq R \max\left\{\max_{1 \leq i,j \leq N} \big|G_{ij}^{\gamma}(E + \mathrm{i}R\eta)\big|, 1\right\}, \quad \text{for any } R > 0, \tag{2.3.11}$$

given as Lemma 2.1 of [23]. Applying (2.3.10), a Markov estimate, the fact that $\sigma = \frac{\varepsilon}{4}$, and the fact that $p\delta \geq D + 30$, we see for any i, j that

$$\begin{aligned}
\sup_{\gamma \in [0,1]} \mathbb{P}\left[\max_{1 \leq i,j \leq N} \left| \operatorname{Im} G_{ij}^{\gamma}(z) \right| > N^{\delta/2}\right] &\leq N^2 \max_{\substack{1 \leq i,j \leq N \\ 0 \leq \gamma \leq 1}} \mathbb{P}\left[\left| \operatorname{Im} G_{ij}^{\gamma}(z) \right| > N^{\delta/2}\right] \\
&\leq N^2 \max_{\substack{1 \leq i,j \leq N \\ 0 \leq \gamma \leq 1}} \frac{\mathbb{E}\left[F_p\left(\operatorname{Im} G_{ij}^{\gamma}\right)\right]}{F_p(N^{\delta/2})} \\
&< \frac{\mathfrak{J}_p}{N^{p\delta - 2}} < B_p N^{-D-25} + B_p N^{B_p} \mathfrak{P}\left(\frac{\varepsilon}{2}, N^{\sigma}\eta\right).
\end{aligned} \tag{2.3.12}$$

Applying a union bound over the i, j and the same reasoning with $\operatorname{Im} G_{ab}^{\gamma}$ replaced by $\operatorname{Re} G_{ab}^{\gamma}$, we deduce the estimate

$$\sup_{\gamma \in [0,1]} \mathbb{P}\left[\max_{1 \leq i,j \leq N} \left| G_{ij}^{\gamma}(z) \right| > N^{\delta/2}\right] < B_p N^{-D-20} + B_p N^{B_p + 2} \mathfrak{P}\left(\frac{\varepsilon}{2}, N^{\sigma}\eta\right). \tag{2.3.13}$$

Now the proposition follows from applying a union bound in (2.3.13) over all $\gamma \in [0,1] \cap N^{-20}\mathbb{Z}$, then extending these range of γ to all of $[0,1]$ through the deterministic estimate

$$\left| G_{ij}^{\gamma}(z) - G_{ij}^{\gamma'}(z) \right| \leq 2|\gamma - \gamma'|^{1/2} N^6 \left(1 + \max_{1 \leq i,j \leq N} |w_{ij}|\right),$$

due to (2.3.1), (2.3.2), the fact that $\eta \geq N^{-2}$, and the bound $\mathbb{P}\big[|w_{ij}| > 2\big] < e^{-N}$. $\square$

We can now establish Proposition 2.2.17.

Proof of Proposition 2.2.17. Set $\kappa = \left\lceil \frac{1 - \varsigma}{\sigma} \right\rceil$. We first claim that, for any integers $D > 0$ and $k \in [-1, \kappa]$, there exists a constant $C = C(D, k) > 0$ such that $\mathfrak{P}\big(\frac{\varepsilon}{2}, N^{-k\sigma}\big) < CN^{-D}$.

To establish this, we proceed by induction on k. Because κ is constant, only finitely many inductive steps are required. Therefore, we may permit the constants $C(D, k)$ to increase at each step.

The base case $k = -1$ is trivial, because (2.3.2) implies that $\mathfrak{P}\big(\frac{\varepsilon}{2}, \eta\big) = 0$ for any $\eta \in (1, N^{\sigma}]$. For the induction step, suppose the claim holds for $k = m \in [-1, \kappa - 1]$, and fix some integer $D > 0$. We must show that there exists a constant $C = C(D, m+1) > 0$ such that $\mathfrak{P}\big(\frac{\varepsilon}{2}, N^{-(m+1)\sigma}\big) \le CN^{-D}$.

To that end, applying Lemma 2.3.3 yields an integer $C_1 = C_1(D, m) > 0$ such that

$$\mathfrak{P}\left(\frac{\varepsilon}{2}, N^{-(m+1)\sigma}\right) \le C_1 N^{C_1} \mathfrak{P}\left(\frac{\varepsilon}{2}, N^{-m\sigma}\right) + C_1 N^{-D}. \tag{2.3.14}$$

Next, recall by the induction hypothesis for $k = m$ that, for any integer $D' > 0$, there exists a constant $C = C(D', m) > 0$ such that $\mathfrak{P}\big(\frac{\varepsilon}{2}, N^{-m\sigma}\big) < CN^{-D'}$. In particular, taking $D' = C_1 + D$, there exists a constant $C_2 = C_2(D, m) > 0$, given by the $C(C_1 + D, m)$ from the induction hypothesis, such that $\mathfrak{P}\big(\frac{\varepsilon}{2}, N^{-m\sigma}\big) < C_2 N^{-C_1 - D}$. Inserting this into (2.3.14) yields $\mathfrak{P}\big(\frac{\varepsilon}{2}, N^{-(m+1)\sigma}\big) \le (C_1 C_2 + C_1) N^{-D}$, which completes the induction after setting $C(D, m+1) = C_1 C_2 + C_1$.

Now fix $\delta, D > 0$. For any $\eta \ge N^{\varsigma - 1}$, applying Lemma 2.3.3 shows there exist constants $B = B(\delta, D) > 0$ and $C = C(\delta, D) > 0$ such that

$$\mathfrak{P}(\delta, \eta) \le CN^C \mathfrak{P}\left(\frac{\varepsilon}{2}, N^{\sigma}\eta\right) + CN^{-D} \le BN^{-D}, \tag{2.3.15}$$

where we used the fact that $N^{\sigma}\eta \ge N^{-\kappa\sigma}$, the bound $\mathfrak{P}\big(\frac{\varepsilon}{2}, N^{-\kappa\sigma}\big) < CN^{-C-D}$, and the monotonicity of $\mathfrak{P}(\delta, \eta)$ in η (which follows from (2.3.11)).

Now let $\mathcal{D}$ denote the set of $z \in \mathbb{H}$ of the form $E + \mathrm{i}\eta$, where $E \in K$ and $N^{\varsigma - 1} \le \eta \le 1$ are both of the form $\frac{k}{N^{10}}$ for some integer k. Then a union bound applied to (2.3.15) shows that

$$\mathbb{P}\left[\sup_{z \in \mathcal{D}} \sup_{\gamma \in [0,1]} \max_{1 \le i,j \le N} \big|G_{ij}^{\gamma}(z)\big| \ge N^{\delta}\right] \le \frac{B_{\delta, D+25}}{N^D}. \tag{2.3.16}$$

Now from (2.3.16) and the deterministic estimate $\big|G_{ij}(z) - G_{ij}(z')\big| \le N^6 |z - z'|$ we deduce that

$$\mathbb{P}\left[\sup_{\eta \geq N^{\varsigma-1}} \sup_{\gamma \in [0,1]} \max_{1 \leq i,j \leq N} \left|G_{ij}^{\gamma}(z)\right| \geq 2N^{\delta}\right] \leq \frac{B_{\delta,D+25}}{N^D}. \tag{2.3.17}$$

Here, we used the fact that bound holds trivially in the region where $\eta \geq 1$ by (2.3.2). Thus (2.2.20) follows by setting $\gamma = 0$ in (2.3.17). □

2.3.4 Outline of the proof of Theorem 2.2.15

For the remainder of Section 2.3, we assume that $m = 1$ in Theorem 2.2.15, and we abbreviate $z_1 = z$, $a_1 = a$, and $b_1 = b$. Since the proof of Theorem 2.2.15 for $m > 1$ is entirely analogous, it is omitted. However, in Section 2.3.8 we briefly outline how to modify the proof in this case.

Observe that

$$\frac{\partial}{\partial \gamma}\mathbb{E}_{\Psi}\Big[F\big(\operatorname{Im} G_{ab}^{\gamma}\big)\Big] = \sum_{1 \leq p,q \leq N} \mathbb{E}_{\Psi}\left[\operatorname{Im}(G_{ap}^{\gamma} G_{qb}^{\gamma})\left(A_{pq} - \frac{\gamma t^{1/2} w_{pq}}{(1-\gamma^2)^{1/2}}\right) F'\big(\operatorname{Im} G_{ab}^{\gamma}\big)\right], \tag{2.3.18}$$

and so it suffices to establish the following proposition. We recall that $\mathfrak{J} = \mathfrak{J}_p(\Psi)$ and Q_0 were defined in Theorem 2.2.15.

Proposition 2.3.4. *Adopt the notation of Theorem 2.2.15. Then there exists a large constant $C = C(\alpha, \nu, \rho) > 0$ such that, for sufficiently large N,*

$$\begin{aligned} \sum_{1 \leq p,q \leq N} \left|\mathbb{E}_{\Psi}\left[\operatorname{Im}(G_{ap}^{\gamma} G_{qb}^{\gamma})\Big(A_{pq} - \frac{\gamma t^{1/2} w_{pq}}{(1-\gamma^2)^{1/2}}\Big) F'\big(\operatorname{Im} G_{ab}^{\gamma}\big)\right]\right| \\ \leq \frac{C}{(1-\gamma^2)^{1/2}}\big(N^{-\omega}(\mathfrak{J}+1) + Q_0 N^{C+C_0}\big). \end{aligned} \tag{2.3.19}$$

To establish Proposition 2.3.4, we estimate each summand on the right side of (2.3.18). Thus, in what follows, let us fix some integer pair $(p,q) \in [1,N] \times [1,N]$.

If $\mathbf{G}^{\gamma}$ were independent from A_{pq} and w_{pq}, then each expectation on the left side of (2.3.19) would be equal to zero, from which the proposition would follow. Since this independence does not hold, we will approximate $\mathbf{G}^\gamma$ with a matrix that is independent from A_{pq} and w_{pq} (after conditioning on Ψ, as we will do throughout the proof of Proposition 2.3.4) and estimate the error incurred by this replacement.

In fact, it will be useful to introduce two matrices. The first will be independent from w_{pq} but not quite independent from A_{pq} (although it will be independent from A_{pq} after additionally conditioning on χ_{pq}); the second will be independent from both w_{pq} and A_{pq}

More specifically, we define the $N \times N$ matrices $\mathbf{D} = \mathbf{D}^{\gamma,p,q} = \{D_{ij}\} = \big\{D_{ij}^{\gamma,p,q}\big\}$ and $\mathbf{E} = \mathbf{E}^{\gamma,p,q} = \big\{E_{ij}\big\} = \big\{E_{ij}^{\gamma,p,q}\big\}$ by setting $D_{ij} = H_{ij}^\gamma = E_{ij}$ if $(i,j) \notin \big\{(p,q),(q,p)\big\}$ and

$$D_{pq} = D_{qp} = X_{pq} = B_{pq} + C_{pq}, \qquad E_{pq} = E_{qp} = C_{pq}.$$

We also define the $N \times N$ matrices $\Gamma = \Gamma^{\gamma,p,q} = \{\Gamma_{ij}\} = \big\{\Gamma_{ij}^{\gamma,p,q}\big\} = \mathbf{H}^\gamma - \mathbf{D}$ and $\Lambda = \Lambda^{\gamma,p,q} = \{\Lambda_{ij}\} = \big\{\Lambda_{ij}^{\gamma,p,q}\big\} = \mathbf{D} - \mathbf{E}$, so that

$$\Gamma_{ij} = \gamma\Theta_{ij} + (1-\gamma^2)^{1/2}\Phi_{ij}, \qquad \Lambda_{ij} = B_{pq}\mathbf{1}_{(i,j)\in\{(p,q),(q,p)\}}, \tag{2.3.20}$$

where

$$\Theta_{ij} = A_{ij}\mathbf{1}_{(i,j)\in\{(p,q),(q,p)\}}, \qquad \Phi_{ij} = t^{1/2}w_{ij}\mathbf{1}_{(i,j)\in\{(p,q),(q,p)\}}. \tag{2.3.21}$$

In addition, we define the resolvent matrices

$$\mathbf{R} = \mathbf{R}^{\gamma,p,q} = \{R_{ij}\} = \big\{R_{ij}^{\gamma,p,q}\big\} = (\mathbf{D}-z)^{-1}, \qquad \mathbf{U} = \mathbf{U}^{\gamma,p,q} = \{U_{ij}\} = \big\{U_{ij}^{\gamma,p,q}\big\} = (\mathbf{E}-z)^{-1}. \tag{2.3.22}$$

Remark 2.3.5. Observe that, after conditioning on Ψ, the matrices Γ and Λ are both independent from $\mathbf{U}$. After further conditioning on χ_{pq}, the matrices $\Theta = \{\Theta_{ij}\}$, $\Phi = \{\Phi_{ij}\}$, and $\mathbf{R}$ become mutually independent.

We would first like to replace the entries G_{ij}^{γ} in the (p,q) summand on the left side of (2.3.19) with the entries $R_{ij} = R_{ij}^{\gamma,p,q}$. To that end, we set

$$\xi_{ij} = \xi_{ij}(\gamma) = \big(\mathbf{G}^{\gamma} - \mathbf{R}\big)_{ij} = \big(- \mathbf{R}\Gamma\mathbf{R} + (\mathbf{R}\Gamma)^2\mathbf{R} - (\mathbf{R}\Gamma)^3\mathbf{G}^{\gamma}\big)_{ij}, \qquad \zeta_{ij} = \operatorname{Im}\xi_{ij}, \tag{2.3.23}$$

for any $1 \le i, j \le N$, where the third equality in (2.3.23) follows from the resolvent identity (2.3.1). We abbreviate $\zeta = \zeta_{ab}$.

By a Taylor expansion, there exists some $\zeta_0 \in [\operatorname{Im} G_{ab}^{\gamma}, \operatorname{Im} R_{ab}]$ such that

$$F'(\operatorname{Im} G_{ab}^{\gamma}) = F'(\operatorname{Im} R_{ab} + \zeta) = F^{(1)}(\operatorname{Im} R_{ab}) + \zeta F^{(2)}(\operatorname{Im} R_{ab}) + \frac{\zeta^2}{2} F^{(3)}(\operatorname{Im} R_{ab}) + \frac{\zeta^3}{6} F^{(4)}(\zeta_0), \tag{2.3.24}$$

where $F^{(i)}(x) = \frac{\partial^i F}{\partial x^i}(x)$ for any $i \in \mathbb{Z}_{\ge 0}$ and $x \in \mathbb{R}$.

Using (2.3.21), (2.3.23) and (2.3.24), we deduce that the (p,q)-summand on the left side of (2.3.19) can be expanded as a finite sum of (consisting of less than 2^{22}) monomials in Θ_{pq} and Φ_{pq}, whose coefficients depend on the entries of $\mathbf{G}^{\gamma}$ and $\mathbf{R}$. We call such a monomial of *degree* k (or a k*-th order term*) if it is of total degree k in Θ_{pq} and Φ_{pq}. We will estimate the (p,q)-summand on the left side of (2.3.19) by bounding the expectation of each such monomial, which will be done in the following sections.

Before proceeding, let us fix an integer pair $(p,q) \in [1,N] \times [1,N]$ throughout this the remainder of section. It will also be useful for us to define some additional parameters that will be fixed throughout this section. In what follows, we define the positive real numbers $\omega > \varepsilon > 0$ through

$$\begin{aligned} \varepsilon &= \frac{\alpha}{100} \min \left\{ (4-\alpha)\nu - 1, (2-\alpha)\nu - \alpha\rho, \nu - \rho, \frac{\rho}{2}, 1 \right\}, \\ \omega &= \min \big\{ (\alpha - 2\varepsilon)\rho - 15\varepsilon, (2-\alpha)\nu - \alpha\rho - 15\varepsilon, (4-\alpha)\nu - 1 - 10\varepsilon, (4-2\alpha)\nu - 15\varepsilon \big\}. \end{aligned} \tag{2.3.25}$$

Moreover, let us fix integers $\vartheta, d > 0$ such that

$$\vartheta(\rho - 2\varepsilon) > C_0\varepsilon + 3, \qquad d > 3\vartheta + 5. \tag{2.3.26}$$

The remainder of this section is organized as follows. We will estimate the contribution to the left side of (2.3.19) resulting from the first, third, and higher degree terms in Section 2.3.6, and we will estimate the contribution from the second degree terms in Section 2.3.7. However, we first require estimates on the entries of $\mathbf{R}$ and $\mathbf{U}$ (from (2.3.22)), which will be provided in Section 2.3.5. We then outline the modifications necessary in the proof of Theorem 2.2.15 in Section 2.3.8.

2.3.5 Estimating the entries of R and U

Recall that the event Ω_0 from (2.2.16) bounds the entries of $\mathbf{G}^\gamma$. In this section we will provide similar estimates on the entries of $\mathbf{R}$ and $\mathbf{U}$ on an event slightly smaller than Ω_0. More specifically, define

$$\Omega_1 = \Omega_1(\rho) = \left\{ \max_{1 \le i,j \le N} |w_{ij}| \le N^{-\rho} \right\}, \qquad \Omega = \Omega(\rho, \varepsilon, z) = \Omega_0 \cap \Omega_1,$$
$$Q = 1 - \mathbb{P}_\Psi\big[\Omega(\rho, \varepsilon, z)\big] = \mathbb{P}_\Psi[\Omega^c],$$

where Ω^c denotes the complement of Ω. Since $\rho < \frac{1}{2}$ and w_{ij} is a Gaussian random variable with variance at most $\frac{2}{N}$, there exists small constant $c = c(\rho) > 0$ such that

$$1 - \mathbb{P}\big[\Omega_1\big] < e^{-cN^c}. \tag{2.3.27}$$

Thus, it suffices to establish (2.3.19) with Q_0 there replaced by Q. The following lemma estimates $|R_{ij}|$ and $|U_{ij}|$ on the event Ω.

Lemma 2.3.6. *For N sufficiently large, we have that*

$$\mathbf{1}_\Omega \sup_{1 \le i,j \le N} \big| R_{ij} \big| \le 2N^\varepsilon, \qquad \mathbf{1}_\Omega \sup_{1 \le i,j \le N} \big| U_{ij}(z) \big| \le 2N^\varepsilon. \tag{2.3.28}$$

Proof. We only establish the second estimate (on $|U_{ij}|$) in (2.3.28), since the proof of the first is entirely analogous. Let us also restrict to the event Ω, since the lemma holds off of Ω.

Recall from the resolvent identity (2.3.1) and the definitions (2.3.20), (2.3.21), and (2.3.22) that

$$\mathbf{U} - \mathbf{G}^\gamma = \sum_{j=1}^{s} \big(\mathbf{G}^\gamma (\Gamma + \Lambda) \big)^j \mathbf{G}^\gamma + \big(\mathbf{G}^\gamma (\Gamma + \Lambda) \big)^{s+1} \mathbf{U}, \tag{2.3.29}$$

for any integer $s > 0$.

Now, set $s = \left\lceil \frac{2}{\rho - 2\varepsilon} \right\rceil$, which is positive by (2.3.25). Observe that $\mathbf{1}_\Omega \max_{1 \le i,j \le N} \big| G_{ij}^\gamma \big| \le N^\varepsilon$ and that the only nonzero entries of $\mathbf{1}_\Omega (\Gamma + \Lambda)$ are $\mathbf{1}_\Omega (\Gamma + \Lambda)_{pq}$ and $\mathbf{1}_\Omega (\Gamma + \Lambda)_{qp}$, which satisfy

$$\mathbf{1}_\Omega (\Gamma + \Lambda)_{pq} = \mathbf{1}_\Omega (\Gamma + \Lambda)_{qp} \le N^{-\rho} + t^{1/2} |w_{pq}| \mathbf{1}_{\Omega_1} \le 2N^{-\rho}. \tag{2.3.30}$$

Thus, (2.3.29) yields

$$\mathbf{1}_\Omega \big| U_{ij} - G_{ij}^\gamma \big| \le \sum_{j=1}^{s} (4N^{2\varepsilon - \rho})^j + (4N^{\varepsilon - \rho})^{(s+1)} \max_{1 \le i', j' \le N} |U_{i'j'}| \le 1, \tag{2.3.31}$$

if N is sufficiently large, where we have also used the deterministic estimate $\big| U_{i'j'} \big| \le \eta^{-1} \le N^2$. Now the estimate (2.3.28) on $|U_{ij}|$ follows from (2.3.31), the choice of s, and the fact that $\mathbf{1}_\Omega \big| G_{ij}^\gamma \big| \le N^\varepsilon$. $\square$

We also require the following lemma, which states that we can approximate quantities near $\big| F^{(k)}(\operatorname{Im} R_{ab}) \big|$ and $\big| F^{(k)}(\operatorname{Im} U_{ab}) \big|$ in terms of derivatives of $F^{(k)}\big(\operatorname{Im} G_{ab}^\gamma \big)$.

Lemma 2.3.7. *Let $\varphi \in \mathbb{R}$ be either such that $\varphi \in \big[\operatorname{Im} G^{\gamma}_{ab}, \operatorname{Im} U_{ab} \big]$ or $\varphi \in \big[\operatorname{Im} G^{\gamma}_{ab}, \operatorname{Im} R_{ab} \big]$. Then there exists a large constant $C = C(\vartheta) > 0$ such that, for any integer $k \geq 0$, we have that*

$$\mathbf{1}_{\Omega} \big| F^{(k)}(\varphi) \big| \leq C \mathbf{1}_{\Omega} \sum_{j=0}^{\vartheta} N^{(2\varepsilon - \rho) j} \big| F^{(k+j)} (\operatorname{Im} G^{\gamma}_{ab}) \big| + \frac{C}{N^3}. \tag{2.3.32}$$

Moreover, if $\varphi \in \big[\operatorname{Im} G^{\gamma}_{ab}, \operatorname{Im} R_{ab} \big]$, then

$$\mathbf{1}_{\Omega} \big| F^{(k)}(\varphi) \big| \leq C \mathbf{1}_{\Omega} \sum_{j=0}^{\vartheta} N^{(2\varepsilon - \rho) j} \big| F^{(k+j)} (\operatorname{Im} R_{ab}) \big| + \frac{C}{N^3}. \tag{2.3.33}$$

Proof. The proof of this lemma will be similar to that of Lemma 2.3.6. We only establish (2.3.32) when $\varphi \in \big[\operatorname{Im} G^{\gamma}_{ab}, \operatorname{Im} U_{ab} \big]$, since the proofs of (2.3.33) and of (2.3.32) when $\varphi \in \big[\operatorname{Im} G^{\gamma}_{ab}, \operatorname{Im} R_{ab} \big]$ are entirely analogous.

Through a Taylor expansion, we have that

$$F^{(k)}(\varphi) - F^{(k)} \big(\operatorname{Im} G^{\gamma}_{ab} \big) = \sum_{j=1}^{\vartheta} \frac{\Upsilon^j}{j!} F^{(j+k)} \big(\operatorname{Im} G^{\gamma}_{ab} \big) + \frac{\Upsilon^{\vartheta+1}}{(\vartheta+1)!} F^{(\vartheta+k)} (\Upsilon_1), \tag{2.3.34}$$

where $\Upsilon_1 \in \big[\operatorname{Im} G^{\gamma}_{ab}, \varphi \big]$, and $\Upsilon = \varphi - \operatorname{Im} G^{\gamma}_{ab}$, which satisfies

$$|\Upsilon| \leq \big| \operatorname{Im} U_{ab} - \operatorname{Im} G^{\gamma}_{ab} \big| = \Big| \operatorname{Im} \big(\mathbf{U} (\Gamma + \Lambda) \mathbf{G}^{\gamma} \big)_{ab} \Big|, \tag{2.3.35}$$

where in (2.3.35) we used the resolvent identity (2.3.1) and the definition (2.3.20) of Γ and Λ.

Recalling that $\Gamma + \Lambda$ has only two nonzero entries, both of which are at most $2N^{-\rho}$ on Ω (due to (2.3.30)), and further recalling that the entries of $\mathbf{G}^{\gamma}$ and $\mathbf{U}$ are bounded by $2N^{\varepsilon}$ on Ω (due to Lemma 2.3.6), we deduce that $\mathbf{1}_{\Omega} \big| \Upsilon \big| \leq 16 N^{2\varepsilon - \rho} \mathbf{1}_{\Omega}$. Inserting this and the first estimate of (2.2.14) into (2.3.34), we deduce the existence of a constant $C = C(\vartheta) > 0$ such that

$$\mathbf{1}_\Omega \Big| F^{(k)}\big(\operatorname{Im} U_{ab}\big) - F^{(k)}\big(\operatorname{Im} G_{ab}^{\gamma}\big)\Big| \le C\mathbf{1}_\Omega \sum_{j=1}^{\vartheta} N^{(2\varepsilon-\rho)j}\big|F^{(j+k)}(\operatorname{Im} G_{ab}^{\gamma})\big| + CN^{(\vartheta+1)(2\varepsilon-\rho)+C_0\varepsilon}. \tag{2.3.36}$$

Now the second estimate in (2.3.32) follows from (2.3.36) and the fact (2.3.26) that $(\rho - 2\varepsilon)\vartheta > C_0\varepsilon + 3$. $\square$

2.3.6 The first, third, and higher order terms

In this section we show that the expectations of the first and third order terms in the expansion of (2.3.19) are equal to 0 through Lemma 2.3.8, and we also estimate the higher order terms through Lemma 2.3.9 and Lemma 2.3.10.

Observe that any degree one or degree three term appearing in the expansion of the (p,q)-summand on the left side of (2.3.19) (using (2.3.23) and (2.3.24)) contains either zero or two factors of Γ. The following lemma indicates that the expectation of any such term is equal to 0.

Lemma 2.3.8. *For any integers $1 \le i,j \le N$ and $k \in \{0,1,2\}$, define $\xi_{ij}^{(k)} = \big((-\mathbf{R}\Gamma)^k\mathbf{R}\big)_{ij}$.*

Let M be a (possibly empty) product of $s \ge 0$ of the $\xi_{ij}^{(k)}$, so that $M = \prod_{r=1}^{s} \xi_{i_r j_r}^{(k_r)}$ for some $1 \le i_r, j_r \le N$ and $k_r \in \{0,1,2\}$. If $\sum_{r=1}^{s} k_r$ is even (in particular, if it is either 0 or 2) and $m \in \{1,2,3\}$, then

$$\mathbb{E}_\Psi\left[F^{(m)}(\operatorname{Im} R_{ab})\bigg(A_{pq} - \frac{\gamma t^{1/2} w_{pq}}{(1-\gamma^2)^{1/2}}\bigg)M\right] = 0. \tag{2.3.37}$$

The same estimate (2.3.37) holds if some of the $\xi_{i_r j_r}^{(k_r)}$ are replaced by $\operatorname{Re}\xi_{i_r j_r}^{(k_r)}$ or $\operatorname{Im}\xi_{i_r j_r}^{(k_r)}$ in the definition of M.

Proof. First observe from the symmetry of the random variables H_{ij} that $\mathbb{E}_\Psi\big[A_{pq}^m \big| \chi_{pq}\big] = 0 = \mathbb{E}_\Psi\big[w_{pq}^m \big| \chi_{pq}\big]$ for any odd integer $m > 0$. Now, recall from Remark 2.3.5 that A_{pq}, w_{pq}, and $\mathbf{R}$ are mutually independent after conditioning on χ_{pq} and Ψ. Therefore,

$$\begin{aligned}\mathbb{E}_\Psi&\left[F^{(k)}(\operatorname{Im} R_{ab})\Big(A_{pq}-\frac{\gamma t^{1/2}w_{pq}}{(1-\gamma^2)^{1/2}}\Big)M\right]\\&=\mathbb{E}^\chi\left[\mathbb{E}_\Psi\left[F^{(k)}(\operatorname{Im} R_{ab})\left(A_{pq}-\frac{\gamma t^{1/2}w_{pq}}{(1-\gamma^2)^{1/2}}\right)M\middle|\chi_{pq}\right]\right]=0,\end{aligned}\tag{2.3.38}$$

where we have used the fact that the term inside the first expectation in the middle of (2.3.38) is a linear combination of products of expressions that each either contain a term of the form $\mathbb{E}\big[A_{pq}^m\big|\chi_{pq}\big]$ or $\mathbb{E}\big[w_{pq}^m\big|\chi_{pq}\big]$ for some odd integer $m>0$ (by (2.3.20), (2.3.21), and the fact that $\sum_{r=1}^s k_r$ is even), and each of these expectations is equal to 0. This establishes (2.3.37). $\square$

Now let us consider the fourth and higher order terms that can occur in (2.3.19) through the expansions (2.3.23) and (2.3.24). Two types of such terms can appear. The first is when the final term in (2.3.24) appears, giving rise to a factor of $\zeta^3F^{(4)}(\zeta_0)$. The second is when $\zeta^3F^{(4)}(\zeta_0)$ does not appear and instead the term is a product of $F^{(m)}(\operatorname{Im} R_{ab})$ (for some $1\le m\le 3$) with at most four expressions of the form $(-\mathbf{R}\Gamma)^k\mathbf{R}$ or $(-\mathbf{R}\Gamma)^k\mathbf{G}^\gamma$ (and their real or imaginary parts).

The following lemma addresses terms of the first type.

Lemma 2.3.9. *There exists a large constant $C=C(\alpha,\nu,\rho,\vartheta)>0$ such that*

$$\begin{aligned}\mathbb{E}_\Psi&\left[\left|\operatorname{Im}(G_{ap}^\gamma G_{qb}^\gamma)\Big(A_{pq}-\frac{\gamma t^{1/2}w_{pq}}{(1-\gamma^2)^{1/2}}\Big)\zeta^3F^{(4)}\big(\zeta_0\big)\right|\right]\\&\le\frac{CN^{10\varepsilon}}{(1-\gamma^2)^{1/2}}\left(N^{(\alpha-4)\nu-1}\mathfrak{J}+\frac{t^2\mathfrak{J}}{N^2}+QN^{C_0+10}+\frac{1}{N^3}\right).\end{aligned}\tag{2.3.39}$$

Proof. We first establish an estimate that holds off of the event Ω. In this case, to bound the left side of (2.3.39), we use the deterministic facts that $|G_{ij}^\gamma|,|R_{ij}|,\zeta\le\eta^{-1}\le N^2$ and $|A_{ij}|<1$, which implies from (2.2.14) that $\big|F(\operatorname{Im} R_{ij})\big|\le N^{C_0}$. This yields for sufficiently large N

$$\mathbb{E}_\Psi\bigg[\bigg|\operatorname{Im}(G^\gamma_{ap}G^\gamma_{qb})\Big(A_{pq}-\frac{\gamma t^{1/2}w_{pq}}{(1-\gamma^2)^{1/2}}\Big)\zeta^3 F^{(4)}\big(\zeta_0\big)\mathbf{1}_{\Omega^c}\bigg|\bigg] \\ \le N^{C_0+10}\mathbb{E}_\Psi\bigg[\mathbf{1}_{\Omega^c}+\frac{\gamma\mathbb{E}\big[|w_{pq}|\mathbf{1}_{\Omega^c}\big]}{(1-\gamma^2)^{1/2}}\bigg]\le\frac{4N^{C_0+10}Q}{(1-\gamma^2)^{1/2}}. \tag{2.3.40}$$

Next we first work on the event Ω. To that end, observe from (2.3.1), (2.3.23), and Lemma 2.3.6 that

$$|\zeta|\mathbf{1}_\Omega\le\Big|\big(\mathbf{G}^\gamma\Gamma\mathbf{R}\big)_{ab}\Big|\mathbf{1}_\Omega=\Big(\big|G^\gamma_{ap}\Gamma_{pq}R_{qb}\big|+\big|G^\gamma_{aq}\Gamma_{qp}R_{pb}\big|\Big)\mathbf{1}_\Omega\le 8N^{2\varepsilon}|\Gamma_{pq}|\mathbf{1}_\Omega.$$

Furthermore, since $\zeta_0\in\big[\operatorname{Im}G^\gamma_{ab},\operatorname{Im}R_{ab}\big]$, (2.3.33) yields that

$$\mathbb{E}_\Psi\bigg[\bigg|\operatorname{Im}(G^\gamma_{ai}G^\gamma_{jb})\Big(A_{pq}-\frac{\gamma w_{ij}}{(1-\gamma^2)^{1/2}}\Big)\zeta^3F^{(4)}\big(\zeta_0\big)\mathbf{1}_\Omega\bigg|\bigg] \\ \le\frac{512N^{10\varepsilon}}{(1-\gamma^2)^{1/2}}\mathbb{E}_\Psi\Big[|\Gamma_{pq}|^3\big(|A_{pq}|+t^{1/2}|w_{ij}|\big)\big|F^{(4)}(\zeta_0)\big|\mathbf{1}_\Omega\Big] \\ \le\frac{CN^{10\varepsilon}}{(1-\gamma^2)^{1/2}}\sum_{j=0}^{2\vartheta}N^{(2\varepsilon-\rho)j}\mathbb{E}_\Psi\bigg[\Big|F^{(j+4)}\big(\operatorname{Im}R_{ab}\big)\Big|\big(|A_{pq}|+t^{1/2}|w_{pq}|\big)^4\bigg]+\frac{C}{N^3} \tag{2.3.41}$$

for some constant $C=C(\vartheta)>0$. To estimate the right side of (2.3.41), we condition on χ_{pq}, and apply Remark 2.3.5 to deduce that

$$\mathbb{E}_\Psi\bigg[\Big|F^{(j+4)}\big(\operatorname{Im}R_{ab}\big)\Big|\big(|A_{pq}|+t^{1/2}|w_{pq}|\big)^4\bigg] \\ \le 8\mathbb{E}^\chi\bigg[\mathbb{E}_\Psi\bigg[\Big|F^{(j+4)}\big(\operatorname{Im}R_{ab}\big)\Big|\big(|A_{pq}|^4+t^2|w_{pq}|^4\big)\Big|\chi_{pq}\bigg]\bigg] \\ =8\mathbb{E}^\chi\bigg[\mathbb{E}_\Psi\bigg[\Big|F^{(j+4)}\big(\operatorname{Im}R_{ab}\big)\Big|\Big|\chi_{pq}\bigg]\mathbb{E}_\Psi\bigg[\big(|A_{pq}|^4+t^2|w_{pq}|^4\big)\Big|\chi_{pq}\bigg]\bigg].$$

Then Lemma 2.3.1 (with $p=4$) and the fact that $\mathbb{E}\big[|w_{pq}|^4\big]\le\frac{60}{N^2}$ yields after enlarging

$C = C(\alpha, \nu, \rho, \vartheta)$ that

$$\mathbb{E}_{\Psi}\left[\left|F^{(j+4)}\big(\operatorname{Im} R_{ab}\big)\right|\big(|A_{pq}|^4 + t^2|w_{pq}|^4\big)\right]$$
$$\leq C\left(N^{(\alpha-4)\nu-1} + \frac{t^2}{N^2}\right)\mathbb{E}^{\chi}\left[\mathbb{E}_{\Psi}\left[\left|F^{(j+4)}\big(\operatorname{Im} R_{ab}\big)\right|\Big|\chi_{pq}\right]\right] \leq C\left(N^{(\alpha-4)\nu-1} + \frac{t^2}{N^2}\right)\mathfrak{J} + \frac{C}{N^3}, \tag{2.3.42}$$

where we used (2.3.32) to deduce the last estimate.

Now (2.3.39) follows from applying (2.3.40) off of Ω and (2.3.41) and (2.3.42) on Ω. $\square$

The following lemma addresses the higher order terms of the second type. Its proof is very similar to that of Lemma 2.3.9 and is therefore omitted.

Lemma 2.3.10. *Recall the definitions of the $\xi_{ij}^{(k)}$ for $k \in \{0, 1, 2\}$ from Lemma 2.3.8, and further define $\xi_{ij}^{(3)} = \big((-\mathbf{R}\Gamma)^3\mathbf{G}^{\gamma}\big)_{ij}$, for each $1 \leq i, j \leq N$.*

There exists a large constant $C = C(\alpha, \nu, \rho, \vartheta) > 0$ such that the following holds. Let M be a product of $s \in \{1, 2, 3, 4\}$ of the $\xi_{ij}^{(k)}$, so that $M = \prod_{r=1}^{s}\xi_{i_r j_r}^{(k_r)}$ for some $1 \leq i_r, j_r \leq N$ and $k_r \in \{1, 2, 3\}$. If $\sum_{r=1}^{s} k_r \geq 3$ and $m \in \{1, 2, 3\}$, then

$$\mathbb{E}_{\Psi}\left[|M|\left|\Big(A_{pq} - \frac{\gamma t^{1/2} w_{pq}}{(1-\gamma^2)^{1/2}}\Big)F^{(m)}\big(\operatorname{Im} R_{ab}\big)\right|\right] \leq \frac{CN^{16\varepsilon}}{(1-\gamma^2)^{1/2}}\left(N^{\nu(\alpha-4)-1}\mathfrak{J} + \frac{t^2\mathfrak{J}}{N^2} + \frac{1}{N^3}\right). \tag{2.3.43}$$

The same estimate holds if some of the $\xi_{i_r j_r}^{(0)}$ are replaced by G_{ij}^{γ}.

2.3.7 Terms of degree 2

In this section we estimate the contribution of terms of degree two to the (p, q)-summand of the left side of (2.3.19). In Section 2.3.7 we will state this bound use it to establish Proposition 2.3.4; we will then establish this estimate in Section 2.3.7.

Estimates on the degree two terms

In this section we bound the contribution of the second order terms to the (p,q)-summand left side of (2.3.19). There are two types of terms to consider. The first corresponds to when the factor of $\zeta F''(\operatorname{Im} R_{ab})$ appears in the expansion (2.3.24) for $F'(\operatorname{Im} G^{\gamma}_{ab})$, and the second corresponds to when either $\operatorname{Im}(-\mathbf{R}\Gamma\mathbf{R})_{ap}$ or $\operatorname{Im}(-\mathbf{R}\Gamma\mathbf{R})_{qb}$ appears in the expansion (2.3.23) for $\operatorname{Im} G^{\gamma}_{ij}$. Both such terms are estimated through the following proposition.

Proposition 2.3.11. *Define*

$$\mathfrak{E}_1 = N^{4\varepsilon+(\alpha-2)\rho-2} t\mathfrak{J} + N^{C_0+6} tQ + \frac{t}{N^2}, \qquad \mathfrak{E}_2 = N^{\alpha\rho+3\varepsilon-1} t\mathfrak{J}.$$

Then, there exists a large constant $C = C(\alpha,\nu,\rho,\vartheta) > 0$ such that

$$\mathbb{E}_{\Psi}\left[\operatorname{Im}\big((\mathbf{R}\Gamma\mathbf{R})_{ap}R_{qb}\big)\left(A_{pq} - \frac{\gamma t^{1/2} w_{pq}}{(1-\gamma^2)^{1/2}}\right) F'(\operatorname{Im} R_{ab})\right] \le C\big(\mathfrak{E}_1 + \mathfrak{E}_2(\psi_{pq} + \mathbf{1}_{p=q})\big), \tag{2.3.44}$$

and similarly if $(\mathbf{R}\Gamma\mathbf{R})_{ap}R_{qb}$ is replaced by $(\mathbf{R}\Gamma\mathbf{R})_{qb}R_{ap}$. Moreover,

$$\mathbb{E}_{\Psi}\left[\operatorname{Im}(R_{ap}R_{qb})\left(A_{pq} - \frac{\gamma t^{1/2} w_{pq}}{(1-\gamma^2)^{1/2}}\right) \operatorname{Im}(\mathbf{R}\Gamma\mathbf{R})_{ab} F''(\operatorname{Im} R_{ab})\right] \le C\big(\mathfrak{E}_1 + \mathfrak{E}_2(\psi_{pq} + \mathbf{1}_{p=q})\big) \tag{2.3.45}$$

We can now establish Theorem 2.2.15 assuming Proposition 2.3.11.

Proof of Proposition 2.3.4 assuming Proposition 2.3.11. As mentioned previously, through (2.3.23) and (2.3.24), the right side of (2.3.19) expands into a sum of expectations of degrees one, two, three, four, and higher. By Lemma 2.3.8, we deduce that the expectation of each term of degree one or three in this expansion is equal to 0. Furthermore, summing Lemma 2.3.9 and Lemma 2.3.10 over all N^2 possibilities for (p,q) yields the existence of a constant $C = C(\alpha,\nu,\rho) > 0$ such that the sum of the fourth and higher order terms is bounded by

$$\frac{C}{(1-\gamma^2)^{1/2}} N^{16\varepsilon} \left(N^{\nu(\alpha-4)+1} \mathfrak{J} + t^2 \mathfrak{J} + Q N^{C_0+10} + \frac{1}{N} \right) < \frac{C}{(1-\gamma^2)^{1/2} N^{\omega}} \big(\mathfrak{J} + 1 + Q N^{C_0+11} \big), \tag{2.3.46}$$

here, we used the definition (2.3.25) of ω and recalled that $t \sim N^{(\alpha-2)\nu}$ from Lemma 3.2.6. Next, summing Proposition 2.3.11 over all N^2 possibilities for (p, q) and using the fact that Ψ has at most $N^{1+\alpha\rho+\varepsilon}$ entries equal to 1, we estimate the second order terms by

$$CN^{4\varepsilon} \left(N^{(\alpha-2)\rho} t \mathfrak{J} + N^{\alpha\rho} t \mathfrak{J} + t + N^{C_0+6} Q + \frac{1}{N} \right) < CN^{-\omega} \big(\mathfrak{J} + 1 + N^{C_0+7} Q \big), \tag{2.3.47}$$

after increasing C if necessary. We have again used the definition (2.3.25) of ω and recalled that $t \sim N^{(\alpha-2)\nu}$.

Now the proposition follows from summing the contributions from (2.3.46) and (2.3.47) and using (2.3.27) to replace Q with Q_0 (up to an additive error that decays exponentially in N). □

Proof of Proposition 2.3.11

In this section we establish Proposition 2.3.11. In fact, we will only establish the first estimate (2.3.44) of that proposition, since the proof of the second estimate (2.3.45) is entirely analogous.

To that end, we will first through Lemma 2.3.12 estimate the error incurred be replacing all entries of $\mathbf{R}$ on the left side of (2.3.44) with those of $\mathbf{U}$. Then, using the mutual independence of $\mathbf{U}$, A_{pq}, and w_{pq} conditional on Ψ (recall Remark 2.3.5) and the definition (3.2.10) of t, we will deduce Proposition 2.3.11.

In order to implement the replacement, first observe that, since $A_{pq}(\mathbf{R} - \mathbf{U}) = 0$ by

(2.3.22),

$$\operatorname{Im}\Big(\big(\mathbf{R}\Gamma\mathbf{R}\big)_{ap}R_{qb}\Big)A_{pq}F'\big(\operatorname{Im}R_{ab}\big) = \operatorname{Im}\Big(\big(\mathbf{U}\Gamma\mathbf{U}\big)_{ap}U_{qb}\Big)A_{pq}F'\big(\operatorname{Im}U_{ab}\big).$$

Now write

$$\begin{aligned}
&\operatorname{Im}\big((\mathbf{R}\Gamma\mathbf{R})_{ap}R_{qb}\big)\left(\frac{\gamma t^{1/2}w_{pq}}{(1-\gamma^2)^{1/2}}\right)F'(\operatorname{Im}R_{ab})\\
&\quad = \operatorname{Im}\big((\mathbf{U}\Gamma\mathbf{U})_{ap}U_{qb}\big)\left(\frac{\gamma t^{1/2}w_{pq}}{(1-\gamma^2)^{1/2}}\right)F'(\operatorname{Im}U_{ab})\\
&\qquad + \left(\frac{\gamma t^{1/2}w_{pq}}{(1-\gamma^2)^{1/2}}\right)\Big(\operatorname{Im}\big((\mathbf{R}\Gamma\mathbf{R})_{ap}R_{qb}\big)F'\big(\operatorname{Im}R_{ab}\big) - \operatorname{Im}\big((\mathbf{U}\Gamma\mathbf{U})_{ap}U_{qb}\big)F'(\operatorname{Im}U_{ab})\Big).
\end{aligned} \tag{2.3.48}$$

Using $\Gamma_{ij} = \gamma\Theta_{ij} + (1-\gamma^2)^{1/2}\Phi_{ij}$ and $A_{pq}(\mathbf{R}-\mathbf{U}) = 0$ again, and recalling from (2.3.20) and (2.3.21) that

$$\Theta_{ij} = A_{ij}\mathbf{1}_{(i,j)\in\{(p,q),(q,p)\}}, \qquad \Phi_{ij} = t^{1/2}w_{ij}\mathbf{1}_{(i,j)\in\{(p,q),(q,p)\}},$$

we can compute the last line in (2.3.48) to find the terms with Θ_{ij} factors vanish, leaving

$$\left(\frac{\gamma t^{1/2}w_{pq}}{(1-\gamma^2)^{1/2}}\right)\Big(\operatorname{Im}\big((\mathbf{R}\Gamma\mathbf{R})_{ap}R_{qb}\big)F'\big(\operatorname{Im}R_{ab}\big) - \operatorname{Im}\big((\mathbf{U}\Gamma\mathbf{U})_{ap}U_{qb}\big)F'(\operatorname{Im}U_{ab})\Big) = -\gamma t w_{pq}^2\mathfrak{Y},$$

where

$$\mathfrak{Y} = \operatorname{Im}\big(U_{ap}U_{qp}U_{qb} + U_{aq}U_{pp}U_{qb}\big)F'\big(\operatorname{Im}U_{ab}\big) - \operatorname{Im}\big(R_{ap}R_{qp}R_{qb} + R_{aq}R_{pp}R_{qb}\big)F'\big(\operatorname{Im}R_{ab}\big). \tag{2.3.49}$$

In total,

$$\begin{aligned}\operatorname{Im}\Big(\big(\mathbf{R}\Gamma\mathbf{R}\big)_{ap}R_{qb}\Big)\Big(A_{pq}-\frac{\gamma t^{1/2}w_{pq}}{(1-\gamma^2)^{1/2}}\Big)F'\big(\operatorname{Im}R_{ab}\big)\\ =\operatorname{Im}\Big(\big(\mathbf{U}\Gamma\mathbf{U}\big)_{ap}U_{qb}\Big)\Big(A_{pq}-\frac{\gamma t^{1/2}w_{pq}}{(1-\gamma^2)^{1/2}}\Big)F'\big(\operatorname{Im}U_{ab}\big)+\gamma t w_{pq}^2\mathfrak{Y},\end{aligned}\tag{2.3.50}$$

and so we would like to estimate $\big|\mathbb{E}_\Psi[\gamma t w_{pq}^2\mathfrak{Y}]\big|$. This will be done through the following lemma.

Lemma 2.3.12. *There exists a large constant $C=C(\alpha,\rho,\varepsilon,\vartheta)>0$ such that*

$$\Big|\mathbb{E}_\Psi\big[\gamma t w_{pq}^2\mathfrak{Y}\big]\Big|\le CN^{4\varepsilon+(\alpha-2)\rho-2}t\mathfrak{J}+\frac{Ct}{N^2}+CN^{C_0+6}tQ.\tag{2.3.51}$$

Proof. Since w_{pq} is independent from $\mathbf{R}$ and $\mathbf{U}$, and since $\mathbb{E}[w_{pq}^2]=\frac{1}{N}$, we have that $\mathbb{E}_\Psi\big[\gamma t w_{pq}^2\mathfrak{Y}\big]=\gamma t N^{-1}\mathbb{E}_\Psi\big[\mathfrak{Y}\big]$, and so it suffices to show that

$$\begin{aligned}\Big|\mathbb{E}_\Psi\Big[\operatorname{Im}\big(U_{ap}U_{qp}U_{qb}\big)F'\big(\operatorname{Im}U_{ab}\big)-\operatorname{Im}\big(R_{ap}R_{qp}R_{qb}\big)F'\big(\operatorname{Im}R_{ab}\big)\Big]\Big|\\ <CN^{4\varepsilon+(\alpha-2)\rho-1}\mathfrak{J}+\frac{C}{N}+CN^{C_0+6}Q,\end{aligned}\tag{2.3.52}$$

and the same estimate if $\operatorname{Im}(U_{ap}U_{qp}U_{qb})$ and $\operatorname{Im}(R_{ap}R_{qp}R_{qb})$ are replaced by $\operatorname{Im}(U_{aq}U_{pp}U_{qb})$ and $\operatorname{Im}(R_{aq}R_{pp}R_{qb})$, respectively. We will only show (2.3.52), since the proof of the second statement is entirely analogous.

To that end, recall that (2.3.1) and the definitions (2.3.20) and (2.3.22) yield

$$\mathbf{R}=\mathbf{U}-\mathbf{U}\Lambda\mathbf{U}+\mathbf{U}\Lambda\mathbf{U}\Lambda\mathbf{R}.\tag{2.3.53}$$

Furthermore, we find from a Taylor expansion

$$F^{(k)}(\operatorname{Im}R_{ab})-F^{(k)}(\operatorname{Im}U_{ab})=\sum_{j=1}^{\vartheta}\frac{1}{j!}\kappa^jF^{(j+k)}(\operatorname{Im}U_{ab})+\frac{1}{(\vartheta+1)!}\kappa^{\vartheta+1}F^{(\vartheta+1)}(\kappa_1),\tag{2.3.54}$$

where

$$\kappa = \operatorname{Im}(R_{ab} - U_{ab}) = -\operatorname{Im}(\mathbf{U}\Lambda\mathbf{R})_{ab}, \tag{2.3.55}$$

by (2.3.1) and (2.3.22), and $\kappa_1 \in (\operatorname{Im} R_{ab}, \operatorname{Im} U_{ab})$.

Applying (2.3.53) and (2.3.54), we find that

$$\begin{aligned}\operatorname{Im}\big(R_{ap}R_{qp}R_{qb}\big)F'\big(\operatorname{Im} R_{ab}\big) = &\left(\sum_{j=0}^{\vartheta} \frac{\kappa^j}{j!} F^{(j+1)}(\operatorname{Im} U_{ab}) + \frac{\kappa^{\vartheta+1}}{(\vartheta+1)!} F^{(\vartheta+1)}(\kappa_1)\right) \\ &\times \operatorname{Im}\Big((\mathbf{U} - \mathbf{U}\Lambda\mathbf{U} + \mathbf{U}\Lambda\mathbf{U}\Lambda\mathbf{R})_{ap}(\mathbf{U} - \mathbf{U}\Lambda\mathbf{U} + \mathbf{U}\Lambda\mathbf{U}\Lambda\mathbf{R})_{qp} \\ &\times (\mathbf{U} - \mathbf{U}\Lambda\mathbf{U} + \mathbf{U}\Lambda\mathbf{U}\Lambda\mathbf{R})_{qb}\Big).\end{aligned} \tag{2.3.56}$$

Using (2.3.55) to express κ in terms of Λ and expanding the right side of (2.3.56) yields a sum of monomials, each of which contains a product of Λ factors. Any such monomial with u factors of Λ will be called an *order u monomial*. Observe that there is only one order 0 monomial on the right side of (2.3.56), which is $F'(\operatorname{Im} U_{ab})U_{ap}U_{qp}U_{qb}$. We would like to estimate the other, higher order, monomials on the right side of (2.3.56).

We first consider the monomials of order 1. Observe that any such monomial is a product of Λ_{pq} with terms of the form $F^{(j+1)}(\operatorname{Im} U_{ab})$ and U_{ij}. Furthermore, recall from Remark 2.3.5 that Λ is independent from $\mathbf{U}$ (conditional on Ψ). Thus, the symmetry of the entries of $\mathbf{H}$ (and therefore the entries of Λ) implies that

$$\mathbb{E}_{\Psi}\big[M\big] = 0, \quad \text{for any monomial } M \text{ of order 1}. \tag{2.3.57}$$

Next let us estimate monomials of order u with $2 \le u \le \vartheta$ on the event Ω. Any such monomial is a product of Λ_{pq}^u with a term of the form $F^{(k)}(\operatorname{Im} U_{ab})$ and at most $2u$ entries of $\mathbf{U}$ or $\mathbf{R}$; Lemma 2.3.6 implies that the latter terms are all bounded by $2N^{\varepsilon}$ on the event

Ω. Thus, if M is a monomial of order $2 \le u \le \vartheta$, we have for some $1 \le k \le \vartheta$ that

$$\begin{aligned} \mathbb{E}_\Psi \big[\mathbf{1}_\Omega |M| \big] &\le 4^u N^{2\varepsilon u} \mathbb{E}_\Psi \Big[\big| F^{(k)} (\operatorname{Im} U_{ab}) \big| |\Lambda_{pq}|^u \Big] \\ &= 4^u N^{2\varepsilon u} \mathbb{E}_\Psi \Big[\big| F^{(k)} (\operatorname{Im} U_{ab}) \big| \Big] \mathbb{E}_\Psi \big[|\Lambda_{pq}|^u \big], \end{aligned} \tag{2.3.58}$$

for some $j \le \vartheta$, where we have used the fact from Remark 2.3.5 that $\mathbf{U}$ and Λ are independent (after conditioning on Ψ).

Now, recalling from (2.3.20) that $\big|\Lambda_{pq}\big| = B_{pq} \le |H_{pq}| \mathbf{1}_{|H_{pq}| \le N^{-\rho}}$, and applying Lemma 2.3.2, the first estimate in (2.3.32), (2.3.58), and the definition (2.2.15) of $\mathfrak{J}$ yields the existence of a constant $C = C(\rho, \vartheta) > 0$

$$\mathbb{E}_\Psi \big[\mathbf{1}_\Omega |M| \big] \le C N^{4\varepsilon + (2\varepsilon - \rho)(u-2) + (\alpha - 2)\rho - 1} \mathfrak{J}, \quad \text{for any monomial } M \text{ of order } 2 \le u \le \vartheta. \tag{2.3.59}$$

The final monomials to estimate on Ω are those of order u, with $u \ge \vartheta + 1$. Since Lemma 2.3.6 implies that $\mathbf{1}_\Omega \big|\kappa_1\big| \le 2N^\varepsilon$, we find from the first estimate of (2.2.14) that $\mathbf{1}_\Omega \big| F^{(k+1)}(\kappa_1) \big| \le N^{C_0 \varepsilon}$ for $0 \le k \le \vartheta$. Moreover, Lemma 2.3.6 and the first estimate of (2.2.14) imply that $\mathbf{1}_\Omega \big| F^{(k+1)}(\operatorname{Im} U_{ab}) \big| \le N^{C_0 \varepsilon}$ for any $0 \le k \le \vartheta$. Combining these estimates, the fact that any monomial of order u is a product of Λ_{pq}^u with one term of the form $F^{(k+1)}(\operatorname{Im} U_{ab})$ or $F^{(k+1)}(\kappa_1)$ and at most $2u$ entries of $\mathbf{U}$ and $\mathbf{R}$, and the fact that $(\rho - 2\varepsilon)\vartheta \ge C_0 \varepsilon + 3$ implies the existence of a constant $C = C(\alpha, \rho, \varepsilon, \vartheta) > 0$ such that

$$\mathbb{E}_\Psi \big[\mathbf{1}_\Omega |M| \big] \le \frac{C}{N^3}, \quad \text{for any monomial } M \text{ of order } u \ge \vartheta + 1. \tag{2.3.60}$$

Off of the event Ω, we apply the estimate

$$\begin{aligned} \bigg| \mathbb{E}_\Psi \Big[\operatorname{Im} \big(U_{ap} U_{qp} U_{qb} \big) F' \big(\operatorname{Im} U_{ab} \big) - \operatorname{Im} \big(R_{ap} R_{qp} R_{qb} \big) F' \big(\operatorname{Im} R_{ab} \big) \Big] \bigg| \\ \le 2 N^{C_0 + 6} \mathbb{E}_\Psi \big[\Phi_{pq}^2 \big] \le 2 N^{C_0 + 6}, \end{aligned} \tag{2.3.61}$$

where we have used the fact that the entries of $\mathbf{R}$ and $\mathbf{U}$ are bounded by $\eta^{-1} \leq N^2$, and also the second estimate in (2.2.14).

Now the lemma follows from applying (2.3.57), (2.3.59), and (2.3.60) on Ω, and applying (2.3.61) off of Ω. $\square$

We can now establish Proposition (2.3.11).

Proof of Proposition 2.3.11. Let us only establish (2.3.44), since the proof of (2.3.45) is entirely analogous.

To that end, observe from (2.3.50) and Lemma 2.3.12 that for some $C = C(\alpha, \nu, \rho, \vartheta) > 0$ we have that

$$\begin{aligned} &\left| \mathbb{E}_\Psi \left[\operatorname{Im} \Big((\mathbf{R}\Gamma\mathbf{R})_{ap} R_{qb} \Big) \Big(A_{pq} - \frac{\gamma t^{1/2} w_{pq}}{(1-\gamma^2)^{1/2}} \Big) F'(\operatorname{Im} R_{ab}) \right] \right| \\ &\qquad \leq \left| \mathbb{E}_\Psi \left[\operatorname{Im} \Big((\mathbf{U}\Gamma\mathbf{U})_{ap} U_{qb} \Big) \Big(A_{pq} - \frac{\gamma t^{1/2} w_{pq}}{(1-\gamma^2)^{1/2}} \Big) F'(\operatorname{Im} U_{ab}) \right] \right| + C\mathfrak{E}_1. \end{aligned} \tag{2.3.62}$$

Now, since A_{pq}, w_{pq}, and $\mathbf{U}$ are mutually independent conditional on ψ_{pq}, and since A_{pq} and w_{pq} are symmetric we have from the definition (2.3.20) of Γ that

$$\begin{aligned} &\mathbb{E}_\Psi \left[\operatorname{Im} \Big((\mathbf{U}\Gamma\mathbf{U})_{ap} U_{qb} \Big) \Big(A_{pq} - \frac{\gamma t^{1/2} w_{pq}}{(1-\gamma^2)^{1/2}} \Big) F'(\operatorname{Im} U_{ab}) \right] \\ &\qquad = \gamma \mathbb{E}_\Psi [A_{pq}^2 - t w_{pq}^2] \mathbb{E}_\Psi \big[\operatorname{Im}(U_{ap} U_{qp} U_{qb} + U_{aq} U_{pp} U_{qb}) F'(\operatorname{Im} U_{ab}) \big]. \end{aligned} \tag{2.3.63}$$

Now there are three cases to consider. If $\psi_{pq} = 0$ and $p \neq q$, then $\mathbb{E}[w_{pq}^2] = \frac{1}{N}$, so by the definition (3.2.10) of t we have that

$$\mathbb{E}_\Psi [A_{pq}^2 - t w_{pq}^2] = \mathbb{E}\big[H_{ij}^2 \mathbf{1}_{|H_{ij}| < N^{-\nu}} \big| |H_{ij}| < N^{-\rho} \big] - \frac{t}{N} = 0, \tag{2.3.64}$$

in which case the left side of (2.3.63) is zero.

If $\psi_{pq} = 1$, then $A_{pq} = 0$ and $\mathbb{E}[w_{pq}^2] \leq \frac{2}{N}$, so

$$\begin{aligned}
&\Big|\mathbb{E}_\Psi[A_{pq}^2 - tw_{pq}^2]\mathbb{E}_\Psi\big[\operatorname{Im}(U_{ap}U_{qp}U_{qb} + U_{aq}U_{pp}U_{qb})F'(\operatorname{Im} U_{ab})\big]\Big| \\
&\qquad \leq \frac{2t}{N}\mathbb{E}_\Psi\Big[\big(|U_{ap}U_{qp}U_{qb}| + |U_{aq}U_{pp}U_{qb}|\big)\big|F'(\operatorname{Im} U_{ab})\big|\Big] \leq \frac{4t}{N}\big(8N^{3\varepsilon}\mathfrak{J} + N^6 Q\big),
\end{aligned} \tag{2.3.65}$$

where we have used Lemma 2.3.6 to bound $\max_{1\leq i,j\leq N}|U_{ij}|$ by $2N^\varepsilon$ on Ω and (2.3.2) and the fact that $\eta \geq N^{-2}$ to bound it off of Ω.

Similarly, if $\psi_{pq} = 0$ and $p = q$, then $\mathbb{E}[w_{pq}^2] = \frac{2}{N}$ and so similar reasoning as applied in (2.3.64) yields $\mathbb{E}_\Psi[A_{pq}^2 - tw_{pq}^2] = -\frac{t}{N}$, and so we again deduce that (2.3.65) holds.

Now the proposition follows from summing (2.3.62), (2.3.63), and either (2.3.64) if $\psi_{pq} = 0$ and $p \neq q$ or (2.3.65) if $\psi_{pq} = 0$ or $p = q$. □

2.3.8 Outline of the proof of Theorem 2.2.15 for $m > 1$

Let us briefly outline the modifications required in the above proof of Theorem 2.2.15 in the case $m > 1$. Then, the analog of (2.3.18) becomes

$$\begin{aligned}
&\frac{\partial}{\partial\gamma}\mathbb{E}_\Psi\Big[F\big(\operatorname{Im} G_{a_1b_1}^\gamma, \ldots, \operatorname{Im} G_{a_mb_m}^\gamma\big)\Big] \\
&\quad = \sum_{k=1}^m \sum_{1\leq p,q\leq N} \mathbb{E}_\Psi\left[\operatorname{Im}(G_{a_kp}^\gamma G_{qb_k}^\gamma)\left(A_{pq} - \frac{\gamma t^{1/2} w_{pq}}{(1-\gamma^2)^{1/2}}\right)\partial_k F\big(\operatorname{Im} G_{a_1b_1}^\gamma, \ldots, \operatorname{Im} G_{a_mb_k}^\gamma\big)\right],
\end{aligned}$$

and so we must show for each integer $k \in [1, m]$ that

$$\begin{aligned}
&\sum_{1\leq p,q\leq N}\left|\mathbb{E}_\Psi\left[\operatorname{Im}(G_{a_kp}^\gamma G_{qb_k}^\gamma)\Big(A_{pq} - \frac{\gamma t^{1/2} w_{pq}}{(1-\gamma^2)^{1/2}}\Big)\partial_k F\big(\operatorname{Im} G_{a_1b_1}^\gamma, \ldots, \operatorname{Im} G_{a_mb_k}^\gamma\big)\right]\right| \\
&\qquad\qquad < \frac{C}{(1-\gamma^2)^{1/2}}\big(N^{-\omega}(\mathfrak{J}+1) + Q_0 N^{C+C_0}\big),
\end{aligned} \tag{2.3.66}$$

for some constants $\omega = \omega(\alpha, \nu, \rho, m) > 0$ and $C = C(\alpha, \nu, \rho, m) > 0$.

Following (2.3.24), for fixed $k \in [1, m]$ we then expand $\partial_k F(\operatorname{Im} G_{a_1b_1}^\gamma, \ldots, \operatorname{Im} G_{a_mb_m}^\gamma)$

as a degree three polynomial in the $\zeta_j = \operatorname{Im}\xi_{a_j b_j}$, whose lower (at most second) degree coefficients are derivatives of $F(\operatorname{Im} R_{a_1 b_1}, \ldots, \operatorname{Im} R_{a_m b_m})$. The degree three coefficients of this polynomial are fourth order derivatives of F, evaluated at some $\big(\zeta_{0;1}, \ldots, \zeta_{0;m}\big)$ with $\zeta_{0;j} \in [\operatorname{Im} G^{\gamma}_{a_j b_j}, \operatorname{Im} R_{a_j b_j}]$. Inserting this expansion into (2.3.66), one can show using Lemma 2.3.8 that the resulting first and third order terms in (2.3.66) will have expectation equal to 0. Following the proofs of Lemma 2.3.9 and Lemma 2.3.10, the fourth and higher order terms in this expansion can further be estimated by $C(1-\gamma^2)^{-1/2}\big(N^{-\omega}(\mathfrak{J}+1) + Q_0 N^{C+C_0}\big)$, for some $\omega = \omega(\alpha, \nu, \rho, m) > 0$ and $C = C(\alpha, \nu, \rho, m) > 0$.

Let us make analogous estimates on the second order terms by following the content in Section 2.3.7. In particular, the analog of (2.3.50) becomes

$$\begin{aligned} &\operatorname{Im}\Big(\big(\mathbf{R}\Gamma\mathbf{R}\big)_{a_k p} R_{q b_k}\Big)\bigg(A_{pq} - \frac{\gamma t^{1/2} w_{pq}}{(1-\gamma^2)^{1/2}}\bigg)\partial_k F(\operatorname{Im} R_{a_1 b_1}, \ldots, \operatorname{Im} R_{a_m b_m}) \\ &\quad = \operatorname{Im}\Big(\big(\mathbf{U}\Gamma\mathbf{U}\big)_{a_k p} U_{q b_k}\Big)\bigg(A_{pq} - \frac{\gamma t^{1/2} w_{pq}}{(1-\gamma^2)^{1/2}}\bigg)\partial_k F(\operatorname{Im} U_{a_1 b_1}, \ldots, \operatorname{Im} U_{a_m b_m}) + \gamma t w_{pq}^2 \mathfrak{Y}_k, \end{aligned} \tag{2.3.67}$$

where

$$\begin{aligned} \mathfrak{Y}_k = &\operatorname{Im}\big(U_{a_k p} U_{qp} U_{q b_k} + U_{a_k q} U_{pp} U_{q b_k}\big)\partial_k F(\operatorname{Im} U_{a_1 b_1}, \ldots, U_{a_m b_m}) \\ &- \operatorname{Im}\big(R_{a_k p} R_{qp} R_{q b_k} + R_{a_k q} R_{pp} R_{q b_k}\big)\partial_k F(\operatorname{Im} R_{a_1 b_1}, \ldots, \operatorname{Im} R_{a_m b_m}). \end{aligned}$$

As in Lemma 2.3.12, $\big|\mathbb{E}_\Psi[\gamma t w_{pq}^2 \mathfrak{Y}_k]\big|$ can be bounded by $CN^{-\omega}(\mathfrak{J}+1) + CQ_0 N^{C+C_0}$. Following (2.3.63), the expectation of the first term on the right side of (2.3.67) is equal to 0 if $\psi_{pq} = 0$ and $p \neq q$ (by (2.3.64)), and so the using the proof of (2.3.65) the total of this expectation over all $(p,q) \in [1,N]^2$ can be bounded by $CN^{-\omega}(\mathfrak{J}+1) + CQ_0 N^{C+C_0}$.

Thus, the second order terms in the expansion of the left side (2.3.66) can also be bounded by $C\big(N^{-\omega}(\mathfrak{J}+1) + Q_0 N^{C+C_0}\big)$, which verifies (2.3.66) and therefore establishes Theorem 2.2.15.

2.4 Intermediate local law for $\alpha \in (1, 2)$

In this section we establish Theorem 2.2.4, which provides a local law for $\mathbf{X}$ (recall Definition 3.2.3) at almost all energies E for $\alpha \in (1, 2)$. We begin by formulating an alternative version of this local law in Section 2.4.1 and showing that it implies Theorem 2.2.4. Its proof is deferred until Section 2.5; the remainder of this section consists of preparatory material. In Section 2.4.2 we recall some preliminary identities and estimates. In Section 2.4.3 we provide an outline of the previous work and of our proof. Finally, we conclude in Section 2.4.4 with a statement for an approximate fixed point equation (given by Proposition 2.4.11), which will be established in Section 2.5.2.

In what follows we fix parameters $\alpha, b, \nu > 0$ satisfying (2.2.1) and $\alpha \in (1, 2)$. We recall the functions $\varphi_{\alpha,z}$, $\psi_{\alpha,z}$, $y(z)$, and $m_\alpha(z)$ from (3.1.5) and (2.1.5); the removal matrix $\mathbf{X}$ and its resolvent $\mathbf{R}$ from Definition 3.2.3; that $m_N(z) = N^{-1} \operatorname{Tr} \mathbf{R}$; and the domain $\mathcal{D}_{K,\varpi,C}$ from (2.2.2). Furthermore, for each $s > 0$ we denote by $\mathbb{K}_s \subset \mathbb{C}$ the set of $z \in \mathbb{C}$ of the form $re^{\mathrm{i}\theta}$, with $r \in \mathbb{R}_{\geq 0}$ and $-\frac{\pi s}{2} \leq \theta \leq \frac{\pi s}{2}$.

2.4.1 An alternative intermediate local law

Through an inductive procedure that has been applied several times for Wigner matrices (see the book [52] and references therein), Theorem 2.2.4 will follow from the following result.

Theorem 2.4.1. *Adopt the notation and hypotheses of Theorem 2.2.4. For each $z \in \mathbb{H}$, define the event*

$$\begin{aligned} \Omega(z) = \left\{ \big| m_N(z) - m_\alpha(z) \big| \leq \frac{1}{N^{\varkappa}} \right\} \cap \left\{ \max_{1 \leq j \leq N} \big| R_{jj}(z) \big| \leq (\log N)^{30/(\alpha - 1)} \right\} \\ \cap \left\{ \max_{1 \leq j \leq N} \left| \mathbb{E}\Big[\big(- \mathrm{i} R_{jj}(z) \big)^{\alpha/2} \Big] - y(z) \right| \leq \frac{1}{N^{\varkappa}} \right\}. \end{aligned} \tag{2.4.1}$$

Then, for sufficiently large N, there exist large constants $C = C(\alpha, b, \nu, \varpi, K) > 0$ and

$\mathfrak{B} = \mathfrak{B}(\alpha) > 0$ *such*

$$\mathbb{P}\big[\Omega(z)^c\big] \le C \exp\left(-\frac{(\log N)^2}{C}\right) \quad \textit{if } \operatorname{Im} z = \mathfrak{B}. \tag{2.4.2}$$

Further, suppose that $z_0, z \in \mathcal{D}_{K,\varpi,\mathfrak{B}}$ *satisfy* $\operatorname{Re} z = \operatorname{Re} z_0$ *and* $\operatorname{Im} z_0 - \frac{1}{N^5} \le \operatorname{Im} z \le \operatorname{Im} z_0$. *If* $\mathbb{P}\big[\Omega(z_0)^c\big] \le \frac{1}{N^{20}}$, *then*

$$\mathbb{P}\big[\mathbf{1}_{\Omega(z)} < \mathbf{1}_{\Omega(z_0)}\big] \le C \exp\left(-\frac{(\log N)^2}{C}\right). \tag{2.4.3}$$

for large enough N.

Proof of Theorem 2.2.4 assuming Theorem 2.4.1. Let $\mathfrak{B}$ be as in Theorem 2.4.1, and let $K = [u, v]$. Now let $A = \big\lfloor N^5(v-u) \big\rfloor$ and let $B = \big\lfloor N^5(\mathfrak{B} - N^{-\varpi}) \big\rfloor$. For each integer $j \in [0, A]$ and $k \in [0, B]$, let $z_{j,k} = u + \frac{j}{N^5} + \mathrm{i}\big(\mathfrak{B} - \frac{k}{N^5}\big)$.

Then, by induction on $M \in [0, B]$, there exists a large constant $C = C(\alpha, b, \nu, \varpi, K) > 0$ such that

$$\mathbb{P}\left[\bigcup_{j=0}^{A} \bigcup_{k=0}^{M} \Omega(z_{j,k})^c\right] \le C(M+1) \exp\left(-\frac{(\log N)^2}{C}\right). \tag{2.4.4}$$

Now, the theorem follows from (2.4.4); the deterministic estimate $\big|R_{ij}(z) - R_{ij}(z_0)\big| < \frac{1}{N}$ and $\big|m_N(z) - m_N(z_0)\big| < \frac{1}{N}$ for $z_0, z \in \mathcal{D}_{[u,v],\delta,\mathfrak{B}}$ with $|z - z_0| < \frac{1}{N^5}$ (due to (2.3.1), (2.3.2), and the fact that $\eta \ge \frac{1}{N}$); and the deterministic estimate $\big|m_\alpha(z) - m_\alpha(z_0)\big| \le \frac{1}{N}$ for z_0 and z subject to the same conditions (which holds since m_α is the Stieltjes transform of the probability measure μ_α). $\square$

2.4.2 Identities and estimates

In this section we recall several facts that will be used throughout the proof of Theorem 2.4.1. In particular, we recall several resolvent identities and related bounds in Section 2.4.2, and we recall several additional estimates in Section 2.4.2.

Resolvent identities and estimates

In this section we collect several resolvent identities and estimates that will be used later.

In what follows, for any index set $\mathcal{I} \subset \{1, 2, \ldots, N\}$, let $\mathbf{X}^{(\mathcal{I})}$ denote the $N \times N$ matrix formed by setting the i-th row and column of $\mathbf{X}$ to zero for each $i \in \mathcal{I}$. Further denote $\mathbf{R}^{(\mathcal{I})} = \{R_{jk}^{(\mathcal{I})}\} = \big(\mathbf{X}^{(\mathcal{I})} - z\big)^{-1}$. If $\mathcal{I} = \{i\}$, we abbreviate $\mathbf{X}^{(\{i\})} = \mathbf{X}^{(i)}$, $\mathbf{R}^{(\{i\})} = \mathbf{R}^{(i)}$, and $R_{jk}^{(\{i\})} = R_{jk}^{(i)}$. Observe that $\mathbb{E}[m_N] = \mathbb{E}[R_{jj}]$, for any $j \in [1, N]$, due to the fact that all entries of $\mathbf{X}$ are identically distributed.

Lemma 2.4.2. *Let $\mathbf{H} = \{H_{ij}\}$ be an $N \times N$ real symmetric matrix, $z \in \mathbb{H}$, and $\eta = \operatorname{Im} z$. Denote $\mathbf{G} = \{G_{ij}\} = (\mathbf{H} - z)^{-1}$.*

1. *We have the Schur complement identity, which states for any $i \in [1, N]$ that*

$$\frac{1}{G_{ii}} = H_{ii} - z - \sum_{\substack{1 \le j,k \le N \\ j,k \ne i}} H_{ij} G_{jk}^{(i)} H_{ki}. \tag{2.4.5}$$

2. *Let $\mathcal{I} \subset [1, N]$. For any $j \in [1, N] \setminus \mathcal{I}$, we have the Ward identity*

$$\sum_{k \in [1,N] \setminus \mathcal{I}} \big| G_{jk}^{(\mathcal{I})} \big|^2 = \frac{\operatorname{Im} G_{jj}^{(\mathcal{I})}}{\eta}. \tag{2.4.6}$$

The estimates (2.4.5) and (2.4.6) can be found as (8.8) and (8.3) in the book [52], respectively.

Observe that (2.3.1), (2.3.2), and the estimate (which holds for any $x, y \in \mathbb{C}$ and $p \in \mathbb{R}$)

$$|x^p - y^p| \le |p||x - y|\big(|x|^{p-1} + |y|^{p-1}\big), \tag{2.4.7}$$

implies that

$$\begin{aligned} \left|R_{ij}^{(\mathcal{I})}(z_1)^p - R_{ij}^{(\mathcal{I})}(z_2)^p\right| &\le |p|\left|R_{ij}^{(\mathcal{I})}(z_1) - R_{ij}^{(\mathcal{I})}(z_2)\right| \left(\frac{1}{(\operatorname{Im} z_0)^{p-1}} + \frac{1}{(\operatorname{Im} z_1)^{p-1}}\right) \\ &\le 2|p||z_1 - z_2| \left(\frac{1}{(\operatorname{Im} z_0)^{p+1}} + \frac{1}{(\operatorname{Im} z_1)^{p+1}}\right) N. \end{aligned} \tag{2.4.8}$$

For each subset $\mathcal{I} \subset \{1, 2, \dots, N\}$ and $i \notin \mathcal{I}$, define

$$S_{i,\mathcal{I}} = \sum_{j \notin \mathcal{I} \cup \{i\}} X_{ij}^2 R_{jj}^{(\mathcal{I} \cup \{i\})}, \qquad T_{i,\mathcal{I}} = X_{ii} - U_{i,\mathcal{I}}, \qquad \mathfrak{S}_{i,\mathcal{I}} = \sum_{j \notin \mathcal{I} \cup \{i\}} Z_{ij}^2 R_{jj}^{(\mathcal{I} \cup \{i\})}, \tag{2.4.9}$$

where we recall the entries H_{ij} of $\mathbf{H}$ are coupled with the entries X_{ij} of $\mathbf{X}$ through the removal coupling of Definition 3.2.3, which also defined the Z_{ij}, and where

$$U_{i,\mathcal{I}} = \sum_{\substack{j,k \notin \mathcal{I} \cup \{i\} \\ j \ne k}} X_{ij} R_{jk}^{(\mathcal{I} \cup \{i\})} X_{ki}. \tag{2.4.10}$$

If $\mathcal{I}$ is empty, we denote $S_i = S_{i,\mathcal{I}}$, $\mathfrak{S}_i = \mathfrak{S}_{i,\mathcal{I}}$, $T_i = T_{i,\mathcal{I}}$, and $U_i = U_{i,\mathcal{I}}$. The Schur complement identity (2.4.5) can be restated as

$$R_{ii} = \frac{1}{T_i - z - S_i}. \tag{2.4.11}$$

Observe that since the matrix $\operatorname{Im} \mathbf{R}^{(\mathcal{I})}$ is positive definite and each X_{ii} is real, we have that

$$\operatorname{Im} S_{i,\mathcal{I}} \ge 0, \qquad \operatorname{Im} \mathfrak{S}_{i,\mathcal{I}} \ge 0, \qquad \operatorname{Im}(S_{i,\mathcal{I}} - T_{i,\mathcal{I}}) = \operatorname{Im}(S_{i,\mathcal{I}} + U_{i,\mathcal{I}}) \ge 0. \tag{2.4.12}$$

Additional estimates

In this section we collect several estimates that mostly appear as (sometimes special cases of) results in [33,34]. The first states that Lipschitz functions of the resolvent entries concentrate around their expectation and appears as Lemma C.3 of [33] (with the f there replaced by Lf here), which was established through the Azuma–Hoeffding estimate.

Lemma 2.4.3 ([33, Lemma C.3]). *Let N be a positive integer, and let $\mathbf{A} = \{a_{ij}\}_{1 \le i,j \le N}$ be an $N \times N$ real symmetric random matrix such that the i-dimensional vectors $A_i = (a_{i1}, a_{i2}, \ldots, a_{ii})$ are mutually independent for $1 \le i \le N$. Let $z = E + \mathrm{i}\eta \in \mathbb{H}$, and denote $\mathbf{B} = \{B_{ij}\} = (\mathbf{A} - z)^{-1}$. Then, for any Lipschitz function f with Lipschitz norm L, we have that*

$$\mathbb{P}\left[\left|\frac{1}{N}\sum_{j=1}^{N} f(B_{jj}) - \frac{1}{N}\sum_{j=1}^{N} \mathbb{E}\big[f(B_{jj})\big]\right| \ge t\right] \le 2\exp\left(-\frac{N\eta^2 t^2}{8L^2}\right).$$

By setting $f(x) = x$ or $f(x) = \operatorname{Im} x$, $L = 1$, and $t = 4(N\eta^2)^{-1/2}\log N$ in Lemma 2.4.3, we obtain

$$\begin{aligned}
\mathbb{P}\left[\Big|m_N(z) - \mathbb{E}\big[m_N(z)\big]\Big| > \frac{4\log N}{(N\eta^2)^{1/2}}\right] &\le 2\exp\big(-(\log N)^2\big), \\
\mathbb{P}\left[\Big|\operatorname{Im} m_N(z) - \mathbb{E}\big[\operatorname{Im} m_N(z)\big]\Big| > \frac{4\log N}{(N\eta^2)^{1/2}}\right] &\le 2\exp\big(-(\log N)^2\big).
\end{aligned} \tag{2.4.13}$$

The next lemma can be deduced from Lemma 2.4.3 by choosing f to be a suitably truncated variant of $x^{\alpha/2}$. It can be found as Lemma C.4 of [33], with their γ equal to our $\frac{\alpha}{2}$.

Lemma 2.4.4 ([33, Lemma C.4]). *Adopt the notation of Lemma 2.4.3, and fix $\alpha \in (0, 2)$. Then there exists a large constant $C = C(\alpha) > 0$ such that, for any $t > 0$,*

$$\mathbb{P}\left[\left|\frac{1}{N}\sum_{j=1}^{N}(-\mathrm{i}B_{jj})^{\alpha/2} - \frac{1}{N}\sum_{j=1}^{N}\mathbb{E}\big[(-\mathrm{i}B_{jj})^{\alpha/2}\big]\right| \ge t\right] \le 2\exp\left(-\frac{N(\eta^{\alpha/2}t)^{4/\alpha}}{C}\right).$$

The following, which is a concentration result for linear combinations of Gaussian random variables, follows from Bernstein's inequality and (2.3.2).

Lemma 2.4.5. *Let $(y_1, y_2, \ldots, y_N)$ be a Gaussian random vector whose covariance matrix* *is given by* Id*, and for each* $1 \leq j \leq N$ *let*

$$f_j = \big(\operatorname{Im} R_{jj} \big)^{\alpha/2} |y_j|^{\alpha}, \qquad g_j = \big(\operatorname{Im} R_{jj} \big)^{\alpha/2} \mathbb{E}\big[|y_j|^{\alpha} \big].$$

Then, there exists a large constant $C > 0$ *such that*

$$\mathbb{P}\left[\left| \frac{1}{N} \sum_{j=1}^{N} (f_j - g_j) \right| > \frac{C (\log N)^4}{N^{1/2} \eta^{\alpha/2}} \right] < C \exp\left(- \frac{(\log N)^2}{C} \right),$$

where the probability is with respect to $(y_1, y_2, \ldots, y_N)$ *and conditional on* $\mathbf{X}^{(i)}$.

The following two results state that the diagonal resolvent entries of $\mathbf{R}$ are close to those of $\mathbf{R}^{(i)}$ on average. The first appears as Lemma 5.5 of [34] and was established by inspecting the singular value decomposition of $\mathbf{R} - \mathbf{R}^{(i)}$ (one could alternatively use the interlacing of eigenvalues between $\mathbf{R}^{(i)}$ and $\mathbf{R}$) and then applying Hölder's inequality. Estimates of this type for $r = 1$ have appeared previously, for example as (2.7) of [59].

Lemma 2.4.6 ([34, Lemma 5.5])**.** *For any* $r \in (0, 1]$*, we have the deterministic estimate*

$$\frac{1}{N} \sum_{j=1}^{N} \big| R_{jj} - R_{jj}^{(i)} \big|^r \leq \frac{4}{(N\eta)^r}. \tag{2.4.14}$$

Corollary 2.4.7. *For any* $r \in [1, 2]$*, we have the deterministic estimate*

$$\frac{1}{N} \sum_{j=1}^{N} \big| R_{jj} - R_{jj}^{(i)} \big|^r \leq \frac{8}{N\eta^r}. \tag{2.4.15}$$

Proof. The estimate (2.3.2) together with the bound $|a - b|^{r-1} \leq |a|^{r-1} + |b|^{r-1}$ for any $a, b \in \mathbb{C}$ yields

$$\big| R_{jj} - R_{jj}^{(i)} \big|^r \leq \big| R_{jj} - R_{jj}^{(i)} \big| \Big(|R_{jj}|^{r-1} + \big| R_{jj}^{(i)} \big|^{r-1} \Big) \leq 2\eta^{1-r} \big| R_{jj} - R_{jj}^{(i)} \big|. \tag{2.4.16}$$

Now combining (2.4.16) with the $r = 1$ case of Lemma 2.4.6 yields (2.4.15). □

We also recall the following Lipschitz estimate for the functions $\varphi_{\alpha,z}$ and $\psi_{\alpha,z}$ (see (3.1.5)), which appears as Lemma 3.6 in [24].

Lemma 2.4.8 ([33, Lemma 3.4])**.** *There exists a large constant $c = c(\alpha)$ such that the following holds. For any $z \in \mathbb{H}$, the functions $\varphi_{\alpha,z}$ and $\psi_{\alpha,z}$ (see* (3.1.5)*) are Lipschitz with constants $c_\varphi = c(\alpha)|z|^{-\alpha}$ and $c_\psi = c(\alpha)|z|^{-\alpha/2}$ on $\mathbb{K}_{\alpha/2}$ and $\mathbb{K}_1$, respectively.*

We conclude this section with the following proposition (which is reminiscent of Lemma 3.2 of [33]) that bounds the quantity T_i from (2.4.9).

Proposition 2.4.9. *Let $z \in \mathbb{H}$ satisfy $\operatorname{Im} z \leq N^{1/\alpha - 1/2}$, and recall the definition of $T_i = T_i(z)$ from* (2.4.9)*. There exists a large constant $C = C(\alpha) > 0$ such that for any $t \geq 1$ we have that*

$$\mathbb{P}\left[|T_i| \geq \frac{Ct}{(N\eta^2)^{1/2}}\right] \leq \frac{C}{t^{\alpha/2}}. \tag{2.4.17}$$

Proof. First, (2.3.3) yields the existence of a large constant $C(\alpha) > 0$ such that

$$\mathbb{P}\left[|X_{ii}| \geq \frac{t}{(N\eta^2)^{1/2}}\right] \leq \frac{C(N\eta^2)^{\alpha/2}}{Nt^\alpha} \leq \frac{C}{t^\alpha}. \tag{2.4.18}$$

Now, from a Markov estimate, we have for any $s > 0$ that

$$\begin{aligned}
\mathbb{P}\left[|U_i| \leq \frac{t}{(N\eta^2)^{1/2}}\right] &\leq \frac{N\eta^2}{t^2}\mathbb{E}\left[\left|\sum_{1\leq j\neq k\leq N} X_j R_{jk}^{(i)} X_k\right|^2 \prod_{j=1}^N \mathbf{1}_{|X_j|\leq s}\right] + \sum_{j=1}^N \mathbb{P}\big[|X_j| \leq s\big] \\
&\leq \frac{N\eta^2}{t^2}\mathbb{E}\left[\sum_{\substack{1\leq j\neq k\leq N \\ 1\leq j'\neq k'\leq N}} X_j X_k \overline{X_{j'} X_{k'}} R_{jk}^{(i)} \overline{R_{j'k'}}^{(i)} \prod_{j=1}^N \mathbf{1}_{|X_j|\leq s}\right] + \frac{C}{s^\alpha} \\
&\leq \frac{2N\eta^2}{t^2} \sum_{1\leq j\neq k\leq N} \big|R_{jk}^{(i)}\big|^2 \mathbb{E}\big[|X_j|^2 \mathbf{1}_{|X_j|\leq s}\big]^2 + \frac{C}{s^\alpha},
\end{aligned} \tag{2.4.19}$$

after increasing C if necessary, where we abbreviated $X_j = X_{ij}$ for each $j \in [1, N]$, used (2.3.3), and recalled the independence and symmetry of the $\{X_j\}$. Then (2.4.19) implies

$$\mathbb{P}\left[|U_i| \le \frac{t}{(N\eta^2)^{1/2}}\right] \le \frac{8C^2 s^{4-2\alpha}\eta^2}{(2-\alpha)^2 t^2 N} \sum_{1\le j\ne k\le N} \big|R_{jk}^{(i)}\big|^2 + \frac{C}{s^\alpha} \le \frac{C^3 s^{4-2\alpha}}{t^2} + \frac{C}{s^\alpha}, \tag{2.4.20}$$

where we used used (2.3.2), (2.4.6), and

$$\mathbb{E}\big[|X_j|^2 \mathbf{1}_{|X_j|\le s}\big] = 2\int_0^s u\mathbb{P}\big[|X_j| \ge u\big]du \le \frac{2C}{N}\int_0^s u^{1-\alpha}du = \frac{2Cs^{2-\alpha}}{(2-\alpha)N}.$$

Setting $s = t^{1/2}$ in (2.4.20) yields

$$\mathbb{P}\left[|U_i| \le \frac{t}{(N\eta^2)^{1/2}}\right] \le \frac{C^3}{t^\alpha} + \frac{C}{t^{\alpha/2}}. \tag{2.4.21}$$

Now the lemma follows from the second identity in (2.4.9), (2.4.18), and (2.4.21). $\square$

Remark 2.4.10. The proof of Proposition 2.4.9 does not require that $\alpha \in (1,2)$ or that $E = \operatorname{Re} z$ is bounded away from 0. Instead, it only uses that the entries of $N^{1/\alpha}\mathbf{X}$ are symmetric random variables satisfying (3.1.3) and that $\operatorname{Im} z = \eta$. Thus, we will also use Proposition 2.4.9 in the proof of the local law in the case $\alpha \in (0,2) \setminus \mathcal{A}$, which appears in Section 2.6.

2.4.3 Outline of proof

In preparation for the next section, we briefly outline the method used in [33] to prove a local law on intermediate scales, and also the way in which we improve on this method. Recalling the notation of Section 2.4.2, we begin with the identity (2.4.11).

Approximating $T_i \approx \mathbb{E}[T_i] = 0$ and replacing each X_{ij} with h_{ij}, we find that $R_{ii} \approx (-\mathrm{i}z - \mathrm{i}\mathfrak{S}_i)^{-1}$. The identity $x^{-s} = \Gamma(s)^{-1}\int_0^\infty t^{s-1}e^{-xt}\,dt$ then yields for any $s > 0$

$$\mathbb{E}\big[(-\mathrm{i}R_{ii})^s\big] \approx \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\mathbb{E}\left[\exp\left(\mathrm{i}tz + \mathrm{i}t\sum_{j\ne i} R_{jj}^{(i)} h_{ij}^2\right)\right] dt. \tag{2.4.22}$$

To linearize the exponential appearing in the integrand on the right side of (2.4.22), we use the fact that, for a standard Gaussian random variable g, $\mathbb{E}\big[\exp(\mathrm{i}zg)\big] = \exp\big(-\frac{z^2}{2}\big)$. Together with the mutual independence of the $\{h_{ij}\}$, this yields

$$\begin{aligned}\mathbb{E}[(-\mathrm{i}R_{ii})^s] &\approx \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}e^{\mathrm{i}tz}\prod_{j\neq i}\mathbb{E}\left[\exp\left(\mathrm{i}\big(-2t\mathrm{i}R_{jj}^{(i)}\big)^{1/2}h_{ij}g_j\right)\right]dt \\ &\approx \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}e^{\mathrm{i}tz}\mathbb{E}\left[\exp\left(-\frac{\sigma^\alpha(2t)^{\alpha/2}}{N}\sum_{j=1}^N(-\mathrm{i}R_{jj})^{\alpha/2}\left|g_j\right|^\alpha\right)\right]dt,\end{aligned} \tag{2.4.23}$$

where we used the explicit formula (3.1.1) for the characteristic function of an α-stable random variable and the $\mathbf{g} = (g_1, g_2, \ldots, g_N)$ is an N-dimensional Gaussian random variable with covariance given by Id.

Approximating $|g_j|^\alpha \approx \mathbb{E}\big[|g_j|^\alpha\big]$, using the identities

$$\mathbb{E}\big[|g_j|^\alpha\big] = \frac{\Gamma(\alpha)}{2^{\alpha/2-1}\Gamma\big(\frac{\alpha}{2}\big)}, \quad \text{and} \quad \Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(1-\frac{\alpha}{2}\right) = \frac{\pi}{\sin\big(\frac{\pi\alpha}{2}\big)}, \tag{2.4.24}$$

recalling the definition of $\varphi_{\alpha,z}$ and $\psi_{\alpha,z}$ from (3.1.5), and applying (2.4.23) first with $s = \frac{\alpha}{2}$ and then with $s = 1$, we deduce

$$Y(z) \approx \varphi_{\alpha,z}\big(Y(z)\big), \qquad X(z) \approx \psi_{\alpha,z}\big(Y(z)\big),$$

where $X(z) = \mathbb{E}\big[-\mathrm{i}R_{jj}(z)\big]$ and $Y(z) = \mathbb{E}\left[(-\mathrm{i}R_{jj}(z))^{\alpha/2}\right]$.

Since the equation $Y(z) = \varphi_{\alpha,z}\big(Y(z)\big)$ is known [18] to have a unique fixed point $y(z)$, we expect from the previous two approximations that there is a global limiting measure $m_\alpha = \mathrm{i}\psi_{\alpha,z}(y(z))$; this matches with (2.1.5).

To obtain an intermediate local law for this measure, one must additionally quantify the error incurred from the above approximations. Among the primary sources of error here is the approximation $R_{ii} \approx \big(-\mathrm{i}z - \mathrm{i}\mathfrak{S}_i\big)^{-1}$. This not only requires that $|T_i|$ be small, but also that $\big|\mathfrak{S}_i + z\big|$ and $\big|S_i - T_i + z\big|$ (which is the denominator of R_{ii}) be bounded below. By analyzing certain Laplace transforms for quadratic forms in heavy-tailed random variables, the work [33] bounded these denominators by $\eta^{2/\alpha-1}$. This bound does not account for the true behavior of these resolvent entries (which should be bounded by N^{δ} for any $\delta > 0$), which causes the loss in scale of the intermediate local law established in [33] for α closer to one.

Thus, the improvement we seek will be to lower bound these denominators by $(\log N)^{-\frac{30}{\alpha-1}}$; see Proposition 2.5.1 below. This will both yield nearly optimal bounds on the diagonal resolvent entries R_{jj} and also allow us to establish an intermediate local law on the smaller scale $\eta = N^{-\varpi}$. Let us mention that the latter improvement (on the scale) is in fact necessary for us to implement our method. Indeed, if for instance α is near one, the results of [33] establish an intermediate local law for $\mathbf{H}$ on scale approximately $\eta \gg N^{-1/5}$. However, in order for us to apply the flow results of [42, 63, 77, 78] we require $\eta < t$, and to apply our comparison result given by Theorem 2.2.15, we require $t \leq N^{1/(\alpha-4)} \sim N^{-1/3}$. Hence in this case we require a local law for $\mathbf{X}$ on a scale $\eta \ll N^{-1/3}$, and this is the scale accessed by Theorem 2.2.4.

We do not know of a direct way to improve such a local law to the nearly optimal scale $\eta = N^{\delta-1}$, which is necessary to establish complete eigenvector delocalization and bulk universality. However, one can instead access such estimates for $\mathbf{H}$ by combining our current local law for $\mathbf{X}$ on scale $\eta^{-\varpi}$ with the comparison result given by Theorem 2.2.15 applied to $\mathbf{V}_t$, for which the estimates hold on the optimal scale by the regularizing effect of Dyson Brownian motion.

2.4.4 Approximate fixed point equations

In light of the outline from Section 2.4.3, let us define the quantities

$$X(z) = \mathbb{E}\big[- \mathrm{i}R_{jj}(z)\big], \qquad Y(z) = \mathbb{E}\big[(-\mathrm{i}R_{jj}(z))^{\alpha/2}\big], \tag{2.4.25}$$

which are independent of the index j, since the entries of $\mathbf{X}$ are identically distributed.

Throughout this section and the next, we use the notation of Theorem 2.2.4 and set the parameters $\theta = \theta(\alpha, b, \nu) > 0$ and $\delta = \delta(\alpha, b, \nu, \varpi) > 0$ by

$$\theta = \frac{2-\alpha}{50}, \qquad \delta = \frac{1}{10} \min \left\{ \theta, \nu - \varpi, \varpi - (2-\alpha)\nu, \frac{1}{2} - \varpi \right\}. \tag{2.4.26}$$

As mentioned in Section 2.4.3, let us now define an event on which the denominators of $R_{jj}(z)$ and $\big(- z - S_j(z) \big)^{-1}$ are bounded below. To that end, for any $z \in \mathbb{H}$, we define

$$\begin{aligned} \Lambda(z) = &\left\{ \min_{1 \le j \le N} \operatorname{Im} \big(S_j + z \big) \ge (\log N)^{-30/(\alpha-1)} \right\} \cap \left\{ \min_{1 \le j \le N} \operatorname{Im} \big(\mathfrak{S}_j + z \big) \ge (\log N)^{-30/(\alpha-1)} \right\} \\ &\cap \left\{ \min_{1 \le j \le N} \operatorname{Im} \big(S_j - T_j + z \big) \ge (\log N)^{-30/(\alpha-1)} \right\}. \end{aligned} \tag{2.4.27}$$

Assuming that $\mathbb{P}\big[\Lambda(z)^c\big]$ has very small probability, the following proposition provides an approximate fixed point equation for $Y(z)$, as explained in Section 2.4.3. Its proof will be provided in Section 2.5.2.

Proposition 2.4.11. *Adopt the notation and hypotheses of Theorem 2.2.4 and recall the parameters δ and θ defined in (2.4.26). Let $z \in \mathcal{D}_{K,\varpi,\mathfrak{B}}$ for some compact interval $K \subset \mathbb{R} \backslash \{0\}$ and some $\mathfrak{B} > 0$. If $\mathbb{P}\big[\Lambda(z)^c\big] < \frac{1}{N^{10}}$, then there exists a large constant $C = C(\alpha, b, \delta, \varepsilon) > 0$ such that*

$$\begin{aligned} \Big| Y(z) - \varphi_{\alpha,z}\big(Y(z)\big) \Big| &\le C(c_\varphi + C)(\log N)^{100/(\alpha-1)} \left(\frac{1}{(N\eta^2)^{\alpha/8}} + \frac{1}{N^{2\theta}} \right), \\ \Big| X(z) - \psi_{\alpha,z}\big(Y(z)\big) \Big| &\le C(c_\psi + C)(\log N)^{100/(\alpha-1)} \left(\frac{1}{(N\eta^2)^{\alpha/8}} + \frac{1}{N^{2\theta}} \right), \end{aligned} \tag{2.4.28}$$

where $c_\varphi = c_\varphi(\alpha, z)$ and $c_\psi = c_\psi(\alpha, z)$ are given by Lemma 2.4.8.

2.5 Proof of Theorem 2.4.1

In this section we establish Theorem 2.4.1 in Section 2.5.2 after bounding the probability $\mathbb{P}\big[\Lambda(z)^c\big]$ in Section 2.5.1.

2.5.1 Estimating $\mathbb{P}\big[\Lambda(z)^c\big]$

In this section, we provide a estimate for $\mathbb{P}\big[\Lambda(z)^c\big]$, given by Proposition 2.5.1. Due to the Schur complement formula, this proposition implies optimal bounds on the resolvent entries that were not present in the previous work [33]. These bounds will in turn allow us to establish the local law on an improved scale.

In Section 2.5.1 we prove Proposition 2.5.1, assuming Proposition 2.5.2 and Proposition 2.5.3 below. These propositions are then established in Section 2.5.1 and Section 2.5.1, respectively.

Proposition 2.5.1. *Assume that $z \in \mathcal{D}_{K,\varpi,\mathfrak{B}}$ for some $\mathfrak{B} > 0$ and that $\varepsilon \le \mathbb{E}\big[\operatorname{Im} m_N(z)\big] \le \frac{1}{\varepsilon}$, for some $\varepsilon > 0$. Then, there exists a large constant $C = C(\alpha, b, \delta, \varepsilon) > 0$ such that*

$$\mathbb{P}\left[\Lambda(z)^c\right] \le C \exp\left(-\frac{(\log N)^2}{C} \right).$$

A heuristic for the proof

We now briefly outline our argument for the lower bound on $\operatorname{Im}(S_i + z)$. The Schur complement formula reads $R_{ii} = (T_i - z - S_i)^{-1}$. For the purposes of this outline, let us assume that $R_{ii} \approx (-z - S_i)^{-1}$, so that a lower bound on $\operatorname{Im} S_i$ implies an upper bound on $|R_{ii}|$.

Letting $\mathbf{A}$ denote the diagonal $(N-1) \times (N-1)$ matrix whose entries are given by $\operatorname{Im} R_{jj}^{(i)}$ with $j \ne i$, we find that $\operatorname{Im} S_i = \langle X, \mathbf{A} X \rangle$, where we defined the $(N-1)$-dimensional vector $\mathbf{X} = (X_{ij})_{j \ne i}$. Thus we obtain from (3.1.1) that, if $Y = (y_1, y_2, \ldots, y_{N-1})$ denotes an $(N-1)$-dimensional Gaussian random variable whose covariance is given by Id, then for any

$t > 0$

$$\mathbb{E}\left[\exp\left(-\frac{t^2}{2}\langle \mathbf{A}X, X\rangle\right)\right] = \mathbb{E}\Big[\exp\big(\mathrm{i}t\langle \mathbf{A}^{1/2}X, Y\rangle\big)\Big] \approx \mathbb{E}\left[\exp\left(-\frac{c|t|^\alpha \|\mathbf{A}^{1/2}Y\|_\alpha^\alpha}{N}\right)\right],$$

for some constant $c > 0$. Assuming that $\|\mathbf{A}^{1/2}Y\|_\alpha^\alpha$ concentrates around its expectation, we obtain after replacing t by $t\sqrt{2}$ and altering c that

$$\mathbb{E}\big[\exp(-t^2 \operatorname{Im} S_i)\big] = \mathbb{E}\left[\exp\left(-t^2\langle \mathbf{A}X, X\rangle\right)\right] < \exp\left(-\frac{c|t|^\alpha}{N}\sum_{j\neq i}\left|\operatorname{Im} R_{jj}^{(i)}\right|^{\alpha/2}\right). \quad (2.5.1)$$

If t is chosen such that

$$\frac{|t|^\alpha}{N}\sum_j \left|\operatorname{Im} R_{jj}^{(i)}\right|^{\alpha/2} = (\log N)^2,$$

then the right side of (2.5.1) is very small. Hence (2.5.1) implies, using Markov's inequality, that

$$\mathbb{P}\left[\operatorname{Im} S_i \leq t^{-2}\right] = \mathbb{P}\left[t^2 \operatorname{Im} S_i \leq 1\right] = \mathbb{P}\left[\exp(-t^2 \operatorname{Im} S_i) \geq \frac{1}{e}\right] \leq C\exp\big(-c(\log N)^2\big).$$

Therefore, using the definition of t and ignoring logarithmic factors, we have with high probability that

$$\operatorname{Im} S_i \geq \left(\frac{1}{N}\sum_{j\neq i}\left|\operatorname{Im} R_{jj}^{(i)}\right|^{\alpha/2}\right)^{2/\alpha}.$$

Since $\frac{\alpha}{2} < 1$, we have

$$\frac{1}{N}\sum_{j\neq i}\left|\operatorname{Im} R_{jj}^{(i)}\right|^{\alpha/2} \geq \operatorname{Im} m_N^{(i)}\left(\max_{j\neq i}|\operatorname{Im} R_{jj}^{(i)}|\right)^{\alpha/2-1}, \quad (2.5.2)$$

where $m_N^{(i)} = N^{-1} \operatorname{Tr} \mathbf{R}^{(i)}$. Proceeding using (2.5.2) yields

$$\operatorname{Im} S_i \geq \left(\operatorname{Im} m_N^{(i)} \left(\max_{j \neq i} \left| \operatorname{Im} R_{jj}^{(i)} \right| \right)^{\alpha/2-1} \right)^{2/\alpha} \approx \left| \operatorname{Im} m_N \right|^{2/\alpha} \left(\max_{1 \leq j \leq N} |R_{jj}| \right)^{1-2/\alpha}. \qquad (2.5.3)$$

Since $|R_{ii}| \leq (ImS_i)^{-1}$, this suggests that

$$|R_{ii}| \leq \left| \operatorname{Im} m_N \right|^{-2/\alpha} \max_{1 \leq j \leq N} |R_{jj}|^{2/\alpha-1},$$

with high probability. Assuming that $\operatorname{Im} m_N$ is bounded below and taking maximum over $i \in [1, N]$, this yields

$$\max_{1 \leq j \leq N} |R_{jj}| \leq C \Big(\max_{1 \leq j \leq N} |R_{jj}| \Big)^{2/\alpha-1},$$

for some constant $C > 0$. Thus, since $\frac{2}{\alpha} - 1 < 1$ (this is where we use $\alpha > 1$), this implies an upper bound on each $|R_{jj}|$ with high probability.

Proof of Proposition 2.5.1

An issue with the outline from Section 2.5.1 is in (2.5.3), where we claimed that $\max_{j \neq i} \left| R_{jj}^{(i)} \right|$ is approximately $\max_j |R_{jj}|$. So, to implement this outline more carefully, we will instead proceed by showing that if one can bound the entries of $\mathbf{R}^{(\mathcal{I})}$ for each $|\mathcal{I}| = k$ (recall Section 2.4.2) by some large $\vartheta > 0$, then we can bound the entries of $\mathbf{R}^{(\mathcal{J})}$ for each $|\mathcal{J}| = k-1$ by $\vartheta^{2/\alpha-1} (\log N)^{20}$. In particular, if $\alpha > 1$, then $\frac{2}{\alpha} - 1 < 1$, so we can repeat this procedure approximately $(\log \log N)^2$ times to obtain nearly optimal estimates on the entries of $\mathbf{R}$.

To that end, we will define generalized versions of the event Λ. Fix the integer $M = \left\lceil (\log \log N)^2 \right\rceil$, and for each $0 \leq k \leq M$, define the positive real numbers $\varsigma_0, \varsigma_1, \ldots, \varsigma_M$ by

$$\varsigma_M = \eta, \quad \text{and} \quad \varsigma_k = \varsigma_{k+1}^{2/\alpha-1} (\log N)^{-20}, \quad \text{for each } 0 \leq k \leq M-1. \qquad (2.5.4)$$

Since $\alpha \in (1,2)$, we have that $\kappa = \frac{2}{\alpha} - 1 \in (0,1)$, and so

$$\varsigma_0 \geq \varsigma_M^{\kappa^M} (\log N)^{-20/(1-\kappa)} \geq N^{-\kappa^M} (\log N)^{-10\alpha/(\alpha-1)},$$

where we have used the fact that $\varsigma_M = \eta \geq N^{-1}$. It therefore follows that $\varsigma_0 \geq (\log N)^{-25/(\alpha-1)}$ for sufficiently large N, since $\alpha \in (1,2)$ and $M = \lfloor (\log\log N)^2 \rfloor$.

Now, for each subset $\mathcal{I} \subset \{1, 2, \ldots, N\}$ with $|\mathcal{I}| = k \leq M$, define the three events

$$\begin{aligned} \Lambda_{S,i,\mathcal{I}}(z) = \Big\{ \operatorname{Im}\big(S_{i,\mathcal{I}}(z) + z\big) \geq \varsigma_k \Big\}, \qquad \Lambda_{\mathfrak{S},i,\mathcal{I}}(z) = \Big\{ \operatorname{Im}\big(\mathfrak{S}_{i,\mathcal{I}}(z) + z\big) \geq \varsigma_k \Big\}, \\ \Lambda_{T,i,\mathcal{I}}(z) = \Big\{ \operatorname{Im}\big(S_{i,\mathcal{I}}(z) - T_{i,\mathcal{I}}(z) + z\big) \geq \varsigma_k \Big\}. \end{aligned} \tag{2.5.5}$$

Furthermore, for each $0 \leq u \leq M$, define the event

$$\Lambda^{(u)}(z) = \bigcap_{k=u}^{M} \bigcap_{\substack{\mathcal{I} \subset \{1,2,\ldots,N\} \\ |\mathcal{I}|=k}} \bigcap_{i \notin \mathcal{I}} \big(\Lambda_{S,i,\mathcal{I}}(z) \cap \Lambda_{\mathfrak{S},i,\mathcal{I}}(z) \cap \Lambda_{T,i,\mathcal{I}}(z) \big).$$

The following propositions estimate the probabilities of the events $\Lambda_{S,i,\mathcal{I}}$, $\Lambda_{\mathfrak{S},i,\mathcal{I}}$, and $\Lambda_{T,i,\mathcal{I}}$. We will establish Proposition 2.5.2 in Section 2.5.1 and Proposition 2.5.3 in Section 2.5.1.

Proposition 2.5.2. *Assume that $z \in \mathcal{D}_{K,\varpi,\mathfrak{B}}$, for some $\mathfrak{B} > 0$, and that $\varepsilon \leq \mathbb{E}\big[\operatorname{Im} m_N(z)\big] \leq \frac{1}{\varepsilon}$, for some $\varepsilon > 0$. Then, there exists a large constant $C = C(\alpha, b, \delta, \varepsilon) > 1$ such that the following holds. For any integer $u \in [0, M-1]$, any subset $\mathcal{I} \subset \{1, 2, \ldots, N\}$ with $|\mathcal{I}| = u$, and any $i \notin \mathcal{I}$, we have that*

$$\begin{aligned} \mathbb{P}\big[\Lambda_{S,i,\mathcal{I}}(z)^c\big] \leq \mathbb{P}\big[\Lambda^{(u+1)}(z)^c\big] + C \exp\left(-\frac{(\log N)^2}{C} \right), \\ \mathbb{P}\big[\Lambda_{\mathfrak{S},i,\mathcal{I}}(z)^c\big] \leq \mathbb{P}\big[\Lambda^{(u+1)}(z)^c\big] + C \exp\left(-\frac{(\log N)^2}{C} \right). \end{aligned} \tag{2.5.6}$$

Proposition 2.5.3. *Adopt the notation and hypotheses of Proposition 2.5.2. Then, there exists a large constant $C = C(\alpha, b, \delta, \varepsilon) > 1$ such that the following holds. For any integer*

$u \in [0, M-1]$, any subset $\mathcal{I} \subset \{1, 2, \ldots, N\}$ with $|\mathcal{I}| = u$, and any $i \notin \mathcal{I}$, we have that

$$\mathbb{P}\big[\Lambda_{T,i,\mathcal{I}}(z)^c\big] \le \mathbb{P}\big[\Lambda^{(u+1)}(z)^c\big] + C \exp\left(-\frac{(\log N)^2}{C}\right). \tag{2.5.7}$$

Assuming Proposition 2.5.2 and Proposition 2.5.3, we can establish Proposition 2.5.1.

Proof of Proposition 2.5.1 assuming Proposition 2.5.2 and Proposition 2.5.3. A union bound over all $\mathcal{I} \subset \{1, 2, \ldots, n\}$ with $|\mathcal{I}| = u$ and $i \notin \mathcal{I}$ in (2.5.6) and (2.5.7) yields (with C as in those estimates)

$$\mathbb{P}\big[\Lambda^{(u)}(z)^c\big] \le 3N^{u+1}\mathbb{P}\big[\Lambda^{(u+1)}(z)^c\big] + 3CN^{u+1} \exp\left(-\frac{(\log N)^2}{C}\right). \tag{2.5.8}$$

The estimate (2.3.2) implies that $\Lambda^{(M)}(z)$ holds deterministically, so (2.5.8) and induction on u yields

$$\mathbb{P}\big[\Lambda^{(u)}(z)^c\big] \le (3CN)^{(M+2)(M-u+1)} \exp\left(-\frac{(\log N)^2}{C}\right), \tag{2.5.9}$$

for each $0 \le u \le M$. Since $M = \lfloor (\log\log N)^2 \rfloor$, it follows from (2.5.9) (after increasing C if necessary) that $\mathbb{P}\big[\Lambda^{(0)}(z)^c\big] \le C \exp\big(-C^{-1}(\log N)^2\big)$, from which the proposition follows since $\Lambda^{(0)}(z) \subseteq \Lambda(z)$. $\square$

Proof of Proposition 2.5.2

In this section we establish Proposition 2.5.2. Before doing so, we require the following estimate on the Laplace transform for quadratic forms of removals of stable laws, which is an extension of Lemma B.1 of [33] to removals of stable laws; this lemma will be established in Section 2.9.

Lemma 2.5.4. *Let $\alpha \in (0, 2)$, $\sigma > 0$ be real, $0 < b < \frac{1}{\alpha}$ be reals, and N be a positive integer. Let $\widetilde{X}$ be a b-removal of a deformed $(0, \sigma)$ α-stable law (recall Definition 2.2.2), and let $X = (X_1, X_2, \ldots, X_N)$ be mutually independent random variables, each having the* *same law as* $N^{-1/\alpha}\widetilde{X}$*. Let* $\mathbf{A} = \{a_{ij}\}$ *be an* $N \times N$ *nonnegative definite, symmetric matrix;* $\mathbf{B} = \{B_{ij}\} = \mathbf{A}^{1/2}$*; and* $Y = (y_1, y_2, \ldots, y_N)$ *be an* N*-dimensional centered Gaussian random variable (independent from* X*) with covariance matrix given by* Id*. Then,*

$$\mathbb{E}\bigg[\exp\Big(-\frac{t^2}{2}\langle \mathbf{A}X, X\rangle\Big)\bigg]$$
$$= \mathbb{E}\Bigg[\exp\Bigg(-\frac{\sigma^\alpha |t|^\alpha \|\mathbf{B}Y\|_\alpha^\alpha}{N}\Bigg)\Bigg]\exp\bigg(O\Big(t^2 N^{(2-\alpha)(b-1/\alpha)-1}(\log N)\operatorname{Tr}\mathbf{A}\Big)\bigg) + Ne^{-(\log N)^2/2},$$

where, for any vector $w = (w_1, w_2, \ldots, w_N) \in \mathbb{C}^N$ *and* $r > 0$*, we define* $\|w\|_r = \big(\sum_{j=1}^N |w_j|^r\big)^{1/r}$*.*

Now we can prove Proposition 2.5.2.

Proof of Proposition 2.5.2. Since the proofs of the two estimates in (2.5.6) are very similar, we only establish the first one (on $\operatorname{Im} S_{i,\mathcal{I}}$). For notational convenience we assume that $i = N$ and $\mathcal{I} = \{N-u, N-u+1, \ldots, N-1\}$. Denote $\mathcal{J} = \mathcal{I} \cup \{N\}$, and set

$$\Lambda_{\mathcal{J}}(z) = \bigcap_{j \notin \mathcal{J}} \big(\Lambda_{S,j,\mathcal{J}}(z) \cap \Lambda_{\mathfrak{S},j,\mathcal{J}}(z) \cap \Lambda_{T,j,\mathcal{J}}(z)\big) \subseteq \Lambda^{(u+1)}(z). \tag{2.5.10}$$

In what follows, let $\mathcal{G}$ denote the event on which

$$\left|\frac{1}{N}\sum_{j=1}^{N-u-1}\operatorname{Im} R_{jj}^{(\mathcal{J})}(z) - \mathbb{E}\big[\operatorname{Im} m_N(z)\big]\right| > \frac{4(u+1)}{(N-u-1)\eta} + \frac{4\log N}{N\eta^2}. \tag{2.5.11}$$

Observe that (2.4.14) (applied with $r = 1$) and the second estimate in (2.4.13) imply that $\mathbb{P}\big[\mathcal{G}\big] \le 2\exp\big(-(\log N)^2\big)$. Now let us apply Lemma 2.5.4 with $X = (X_{Nj})_{1 \le j \le N-u-1}$ and $\mathbf{A} = \{A_{ij}\}$ given by the $(N-u-1)\times(N-u-1)$ diagonal matrix whose (j,j)-entry is equal to $A_{jj} = \operatorname{Im} R_{jj}^{(\mathcal{J})}$. Then $\operatorname{Im} S_{N,\mathcal{I}} = \langle X, \mathbf{A}X\rangle$, so taking $t = (2\log 2)^{1/2}\varsigma_u^{-1/2}$ in Lemma 2.5.4

yields from a Markov estimate that for sufficiently large N,

$$
\begin{aligned}
&\mathbb{P}\big[\operatorname{Im} S_{N,\mathcal{I}} \le \varsigma_u \mathbf{1}_{\Lambda^{(u+1)}(z)}\big] \\
&\quad \le 2\mathbb{E}\Bigg[\exp\bigg(-\frac{t^2}{2}\langle \mathbf{A}X, X\rangle\bigg)\mathbf{1}_{\Lambda_{\mathcal{J}}(z)}\mathbf{1}_{\mathcal{G}}\Bigg] + 2\mathbb{P}[\mathcal{G}] \\
&\quad = 2\mathbb{E}\Bigg[\mathbb{E}\Bigg[\exp\bigg(-\frac{t^2}{2}\langle \mathbf{A}X, X\rangle\bigg)\bigg|\{X_{jk}\}_{j,k\notin\mathcal{J}}\Bigg]\mathbf{1}_{\Lambda_{\mathcal{J}}(z)}\mathbf{1}_{\mathcal{G}}\Bigg] + 2\mathbb{P}[\mathcal{G}] \\
&\quad \le 2\mathbb{E}\Bigg[\exp\bigg(-\frac{\sigma^\alpha \|\mathbf{A}^{1/2}Y\|_\alpha^\alpha}{2(N-u-1)\varsigma_u^{\alpha/2}}\bigg)\exp\Big(O\big(\varsigma_u^{-1}N^{(2-\alpha)(b-1/\alpha)-1}\operatorname{Tr}\mathbf{A}\big)\Big)\mathbf{1}_{\Lambda_{\mathcal{J}}(z)}\mathbf{1}_{\mathcal{G}}\Bigg] \\
&\qquad + 6N\exp\bigg(-\frac{(\log N)^2}{4}\bigg),
\end{aligned}
\tag{2.5.12}
$$

where $Y = (y_1, y_2, \ldots, y_{N-u-1})$ is an $(N-u-1)$-dimensional Gaussian vector whose covariance is given by Id. On the right side of the equality in (2.5.12), the inner expectation is over the $\{X_{jk}\}$ with either $i \in \mathcal{J}$ or $j \in \mathcal{J}$, conditional on the remaining $\{X_{jk}\}$; the outer expectation is over these remaining $\{X_{jk}\}$ (with $j, k \notin \mathcal{J}$).

To estimate the terms on the right side of (2.5.12), first observe from the definition (2.5.11) of the event $\mathcal{G}$ that $\mathbf{1}_{\mathcal{G}} N^{-1}\operatorname{Tr}\mathbf{A} < \mathbb{E}\big[\operatorname{Im} m_N(z)\big] + N^{-\delta}$ for sufficiently large N. Applying this, our assumption $\mathbb{E}\big[\operatorname{Im} m_N(z)\big] \le \frac{1}{\varepsilon}$, and the fact that $\varsigma_u \ge \varsigma_{M-1} \ge \eta^{2/\alpha-1}(\log N)^{-20}$ yields for sufficiently large N

$$
\begin{aligned}
\mathbf{1}_{\mathcal{G}}\varsigma_u^{-1}N^{(2-\alpha)(b-1/\alpha)-1}\operatorname{Tr}\mathbf{A} &\le 2\varepsilon^{-1}\eta^{1-2/\alpha}N^{(2-\alpha)(b-1/\alpha)}(\log N)^{20} \\
&\le 2\varepsilon^{-1}N^{(2-\alpha)(b-1/\alpha+\varpi/\alpha)}(\log N)^{20} \le N^{5\delta(\alpha-2)} \le 1,
\end{aligned}
\tag{2.5.13}
$$

where we have recalled that $\eta \ge N^{-\varpi}$ and used (2.2.1) and (2.4.26). Inserting (2.5.13) into (2.5.12) yields the existence of a large constant $C = C(\alpha, b, \delta, \varepsilon) > 0$ such that

$$
\mathbb{P}\big[\operatorname{Im} S_{N,\mathcal{I}} < \varsigma_u \mathbf{1}_{\Lambda_{\mathcal{J}}(z)}\big] \le C\mathbb{E}\Bigg[\exp\bigg(-\frac{\|\mathbf{A}^{1/2}Y\|_\alpha^\alpha}{CN\varsigma_u^{\alpha/2}}\bigg)\mathbf{1}_{\Lambda_{\mathcal{J}}(z)}\Bigg] + C\exp\bigg(-\frac{(\log N)^2}{C}\bigg).
\tag{2.5.14}
$$

Therefore it suffices to lower bound $N^{-1}\|\mathbf{A}^{1/2}Y\|_\alpha^\alpha$. To that end, we apply Lemma 2.4.5 to deduce (after increasing C if necessary) that

$$\begin{aligned}\mathbb{P}\Bigg[\bigg|\frac{1}{N}\sum_{j=1}^{N-u-1}\big(\operatorname{Im} R_{jj}^{(\mathcal{J})}\big)^{\alpha/2}|y_j|^\alpha - \frac{1}{N}\sum_{j=1}^{N-u-1}\big(\operatorname{Im} R_{jj}^{(\mathcal{J})}\big)^{\alpha/2}\mathbb{E}\big[|y_j|^\alpha\big]\bigg| > \frac{C(\log N)^4}{N^{1/2}\eta^{\alpha/2}}\Bigg] \\ \le C\exp\bigg(-\frac{(\log N)^2}{C}\bigg),\end{aligned}$$

from which we find (again, after increasing C if necessary) that

$$\mathbb{P}\Bigg[\frac{\|\mathbf{A}^{1/2}Y\|_\alpha^\alpha}{N} < \frac{1}{CN}\sum_{j=1}^{N-u-1}\big(\operatorname{Im} R_{jj}^{(\mathcal{J})}\big)^{\alpha/2} - \frac{C(\log N)^4}{N^{1/2}\eta^{\alpha/2}}\Bigg] \le C\exp\bigg(-\frac{(\log N)^2}{C}\bigg). \quad (2.5.15)$$

Now, observe that by (2.4.5) and the definition (2.5.10) of the event $\Lambda_{S,N,\mathcal{J}}(z)$ that

$$\mathbf{1}_{\Lambda_\mathcal{J}(z)}\big|R_{jj}^{(\mathcal{J})}(z)\big| = \mathbf{1}_{\Lambda_\mathcal{J}(z)}\big|S_{j,\mathcal{J}}(z) - T_{j,\mathcal{J}}(z) + z\big|^{-1} \le \varsigma_{u+1}^{-1}$$

for each $j \notin \mathcal{J}$. Therefore,

$$\frac{\mathbf{1}_{\Lambda_\mathcal{J}(z)}}{N}\sum_{j=1}^{N-u-1}\big(\operatorname{Im} R_{jj}^{(\mathcal{J})}\big)^{\alpha/2} \ge \frac{\varsigma_{u+1}^{1-\alpha/2}\mathbf{1}_{\Lambda_\mathcal{J}(z)}}{N}\sum_{j=1}^{N-u-1}\operatorname{Im} R_{jj}^{(\mathcal{J})}. \quad (2.5.16)$$

Furthermore we have by (2.4.14) (applied with $r = 1$) that

$$\frac{\mathbf{1}_{\Lambda_\mathcal{J}(z)}}{N}\sum_{j=1}^{N-u-1}\operatorname{Im} R_{jj}^{(\mathcal{J})}(z) \ge \mathbf{1}_{\Lambda_\mathcal{J}(z)}\bigg(m_N(z) - \frac{4(u+1)}{(N-u-1)\eta}\bigg). \quad (2.5.17)$$

It then follows from the second estimate in (2.4.13), the assumption $\mathbb{E}\big[\operatorname{Im} m_N(z)\big] \ge \varepsilon$, and $0 \le u \le M = \big\lfloor(\log\log N)^2\big\rfloor$ that

$$\mathbb{P}\Bigg[\frac{\mathbf{1}_{\Lambda_\mathcal{J}(z)}}{N}\sum_{j=1}^{N-u-1}\operatorname{Im} R_{jj}^{(\mathcal{J})}(z) \le \frac{\varepsilon\mathbf{1}_{\Lambda_\mathcal{J}(z)}}{2}\Bigg] \le 2\exp\big(-(\log N)^2\big), \quad (2.5.18)$$

for sufficiently large N. Inserting (2.5.16) and (2.5.18) into (2.5.15) (upon observing that

$\varsigma_{u+1}^{1-\alpha/2} \geq \eta^{1-\alpha/2} \geq \frac{1}{N^{1/2-\delta}\eta^{\alpha/2}}$) yields

$$\mathbb{P}\left[\frac{\|\mathbf{A}^{1/2}Y\|_\alpha^\alpha}{N} < \frac{\varepsilon\varsigma_{u+1}^{1-\alpha/2}\mathbf{1}_{\Lambda_{\mathcal{J}(z)}}}{C}\right] \leq C\exp\left(-\frac{(\log N)^2}{C}\right) \tag{2.5.19}$$

for sufficiently large N, again after increasing C if necessary. Therefore, inserting (2.5.19) into (2.5.14) yields

$$\mathbb{P}\big[\operatorname{Im} S_{N,\mathcal{I}} < \varsigma_u \mathbf{1}_{\Lambda_{\mathcal{J}(z)}}\big] \leq C\mathbb{E}\left[\exp\left(-\frac{\varepsilon\varsigma_{u+1}^{1-\alpha/2}}{C\varsigma_u^{\alpha/2}}\right)\right] + 2C\exp\left(-\frac{(\log N)^2}{C}\right),$$

from which the proposition follows since $\varsigma_u^{\alpha/2} = \varsigma_{u+1}^{1-\alpha/2}(\log N)^{-10\alpha}$ (due to (2.5.4)), and we may increase C so that the bound holds for all N. □

Proof of Proposition 2.5.3

In this section we establish Proposition 2.5.3. We first require the following lemma that will be established in Section 2.5.1.

Lemma 2.5.5. *Let N be a positive integer and $0 < r < 2 < a \leq 4$ be positive real numbers. Denote by $\mathbf{w} = (w_1, w_2, \ldots, w_N)$ a centered N-dimensional Gaussian random variable with covariance $U_{ij} = \mathbb{E}[w_i w_j]$ for each $1 \leq i, j \leq N$. Define $V_j = \mathbb{E}[w_j^2]$ for each $1 \leq j \leq N$, and define*

$$U = \frac{1}{N}\sum_{1\leq i,j\leq N} U_{ij}^2, \quad V = \frac{\mathbb{E}\big[\|\mathbf{w}\|_2^2\big]}{N} = \frac{1}{N}\sum_{j=1}^N V_j, \quad \mathcal{X} = \frac{1}{N}\sum_{j=1}^N V_j^{a/2},$$
$$p = \frac{a-r}{a-2}, \qquad q = \frac{a-r}{2-r}.$$

If $V > 100(\log N)^{10}U^{1/2}$, then there exists a large constant $C = C(a,r) > 0$ such that

$$\mathbb{P}\left[\frac{\|\mathbf{w}\|_r^r}{N} \leq \frac{V^p}{C\big(\mathcal{X}(\log N)^8\big)^{p/q}}\right] \leq C\exp\left(-\frac{(\log N)^2}{2}\right).$$

Observe that Lemma 2.5.5 is a certain type of Hölder estimate for correlated Gaussian random variables. The exponents p and q in that lemma come from such a bound (see (2.5.29)). With this lemma, we can now establish Proposition 2.5.3.

Proof of Proposition 2.5.3. For notational convenience we assume that $i = N$ and $u = 0$ (in which case $\mathcal{I}$ is empty); in what follows, we abbreviate the event

$$\Lambda_N(z) = \bigcap_{j=1}^{N-1} \big(\Lambda_{S,j,\{N\}}(z) \cap \Lambda_{\mathfrak{S},j,\{N\}}(z) \cap \Lambda_{T,j,\{N\}}(z) \big). \tag{2.5.20}$$

Now let us apply Lemma 2.5.4 with $X = (X_{Nj})_{1 \le j \le N-1}$ and the $(N-1) \times (N-1)$ matrix $\mathbf{A} = \{A_{ij}\}$, where we define $A_{ij} = \operatorname{Im} R_{ij}^{(N)}$ for $1 \le i, j \le N-1$ (the superscript refers to the removal of the Nth row.) Then $\operatorname{Im}(S_{N,\mathcal{I}} - T_{i,\mathcal{I}}) = \langle X, \mathbf{A} X \rangle$. Therefore, taking $t = (2 \log 2)^{1/2} \varsigma_0^{-1/2}$ in Lemma 2.5.4 yields by following the beginning of the proof of Proposition 2.5.2 until (2.5.14) the existence of a large constant $C = C(\alpha, b, \delta, \varepsilon) > 0$ such that

$$\mathbb{P}\big[\operatorname{Im}(S_N - T_N) < \varsigma_0 \mathbf{1}_{\Lambda^{(1)}(z)} \big] \le C \mathbb{E}\left[\exp\left(- \frac{\| \mathbf{A}^{1/2} Y \|_\alpha^\alpha}{C N \varsigma_0^{\alpha/2}} \right) \mathbf{1}_{\Lambda_N(z)} \right] + C \exp\left(- \frac{(\log N)^2}{C} \right), \tag{2.5.21}$$

where $Y = (y_1, y_2, \ldots, y_{N-1})$ is an $(N-1)$-dimensional centered Gaussian random variable whose covariance is given by Id.

Now let us apply Lemma 2.5.5 with $w_i = (\mathbf{A}^{1/2} Y)_i$, $r = \alpha$, and $a = 4 - \alpha$. Then we find that $p = 2 = q$, $V_j = \operatorname{Im} R_{jj}^{(N)}(z)$, and $U_{jk} = \operatorname{Im} R_{jk}^{(N)}(z)$ for each $1 \le j, k \le N-1$. We must next estimate the quantities V, $\boldsymbol{\mathcal{X}}$, and U from that lemma.

To that end, observe from (2.4.6) and (2.3.2) that

$$U \le \frac{4}{N^2} \sum_{1 \le i,j \le N-1} \big| \operatorname{Im} R_{ij}^{(N)}(z) \big|^2 \le \frac{4}{N^2 \eta} \sum_{j=1}^{N-1} \operatorname{Im} R_{jj}^{(N)}(z) \le \frac{4}{N \eta^2}. \tag{2.5.22}$$

Furthermore, since (2.4.14) (with $r = 1$) and (2.3.2) together imply (2.5.17), we obtain from the second estimate in (2.4.13), the assumption $\mathbb{E}\big[\operatorname{Im} m_N(z)\big] \geq \varepsilon$, and the fact that $V \geq N^{-1} \sum_{j=1}^{N-1} \operatorname{Im} R_{jj}^{(N)}(z)$ that

$$\mathbb{P}\left[V \leq \frac{\varepsilon}{2} \mathbf{1}_{\Lambda_N(z)}\right] \leq 2 \exp\big(- (\log N)^2\big), \tag{2.5.23}$$

for sufficiently large N (depending on ε), which in particular by (2.5.22) implies that

$$\mathbb{P}\big[V \leq 100 (\log N)^{10} U^{1/2}\big] \leq 2 \exp\big(- (\log N)^2\big). \tag{2.5.24}$$

To upper bound $\mathcal{X}$, first observe from (2.4.5) and the definition (2.5.20) of the event $\Lambda_N(z)$ that $\big|R_{jj}^{(N)}(z)\big| \mathbf{1}_{\Lambda_N(z)} \leq \varsigma_1^{-1}$. Therefore, for sufficiently large N,

$$\mathcal{X} \mathbf{1}_{\Lambda_N(z)} = \frac{\mathbf{1}_{\Lambda_N(z)}}{N-1} \sum_{j=1}^{N-1} \big(\operatorname{Im} R_{jj}^{(N)}(z)\big)^{2-\alpha/2} \leq \frac{2 \mathbf{1}_{\Lambda_N(z)}}{N \varsigma_1^{1-\alpha/2}} \sum_{j=1}^{N-1} \operatorname{Im} R_{jj}^{(N)}(z). \tag{2.5.25}$$

Therefore, (2.5.25), (2.4.14) (applied with $r = 1$), the second estimate in (2.4.13), and the assumption that $\mathbb{E}\big[\operatorname{Im} m_N(z)\big] \leq \frac{1}{\varepsilon}$ imply that for sufficiently large N

$$\mathbb{P}\left[\mathcal{X} \mathbf{1}_{\Lambda_N(z)} > \frac{4}{\varepsilon \varsigma_1^{1-\alpha/2}}\right] \leq 2 \exp\big(- (\log N)^2\big). \tag{2.5.26}$$

Now (2.5.23), (2.5.24), (2.5.26), and Lemma 2.5.5 yield (after increasing C if necessary) that

$$\mathbb{P}\left[\frac{\|\mathbf{A}^{1/2} Y\|_\alpha^\alpha}{N} \leq \frac{\varepsilon^3 \varsigma_1^{1-\alpha/2} \mathbf{1}_{\Lambda_N(z)}}{C (\log N)^8}\right] \leq C \exp\left(- \frac{(\log N)^2}{2}\right). \tag{2.5.27}$$

Inserting (2.5.27) into (2.5.21), we obtain (again after increasing C if necessary) that

$$\begin{aligned} &\mathbb{P}\big[\operatorname{Im}(S_N - T_N) < \varsigma_0 \mathbf{1}_{\Lambda_N(z)}\big] \\ &\qquad \leq C \exp\left(- \frac{\varepsilon^3 \varsigma_1^{1-\alpha/2}}{C \varsigma_0^{\alpha/2} (\log N)^8}\right) + C \exp\left(- \frac{(\log N)^2}{C}\right) + C \exp\left(- \frac{(\log N)^2}{2}\right), \end{aligned}$$

from which we deduce the proposition since $\varsigma_0^{\alpha/2} = \varsigma_1^{1-\alpha/2} (\log N)^{-10\alpha}$ (due to (2.5.4)) (after increasing C so the bound holds for all N). $\square$

Proof of Lemma 2.5.5

In this section we establish Lemma 2.5.5. Before doing so, however, we require the following (likely known) estimate for sums of squares of correlated Gaussian random variables.

Lemma 2.5.6. *Let N be a positive integer, and let $\mathbf{g} = (g_1, g_2, \ldots, g_N)$ denote an N-dimensional centered Gaussian random variable with covariance matrix $\mathbf{C} = \{c_{ij}\}$. Define $\mathbf{a} = (a_1, a_2, \ldots, a_N) \in \mathbb{R}_{\geq 0}$ by $a_j^2 = c_{jj}$ for each $j \in [1, N]$. Then, for sufficiently large N,*

$$\mathbb{P}\Bigg[\big| \|\mathbf{g}\|_2^2 - \|\mathbf{a}\|_2^2 \big| \geq 50 (\log N)^{10} \bigg(\sum_{1 \leq j,k \leq N} c_{jk}^2 \bigg)^{1/2} \Bigg] \leq \exp\big(- (\log N)^2 \big).$$

Proof. Let $\mathbf{w} = (w_1, w_2, \ldots, w_N)$ be an N-dimensional centered Gaussian random variable with covariance matrix given by Id. Let $\mathbf{D}$ and $\mathbf{U}$ be diagonal and orthogonal matrices, respectively, such that $\mathbf{C} = \mathbf{U}\mathbf{D}\mathbf{U}^{-1}$. Then $\mathbf{g}$ has the same law as $\mathbf{U}\mathbf{D}^{1/2}\mathbf{w}$, which implies that $\|\mathbf{g}\|_2^2$ has the same law as $\sum_{j=1}^N d_j w_j^2$. Moreover,

$$\sum_{j=1}^N a_j^2 = \operatorname{Tr} \mathbf{C} = \operatorname{Tr} \mathbf{D} = \sum_{j=1}^N d_j, \qquad \sum_{1 \leq j,k \leq N} c_{jk}^2 = \operatorname{Tr} \mathbf{C}^2 = \operatorname{Tr} \mathbf{D}^2 = \sum_{j=1}^N d_j^2,$$

so that

$$\begin{aligned} \mathbb{P}\Bigg[\big| \|\mathbf{g}\|_2^2 - \|\mathbf{a}\|_2^2 \big| &\geq 50 (\log N)^{10} \bigg(\sum_{1 \leq j,k \leq N} c_{jk}^2 \bigg)^{1/2} \Bigg] \\ &= \mathbb{P}\Bigg[\bigg| \sum_{j=1}^N d_j (w_j^2 - 1) \bigg| \geq 50 (\log N)^{10} \bigg(\sum_{j=1}^N d_j^2 \bigg)^{1/2} \Bigg]. \end{aligned} \tag{2.5.28}$$

Now, since the $\{w_j^2 - 1\}$ are mutually independent, the fact that the right side of (2.5.28) is bounded by $\exp\big(- (\log N)^2 \big)$ is standard. For instance, it can be deduced by truncating each $d_j(w_j^2 - 1)$ at $4 d_j \log N$ and then applying the Azuma–Hoeffding inequality. $\square$

Proof of Lemma 2.5.5. First observe that

$$\left(\frac{1}{N} \sum_{j=1}^{N} |w_j|^r \right)^{1/p} \left(\frac{1}{N} \sum_{j=1}^{N} |w_j|^a \right)^{1/q} \geq \frac{1}{N} \sum_{j=1}^{N} |w_j|^2. \tag{2.5.29}$$

We must therefore provide an upper bound on the a-th moments of the w_j and a lower bound on the second moments. To that end, observe that since each w_j is a Gaussian random variable of variance V_j, we have that

$$\mathbb{P}\left[\frac{1}{N} \sum_{j=1}^{N} |w_j|^a \geq 16 \mathcal{X} (\log N)^8 \right] \leq \sum_{j=1}^{N} \mathbb{P}\big[|w_j| \geq 2 (\log N)^2 V_j^{1/2} \big] \leq CN \exp\big(- (\log N)^2 \big). \tag{2.5.30}$$

Furthermore, by Lemma 2.5.6, we have that

$$\mathbb{P}\left[\left| \frac{1}{N} \sum_{j=1}^{N} |w_j|^2 - V \right| \geq 50 (\log N)^{10} U^{1/2} \right] \leq \exp\big(- (\log N)^2 \big). \tag{2.5.31}$$

Now the lemma follows from combining (2.5.29), (2.5.30), (2.5.31), and the assumption that $V > 100(\log N)^{10} U^{1/2}$. □

2.5.2 Establishing Theorem 2.4.1

In this section we prove Theorem 2.4.1. We first establish Proposition 2.4.11 in Section 2.5.2. Then, we will show that Theorem 2.4.1 holds when $|z|$ is sufficiently large in Section 2.5.2; we will establish Theorem 2.4.1 for more general z in Section 2.5.2.

Proof of Proposition 2.4.11

In this section we establish Proposition 2.4.11. To that end, denote

$$J(z) = \mathbb{E}\Big[\big(- \mathrm{i} z - \mathrm{i} S_j(z) \big)^{-1} \Big], \quad I(z) = \mathbb{E}\Big[\big(- \mathrm{i} z - \mathrm{i} S_j(z) \big)^{-\alpha/2} \Big].$$

We begin by showing that $Y(z)$ is approximately equal to $I(z)$ and that $X(z)$ is approximately equal to $J(z)$ (recall (2.4.25)), assuming that $\mathbb{P}\big[\Lambda(z)^c\big]$ is small.

Lemma 2.5.7. *Let $z \in \mathcal{D}_{K,\varpi,\mathfrak{B}}$ for some compact interval $K \subset \mathbb{R}$ and some $\mathfrak{B} > 0$. If $\mathbb{P}\big[\Lambda(z)^c\big] < \frac{1}{N^{10}}$, then there exists a large constant $C = C(\alpha, b, \delta) > 0$ such that*

$$\big|Y(z) - I(z)\big| \le \frac{C(\log N)^{70/(\alpha-1)}}{(N\eta^2)^{\alpha/8}}, \quad \big|X(z) - J(z)\big| \le \frac{C(\log N)^{70/(\alpha-1)}}{(N\eta^2)^{\alpha/8}}. \tag{2.5.32}$$

Proof. In this proof, we will abbreviate $S_1 = S_1(z)$, $T_1 = T_1(z)$, and $R_{11} = R_{11}(z)$. To bound $|Y(z) - I(z)|$, we apply (2.4.11), (2.4.7) (with $x = z + S_1$, $y = z + S_1 - T_1$, and $p = -\frac{\alpha}{2}$), and (2.3.2) to obtain for any $v > 0$ that

$$\begin{aligned}
&\Big|\big(-\mathrm{i}z - \mathrm{i}S_1\big)^{-\alpha/2} - \big(-\mathrm{i}R_{11}\big)^{\alpha/2}\Big| \\
&\quad \le \frac{\alpha}{2}|T_1|\left(\left|\frac{1}{z+S_1}\right|^{\alpha/2+1} + \left|\frac{1}{z+S_1-T_1}\right|^{\alpha/2+1}\right)\mathbf{1}_{|T_1|\le v}\mathbf{1}_{\Lambda(z)} \\
&\quad + \left(\left|\frac{1}{z+S_1}\right|^{\alpha/2} + \left|\frac{1}{z+S_1-T_1}\right|^{\alpha/2}\right)\mathbf{1}_{|T_1|>v}\mathbf{1}_{\Lambda(z)} + \left(\left|\frac{1}{z+S_1}\right|^{\alpha/2} + \left|\frac{1}{z+S_1-T_1}\right|^{\alpha/2}\right)\mathbf{1}_{\Lambda(z)^c} \\
&\quad \le \alpha v(\log N)^{60/(\alpha-1)}\mathbf{1}_{\Lambda(z)} + 2(\log N)^{30/(\alpha-1)}\mathbf{1}_{|T_1|>v}\mathbf{1}_{\Lambda(z)} + \frac{2}{\eta}\mathbf{1}_{\Lambda(z)^c}.
\end{aligned} \tag{2.5.33}$$

Setting $v = (N\eta^2)^{-1/4}$ in (2.5.33), taking expectations, using Proposition 2.4.9 to bound $\mathbb{P}\big[|T_1| > v\big]$, and applying our assumed estimate $\mathbb{P}\big[\Lambda(z)^c\big] < \frac{1}{N^{10}}$ yields

$$\mathbb{E}\left[\Big|\big(-\mathrm{i}z - \mathrm{i}S_1\big)^{-\alpha/2} - \big(-\mathrm{i}R_{11}\big)^{\alpha/2}\Big|\mathbf{1}_{\Lambda(z)}\right] \le \frac{6(\log N)^{70/(\alpha-1)}}{(N\eta^2)^{\alpha/8}},$$

from which we deduce the first estimate in (2.5.32). The proof of the second estimate in (2.5.32) is entirely analogous and therefore omitted. □

We now estimate the error resulting in replacing the entries of $\mathbf{X}$ with those of $\mathbf{H}$.

Lemma 2.5.8. *There exists a large constant $C = C(\alpha) > 0$ such that*

$$\mathbb{P}\big[|S_i - \mathfrak{S}_i| \geq N^{-4\theta}\big] < CN^{-4\theta}\Big(1 + \mathbb{E}\big[|R_{11}|\big]\Big).$$

Proof. Let $q > 0$ be a real parameter, which will be chosen later. Fix i, and let $\mathcal{B}$ denote the event that for every $1 \leq j \leq N$ with $j \neq i$, $|H_{ij}| < N^q$ and $|Z_{ij}| < N^q$. By the hypotheses on the tail behavior of the H_{ij} stated in Definition 3.1.1 and a union bound,

$$\mathbb{P}\left[\mathcal{B}\right] \geq 1 - CN^{-q\alpha}, \tag{2.5.34}$$

for some constant $C = C(\alpha) > 0$. We now work on the set $\mathcal{B}$. Due to the coupling between $\mathbf{X}$ and $\mathbf{H}$ (of Definition 3.2.3),

$$\begin{aligned}\mathbb{E}\big[\mathbf{1}_{\mathcal{B}}|S_i - \mathfrak{S}_i|\big] &\leq \mathbb{E}\left[\mathbf{1}_{\mathcal{B}}\sum_{j\neq i}|Z_{ij}^2 - X_{ij}^2|\big|R_{jj}^{(i)}\big|\right] \\ &\leq \sum_{j\neq i}\mathbb{E}\left[\mathbf{1}_{\mathcal{B}}|Z_{ij}^2 - H_{ij}^2| + \mathbf{1}_{\mathcal{B}}|H_{ij}^2 - X_{ij}^2|\right]\mathbb{E}\Big[\big|R_{jj}^{(i)}\big|\Big].\end{aligned} \tag{2.5.35}$$

In this calculation, we used the independence of H_{ij}, Z_{ij}, and X_{ij} from $R_{jj}^{(i)}$. To estimate the right side of (2.5.35), we take expectations in (2.4.14) applied with $r = 1$ to obtain

$$\left|\mathbb{E}\Big[\big|R_{jj}^{(i)}\big|\Big] - \mathbb{E}\big[|R_{jj}|\big]\right| \leq \frac{10}{N\eta},$$

where we used the exchangeability of the R_{jj} and (2.3.2). Also, from (3.1.3) and Definition 3.2.3 we compute (after increasing C if necessary)

$$\mathbb{E}\left[|H_{ij}^2 - X_{ij}^2|\right] = \mathbb{E}\left[H_{ij}^2\mathbf{1}_{H_{ij}<N^{b-1/\alpha}}\right] \leq CN^{2b-2/\alpha-b\alpha}.$$

Similarly, we compute (again after increasing C if necessary)

$$\begin{aligned}\mathbb{E}\left[\mathbf{1}_{\mathcal{B}}|Z_{ij}^2 - H_{ij}^2|\right] \le \mathbb{E}\Big[\mathbf{1}_{\mathcal{B}}\big(2|Z_{ij}||J_{ij}| + |J_{ij}|^2\big)\Big] &\le C\left(N^{q-q\alpha/2-1/2-1/\alpha} + N^{-2/\alpha}\right)\\ &\le 2CN^{(1-\alpha/2)(q-1/\alpha)-1},\end{aligned}$$

where in the second inequality we used $\mathbb{E}\left[\mathbf{1}_{\mathcal{B}}|Z||J|\right] \le \sqrt{\mathbb{E}\left[\mathbf{1}_{\mathcal{B}}|Z|^2\right]\mathbb{E}\left[\mathbf{1}_{\mathcal{B}}|J|^2\right]}$. We therefore deduce from (2.4.26) and (2.5.35) that, after gaining a factor of N due to the sum over j and choosing $q = \frac{1}{4}$,

$$\mathbb{E}\big[\mathbf{1}_{\mathcal{B}}|S_i - \mathfrak{S}_i|\big] \le CN^{-10\theta}\left(\frac{10}{N\eta} + \mathbb{E}\big[|R_{11}|\big]\right).$$

We conclude from a Markov estimate that

$$\mathbb{P}\big[\mathbf{1}_{\mathcal{B}}|S_i - \mathfrak{S}_i| \ge N^{-4\theta}\big] < CN^{-4\theta}\Big(1 + \mathbb{E}\big[|R_{11}|\big]\Big), \tag{2.5.36}$$

for sufficiently large N. The claim now follows from (2.5.34) and (2.5.36). □

Given Lemma 2.5.8, the proof of the following corollary is very similar to that of Lemma 2.5.7 given Proposition 2.4.9. Therefore, we omit its proof.

Corollary 2.5.9. *Let $p \in (0,1]$ and $z \in \mathcal{D}_{K,\varpi,\mathfrak{B}}$ for some compact interval $K \subset \mathbb{R}$ and some $\mathfrak{B} > 0$. If $\mathbb{P}\big[\Lambda(z)^c\big] < \frac{1}{N^{10}}$, then there exists a large constant $C = C(\alpha, b, \delta, \varepsilon, p) > 0$ such that*

$$\mathbb{E}\Big[\Big|\big(-\mathrm{i}z - \mathrm{i}S_1(z)\big)^{-p} - \big(-\mathrm{i}z - i\mathfrak{S}_1(z)\big)^{-p}\Big|\Big] \le \frac{C(\log N)^{100/(\alpha-1)}}{N^{2\theta}}.$$

We can now establish Proposition 2.4.11.

Proof of Proposition 2.4.11. Given what we have done, the proof of this proposition will be similar to that of Proposition 3.1 in [33]. Specifically, defining

$$\gamma(z) = \mathbb{E}[(-\mathrm{i}z - i\mathfrak{S}_1)^{-\alpha/2}], \qquad \Xi(z) = \mathbb{E}[(-\mathrm{i}z - i\mathfrak{S}_1)^{-1}],$$

we have from Corollary B.2 of (see in particular equation (31) of) [33] that

$$\begin{aligned} \gamma(z) &= \mathbb{E}\Bigg[\varphi_{\alpha,z}\bigg(\frac{1}{N}\sum_{j=2}^{N}\big(-\mathrm{i}R_{jj}^{(1)}\big)^{\alpha/2}\frac{|g_j|^\alpha}{\mathbb{E}\big[|g_j|^\alpha\big]}\bigg)\Bigg], \\ \Xi(z) &= \mathbb{E}\Bigg[\psi_{\alpha,z}\bigg(\frac{1}{N}\sum_{j=2}^{N}\big(-\mathrm{i}R_{jj}^{(1)}\big)^{\alpha/2}\frac{|g_j|^\alpha}{\mathbb{E}\big[|g_j|^\alpha\big]}\bigg)\Bigg], \end{aligned} \tag{2.5.37}$$

where $\mathbf{g} = (g_2, g_3, \ldots, g_N)$ denotes an $(N-1)$-dimensional centered Gaussian random variable with covariance matrix given by Id that is independent from $\mathbf{H}$, and $\mathbb{E}$ denotes the expectation with respect to both $\mathbf{H}$ and $\mathbf{g}$.

We will only establish the first estimate in (2.4.28) (on $\big|Y(z) - \varphi_{\alpha,z}\big(Y(z)\big)\big|$). The proof of the second is entirely analogous and is therefore omitted. To that end, set $\rho_j = \big(-\mathrm{i}R_{jj}^{(1)}(z)\big)^{\alpha/2}$ for each $2 \le j \le N$.

We will show that $\gamma(z) \approx \varphi_{\alpha,z}\big(\mathbb{E}[\rho_2]\big)$ and $\mathbb{E}[\rho_2] \approx Y$, and then use Corollary 2.5.9 and Lemma 2.5.7 to deduce that $I(z) \approx \gamma(z)$ and $Y(z) \approx I(z)$, respectively. To implement the first task, observe from (2.5.37) that

$$\begin{aligned}
&\Big|\gamma(z) - \varphi_{\alpha,z}\big(\mathbb{E}[\rho_2]\big)\Big| \\
&\quad = \Bigg|\mathbb{E}\Bigg[\varphi_{\alpha,z}\Big(\frac{1}{N}\sum_{j=2}^{N}\big(-\mathrm{i}R_{jj}^{(1)}\big)^{\alpha/2}\frac{|g_j|^\alpha}{\mathbb{E}\big[|g_j|^\alpha\big]}\Big)\Bigg] - \varphi_{\alpha,z}\big(\mathbb{E}[\rho_2]\big)\Bigg| \\
&\quad \le \mathbb{E}\Bigg[c_\varphi\Bigg|\frac{1}{N}\sum_{j=2}^{N}\big(-\mathrm{i}R_{jj}^{(1)}\big)^{\alpha/2}\frac{|g_j|^\alpha}{\mathbb{E}\big[|g_j|^\alpha\big]} - \mathbb{E}[\rho_2]\Bigg|\Bigg] \\
&\quad \le \frac{c_\varphi}{\mathbb{E}\big[|g_j|^\alpha\big]}\mathbb{E}\Bigg[\Bigg|\frac{1}{N-1}\sum_{j=2}^{N}\rho_j|g_j|^\alpha - \frac{1}{N-1}\mathbb{E}\Big[\sum_{j=2}^{N}\rho_j|g_j|^\alpha\Big]\Bigg|\Bigg] + \frac{c_\varphi\mathbb{E}\big[|\rho_2|\big]}{N} \\
&\quad \le \frac{2c_\varphi}{N\mathbb{E}\big[|g_j|^\alpha\big]}\Bigg(\mathbb{E}\Bigg[\Big|\sum_{j=2}^{N}\rho_j|g_j|^\alpha - \sum_{j=2}^{N}\rho_j\mathbb{E}\big[|g_j|^\alpha\big]\Big|\Bigg] + \mathbb{E}\big[|g_j|^\alpha\big]\mathbb{E}\Bigg[\Big|\sum_{j=2}^{N}\big(\rho_j - \mathbb{E}[\rho_j]\big)\Big|\Bigg]\Bigg) + \frac{c_\varphi}{N\eta},
\end{aligned} \tag{2.5.38}$$

where to deduce the first estimate we used the fact (from Lemma 2.4.8) that $\varphi_{\alpha,z}$ is Lipschitz with constant c_φ and the fact that $\mathbb{E}[\rho_j]$ is independent of $j \in [2, N]$, and to deduce the third estimate we used (2.3.2).

Now recall that by the Cauchy–Schwarz inequality, $\mathbb{E}\,[|X|] \le \mathbb{E}[X^2]^{1/2}$ for a centered random variable X, so

$$\frac{1}{N}\Bigg(\mathbb{E}\Bigg[\Big|\sum_{j=2}^{N}\rho_j|g_j|^\alpha - \sum_{j=2}^{N}\rho_j\mathbb{E}\big[|g_j|^\alpha\big]\Big|\Bigg]\Bigg) \le \frac{1}{N}\Bigg(\Big|\sum_{j=2}^{N}|\rho_j|^2\mathbb{E}\big[|g_j|^{2\alpha}\big] - \sum_{j=2}^{N}|\rho_j|^2\mathbb{E}\big[|g_j|^\alpha\big]^2\Big|\Bigg)^{1/2}. \tag{2.5.39}$$

Furthermore, Lemma 2.4.4 with t replaced by $(N\eta^2)^{-\alpha/4}t(\log N)^2$ yields the existence of a large constant $C = C(\alpha) > 0$ such that

$$\mathbb{P}\Bigg[\frac{1}{N}\sum_{j=2}^{N}\big(\rho_j - \mathbb{E}[\rho_j]\big) > \frac{t(\log N)^2}{(N\eta^2)^{\alpha/4}}\Bigg] \le C\exp\Bigg(-\frac{t^2(\log N)^2}{C}\Bigg), \tag{2.5.40}$$

for each $t \geq 1$. Integrating (2.5.40) yields

$$\mathbb{E}\left[\frac{1}{N}\sum_{j=2}^{N}\big(\rho_j - \mathbb{E}[\rho_j]\big)\right] \leq \frac{C(\log N)^2}{(N\eta^2)^{\alpha/4}}, \tag{2.5.41}$$

after increasing C if necessary. Combining (2.5.38), (2.5.39), and (2.5.41) yields (again upon increasing C if necessary)

$$\Big|\gamma(z) - \varphi_{\alpha,z}\big(\mathbb{E}[\rho_2]\big)\Big| \leq \frac{Cc_\varphi \mathbb{E}\big[|\rho_2|^2\big]^{1/2}}{N^{1/2}} + \frac{Cc_\varphi(\log N)^2}{(N\eta^2)^{\alpha/4}} + \frac{c_\varphi}{N\eta} \leq \frac{2Cc_\varphi(\log N)^2}{(N\eta^2)^{\alpha/4}}, \tag{2.5.42}$$

where in the second estimate we used the fact that $\big|\rho_2\big|^2 \leq \eta^{-\alpha} \leq \eta^{-2}$ (due to (2.3.2)).

To show that $\mathbb{E}[\rho_2] \approx Y(z)$, we apply (2.3.2), (2.4.14) with $r = \frac{\alpha}{2}$, and the exchangeability of the entries of $\mathbf{X}$, and then take expectations to find

$$\big|\mathbb{E}[\rho_2] - Y(z)\big| \leq \frac{5}{(N\eta)^{\alpha/2}}. \tag{2.5.43}$$

From (2.5.42), (2.5.43), and the fact that $\varphi_{\alpha,z}$ is Lipschitz with constant c_φ, we deduce that

$$\Big|\gamma(z) - \varphi_{\alpha,z}\big(Y(z)\big)\Big| \leq \frac{c_\varphi C(\log N)^2}{(N\eta^2)^{\alpha/8}}, \tag{2.5.44}$$

upon increasing C if necessary.

Now, by Corollary 2.5.9 (with $p = \frac{\alpha}{2}$) and Lemma 2.5.7 we have (again after increasing C if necessary) that

$$\big|I(z) - \gamma(z)\big| \leq \frac{C(\log N)^{100/(\alpha-1)}}{N^{4\theta}}, \qquad \big|Y(z) - I(z)\big| \leq \frac{C(\log N)^{70/(\alpha-1)}}{(N\eta^2)^{\alpha/4}}. \tag{2.5.45}$$

Now the first estimate in (2.4.28) follows from (2.5.44) and (2.5.45). $\square$

Proof of Theorem 2.4.1 for large $|z|$

In this section we establish Theorem 2.4.1 if $|z|$ is sufficiently large. We begin by addressing the case of large η, given by the following lemma.

Lemma 2.5.10. *Adopt the notation of Theorem 2.2.4. There exist constants $C = C(\alpha, b) > 0$ and $\mathfrak{B} = \mathfrak{B}(\alpha) > 0$ such that (2.4.2) holds for some $\varkappa > 0$.*

Proof. From the definition (2.1.5) of $m_\alpha(z)$, we deduce that

$$\Big|\mathbb{E}\big[m_N(z)\big] - m_\alpha(z)\Big| \le \Big|X(z) - \psi_{\alpha,z}\big(Y(z)\big)\Big| + \Big|\psi_{\alpha,z}\big(Y(z)\big) - \psi_{\alpha,z}\big(y(z)\big)\Big|. \tag{2.5.46}$$

In view of Lemma 2.4.8, there exists a large constant $\mathfrak{B} = \mathfrak{B}(\alpha) > 1$ such that for any $z \in \mathbb{H}$ with $|z| \ge \mathfrak{B}$ we have that $\max\{c_\varphi, c_\psi\} < \frac{1}{2}$. Thus, let $E \in \mathbb{R}$ and let $z = E + \mathrm{i}\mathfrak{B}$. Then,

$$\begin{aligned} \big|Y(z) - y(z)\big| &\le \big|Y(z) - \varphi_{\alpha,z}(Y(z))\big| + \big|\varphi_{\alpha,z}(Y(z)) - \varphi_{\alpha,z}(y(z))\big| \\ &\le \big|Y(z) - \varphi_{\alpha,z}(Y(z))\big| + \frac{\big|Y(z) - y(z)\big|}{2}, \end{aligned}$$

which implies that

$$\big|Y(z) - y(z)\big| \le 2\big|Y(z) - \varphi_{\alpha,z}\big(Y(z)\big)\big|. \tag{2.5.47}$$

By (2.4.12), $\Lambda(z)$ holds deterministically. Thus we can apply Proposition 2.4.11 (and (2.5.47)) to bound the right side of (2.5.46). This yields the existence of a large constant $C = C(\alpha, b, \varkappa) > 0$ such that

$$\begin{aligned} \big|Y(z) - y(z)\big| &\le C(\log N)^{100/(\alpha-1)} \left(\frac{1}{(N\eta^2)^{\alpha/8}} + \frac{1}{N^{2\theta}}\right) \le \frac{2C(\log N)^{100/(\alpha-1)}}{N^{\varkappa}}, \\ \Big|\mathbb{E}\big[m_N(z)\big] - m_\alpha(z)\Big| &\le C(\log N)^{100/(\alpha-1)} \left(\frac{1}{(N\eta^2)^{\alpha/8}} + \frac{1}{N^{2\theta}}\right) \le \frac{2C(\log N)^{100/(\alpha-1)}}{N^{\varkappa}}. \end{aligned} \tag{2.5.48}$$

Now the lemma follows from (2.5.48), the first estimate in (2.4.13), and the deterministic estimate $\big|R_{ij}(z)\big| \le \frac{1}{\eta} < 1$. $\square$

The following proposition analyzes the case when $\operatorname{Re} E$ is large.

Proposition 2.5.11. *Let $\mathfrak{B}$ be as in Lemma 2.5.10. There exists a large constant $E_0 = E_0(\alpha) > 0$ such that, for any compact interval $K = [u, v]$ disjoint from $[-E_0, E_0]$, there exists a large constant $C = C(\alpha, b, u, v, \delta) > 0$ and absolute constant $c > 0$ such that the following holds. Suppose $E \in [u, v]$ and $z_0, z \in \mathcal{D}_{[u,v],\varpi,\mathfrak{B}}$ satisfy $\operatorname{Re} z_0 = E = \operatorname{Re} z$ and $\operatorname{Im} z_0 - \frac{1}{N^5} \le \operatorname{Im} z \le \operatorname{Im} z_0$, and $\varkappa < c$. If $\mathbb{P}\big[\Omega(z_0)^c\big] < \frac{1}{N^{20}}$, then*

$$\mathbb{P}\big[\mathbf{1}_{\Omega(z)} < \mathbf{1}_{\Omega(z_0)}\big] \le C \exp\left(-\frac{(\log N)^2}{C}\right) \tag{2.5.49}$$

for large enough N.

Proof. First, recall that, since $m_\alpha(z)$ is the Stieltjes transform of a probability measure μ_α whose density is bounded and whose support is $\mathbb{R}$ (see Proposition 1.1 of [24]), for any $\mathfrak{B} > 0$ there exists a small constant $\varepsilon = \varepsilon(u, v, \mathfrak{B}) > 0$ such that

$$\varepsilon < \sup_{w \in \mathcal{D}_{[u,v],\varpi,\mathfrak{B}}} \operatorname{Im} m_\alpha(w) < \frac{1}{\varepsilon}. \tag{2.5.50}$$

Now, we claim that

$$\mathbb{P}\left[\mathbf{1}_{\Omega(z_0)}\Big|\operatorname{Im} m_N(z) - \operatorname{Im} m_\alpha(z)\Big| > \frac{2}{N^\varkappa}\right] \le 2\exp\big(-(\log N)^2\big). \tag{2.5.51}$$

Indeed, (2.5.51) follows from the fact that $\mathbf{1}_{\Omega(z_0)}\big|m_N(z_0) - \operatorname{Im} m_\alpha(z_0)\big| < N^{-\varkappa}$, the fact that $\big|m_N(z) - m_N(z_0)\big| < \frac{2}{N}$ since $|z - z_0| < \frac{1}{N^5}$ (from (2.4.8)), the fact that $\big|m_\alpha(z) - m_\alpha(z_0)\big| \le \frac{2}{N}$ (since m_α is the Stieltjes transform of the probability measure μ_α), and the second estimate in (2.4.13).

In particular, (2.5.50) and (2.5.51) imply that

$$\mathbb{P}\left[2\varepsilon \mathbf{1}_{\Omega(z_0)} \leq \mathbf{1}_{\Omega(z_0)} \operatorname{Im} m_N(z) < \frac{1}{2\varepsilon}\right] \geq 1 - 2\exp\big(-(\log N)^2\big) \tag{2.5.52}$$

Next, as in the proof of Lemma 2.5.10, Lemma 2.4.8 implies the existence of a large constant $E_0 = E_0(\alpha) > 0$ such that if $z \in \mathbb{H}$ satisfies $|z| > E_0$ then $\max\{c_\varphi, c_\psi\} < \frac{1}{2}$. Recalling $X(z)$ and $Y(z)$ from (2.4.25) and following the proof of Lemma 2.5.10, we deduce that

$$\begin{aligned} \Big|\mathbb{E}\big[m_N(z)\big] - m_\alpha(z)\Big| &\leq \Big|X(z) - \psi_{\alpha,z}\big(Y(z)\big)\Big| + \Big|\psi_{\alpha,z}\big(Y(z)\big) - \psi_{\alpha,z}\big(y(z)\big)\Big|, \\ \big|Y(z) - y(z)\big| &\leq 2\big|Y(z) - \varphi_{\alpha,z}\big(Y(z)\big)\big|. \end{aligned} \tag{2.5.53}$$

Observe that the hypotheses of Proposition 2.4.11 are satisfied for z; this is because (2.5.52) and Proposition 2.5.1, together with the trivial bound $|m_N(z)| \leq \eta^{-1}$ on the set $\Omega(z_0)^c$, imply that $\mathbb{P}[\Lambda(z)^c] < \mathbb{P}\big[\Omega(z_0)\big] + \frac{1}{N^{20}} < \frac{1}{N^{10}}$ for large enough N. Then we can use Proposition 2.4.11 to estimate the terms appearing on the right side of (2.5.53). Since $c_\psi < \frac{1}{2}$, this yields the existence of a large constant $C = C(\alpha, b, u, v, \varkappa) > 0$ such that

$$\begin{aligned} \mathbf{1}_{\Omega(z_0)}\big|Y(z) - y(z)\big| &\leq C(\log N)^{100/(\alpha-1)}\left(\frac{1}{(N\eta^2)^{\alpha/8}} + \frac{1}{N^{2\theta}}\right) \leq \frac{2C(\log N)^{100/(\alpha-1)}}{N^{\varkappa}}, \\ \mathbf{1}_{\Omega(z_0)}\Big|\mathbb{E}\big[m_N(z)\big] - m_\alpha(z)\Big| &\leq C(\log N)^{100/(\alpha-1)}\left(\frac{1}{(N\eta^2)^{\alpha/8}} + \frac{1}{N^{2\theta}}\right) \leq \frac{2C(\log N)^{100/(\alpha-1)}}{N^{\varkappa}}. \end{aligned} \tag{2.5.54}$$

Therefore, the first estimate in (2.4.13) and the second estimate in (2.5.54) together imply that

$$\mathbb{P}\left[\mathbf{1}_{\Omega(z_0)}\Big|m_N(z) - m_\alpha(z)\Big| > \frac{1}{N^{\varkappa}}\right] \leq 2\exp\left(-\frac{(\log N)^2}{8}\right). \tag{2.5.55}$$

Furthermore, observe that (2.4.11), (2.5.52), and Proposition 2.5.1 together yield

$$\mathbb{P}\left[\mathbf{1}_{\Omega(z_0)} \max_{1\le j\le N} \big|R_{jj}(z)\big| > (\log N)^{30/(\alpha-1)}\right] < C\exp\left(-\frac{(\log N)^2}{C}\right). \tag{2.5.56}$$

Now (2.5.49) follows from the first estimate in (2.5.54), (2.5.55), and (2.5.56). □

Bootstrap for small energies

Let E_0 and $\mathfrak{B}$ be as in Proposition 2.5.11; in this section we establish the analog of that proposition when $|E| \le E_0 + 1$. To that end, let $\mathcal{S} = \mathcal{S}_\alpha$ denote the set of $x \in \mathbb{C}$ with $\operatorname{Re} x \in K$ and $\operatorname{Im} x \in [0, \mathfrak{B}]$ such that $\varphi'_{\alpha,z}(x) - 1 = 0$. Recall from either Lemma 6.2 of [18] or equation (3.17) of [24] that if $z \ne 0$ there exists an entire function $g(x) = g_\alpha(x)$ such that $\varphi_{\alpha,z}(x) = Cz^{-\alpha}g(x)$. Therefore, since K is a compact interval that does not contain 0, $\mathcal{S}_\alpha$ is finite.

Thus the implicit function theorem yields the existence of some integer $M = M(\alpha, K) > 0$ (corresponding to the order of the largest zero of $\varphi'_{\alpha,z} - 1$ in $\mathcal{S}_\alpha$), a small constant $c = c(\alpha, K, \mathfrak{B}) > 0$, and a large constant $C = C(\alpha, K, \mathfrak{B}) > 0$ such that the following holds. If $z \in \mathbb{H}$ satisfies $\operatorname{Re} z \in K$ and $\operatorname{Im} z \le \mathfrak{B}$, then for any $t > 0$ and $w \in \mathbb{C}$,

$$\big|w - y(z)\big| \le c \quad \text{and} \quad \big|w - \varphi_{\alpha,z}(w)\big| \le t \quad \text{together imply} \quad \big|w - y(z)\big| \le Ct^{1/M}. \tag{2.5.57}$$

Now we can establish the following proposition that establishes (2.4.3) when $|\operatorname{Re} z| \le E_0 + 1$.

Proposition 2.5.12. *Let $\varkappa = \varkappa(\alpha, \delta, K) = \frac{\delta}{20M}$. For any compact interval $K = [u, v] \subset \mathbb{R}$ that does not contain 0, there exists a large constant $C = C(\alpha, b, u, v, \varkappa) > 0$ such that the following holds. Suppose $E \in [u, v]$ and $z_0, z \in \mathcal{D}_{[u,v],\varpi,\mathfrak{B}}$ satisfy* $\operatorname{Re} z_0 = E = \operatorname{Re} z$ *and*

$\operatorname{Im} z_0 - \frac{1}{N^5} \leq \operatorname{Im} z \leq \operatorname{Im} z_0$. *If* $\mathbb{P}\big[\Omega(z_0)\big] < \frac{1}{N^{20}}$, *then*

$$\mathbb{P}\big[\mathbf{1}_{\Omega(z)} < \mathbf{1}_{\Omega(z_0)}\big] < C \exp\left(-\frac{(\log N)^2}{C}\right).$$

Proof. Since $\operatorname{Re} z = \operatorname{Re} z_0$, $\operatorname{Im} z_0 - \frac{1}{N^5} \leq \operatorname{Im} z \leq \operatorname{Im} z_0$, continuity estimates for $Y(z)$ and $y(z)$ (see, for instance, equation (39) of [33]) imply that $\big|Y(z) - Y(z_0)\big| + \big|y(z) - y(z_0)\big| \leq \frac{1}{N}$. Therefore, since $\big|Y(z_0) - y(z_0)\big| \mathbf{1}_{\Omega(z_0)} \leq N^{-\varkappa}$, it follows that $\big|Y(z) - y(z)\big| \mathbf{1}_{\Omega(z_0)} \leq 2N^{-\varkappa}$ for N sufficiently large.

Thus (2.5.57) implies the existence of a large constant $C = C(\alpha, u, v) > 0$ such that

$$\mathbf{1}_{\Omega(z_0)} \big|Y(z) - y(z)\big| \leq C \Big|Y(z) - \varphi_{\alpha,z}\big(Y(z)\big)\Big|^{\frac{1}{M}} \mathbf{1}_{\Omega(z_0)}. \tag{2.5.58}$$

Following the reasoning used to establish (2.5.53) in the proof of Proposition 2.5.11, we obtain

$$\begin{aligned} \mathbf{1}_{\Omega(z_0)} \Big|\mathbb{E}\big[m_N(z)\big] - m_\alpha(z)\Big| &\leq \left(\Big|X(z) - \psi_{\alpha,z}\big(Y(z)\big)\Big| + \Big|\psi_{\alpha,z}\big(Y(z)\big) - \psi_{\alpha,z}\big(y(z)\big)\Big|\right) \mathbf{1}_{\Omega(z_0)} \\ &\leq \left(\Big|X(z) - \psi_{\alpha,z}\big(Y(z)\big)\Big| + c_\psi \big|Y(z) - y(z)\big|\right) \mathbf{1}_{\Omega(z_0)}. \end{aligned} \tag{2.5.59}$$

Having established the estimates (2.5.58) and (2.5.59), the remainder of the proof of this proposition is very similar to that of Proposition 2.5.11 after (2.5.53) and is therefore omitted. $\square$

Using the results above, we can now establish Theorem 2.4.1.

Proof of Theorem 2.4.1. The estimate (2.4.2) follows from Lemma 2.5.10. Furthermore, Proposition 2.5.11 establishes the existence of a large constant $E_0 = E_0(\alpha)$ such that (2.4.3) holds when $|\operatorname{Im} z| = |\operatorname{Im} z_0| > E_0$ and $\varkappa < c$. Then Proposition 2.5.12 implies (2.4.3) when $|\operatorname{Re} z| = |\operatorname{Re} z_0| \leq E_0 + 1$ and $\varkappa = \frac{\delta}{20M}$. Together these yield Theorem 2.4.1. $\square$

2.6 Intermediate local law for almost all $\alpha \in (0, 2)$ at small energies

In this section and in Section 2.7 we establish Theorem 2.2.5 (in fact the slightly more general Theorem 2.6.6 below), which provides a local law at sufficiently small energies for the removal matrix $\mathbf{X}$ for almost all $\alpha \in (0, 2)$. In Section 2.6.1 we state the local law (given by Theorem 2.6.6 below) and an estimate (Theorem 2.6.8) that implies the local law. We will then establish Theorem 2.6.8 in Section 2.6.2.

However, before doing this, let us recall some notation. In what follows we fix parameters $\alpha \in (1, 2)$ and $0 < b < \frac{1}{\alpha}$; we recall the removal matrix $\mathbf{X}$ and its resolvent $\mathbf{R}$ from Definition 3.2.3; we recall $m_N(z) = N^{-1} \operatorname{Tr} \mathbf{R}$; and we recall the domain $\mathcal{D}_{C,\delta}$ from (2.2.4). Furthermore, we denote by $\mathbb{K}$ the set of $z \in \mathbb{C}$ with $\operatorname{Re} z > 0$, and we set $\mathbb{K}^+ = \overline{\mathbb{K} \cap \mathbb{H}}$ to be the closure of the positive quadrant of the complex plane. We also let $\mathbb{S}^1$ be the unit circle, consisting of all $z \in \mathbb{C}$ with $|z| = 1$, and we define the closure $\mathbb{S}^1_+ = \overline{\mathbb{K}^+ \cap \mathbb{S}}$.

2.6.1 An estimate for the intermediate local law

In this section we state the local law for $\mathbf{X}$ on scales $N^{\delta - 1/2}$ (Theorem 2.6.6 below) and an estimate (Theorem 2.6.8) that implies it; this will be done in Section 2.6.1. However, we will first define a certain inner product and metric in Section 2.6.1 that will be required to define a family of fixed point equations in Section 2.6.1.

Inner Product and Metric

In order to establish a convergence result for $m_N(z)$ (which is approximately equal to $\mathbb{E}[R_{ii}]$), we in fact must understand the convergence of more general expectations, including the fractional moments $\mathbb{E}\big[(-\mathrm{i}R_{jj})^p\big]$, the absolute moments $\mathbb{E}\big[|R_{jj}|^p\big]$, and the imaginary moments

$\mathbb{E}\big[|\operatorname{Im} R_{jj}|^p\big]$. To facilitate this, we define for any $u, v \in \mathbb{C}$, the inner product

$$(u \,|\, v) = u \operatorname{Re} v + \overline{u} \operatorname{Im} v = \operatorname{Re} u\big(\operatorname{Re} v + \operatorname{Im} v\big) + \mathrm{i} \operatorname{Im} u\big(\operatorname{Re} v - \operatorname{Im} v\big).$$

In particular, for any $u, v \in \mathbb{C}$, we have that

$$(u \,|\, 1) = u, \qquad (-\mathrm{i}u \,|\, e^{\pi \mathrm{i}/4}) = \operatorname{Im} u \sqrt{2}, \qquad \Big|\big(u \,|\, v\big)\Big| \le 2|u||v|. \tag{2.6.1}$$

We will attempt to simultaneously understand the quantities $A_z(u) = \mathbb{E}\Big[\big((-\mathrm{i}R_{ii})^{\alpha/2} \,|\, u\big)\Big]$ for all $u \in \mathbb{K}^+$. Our reason for this (as opposed to only considering the cases $u = 1$ and $u = e^{\pi \mathrm{i}/4}$) is that the absolute moments $\mathbb{E}\big[|R_{jj}|^p\big]$ will be expressed as an integral of a function of $A_z(u)$ over u (see the definitions (2.6.7) of J_p and $r_{p,z}$ and also the second estimate in (2.6.16) below); this was implemented in [34].

To explain this fixed point equation further, we require a metric space of functions. To that end, for any $w \in \mathbb{C}$, we let $\mathcal{H}_w$ denote the space of $\mathcal{C}^1$ functions $g : \mathbb{K}^+ \to \mathbb{C}$ such that $g(\lambda u) = \lambda^w g(u)$ for each $\lambda \in \mathbb{R}_{\ge 0}$. Following equation (10) of [34], we define for any $r \in [0, 1)$ a norm on $\mathcal{H}_r$ by

$$\|g\|_\infty = \sup_{u \in \mathbb{S}^1_+} \big|g(u)\big|, \qquad \|g\|_r = \|u\|_\infty + \sup_{u \in \mathbb{S}^1_+} \sqrt{\big|(\mathrm{i} \,|\, u)^r \partial_1 g(u)\big|^2 + \big|(\mathrm{i} \,|\, u)^r \partial_2 g(u)\big|^2},$$

where $\partial_1 g(x + \mathrm{i}y) = \partial_x g(x + \mathrm{i}y)$ and $\partial_2 g(x + \mathrm{i}y) = \partial_y g(x + \mathrm{i}y)$. Observe in particular that

$$\sup_{u \in \mathbb{S}^1_+} \big|g(u)\big| \le \big\|g\|_r, \qquad \text{for any } r > 0. \tag{2.6.2}$$

We let $\mathcal{H}_{w,r}$ be the completion of $\mathcal{H}_w$ with respect to the $\|g\|_r$ norm. Further define for any $\delta > 0$ the subset $\mathcal{H}^\delta_{w,r} \subset \mathcal{H}_{w,r}$ consisting of all $g \in \mathcal{H}_{w,r}$ such that $\operatorname{Re} g(u) > \delta$ for all $u \in \mathbb{S}^1_+$, and define $\mathcal{H}^0_{w,r} = \bigcup_{\delta > 0} \mathcal{H}^\delta_{w,r}$. Further abbreviate $\mathcal{H}^\delta_w = \mathcal{H}^\delta_{w,0}$.

The following stability lemma, which appears as Lemma 5.2 of [34], will be useful to us.

Lemma 2.6.1 ([34, Lemma 5.2])**.** *Assume that $r \in (0,1)$ and $u \in \mathbb{S}^1_+$. Let $x_1, x_2 \in \mathbb{K}^+$, and let $a \in (0,1)$ be such that $|x_1|, |x_2| \le a^{-1}$. Set $F_k(u) = \big(x_k \,|\, u\big)^r$ for each $k \in \{1,2\}$. Then, there exists a constant $C = C(r) > 0$ such that for any $s \in (0,r)$, we have that for any $k \in \{1,2\}$,*

$$\|F_k\|_{1-r+s} \le C|x_k|^r, \qquad \|F_1 - F_2\|_{1-r+s} \le Ca^{-r}\big(|x_1 - x_2|^r + a^s|x_1 - x_2|^s\big). \tag{2.6.3}$$

If we further assume that $\operatorname{Re} x_1, \operatorname{Re} x_2 \ge t$ *for some $t > 0$, and we set $G_k(u) = \big(x_k^{-1} \,|\, u\big)^r$ for each $k \in \{1,2\}$, then there exists a constant $C = C(r) > 0$ such that*

$$\|G_1 - G_2\|_{1-r+s} \le Ct^{r-2}a^{2r-1}|x_1 - x_2|. \tag{2.6.4}$$

Equations for m

Following Section 3.2 of [34] (or Section 5.1 of [33]), define for any complex numbers $u \in \mathbb{S}^1_+$ and $h \in \overline{\mathbb{K}}$, and any function $g \in \mathcal{H}_{\alpha/2}$, the function

$$F_{h,g}(u) = \int_0^{\pi/2} \left(\int_0^{\infty} \left(\int_0^{\infty} \left(e^{-r^{\alpha/2} g(e^{\mathrm{i}\theta}) - (rh \,|\, e^{\mathrm{i}\theta})} - e^{-r^{\alpha/2} g(e^{\mathrm{i}\theta} + uy) - (yrh \,|\, u) - (rh \,|\, e^{\mathrm{i}\theta})} \right) r^{\alpha/2-1} dr \right) \times y^{-\alpha/2-1} dy \right) (\sin 2\theta)^{\alpha/2-1} d\theta.$$

It was shown as Lemma 4.1 of [34] that $F_{h,g} \in \mathcal{H}_{\alpha/2,r}$ if $g \in \mathcal{H}^0_{\alpha/2,r}$, and also that it is in the closure $\overline{H}^0_{\alpha/2,r}$ for any $g \in \overline{\mathcal{H}}^0_{\alpha/2,r}$ if $\operatorname{Re} h > 0$. As in equation (13) of [34], define the function

$$\Upsilon_f(u) = \Upsilon_{z,f}(u) = c_\alpha F_{-\mathrm{i}z,f}(\widetilde{u}), \quad \text{where } c = c_\alpha = \frac{\alpha}{2^{\alpha/2}\Gamma(\alpha/2)^2} \text{ and } \widetilde{u} = \mathrm{i}\overline{u}. \tag{2.6.5}$$

Observe that $(\overline{u}, v) = \overline{u} \operatorname{Re} v + u \operatorname{Im} v = (u, \widetilde{v})$. Now, for any $u \in \mathbb{C}$, define

$$\vartheta_z(u) = \Gamma\left(1 - \frac{\alpha}{2}\right) (-\mathrm{i} R_{jj} \,|\, u)^{\alpha/2}, \qquad \gamma_z(u) = \mathbb{E}\big[\vartheta_z(u)\big] = \Gamma\left(1 - \frac{\alpha}{2}\right) \mathbb{E}\big[(-\mathrm{i} R_{jj} \,|\, u)^{\alpha/2}\big], \tag{2.6.6}$$

for any $j \in [1, N]$; observe that $\gamma_z(u)$ does not depend on j due to the fact that the entries of $\mathbf{X}$ are identically distributed.

Furthermore, for any $p > 0$ and $f \in \mathcal{H}_{\alpha/2}$, define $I_p, J_p, r_{p,z}(f) \in \mathbb{C}$ and $s_{p,z} : \mathbb{K} \to \mathbb{C}$ by

$$\begin{aligned} I_p &= I_p(z) = \mathbb{E}\big[(-\mathrm{i} R_{jj})^p\big], \quad s_{p,z}(x) = \frac{1}{\Gamma(p)} \int_0^\infty y^{p-1} e^{-\mathrm{i} y z - x y^{\alpha/2}} \, dy, \\ J_p &= J_p(z) = \mathbb{E}\big[|\mathrm{i} R_{jj}|^p\big], \quad r_{p,z}(f) = \frac{2^{1-p/2}}{\Gamma(p/2)^2} \int_0^{\pi/2} \int_0^\infty y^{p-1} e^{(\mathrm{i} y z \,|\, e^{\mathrm{i}\theta}) - y^{\alpha/2} f(e^{\mathrm{i}\theta})} (\sin 2\theta)^{p/2-1} \, dy \, d\theta, \end{aligned} \tag{2.6.7}$$

for any $x \in \mathbb{K}$. The convergence of these integrals can quickly be deduced from the fact that $\operatorname{Re}(\mathrm{i}z) < 0$.

We now state four lemmas that can be found in [34]. The first two provide existence, stability, and estimates for the solution to a certain fixed point equation, while the latter two provide bounds and stability estimates for the functions F, $s_{p,z}$, $r_{p,z}$, and Υ.

Lemma 2.6.2 ([34, Proposition 3.3], [32, Lemma 4.3])**.** *There exists a countable subset $\mathcal{A} \subset (0, 2)$ with no accumulation points on $(0, 2)$ such that, for any $r \in (0, 1]$ and $\alpha \in (0, 2) \setminus \mathcal{A}$, there exists a constant $c = c(\alpha, r) > 0$ with the following property.*

There exists a unique function $\Omega_0 \in \mathcal{H}_{\alpha/2}$ such that $\Omega_0 = \Upsilon_{0,\Omega_0}$. Additionally, if $\operatorname{Im} z > 0$ and $|z| \le c$, then there is a unique function $f = \Omega_z \in \mathcal{H}_{\alpha/2,r}$ that solves $f = \Upsilon_{z,f}$ with $\|f - \Omega_0\|_r \le c$. Moreover, this function satisfies $\Omega_z(e^{\pi \mathrm{i}/4}) \ge c$ and, for any $p > 0$, there exists a constant $C = C(\alpha, p) > 0$ such that $r_{p,z}(\Omega_z) \le C$.

Lemma 2.6.3 ([34, Proposition 3.4])**.** *Adopt the notation of Lemma 2.6.2. After decreasing c if necessary, there exists a constant $C > 0$ such that the following holds. If $\operatorname{Im} z > 0$,*

$|z| \leq c$, and $\|f - \Omega_z\|_r \leq c$, then

$$\|f - \Omega_z\|_r \leq C \big\| f - \Upsilon_{z;f} \big\|_r.$$

The following stability properties of F_h and Υ will be useful to us later.

Lemma 2.6.4 ([34, Lemma 4.1]). *Let $r \in (0,1)$ and $p > 0$. There exists a constant $C = C(\alpha, p, r) > 0$ such that, for any $g \in \overline{\mathcal{H}}^0_{\alpha/2,r}$ and $h \in \mathbb{K}$, we have that*

$$\begin{gathered} \|F_h(g)\|_r \leq C(\operatorname{Re} h)^{-\alpha/2} + C\|g\|_r (\operatorname{Re} h)^{-\alpha/2}, \\ \big| r_{p,\mathrm{i}h}(g) \big| \leq C(\operatorname{Re} h)^{-p}, \qquad \big| s_{p,\mathrm{i}h}\big(g(1)\big| \leq C(\operatorname{Re} h)^{-p}. \end{gathered} \tag{2.6.8}$$

Lemma 2.6.5 ([34, Lemma 4.3]). *For any fixed $a, r > 0$, there exists a constant $C = C(\alpha, a, r)$ such that for any $f, g \in \mathcal{H}^a_{\alpha/2,r}$ and $z \in \mathbb{C}$, we have that*

$$\|\Upsilon_f - \Upsilon_g\|_r \leq C\|f - g\|_r + \|f - g\|_\infty \big(\|f\|_r + \|g\|_r \big). \tag{2.6.9}$$

Furthermore, for any $p > 0$ there exists a constant $C' = C'(\alpha, a, r, p)$ such that for any $f, g \in \mathcal{H}^a_{\alpha/2,r}$ and any $z \in \mathbb{C}$ and $x, y \in \mathbb{K}$ with $\operatorname{Re} x, \operatorname{Re} y \geq a$, we have that

$$\big| r_{p,z}(f) - r_{p,z}(g) \big| \leq C' \|f - g\|_\infty, \qquad \big| s_{p,z}(x) - s_{p,z}(y) \big| \leq C' |x - y|. \tag{2.6.10}$$

An intermediate local law for X

The following theorem provides a local law for $\mathbf{X}$.

Theorem 2.6.6. *There exists a countable set $\mathcal{A} \subset (0,2)$, with no accumulation points in $(0,2)$, such that the following holds. Fix $\alpha \in (0,2) \backslash \mathcal{A}$ and $0 < b < \frac{1}{\alpha}$. Denote $\theta = \frac{(b-1/\alpha)(2-\alpha)}{20}$ and fix some $\delta \in (0, \theta)$ with $\delta < \frac{1}{2}$. Then, there exists a constant $C = C(\alpha, b, \delta, p) > 0$ such* *that*

$$\mathbb{P}\left[\sup_{z\in\mathcal{D}_{C,\delta}}\left|m_N(z)-\mathrm{i}s_{1,z}\big(\Omega_z(1)\big)\right|>\frac{1}{N^{\alpha\delta/8}}\right]<C\exp\left(-\frac{(\log N)^2}{C}\right), \tag{2.6.11}$$

where we recall the definition of Ω_z *from Lemma 2.6.2. Furthermore, we have that*

$$\sup_{u\in\mathbb{S}^1_+}\left|\gamma_z(u)-\Omega_z(u)\right|\leq\frac{C}{N^{\alpha\delta/8}},\qquad\left|J_2-r_{2,z}(\Omega_z)\right|\leq\frac{C}{N^{\alpha\delta/8}}, \tag{2.6.12}$$

and

$$\mathbb{P}\left[\sup_{z\in\mathcal{D}_{C,\delta}}\max_{1\leq j\leq N}\left|R_{jj}(z)\right|>(\log N)^C\right]<C\exp\left(-\frac{(\log N)^2}{C}\right). \tag{2.6.13}$$

Remark 2.6.7. One can show that the fixed point equations (2.1.5) and Lemma 2.6.2 defining $m_\alpha(z)$ and Ω_z, respectively, are equivalent when $u=1$; this implies that $\mathrm{i}s_{1,z}\big(\Omega_z(1)\big)=m_\alpha(z)$.

Theorem 2.6.6 is a consequence of the following theorem (whose proof will be given in Section 2.6.2 below), which is similar to Proposition 3.2 of [34] but with two main differences. The first is that Theorem 2.6.8 below establishes estimates on the scale $\eta\gg N^{-1/2}$, while the corresponding estimate in [34] was shown with $\eta\gg N^{-\alpha/(2+\alpha)}$. The second is that we also establish estimates on each $\left|R_{jj}\right|$, which was not pursued in [34]. In fact, these bounds on the resolvent entries (which follow as consequences of Proposition 2.6.9 and Proposition 2.6.10 below) are partially what allow us to improve the scale from $\eta\gg N^{-\alpha/(2+\alpha)}$ in [34] to $\eta\gg N^{-1/2}$ here.

Theorem 2.6.8. *Fix* $\alpha\in(0,2)$, $0<b<\frac{1}{\alpha}$, $s\in\left(0,\frac{\alpha}{2}\right)$, $p>0$, $\varepsilon\in(0,1]$, *and a positive integer* N. *Define* $\theta=\frac{(b-1/\alpha)(2-\alpha)}{10}$, *and suppose that* $z=E+\mathrm{i}\eta\in\mathbb{H}$ *with* $E,\eta\in\mathbb{R}$. *Assume that*

$$\eta\geq N^{\varepsilon-s/\alpha},\quad|z|<\frac{1}{\varepsilon},\quad\mathbb{E}\big[(\operatorname{Im}R_{11})^{\alpha/2}\big]\geq\varepsilon,\quad\mathbb{E}\big[|R_{11}|^2\big]\leq\varepsilon^{-1}. \tag{2.6.14}$$

Then, there exists a constant $C = C(\alpha, \varepsilon, b, s, p) > 0$ such that

$$\|\gamma_z - \Upsilon_{\gamma_z}\|_{1-\alpha/2+s} \le C(\log N)^C \left(\frac{1}{(N\eta^2)^{\alpha/8}} + \frac{1}{N^\theta} + \frac{1}{N^s \eta^{\alpha/2}} \right), \tag{2.6.15}$$

and

$$\begin{aligned} \big| I_p - s_{p,z}\big(\gamma_z(1)\big)\big| &\le C(\log N)^C \left(\frac{1}{(N\eta^2)^{\alpha/8}} + \frac{1}{N^\theta} + \frac{1}{N^s \eta^{\alpha/2}} \right), \\ \big| J_p - r_{p,z}(\gamma_z)\big| &\le C(\log N)^C \left(\frac{1}{(N\eta^2)^{\alpha/8}} + \frac{1}{N^\theta} + \frac{1}{N^s \eta^{\alpha/2}} \right). \end{aligned} \tag{2.6.16}$$

Furthermore,

$$\inf_{u \in \mathbb{S}^1_+} \operatorname{Im} \gamma_z(u) \ge \frac{1}{C}, \tag{2.6.17}$$

and

$$\mathbb{P}\left[\max_{1 \le j \le N} \big|R_{jj}\big| > C(\log N)^C \right] < C \exp\left(-\frac{(\log N)^2}{C} \right). \tag{2.6.18}$$

Given Lemma 2.6.2, Lemma 2.6.3, and Theorem 2.6.8, the proof of Theorem 2.6.6 is very similar to the proof of Theorem 5.11 in Section 5.4 of [34] and is therefore omitted. However, let us briefly explain the idea of the proof, referring to [34] for the remaining details.

To that end, after proving Theorem 2.6.6 in the case when $\eta = \frac{1}{C}$ is of order 1, one first observes by (2.4.8) that it suffices to establish Theorem 2.6.6 for any individual z on a certain lattice. In particular, for a constant $C > 0$, let $A = A(C) = \lfloor \frac{2N^C}{C} \rfloor$ and $B = B(C) = \lfloor \frac{N^C - N^{C+\delta-1/2}}{C} \rfloor$, and define $z_{jk} = z_{j,k} = \frac{j}{N^C} - \frac{1}{C} + \mathrm{i}\big(\frac{1}{C} - \frac{k}{N^C}\big)$ for each $0 \le j \le A$ and $0 \le k \le B$.

If C is sufficiently large, it suffices to verify (2.6.11) and (3.3.12) for each z_{jk}. We will induct on k; the initial estimate states that they are true for $k = 0$. So let $M \in [1, B]$ be an integer, assume that the theorem holds for $k \le M - 1$, and let us establish it for $k = M$. To that end, we will apply Theorem 2.6.8 with $s = \frac{\alpha - \alpha\delta}{2}$ and $\varepsilon \leq \frac{\delta}{2}$.

To apply this theorem, we must verify (2.6.14). The first estimate there holds since $\eta \geq N^{\delta - 1/2}$, and the second holds for sufficiently small ε if $z \in \mathcal{D}_{C,\delta}$. The third follows from (2.4.8), the second statement of (2.6.1), the first estimate in (2.6.12) (applied with $z = z_{j,M-1}$ on the previous scale), and the lower bound on $\Omega_z(e^{\pi \mathrm{i}/4})$ provided by Lemma 2.6.2. The fourth estimate in (2.6.14) similarly follows from (2.4.8), the second estimate in (2.6.12) (applied with $z = z_{j,M-1}$ on the previous scale), and the upper bound on $r_{2,z}(\Omega_z)$ given by Lemma 2.6.2.

Thus, applying Theorem 2.6.8 yields that (2.6.15), (2.6.16), (2.6.17), and (2.6.18) all hold for $z = z_{jM}$; the last estimate implies (3.3.12). Furthermore, (2.6.15), (2.6.2), (2.4.8), the estimate (2.6.11) (applied with $z = z_{j,M-1}$ on the previous scale), and Lemma 2.6.3 together imply the first estimate in (2.6.12) for $z = z_{jM}$. Now (2.6.11) for $z = z_{jM}$ follows from the first estimate in (2.6.16) (applied with $p = 1$), the first estimate in (2.6.12), the first identity in (2.6.1), (2.6.17), the second estimate in (2.6.10), and the first estimate in (2.4.13). The second estimate in (2.6.12) for $z = z_{jM}$ follows from the second estimate in (2.6.16) (applied with $p = 2$), the first estimate in (2.6.12), (2.6.17), and the first estimate in (2.6.10).

2.6.2 Establishing Theorem 2.6.8

In this section we establish Theorem 2.6.8 assuming Proposition 2.6.9, Proposition 2.6.10, and Proposition 2.6.17; the latter results will be proven later, in Section 2.7.

Define

$$S_i = \sum_{j \neq i} X_{ij}^2 R_{jj}^{(i)}, \quad \text{and} \quad T_i = X_{ii} - U_i, \quad \text{where} \quad U_i = \sum_{\substack{j,k \neq i \\ j \neq k}} X_{ij} R_{jk}^{(i)} X_{ki}, \tag{2.6.19}$$

and observe that R_{ii} can be expressed in terms of T_i, z, and S_i through (2.4.11).

We begin in Section 2.6.2 by "removing T_i" from the equations defining R_{ii} by approximating functions of the R_{ii} by analogous functions of $\big(-z - S_i \big)^{-1}$. Next, in Section 2.6.2

we analyze the error in replacing all of the removal entries X_{ij} in the expression defining S_i with the original α-stable entries Z_{ij} (recall Definition 3.2.3). This will be useful for deriving approximate fixed point equations in Section 2.6.2, which we will use to conclude the proof of Theorem 2.6.8 in Section 2.6.2.

Removing T_i

Denoting

$$\omega_z(u) = \big((-\mathrm{i}z - \mathrm{i}S_i)^{-1} \,|\, u\big)^{\alpha/2}, \qquad \varpi_z(u) = \mathbb{E}\Big[\big((-\mathrm{i}z - \mathrm{i}S_i)^{-1} \,|\, u\big)^{\alpha/2}\Big], \tag{2.6.20}$$

we would like to show that $\gamma_z \approx \varpi_z$ and that other similar approximations hold; see Proposition 2.6.11 below. Such estimate which would follow from Proposition 2.4.9 if one could show that $\operatorname{Im}(S_i - T_i)$ and $\operatorname{Im} S_i$ could be bounded from below with overwhelming probability. The following two propositions, which will be proven in Section 2.7.2 and Section 2.7.3, establish the latter statement.

Proposition 2.6.9. *Adopt the notation of Theorem 2.6.8. There exists a large constant $C = C(\alpha, \varepsilon, b) > 1$ such that*

$$\mathbb{P}\left[\operatorname{Im} S_i < \frac{1}{C(\log N)^C}\right] < C \exp\left(-\frac{(\log N)^2}{C}\right). \tag{2.6.21}$$

Proposition 2.6.10. *Adopt the notation of Theorem 2.6.8. There exists a large constant $C = C(\alpha, \varepsilon, b) > 1$ such that*

$$\mathbb{P}\left[\operatorname{Im}(S_i - T_i) < \frac{1}{C(\log N)^C}\right] < C \exp\left(-\frac{(\log N)^2}{C}\right). \tag{2.6.22}$$

In particular, we have that

$$\mathbb{P}\left[\max_{1 \le j \le N} |R_{jj}| < C(\log N)^C\right] < C \exp\left(-\frac{(\log N)^2}{2C}\right). \tag{2.6.23}$$

The following proposition is a consequence of Proposition 2.4.9, Proposition 2.6.9, and Proposition 2.6.10; its proof will be similar to that of Lemma 2.5.7.

Proposition 2.6.11. *Adopt the notation of Theorem 2.6.8. Then, there exists a large constant $C = C(\alpha, \varepsilon, b, s, p) > 0$ such that*

$$\mathbb{E}\Big[\Big||R_{ii}|^p - \big|(-z - S_i)^{-1}\big|^p\Big|\Big] \le \frac{C(\log N)^C}{(N\eta^2)^{\alpha/8}}, \quad \mathbb{E}\Big[\big|(-\mathrm{i}R_{ii})^p - (-\mathrm{i}z - \mathrm{i}S_i)^{-p}\big|\Big] \le \frac{C(\log N)^C}{(N\eta^2)^{\alpha/8}}, \tag{2.6.24}$$

and

$$\big\|\gamma_z - \varpi_z\big\|_{1-\alpha/2+s} < \frac{C(\log N)^C}{(N\eta^2)^{\alpha/8}}, \tag{2.6.25}$$

where γ_z and ϖ_z are defined in (2.6.6) *and* (2.6.20)*, respectively.*

Proof. Let us first establish the first estimate in (2.6.24). The proof of the second is entirely analogous and is therefore omitted. To that end, observe from (2.4.5) and (2.4.7) that, for any $v > 0$, we have that

$$\begin{aligned}\Big||R_{ii}|^p - \big|(-z - S_i)^{-1}\big|^p\Big| \le (p-1)v\bigg(\bigg|\frac{1}{\operatorname{Im}(S_i - T_i + z)}\bigg|^{p+1} + \bigg|\frac{1}{\operatorname{Im}(z + S_i)}\bigg|^{p+1}\bigg)\mathbf{1}_{|T_i|<v} \\ + \bigg(\bigg|\frac{1}{\operatorname{Im}(S_i - T_i + z)}\bigg|^{p} + \bigg|\frac{1}{\operatorname{Im}(z + S_i)}\bigg|^{p}\bigg)\mathbf{1}_{|T_i|\ge v}.\end{aligned} \tag{2.6.26}$$

We will use Proposition 2.4.9, Proposition 2.6.9, and Proposition 2.6.10 to bound the expectation of the right side of (2.6.26). Let C_1, C_2, and C_3 denote the constants C from Proposition 2.4.9, Proposition 2.6.9, and Proposition 2.6.10 respectively. Also let E_1 denote the event on which $\inf_{1\le i\le N} \operatorname{Im} S_i < C_2^{-1}(\log N)^{-C_2}$, let E_2 denote the event on which $\inf_{1\le i\le N} \operatorname{Im}(S_i - T_i) < C_3^{-1}(\log N)^{-C_3}$, and let $E = E_1 \cup E_2$.

Now, using the deterministic estimate (2.4.12) and the fact that $\operatorname{Im} z = \eta \le N^{-1/2}$ to estimate the expectation of the right side of (2.6.26) on E, and using Proposition 2.6.9 and

Proposition 2.6.10 to estimate it off of E, yields

$$\begin{aligned}\mathbb{E}\Big[\Big||R_{ii}|^p - \big|(-z-S_i)^{-1}\big|^p\Big|\Big] \\ &\le (p-1)v\big(C_2^{p+1}(\log N)^{(p+1)C_2} + C_3^{p+1}(\log N)^{(p+1)C_3}\big) \\ &\quad + (p-1)v\big(C_2^{p+1}(\log N)^{(p+1)C_2} + C_3^{p+1}(\log N)^{(p+1)C_3}\big)\mathbb{P}\big[|T_i| \ge v\big] \\ &\quad + \big(N^{p/2} + (p-1)vN^{(p+1)/2}\big)\Bigg(\exp\bigg(-\frac{(\log N)^2}{C_2}\bigg) + \exp\bigg(-\frac{(\log N)^2}{C_3}\bigg)\Bigg).\end{aligned} \tag{2.6.27}$$

Setting $v = (N\eta^2)^{-1/4}$ in (2.6.27) together with the estimate on $\mathbb{P}\big[|T_i| \ge s\big]$ given by (2.4.17) (applied with $t = (N\eta^2)^{1/4}$) yields (2.6.24).

The proof of (2.6.25) is similar, except we now use Lemma 2.6.1. Recall the functions ϑ_z and ω_z from (2.6.6) and (2.6.20), respectively. Furthermore, recall the event E from above, and let F denote the complement of E. On F, we apply (2.6.4) with $x_1 = \mathrm{i}T_i - \mathrm{i}z - \mathrm{i}S_i$, $x_2 = -\mathrm{i}z - \mathrm{i}S_i$, $r = \alpha/2$, and $t = a = (C_2 + C_3)^{-1}(\log N)^{-C_2 - C_3}$ to obtain that

$$\begin{aligned}\big\|\vartheta_z - \omega_z\big\|_{1-\alpha/2+s}\mathbf{1}_F\mathbf{1}_{|T_i|\le v} &\le C(C_2+C_3)^{3-3\alpha/2}(\log N)^{(3-3\alpha/2)(C_2+C_3)}\left|\frac{1}{z+S_i-T_i} - \frac{1}{z+S_i}\right|\mathbf{1}_F \\ &\le C(C_2+C_3)^{5-3\alpha/2}(\log N)^{(5-3\alpha/2)(C_2+C_3)}v\mathbf{1}_F.\end{aligned} \tag{2.6.28}$$

Similarly, applying the first estimate in (2.6.3) with $x_1 = (\mathrm{i}T_i - \mathrm{i}z - \mathrm{i}S_i)^{-1}$, $x_2 = (-\mathrm{i}z - \mathrm{i}S_i)^{-1}$, $r = \alpha/2$, and $a = (C_2 + C_3)^{-1}(\log N)^{-C_2-C_3}$ yields the existence of a constant $C = C(\alpha) > 0$ such that

$$\begin{aligned}\big\|\vartheta_z - \omega_z\big\|_{1-\alpha/2+s}\mathbf{1}_F\mathbf{1}_{|T_i|>v} \\ &\le C\Big(\big|z+S_i-T_i\big|^{-1} + \big|z+S_i\big|^{-1}\Big)\mathbf{1}_F\mathbf{1}_{|T_i|>v} \le 2C(C_2+C_3)^{\alpha/2}(\log N)^{C_2+C_3}\mathbf{1}_F\mathbf{1}_{|T_i|>v}.\end{aligned} \tag{2.6.29}$$

Moreover, again using the first estimate in (2.6.3) with the same x_1, x_2, and r as above, but now with $a = \eta \geq N^{-1/2}$, we obtain that

$$\left\| \vartheta_z - \omega_z \right\|_{1-\alpha/2+s} \mathbf{1}_E \leq 2CN^{\alpha/4} \mathbf{1}_E. \tag{2.6.30}$$

Now (2.6.25) follows similarly to (2.6.24) as explained above. Set $v = (N\eta^2)^{-1/4}$ and sum (2.6.28), (2.6.29), and (2.6.30). Then apply (2.4.17) (with $t = (N\eta^2)^{1/4}$), Proposition 2.6.9, and Proposition 2.6.10. Finally, use the facts that $\varpi_z = \mathbb{E}\big[\omega_z(u)\big]$ and $\gamma_z(u) = \mathbb{E}\big[\vartheta_z(u)\big]$. $\square$

Replacing X

To facilitate the proof of Theorem 2.6.8, it will be useful to replace all of the X_{ij} with Z_{ij} (which we recall are coupled from Definition 3.2.3). To that end, we define

$$\begin{aligned} \mathfrak{S}_i = \sum_{j \neq i} Z_{ij}^2 R_{jj}^{(i)}, \qquad \Psi_z(u) = \Gamma\left(1 - \frac{\alpha}{2}\right) \left((-\mathrm{i}z - \mathrm{i}\mathfrak{S}_i)^{-1} \mid u \right)^{\alpha/2}, \\ \psi_z(u) = \mathbb{E}[\Psi_z] = \Gamma\left(1 - \frac{\alpha}{2}\right) \mathbb{E}\Big[\left((-\mathrm{i}z - \mathrm{i}\mathfrak{S}_i)^{-1} \mid u \right)^{\alpha/2} \Big]. \end{aligned} \tag{2.6.31}$$

We now have the following lemma that compares S_i and $\mathfrak{S}_i$. It is a quick consequence of Lemma 2.5.8 in Section 2.5.2 and our assumption that $\mathbb{E}\big[|R_{jj}|\big] \leq \mathbb{E}\big[|R_{jj}|^2\big]^{1/2} < \varepsilon^{-1/2}$.

Lemma 2.6.12. *Adopt the notation of Theorem 2.6.8. There exists a large constant $C = C(\alpha, b, \varepsilon) > 0$ such that*

$$\mathbb{P}\Big[\big|\mathfrak{S}_i - S_i\big| > N^{-4\theta}\Big] < CN^{-4\theta}. \tag{2.6.32}$$

The proof of the following proposition, which lower bounds $\operatorname{Im} \mathfrak{S}_i$, is very similar to that of Proposition 2.6.9 and is therefore omitted.

Proposition 2.6.13. *Adopt the notation of Theorem 2.6.8. There exists a large constant*

$C = (\alpha, \varepsilon) > 0$ *such that*

$$\mathbb{P}\left[\operatorname{Im} \mathfrak{S}_i < \frac{1}{C(\log N)^C}\right] \leq C \exp\left(-\frac{(\log N)^2}{C}\right). \tag{2.6.33}$$

Given Proposition 2.6.9, Lemma 2.6.12, and Proposition 2.6.13, the proof of the following proposition is similar to that of Proposition 2.6.11 and is therefore omitted.

Proposition 2.6.14. *Adopt the notation of Theorem 2.6.8. Then, there exists a constant* $C = C(\alpha, \varepsilon, b, s, p) > 0$ *such that*

$$\begin{aligned} \mathbb{E}\left[\left|\left|(-z - \mathfrak{S}_i)^{-1}\right|^p - \left|(-z - S_i)^{-1}\right|^p\right|\right] &\leq C(\log N)^C N^{-4\theta}, \\ \mathbb{E}\left[\left|(-\mathrm{i}z - \mathrm{i}\mathfrak{S}_i)^{-p} - (-\mathrm{i}z - \mathrm{i}S_i)^{-p}\right|\right] &\leq C(\log N)^C N^{-4\theta}, \end{aligned} \tag{2.6.34}$$

and

$$\left\|\psi_z - \varpi_z\right\|_{1-\alpha/2+s} < C(\log N)^C N^{-4\theta}, \tag{2.6.35}$$

where ϖ_z *and* ψ_z *are defined in* (2.6.20) *and* (2.6.31), *respectively.*

Approximate fixed point equations

In this section we establish several approximate fixed point equations for ψ_z. To that end, we begin with the following lemma, which appears as Corollary 5.8 of [34].

Lemma 2.6.15 ([34, Corollary 5.8])**.** *Fix* $\sigma > 0$, $\alpha \in (0, 2)$, $p > 0$, *and a positive integer* N. *Let* Z *be a* $(0, \sigma)$ α*-stable law, and let* $h_1, h_2, \ldots, h_N$ *be mutually independent, identically distributed random variables with laws given by* $N^{-1/\alpha} Z$. *Suppose that* $A_1, A_2, \ldots, A_N \in \mathbb{C}$

are complex numbers with nonnegative real part. Then, denoting

$$\mathcal{F}(u) = \Gamma\left(1 - \frac{\alpha}{2}\right) \mathbb{E}\left[\left(\left(\sum_{j=1}^{N} h_j^2 A_j - \mathrm{i}z\right)^{-1} \middle| u\right)^{\alpha/2}\right],$$

$$S_p = \mathbb{E}\left[\left(\sum_{j=1}^{N} h_j^2 A_j - \mathrm{i}z\right)^{-p}\right], \qquad R_p = \mathbb{E}\left[\left|\sum_{j=1}^{N} h_j^2 A_j - \mathrm{i}z\right|^{-p}\right],$$

we have that $\mathcal{F}(u) = \mathbb{E}\big[\Upsilon_{\mathfrak{Z}}\big]$, $S_p = \mathbb{E}\big[s_{p,z}(\mathcal{Z})\big]$, *and* $R_p = \mathbb{E}\big[r_{p,z}(\mathfrak{Z})\big]$, *where* Υ *is given by* (2.6.5) *and*

$$\mathfrak{Z} = \mathfrak{Z}(u) = \frac{2^{\alpha/2}\sigma^\alpha}{N} \sum_{j=1}^{N} \big(A_j \,|\, u\big)^{\alpha/2} |y_j|^\alpha, \qquad \mathcal{Z} = \mathfrak{Z}(1) = \frac{2^{\alpha/2}\sigma^\alpha}{N} \sum_{j=1}^{N} A_j^{\alpha/2} |y_j|^\alpha,$$

where $(y_1, y_2, \ldots, y_N)$ *is an* N*-dimensional centered Gaussian random variable whose covariance matrix is* Id.

Using Lemma 2.6.15, we can express a number of quantities of interest in terms of the function $\mathfrak{Z}$ above.

Corollary 2.6.16. *Recalling the definition of* Υ *from* (2.6.5) *and* Ψ_z *from* (2.6.31)*, we have that*

$$\Psi_z(u) = \mathbb{E}_{\mathfrak{Y}}\big[\Upsilon_{\mathfrak{Z}}\big], \tag{2.6.36}$$

where

$$\mathfrak{Z} = \mathfrak{Z}(u) = \frac{\Gamma\big(1 - \frac{\alpha}{2}\big)}{N-1} \sum_{j \neq i} \big(-\mathrm{i} R_{jj}^{(i)} \,|\, u\big)^{\alpha/2} \frac{|y_j|^\alpha}{\mathbb{E}\big[|y_j|^\alpha\big]}, \tag{2.6.37}$$

where $\mathfrak{Y} = (y_j)_{j \neq i}$ *is an* $(N-1)$*-dimensional centered real Gaussian random variable with covariance matrix given by* Id*. In* (2.6.36)*, the expectation is with respect to* $\mathfrak{Y}$.

Moreover, denoting $\mathcal{Z} = \mathfrak{Z}(1)$, *we have that*

$$\mathbb{E}\Big[\big(-\mathrm{i}z - \mathrm{i}\mathfrak{S}_i\big)^{-p}\Big] = \mathbb{E}_{\mathfrak{Y}}\big[s_{p,z}(\mathcal{Z})\big], \qquad \mathbb{E}\Big[\big|-z - \mathfrak{S}_i\big|^{-p}\Big] = \mathbb{E}_{\mathfrak{Y}}\big[r_{p,z}(\mathfrak{Z})\big]. \tag{2.6.38}$$

Proof. The identity (2.6.36) follows from the first statement of Lemma 2.6.15, applied with $h_j = X_{ij}$ and $A_j = -\mathrm{i}R_{jj}^{(i)}$, and also the fact that

$$2^{\alpha/2}\sigma^{\alpha} = \frac{2^{\alpha/2-1}\pi}{\sin\left(\frac{\pi\alpha}{2}\right)\Gamma(\alpha)} = \frac{\pi}{\sin\left(\frac{\pi\alpha}{2}\right)\Gamma\left(\frac{\alpha}{2}\right)\mathbb{E}\big[|y_j|^{\alpha}\big]} = \frac{\Gamma\left(1-\frac{\alpha}{2}\right)}{\mathbb{E}\big[|y_j|^{\alpha}\big]}. \tag{2.6.39}$$

To establish the first identity in (2.6.39) we used the definition (3.1.2) of σ, and to establish the second and third we used (2.4.24). The proof of (2.6.38) is entirely analogous, as a consequence of the second and third statements of Lemma 2.6.15, as well as (2.6.39). □

The following proposition, which will be proven in Section 2.7.4, states that $\mathfrak{Z}$ is approximately equal to γ_z. Thus, taking the expectation of both sides of (2.6.36), using the facts that $\psi_z = \mathbb{E}\big[\Psi_z\big]$ (recall (2.6.31)) and that γ_z is approximately equal to ψ_z (recall (2.6.25) and (2.6.35)), (2.6.36) yields an approximate fixed point equation for ψ_z.

Proposition 2.6.17. *Adopt the notation of Theorem 2.6.8. There exists a constant* $C = C(\alpha, \varepsilon, s) > 1$ *such that*

$$\mathbb{P}\left[\|\mathfrak{Z} - \gamma_z\|_{1-\alpha/2+s} > \frac{C(\log N)^C}{N^{s/2}\eta^{\alpha/2}}\right] < C\exp\left(-\frac{(\log N)^2}{C}\right). \tag{2.6.40}$$

Convergence to fixed points

In this section we establish Theorem 2.6.8. To that end, recall that (2.6.36) can be viewed as a fixed point equation for ψ_z. In order to analyze this fixed point equation, we require the following lemma.

Lemma 2.6.18. *Adopt the notation and assumptions of Theorem 2.6.8. There exists a* *constant* $C = C(\alpha, \varepsilon, s) > 1$ *such that*

$$\|\gamma_z\|_{1-\alpha/2+s} < C, \qquad \inf_{u \in \mathbb{S}^1_+} \operatorname{Re} \gamma_z(u) > \frac{1}{C}, \qquad \mathbb{P}\bigg[\inf_{u \in \mathbb{S}^1_+} \operatorname{Re} \mathfrak{Z}(u) < \frac{1}{C} \bigg] < C \exp \bigg(- \frac{(\log N)^2}{C} \bigg). \tag{2.6.41}$$

Proof. In view of (2.6.2) and Proposition 2.6.17, it suffices to only establish the first two estimates in (2.6.41) on γ_z. Let us first establish the upper bound. To that end, observe that the first statement of (2.6.3) implies the existence of a constant $C = C(s)$ such that

$$\Big\| \big(- \mathrm{i} R_{ii} \,|\, u \big)^{\alpha/2} \Big\|_{1-\alpha/2+s} \le C \big| R_{ii} \big|^{\alpha/2}. \tag{2.6.42}$$

Taking expectations in (2.6.42), using the definition (2.6.6) of γ_z, and using the fact that $\mathbb{E}\big[|R_{ii}|^2 \big] < \varepsilon^{-1}$, we deduce that

$$\begin{aligned} \|\gamma_z\|_{1-\alpha/2+s} &\le \Gamma \left(1 - \frac{\alpha}{2} \right) \mathbb{E} \Big[\big\| \big(- \mathrm{i} R_{ii} \,|\, u \big)^{\alpha/2} \big\|_{1-\alpha/2+s} \Big] \\ &\le C \Gamma \left(1 - \frac{\alpha}{2} \right) \mathbb{E} \big[|R_{ii}|^{\alpha/2} \big] \le C \Gamma \left(1 - \frac{\alpha}{2} \right) \mathbb{E} \big[|R_{ii}|^2 \big]^{\alpha/2} \le C \Gamma \left(1 - \frac{\alpha}{2} \right) \varepsilon^{-\alpha/2}, \end{aligned}$$

from which we deduce the first estimate in (2.6.41).

Now let us verify the lower bound on $\operatorname{Re} \gamma_z$. In that direction, observe that for any $u \in \mathbb{S}^1_+$, we have that

$$\begin{aligned} \operatorname{Re} \gamma_z(u) &= \Gamma \left(1 - \frac{\alpha}{2} \right) \mathbb{E} \big[\operatorname{Re} \big(- \mathrm{i} R_{ii} \,|\, u \big)^{\alpha/2} \big] \\ &\ge \Gamma \left(1 - \frac{\alpha}{2} \right) \mathbb{E} \big[\big(\operatorname{Re} (-\mathrm{i} R_{jj} \,|\, u) \big)^{\alpha/2} \big] \ge \Gamma \left(1 - \frac{\alpha}{2} \right) \mathbb{E} \big[(\operatorname{Im} R_{jj})^{\alpha/2} \big] \ge \Gamma \left(1 - \frac{\alpha}{2} \right) \varepsilon. \end{aligned}$$

The first identity above follows from the definition (2.6.6) of γ_z; the second follows from the fact that $\operatorname{Re} a^r \ge (\operatorname{Re} a)^r$ for any $a \in \mathbb{K}$ and $r \in (0, 1)$ (see Lemma 5.10 of [34]); the third follows from the fact that $\operatorname{Re} \big(a \,|\, u \big) \ge \operatorname{Re} a$ for any $u \in \mathbb{S}^1_+$ and $a \in \mathbb{K}^+$; and the fourth follows from our assumed lower bound on $\mathbb{E}\big[(\operatorname{Im} R_{jj})^{\alpha/2} \big]$. $\square$

Now we can deduce the following consequence of (2.6.36).

Corollary 2.6.19. *Adopt the notation of Theorem 2.6.8. There exists a constant* $C = C(\alpha, \varepsilon, s) > 0$ *such that*

$$\mathbb{P}\left[\|\psi_z - \Upsilon_{\gamma_z}\|_{1-\alpha/2+s} < \frac{C(\log N)^C}{N^{s/2}\eta^{\alpha/2}} \right] < C \exp\left(-\frac{(\log N)^2}{C} \right). \tag{2.6.43}$$

Proof. Let us first show that Υ_{γ_z} is approximately equal to $\Upsilon_{\mathfrak{z}}$ using Lemma 2.6.5. To verify the conditions of that lemma, first observe that $\gamma_z, \mathfrak{z} \in \mathcal{H}_{\alpha/2}$ since the inner product $(x \,|\, y)$ is bilinear. Furthermore, let C_1 denote the constant C from Proposition 2.6.17, and let C_2 denote the constant C from Lemma 2.6.18. Define the events

$$\begin{aligned} E_1 &= \left\{ \|\mathfrak{z} - \gamma_z\|_{1-\alpha/2+s} \geq \frac{C_1(\log N)^{C_1}}{N^{s/2}\eta^{\alpha/2}} \right\} \\ E_2 &= \left\{ \inf_{u \in \mathbb{S}^1_+} \operatorname{Re} \mathfrak{z}(u) \leq \frac{1}{C_2} \right\} \cup \left\{ \inf_{u \in \mathbb{S}^1_+} \operatorname{Re} \gamma_z(u) \leq \frac{1}{C_2} \right\} \cup \big\{ \|\gamma_z\|_{1-\alpha/2+s} \geq C_2 \big\}. \end{aligned} \tag{2.6.44}$$

Denoting $E = E_1 \cup E_2$, Proposition 2.6.17 and Lemma 2.6.18 together imply that

$$\mathbb{P}[E] \leq (C_1 + C_2) \exp\left(-\frac{(\log N)^2}{C_1 + C_2} \right). \tag{2.6.45}$$

Therefore, denoting the complement of E by F and applying (2.6.9) and (2.6.2) yields a constant $C > 1$ (only dependent on C_2 and s) such that

$$\begin{aligned} \mathbf{1}_F \big\| \Upsilon_{\mathfrak{z}} - \Upsilon_{\gamma_z} \big\|_{1-\alpha/2+s} &\leq C \mathbf{1}_F \|\mathfrak{z} - \gamma_z\|_{1-\alpha/2+s} + \mathbf{1}_F \|\mathfrak{z} - \gamma_z\|_\infty \big(\|\mathfrak{z}\|_{1-\alpha/2+s} + \|\gamma_z\|_{1-\alpha/2+s} \big) \\ &\leq \frac{CC_1(\log N)^{C_1}}{N^{s/2}\eta^{\alpha/2}} \left(1 + C_2 + \frac{C_1 C_2 (\log N)^{C_1}}{N^{s/2}\eta^{\alpha/2}} \right), \end{aligned} \tag{2.6.46}$$

where we have used the fact that $\mathfrak{z}$ and γ_z are in $\mathcal{H}^{1/C_2}_{\alpha/2, 1-\alpha/2+s}$ on the event F.

The estimate (2.6.46) bounds $\|\Upsilon_{\mathfrak{z}} - \Upsilon_{\gamma_z}\|_{1-\alpha/2+s}$ away from the event E; now let us bound it on E through a deterministic estimate. Using the first bound in Lemma 2.6.4 and the definition (2.6.5) of Υ in terms of F, we deduce that

$$\|\Upsilon_{\gamma_z}\|_{1-\alpha/2+s} \le C\eta^{-\alpha/2}\big(1 + \|\gamma_z\|_{1-\alpha/2+s}\big), \tag{2.6.47}$$

after enlarging C if necessary. Now, applying the first statement of (2.6.3) (with $x_1 = R_{ii}$, $r = \alpha/2$, and $a = \eta$) and (2.3.2), we have that

$$\|\gamma_z\|_{1-\alpha/2+s} \le C\Gamma\left(1 - \frac{\alpha}{2}\right)\left|R_{jj}\right|^{\alpha/2} \le C\Gamma\left(1 - \frac{\alpha}{2}\right)\eta^{-\alpha/2}. \tag{2.6.48}$$

Inserting (2.6.48) into (2.6.47) yields

$$\|\Upsilon_{\gamma_z}\|_{1-\alpha/2+s} \le 2C^2\eta^{-\alpha}\Gamma\left(1 - \frac{\alpha}{2}\right). \tag{2.6.49}$$

Furthermore, applying the definition (2.6.31) of Ψ_z, (2.3.2), and the first statement of (2.6.3) (now with $x_1 = -\mathrm{i}z - \mathrm{i}\mathfrak{S}$, $r = \alpha/2$, and $a = \eta$) yields that

$$\|\Psi_z\|_{1-\alpha/2+s} \le C\Gamma\left(1 - \frac{\alpha}{2}\right)\eta^{-\alpha/2}. \tag{2.6.50}$$

Combining (2.6.31), (2.6.46), (2.6.49), and (2.6.50) yields

$$\begin{aligned} \|\psi_z - \Upsilon_{\gamma_z}\|_{1-\alpha/2+s} &\le \mathbb{E}\big[\|\Psi_z - \Upsilon_{\gamma_z}\|_{1-\alpha/2+s}\big] \\ &\le \mathbb{E}\big[\mathbf{1}_F\|\Psi_z - \Upsilon_{\gamma_z}\|_{1-\alpha/2+s}\big] + \mathbb{E}\big[\mathbf{1}_E\|\Psi_z\|_{1-\alpha/2+s}\big] + \mathbb{E}\big[\mathbf{1}_E\|\gamma_z\|_{1-\alpha/2+s}\big] \\ &\le \frac{CC_1^2(C_2+2)(\log N)^{C_1}}{N^{s/2}\eta^{\alpha/2}} + C\Gamma\left(1 - \frac{\alpha}{2}\right)\eta^{-\alpha}(2C+1)\mathbb{P}[E]. \end{aligned} \tag{2.6.51}$$

For the first inequality, we used Jensen's inequality and the fact that all norms, in particular $\|\cdot\|_{1-\alpha/2+s}$, are convex. Now (2.6.43) follows from (2.6.45) and (2.6.51). $\square$

Now we can establish Theorem 2.6.8.

Proof of Theorem 2.6.8. The first estimate (2.6.15) follows from (2.6.25), (2.6.35), and (2.6.43). Furthermore, the fourth estimate (2.6.17) follows from the second estimate in (2.6.41); the fifth estimate (2.6.18) follows from (2.6.23).

The proofs of the two estimates given in (2.6.16) are similar, so let us only establish the latter. To that end, recall the notation from the proof of Corollary 2.6.19, and define the events E_1 and E_2 as in (2.6.44). As in the proof of Corollary 2.6.19, we let $E = E_1 \cup E_2$ and F be the complement of E.

Then, γ_z and $\mathbf{1}_F \mathfrak{z}$ are both in $\mathcal{H}^{1/C_2}_{\alpha/2,1-\alpha/2+s}$, so applying the first estimate in (2.6.10) and (2.6.2) yields a constant C' (only dependent on C_2, s, and p) such that

$$\mathbf{1}_F \big| r_{p,z}(\gamma_z) - r_{p,z}(\mathfrak{z}) \big| \le C' \sup_{u \in \mathbb{S}^1_+} \big| \gamma_z - \mathfrak{z} \big| \mathbf{1}_F \le C' \big\| \gamma_z - \mathfrak{z} \big\|_{1-\alpha/2+s} \mathbf{1}_F \le \frac{C' C_1 (\log N)^{C_1}}{N^{s/2} \eta^{\alpha/2}}. \tag{2.6.52}$$

The estimate (2.6.52) bounds $\big| r_{p,z}(\gamma_z) - r_{p,z}(\mathfrak{z}) \big|$ off of E. To bound it on E, we use the deterministic estimate given by the second inequality in (2.6.8). This yields the existence of a constant $C = C(\alpha, p, s)$ such that

$$\mathbf{1}_E \big| r_{p,z}(\gamma_z) - r_{p,z}(\mathfrak{z}) \big| \le \mathbf{1}_E \big| r_{p,z}(\gamma_z) \big| + \big| r_{p,z}(\mathfrak{z}) \big| \le 2C \eta^{-p} \mathbf{1}_E. \tag{2.6.53}$$

Combining the second equality in (2.6.38), (2.6.52), and (2.6.53) yields

$$\begin{aligned} \Big| r_{p,z}(\gamma_z) - \mathbb{E}\big[| -z - \mathfrak{S}_i |^p \big] \Big| &\le \mathbb{E}_{\mathfrak{Y}} \Big[\big| r_{p,z}(\gamma_z) - r_{p,z}(\mathfrak{z}) \big| \Big] \\ &= \mathbb{E}_{\mathfrak{Y}} \Big[\mathbf{1}_F \big| r_{p,z}(\gamma_z) - r_{p,z}(\mathfrak{z}) \big| \Big] + \mathbb{E}_{\mathfrak{Y}} \Big[\mathbf{1}_E \big| r_{p,z}(\gamma_z) - r_{p,z}(\mathfrak{z}) \big| \Big] \\ &\le \frac{C' C_1 (\log N)^{C_1}}{N^{s/2} \eta^{\alpha/2}} + 2C \eta^{-p} \mathbb{P}[E]. \end{aligned} \tag{2.6.54}$$

The second statement of (2.6.16) now follows from the first statement of (2.6.24), the first statement of (2.6.34), (2.6.45), and (2.6.54). $\square$

2.7 Estimates for the fixed point quantities

In this section we establish the estimates stated in the proof of Theorem 2.6.8 in Section 2.6.2. To that end, we first require some concentration estimates, which will be given in Section 2.7.1. We will then establish Proposition 2.6.9, Proposition 2.6.10, and Proposition 2.6.17 in Section 2.7.2, Section 2.7.3, and Section 2.7.4, respectively.

2.7.1 Concentration results

In this section, we collect concentration statements that will be used in the proofs of the estimates stated in Section 2.6.2. The first (which is an analog of Lemma 2.4.4) is Lemma 5.3 of [34], applied with their β equal to our $\frac{\alpha}{2}$ and their δ equal to our s.

Lemma 2.7.1 ([34, Lemma 5.3]). *Let N be a positive integer, let r and s be positive real numbers, and let $\mathbf{A} = \{a_{ij}\}_{1 \le i,j \le N}$ be an $N \times N$ symmetric random matrix such that the i-dimensional vectors $A_i = (a_{i1}, a_{i2}, \ldots, a_{ii})$ are mutually independent for $1 \le i \le N$. Let $z = E + \mathrm{i}\eta \in \mathbb{H}$, and denote $\mathbf{B} = \{B_{ij}\} = (\mathbf{A} - z)^{-1}$. Fix $u \in \mathbb{S}^1_+$, $\alpha \in (0,2)$, and $s \in \big(0, \frac{\alpha}{2}\big)$.*

Then, if we denote $f = f_u : \mathbb{C} \to \mathbb{C}$ by $f_u(z) = (\mathrm{i}z \,|\, u)^{\alpha/2}$, there exists a constant $C = C(\alpha) > 0$ such that

$$\mathbb{P}\left[\left\| \frac{1}{N}\sum_{j=1}^{N} f(B_{jj}) - \frac{1}{N}\sum_{j=1}^{N} \mathbb{E}\big[f(B_{jj})\big] \right\|_{1-\alpha/2+s} \ge t\right] \le C(\eta^{\alpha/2}t)^{-1/s} \exp\left(-\frac{N(\eta^{\alpha/2}t)^{2/s}}{C}\right).$$

The following (which is analog of Lemma 2.4.5) is a special case of Lemma 5.4 of [34], applied with their $\{g_j\}$ equal to our $\{y_j\}$; their $\{h_j\}$ equal to our $-\mathrm{i}R_{jj}$; their β equal to our $\frac{\alpha}{2}$; their δ equal to our s; and their t equal to $CN^{-s/2}\eta^{-\alpha/2}(\log N)^s$.

Lemma 2.7.2 ([34, Lemma 5.4]). *Let $(y_1, y_2, \ldots, y_N)$ be a Gaussian random vector whose covariance matrix is given by Id, let $s \in \big(0, \frac{\alpha}{2}\big)$, and for each $1 \le j \le N$ let*

$$f_j(u) = \big(-\mathrm{i}R_{jj}^{(i)} \,|\, u\big)^{\alpha/2} |y_j|^{\alpha}, \qquad g_j(u) = \big(-\mathrm{i}R_{jj}^{(i)} \,|\, u\big)^{\alpha/2} \mathbb{E}\big[|y_j|^{\alpha}\big].$$

Then, there exists a constant $C = C(\alpha) > 0$ that

$$\mathbb{P}\left[\left\| \frac{1}{N} \sum_{j=1}^{N} (f_j - g_j) \right\|_{1-\alpha/2+s} > \frac{C(\log N)^s}{N^{s/2}\eta^{\alpha/2}}\right] < \frac{CN^{1/2}}{\log N} \exp\left(-\frac{(\log N)^2}{C}\right), \tag{2.7.1}$$

where the expectation is with respect to $(y_1, y_2, \ldots, y_N)$ and conditional on $\mathbf{X}^{(i)}$.

2.7.2 Proof of Proposition 2.6.9

In this section we establish Proposition 2.6.9. Its proof will be similar to that of Proposition 2.5.2 in Section 2.5.1.

Proof of Proposition 2.6.9. Since all entries of $\mathbf{R}$ are identically distributed, we may assume that $i = N$. In what follows, let $\mathcal{E}$ denote the event on which

$$\big| \operatorname{Tr} \operatorname{Im} \mathbf{R}^{(N)} - \mathbb{E}[\operatorname{Im} R_{11}] \big| \leq \frac{4 \log N}{(N\eta^2)^{1/2}} + \frac{8}{N\eta}. \tag{2.7.2}$$

In view of Lemma 2.4.6 (applied with $r = 1$) and the second estimate in (2.4.13), we deduce that $\mathbb{P}\big[\mathcal{E}^c\big] \leq 2 \exp\big(-(\log N)^2\big)$, where $\mathcal{E}^c$ denotes the complement of $\mathcal{E}$.

We now apply Lemma 2.5.4 with $X = \big(X_{Nj}\big)_{j \neq N}$ and $\mathbf{A} = \{A_{ij}\}$ equal to the $(N-1) \times (N-1)$ diagonal matrix with $A_{jj} = \operatorname{Im} R_{jj}^{(N)}$. Then, $\operatorname{Im} S_N = \langle \mathbf{A}X, X \rangle$. Inserting $t = (\log N)^{2/\alpha}(2 \log 2)^{1/2}$ into Lemma 2.5.4, we find from a Markov estimate that

$$\begin{aligned}
&\mathbb{P}\big[\operatorname{Im} S_N < \mathbf{1}_{\mathcal{E}} (\log N)^{-4/\alpha}\big] \\
&\quad \leq 2\mathbb{E}\left[\mathbf{1}_{\mathcal{E}} \exp\left(-\frac{t^2}{2}\langle \mathbf{A}X, X \rangle\right)\right] \\
&\quad \leq 2\mathbb{E}\left[\mathbf{1}_{\mathcal{E}} \exp\left(-\frac{\sigma^\alpha (2\log 2)^{\alpha/2} (\log N)^2 \|\mathbf{A}^{1/2} Y\|_\alpha^\alpha}{N-1}\right) \exp\Big(O\Big((\log N)^{4/\alpha+1} N^{-10\theta-1} \operatorname{Tr} \mathbf{A}\Big)\Big)\right] \\
&\qquad + 2N \exp\left(-\frac{(\log N)^2}{4}\right) + 2\mathbb{P}\big[\mathcal{E}^c\big],
\end{aligned} \tag{2.7.3}$$

where $Y = (y_1, y_2, \ldots, y_{N-1})$ is a Gaussian random variable whose covariance matrix is given by Id, and we recall the definition $\theta = (b - 1/\alpha)(2 - \alpha)/10$ from Theorem 2.6.8.

Now, in view of the definition (2.7.2) of the event $\mathcal{E}$ and our assumption that $\mathbb{E}[\operatorname{Im} R_{11}] < \mathbb{E}\big[|R_{11}|^2\big]^{-1/2} \le \varepsilon^{-1/2}$, we have that $\mathbf{1}_{\mathcal{E}}\big|\operatorname{Tr}\mathbf{A}\big| < 2\varepsilon^{-1/2}$ for sufficiently large N. This (and our previous estimate $\mathbb{P}\big[\mathcal{E}^c\big] \le 2\exp\big(-(\log N)^2\big)$) guarantees the existence of a constant $C = C(\alpha, b, \varepsilon) > 0$ such that

$$\mathbb{P}\big[\operatorname{Im} S_i < (\log N)^{-4/\alpha}\big] \le C\mathbb{E}\left[\exp\left(-\frac{(\log N)^2\|\mathbf{A}^{1/2}Y\|_\alpha^\alpha}{CN}\right)\right] + C\exp\left(-\frac{(\log N)^2}{C}\right). \tag{2.7.4}$$

Thus, to provide a lower bound on $\operatorname{Im} S_N$, it suffices to establish a lower bound on

$$\frac{\|\mathbf{A}^{1/2}Y\|_\alpha^\alpha}{N} = \frac{1}{N}\sum_{j=1}^{N-1}\big|\operatorname{Im} R_{jj}^{(N)}\big|^{\alpha/2}|y_j|^\alpha. \tag{2.7.5}$$

To that end, we apply Lemma 2.4.5 (with $\mathbf{A} = \mathbf{H}^{(N)}$ and $t = (\log N)^{\alpha/2}(N\eta^2)^{\alpha/4}$) to obtain that

$$\begin{aligned}\mathbb{P}\left[\left|\frac{1}{N}\sum_{j=1}^{N-1}\big|\operatorname{Im} R_{jj}^{(N)}\big|^{\alpha/2}|y_j|^\alpha - \frac{1}{N}\sum_{j=1}^{N-1}\big|\operatorname{Im} R_{jj}^{(N)}\big|^{\alpha/2}\mathbb{E}\big[|y_j|^\alpha\big]\right| > \frac{C(\log N)^4}{N^{\alpha/4}\eta^{\alpha/2}}\right] \\ < C\exp\left(-\frac{(\log N)^2}{C}\right),\end{aligned} \tag{2.7.6}$$

after increasing C if necessary. Next, applying Lemma 2.4.6 with $r = \frac{\alpha}{2}$ yields the deterministic estimate

$$\frac{1}{N}\sum_{j=1}^{N}\left|\big(\operatorname{Im} R_{jj}\big)^{\alpha/2} - \big(\operatorname{Im} R_{jj}^{(N)}\big)^{\alpha/2}\right| < \frac{4}{(N\eta)^{\alpha/2}}. \tag{2.7.7}$$

The estimate (2.6.2) and Lemma 2.7.1 yield, after increasing C if necessary, that

$$\mathbb{P}\left[\left|\frac{1}{N}\sum_{j=1}^{N-1}\big|\operatorname{Im}R_{jj}\big|^{\alpha/2}-\frac{1}{N}\sum_{j=1}^{N-1}\mathbb{E}\big[|\operatorname{Im}R_{jj}|^{\alpha/2}\big]\right|>\frac{(\log N)^{\alpha/2}}{N^{\alpha/4}\eta^{\alpha/2}}\right]<C\exp\left(-\frac{(\log N)^2}{C}\right). \tag{2.7.8}$$

Combining the lower bound $\mathbb{E}\big[|\operatorname{Im}R_{jj}|^{\alpha/2}\big]\geq\varepsilon$ (see the second estimate in (2.6.14)), (2.3.2), (2.7.5), (2.7.6), (2.7.7), (2.7.8), and the fact that all entries of $\mathbf{R}$ are identically distributed yields (again, after increasing C if necessary)

$$\mathbb{P}\left[\frac{\|\mathbf{A}^{1/2}Y\|_\alpha^\alpha}{N}\leq\frac{\varepsilon}{C}\right]\leq C\exp\left(-\frac{(\log N)^2}{C}\right),$$

from which we deduce the lemma upon insertion into (2.6.32). □

2.7.3 Proof of Proposition 2.6.10

In this section we establish Proposition 2.6.10. Its proof will be similar to that of Proposition 2.5.3 in Section 2.5.1.

Proof of Proposition 2.6.10. Since all entries of $\mathbf{R}$ are identically distributed, we may assume that $i=N$.

As in the proof of Proposition 2.6.9, we begin by applying Lemma 2.5.4, now with $\mathbf{A}=\operatorname{Im}\mathbf{R}^{(N)}$, $X=\big(X_{Nj}\big)_{1\leq j\leq N-1}$, and $t=(\log N)^{2/\alpha}(2\log 2)^{1/2}$. Then, $\operatorname{Im}(S_N-T_N)=\langle\mathbf{A}X,X\rangle$. Following the proof of Proposition 2.6.9 yields a constant $C=C(\alpha,b,\varepsilon)>0$ such that

$$\begin{aligned}\mathbb{P}\big[\operatorname{Im}(S_N-T_N)<(\log N)^{-4/\alpha}\big]&\\ \leq C\mathbb{E}\left[\exp\left(-\frac{C(\log N)^2\|\mathbf{A}^{1/2}Y\|_\alpha^\alpha}{N}\right)\right]&+C\exp\left(-\frac{(\log N)^2}{C}\right),\end{aligned} \tag{2.7.9}$$

where $Y=(y_1,y_2,\ldots,y_{N-1})$ is a Gaussian random variable whose covariance is given by Id. Thus, it again suffices to establish a lower bound on $N^{-1}\|\mathbf{A}Y\|_\alpha^\alpha$.

To that end, we apply Lemma 2.5.5 with $w_i = (\mathbf{A}^{1/2} Y)_i$, $r = \alpha$, and $a = 2 + \varepsilon$. Then we find that $V_j = \operatorname{Im} R_{jj}^{(N)}(z)$, and $U_{jk} = \operatorname{Im} R_{jk}^{(N)}(z)$ for each $1 \le j, k \le N-1$. We must next estimate the quantities V, $\mathcal{X}$, and U from that lemma. These are given by $V = (N-1)^{-1} \sum_{i=1}^{N-1} V_j$, $\mathcal{X} = (N-1)^{-1} \sum_{i=1}^{N-1} V_j^{a/2}$, and $U = (N-1)^{-2} \sum_{1 \le j,k \le N-1} c_{jk}$.

To do this, observe from (2.4.6) and (2.3.2) that

$$U \le \frac{4}{N^2} \sum_{j=1}^{N-1} \sum_{k=1}^{N-1} c_{jk}^2 = \frac{4}{N^2} \sum_{j=1}^{N-1} \sum_{k=1}^{N-1} \big| \operatorname{Im} R_{jk}^{(N)} \big|^2 \le \frac{4}{N^2 \eta} \sum_{j=1}^{N-1} \operatorname{Im} R_{jj}^{(N)} \le \frac{4}{N\eta^2}. \tag{2.7.10}$$

To bound V, we apply the first estimate in (2.4.13) to deduce that

$$\mathbb{P} \left[\left| \frac{1}{N} \sum_{j=1}^{N} \operatorname{Im} R_{jj} - \mathbb{E}\big[\operatorname{Im} R_{jj} \big] \right| > \frac{4 \log N}{(N\eta^2)^{1/2}} \right] < 2 \exp \left(- \frac{(\log N)^2}{8} \right). \tag{2.7.11}$$

Therefore, Lemma 2.4.6 (applied with $r = 1$), (2.7.11), and the assumption (2.6.14) that $\mathbb{E}\big[\operatorname{Im} R_{jj} \big] \ge \mathbb{E}\big[(\operatorname{Im} R_{jj})^{\alpha/2} \big]^{2/\alpha} \ge \varepsilon^{2/\alpha}$ together imply that

$$\mathbb{P} \left[|V| < \frac{1}{C} \right] < C \exp \left(- \frac{(\log N)^2}{C} \right), \tag{2.7.12}$$

after increasing C if necessary. In particular, $\mathbb{P}\big[|V| \le 100 (\log N)^{10} U^{1/2} \big] < 2C \exp\big(- C^{-1} (\log N)^2 \big)$ for sufficiently large N.

Now let us estimate $\mathcal{X} = (N-1)^{-1} \sum_{j=1}^{N} V_j^{a/2}$. To that end, observe by (2.3.2) and Corollary 2.4.7 (applied with $r = \frac{a}{2} \le 2$), we find that

$$\begin{aligned} \left| \frac{1}{N} \sum_{j=1}^{N-1} \big| \operatorname{Im} R_{jj} \big|^{a/2} - \frac{(N-1)\mathcal{X}}{N} \right| &\le \left| \frac{1}{N} \sum_{j=1}^{N} \big| \operatorname{Im} R_{jj} \big|^{a/2} - \frac{1}{N} \sum_{j=1}^{N} \big| \operatorname{Im} R_{jj}^{(i)} \big|^{a/2} \right| + \frac{4}{N\eta^{a/2}} \\ &\le \frac{12}{N\eta^{a/2}}. \end{aligned} \tag{2.7.13}$$

Now let $f(y) = \mathbf{1}_{|\operatorname{Im} y| \le \eta^{-1}} |\operatorname{Im} y|^{a/2} + \mathbf{1}_{|\operatorname{Im} y| > \eta^{-1}} (2\eta)^{-a/2}$, and observe that f is Lipschitz with constant $L = a\eta^{1-a/2}$. Applying Lemma 2.4.3 with $t = N^{-1/2}\eta^{-a/2}\log N$ and using (2.3.2) yields

$$\mathbb{P}\left[\left|\frac{1}{N}\sum_{j=1}^{N}\big|\operatorname{Im} R_{jj}\big|^{a/2} - \mathbb{E}\big[\big|\operatorname{Im} R_{jj}\big|^{a/2}\big]\right| \geq \frac{\log N}{N^{1/2}\eta^{a/2}}\right] \leq 2\exp\left(-\frac{(\log N)^2}{8a^2}\right). \tag{2.7.14}$$

Combining (2.7.13), (2.7.14), the fact that $\eta \geq N^{\varepsilon - s/\alpha} \geq N^{\varepsilon - 1/2}$, and the fact (due to (2.6.14)) that $\mathbb{E}\big[|R_{jj}|^{a/2}\big] \leq \mathbb{E}\big[|R_{jj}|^2\big]^{a/4} \leq \varepsilon^{-a/4}$ yields that

$$\mathbb{P}\big[|\mathcal{X}| > C\big] < C\exp\left(-\frac{(\log N)^2}{C}\right), \tag{2.7.15}$$

after increasing C if necessary. Now Lemma 2.5.5 with (2.7.10), (2.7.12), and (2.7.15) together yield that

$$\mathbb{P}\left[\frac{\|\mathbf{A}^{1/2}Y\|_\alpha^\alpha}{N} < (\log N)^{-C}\right] < C\exp\left(-\frac{(\log N)^2}{C}\right), \tag{2.7.16}$$

after increasing C if necessary. Now the lemma follows from combining (2.7.9) and (2.7.16).

□

2.7.4 Proof of Proposition 2.6.17

In this section we establish Proposition 2.6.17.

Proof of Proposition 2.6.17. Let us define

$$\begin{aligned}
\mathcal{Z} = \mathcal{Z}(u) = \mathbb{E}[\mathfrak{Z}] &= \frac{\Gamma\big(1-\frac{\alpha}{2}\big)}{N}\sum_{j\neq i}\big(-\mathrm{i}R_{jj}^{(i)}\,|\,u\big)^{\alpha/2}, \\
\Phi_z = \Phi_z(u) &= \frac{\Gamma\big(1-\frac{\alpha}{2}\big)}{N}\sum_{j\neq i}\mathbb{E}\Big[\big(-\mathrm{i}R_{jj}^{(i)}\,|\,u\big)^{\alpha/2}\Big], \\
\xi_z = \xi_z(u) &= \frac{\Gamma\big(1-\frac{\alpha}{2}\big)}{N}\sum_{j\neq i}\mathbb{E}\Big[\big(-\mathrm{i}R_{jj}\,|\,u\big)^{\alpha/2}\Big].
\end{aligned}$$

To establish this proposition, we will first show that $\mathfrak{Z}$, $\mathcal{Z}$, ξ_z, and γ_z are all approximately equal. To that end, first observe that Lemma 2.7.2 implies the existence of a constant $C = C(\alpha) > 0$ such that

$$\mathbb{P}\left[\big\| \mathcal{Z} - \mathfrak{Z} \big\|_{1-\alpha/2+s} \geq \frac{(\log N)^s}{N^{s/2}\eta^{\alpha/2}} \right] \leq C \exp\left(-\frac{(\log N)^2}{C} \right). \tag{2.7.17}$$

Next, applying Lemma 2.7.1 with $\mathbf{A} = \mathbf{X}^{(i)}$ and $t = N^{-s/2}\eta^{-\alpha/2}(\log N)^s$ yields (after increasing C if necessary)

$$\mathbb{P}\left[\big\| \mathcal{Z} - \Phi_z \big\|_{1-\alpha/2+s} \geq \frac{(\log N)^s}{N^{s/2}\eta^{\alpha/2}} \right] \leq C \exp\left(-\frac{(\log N)^2}{C} \right). \tag{2.7.18}$$

Now we apply the second estimate in (2.6.3) with $x_1 = R_{jj}$, $x_2 = R_{jj}^{(i)}$, $r = \frac{\alpha}{2}$, and $a = \eta$ to obtain (again, after increasing C if necessary)

$$\Big\| \big(R_{jj} \,|\, u \big)^{\alpha/2} - \big(R_{jj}^{(i)} \,|\, u \big)^{\alpha/2} \Big\|_{1-\alpha/2+s} \leq C\eta^{-\alpha/2} \Big(\big| R_{jj} - R_{jj}^{(i)} \big|^{\alpha/2} + \eta^s \big| R_{jj} - R_{jj}^{(i)} \big|^s \Big). \tag{2.7.19}$$

To estimate the right side of (2.7.19) we apply apply Lemma 2.4.6 to deduce that

$$\frac{1}{N} \sum_{j=1}^{N} \big| R_{jj} - R_{jj}^{(i)} \big|^{\alpha/2} \leq \frac{4}{(N\eta)^{\alpha/2}}, \qquad \frac{1}{N} \sum_{j=1}^{N} \big| R_{jj} - R_{jj}^{(i)} \big|^{s} \leq \frac{4}{(N\eta)^{s}}. \tag{2.7.20}$$

Summing (2.7.19) over all $j \neq i$, taking expectations, applying (2.7.20), and (2.3.2) yields (after increasing C if necessary) that

$$\| \Phi_z - \xi_z \|_{1-\alpha/2+s} \leq C \left(\frac{1}{N^{\alpha/2}\eta^{\alpha}} + \frac{1}{N^s \eta^{\alpha/2}} \right). \tag{2.7.21}$$

Furthermore, since the entries of $\mathbf{R}$ are identically distributed, we have (after increasing C

if necessary) that

$$\big\|\xi_z - \gamma_z\big\|_{1-\alpha/2+s} = \frac{\Gamma\big(1-\frac{\alpha}{2}\big)}{N}\bigg\|\mathbb{E}\Big[\big(-\mathrm{i}R_{jj}\,|\,u\big)^{\alpha/2}\Big]\bigg\|_{1-\alpha/2+s} \le \frac{C}{N\eta^{\alpha/2}}, \tag{2.7.22}$$

where we have used (2.3.2) and the first estimate in (2.6.3).

Now the proposition follows from (2.7.17), (2.7.18), (2.7.21), (2.7.22), and the fact that $N > \eta^{-2}$. $\square$

2.8 Estimating the entries of $\mathbf{G}_t$

In this section we establish Proposition 2.2.9. To that end, we first require some additional notation. Recalling the definitions of $\mathbf{H}_s$ and $\mathbf{G}_s$ from the beginning of Section 2.2.2, let $\big\{\lambda_j(s)\big\}_{j\in[1,N]}$ denote the N eigenvalues of $\mathbf{H}_s$, and define $m_s = m_s(z) = N^{-1}\operatorname{Tr}\mathbf{G}_s = N^{-1}\sum_{j=1}^{N}\big(\lambda_j(s)-z\big)^{-1}$.

Further let $m_{\mathrm{fc},s}(z) \in \mathbb{H}$ denote the unique solution in the upper half plane to the equation

$$m_{\mathrm{fc},s}(z) = m_0\big(z + t m_{\mathrm{fc},s}(z)\big) = \frac{1}{N}\sum_{j=1}^{N} g_j(s,z), \quad \text{where} \quad g_j(s,z) = \frac{1}{\lambda_j - z - t m_{\mathrm{fc},s}(z)}. \tag{2.8.1}$$

The quantity $m_{\mathrm{fc},s}$ denotes the Stieltjes transform of the free convolution (see directly before Proposition 2.2.11) of the empirical spectral distribution of $\mathbf{H}_0$ with a suitable multiple of the semicircle law [29].

We require the following two results, which appear as Theorem 2.1 and Theorem 2.2 of [42].

Proposition 2.8.1 ([42, Theorem 2.1]). *Adopt the notation of Definition 2.2.8, and further assume that $\mathbf{H}_0$ is (η_0,γ,r)-regular with respect to E_0. Let $\mathbf{U} = \{u_{ij}\}$ and $\mathbf{D} = \{d_{jj} = d_j\}$ denote orthogonal and diagonal matrices, respectively, so that $\mathbf{H}_0 = \mathbf{U}\mathbf{D}\mathbf{U}^{-1}$.*

Fix $s \in [0,1]$ satisfying $N^{\delta}\eta \le s \le N^{-\delta}\gamma$. Then, for any $D > 1$ and $\kappa \in (0,1)$, there exists a constant $C = C(\delta, \kappa, D, A) > 0$ such that

$$\mathbb{P}\left[\sup_{z \in \mathcal{D}} \left(\left| \langle \mathbf{q}\mathbf{G}_s(z), \mathbf{q} \rangle - \sum_{j=1}^{N} \langle \mathbf{u}_j, \mathbf{q} \rangle^2 g_j(s,z) \right| - \frac{N^{2\delta}}{(N\eta)^{1/2}} \operatorname{Im}\Big(\sum_{j=1}^{N} \langle \mathbf{u}_j, \mathbf{q} \rangle^2 g_j(s,z) \Big) \right) > 0 \right] < CN^{-D}, \tag{2.8.2}$$

for any vector $\mathbf{q} \in \mathbb{R}^N$ such that $\|\mathbf{q}\|_2 = 1$. In (2.8.2), we have abbreviated $\mathcal{D} = \mathcal{D}(E_0, r, N^{4\delta-1}, 1-\kappa r, \kappa)$ (recall (2.2.9)).

Proposition 2.8.2 ([42, Proposition 2.2])**.** *Adopt the notation and assumptions of Proposition 2.8.1. Then, there exists a constant $C = C(\delta, \kappa, D, A) > 0$ such that*

$$\big| m_{\mathrm{fc},s}(z) \big| \le \frac{1}{N} \sum_{j=1}^{N} \big| g_j(s,z) \big| \le C \log N, \qquad \frac{1}{C} \le \operatorname{Im} m_{\mathrm{fc},s}(z) \le C, \tag{2.8.3}$$

for any $z \in \mathcal{D}$.

Now we can establish Proposition 2.2.9.

Proof of Proposition 2.2.9. Recall that u_{jk} denotes the j-th entry of the eigenvector corresponding to λ_k. Applying (2.8.2) with $\mathbf{q} = (q_1, q_2, \ldots, q_N)$ satisfying $q_k = \mathbf{1}_{k=j}$ for each $k \in [1, N]$ yields the existence of a constant $C = C(\delta, \kappa, D, A) > 0$ such that

$$\mathbb{P}\left[\sup_{z \in \mathfrak{D}} \left(\left| G_{jj}(s,z) - \sum_{k=1}^{N} u_{jk}^2 g_k(s,z) \right| - \frac{N^{\delta/2}}{(N\eta)^{1/2}} \operatorname{Im}\left(\sum_{k=1}^{N} u_{jk}^2 g_k(s,z) \right) \right) > 0 \right] < CN^{-10D}. \tag{2.8.4}$$

Let us estimate the terms $g_k(s,z)$ appearing in (2.8.4). To that end, we define $\mathcal{A}_0 = \mathcal{A}_0(E_0) = [E_0 - \eta_0, E_0 + \eta_0]$ and $\mathcal{A}_m = \mathcal{A}_m(E_0) = \big[E_0 - 2^m \eta_0, E_0 - 2^{m-1}\eta_0\big] \cup \big[E_0 + 2^{m-1}\eta_0, E_0 + 2^m \eta_0\big]$, for each integer $m \ge 1$. Since (2.8.3) implies the existence a constant $C = C(\delta, \kappa, E_0, D, A) > 1$ such that $\big| m_{\mathrm{fc},s}(z) \big| \le C \log N$ and $\frac{1}{C} \le \operatorname{Im} m_{\mathrm{fc},s}(z) \le C$, the definition (2.8.1) of the g_k implies that

$$\max_{\lambda_k \in \mathcal{A}_m} \big| g_k(s, E_0 + \mathrm{i}\eta) \big| \leq \left(\frac{C^2}{\big(\min\{2^{m-1}\eta_0 - C^2 s \log N, 0\} \big)^2 + s^2} \right)^{1/2}. \tag{2.8.5}$$

for any integer $m \geq 1$.

Next let us estimate the entries of $\mathbf{U}$, where we recall from Proposition 2.8.1 that $\mathbf{H}_0 = \mathbf{U}\mathbf{D}\mathbf{U}^{-1}$. The assumed bound on the entries of $\mathbf{G}_0(z)$ implies

$$\sup_{z \in \mathcal{D}(E_0, r, \eta_0, \gamma, 0)} \left| \sum_{k=1}^{N} \frac{u_{jk}^2}{z - \lambda_k} \right| = \sup_{z \in \mathcal{D}(E_0, r, \eta_0, \gamma, 0)} \big| G_{jj}(z) \big| \leq B, \tag{2.8.6}$$

where we have denoted $\lambda_j = \lambda_j(0)$ as the eigenvalues of $\mathbf{H}_0$. Thus, setting $z = E_0 + \mathrm{i}\eta_0$ in (2.8.6) yields

$$\max_{1 \leq j \leq N} \sum_{\lambda_k \in \mathcal{A}_m} u_{jk}^2 \leq \min\big\{ 2^m \eta_0 B, 1 \big\}, \quad \text{for any integer } m \geq 0. \tag{2.8.7}$$

Now we can bound the terms appearing in (2.8.4). We define

$$M = \left\lceil \log_2 \left(\frac{s (\log N)^2}{\eta_0} \right) \right\rceil,$$

and write

$$\begin{aligned} \left| \sum_{k=1}^{N} u_{jk}^2 g_k(s, z) \right| &\leq \sum_{m=0}^{\infty} \sum_{\lambda_k \in \mathcal{A}_m(E)} u_{jk}^2 \big| g_k(s, z) \big| \\ &\leq \sum_{m=0}^{M} \sum_{\lambda_k \in \mathcal{A}_m(E)} u_{jk}^2 \big| g_k(s, z) \big| + \sum_{m=M+1}^{\lceil 4 \log N \rceil} \sum_{\lambda_k \in \mathcal{A}_m(E)} u_{jk}^2 \big| g_k(s, z) \big| \\ &\quad + \sum_{m=\lceil 4 \log N \rceil}^{\infty} \sum_{\lambda_k \in \mathcal{A}_m(E)} \big| g_k(s, z) \big|. \end{aligned}$$

We bound these three sums by combining (2.8.5), (2.8.7), and the facts that $\eta_0 > N^{-1}$, $1 < B < N$, and $s \in \big(\eta_0, N^{-\delta} \big)$. For the first sum, we apply (2.8.5) – noting the minimum in the denominator of the right side takes the value 0 – and the first argument of the minimum in the left side of (2.8.7). For the second sum, we apply (2.8.5), with the minimum on the right side taking the nonzero value, and (2.8.7). The third sum is bounded using (2.8.5) only.

We deduce for sufficiently large N that

$$\left| \sum_{k=1}^{N} u_{jk}^2 g_k(s,z) \right| \le C s^{-1} 2^M \eta_0 B + CB \log N + \frac{C}{N} \le CB (\log N)^3, \tag{2.8.8}$$

after increasing C (in a way that only depends on δ, κ, D, and A) if necessary.

Therefore, combining (2.8.4), (2.8.8), the fact that $N\eta \ge N^{\delta}$, and a union bound over $j \in [1, N]$ yields (again after increasing C if necessary, in a way that only depends on δ, κ, D, and A)

$$\mathbb{P}\left[\sup_{z \in \mathfrak{D}} \max_{1 \le j \le N} \big| G_{jj}(s,z) \big| > CB (\log N)^3 \right] < CN^{-5D}. \tag{2.8.9}$$

To estimate the remaining entries of $\mathbf{G}_s$, we apply (2.8.2) with $\mathbf{q} = (q_1, q_2, \ldots, q_N)$ satisfying $q_k = 2^{-1/2} \left(\mathbf{1}_{k=i} + \mathbf{1}_{k=j} \right)$ for some fixed $i, j \in [1, N]$. Using (2.8.8), this yields (after increasing C if necessary, in a way that only depends on δ, κ, D, and A)

$$\mathbb{P}\left[\sup_{z \in \mathfrak{D}} \big| G_{jj}(s,z) + G_{ii}(s,z) + 2G_{ij}(s,z) \big| > CB (\log N)^3 \right] < CN^{-5D}. \tag{2.8.10}$$

Now the corollary follows from combining (2.8.9), (2.8.10), and a union bound over all $i, j \in [1, N]$. $\square$

2.9 Comparing deformed stable laws to their removals

In this section we establish Lemma 2.5.4. However, we first require the following lemma that estimates the characteristic functions of removals of stable laws.

Lemma 2.9.1. *Fix $\sigma > 0$, $\alpha \in (0,2)$, a positive integer N, and $0 < b < \frac{1}{\alpha}$. Let X denote the random variable given by the b-removal of a deformed $(0,\sigma)$ α-stable law, as in Definition 2.2.2. Let $X_1, X_2, \ldots, X_N$ be mutually independent random variables, each with law $N^{-1/\alpha}X$, and let $c_1, c_2, \ldots, c_N \in \mathbb{R}$ be constants.*

Then, for any $t \in \mathbb{R}$, we have that

$$\mathbb{E}\Bigg[\exp\Bigg(\mathrm{i}t\sum_{j=1}^{N} c_j X_j\Bigg)\Bigg] = \exp\Bigg(-\frac{\sigma^\alpha |t|^\alpha}{N}\sum_{j=1}^{N}|c_j|^\alpha\Bigg)\exp\Bigg(O\bigg(t^2 N^{(2-\alpha)(b-1/\alpha)-1}\sum_{j=1}^{N}|c_j|^2\bigg)\Bigg),$$

where the implicit constant on the right side only depends on α.

Proof. Let Z be a $(0,\sigma)$ α-stable law and J be a random variable satisfying Definition 3.1.1. Let $Y = (Z+J)\mathbf{1}_{|Z+J|<N^b}$, so that $X = Z - Y$. Let $Y_1, Y_2, \ldots, Y_N$ be mutually independent random variables with law $N^{-1/\alpha}Y$, let $Z_1, Z_2, \ldots, Z_N$ be mutually independent random variables with law $N^{-1/\alpha}Z$, and let $J_1, J_2, \ldots, J_N$ be mutually independent variables with law $N^{-1/\alpha}J$. Then the random variables X_j have laws $N^{-1/\alpha}X$, where we assume that the X_j, Y_j, Z_j, and J_j are coupled so that $X_j = Z_j + J_j - Y_j$ for each $1 \le j \le N$.

Observe that, for any $t \in \mathbb{R}$, we have that

$$\begin{aligned}\mathbb{E}\big[e^{\mathrm{i}tX}\big] &= \mathbb{E}\big[e^{\mathrm{i}t(Z+J)}\big] + \mathbb{E}\big[e^{\mathrm{i}t(Z+J)}(e^{-\mathrm{i}tY}-1)\big] \\ &= \mathbb{E}\big[e^{\mathrm{i}t(Z+J)}\big] - \mathrm{i}t\mathbb{E}\big[e^{\mathrm{i}t(Z+J)}Y\big] + O\Big(\mathbb{E}\big[t^2Y^2\big]\Big) \\ &= \mathbb{E}\big[e^{\mathrm{i}t(Z+J)}\big] - \mathrm{i}t\mathbb{E}\big[e^{\mathrm{i}t(Z+J)}(Z+J)\mathbf{1}_{|Z+J|<N^b}\big] + O\Big(\mathbb{E}\big[t^2Y^2\big]\Big) \\ &= \mathbb{E}\big[e^{\mathrm{i}t(Z+J)}\big] - \mathrm{i}t\mathbb{E}\big[(Z+J)\mathbf{1}_{|Z+J|<N^b}\big] + O\Big(\mathbb{E}\big[t^2Y^2\big]\Big) = \mathbb{E}\big[e^{\mathrm{i}t(Z+J)}\big] + O\Big(\mathbb{E}\big[t^2Y^2\big]\Big),\end{aligned} \tag{2.9.1}$$

where the second equality above follows from a Taylor expansion, the third from the definition of Y, the fourth from another Taylor expansion, and the fifth from the fact that $Z+J$ is symmetric. A similar argument shows that

$$\mathbb{E}\big[e^{\mathrm{i}t(Z+J)}\big] = \mathbb{E}\big[e^{\mathrm{i}tZ}\big] + O\Big(\mathbb{E}\big[t^2 J^2\big]\Big). \tag{2.9.2}$$

Replacing t with $c_j N^{-1/\alpha} t$ in (2.9.1) and (2.9.2), we find that

$$\begin{aligned}\mathbb{E}\big[e^{\mathrm{i}c_j t X_j}\big] &= \mathbb{E}\big[e^{\mathrm{i}c_j t N^{-1/\alpha} Z}\big] + \frac{c_j^2 t^2}{N^{2/\alpha}} O\Big(\mathbb{E}\big[|Z+J|^2 \mathbf{1}_{|Z+J| \le N^b}\big] + \mathbb{E}[J^2]\Big) \\ &= \exp\left(-\frac{\sigma^\alpha |c_j t|^\alpha}{N}\right) + O\big(N^{(2-\alpha)(b-1/\alpha)-1} |c_j t|^2\big),\end{aligned}$$

where in the second estimate above we used (3.1.1) and integrated (3.1.3). Now, let $R = R_j = N^{-1}|c_j t|^\alpha$. Then, we find that

$$\mathbb{E}\big[e^{\mathrm{i}c_j t X_j}\big] \le \exp\left(-\sigma^\alpha R\right) + O\big(N^{(2-\alpha)b} R^{2/\alpha}\big) \le \exp\big(-\sigma^\alpha R\big) \exp\Big(O\big(N^{(2-\alpha)b} R^{2/\alpha}\big)\Big). \tag{2.9.3}$$

Indeed, if $R \le 1$ then (2.9.3) follows from the estimate $y \le e^y - 1$. Otherwise, if $R > 1$ and N is sufficiently large, we have that $N^{(2-\alpha)b} R^{2/\alpha} > 2\sigma^\alpha R$ (since $\alpha < 2$), from which we again deduce (2.9.3) from the estimate $y \le e^y - 1$. Inserting the definition of $R = N^{-1}|c_j t|^\alpha$ into (2.9.3) yields

$$\mathbb{E}\big[e^{\mathrm{i}c_j t X_j}\big] \le \exp\left(-\frac{\sigma^\alpha |c_j t|^\alpha}{N}\right) \exp\Big(O\big(N^{(2-\alpha)(b-1/\alpha)-1} |c_j t|^2\big)\Big). \tag{2.9.4}$$

Now the lemma follows from taking the product of (2.9.4) over all $j \in [1, N]$. $\square$

Now we can establish Lemma 2.5.4.

Proof of Lemma 2.5.4. The proof of this lemma will follow a similar method as the one used to establish Lemma B.1 of [33]. To that end, observe that

$$\mathbb{E}\bigg[\exp\Big(-\frac{t^2}{2}\langle \mathbf{A}X, X\rangle\Big)\bigg] = \mathbb{E}\bigg[\exp\Big(-\frac{t^2}{2}\langle \mathbf{B}X, \mathbf{B}X\rangle\Big)\bigg] = \mathbb{E}\bigg[\exp\Big(-\mathrm{i}t\langle \mathbf{B}X, Y\rangle\Big)\bigg]$$
$$= \mathbb{E}\bigg[\exp\Big(-\mathrm{i}t\langle X, \mathbf{B}Y\rangle\Big)\bigg].$$

Denote $W = \mathbf{B}Y = (w_1, w_2, , \ldots, w_N)$. In view of Lemma 2.9.1, we have the conditional expectation estimate

$$\mathbb{E}\bigg[\exp\Big(-\mathrm{i}t\langle X, W\rangle\Big)\bigg|W\bigg] = \exp\Bigg(-\frac{\sigma^\alpha|t|^\alpha}{N}\sum_{j=1}^N |w_j|^\alpha\Bigg)\exp\Bigg(O\bigg(t^2N^{(2-\alpha)(b-1/\alpha)-1}\sum_{j=1}^N |w_j|^2\bigg)\Bigg). \tag{2.9.5}$$

Now, observe that since each w_j is a Gaussian random variable with variance $\sum_{i=1}^N b_{ij}^2$, we have from a union bound that

$$\mathbb{P}\Bigg[\sum_{j=1}^N w_j^2 > (\log N)\operatorname{Tr}\mathbf{A}\Bigg] \le \sum_{j=1}^N \mathbb{P}\Bigg[w_j^2 > (\log N)\sum_{i=1}^N b_{ij}^2\Bigg] \le Ne^{-(\log N)^2/2}, \tag{2.9.6}$$

where in the first estimate we used the fact that $\operatorname{Tr}\mathbf{A} = \operatorname{Tr}\mathbf{B}^2 = \sum_{1\le i,j\le N} b_{ij}^2$.

Taking the expectation on both sides of (2.9.5) over the events where $\sum_{j=1}^N |w_j|^2$ is at most or at least $(\log N)\operatorname{Tr}\mathbf{A}$ and further using the fact that the exponential inside the expectation on the left side of (2.9.5) is bounded by 1, we deduce that

$$\begin{aligned}\mathbb{E}\bigg[\exp\Big(-\mathrm{i}t\langle X, W\rangle\Big)\bigg] = \\ \mathbb{E}\Bigg[\exp\Bigg(-\frac{\sigma^\alpha|t|^\alpha}{N}\sum_{j=1}^N |w_j|^\alpha\Bigg)\Bigg]\exp\Bigg(O\bigg(t^2N^{(2-\alpha)(b-1/\alpha)-1}(\log N)\operatorname{Tr}\mathbf{A}\bigg)\Bigg) \\ + Ne^{-(\log N)^2/2},\end{aligned} \tag{2.9.7}$$

from which we deduce the lemma. $\square$

Chapter 3

Eigenvector Statistics of Lévy Matrices

3.1 Results

3.1.1 Definitions

Denote the upper half plane by $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$. Set $\mathbb{R}_+ = [0, \infty)$, set $\mathbb{K} = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$, and set $\mathbb{K}^+ = \overline{\mathbb{K} \cap \mathbb{H}}$ to be the closure of the positive quadrant of the complex plane. We also let $\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$ be the unit circle and define $\mathbb{S}^1_+ = \overline{\mathbb{K}^+ \cap \mathbb{S}}$.

Fix a parameter $\alpha \in (0, 2)$, and let $\sigma > 0$ and $\beta \in [-1, 1]$ be real numbers. A random variable Z is a (β, σ) *α-stable law* if it has the characteristic function

$$\mathbb{E}\left[e^{\mathrm{i}tZ}\right] = \exp\Big(- \sigma^\alpha |t|^\alpha \big(1 - \mathrm{i}\beta \operatorname{sgn}(t) u\big)\Big), \quad \text{for all } t \in \mathbb{R}, \tag{3.1.1}$$

where $u = u_\alpha = \tan\left(\frac{\pi\alpha}{2}\right)$ if $\alpha \neq 1$ and $u = u_1 = -\frac{2}{\pi} \log |t|$ if $\alpha = 1$. Note $\beta = 0$ ensures that Z is symmetric. The case $\beta = 1$ is known as a *one-sided α-stable law* and is always positive.

We now define the entry distributions we consider in this paper. Our proofs and results should also apply to wider classes of distributions, but we will not pursue this here (see the similar remark in [6, Section 2] for more on this point).

Definition 3.1.1. Let Z be a $(0, \sigma)$ α-stable law with

$$\sigma = \left(\frac{\pi}{2 \sin\left(\frac{\pi\alpha}{2}\right) \Gamma(\alpha)} \right)^{1/\alpha} > 0. \tag{3.1.2}$$

Let J be a symmetric[1] random variable (not necessarily independent from Z) such that $\mathbb{E}[J^2] < \infty$, $Z + J$ is symmetric, and

$$\frac{C_1}{\big(|t|+1\big)^\alpha} \le \mathbb{P}\big[|Z+J| \ge t\big] \le \frac{C_2}{\big(|t|+1\big)^\alpha} \quad \text{for each } t \ge 0 \text{ and some constants } C_1, C_2 > 0. \tag{3.1.3}$$

Denoting $\mathfrak{z} = Z + J$, the symmetry of J and the condition $\mathbb{E}[J^2] < \infty$ are equivalent to imposing a coupling between $\mathfrak{z}$ and Z such that $\mathfrak{z} - Z$ is symmetric and has finite variance, respectively.

For each positive integer N, let $\{H_{ij}\}_{1 \le i \le j \le N}$ be mutually independent random variables that each have the same law as $N^{-1/\alpha}(Z + J) = N^{-1/\alpha}\mathfrak{z}$. Set $H_{ij} = H_{ji}$ for each i, j, and define the $N \times N$ random matrix $\mathbf{H} = \mathbf{H}_N = \{H_{ij}\} = \{H_{i,j}^{(N)}\}$, which we call an *$\alpha$-Lévy matrix*.

The $N^{-1/\alpha}$ scaling of the entries H_{ij} is different from the usual $N^{-1/2}$ scaling for Wigner matrices. It makes the typical row sum of $\mathbf{H}$ of order one. The constant σ is chosen so that our notation is consistent with previous works [18, 33, 34], but can be altered by rescaling $\mathbf{H}$ without affecting our main results.

By [18, Theorem 1.1], the empirical spectral distribution of $\mathbf{H}$ converges to a deterministic measure that we denote μ_α, which is absolutely continuous with respect to the Lebesgue measure and symmetric about 0. We denote its probability density function and Stieltjes

[1] By *symmetric*, we mean that J has the same law as $-J$.

transform by $\varrho_\alpha(x)$ and

$$m_\alpha(z) = \int_{\mathbb{R}} \frac{\varrho_\alpha(x)\, dx}{x - z}, \tag{3.1.4}$$

defined for $z \in \mathbb{H}$, respectively.

The Siteltjes transform $m_\alpha(z)$ may be characterized as the solution to a certain self-consistent equation [33, Section 3.1]. We note it here, although we will not need this representation for our work. For any $z \in \mathbb{H}$, define the functions $\varphi = \varphi_{\alpha,z} \colon \mathbb{K} \to \mathbb{C}$ and $\psi = \psi_{\alpha,z} \colon \mathbb{K} \to \mathbb{C}$ by

$$\varphi_{\alpha,z}(x) = \frac{1}{\Gamma(\alpha/2)} \int_{\mathbb{R}_+} t^{\alpha/2-1} e^{\mathrm{i}tz} e^{-\Gamma(1-\alpha/2) t^{\alpha/2} x} dt, \quad \psi_{\alpha,z}(x) = \int_{\mathbb{R}_+} e^{\mathrm{i}tz} e^{-\Gamma(1-\alpha/2) t^{\alpha/2} x} dt, \tag{3.1.5}$$

for any $x \in \mathbb{K}$. For each $z \in \mathbb{H}$, there exists a unique solution $y = y(z) \in \mathbb{K}$ to the equation $y(z) = \varphi_{\alpha,z}\big(y(z)\big)$. Then, the Stieltjes transform $m_\alpha(z) : \mathbb{H} \to \mathbb{H}$ is defined by setting $m_\alpha(z) = \mathrm{i}\psi_{\alpha,z}\big(y(z)\big)$.

We recall that, like any Stieltjes transform of an absolutely continuous measure, $\operatorname{Im} m_\alpha(z)$ extends to the real line with

$$\lim_{\eta \to 0} \operatorname{Im} m_\alpha(E + \mathrm{i}\eta) = \pi \varrho_\alpha(E) \tag{3.1.6}$$

for $E \in \mathbb{R}$. It is known that $\varrho_\alpha(x) \sim \frac{\alpha}{2x^{\alpha+1}}$ as x tends to ∞ [32, Theorem 1.6].

Definition 3.1.2. The classical eigenvalue locations $\gamma_i = \gamma_i^{(\alpha)}$ for $\varrho_\alpha(x)$ are defined by the quantiles

$$\gamma_i = \inf \left\{ y \in \mathbb{R} : \int_{-\infty}^{y} \varrho_\alpha(x)\, dx \geq \frac{i}{N} \right\}. \tag{3.1.7}$$

Given a random matrix $\mathbf{A}$, it is common to study its resolvent $(\mathbf{A} - z)^{-1}$. Contrary to those for the Wigner model, the diagonal entries $G_{ii}(z)$ of the resolvent $\mathbf{G}(z) = (\mathbf{H} - z)^{-1}$ of a Lévy matrix do not converge to a constant value but instead converge to a nontrivial limiting distribution as N tends to infinity and $z \in \mathbb{H}$ remains fixed. This was shown in [32], where the limit $R_\star(z)$ was identified as the resolvent of a random operator defined on a space known as the Poisson Weighted Infinite Tree [13,14], which is a weighted and directed rooted tree, evaluated at its root. We note the basic construction here and refer to [32, Section 2.3] for details.

Set $d\nu = (1/2)\, d\mu$, where μ is the Lebesgue measure on $\mathbb{R}$. The vertex set of the tree is given by $V = \bigcup_{k \in \mathbb{N}} \mathbb{N}^k$, where the root is $\mathbb{N}^0 = \varnothing$, and the children of $\mathbf{v} \in \mathbb{N}^k$ are denoted $(\mathbf{v}, 1), (\mathbf{v}, 2), \ldots \in \mathbb{N}^{k+1}$. To determine the weights, let $\{\Xi_{\mathbf{v}}\}_{\mathbf{v} \in V}$ be a collection of independent Poisson point processes with intensity measure ν on $\mathbb{R}$. Let $\Xi_{\varnothing} = \{y_1, y_2, \ldots\}$ be ordered so that $|y_1| \le |y_2| \le \cdots$, and set y_i to be the weight of edge connecting $\varnothing$ to the vertex (i). This process is repeated for all vertices so that, for any $\mathbf{v} \in \mathbb{N}^k$, the edge between vertices $\mathbf{v}$ and $(\mathbf{v}, i)$ is weighted with the value $y_{(\mathbf{v},i)}$, where $\Xi_{\mathbf{v}} = \{y_{(\mathbf{v},1)}, y_{(\mathbf{v},2)}, \ldots\}$ is labeled so that $|y_{(\mathbf{v},1)}| \le |y_{(\mathbf{v},2)}| \le \cdots$.

Let $\mathcal{F}$ be the (dense) subset of $L^2(V)$ of vectors with finite support and, for any $\mathbf{v} \in V$, let $\delta_{\mathbf{v}} \in \mathcal{F}$ denote the unit vector supported on $\mathbf{v}$. Then, define the linear operator $\mathbf{T} \colon \mathcal{F} \to L^2(V)$ by setting

$$\langle \delta_{\mathbf{v}}, \mathbf{T}\delta_{\mathbf{w}} \rangle = \begin{cases} \operatorname{sign}(y_{\mathbf{w}})|y_{\mathbf{w}}|^{-1/\alpha} & \text{if } \mathbf{w} = (\mathbf{v}, k) \text{ for some } k, \\ \operatorname{sign}(y_{\mathbf{v}})|y_{\mathbf{v}}|^{-1/\alpha} & \text{if } \mathbf{v} = (\mathbf{w}, k) \text{ for some } k, \\ 0 & \text{otherwise.} \end{cases} \tag{3.1.8}$$

We identify $\mathbf{T}$ with its closure, which is self-adjoint [32, Section 2.3]. It can be considered a weak limit of the matrix $\mathbf{H}$, as N tends to ∞.

Definition 3.1.3. For any $z \in \mathbb{H}$, we define $R_\star(z) \colon \mathbb{H} \to \mathbb{H}$ to be the resolvent entry $\langle \delta_\varnothing, (\mathbf{T} - z)^{-1} \delta_\varnothing \rangle$.

For any $z \in \mathbb{H}$, define the $N \times N$ matrix $\mathbf{G}(z) = \big\{ G_{ij}(z) \big\}$ by $\mathbf{G}(z) = (\mathbf{H} - z)^{-1}$, which is the resolvent of $\mathbf{H}$. It is known from [32, Section 2] that any diagonal entry $G_{jj}(z)$ converges to $R_\star(z)$ in distribution for fixed $z \in \mathbb{H}$, as N tends to ∞.

Next, we require the following result and definition concerning the limit of $R_\star(z)$ as $\operatorname{Im} z$ tends to 0. The following proposition will be proved in Section 3.6 below.

Proposition 3.1.4. *There exists a (deterministic) countable set $\mathcal{A} \subset (0,2)$ with no accumulation points in $(0,2)$ such that the following two statements hold. First, for all $\alpha \in (0,2) \backslash \mathcal{A}$, there exists a constant $c = c(\alpha) > 0$ such that, for every real number $E \in [-c, c]$, the sequence of random variables $\big\{ \operatorname{Im} R_\star(E + \mathrm{i}\eta) \big\}_{\eta > 0}$ is tight as η tends to 0. Second, for any fixed $p \in \mathbb{N}$, all limit points $\mathcal{R}(E)$ of this sequence under the weak topology have the same moment $\mathbb{E}\big[\mathcal{R}(E)^p\big]$.*

The set $\mathcal{A}$ is non-explicit and originally appeared in [34] from an application of the implicit function theorem for Banach spaces to a certain self-consistent equation. In our context, $\mathcal{A}$ will come from a local law, given by Lemma 3.3.2 below.

Definition 3.1.5. Let $\mathcal{R}_\star(E)$ be an arbitrary limit point (under the weak topology) as η tends to 0 of the sequence $\big\{ \operatorname{Im} R_\star(E + \mathrm{i}\eta) \big\}_{\eta > 0}$. By Proposition 2.4 and Prokhorov's theorem, there exists at least one. Given $\mathcal{R}_\star(E)$, define the random variable $\mathcal{U}_\star(E) = \big(\pi \varrho_\alpha(E)\big)^{-1} \mathcal{R}_\star(E)$.

We also need the following definition to state our results.

Definition 3.1.6. Let $\mathbf{w} = (w_i)_{1 \le i \le n} \in \mathbb{R}^n$ be a random vector and $\mathbf{w}^{(j)} = (w_i^{(j)})_{1 \le i \le n}$, defined for $j \ge 1$, be a sequence of random vectors in $\mathbb{R}^n$. We say that $\mathbf{w}^{(j)}$ *converges in moments* to $\mathbf{w}$ if for every polynomial $P \colon \mathbb{R}^n \to \mathbb{R}$ in n variables, we have

$$\lim_{N \to \infty} \mathbb{E}\Big[P\big(\mathbf{w}^{(N)}\big) \Big] = \mathbb{E}\big[P(\mathbf{w}) \big]. \tag{3.1.9}$$

3.1.2 Results

In this section, we state our results, which are proved in Section 3.2. Our first, Theorem 3.1.7, identifies the joint moments of different entries of the same eigenvector. Our second, Theorem 3.1.8, does this for the same entries of different eigenvectors. We let $\lambda_1(\mathbf{H}) \le \lambda_2(\mathbf{H}) \le \cdots \le \lambda_N(\mathbf{H})$ denote the eigenvalues of $\mathbf{H}$ in non-decreasing order and, for each $k \in [1, N]$, we write $\mathbf{u}_k = \big(u_k(1), u_k(2), \ldots, u_k(N)\big)$ for a unit eigenvector of $\mathbf{H}$ corresponding to $\lambda_k(\mathbf{H})$.

In the theorem statements, certain index parameters (for instance i and k) may depend on N. For brevity, we sometimes suppress this dependence in the notation, writing for example $\mathbf{u}_k$ instead of $\mathbf{u}_{k(N)}$. Throughout, we recall the countable set $\mathcal{A} \subset (0, 2)$ from Proposition 3.1.4.

Theorem 3.1.7. *For all $\alpha \in (0, 2) \setminus \mathcal{A}$, there exists a constant $c = c(\alpha) > 0$ such that the following holds. Fix an integer $n > 0$ (independently of N) and index sequences $\{i_j(N)\}_{1 \le j \le n}$ such that for every N, $\{i_j(N)\}_{1 \le j \le n}$ are distinct integers in $[1, N]$. Further let $k = k(N) \in [1, N]$ be an index sequence such that $\lim_{N \to \infty} \gamma_k = E$ for some $E \in [-c, c]$. Then the vector*

$$\big(N\mathbf{u}_k(i_1)^2, N\mathbf{u}_k(i_2)^2, \ldots, N\mathbf{u}_k(i_n)^2\big) \tag{3.1.10}$$

converges in moments to

$$\big(\mathcal{N}_1^2 \cdot \mathcal{U}_1(E), \mathcal{N}_2^2 \cdot \mathcal{U}_2(E), \ldots, \mathcal{N}_n^2 \cdot \mathcal{U}_n(E)\big), \tag{3.1.11}$$

where the $\mathcal{N}_j$ are are independent, identically distributed (i.i.d.) standard Gaussians and the $\mathcal{U}_j(E)$ are i.i.d. random variables with law $\mathcal{U}_\star(E)$ that are independent from the $\mathcal{N}_j$.

Theorem 3.1.8. *For all $\alpha \in (0, 2) \setminus \mathcal{A}$, there exists a constant $c = c(\alpha) > 0$ such that the following holds. Fix an integer $n > 0$ (independently of N) and index sequences $\{k_j(N)\}_{1 \le j \le n}$ such that for every N, $\{k_j(N)\}_{1 \le j \le n}$ are distinct integers in $[1, N]$ and $|k_1 - k_j| < N^{1/2}$ for each $j \in [2, n]$. Suppose that $\lim_{N \to \infty} \gamma_{k_1} = E$ for some $E \in [-c, c]$. Further let $i = i(N) \in [1, N]$ be an index sequence. Then the vector*

$$\big(N\mathbf{u}_{k_1}(i)^2, N\mathbf{u}_{k_2}(i)^2, \ldots, N\mathbf{u}_{k_n}(i)^2\big) \tag{3.1.12}$$

converges in moments to

$$\big(\mathcal{N}_1^2 \cdot \mathcal{U}_\star(E), \mathcal{N}_2^2 \cdot \mathcal{U}_\star(E), \ldots, \mathcal{N}_n^2 \cdot \mathcal{U}_\star(E)\big), \tag{3.1.13}$$

where the $\mathcal{N}_j$ are i.i.d. standard Gaussians that are independent from $\mathcal{U}_\star(E)$.

Theorem 3.1.7 shows that entries of the same eigenvector are asymptotically independent, as in the Wigner case [44, Corollary 1.3]. However, unlike in the Wigner case [44, Theorem 1.2], Theorem 3.1.8 indicates that entries of different eigenvectors with the same index can be asymptotically correlated. This can be seen by taking $n = 2$ and $k_2 = k_1 + 1$ in that result, in which case $N\mathbf{u}_{k_1}(m)^2$ and $N\mathbf{u}_{k_2}(m)^2$ are correlated through $\mathcal{U}_\star(E)$.

For almost all $E \in [-c, c]$, the random variable $\mathcal{U}_\star(E)$ is not explicit. However, as a consequence of [32, Theorem 4.3], an exception occurs at $E = 0$, where $\mathcal{U}_\star(0)$ is given by the inverse of a stable law. In this case, the $n = 1$ cases of Theorem 3.1.7 and Theorem 3.1.8 reduce to the following corollary.

Corollary 3.1.9. *Retain the notation of Theorem 3.1.7. Choose k so that $E = 0$, and set $n = 1$ and $m = i_1$. Then $N\mathbf{u}_k(m)^2$ converges in moments to*

$$\frac{1}{\Gamma\left(1 + \frac{2}{\alpha}\right)} \cdot \mathcal{N}^2 \cdot \vartheta, \tag{3.1.14}$$

where $\mathcal{N}$ is a standard Gaussian and ϑ is independent with law S^{-1}, where S is a $(1, 1)$ $\frac{\alpha}{2}$-stable law.

The non-triviality of the random variable ϑ shows that the entries of $\mathbf{u}_k$ are asymptotically non-Gaussian; this is again different from the eigenvector behavior in the Wigner case. It is natural to wonder whether $\mathcal{R}_\star(E)$ is non-constant for $E \neq 0$. As a consequence of the last statement of Lemma 3.6.8 below, for all $p \in \mathbb{N}$, the moments $\mathbb{E}\big[\big(\mathcal{R}_\star(E)\big)^p\big]$ are continuous in E, for $|E|$ sufficiently small. This implies that moments of $\mathcal{U}_\star(E)$ are non-constant for all E in a neighborhood of 0, so the eigenvectors of $\mathbf{H}$ corresponding to sufficiently small eigenvalues are also non-Gaussian.

It is also natural to ask whether our results hold for convergence in distribution. In the case $\alpha \in (1,2) \setminus \mathcal{A}$ we will address this in Section 3.8 through Proposition 3.8.1 by studying the rate of growth of the moments of the limiting distribution. If $\alpha < 1$, then the moments of $\mathcal{U}_\star(E)$ grow too quickly for this to determine the law of $\mathcal{N}^2 \cdot \mathcal{U}_\star(E)$.

Finally, we note that we consider the squared eigenvector entries $\mathbf{u}_k(i)^2$ to avoid ambiguity in the choice of sign for $\mathbf{u}_k(i)$, since given an eigenvalue λ_k of a real symmetric matrix and a corresponding eigenvector $\mathbf{v}_k$, the vector $-\mathbf{v}_k$ is also an eigenvector. In the context of Lévy random matrices, if one chooses this sign independently with probability 1/2 for each possibility, then our methods show the above results hold with the conclusion of Theorem 3.1.7 replaced by the convergence in moments of $\big(\sqrt{N}\mathbf{u}_k(i_1), \sqrt{N}\mathbf{u}_k(i_2), \dots, \sqrt{N}\mathbf{u}_k(i_n)\big)$ to $\big(\mathcal{N}_1 \cdot \mathcal{U}_1^{1/2}(E), \mathcal{N}_2 \cdot \mathcal{U}_2^{1/2}(E), \dots, \mathcal{N}_n \cdot \mathcal{U}_n^{1/2}(E)\big)$, where the $\mathcal{N}_k$ remain i.i.d. standard Gaussians, and similarly for Theorem 3.1.8.

3.2 Proofs of main results

Assuming some claims proven in later parts of this paper, we will in this section establish the results stated in Section 3.1.2. This will proceed through the following steps.

1. We define a matrix $\mathbf{X}$, obtained by setting the small entries of the original Lévy matrix $\mathbf{H}$ to zero, and the Gaussian perturbation $\mathbf{X}_s = \mathbf{X} + \sqrt{s}\mathbf{W}$, where $\mathbf{W}$ is a GOE matrix. For a specific choice of $s = t$, with $N^{-1/2} \ll t \ll 1$, we show as Theorem 3.2.7 that the eigenvector statistics of $\mathbf{H}$ (corresponding to small eigenvalues) are approximated by those of $\mathbf{X}_t$.

2. We show as Theorem 3.2.8 that moments of the eigenvector entries of $\mathbf{X}_t$ (corresponding to small eigenvalues) can be identified through resolvent entries of $\mathbf{X}_t$.

3. We compute as Theorem 3.2.9 the limits of these resolvent entries as N and η tend to ∞ and 0, respectively.

In Section 3.2.5, we prove Theorem 3.1.7, Theorem 3.1.8, and Corollary 3.1.9, given that the results enumerated above and Proposition 3.1.4 hold. The remaining sections of the paper verify these prerequisite results.

3.2.1 Notation

Throughout, we write C for a large constant and c for a small constant. These may depend on other constants and may change line to line, but only finitely many times, so that they remain finite. We say $X \ll Y$ if there exists a small constant $c > 0$ such that $N^c|X| \le Y$. Constants in this paper may depend on the constant $c > 0$ implicit in the claim $X \ll Y$, but we suppress this in the notation. We write $X \lesssim Y$ if there exists $C > 0$ such that $|X| \le CY$; we also say $X \lesssim_u Y$, or equivalently $X = O_u(Y)$, if $|X| \le C_u|Y|$ for some constant $C_u > 0$ depending on a parameter u.

In what follows, for any function (or vector) f, we let $\|f\|_\infty$ denote the L^∞-norm of f. We also denote $\mathrm{Mat}_{N\times N}$ by the set of $N \times N$ real, symmetric matrices. Given $\mathbf{M} \in \mathrm{Mat}_{N\times N}$, we denote its eigenvalues by $\lambda_1(\mathbf{M}), \lambda_2(\mathbf{M}), \ldots, \lambda_N(\mathbf{M})$ in non-decreasing order. We further let $\mathbf{u}_i(\mathbf{M})$ denote the unit eigenvector corresponding to the eigenvalue $\lambda_i(\mathbf{M})$ for each i. We also make the following definition.

Definition 3.2.1. We say a (sequence of) vectors $\mathbf{q} = \mathbf{q}(N) = (q_1, q_2, \ldots q_N) \in \mathbb{R}^N$ has *stable support* if there exists a constant $C > 0$ such that the set $\{(i, q_i) : q_i \neq 0\}$ does not change for $N > C$. We let $\operatorname{supp} \mathbf{q} = \{i : q_i \neq 0\}$ denote the support of $\mathbf{q}$.

We next introduce the notion of overwhelming probability.

Definition 3.2.2. We say that a family of events $\{\mathcal{F}(u)\}$ indexed by some parameter(s) $u \in U^{(N)}$, where $U^{(N)}$ is a parameter set which may depend on N, holds with *overwhelming probability* if, for any $D > 0$, there exists $N\big(D, U^{(N)}\big) > 0$ such that for $N \ge N\big(D, U^{(N)}\big)$,

$$\inf_{u \in U^{(N)}} \mathbb{P}\big(\mathcal{F}(u)\big) \ge 1 - N^{-D}. \tag{3.2.1}$$

Next, given $\alpha \in (0,2)$ we may select positive real numbers $b = b(\alpha) > 0$; $\nu = \nu(\alpha) > 0$; $\mathfrak{a} = \mathfrak{a}(\alpha) > 0$; and $\rho = \rho(\alpha) > 0$ such that

$$\nu = \frac{1}{\alpha} - b > 0; \quad \frac{1}{4-\alpha} < \nu < \frac{1}{4-2\alpha}; \quad (2-\alpha)\nu < \mathfrak{a} < \frac{1}{2}; \quad 0 < \rho < \nu < \frac{1}{2}; \quad \alpha\rho < (2-\alpha)\nu. \tag{3.2.2}$$

These parameters will be fixed throughout the paper, and we will let other constants depend on them (and on α), even when not explicitly noted. We always assume $\alpha \in (0,2) \setminus \mathcal{A}$, where $\mathcal{A}$ is the set from Lemma 3.3.2 below (or, equivalently, the one from Proposition 3.1.4).

3.2.2 Comparison

We first recall the definition of the removed model $\mathbf{X}$ from [6, Definition 3.2].

Definition 3.2.3. Recalling the notation of Definition 3.1.1, let $X = (Z+J)\mathbf{1}_{|Z+J|>N^b}$. We call X the *b-removal of $Z+J$*. Further, let $\{X_{ij}\}_{1 \le i \le j \le N}$ be mutually independent random variables that each have the same law as $N^{-1/\alpha}X$. Set $X_{ij} = X_{ji}$ for each $1 \le j < i \le N$, and define the $N \times N$ symmetric matrix $\mathbf{X} = \{X_{ij}\}$. We call $\mathbf{X}$ a *b-removed α-Lévy matrix*.

We also recall a resampling and coupling of $\mathbf{X}$ and $\mathbf{H}$ that was described in [6, Section 3.3.1].

Definition 3.2.4. We define mutually independent random variables $\{a_{ij}, b_{ij}, c_{ij}, \psi_{ij}, \chi_{ij}\}_{1 \le i \le j \le N}$ as follows. Let ψ_{ij} and χ_{ij} denote $0-1$ Bernoulli random variables with distributions

$$\mathbb{P}\big[\psi_{ij} = 1\big] = \mathbb{P}\big[|H_{ij}| \ge N^{-\rho}\big], \quad \mathbb{P}\big[\chi_{ij} = 1\big] = \frac{\mathbb{P}\big[|H_{ij}| \in [N^{-\nu}, N^{-\rho})\big]}{\mathbb{P}\big[|H_{ij}| < N^{-\rho}\big]}. \tag{3.2.3}$$

Additionally, let a_{ij}, b_{ij}, and c_{ij} be random variables such that

$$\mathbb{P}[a_{ij} \in I] = \frac{\mathbb{P}\Big[H_{ij} \in (-N^{-\nu}, N^{-\nu}) \cap I\Big]}{\mathbb{P}\big[|H_{ij}| < N^{-\nu})\big]}, \tag{3.2.4}$$

$$\mathbb{P}[b_{ij} \in I] = \frac{\mathbb{P}\Big[H_{ij} \in \big((-N^{-\rho}, -N^{-\nu}] \cup [N^{-\nu}, N^{-\rho})\big) \cap I\Big]}{\mathbb{P}\big[|H_{ij}| \in [N^{-\nu}, N^{-\rho})\big]}, \tag{3.2.5}$$

$$\mathbb{P}[c_{ij} \in I] = \frac{\mathbb{P}\Big[H_{ij} \in \big((-\infty, -N^{-\rho}] \cup [N^{-\rho}, \infty)\big) \cap I\Big]}{\mathbb{P}\big[|H_{ij}| \geq N^{-\rho}\big]} \tag{3.2.6}$$

for any interval $I \subset \mathbb{R}$. For each $1 \leq j < i \leq N$, define $a_{ij} = a_{ji}$ by symmetry, and similarly for each of b_{ij}, c_{ij}, ψ_{ij}, and χ_{ij}.

Because a_{ij}, b_{ij}, c_{ij}, ψ_{ij}, and χ_{ij} are mutually independent, H_{ij} has the same law as

$$(1 - \psi_{ij})(1 - \chi_{ij})a_{ij} + (1 - \psi_{ij})\chi_{ij}b_{ij} + \psi_{ij}c_{ij} \tag{3.2.7}$$

and X_{ij} has the same law as $(1 - \psi_{ij})\chi_{ij}b_{ij} + \psi_{ij}c_{ij}$. Therefore, although the random variables $H_{ij}\mathbf{1}_{|H_{ij}| \geq N^{-\rho}}$, $H_{ij}\mathbf{1}_{N^{-\nu} \leq |H_{ij}| < N^{-\rho}}$, and $H_{ij}\mathbf{1}_{|H_{ij}| < N^{-\nu}}$ are correlated, this decomposition expresses their dependence through the Bernoulli random variables ψ_{ij} and χ_{ij}.

Definition 3.2.5. For each $1 \leq i, j \leq N$, set

$$A_{ij} = (1 - \psi_{ij})(1 - \chi_{ij})a_{ij}, \quad B_{ij} = (1 - \psi_{ij})\chi_{ij}b_{ij}, \quad C_{ij} = \psi_{ij}c_{ij}, \tag{3.2.8}$$

and define the four $N \times N$ matrices $\mathbf{A} = \{A_{ij}\}$, $\mathbf{B} = \{B_{ij}\}$, $\mathbf{C} = \{C_{ij}\}$, and $\Psi = \{\psi_{ij}\}$.

For the remainder of the paper we sample $\mathbf{H}$ and $\mathbf{X}$ by setting $\mathbf{H} = \mathbf{A} + \mathbf{B} + \mathbf{C}$ and $\mathbf{X} = \mathbf{B} + \mathbf{C}$, inducing a coupling between the two matrices. We commonly refer to Ψ as the *label* of $\mathbf{H}$ (or of $\mathbf{X}$). Defining $\mathbf{H}$ and $\mathbf{X}$ in this way ensures that their entries have the same laws as in Definition 3.1.1 and Definition 3.2.3, respectively.

For any $s \in \mathbb{R}_+$, we define the matrix $\mathbf{X}_s \in \mathrm{Mat}_{N\times N}$ by setting

$$\mathbf{X}_s = \mathbf{X} + \mathbf{W}_s, \tag{3.2.9}$$

where $\mathbf{W}_s = \big(w_{ij}(s)\big)_{1\le i,j\le N} \in \mathrm{Mat}_{N\times N}$ and w_{ij} are mutually independent Brownian motions with symmetry constraint $w_{ij} = w_{ji}$ and variance $(1 + \mathbf{1}_{i=j})N^{-1}$.

We now make a specific choice of the time t to enable our comparison argument. Define t by

$$t = N\mathbb{E}\Big[H_{11}^2 \mathbf{1}_{|H_{11}|<N^{-\nu}} \big| |H_{11}| < N^{-\rho}\Big] = \frac{N\mathbb{E}\big[H_{11}^2\mathbf{1}_{|H_{11}|<N^{-\nu}}\big]}{\mathbb{P}\big[|H_{11}| < N^{-\rho}\big]}. \tag{3.2.10}$$

The following estimate is [6, Lemma 3.5] and can be quickly deduced from (3.1.3) and (3.2.10).

Lemma 3.2.6 ([6, Lemma 3.5])**.** *Under the choice of* (3.2.10)*, we have that*

$$cN^{(\alpha-2)\nu} \le t \le CN^{(\alpha-2)\nu}. \tag{3.2.11}$$

Observe in particular that (3.2.11) implies that $N^{-1/2} \ll t \ll 1$, by the third inequality in (3.2.2). The next theorem is proved in Section 3.4 and completes the first step of the outline given in the beginning of Section 3.2.

Theorem 3.2.7. *There exist constants $c_1, c_2 > 0$ such that the following holds. Let t be as in* (3.2.10)*, $P\colon \mathbb{R}^n \to \mathbb{R}^n$ be a polynomial in n variables, and $\mathbf{q} \in \mathbb{R}^N$ be a unit vector with stable support. Then there exists a constant $C = C\big(P, |\operatorname{supp}\mathbf{q}|\big) > 0$ such that, for indices $i_1, i_2, \ldots, i_n \in \big[(1/2 - c_1)N, (1/2 + c_1)N\big]$,*

$$\left|\mathbb{E}\left[P\bigg(\Big(N\langle \mathbf{q}, \mathbf{u}_{i_k}(\mathbf{X}_t)\rangle^2\Big)_{1\le k\le n}\bigg)\right] - \mathbb{E}\left[P\bigg(\Big(N\langle \mathbf{q}, \mathbf{u}_{i_k}(\mathbf{H})\rangle^2\Big)_{1\le k\le n}\bigg)\right]\right| \le CN^{-c_2}. \tag{3.2.12}$$

3.2.3 Short-time universality

For each integer $k \in [1, N]$ and real number $s \geq 0$, abbreviate $\lambda_k(s) = \lambda_k(\mathbf{X}_s)$, and set $\boldsymbol{\lambda}(s) = \big(\lambda_1(s), \lambda_2(s), \ldots, \lambda_N(s)\big)$. Further let $\mathbf{u}_k(s) \in \mathbb{R}^N$ denote the unit eigenvector of $\mathbf{X}_s$ associated with $\lambda_k(s)$, and set $\mathbf{u}(s) = \big(\mathbf{u}_1(s), \ldots, \mathbf{u}_N(s)\big)$.

Next, for any unit vector $\mathbf{q} \in \mathbb{R}^N$ and $k \in [1, n]$, set $z_k(s) = z_k(s, \mathbf{q}) = \sqrt{N}\big\langle \mathbf{q}, \mathbf{u}_k(s) \big\rangle$. For any integer $m \geq 1$; indices $i_1, i_2, \ldots, i_m \in [1, N]$; and integers $j_1, j_2, \ldots, j_m \geq 0$, define

$$Q^{j_1, \ldots, j_m}_{i_1, \ldots, i_m}(s) = \prod_{l=1}^{m} z_{i_l}(s)^{2 j_l} \prod_{l=1}^{m} a(2 j_l)^{-1}, \quad \text{where} \quad a(2j) = (2j-1)!!. \tag{3.2.13}$$

The normalization factors $a(2j)$ are chosen because they are the moments of a standard Gaussian.

To any index set $\{(i_1, j_1), \ldots, (i_m, j_m)\}$ with distinct $i_k \in [1, N]$ and positive j_k, we may associate the vector $\boldsymbol{\xi} = (\xi_1, \xi_2, \ldots, \xi_N) \in \mathbb{N}^N$ with $\xi_{i_k} = j_k$ for $1 \leq k \leq m$ and $\xi_p = 0$ for $p \notin \{i_1, \ldots, i_m\}$. We think of $\boldsymbol{\xi}$ as a particle configuration on the integers, with j_k particles at site i_k for all k and zero particles on the sites not in $\{i_1, \ldots, i_m\}$. We call the set $\{i_1, \ldots, i_m\}$ the support of $\boldsymbol{\xi}$, denoted $\operatorname{supp} \boldsymbol{\xi}$. Denote $\mathcal{N}(\boldsymbol{\xi}) = \sum_{j=1}^{m} j_k$, the total number of particles. The configuration $\boldsymbol{\xi}^{ij}$ is defined as the result of moving one particle in $\boldsymbol{\xi}$ from i to j, that is, if $i \neq j$ then ξ^{ij}_k equals $\xi_k + 1$, $\xi_k - 1$, or ξ_k for $k = j$, $k = i$, and $k \notin \{i, j\}$, respectively. Under this notation, we define an observable $F_s(\boldsymbol{\xi})$ by the expectation

$$F_s(\boldsymbol{\xi}) = \mathbb{E}\big[Q^{j_1, \ldots, j_m}_{i_1, \ldots, i_m}(s)\big]. \tag{3.2.14}$$

Now fix $\mathfrak{c} \in \mathbb{R}_{>0}$, later chosen to be sufficiently small. Recalling $\mathfrak{a}$ from (3.2.2) and t from (3.2.10), define

$$\psi = N^{\mathfrak{c}}, \quad \eta = N^{-\mathfrak{a}} \psi, \quad \text{so that} \quad N^{-1/2} \ll \eta \ll t, \tag{3.2.15}$$

where the last inequality in (3.2.15) follows from the third bound in (3.2.2), (3.2.11), and the fact that $\mathfrak{c}$ is small. For each $s \in \mathbb{R}_{>0}$ we define the resolvent $\mathbf{R}(s, z)$, Stieltjes transform

$m_N(s,z)$ of $\mathbf{X}_s$, and the expectation of $m_N(s,z)$ by

$$\mathbf{R}(s,z) = (\mathbf{X}_s - z)^{-1}, \qquad m_N(s,z) = N^{-1} \operatorname{Tr} \mathbf{R}(s,z), \qquad \widehat{m}_N(s,z) = \mathbb{E}\big[m_N(s,z)\big]. \tag{3.2.16}$$

We also define the (random) empirical spectral measure for $\mathbf{X}_s$ by

$$\mu_s = \frac{1}{N} \sum_{i=1}^{N} \delta_{\lambda_i(s)}, \tag{3.2.17}$$

where δ_x is the discrete probability measure that places all its mass at x. Further, we define $\widehat{\mu}_s = \mathbb{E}[\mu_s]$ and observe that the Stieltjes transform of $\widehat{\mu}_s$ is $\widehat{m}_N(s,z)$. The classical eigenvalue locations for $\widehat{\mu}_s$ are given by

$$\widehat{\gamma}_i(s) = \inf \left\{ y \in \mathbb{R} : \widehat{\mu}_s\big((-\infty, y]\big) \geq \frac{i}{N} \right\}. \tag{3.2.18}$$

Recalling the specific choice of t from (3.2.10), we abbreviate $\widehat{\gamma}_i = \widehat{\gamma}_i(t)$.

The following theorem is proved in Section 3.5 and completes the second step of the above outline. We recall from (3.1.7) the notation $\gamma_k = \gamma_k^{(\alpha)}$.

Theorem 3.2.8. *Fix $m \in \mathbb{N}$, let $\mathbf{q} \in \mathbb{R}^N$ be a unit vector with stable support, and let t be the time defined in* (3.2.10). *There exist constants $c_1 > 0$, $c_2 = c_2(m) > 0$, and $C = C(m, |\operatorname{supp} \mathbf{q}|) > 0$ such that, if $\mathfrak{c} < c_2$, then*

$$\max_{\substack{\boldsymbol{\xi} : \mathcal{N}(\boldsymbol{\xi}) = m \\ \operatorname{supp} \boldsymbol{\xi} \in [(1/2 - c_1)N, (1/2 + c_1)N]}} \left| F_t(\boldsymbol{\xi}) - \mathbb{E}\left[\prod_{k=1}^{N} \left(\frac{\operatorname{Im} \langle \mathbf{q}, \mathbf{R}(t, \widehat{\gamma}_k + \mathrm{i}\eta) \mathbf{q} \rangle}{\operatorname{Im} m_\alpha(\gamma_k + \mathrm{i}\eta)} \right)^{\xi_k} \right] \right| \leq C N^{-c_2}, \tag{3.2.19}$$

where $\boldsymbol{\xi} = (\xi_1, \xi_2, \dots, \xi_N)$.

3.2.4 Scaling limit

The next theorem will be proven in Section 3.6 and establishes the scaling limit of the quantity compared to F_t on the left side of (3.2.19), completing step 3 of the above outline. Here, we recall $\mathcal{U}_\star(E)$ from Definition 3.1.5.

Theorem 3.2.9. *There exist constants $c_1, c_2 > 0$ such that the following holds. Fix an integer $n > 0$ (independently of N) and index sequences $\{k_j(N)\}_{1 \le j \le n}$ such that for every N, $\{k_j(N)\}_{1 \le j \le n}$ are distinct integers in $[1, N]$ and $|k_1 - k_j| < N^{1/2}$ for each $j \in [2, n]$. Let $\mathbf{q} = (q_1, \dots, q_N) \in \mathbb{R}^N$ be a unit vector with stable support; let t be as in* (3.2.10)*; and assume that $\lim_{N \to \infty} \gamma_{k_1} = E$, for some $E \in [-c_1, c_1]$, and $\mathfrak{c} < c_2$. Then the vector*

$$\left(\frac{\operatorname{Im} \langle \mathbf{q}, \mathbf{R}(t, \widehat{\gamma}_{k_j} + \mathrm{i}\eta) \mathbf{q} \rangle}{\operatorname{Im} m_\alpha(\gamma_{k_j} + \mathrm{i}\eta)} \right)_{1 \le j \le n} \tag{3.2.20}$$

converges in moments to

$$(1, 1, \dots, 1) \cdot \sum_{i \in \operatorname{supp} \mathbf{q}} q_i^2 \mathcal{U}_i(E), \tag{3.2.21}$$

where the random variables $\mathcal{U}_i(E)$ are independent and identically distributed with law $\mathcal{U}_\star(E)$.

3.2.5 Proofs

In this section we establish Theorem 3.1.8, Theorem 3.1.7, and Corollary 3.1.9.

Proof of Theorem 3.1.8. By symmetry, we may suppose $i = 1$ in the theorem statement. Recalling t from (3.2.10) and applying Theorem 3.2.7 with $\mathbf{q} = \mathbf{e}_1 = (1, 0, 0, \dots, 0)$ gives

$$\lim_{N \to \infty} \left| \mathbb{E}\left[P\left(\left(N \langle \mathbf{u}_{k_j}(\mathbf{H}), \mathbf{e}_1 \rangle^2 \right)_{1 \le j \le n} \right) \right] - \mathbb{E}\left[P\left(\left(N \langle \mathbf{u}_{k_j}(\mathbf{X}_t), \mathbf{e}_1 \rangle^2 \right)_{1 \le j \le n} \right) \right] \right| = 0, \tag{3.2.22}$$

Next, Theorem 3.2.8 yields

$$\lim_{N\to\infty}\left|\mathbb{E}\left[P\bigg(\Big(N\big\langle\mathbf{u}_{k_j}(\mathbf{X}_t),\mathbf{e}_1\big\rangle^2\Big)_{1\le j\le n}\bigg)\right]-\mathbb{E}\left[P\left(\left(\mathcal{N}_j^2\cdot\frac{\operatorname{Im}R_{11}(t,\widehat{\gamma}_{k_j}+\mathrm{i}\eta)}{\operatorname{Im}m_\alpha(\gamma_{k_j}+\mathrm{i}\eta)}\right)_{1\le j\le n}\right)\right]\right|=0, \tag{3.2.23}$$

where the $\mathcal{N}_j$ are i.i.d. standard Gaussians that are independent from $\operatorname{Im}R_{11}(t,\widehat{\gamma}_{k_j}+\mathrm{i}\eta)$. Here we used (3.2.13) and the fact that $a(2j)=\mathbb{E}\,[\mathcal{N}^{2j}]$ for a standard Gaussian $\mathcal{N}$.

Now the theorem follows from (3.2.22), (3.2.23), and Theorem 3.2.9. □

Proof of Theorem 3.1.7. By symmetry, we may suppose that $i_j=j$ for each $j\in[1,n]$. Let

$$\mathbf{v}=\big(\mathcal{U}_1(E),\mathcal{U}_2(E),\ldots,\mathcal{U}_n(E)\big) \tag{3.2.24}$$

be a vector of i.i.d. random variables with distribution $\mathcal{U}_\star(E)$, where $\mathcal{U}_\star(E)$ is as in Definition 3.1.5.

For any vector $\mathbf{q}$ with stable support such that $q_i=0$ for $i\notin[1,n]$, let $\mathbf{w}\in\mathbb{R}^N$ denote the vector $\mathbf{w}=(q_1^2,q_2^2,\ldots,q_N^2)$. Fix $m\in\mathbb{N}$, recall $a(2m)=(2m-1)!!$ from (3.2.13), abbreviate $\mathbf{u}_k=\mathbf{u}_k(\mathbf{H})$, and consider the polynomial

$$Q(q_1,\ldots,q_n)=\mathbb{E}\Big[\big(N\langle\mathbf{q},\mathbf{u}_k\rangle^2\big)^m\Big]-a(2m)\mathbb{E}\big[\langle\mathbf{w},\mathbf{v}\rangle^m\big]. \tag{3.2.25}$$

Then together Theorem 3.2.7, Theorem 3.2.8, and (the $n=1$ case of) Theorem 3.2.9 imply for any unit vector $\mathbf{q}\in\mathbb{R}^N$ with $\operatorname{supp}\mathbf{q}\subseteq\{1,2,\ldots,n\}$ that

$$\lim_{N\to\infty}Q(q_1,\ldots,q_n)=0. \tag{3.2.26}$$

Here we recalled (3.2.13) and the fact that $a(2j)=\mathbb{E}\,[\mathcal{N}^{2j}]$ for a standard Gaussian $\mathcal{N}$. Now observe that Q is a polynomial of degree $2m$ in the q_i, that is, there exists coefficients $B_{\mathbf{d}}\in\mathbb{R}$

such that

$$Q(q_1, q_2, \ldots, q_n) = \sum_{|\mathbf{d}|=2m} B_{\mathbf{d}} \prod_{j=1}^{n} q_j^{d_j}, \tag{3.2.27}$$

where $\mathbf{d} = (d_1, d_2, \ldots, d_n) \in \mathbb{Z}_{\geq 0}^n$ is summed over all n-tuples of nonnegative integers with $|\mathbf{d}| = \sum_{j=1}^n d_j = 2m$. Thus, since (3.2.26) holds for all $(q_1, q_2, \ldots, q_n)$ with $\sum_{j=1}^n q_j^2 = 1$, we have

$$\lim_{N\to\infty} \max_{|\mathbf{d}|=2m} |B_{\mathbf{d}}| = 0, \tag{3.2.28}$$

where again $\mathbf{d}$ ranges over all n-tuples of nonnegative integers summing to $2m$. In particular, fixing some n-tuple $(m_1, m_2, \ldots, m_n)$ of nonnegative integers summing to m and taking $\mathbf{d} = (2m_1, 2m_2, \ldots, 2m_n)$ gives

$$\frac{(2m)!}{\prod_{j=1}^n (2m_i)!} \lim_{N\to\infty} \mathbb{E}\left[\prod_{j=1}^n \big(N\mathbf{u}_k(j)^2\big)^{m_j}\right] = \frac{m!(2m-1)!!}{\prod_{j=1}^n m_i!} \mathbb{E}\left[\prod_{j=1}^n \mathbf{v}(j)^{m_j}\right], \tag{3.2.29}$$

which implies that

$$\lim_{N\to\infty} \mathbb{E}\left[\prod_{j=1}^n \big(N\mathbf{u}_k(j)^2\big)^{m_j}\right] = \mathbb{E}\left[\prod_{j=1}^n a(2m_j)\mathbf{v}(j)^{m_j}\right]. \tag{3.2.30}$$

This yields the desired conclusion, since (3.2.30) holds for all $(m_1, m_2, \ldots, m_n) \in \mathbb{Z}_{\geq 0}^n$ and $\mathbb{E}[\mathcal{N}^{2j}] = a(2j)$, for any integer $j \geq 0$, if $\mathcal{N}$ is a standard Gaussian random variable. $\square$

Proof of Corollary 3.1.9. By [32, Theorem 1.6(ii)],

$$\varrho_\alpha(0) = \frac{1}{\pi}\Gamma\left(1 + \frac{2}{\alpha}\right)\left(\frac{\Gamma\big(1 - \frac{\alpha}{2}\big)}{\Gamma\big(1 + \frac{\alpha}{2}\big)}\right)^{1/\alpha}. \tag{3.2.31}$$

By [32, Lemma 4.3(ii)], $\mathcal{R}_\star(0)$ has the same law as Υ^{-1}, where Υ is a one-sided $\frac{\alpha}{2}$-stable law with Laplace transform

$$\mathbb{E}\big[\exp(-t\Upsilon)\big] = \exp\left(- t^{\alpha/2}\left(\frac{\Gamma\left(1+\frac{\alpha}{2}\right)}{\Gamma\left(1-\frac{\alpha}{2}\right)}\right)^{1/2}\right), \quad \text{for } t \geq 0. \tag{3.2.32}$$

Since a $(1,1)$ $\frac{\alpha}{2}$-stable law S has Laplace transform $\exp\left(-t^{\alpha/2}\right)$, Υ has the same law as

$$\left(\frac{\Gamma\left(1+\frac{\alpha}{2}\right)}{\Gamma\left(1-\frac{\alpha}{2}\right)}\right)^{1/\alpha} S, \tag{3.2.33}$$

so the conclusion follows from Theorem 3.1.7, (3.2.31), and the fact (see Definition 3.1.5) that $\mathcal{U}_\star(E) = \big(\pi \varrho_\alpha(E)\big)^{-1}\mathcal{R}_\star(E)$. $\square$

3.3 Preliminary results

In this section we collect several miscellaneous known results that will be used throughout the paper. After recalling general estimates and identities on resolvent matrices in Section 3.3.1, we state several facts about the density ρ_α in Section 3.3.2. In Section 3.3.3, we recall several results about the removed model $\mathbf{X}_s$. Finally, in Section 3.3.4 we recall properties of a certain matrix interpolating between $\mathbf{X}_0$ and $\mathbf{H}$.

3.3.1 Resolvent identities and estimates

For any invertible $\mathbf{K}, \mathbf{M} \in \operatorname{Mat}_{N\times N}$, we have

$$\mathbf{K}^{-1} - \mathbf{M}^{-1} = \mathbf{K}^{-1}(\mathbf{M} - \mathbf{K})\mathbf{M}. \tag{3.3.1}$$

Next, assume $z = E + \mathrm{i}\eta \in \mathbb{H}$ and $\mathbf{K} = \{K_{ij}\} = (\mathbf{M} - z)^{-1}$. Then, we have the bound

$$\max_{1\leq i,j\leq N} |K_{ij}| \leq \frac{1}{\eta}, \tag{3.3.2}$$

and the Ward identity

$$\sum_{j=1}^{N} |K_{ij}|^2 = \frac{\operatorname{Im} K_{ii}}{\eta}. \tag{3.3.3}$$

3.3.2 The density ϱ_α

The following properties of the density ϱ_α are proved in Section 3.7; here, we recall the γ_i from (3.1.7).

Lemma 3.3.1. *There exists a (deterministic) countable set $\mathcal{A} \subset (0,2)$ with no accumulation points in $(0,2)$ and constants $C, c > 0$ such that the following statements hold for $\alpha \in (0,2) \setminus \mathcal{A}$.*

1. *For real numbers $E_1, E_2 \in [-c, c]$,*

$$\big|\varrho_\alpha(E_1) - \varrho_\alpha(E_2)\big| \le C|E_1 - E_2|, \qquad c \le \varrho_\alpha(E_1) \le C. \tag{3.3.4}$$

2. *For real numbers $E_1, E_2 \in [-c, c]$, and any $\eta > 0$, we have*

$$\big|\operatorname{Im} m_\alpha(E_1 + \mathrm{i}\eta) - \operatorname{Im} m_\alpha(E_2 + \mathrm{i}\eta)\big| \le C|E_1 - E_2| + C\eta. \tag{3.3.5}$$

3. *For real numbers $|E| < c$ and $\eta \in (0, c]$, and any integer $j \in \big[(1/2 - c)N, (1/2 + c)N\big]$,*

$$c \le \big|\operatorname{Im} m_\alpha(E + \mathrm{i}\eta)\big| \le C, \quad c \le \big|\operatorname{Im} m_\alpha(\gamma_j + \mathrm{i}\eta)\big| \le C. \tag{3.3.6}$$

3.3.3 Removed model

In this section we recall several results concerning the resolvent $\mathbf{R}(s,z)$ and Stieltjes transform $m_N(s,z)$ of $\mathbf{X}_s$ (recall (3.2.9) and (3.2.16)). In what follows, we recall that the i-th eigenvalue of $\mathbf{X}_s$ is denoted by $\lambda_i(s)$ and its associated unit eigenvector is denoted by $\mathbf{u}_i(s)$.

For any constants $C, \delta > 0$, we define the two spectral domains

$$\mathcal{D}_{C,\delta} = \left\{ z = E + \mathrm{i}\eta \colon |E| \le \frac{1}{C},\ N^{-1+\delta} \le \eta \le \frac{1}{C} \right\}, \tag{3.3.7}$$

$$\widetilde{\mathcal{D}}_{C,\delta} = \left\{ z = E + \mathrm{i}\eta \colon |E| \le \frac{1}{C},\ N^{-1/2+\delta} \le \eta \le \frac{1}{C} \right\}. \tag{3.3.8}$$

We also recall the free convolution of $\mathbf{X}$ with the semicircle law is defined to be the probability measure on $\mathbb{R}$ whose Stiletjes transform satisfies the equation

$$m_{\mathrm{fc},t}(s,z) = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{\lambda_i(0) - z - s m_{\mathrm{fc},t}(s,z)}. \tag{3.3.9}$$

Basic facts about the free convolution, including its existence and uniqueness, may be found in [29]. It has a density, $\rho_{\mathrm{fc},t}(s,x)\,dx$, and its classical eigenvalue locations are defined for $1 \le i \le N$ by

$$\gamma_i(s) = \inf \left\{ y \in \mathbb{R} \colon \int_{-\infty}^{y} \rho_{\mathrm{fc},t}(s,x)\,dx \ge \frac{i}{N} \right\}. \tag{3.3.10}$$

These are random variables which depend on the initial data $\big(\lambda_i(0)\big)_{i=1}^{N}$ determined by $\mathbf{X}_0$.

The following intermediate local law for $\mathbf{R}(s,z)$ (on scale $\eta \gg N^{-1/2+\delta}$) was essentially shown as [6, Theorem 3.5].

Lemma 3.3.2 ([6, Theorem 3.5])**.** *There exists a (deterministic) countable set $\mathcal{A} \subset (0,2)$ with no accumulation points in $(0,2)$ such that the following holds for $\alpha \in (0,2) \setminus \mathcal{A}$. For any fixed real number $\delta > 0$ with $\delta < \max\left\{ \frac{(b-1/\alpha)(2-\alpha)}{20}, \frac{1}{2} \right\}$, there exists a constant $C = C(\delta) > 0$ such that for $s \in [0, N^{-\delta}]$, we have with overwhelming probability that*

$$\sup_{z \in \widetilde{\mathcal{D}}_{C,\delta}} \big| m_N(s,z) - m_\alpha(z) \big| < C N^{-\alpha\delta/8}, \qquad \sup_{z \in \widetilde{\mathcal{D}}_{C,\delta}} \max_{1 \le j \le N} \big| R_{jj}(s,z) \big| < (\log N)^C, \tag{3.3.11}$$

where we recall $\widetilde{D}_{C,\delta}$ from (3.3.8)*.*

In fact, [6, Theorem 3.5] was only stated in the case $s = 0$, but it is quickly verified that the same proof applies for arbitrary $s \in [0, N^{-\delta}]$, especially since $\mathbf{H} + s^{1/2}\mathbf{W}$ satisfies the conditions in Definition 3.1.1 for $s \in [0, N^{-\delta}]$ if $\mathbf{H}$ does.

From now on, we always assume $\alpha \in (0,2) \backslash \mathcal{A}$, where $\mathcal{A}$ is the set from Lemma 3.3.2, even when this is not noted explicitly. The next lemma provides more local estimates on $\mathbf{R}(s,z)$ and $\mathbf{X}_s$ (on scales around N^{-1}), if $N^{-1/2} \ll s \ll 1$; they are consequences of Lemma 3.3.2 using results of [78]. Specifically, the first bound in (3.3.12) follows from [78, Theorem 3.3] and the second follows from the first and the first estimate in (3.3.11); (3.3.14) follows from [78, Theorem 3.5]; and (3.3.15) follows from [78, Theorem 3.6]. The hypotheses of these statements from [78] are all verified by the first bound in (3.3.11). The final estimate is an immediate consequence of (3.3.12), (3.3.13), and (3.3.5).

In the following, we recall the $\gamma_i(s)$ defined in (3.3.10).

Lemma 3.3.3 ([78])**.** *There exists a constant $K > 0$ such that the following holds for any real numbers $r, \delta > 0$.*

1. *Set $\mathcal{D} = \mathcal{D}_{K,\delta}$ and $\widetilde{\mathcal{D}} = \widetilde{\mathcal{D}}_{K,\delta}$, where we recall the definitions* (3.3.7) *and* (3.3.8)*, respectively. With overwhelming probability, we have that*

$$\sup_{s \in [N^{-1/2+\delta}, N^{-\delta}]} \sup_{z \in \mathcal{D}} \big| m_N(s,z) - m_{\mathrm{fc},t}(s,z) \big| < \frac{N^{\delta}}{N\eta}, \tag{3.3.12}$$

$$\sup_{s \in [N^{-1/2+\delta}, N^{-\delta}]} \sup_{z \in \widetilde{\mathcal{D}}} \big| m_\alpha(z) - m_{\mathrm{fc},t}(s,z) \big| < N^{-\alpha\delta/16}. \tag{3.3.13}$$

2. *With overwhelming probability, we have that*

$$\sup_{s \in [N^{-1/2+\delta}, N^{-\delta}]} \big| \lambda_i(s) - \gamma_i(s) \big| \le N^{-1+\delta}. \tag{3.3.14}$$

3. *For any $\varepsilon \in (0,1)$ and $s \in [N^{-1/2+\delta}, N^{-\delta}]$, we have for sufficiently large N that*

$$\mathbb{P}\left(\big| \lambda_i(s) - \lambda_{i+1}(s) \big| \le \frac{\varepsilon}{N} \right) \le N^{\delta} \varepsilon^{2-r}. \tag{3.3.15}$$

4. For $E_1, E_2 \in [-K^{-1}, K^{-1}]$ and $\eta \in [N^{-1/2+\delta}, N^{-\delta}]$, we have

$$\big| \operatorname{Im} m_N(E_1 + \mathrm{i}\eta) - \operatorname{Im} m_N(E_2 + \mathrm{i}\eta) \big| \le C|E_1 - E_2| + C\eta + CN^{-\alpha\delta/16} \tag{3.3.16}$$

with overwhelming probability.

The following lemma provides resolvent and delocalization estimates for $\mathbf{X}_s$. The first estimate in (3.3.17) below follows from [6, Proposition 3.9], whose hypotheses are verified by the second bound in (3.3.11). We omit the proof of the second since, given the first, it follows by standard arguments (for example, see the proof of [26, Theorem 2.10]).

Lemma 3.3.4 ([6, Proposition 3.9])**.** *There exists a constant $K > 0$ such that the following holds. Fix real numbers $\delta > 0$ and $s \in [N^{-1/2+\delta}, N^{-\delta}]$, and a unit vector $\mathbf{q} \in \mathbb{R}^N$ with stable support. For each index $i \in [1, N]$ such that $|\gamma_i(s)| < K^{-1}$, we have with overwhelming probability that*

$$\sup_{z \in \mathcal{D}} \max_{1 \le j,k \le N} \big| R_{jk}(s, z) \big| < N^{\delta}, \qquad \big\langle \mathbf{u}_i(s), \mathbf{q} \big\rangle^2 \le N^{-1+\delta}. \tag{3.3.17}$$

3.3.4 Interpolating matrix

Recalling t from (3.2.10), define the interpolating matrix

$$\mathbf{H}^{\gamma} = \{H_{ij}^{\gamma}\} = \gamma \mathbf{A} + \mathbf{X} + (1 - \gamma^2)^{1/2} t^{1/2} \mathbf{W}, \tag{3.3.18}$$

where $\mathbf{W}$ is an independent $N \times N$ GOE matrix. Namely, it is an $N \times N$ real symmetric random matrix $\mathbf{W}_N = \{w_{ij}\}$, whose upper triangular entries w_{ij} are mutually independent Gaussian random variables with variances $(1 + \mathbf{1}_{i=j})N^{-1}$.

The following lemma estimates the entries of the resolvent matrix $\mathbf{G}^{\gamma} = \big\{ G_{ij}^{\gamma}(z) \big\} = (\mathbf{H}^{\gamma} - z)^{-1}$ and provides complete eigenvector delocalization for $\mathbf{H}^{\gamma}$. The first bound in (3.3.19) below was obtained as [6, Theorem 3.16]; given this, the second bound there follows by standard arguments (again, see the proof of [26, Theorem 2.10]).

Lemma 3.3.5 ([6, Theorem 3.16])**.** *There exists a constant $K > 0$ such that the following holds. Fix real numbers $\delta > 0$ and $\gamma \in [0, 1]$, and abbreviate $\mathcal{D} = \mathcal{D}_{K,\delta}$ (recall (3.3.7)). Let $\mathbf{u}_i(\mathbf{H}^\gamma)$ be a unit eigenvector of $\mathbf{H}^\gamma$ such that the corresponding eigenvalue $\lambda_i(\mathbf{H}^\gamma)$ satisfies $\big|\lambda_i(\mathbf{H}^\gamma)\big| \le K^{-1}$. Then, with overwhelming probability we have the bounds*

$$\sup_{0 \le \gamma \le 1} \sup_{z \in \mathcal{D}} \max_{1 \le j,k \le N} \big|G_{jk}^\gamma(z)\big| < N^\delta; \qquad \big\|\mathbf{u}_i(\mathbf{H}^\gamma)\big\|_\infty \le N^{-1/2+\delta}. \tag{3.3.19}$$

In view of the identity

$$\eta \sum_{j=1}^{N} \Big(\big|\lambda_i(\mathbf{M}) - E\big|^2 + \eta^2\Big)^{-1} = N^{-1} \operatorname{Im} \operatorname{Tr}\big(\mathbf{M} - z\big)^{-1}, \tag{3.3.20}$$

which holds for any $N \times N$ matrix $\mathbf{M}$ and complex number $z = E + \mathrm{i}\eta \in \mathbb{H}$, Lemma 3.3.3 and Lemma 3.3.5 together quickly imply the following lemma that bounds the number of eigenvalues of $\mathbf{H}^\gamma$ or $\mathbf{X}_s$ in a given interval.

Lemma 3.3.6. *For any real number $\delta > 0$, there exist constants $K > 0$ and $C = C(\delta) > 0$ such that the following holds. For any interval $I \subseteq [-K^{-1}, K^{-1}]$ of length $|I| \ge N^{-1+\delta}$, we have with overwhelming probability that*

$$\sup_{\gamma \in [0,1]} \Big|\big\{i : \lambda_i(\mathbf{H}^\gamma) \in I\big\}\Big| \le C|I|N^{1+\delta}; \qquad \sup_{s \in [N^{-1/2+\delta}, N^{-\delta}]} \Big|\big\{i : \lambda_i(\mathbf{X}_s) \in I\big\}\Big| \le C|I|N. \tag{3.3.21}$$

The following result states that the i-th eigenvalue of $\mathbf{H}^\gamma$ and $\mathbf{X}_s$ is close to 0 if i is close to $\frac{N}{2}$. Its proof will be given in Section 3.7.

Lemma 3.3.7. *For each real number $c_1 > 0$, there exists a constant $c_2 > 0$ such that the* *eigenvalues* $\lambda_i(\mathbf{H}^\gamma)$ *of* $\mathbf{H}^\gamma$ *and* $\lambda_i(\mathbf{X}_s)$ *of* $\mathbf{X}_s$ *satisfy*

$$\sup_{\gamma\in[0,1]} \big|\lambda_i(\mathbf{H}^\gamma)\big| < c_1; \qquad \sup_{s\in[0,1]} \big|\lambda_i(\mathbf{X}_s)\big| < c_1; \qquad \sup_{s\in[0,1]} \big|\gamma_i(s)\big| < c_1, \tag{3.3.22}$$

for each $i \in \big[(1/2-c_2)N, (1/2+c_2)N\big]$*, with overwhelming probability.*

In Section 3.7, we use Lemma 3.3.7 and Lemma 3.3.3 to deduce the following rigidity statements comparing the classical locations $\widehat{\gamma}_i(s)$ to the γ_i, and the $\widehat{\gamma}_i(s)$ to the $\gamma_i(s)$ (recall (3.1.7), (3.3.10), and (3.2.18)).

Lemma 3.3.8. *Fix* $\delta > 0$*. There exist constants* $C, c_1 > 0$ *and* $c = c(\delta) > 0$ *such that for each* $i \in \big[(1/2-c_1)N, (1/2+c_1)N\big]$*, we deterministically have the bound*

$$\sup_{s\in[N^{-1/2+\delta}, N^{-\delta}]} \big|\widehat{\gamma}_i(s) - \gamma_i\big| \le CN^{-c}, \tag{3.3.23}$$

and with overwhelming probability the bound

$$\sup_{s\in[N^{-1/2+\delta}, N^{-\delta}]} \big|\widehat{\gamma}_i(s) - \gamma_i(s)\big| \le CN^{-1/2+\delta}. \tag{3.3.24}$$

3.4 Comparison

This section establishes Theorem 3.2.7, which compares the eigenvector statistics of $\mathbf{X}_t$ to those of $\mathbf{H}$. Section 3.4.1 establishes this result assuming a general comparison estimate, certain derivative bounds, and a level repulsion estimate for $\mathbf{H}^\gamma$. We then prove the comparison estimate in Section 3.4.2; the necessary derivative bounds in Section 3.4.3; and the level repulsion estimate in Section 3.4.4.

3.4.1 Proof of Theorem 3.2.7

In this section we establish Theorem 3.2.7 assuming Lemma 3.4.1, Lemma 3.4.3, Lemma 3.4.4, and Lemma 3.4.5 below. In what follows, for any $\kappa \in [0,1]$, $\mathbf{M} = \{m_{ij}\} \in \mathrm{Mat}_{N\times N}$, and $a, b \in [1, N]$, we define $\Theta_\kappa^{(a,b)} \mathbf{M} \in \mathrm{Mat}_{N \times N}$ as follows. Recalling ρ from (3.2.2), if $|m_{ab}| = |m_{ba}| \geq N^{-\rho}$, then set $\Theta_\kappa^{(a,b)} \mathbf{M} = \mathbf{M}$. Otherwise, if $|m_{ab}| = |m_{ba}| < N^{-\rho}$, set $\Theta_\kappa^{(a,b)} \mathbf{M}$ to be the $N \times N$ matrix whose (i, j) entry is equal to m_{ij} if $(i, j) \notin \big\{ (a, b), (b, a) \big\}$ and is equal to $\kappa m_{ab} = \kappa m_{ba}$ otherwise. Moreover, for any differentiable function $F : \mathrm{Mat}_{N \times N} \to \mathbb{C}$ and indices $a, b \in [1, N]$, we define $\partial_{ab} F$ to be the derivative of F with respect to m_{ab}.

We first state the following comparison theorem between functions of $\mathbf{H}^0 = \mathbf{X}_t$ and $\mathbf{H}^\gamma$ (recall (3.3.18)), which will be established in Section 3.4.2 below.

Lemma 3.4.1. *There exists a constant $c > 0$ such that the following holds. Let $F \colon \mathrm{Mat}_{N \times N} \to \mathbb{C}$ denote a smooth function, and suppose $K, L > 1$ are such that*

$$\max_{0 \leq j \leq 4} \sup_{0 \leq \gamma \leq 1} \max_{1 \leq a, b \leq N} \sup_{0 \leq \kappa \leq 1} \Big| \partial_{ab}^{(j)} F \big(\Theta_\kappa^{(a,b)} \mathbf{H}^\gamma \big) \Big| \leq K \tag{3.4.1}$$

holds with overwhelming probability, and

$$\max_{0 \leq j \leq 4} \sup_{0 \leq \gamma \leq 1} \max_{1 \leq a, b \leq N} \sup_{0 \leq \kappa \leq 1} \Big| \partial_{ab}^{(j)} F \big(\Theta_\kappa^{(a,b)} \mathbf{H}^\gamma \big) \Big| \leq L \tag{3.4.2}$$

holds deterministically. Then, for any $D > 0$, there exists a constant $C = C(D) > 0$ such that

$$\sup_{0 \leq \gamma \leq 1} \big| F(\mathbf{H}^\gamma) - F(\mathbf{H}^0) \big| \leq K N^{-c} + C L N^{-D}. \tag{3.4.3}$$

Next we require the following function, originally introduced in [96, Section 3.2], that measures how close eigenvalues of some matrix $\mathbf{A}$ are to a given eigenvalue.

Definition 3.4.2. Let $\mathbf{A} \in \mathrm{Mat}_{N \times N}$. If $1 \leq i \leq N$ is such that $\lambda_i(\mathbf{A})$ is an eigenvalue of a matrix $\mathbf{A}$ with multiplicity one, we define

$$Q_i(\mathbf{A}) = \frac{1}{N^2} \sum_{\substack{1 \leq j \leq N \\ j \neq i}} \big| \lambda_j(\mathbf{A}) - \lambda_i(\mathbf{A}) \big|^{-2}. \tag{3.4.4}$$

To deal with the case of multiplicity greater than one, we introduce a cutoff. For any $M > 0$, we fix a smooth function $f_M : \mathbb{R}_{\geq 0} \to \mathbb{R}$ such that there exists a constant $C > 0$ (independent of M and N) satisfying the following two properties.

1. For any $x \in \mathbb{R}_{>0}$, we have that $\left|f_M'(x)\right| + \left|f_M''(x)\right| + \left|f_M'''(x)\right| \leq C$.
2. If $x \in [0, M]$ then $\left|f_M(x) - x\right| \leq 1$, and if $x \geq M$, then $f_M(x) = M$.

The function $f_M\big(Q_i(\mathbf{A})\big)$ is then well-defined and smooth on real symmetric matrices.

The following two lemmas control the derivatives of the $Q_i(\mathbf{H})$ and of the eigenvector entries of $\mathbf{H}^\gamma$ with respect to the matrix entries of $\mathbf{H}$; the first is an overwhelming probability bound, and the second is a deterministic bound. They will be established in Section 3.4.3 below.

Lemma 3.4.3. *There exists a constant* $c > 0$ *such that the following holds. Fix real numbers* $\gamma, \kappa \in [0, 1]$*, a constant* $\omega > 0$*, and integers* $i, a, b \in [1, N]$*. Set* $M = N^{2\omega}$*, and assume that*

$$\left|\lambda_i\big(\Theta_\kappa^{(a,b)}\mathbf{H}^\gamma\big)\right| < c, \quad \textit{and} \quad Q_i\big(\Theta_\kappa^{(a,b)}\mathbf{H}^\gamma\big) \leq M = N^{2\omega} \tag{3.4.5}$$

both hold with overwhelming probability. Then,

$$\left|\partial_{ab}^{(k)}\Big(Q_i\big(\Theta_\kappa^{(a,b)}\mathbf{H}^\gamma\big)\Big)\right| \leq CN^{10k(\omega+\delta)}, \tag{3.4.6}$$

also holds with overwhelming probability, for any integer $0 \leq k \leq 4$.

Moreover, for any $\mathbf{q} \in \mathbb{R}^N$*, there exists a constant* $C = C\big(|\operatorname{supp}\mathbf{q}|\big) > 0$ *such that*

$$\left|\partial_{ab}^{(k)}\left(\Big\langle \mathbf{q}, \mathbf{u}_i\big(\Theta_\kappa^{(ab)}\mathbf{H}^\gamma\big)\Big\rangle^2\right)\right| \leq CN^{-1+10k(\omega+\delta)}, \tag{3.4.7}$$

also holds with overwhelming probability, for any integer $0 \leq k \leq 4$.

Lemma 3.4.4. *Fix real numbers $\gamma, \kappa \in [0,1]$, a constant $\omega > 0$, and integers $i, a, b \in [1, N]$; assume that $Q_i\big(\Theta_\kappa^{(a,b)} \mathbf{H}^\gamma\big) < N^{2\omega}$. Then, for any integers $k \in [0,4]$ and $1 \le i \le N$, we have the deterministic bounds*

$$\left| \partial_{ab}^{(k)} \Big(Q_i\big(\Theta_\kappa^{(a,b)} \mathbf{H}^\gamma\big) \Big) \right| \le C N^{10+6\omega}; \qquad \left| \partial_{ab}^{(k)} \bigg(\Big\langle \mathbf{q}, \mathbf{u}_i\big(\Theta_\kappa^{(ab)} \mathbf{H}^\gamma\big) \Big\rangle^2 \bigg) \right| \le C N^{15+10\omega}. \tag{3.4.8}$$

Next we state a level repulsion estimate, which will be established in Section 3.4.4 below.

Lemma 3.4.5. *There exist constants $c, \upsilon > 0$ such that, for any fixed index $i \in \big[(1/2 - c)N, (1/2 + c)N\big]$ and real number $\gamma \in [0,1]$, we have that*

$$\mathbb{P}\big(Q_i(\mathbf{H}^\gamma) \ge N^\upsilon\big) \le 2N^{-\upsilon/4}. \tag{3.4.9}$$

Given these statements, we now prove Theorem 3.2.7. The argument follows [42, Theorem 1.1].

Proof of Theorem 3.2.7. For brevity we consider just $n = 1$; the general case is no harder.

By Lemma 3.4.5, there exists some $\omega > 0$ such that, for each $\gamma \in \{0, 1\}$,

$$\mathbb{P}\big(Q_i(\mathbf{H}^\gamma) \ge N^\omega\big) \le 2N^{-\omega/4}. \tag{3.4.10}$$

Denote the degree of P by m, so that $P(x) \le C(x^m + 1)$ for $x \ge 0$. Delocalization for $\mathbf{H}^\gamma$, (3.3.19), implies that $N\big\langle \mathbf{q}, \mathbf{u}_i(\mathbf{H}^\gamma)\big\rangle^2 \le N^\delta$ with overwhelming probability for each $\delta > 0$ and $\gamma \in [0,1]$, if N is sufficiently large. Therefore,

$$\mathbb{E}\bigg[P\Big(N \big\langle \mathbf{q}, \mathbf{u}_i(\gamma) \big\rangle^2 \Big)^2 \bigg] \le C N^{2m\delta}. \tag{3.4.11}$$

Now set $M = N^{2\omega}$, and let $g = g_M$ be a smooth function with uniformly bounded derivatives such that $0 \le g(x) \le 1$ for each $x \in \mathbb{R}_{>0}$; $g(x) = 1$ for $x \le M$; and $g(x) = 0$ for $x \ge 2M$.

Then,

$$\left|\mathbb{E}\Big[P\Big(N\big\langle\mathbf{q},\mathbf{u}_i(\mathbf{H}^1)\big\rangle^2\Big)\Big]-\mathbb{E}\Big[P\Big(N\big\langle\mathbf{q},\mathbf{u}_i(\mathbf{H}^0)\big\rangle^2\Big)\Big]\right| \tag{3.4.12}$$

$$\leq\left|\mathbb{E}\Big[P\Big(N\big\langle\mathbf{q},\mathbf{u}_i(\mathbf{H}^1)\big\rangle^2\Big)g\big(Q_i(\mathbf{H}^1)\big)\Big]-\mathbb{E}\Big[P\Big(N\big\langle\mathbf{q},\mathbf{u}_i(\mathbf{H}^0)\big\rangle^2\Big)g\big(Q_i(\mathbf{H}^0)\big)\Big]\right| \tag{3.4.13}$$

$$+\mathbb{E}\Big[P\Big(N\big\langle\mathbf{q},\mathbf{u}_i(\mathbf{H}^1)\big\rangle^2\Big)^2\Big]^{1/2}\mathbb{P}\big(Q_i(\mathbf{H}^1)\geq M\big)+\mathbb{E}\Big[P\Big(N\big\langle\mathbf{q},\mathbf{u}_i(\mathbf{H}^0)\big\rangle^2\Big)^2\Big]^{1/2}\mathbb{P}\big(Q_i(\mathbf{H}^0)\geq M\big) \tag{3.4.14}$$

$$\leq\left|\mathbb{E}\Big[P\Big(N\big\langle\mathbf{q},\mathbf{u}_i(\mathbf{H}^1)\big\rangle^2\Big)g\big(Q_i(\mathbf{H}^1)\big)\Big]-\mathbb{E}\Big[P\Big(N\big\langle\mathbf{q},\mathbf{u}_i(\mathbf{H}^0)\big\rangle^2\Big)g\big(Q_i(\mathbf{H}^0)\big)\Big]\right|+CN^{-\omega/4+m\delta}, \tag{3.4.15}$$

where in the last estimate we applied (3.4.10) and (3.4.11).

Now let us define the function $h:\mathrm{Mat}_{N\times N}\to\mathbb{R}$ by setting

$$h(\mathbf{A})=h_i(\mathbf{A})=P\Big(N\big\langle\mathbf{q},\mathbf{u}_i(\mathbf{A})\big\rangle^2\Big)g\big(Q_i(\mathbf{A})\big), \tag{3.4.16}$$

for any $\mathbf{A}\in\mathrm{Mat}_{N\times N}$. By Lemma 3.4.3, Lemma 3.4.4; a union bound over $1\leq i,a,b\leq N$ and γ and κ in an N^{-30}-net of $[0,1]$; and the fact that $h(\mathbf{A})=0$ if $Q_i(\mathbf{A})\geq 2M$, we have that h deterministically satisfies

$$\sup_{0\leq k\leq 4}\sup_{\gamma\in[0,1]}\max_{1\leq a,b\leq N}\sup_{\kappa\in[0,1]}\left|\partial_{ab}^{(k)}h\big(\Theta_\kappa^{(a,b)}\mathbf{H}^\gamma\big)\right|\leq CN^{15+15m\omega}, \tag{3.4.17}$$

and with overwhelming probability satisfies

$$\sup_{0\leq k\leq 4}\sup_{\gamma\in[0,1]}\max_{1\leq a,b\leq N}\sup_{\kappa\in[0,1]}\left|\partial_{ab}^{(k)}h\big(\Theta_\kappa^{(a,b)}\mathbf{H}^\gamma\big)\right|\leq CN^{20m(\omega+\delta)}. \tag{3.4.18}$$

Therefore, upon setting ω and δ sufficiently small, Lemma 3.4.1 implies $\big|\mathbb{E}\big[h(\mathbf{H}^1)\big]-\mathbb{E}\big[h(\mathbf{H}^0)\big]\big|$

is bounded by CN^{-c}. Inserting this into (3.4.15) yields

$$\left| \mathbb{E}\Big[P\Big(N\langle \mathbf{q}, \mathbf{u}_i(1) \rangle^2 \Big) \Big] - \mathbb{E}\Big[P\Big(N\langle \mathbf{q}, \mathbf{u}_i(0) \rangle^2 \Big) \Big] \right| \le CN^{-c} + CN^{-\omega/4 + m\delta}. \tag{3.4.19}$$

The lemma follows from further imposing that $5m\delta < \omega$. □

3.4.2 Proof of Lemma 3.4.1

In this section we establish Lemma 3.4.1.

Proof of Lemma 3.4.1. Observe (by (3.3.1), for instance) that

$$\partial_\gamma \mathbb{E}\big[F(\mathbf{H}^\gamma) \big] = \sum_{1 \le i,j \le N} \mathbb{E}\left[\partial_{ij} F(\mathbf{H}^\gamma) \bigg(A_{ij} - \frac{\gamma t^{1/2}}{(1-\gamma^2)^{1/2}} w_{ij} \bigg) \right]. \tag{3.4.20}$$

Now, we condition on the label Ψ of $\mathbf{H}$ (recall Definition 3.2.5) and denote the associated conditional expectation by $\mathbb{E}_\Psi$. We first consider the case $\psi_{ij} = 1$. This implies $A_{ij} = B_{ij} = 0$, and Gaussian integration by parts (see for instance [93, Appendix A.4]) yields

$$\mathbb{E}_\Psi \left[\partial_{ij} F(\mathbf{H}^\gamma) \bigg(\frac{\gamma t^{1/2}}{(1-\gamma^2)^{1/2}} w_{ij} \bigg) \right] = \frac{t\gamma}{N} \mathbb{E}_\Psi \left[\partial_{ij}^2 F(\mathbf{H}^\gamma) \right], \quad \text{whenever } \psi_{ij} = 1. \tag{3.4.21}$$

Hoeffding's inequality applied to the Bernoulli random variable ψ_{ij}, whose distribution was defined in (3.2.3), implies that there are likely at most $CN^{1+\alpha\rho}$ pairs (i, j) such that $\psi_{ij} = 1$. Specifically,

$$\mathbb{P}\left[\Big| \big\{ (i,j) \in [1,N] \times [1,N] : \psi_{ij} = 1 \big\} \Big| < CN^{1+\alpha\rho} \right] \ge 1 - C \exp\big(- N^{\alpha\rho} \big). \tag{3.4.22}$$

By (3.4.2), the contribution of (3.4.21) over the complement of the event described in (3.4.1) or (3.4.22) is bounded by CLN^{-D}, for some constant $C = C(D) > 0$. This, together with (3.4.1), (3.4.21), (3.4.22), (3.2.11), and (3.2.2) imply that the sum of (3.4.21) over all

(i, j) such that $\psi_{ij} = 1$ or $i = j$ is at most

$$CKtN^{-1}N^{\alpha\rho+1} + CLN^{-D} \leq CKN^{\alpha\rho-(2-\alpha)\nu} + CLN^{-D} < KN^{-c} + CLN^{-D}, \qquad (3.4.23)$$

for some constants $c > 0$ (only dependent on the fixed parameters α, ρ, and ν) and $C = C(D) > 0$.

We next consider the case when $\psi_{ij} = 0$ and $i \neq j$. Then, $A_{ij} = a_{ij}(1 - \chi_{ij})$ and $B_{ij} = b_{ij}\chi_{ij}$; abbreviate $a_{ij} = a$, $b_{ij} = b$, $\chi_{ij} = \chi$, and $w_{ij} = w$. Set

$$h = \gamma(1 - \chi)a + \chi b + (1 - \gamma^2)^{1/2}t^{1/2}w. \qquad (3.4.24)$$

Fix $(i, j) \in [1, N]^2$ such that $\psi_{ij} = 0$, abbreviate $F^{(k)} = \partial_{ij}^{(k)} F$, and abbreviate $\mathbf{S} = \Theta_0^{(i,j)}\mathbf{H}$. Then a Taylor expansion yields

$$F'(\mathbf{H}^\gamma) = F'(\mathbf{S}) + hF''(\mathbf{S}) + h^2 F^{(3)}(\mathbf{S}) + h^3 F^{(4)}\big(\Theta_\kappa^{(i,j)}\mathbf{H}^\gamma\big), \qquad (3.4.25)$$

for some $\kappa \in [0, 1]$. Hence, the (i, j) term in the sum on the right side of (3.4.20) is equal to

$$\mathbb{E}_\Psi\left[\left((1-\chi)a - \frac{\gamma t^{1/2}}{(1-\gamma^2)^{1/2}}w\right)\Big(F'(\mathbf{S}) + hF''(\mathbf{S}) + h^2F^{(3)}(\mathbf{S}) + h^3F^{(4)}\big(\Theta_\kappa^{(i,j)}\mathbf{H}^\gamma\big)\Big)\right]. \quad (3.4.26)$$

Using the mutual independence between $\mathbf{S}$, a, b, χ, and w, and the fact that a, b, and w are all symmetric, we conclude that (3.4.26) is equal to

$$\mathbb{E}_\Psi\left[\left((1-\chi)a - \frac{\gamma t^{1/2}}{(1-\gamma^2)^{1/2}}w\right)hF''(\mathbf{S})\right] + \mathbb{E}_\Psi\left[\left((1-\chi)a - \frac{\gamma t^{1/2}}{(1-\gamma^2)^{1/2}}w\right)h^3F^{(4)}\big(\Theta_\kappa^{(i,j)}\mathbf{H}^\gamma\big)\right]. \qquad (3.4.27)$$

Again using the mutual independence between $\mathbf{S}$, a, b, χ, and w; the fact that a, b, and w

are all symmetric; and (3.4.24), we find that the first term in (3.4.27) is

$$\mathbb{E}_\Psi\left[F''(\mathbf{S})\left((1-\chi)a - \frac{\gamma t^{1/2}}{(1-\gamma^2)^{1/2}}w\right)h\right] = \gamma\mathbb{E}\big[F''(\mathbf{S})\big]\mathbb{E}\big[a^2(1-\chi) - tw^2\big] = 0, \quad (3.4.28)$$

where the final equality follows from the choice of t in (3.2.10).

The second term in (3.4.27) is bounded above by

$$C\mathbb{E}_\Psi\left[\left|F^{(4)}\big(\Theta_\kappa^{(i,j)}\mathbf{H}^\gamma\big)\right|\big((1-\chi)a^4 + t^2w^4 + \chi tw^2b^2\big)\right]. \quad (3.4.29)$$

On the complement of the event in (3.4.1), this expectation is bounded by CLN^{-D-2}, for some constant $C = C(D) > 0$. On this event, we use (3.2.11); the facts that $\mathbb{E}[w^2] \le N^{-1}$ and $\mathbb{E}[w^4] \le N^{-2}$; and the estimates (which can be quickly deduced from Definition 3.2.4; see [6, Section 3.3.2] for details)

$$\mathbb{E}_\Psi\big[(1-\chi)a^4\big] \le CN^{\nu(\alpha-4)-1}, \quad \mathbb{E}_\Psi\big[\chi b^2\big] \le CN^{\rho(\alpha-2)-1}, \quad (3.4.30)$$

to bound it by

$$CK\big(N^{\nu(\alpha-4)-1} + N^{2\nu(\alpha-2)-2} + N^{(\rho+\nu)(\alpha-2)-2}\big) \le CKN^{-2-c}, \quad (3.4.31)$$

for some constant $c > 0$ (only dependent on the fixed parameters α, ν, and ρ), where we have used (3.2.2) in the last inequality. So, the sum of (3.4.29) over all $(i,j) \in [1,N]^2$ such that $i \neq j$ and $\psi_{ij} = 0$ is at most

$$KN^{-c} + CLN^{-D}. \quad (3.4.32)$$

Now the lemma follows from the fact that the contribution to (3.4.20) from all terms corresponding to (i,j) with $i = j$ or $\psi_{ij} = 1$ is bounded by (3.4.23) and the fact that the contribution of all terms coming from (i,j) with $i \neq j$ and $\psi_{ij} = 0$ is bounded by (3.4.32). $\square$

3.4.3 Proof of Lemma 3.4.3 and Lemma 3.4.4

In this section we prove Lemma 3.4.3 and Lemma 3.4.4. We begin with the following estimate on the resolvent entries of $\Theta_\kappa^{(a,b)}\mathbf{H}^\gamma$. In the below, we recall $\mathcal{D}_{C;\delta}$ from (3.3.7).

Lemma 3.4.6. *There exists a constant $K > 0$ such that following holds. For any $\delta > 0$, the bound*

$$\sup_{0\le\kappa\le 1}\max_{1\le a,b\le N}\sup_{0\le\gamma\le 1}\max_{1\le i,j\le N}\sup_{z\in\mathcal{D}_{K;\delta}}\left|\Big(\big(\Theta_\kappa^{(a,b)}\mathbf{H}^\gamma - z\big)^{-1}\Big)_{ij}\right| < N^\delta \tag{3.4.33}$$

holds with overwhelming probability. Moreover, for each $c_1 > 0$, there exists some $c_2 > 0$ such that

$$\begin{aligned}&\sup_{0\le\kappa\le 1}\max_{1\le a,b\le N}\sup_{0\le\gamma\le 1}\left\|\mathbf{u}_i\big(\Theta_\kappa^{(a,b)}\mathbf{H}^\gamma\big)\right\|_\infty < N^{\delta-1/2};\\ &\sup_{0\le\kappa\le 1}\max_{1\le a,b\le N}\sup_{0\le\gamma\le 1}\left|\lambda_i\big(\Theta_\kappa^{(a,b)}\mathbf{H}^\gamma\big)\right| < c_1,\end{aligned} \tag{3.4.34}$$

both hold for each $(1/2-c_2)N \le i \le (1/2+c_2)N$ with overwhelming probability. Additionally, for any interval $I \subset [-c_1, c_1]$ of length $|I| \ge N^{-1+\delta}$,

$$\sup_{0\le\kappa\le 1}\max_{1\le a,b\le N}\sup_{0\le\gamma\le 1}\left|\Big\{i\in[1,N]\colon \lambda_i\big(\Theta_\kappa^{(a,b)}\mathbf{H}^\gamma\big)\in I\Big\}\right| \le C|I|N^{1+\delta}, \tag{3.4.35}$$

holds with overwhelming probability.

Proof. The second bound in (3.4.34) follows from Lemma 3.3.7 and the Weyl interlacing inequality for eigenvalues of symmetric matrices. Furthermore, the proofs of the first bound in (3.4.34) and (3.4.35) given (3.4.33) follow from standard arguments (see for example the proofs of [26, Theorem 2.10] and [78, Lemma 7.4]). So, we only establish (3.4.33).

To that end, let K be as in Lemma 3.3.5, and fix indices $a, b \in [1, N]$; real numbers $\kappa, \gamma \in [0, 1]$; and a complex number $z \in \mathcal{D}_{K;\delta}$. Set $\mathbf{E} = \Theta_\kappa^{(a,b)}\mathbf{H}^\gamma$, $\mathbf{V} = (\mathbf{E} - z)^{-1} = \{V_{ij}\}$, and $\Delta = \mathbf{H}^\gamma - \mathbf{E}$. We may assume throughout this proof that $|h_{ab}| \le N^{-\rho}$, for otherwise $\mathbf{E} = \mathbf{H}^\gamma$, and the result follows from Lemma 3.3.5.

For any $M \in \mathbb{N}$, the resolvent identity (3.3.1) gives

$$\mathbf{V} - \mathbf{G}^\gamma = \sum_{k=0}^{M} (\mathbf{G}^\gamma \Delta)^k \mathbf{G}^\gamma + (\mathbf{G}^\gamma \Delta)^{M+1} \mathbf{V}. \tag{3.4.36}$$

Now select M in (3.4.36) such that $M\rho > 10$. Then (3.3.1); Lemma 3.3.5; the deterministic bound (3.3.2); and the fact that Δ is supported on at most two entries, each of which is bounded by $N^{-\rho}$, implies for sufficiently small $\delta > 0$ that

$$\max_{1 \le i,j \le N} |V_{ij}| \le \max_{1 \le i,j \le N} \big| G_{ij}^\gamma \big| + \sum_{k=0}^{M} 2^k N^{(k+1)\delta - k\rho} + 2^{M+1} N^{(M+1)\delta - \rho - 10} \eta^{-1} \le N^\delta, \tag{3.4.37}$$

for sufficiently large N, with overwhelming probability. Taking a union bound of (3.4.37) over all $a, b \in [1, N]$; κ and γ in a N^{-10}-net of $[0,1]$; and z in a N^{-10}-net of $\mathcal{D}$, and also applying (3.3.1), then yields (3.4.33). $\square$

Next we require the following result that essentially provides level repulsion estimates for $\mathbf{X}_t$.

Lemma 3.4.7. *For all $\omega > 0$, there exist constants $c > 0$ (independent of ω) and $C = C(\omega) > 0$ such that the following holds. Set $M = N^{2\omega}$; recall t from (3.2.10); and fix an index $i \in \big[(1/2 - c)N, (1/2 + c)N\big]$. Then,*

$$\mathbb{E}\Big[f_M\big(Q_i(\mathbf{X}_t)\big) \Big] \le C N^{3\omega/2}. \tag{3.4.38}$$

Further fix $\delta > 0$ and, for fixed real numbers $\kappa, \gamma \in [0,1]$ and indices $1 \le a, b \le N$, abbreviate $\mu_j = \lambda_j\big(\Theta_\kappa^{(a,b)} \mathbf{H}^\gamma\big)$ for each $j \in [1, N]$. Then we have with overwhelming probability that

$$\mathbf{1}_{Q_i(\Theta_\kappa^{(a,b)} \mathbf{H}^\gamma) < M} \sum_{j \ne i} \frac{1}{|\mu_j - \mu_i|} \le N^{1 + \omega + \delta}. \tag{3.4.39}$$

Proof. Throughout this proof, we may assume that $\delta < \frac{\omega}{4}$. Define the sets

$$U_0 = \Big\{ j \in [1, N] \setminus \{i\} \colon \big| \lambda_j(t) - \lambda_i(t) \big| \le N^{-1+\delta/2} \Big\}. \tag{3.4.40}$$

and

$$U_n = \Big\{ j \in [1, N] \colon 2^{n-1} N^{-1+\delta/2} < \big| \lambda_j(t) - \lambda_i(t) \big| \le 2^n N^{-1+\delta/2} \Big\}. \tag{3.4.41}$$

for each integer $n \ge 1$.

Now choose the $c > 0$ here with respect to the K from Lemma 3.3.6 to satisfy $c < \frac{1}{4K}$, and define $L = \big\lfloor \log_2 (2cN^{1-\delta/2}) \big\rfloor$. Then Lemma 3.3.6 and Lemma 3.3.7 together imply (after further decreasing c if necessary) that

$$|U_n| \le C 2^n N^{\delta}. \tag{3.4.42}$$

holds with overwhelming probability, for each $n \in [0, L]$. Next, for any $\theta \in (0, 1)$, also define the event

$$E(\theta) = \left\{ \min \big\{ \lambda_i(t) - \lambda_{i-1}(t), \lambda_{i+1}(t) - \lambda_i(t) \big\} > \frac{\theta}{N} \right\}, \tag{3.4.43}$$

and let $E(\theta)^c$ denote the complement of $E(\theta)$. Then (3.4.42) implies with overwhelming probability that

$$\frac{\mathbf{1}_{E(\theta)}}{N^2} \sum_{n=0}^{L} \sum_{j \in U_n} \big| \lambda_j(t) - \lambda_i(t) \big|^{-2} \le C N^{\delta} \theta^{-2}. \tag{3.4.44}$$

Further, we deterministically have that

$$\frac{1}{N^2} \sum_{n=L+1}^{\infty} \sum_{j \in U_n} \big| \lambda_j(t) - \lambda_i(t) \big|^{-2} \le C N^{-1}. \tag{3.4.45}$$

Then combining (3.4.44) and (3.4.45) bounds

$$\mathbb{E}\Big[Q_i\big(\Theta_\kappa^{(a,b)}\mathbf{X}_t\big)\mathbf{1}_{E(\theta)}\Big] < CN^{\delta}\theta^{-2}. \tag{3.4.46}$$

On $E(\theta)^c$, we use the third part of Lemma 3.3.3 and the fact that $\big|f_M(x)\big| < M$ holds for all $x > 0$ to deduce that

$$\mathbb{E}\Bigg[f_M\Big(Q_i\big(\Theta_\kappa^{(a,b)}\mathbf{X}_t\big)\Big)\mathbf{1}_{E(\theta)^c}\Bigg] < CN^{2\omega}\theta. \tag{3.4.47}$$

Then selecting $\theta = N^{-\omega/2}$, using the fact that $\delta < \frac{\omega}{4}$, and combining (3.4.46) and (3.4.47) yields (3.4.38). We omit the proof of (3.4.39), as it is entirely analogous, and follows from replacing the above application of Lemma 3.3.6 and Lemma 3.3.7 (to establish (3.4.42)) with (3.4.35) and the second bound in (3.4.34), respectively, and using the fact that $\mu_i - \mu_{i-1}, \mu_{i+1} - \mu_i \geq N^{-\omega}$ holds on the event that $Q_i\big(\Theta_\kappa^{(a,b)}\mathbf{H}^\gamma\big) < M$. $\square$

Now we can establish the derivative bounds given by Lemma 3.4.3 and Lemma 3.4.4.

Proof of Lemma 3.4.3 (Outline). In outline, the bound (3.4.6) is proven by expanding $\partial_{ab}^{(k)} Q_i(\mathbf{H}^\gamma)$ using contour integration into a sum of terms which are then bounded individually. Since the proof of (3.4.7) uses a similar expansion, we only discuss that of (3.4.6) here (for the former, see the proof of [42, Proposition 4.2] for further details).

Our claim (3.4.6) is essentially the same as that of [70, Proposition 4.6], but there are two differences. First, one of our hypotheses is weaker: we only have complete delocalization at small energies and not throughout the spectrum. Second, our conclusion is stronger: [70, Proposition 4.6] bounded derivatives up to third order, but we here we bound fourth order derivatives. This extension to fourth order derivatives parallels the proof for the third order derivatives and requires no new ideas. Therefore, let us only show how the proof in [70, Proposition 4.6] may be modified to accommodate the fact that our delocalization estimate is weaker than the one used in that reference. In what follows, we also assume for notational convenience that $\kappa = 1$, so that $\Theta_\kappa^{(a,b)}(\mathbf{H}^\gamma) = \mathbf{H}^\gamma$, as the proof for general $\kappa \in [0,1]$ is entirely analogous by replacing our use of (3.3.19) below by Lemma 3.4.6.

For any vector $\mathbf{v}$, let $\mathbf{v}^*$ denote its transpose. Set $\theta_{jk} = \mathbf{u}_j^* \mathbf{V} \mathbf{u}_k$, where $\mathbf{V} = \mathbf{V}^{(a,b)} = \{V_{ij}\}$ is the $N \times N$ matrix whose entries are zero except for $V_{ab} = V_{ba} = 1$. In the proof of [70, Proposition 4.6], $\partial_{ab}^{(k)} Q_i(\mathbf{H}^\gamma)$ was expanded into a sum of certain terms using a contour integral representation and Green's function identities. For instance, in the expansion of $\partial_{ab}^{(3)} Q_i(\mathbf{H}^\gamma)$ there are 13 distinct terms, which are listed after line (4.18) in [70, Proposition 4.6]. Setting $\lambda_i = \lambda_i(\mathbf{H}^\gamma)$, one such term is

$$\frac{1}{N^2} \sum_{\substack{1 \le j_1, j_2, j_3 \le N \\ j_1, j_2, j_3 \ne i}} \frac{\theta_{j_1 j_2} \theta_{j_2 j_3} \theta_{j_3 j_1}}{(\lambda_i - \lambda_{j_1})^3 (\lambda_i - \lambda_{j_2})(\lambda_i - \lambda_{j_3})}. \tag{3.4.48}$$

The terms produced by expanding $\partial_{ab}^{(k)} Q_i(\mathbf{H}^\gamma)$ are fractions with a product of k $\theta_{\alpha\beta}$ terms in the numerator, where each of α, β may be a summation index or i, and a product of $k+2$ eigenvalue differences $\lambda_i - \lambda_j$ in the denominator, where j is a summation index. We call k the *order* of such a term. The proof of [70, Proposition 4.6] shows that to prove the claim (3.4.6), it suffices to bound each of the the order k terms appearing in its expansion by $CN^{(2k+2)\delta + (k+2)\omega}$.

For illustrative purposes, we consider just the term (3.4.48) in the $k = 3$ case here; other terms of the same order and the cases $k \in \{1, 2, 4\}$ are analogous. So, let us show that

$$\left| \frac{1}{N^2} \sum_{\substack{1 \le j_1, j_2, j_3 \le N \\ j_1, j_2, j_3 \ne i}} \frac{\theta_{j_1 j_2} \theta_{j_2 j_3} \theta_{j_3 j_1}}{(\lambda_i - \lambda_{j_1})^3 (\lambda_i - \lambda_{j_2})(\lambda_i - \lambda_{j_3})} \right| \le CN^{8\delta + 5\omega}. \tag{3.4.49}$$

To prove (3.4.49), we consider various cases depending on the locations of the eigenvalues $\lambda_{j_3}, \lambda_{j_2}, \lambda_{j_3}$. Let K be the constant from Lemma 3.3.5, and set $c = (2K)^{-1}$. When $\lambda_{j_3}, \lambda_{j_2}, \lambda_{j_3} \in [-2c, 2c]$, (3.3.19) shows the corresponding eigenvectors are completely delocalized, and the proof of [70, Proposition 4.6] requires no modification. There are three remaining cases: exactly one of the j_ℓ is such that $|\lambda_{j_\ell}| > 2c$, exactly two are, or all three are.

In the first case, suppose for example that $|\lambda_{j_2}| > 2c$. Then, (3.3.19) implies

$$|\theta_{j_1 j_2}| \le \Big(\big|u_{j_1}(a)\big|\big|u_{j_2}(b)\big| + \big|u_{j_1}(b)\big|\big|u_{j_2}(a)\big|\Big) \le N^{-1/2+\delta}\Big(\big|u_{j_2}(a)\big| + \big|u_{j_2}(b)\big|\Big), \tag{3.4.50}$$

$$|\theta_{j_2 j_3}| \le \Big(\big|u_{j_2}(a)\big|\big|u_{j_3}(b)\big| + \big|u_{j_2}(b)\big|\big|u_{j_3}(a)\big|\Big) \le N^{-1/2+\delta}\Big(\big|u_{j_2}(a)\big| + \big|u_{j_2}(b)\big|\Big), \tag{3.4.51}$$

$$|\theta_{j_3 j_1}| \le \Big(\big|u_{j_1}(a)\big|\big|u_{j_3}(b)\big| + \big|u_{j_1}(b)\big|\big|u_{j_3}(a)\big|\Big) \le N^{-1+\delta}, \tag{3.4.52}$$

with overwhelming probability. Inserting these bounds in (3.4.48) decouples the sum into a product of a sum over j_2 and a sum over j_1, j_3:

$$(3.4.48) \le c^{-1} N^{-4+3\delta}\left(\sum_{j_2=1}^{N}\Big(\big|u_{j_2}(a)\big| + \big|u_{j_2}(b)\big|\Big)^2\right)\left(\sum_{1\le j_1, j_3\le N}\frac{1}{|\lambda_i-\lambda_{j_1}|^3|\lambda_i-\lambda_{j_3}|}\right). \tag{3.4.53}$$

Here we used $|\lambda_i - \lambda_{j_2}| > c$. The first factor is at most a constant, since the matrix of eigenvectors is orthonormal:

$$\sum_{j_2=1}^{N}\Big(\big|u_{j_2}(a)\big| + \big|u_{j_2}(b)\big|\Big)^2 \le 2\sum_{j_2=1}^{N}\Big(\big|u_{j_2}(a)\big|^2 + \big|u_{j_2}(b)\big|^2\Big) = 4. \tag{3.4.54}$$

For the second factor, the assumption (3.4.5) implies that $|\lambda_i - \lambda_{i\pm1}| \ge N^{-1-\omega}$, and so (3.4.39) yields

$$\sum_{j\ne i}|\lambda_j - \lambda_i|^{-1} \le CN^{1+\omega+\delta}. \tag{3.4.55}$$

Further, for $k \ge 2$, the hypothesis (3.4.5) yields

$$\sum_{j\ne i}|\lambda_j-\lambda_i|^{-k} \le \left(\sum_{j\ne i}|\lambda_j-\lambda_i|^{-2}\right)^{k/2} \le C^{k/2}N^{k(1+\omega+\delta)}. \tag{3.4.56}$$

These inequalities imply the second factor of (3.4.53) is at most $CN^{4+4\omega+4\delta}$, and so (3.4.48) is at most $CN^{7\delta+4\omega}$.

In the second case, suppose for example that $|\lambda_{j_2}| > c$ and $|\lambda_{j_3}| > c$. Since then $|\lambda_i - \lambda_{j_2}|$ and $|\lambda_i - \lambda_{j_3}|$ are bounded below by c, in this case (3.4.48) is bounded above by

$$\frac{1}{c^2 N^2} \sum_{\substack{1 \le j_1, j_2, j_3 \le N \\ j_1, j_2, j_3 \neq i}} \left| \frac{\theta_{j_1 j_2} \theta_{j_2 j_3} \theta_{j_3 j_1}}{(\lambda_i - \lambda_{j_1})^3} \right|. \tag{3.4.57}$$

Proceeding as in the previous case, we find that (3.4.48) is bounded above by

$$c^{-2} N^{2\delta - 3} \Bigg(\sum_{1 \le j_2, j_3 \le N} \Big(|u_{j_2}(a)| + |u_{j_2}(b)| \Big) \Big(|u_{j_3}(a)| + |u_{j_3}(b)| \Big) \tag{3.4.58}$$

$$\times \Big(|u_{j_3}(a)||u_{j_2}(b)| + |u_{j_3}(b)||u_{j_2}(a)| \Big) \Bigg) \left(\sum_{j_1 \neq i} \frac{1}{|\lambda_i - \lambda_{j_1}|^3} \right). \tag{3.4.59}$$

By (3.4.56), the sum over j_1 is at most $CN^{3+3\omega+3\delta}$. Moreover, by the orthogonality of the eigenvectors of $\mathbf{H}^\gamma$ (following (3.4.54)), the sum over j_2 and j_3 is bounded by 8. Thus, (3.4.48) is bounded above by $CN^{5\delta+3\omega}$.

Finally, when $|\lambda_{j_\ell}| > c$ for all ℓ, (3.4.48) is bounded by

$$\frac{1}{N^2 c^5} \sum_{1 \le j_1, j_2, j_3 \le N} |\theta_{j_1 j_2} \theta_{j_2 j_3} \theta_{j_3 j_1}| \le \frac{1}{N^2 c^5} \sum_{1 \le j_1, j_2, j_3 \le N} \Big(|u_{j_1}(a)||u_{j_2}(b)| + |u_{j_1}(b)||u_{j_2}(a)| \Big) \tag{3.4.60}$$

$$\times \Big(|u_{j_2}(a)||u_{j_3}(b)| + |u_{j_2}(b)||u_{j_3}(a)| \Big) \tag{3.4.61}$$

$$\times \Big(|u_{j_1}(a)||u_{j_3}(b)| + |u_{j_1}(b)||u_{j_3}(a)| \Big). \tag{3.4.62}$$

Again by the orthogonality of the eigenvectors of $\mathbf{H}^\gamma$ (following (3.4.54)), the latter is bounded by 8, and so (3.4.48) is bounded above by obtain $8c^{-5}N^{-2}$.

This completes our demonstration of how to bound the sum (3.4.48) and concludes the proof. □

Proof of Lemma 3.4.4 (Outline). We again only discuss the first bound in (3.4.8), as the proof of the second is similar. To that end observe, since we have assumed $Q_i(\mathbf{H}^\gamma) \le N^{2\omega}$, we must have $|\lambda_i - \lambda_j| \ge N^{-1-\omega}$, for each $j \in [1, N] \setminus \{i\}$. As in the proof of Lemma 3.4.3, the derivative $\partial_{ab}^{(k)} \big(Q_i(\Theta_\kappa^{(a,b)} \mathbf{H}^\gamma) \big)$ for $k \le 4$ can be expressed as a sum of a uniformly bounded number of terms similar to (3.4.48), in which at most four indices j_k are being summed over and in which the denominator is of degree at most six in the gaps $\lambda_i - \lambda_{j_k}$. Thus, each such term is bounded by at most $N^{6+6\omega}$, and so their sum is bounded by a multiple of $N^{10+6\omega}$. This yields the first estimate in (3.4.8) and, as mentioned previously, the proof of the second is omitted. $\square$

3.4.4 Proof of Lemma 3.4.5

Now we can establish Lemma 3.4.5.

Proof of Lemma 3.4.5. It suffices to show that there exists some $\omega > 0$ such that, if $M = N^{2\omega}$, then

$$\mathbb{E}\Big[f_M\big(Q_i(\mathbf{H}^\gamma) \big) \Big] \le 2N^{3\omega/2}, \tag{3.4.63}$$

since then a Markov inequality would imply

$$\mathbb{P}\big(Q_i(\mathbf{H}^\gamma) \ge N^{2\omega} \big) \le \mathbb{P}\Big(f_M\big(Q_i(\mathbf{H}^\gamma) \big) \ge N^{2\omega} \Big) \le N^{-2\omega} \mathbb{E}\Big[f_M\big(Q_i(\mathbf{H}^\gamma) \big) \Big] \le 2N^{-\omega/2}, \tag{3.4.64}$$

and the lemma follows after setting $\upsilon = 2\omega$. To prove (3.4.63), we apply Lemma 3.4.1 to interpolate between $\mathbf{X}_t = \mathbf{H}^0$ and $\mathbf{H}^\gamma$, carrying the level repulsion estimate (3.4.38) from the former to the latter. To implement this argument, let us take $c_1 > 0$ sufficiently small so that $i \in \big[(1/2 - c_1)N, (1/2 + c_1)N \big]$ implies that $\big| \lambda_i(\Theta_\kappa^{(a,b)} \mathbf{H}^\gamma) \big|$ is less than the c_1 from Lemma 3.4.6 for each $\gamma, \kappa \in [0, 1]$ and $1 \le a, b \le N$ with overwhelming probability; such a c_1 exists by the second bound of (3.4.34). Further take $\omega, \delta > 0$ sufficiently small so that $100(\omega + \delta)$ is less than the constant c from Lemma 3.4.1, and set $M = N^{2\omega}$.

Then, we may apply Lemma 3.4.1, with the $F(\mathbf{A})$ there equal to $f_M\big(Q_i(\mathbf{A})\big)$ here; the K there equal to $CN^{40(\omega+\delta)}$ here; and the L there equal to $CN^{15+7\omega}$ here. Then (3.4.1) and (3.4.2) follow from Lemma 3.4.3 and Lemma 3.4.4, respectively, and so Lemma 3.4.1 implies that

$$\mathbb{E}\Big[f_M\big(Q_i(\mathbf{H}^\gamma)\big)\Big] \le \mathbb{E}\Big[f_M\big(Q_i(\mathbf{X}_t)\big)\Big] + CN^{-c/2} < 2N^{3\omega/2},$$

where we used (3.4.38) to deduce the second inequality This verifies (3.4.63). □

3.5 Dynamics

This section determines the eigenvector statistics of $\mathbf{X}_t$. Our main goal is a proof of Theorem 3.2.8. In Section 3.5.1 we recall the definition of the eigenvector moment flow from [44] and some of its properties. Section 3.5.2 contains continuity estimates used in the proof of Theorem 3.2.8. In Section 3.5.3 we use the eigenvector moment flow to establish Theorem 3.2.8, assuming several results that will be shown in Section 3.5.4 and Section 3.5.5.

3.5.1 Eigenvector moment flow

Recall the matrix $\mathbf{X}_s$ from (3.2.9) and that its eigenvalues are given by $\lambda_1(s) \le \lambda_2(s) \le \cdots \le \lambda_N(s)$ with associated unit eigenvectors $\mathbf{u}_1(s), \mathbf{u}_2(s), \ldots, \mathbf{u}_N(s)$, respectively. By [44, Theorem 2.3], these eigenvalues and eigenvectors are governed by two stochastic differential equations (SDEs):

$$d\lambda_k(s) = \frac{db_{kk}(s)}{\sqrt{N}} + \frac{1}{N}\sum_{l\neq k}\frac{ds}{\lambda_k(s)-\lambda_l(s)}, \tag{3.5.1}$$

$$d\mathbf{u}_k(s) = \frac{1}{\sqrt{N}}\sum_{l\neq k}\frac{db_{kl}(s)}{\lambda_k(s)-\lambda_l(s)}\mathbf{u}_l(s) - \frac{1}{2N}\sum_{l\neq k}\frac{ds}{(\lambda_k-\lambda_l)^2}\mathbf{u}_k(s), \tag{3.5.2}$$

where $\big(b_{ij}(s)\big)_{1 \le i \le j \le N}$ are mutually independent Brownian motions with variance $1 + \mathbf{1}_{i=j}$. The first equation, for the eigenvalues, is called Dyson Brownian motion. The second, for the eigenvectors, is called the Dyson vector flow. Using these SDEs, we define the stochastic processes $\boldsymbol{\lambda} = \big(\boldsymbol{\lambda}(s)\big)_{0 \le s \le 1}$ and $\mathbf{u} = \big(\mathbf{u}(s)\big)_{0 \le s \le 1}$.

A key tool for analyzing the Dyson vector flow is the *eigenvector moment flow*, introduced in [44, Section 3.1], which characterizes the time evolution of the observable $f_s = f_{\boldsymbol{\lambda},s}$ defined by

$$f_s(\boldsymbol{\xi}) = f_{\boldsymbol{\lambda},s}(\boldsymbol{\xi}) = \mathbb{E}\big[Q_{i_1,\ldots,i_m}^{j_1,\ldots,j_m}(s) \mid \boldsymbol{\lambda}\big], \tag{3.5.3}$$

where we recall $Q_{i_1,\ldots,i_m}^{j_1,\ldots,j_m}(s)$ was defined in (3.2.13).

Theorem 3.5.1 ([44, Theorem 3.1])**.** *Let $\mathbf{q} \in \mathbb{R}^N$ be a unit vector and, for each $s \in [0,1]$, set*

$$c_{ij}(s) = N^{-1}\big(\lambda_i(s) - \lambda_j(s)\big)^{-2}. \tag{3.5.4}$$

Then,

$$\partial_s f_s = \mathcal{B}(s) f_s, \quad \textit{where} \quad \mathcal{B}(s) f_s(\boldsymbol{\xi}) = \sum_{i \neq j} c_{ij}(s) 2\xi_i (1 + 2\xi_j)\big(f_s(\boldsymbol{\xi}^{ij}) - f_s(\boldsymbol{\xi})\big). \tag{3.5.5}$$

Recall from Section 3.2.3 that we view N-tuples $\boldsymbol{\xi} = (\xi_1, \xi_2, \ldots, \xi_N) \in \mathbb{N}^N$ as particle configurations, with ξ_k particles at location k for each $k \in [1, N]$; the total number of particles in this configuration is $n = \mathcal{N}(\boldsymbol{\xi}) = \sum_{j=1}^N \xi_j$. We label the locations of these particles in non-decreasing order by

$$x_1(\boldsymbol{\xi}) \le \cdots \le x_n(\boldsymbol{\xi}). \tag{3.5.6}$$

Given another particle configuration $\boldsymbol{\zeta}$ with the same number n of particles, whose locations are labeled by $\big(y_j(\boldsymbol{\zeta})\big)$ in non-decreasing order, we define the distance

$$d(\boldsymbol{\xi}, \boldsymbol{\zeta}) = \sum_{j=1}^n \big|x_j(\boldsymbol{\xi}) - y_j(\boldsymbol{\zeta})\big|. \tag{3.5.7}$$

Next, we recall $\mathfrak{c}, \eta, \psi$ from (3.2.15), fix $n \in \mathbb{N}$, and define the parameter

$$\ell = \ell(n) = \psi^{4n+1} N^{1+\mathfrak{d}} \eta, \quad \text{where} \quad \mathfrak{d} = \mathfrak{d}(n) = 50(n+1)\mathfrak{c}. \tag{3.5.8}$$

Recalling t from (3.2.10) (which satisfies (3.2.11)), we also set

$$\tau = \tau(n) = t - N^{7\mathfrak{d}} \psi \eta, \quad t_0 = t - \psi \eta. \tag{3.5.9}$$

These are chosen so that for fixed $n > 0$, recalling the choices from (3.2.15) (after selecting $\mathfrak{c} = \mathfrak{c}(n) > 0$ to be sufficiently small),

$$N^{-1/2} \ll \psi \eta \ll N^{7\mathfrak{d}} \psi \eta \ll \tau < t_0 < t < t_0 + \frac{\ell}{N} \ll 1. \tag{3.5.10}$$

Recalling the operator $\mathcal{B} = \mathcal{B}(s)$ from (3.5.5), we decompose it into the sum of a "short range" and "long range" operator, given explicitly by $\mathcal{B} = \mathcal{S} + \mathcal{L}$, where $\mathcal{S} = \mathcal{S}_n = \mathcal{S}_n(s)$ and $\mathcal{L} = \mathcal{L}_n = \mathcal{L}_n(s)$ are defined by

$$(\mathcal{S} f_s)(\boldsymbol{\xi}) = \sum_{0 < |j-k| \le \ell} c_{jk}(s) 2\xi_j (1 + 2\xi_k) \left(f_s(\boldsymbol{\xi}^{jk}) - f_s(\boldsymbol{\xi}) \right), \tag{3.5.11}$$

$$(\mathcal{L} f_s)(\boldsymbol{\xi}) = \sum_{|j-k| > \ell} c_{jk}(s) 2\xi_j (1 + 2\xi_k) \left(f_s(\boldsymbol{\xi}^{jk}) - f_s(\boldsymbol{\xi}) \right). \tag{3.5.12}$$

We let $\mathcal{U}_{\mathcal{B}}(s_1, s_2)$ be the semigroup associated with $\mathcal{B}$ and likewise define $\mathcal{U}_{\mathcal{S}}(s_1, s_2)$ and $\mathcal{U}_{\mathcal{L}}(s_1, s_2)$.

We also let $\mathcal{F}_{t_0}$ denote the σ-algebra generated by $\{\mathbf{X}_s\}_{0 \le s \le t_0}$, and define

$$h_s(\boldsymbol{\xi}) = h_{\boldsymbol{\lambda}, s}(\boldsymbol{\xi}) = \mathbb{E}\big[Q^{j_1, \dots, j_m}_{i_1, \dots, i_m}(s) \mid \boldsymbol{\lambda}, \mathcal{F}_{t_0} \big] = \mathbb{E}\big[Q^{j_1, \dots, j_m}_{i_1, \dots, i_m}(s) \mid \boldsymbol{\lambda}, \mathbf{X}_{t_0} \big]. \tag{3.5.13}$$

In the last equality, we used the Markov properties for Dyson Brownian motion and the eigenvector moment flow. Informally, $h_s(\boldsymbol{\xi})$ corresponds to a solution of the eigenvector moment flow dynamics, run for time $s - t_0$, with initial data $\mathbf{X}_{t_0}$.

For consistency with [42] we introduce the following notation. Let $K > 1$ be such that K^{-1} is less than the constant c from Lemma 3.3.1 and K is greater than those from Lemma 3.3.3, Lemma 3.3.4, Lemma 3.3.5, and Lemma 3.3.6; and define

$$r = \frac{1}{2K}; \qquad \mathcal{D}_r = \left\{ z = E + \mathrm{i}\eta \colon |E| \leq r, \frac{\psi^4}{N} \leq \eta \leq r \right\}. \tag{3.5.14}$$

We define the function $\widetilde{d} = \widetilde{d}_n$ on n-particle configurations by

$$\widetilde{d}(\boldsymbol{\xi}, \boldsymbol{\zeta}) = \max_{1 \leq \beta \leq n} \left| \left\{ 1 \leq i \leq N \colon \left| \gamma_i(t_0) \right| \leq r, \min \left\{ x_\beta(\boldsymbol{\xi}), y_\beta(\boldsymbol{\zeta}) \right\} \leq i \leq \max \left\{ x_\beta(\boldsymbol{\xi}), y_\beta(\boldsymbol{\zeta}) \right\} \right\} \right|. \tag{3.5.15}$$

The next lemma provides several estimates necessary to analyze the eigenvector moment flow. Its first and second parts follow from Lemma 3.3.3 and Lemma 3.3.4, respectively, together with a standard stochastic continuity argument. Its third part constitutes a special case of [42, Corollary 3.3], whose assumptions are verified by Lemma 3.3.2. Its fourth part is a consequence of [42, (3.48)]. In what follows, we recall $m_{\mathrm{fc}}(s, z)$ from (3.3.9), $\gamma_i(s)$ from (3.3.10), and $\mathbf{R}(s, z)$ from (3.2.16).

Lemma 3.5.2 ([42])**.** *The initial data $\mathbf{X}_0$ and the Dyson Brownian motion $\{\mathbf{X}_s\}_{0 < s \leq t}$ together induce a measure $\mathcal{M}$ on the space of eigenvalue and eigenvector trajectories $\big(\boldsymbol{\lambda}(s), \mathbf{u}(s)\big)_{0 \leq s \leq t}$ on which the following event of trajectories holds with overwhelming probability.*

1. *Eigenvalue rigidity holds: $\sup_{t_0 \leq s \leq t} \left| m_N(s, z) - m_{\mathrm{fc},t}(s, z) \right| \leq \psi (N\eta)^{-1}$ uniformly for $z \in \mathcal{D}_r$ and $\sup_{t_0 \leq s \leq t} \left| \lambda_i(s) - \gamma_i(s) \right| \leq \psi N^{-1}$ uniformly for indices i such that $\left| \gamma_i(s) \right| \leq r$.*

2. *Delocalization holds: Conditional on $\boldsymbol{\lambda}$, for any $\mathbf{q} \in \mathbb{R}^N$ with stable support we have that*

$$\sup_{z \in \mathcal{D}_r} \sup_{t_0 \leq s \leq t} \left| \left\langle \mathbf{q}, \mathbf{R}(s, z) \mathbf{q} \right\rangle \right| \leq C(q) \psi; \qquad \sup_{z \in \mathcal{D}_r} \sup_{t_0 \leq s \leq t} N \left\langle \mathbf{u}_i(s), \mathbf{q} \right\rangle^2 \leq C(q) \psi, \tag{3.5.16}$$

where $C(q) > 0$ is a constant depending on $|\operatorname{supp}\mathbf{q}|$.

3. *Finite speed of propagation holds: Let $n > 0$ be an integer, and abbreviate $\ell = \ell(n)$ and $\mathcal{S} = \mathcal{S}_n$. Conditional on $\boldsymbol{\lambda}$, we have the following estimate that is uniform in any function $g\colon \{\boldsymbol{\xi} \in \mathbb{N}^N : \mathcal{N}(\boldsymbol{\xi}) = n\} \to \mathbb{R}$. For a particle configuration $\boldsymbol{\xi} \in \mathbb{N}^N$ with $\mathcal{N}(\boldsymbol{\xi}) = n$ such that $\widetilde{d}_n(\boldsymbol{\xi}, \boldsymbol{\zeta}) \geq \psi\ell$ for each $\boldsymbol{\zeta}$ in the support of g, we have*

$$\sup_{t_0 \leq s \leq t} \big|\mathcal{U}_{\mathcal{S}}(t_0, s) g(\boldsymbol{\xi})\big| \leq N^n e^{-c\psi} \|g\|_\infty. \tag{3.5.17}$$

4. *For any interval $I \subset \big[-r, r\big]$ of length $|I| \geq \psi N^{-1}$, we have*

$$C^{-1}|I|N \leq \Big|\big\{i\colon \lambda_i(s) \in I\big\}\Big| \leq C|I|N, \tag{3.5.18}$$

uniformly in $s \in [t_0, t]$.

The next estimate on the short range operator $\mathcal{S}$ is a consequence of [42, Lemma 3.5] (whose conditions are verified by Lemma 3.3.2 and Lemma 3.3.7).

Lemma 3.5.3 ([42, Lemma 3.5])**.** *There exists a constant $C > 0$ such that the following holds with overwhelming probability with respect to $\mathcal{F}_{t_0}$. Fix an integer $n > 0$, and abbreviate $\ell = \ell(n)$ (recall (3.5.8)) and $\mathcal{S} = \mathcal{S}_n$. There exists an event $\mathcal{E}$ of trajectories $\big(\boldsymbol{\lambda}(s), \mathbf{u}(s)\big)_{t_0 \leq s \leq t}$ of overwhelming probability on which we have*

$$\sup_{s \in [t_0, t]} \Big|\big(\mathcal{U}_{\mathcal{B}}(t_0, s) h_{t_0} - \mathcal{U}_{\mathcal{S}}(t_0, s) h_{t_0}\big)(\boldsymbol{\xi})\Big| \leq C \frac{\psi^n N(t - t_0)}{\ell} \tag{3.5.19}$$

for any configuration $\boldsymbol{\xi} \in \mathbb{N}^N$ such that $\mathcal{N}(\boldsymbol{\xi}) = n$ and $\operatorname{supp}\boldsymbol{\xi} \subset \big[(1/2 - c)N - 2\psi\ell, (1/2 + c)N + 2\psi\ell\big]$.

3.5.2 Continuity estimates

To prepare for the proof of Theorem 3.2.8, we require the following continuity estimates for entries of $\mathbf{R}(t, z)$. We recall $\mathfrak{a}$ from (3.2.2); t_0 and τ from (3.5.9); and r from (3.5.14), and define

$$\widehat{\mathcal{D}} = \left\{ z = E + \mathrm{i}\eta \colon \; |E| \le \frac{r}{4}, \, N^{-\mathfrak{a}} \le \eta \le \frac{r}{4} \right\}. \tag{3.5.20}$$

Lemma 3.5.4. *Fix an integer $n > 0$, a real number $\delta > 0$, and a unit vector $\mathbf{q}$ with stable support; set $q = |\operatorname{supp} \mathbf{q}|$, and abbreviate $\tau = \tau(n)$. Then, there exist constants $c > 0$ (independent of n, δ, and q) and $C = C(\delta, q, n) > 0$ such that, uniformly in $t_1, t_2 \in [t_0, t]$ with $t_1 < t_2$, we have*

$$\sup_{z \in \widehat{\mathcal{D}}} \Big| \big\langle \mathbf{q}, \mathbf{R}(t_1, z) \mathbf{q} \big\rangle - \big\langle \mathbf{q}, \mathbf{R}(t_2, z) \mathbf{q} \big\rangle \Big| \le C \frac{\psi^4}{\sqrt{N\eta}} + C N^{\delta} \bigg(\frac{t_2 - t_1}{t_2 - \tau} + N^{-c} \bigg); \tag{3.5.21}$$

$$\sup_{\substack{z_1, z_2 \in \widehat{\mathcal{D}} \\ \operatorname{Im} z_1 = \operatorname{Im} z_2}} \Big| \big\langle \mathbf{q}, \mathbf{R}(t_1, z_1) \mathbf{q} \big\rangle - \big\langle \mathbf{q}, \mathbf{R}(t_1, z_2) \mathbf{q} \big\rangle \Big| \le C \frac{\psi^4}{\sqrt{N\eta}} + C N^{\delta} \bigg(N^{-c} + \frac{|z_1 - z_2| + \operatorname{Im} z_1}{t_1 - \tau} \bigg), \tag{3.5.22}$$

both with overwhelming probability.

Proof of Lemma 3.5.4. For $s \ge \tau$, define

$$r_i(s, z) = \frac{1}{\lambda_i(\tau) - z - (s - \tau) m_{\mathrm{fc}, t}(s, z)}, \tag{3.5.23}$$

which is similar to the terms appearing in the definition of the free convolution (3.3.9) (but, in a sense, "started" at $\boldsymbol{\lambda}(\tau)$ instead of at $\boldsymbol{\lambda}(0)$).

Now let us apply [42, Theorem 2.1], with the t there equal to $s - \tau$ here and the H_0 given by $\mathbf{X}_\tau$ here; the assumptions of that theorem are verified by (3.3.12), (3.3.13), and the facts that $\tau \gg N^{-1/2}$ and $s - \tau \ge t_0 - \tau \gg N^{-1/2}$. For any $s \in [t_0, t]$, this gives with overwhelming probability that

$$\left|\big\langle \mathbf{q}, \mathbf{R}(s,z)\mathbf{q}\big\rangle - \sum_{i=1}^{N} \big\langle \mathbf{u}_i(\tau), \mathbf{q}\big\rangle^2 r_i(s,z)\right| \le \frac{\psi^2}{\sqrt{N\eta}} \operatorname{Im}\left(\sum_{i=1}^{N} \big\langle \mathbf{u}_i(\tau), \mathbf{q}\big\rangle^2 \big|r_i(s,z)\big|\right). \quad (3.5.24)$$

Thus, (3.3.1), (3.3.2), and a union bound over s in an N^{-10}-net of $[t_0, t]$ together yield that (3.5.24) holds with overwhelming probability, uniformly in $s \in [t_0, t]$.

To bound the right side of (3.5.24), observe that (3.3.17), [42, (2.3)], and the exchangeability of the eigenvector entries together yield the bound

$$\sum_{i=1}^{N} \big\langle \mathbf{u}_i(\tau), \mathbf{q}\big\rangle^2 \big|r_i(s,z)\big| \le C\psi(\log N)^2, \quad (3.5.25)$$

uniformly for any standard basis vector $\mathbf{q} \in \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_N\}$. Therefore, (3.5.25) also holds for unit vectors $\mathbf{q} \in \mathbb{R}^N$ of stable support (where the C there now depends on $q = |\operatorname{supp} \mathbf{q}|$), by expanding $\mathbf{q}$ in the standard basis and using the inequality $(a+b)^2 \le 2(a^2+b^2)$ on the products $\langle \mathbf{u}_i, \mathbf{e}_j\rangle\langle \mathbf{u}_i, \mathbf{e}_k\rangle$ that appear in the corresponding expansion of $\big\langle \mathbf{u}_i(\tau), \mathbf{q}\big\rangle^2$. Thus, (3.5.24) and (3.5.25) together imply

$$\left|\big\langle \mathbf{q}, \mathbf{R}(s,z)\mathbf{q}\big\rangle - \sum_{i=1}^{N} \big\langle \mathbf{u}_i(\tau), \mathbf{q}\big\rangle^2 r_i(s,z)\right| \le \frac{C(q)\psi^4}{\sqrt{N\eta}}. \quad (3.5.26)$$

We now establish (3.5.21) by subtracting (3.5.26) evaluated at $s = t_2$ from that equation evaluated at $s = t_1$. To that end, we have from (3.5.23) that

$$\big|r_i(t_1,z) - r_i(t_2,z)\big| = \frac{\big|(t_2-\tau)m_{\mathrm{fc},t}(t_2,z) - (t_1-\tau)m_{\mathrm{fc},t}(t_1,z)\big|}{\Big|\big(\lambda_i(\tau) - z - (t_1-\tau)m_{\mathrm{fc},t}(t_1,z)\big)\big(\lambda_i(\tau) - z - (t_2-\tau)m_{\mathrm{fc},t}(t_2,z)\big)\Big|}. \quad (3.5.27)$$

The numerator of the right side of (3.5.27) is with overwhelming probability bounded by

$$\begin{aligned}&\big|(t_1-\tau)m_{\mathrm{fc},t}(t_1,z)-(t_2-\tau)m_{\mathrm{fc},t}(t_2,z)\big|\\&\quad\le|t_1-t_2|\big|m_{\mathrm{fc},t}(t_2,z)\big|+|t_1-\tau|\big|m_{\mathrm{fc},t}(t_1,z)-m_{\mathrm{fc},t}(t_2,z)\big|\le C(\log N)^2(t_2-t_1)+C(t_1-\tau)N^{-c},\end{aligned}\tag{3.5.28}$$

Here, in the last inequality we used the overwhelming probability bound $\big|m_{\mathrm{fc},t}(t_2,z)\big|\le C(\log N)^2$ (which follows from [42, (2.3)], whose hypotheses are satisfied by (3.3.11) with $s=0$) and also the overwhelming probability estimate

$$\big|m_{\mathrm{fc},t}(t_1,z)-m_{\mathrm{fc},t}(t_2,z)\big|\le\big|m_{\mathrm{fc},t}(t_1,z)-m_\alpha(z)\big|+\big|m_{\mathrm{fc},t}(t_2,z)-m_\alpha(z)\big|\le CN^{-c},\tag{3.5.29}$$

where the last inequality follows from (3.3.13) (with the δ there equal to $\frac{1}{2}\big(\frac{1}{2}-\mathfrak{a}\big)$ here).

To bound the denominator of the right side of (3.5.27) observe, since $\operatorname{Im}m_{\mathrm{fc},t}(t_1,z)\ge c'$ for some uniform constant $c'>0$ (which is a consequence of (3.3.13) and (3.3.6)), we have with overwhelming probability that

$$\frac{1}{\big(\lambda_i(\tau)-z-(t_1-\tau)m_{\mathrm{fc},t}(t_1,z)\big)\big(\lambda_i(\tau)-z-(t_2-\tau)m_{\mathrm{fc},t}(t_2,z)\big)}\le C\frac{\big|r_i(t_1,z)\big|}{t_2-\tau}.\tag{3.5.30}$$

Altogether, we obtain with overwhelming probability that

$$\begin{aligned}\big|r_i(t_1,z)-r_i(t_2,z)\big|&\le C\frac{\big|r_i(t_1,z)\big|}{t_2-\tau}\big((\log N)^2(t_2-t_1)+(t_1-\tau)N^{-c}\big)\\&\le C\big|r_i(t_1,z)\big|\left((\log N)^2\frac{t_2-t_1}{t_2-\tau}+N^{-c}\right),\end{aligned}\tag{3.5.31}$$

where we used $t_1 - \tau \leq t_2 - \tau$ in the last inequality. It follows that

$$\begin{aligned}\left| \sum_{i=1}^{N} \langle \mathbf{u}_i(\tau), \mathbf{q} \rangle^2 r_i(t, z) - \sum_{i=1}^{N} \langle \mathbf{u}_i(\tau), \mathbf{q} \rangle^2 r_i(\tau, z) \right| \\ \leq C(\log N)^2 \left(\frac{t_2 - t_1}{t_2 - \tau} + N^{-c} \right) \sum_{i=1}^{N} \langle \mathbf{u}_i(\tau), \mathbf{q} \rangle^2 \big| r_i(t_1, z) \big|.\end{aligned} \tag{3.5.32}$$

To bound the right side of (3.5.32), observe that

$$\sum_{i=1}^{N} \langle \mathbf{u}_i(\tau), \mathbf{q} \rangle^2 \big| r_i(t_1, z) \big| = \sum_{\substack{1 \leq i \leq N \\ |\lambda_i(\tau)| < r}} \langle \mathbf{u}_i(\tau), \mathbf{q} \rangle^2 \big| r_i(t_1, z) \big| + \sum_{\substack{1 \leq i \leq N \\ |\lambda_i(\tau)| \geq r}} \langle \mathbf{u}_i(\tau), \mathbf{q} \rangle^2 \big| r_i(t_1, z) \big|. \tag{3.5.33}$$

We bound the first term on the right side of (3.5.33) using Lemma 3.3.4, which yields

$$\sum_{\substack{1 \leq i \leq N \\ |\gamma_i(\tau)| < r}} \langle \mathbf{u}_i(\tau), \mathbf{q} \rangle^2 \big| r_i(t_1, z) \big| \leq N^{\delta/2 - 1} \sum_{i=1}^{N} \big| r_i(t_1, z) \big| \leq N^{\delta}, \tag{3.5.34}$$

for any $\delta > 0$, with overwhelming probability. Here, in the last bound we used the fact that $\sum_{i=1}^{N} \big| r_i(t_1, z) \big| \leq N(\log N)^2$ (which is a consequence of [78, Lemma 7.5], whose assumptions are verified by (3.3.12) and the fact that $t_1 - \tau \gg N^{-1/2}$).

To bound the second term in (3.5.33), observe for $\big|\gamma_i(\tau)\big| > r$ we have $\big|\lambda_i(\tau)\big| > \frac{3r}{4}$ (by (3.3.14)), so

$$\big| \lambda_i(\tau) - z - (t_1 - \tau) m_{\mathrm{fc},t}(t_1, z) \big| \geq c, \tag{3.5.35}$$

for some constant $c > 0$, where we used $t_1 - \tau \ll 1$ and $|m_{\mathrm{fc},t}(t_1, z)| < C(\log N)^2$ (again by [42, (2.3)]). We then obtain by (3.5.23) that

$$\sum_{\substack{1 \leq i \leq N \\ |\gamma_i(\tau)| \geq r}} \langle \mathbf{u}_i(\tau), \mathbf{q} \rangle^2 \big| r_i(t_1, z) \big| \leq C \sum_{i=1}^{N} \langle \mathbf{u}_i(\tau), \mathbf{q} \rangle^2 \leq C. \tag{3.5.36}$$

Now the first bound (3.5.21) of the lemma follows from (3.5.26), (3.5.32), (3.5.33), (3.5.34), and (3.5.36), after absorbing the $(\log N)^2$ prefactor into N^δ and adjusting δ appropriately. We omit the proof of the second as it is analogous, but obtained by replacing (3.5.27) with the bound

$$\big|r_i(t_1,z_1)-r_i(t_1,z_2)\big| \le \frac{(t_1-\tau)\big|m_{\mathrm{fc},t}(t_1,z_1)-m_{\mathrm{fc},t}(t_1,z_2)\big| + |z_1-z_2|}{\Big|\big(\lambda_i(\tau)-z_1-(t_1-\tau)m_{\mathrm{fc},t}(t_1,z_1)\big)\big(\lambda_i(\tau)-z_2-(t_1-\tau)m_{\mathrm{fc},t}(t_1,z_2)\big)\Big|}, \tag{3.5.37}$$

and (3.5.28) with the bound

$$(t_1-\tau)\big|m_{\mathrm{fc},t}(t_1,z_1)-m_{\mathrm{fc},t}(t_1,z_2)\big| \tag{3.5.38}$$

$$\le (t_1-\tau)\Big(\big|m_{\mathrm{fc},t}(t_1,z_1)-m_\alpha(z_1)\big| + \big|m_{\mathrm{fc},t}(t_1,z_2)-m_\alpha(z_2)\big| + \big|m_\alpha(z_1)-m_\alpha(z_2)\big|\Big) \tag{3.5.39}$$

$$\le C(t_1-\tau)\big(N^{-c}+|z_1-z_2|+\operatorname{Im} z_1\big), \tag{3.5.40}$$

where (3.5.40) follows from (3.3.13) and (3.3.5). □

3.5.3 Short-time relaxation

The proof of short-time relaxation here is similar to that of [42, Theorem 3.6]. However, certain changes are necessary, since the diagonal resolvent entries $R_{ii}(t,z)$ for the removed model $\mathbf{X}_t$ do not converge to a deterministic quantity, unlike those of the matrix model considered in [42]. This causes the observable $f_{\boldsymbol{\lambda},t}(\boldsymbol{\xi})$ from (3.5.3) to now converge to the random variable $A(\mathbf{q},\boldsymbol{\xi})$, which is defined as follows.

Recall t and t_0 from (3.2.10) and (3.5.9), respectively; recall that $\{\lambda_j\}$ are the eigenvalues of $\mathbf{X}_s$ and that $\big\{\gamma_j(s)\big\}$ are given by (3.3.10); fix a unit vector $\mathbf{q}\in\mathbb{R}^N$ with stable support; and set $q = |\operatorname{supp}\mathbf{q}|$. For any integer $k\in[1,N]$; particle configuration $\boldsymbol{\xi}=(\xi_1,\xi_2,\ldots,\xi_N)$;

and eigenvalue trajectory $\boldsymbol{\lambda}$, define $A(\mathbf{q}, k) = A_{t,\boldsymbol{\lambda}}(\mathbf{q}, k)$ and $A(\mathbf{q}, \boldsymbol{\xi}) = A_{t,\boldsymbol{\lambda}}(\mathbf{q}, \boldsymbol{\xi})$ by

$$A(\mathbf{q}, k) = \frac{\operatorname{Im} \big\langle \mathbf{q}, \mathbf{R}(s, \widehat{\gamma}_k + \mathrm{i}\eta)\mathbf{q} \big\rangle}{\operatorname{Im} m_\alpha(\gamma_k + \mathrm{i}\eta)}, \quad A(\mathbf{q}, \boldsymbol{\xi}) = \prod_{k=1}^{N} A(\mathbf{q}, k)^{\xi_k}, \tag{3.5.41}$$

where we have recalled $\gamma_k = \gamma_k^{(\alpha)}$ from (3.1.7) and $\widehat{\gamma}_k$ from (3.2.18).

The initial data $\mathbf{X}_0$ and Dyson Brownian motion $\mathbf{X}_s$ for $0 \le s \le t$ together induce a measure on the space of eigenvalues and eigenvectors $\big(\boldsymbol{\lambda}(s), \mathbf{u}(s)\big)_{0 \le s \le t}$, which we denote by $\mathcal{M}$. Let $\boldsymbol{\lambda} = \big(\boldsymbol{\lambda}(s)\big)_{0 \le s \le t}$ be an eigenvalue trajectory with initial data given by a realization of the spectrum of $\mathbf{X}_0$, and recall the observable $h_s(\boldsymbol{\xi})$ from (3.5.13) that is associated to an eigenvalue trajectory $\boldsymbol{\lambda}$ and "starts" at time t_0.

Before proceeding, we first fix a small constant $c_0 = c_0(\alpha) > 0$ such that the conclusions of Lemma 3.3.1, Lemma 3.3.3, Lemma 3.5.2, Lemma 3.5.3, and Lemma 3.5.4 apply to any $z = E + \mathrm{i}\eta \in \mathcal{D}_r$ with $|E| \le 16c_0$ and any $i \in [1, N]$ with $|\gamma_i(s)| < 16c_0$. By Lemma 3.3.7, we may choose $c_1 > 0$ such that, for any fixed real number $s \in [t_0, t]$ and index $i \in \big[(1/2 - c_1)N, (1/2 + c_1)N\big]$, we have that $\big|\gamma_i(s)\big| < c_0$ with overwhelming probability. Hence, we will fix this choice of $c_1 > 0$ in what follows and apply the five lemmas listed above without further comment.

Then, to establish Theorem 3.2.8, it suffices prove the following proposition.

Proposition 3.5.5. *For any integer $m \ge 0$, there exist constants $c_2 = c_2(m) > 0$ and $C = C(m, q) > 0$ such that for $\mathfrak{c} < c_2$ we have*

$$\max_{\substack{\boldsymbol{\xi} \in \mathbb{N}^N : \mathcal{N}(\boldsymbol{\xi}) = m \\ \operatorname{supp} \boldsymbol{\xi} \in [(1/2 - c_1)N, (1/2 + c_1)N]}} \big| h_t(\boldsymbol{\xi}) - A(\mathbf{q}, \boldsymbol{\xi}) \big| \le C N^{-c_2}, \tag{3.5.42}$$

with overwhelming probability with respect to $\mathcal{M}$.

Proof of Theorem 3.2.8. Recall from (3.2.14) and (3.5.13) that $F_t(\boldsymbol{\xi}) = \mathbb{E}\big[h_t(\boldsymbol{\xi})\big]$, where the expectation is over $\boldsymbol{\lambda}$ and $\mathbf{X}_{t_0}$. Therefore, the theorem follows from applying (3.5.42) on an event of overwhelming probability, and applying the deterministic bounds $\big|h_t(\boldsymbol{\xi})\big| \le N^m$ and $\big|A(\mathbf{q}, \boldsymbol{\xi})\big| \le C^m \eta^{-m}$ (which holds by applying (3.3.2) to bound the numerator of $A(\mathbf{q}, \boldsymbol{\xi})$ by η^{-m} and (3.3.6) to bound its denominator by c^m) off of this event. □

We now introduce some notation. Fix a particle configuration $\boldsymbol{\zeta} = (\zeta_1, \zeta_2, \ldots, \zeta_N) \in \mathbb{N}^N$ with $m = \mathcal{N}(\boldsymbol{\zeta})$ particles such that $\operatorname{supp} \boldsymbol{\zeta} \in \big[(1/2 - c_1)N, (1/2 + c_1)N\big]$. We must verify (3.5.42) for $\boldsymbol{\xi} = \boldsymbol{\zeta}$.

For notational simplicity, we assume that $|\operatorname{supp} \boldsymbol{\zeta}| = 2$. The cases where $\boldsymbol{\zeta}$ is supported on one site and on more than two sites constitute straightforward modifications of the following two-site argument and will be briefly outlined in Section 3.5.6. Denote $\operatorname{supp} \boldsymbol{\zeta} = \{i_1, i_2\}$, with $i_1 < i_2$, and let j_1 and j_2 denote the number of particles in $\boldsymbol{\zeta}$ at sites i_1 and i_2, respectively. Thus $\zeta_{i_1} = j_1$, $\zeta_{i_2} = j_2$, and $j_1 + j_2 = m$.

Recalling $\mathfrak{d} = \mathfrak{d}(m) = 50(1+m)\mathfrak{c}$ and $\ell = \ell(m) = \psi^{4m+1} N^{1+\mathfrak{d}} \eta$ from (3.5.8), we define a "short-range averaging parameter"

$$\widetilde{d} = \lfloor \ell \psi^{5m} N^{\mathfrak{d}} \rfloor, \tag{3.5.43}$$

which by (3.2.15) and (3.5.9) satisfies $\psi^2 \ell \ll \widetilde{d} \ll N t_0$ (assuming $\mathfrak{c} = \mathfrak{c}(m) > 0$ is sufficiently small). For $a \in \mathbb{R}$ and $b \in \mathbb{N}$, we further define the interval $I_a^{(b)} = I_a^{(b)}(\boldsymbol{\zeta})$ by

$$I_a^{(b)} = I_{a,1}^{(b)} \cup I_{a,2}^{(b)}, \tag{3.5.44}$$

where the intervals $I_{a,1}^{(b)} = I_{a,1}^{(b)}(\boldsymbol{\zeta})$ and $I_{a,2}^{(b)} = I_{a,2}^{(b)}(\boldsymbol{\zeta})$ are given by

$$I_{a,1}^{(b)} = \big[i_1 - 10b\widetilde{d} - a, i_1 + 10b\widetilde{d} + a\big], \qquad I_{a,2}^{(b)} = \big[i_2 - 10b\widetilde{d} - a, i_2 + 10b\widetilde{d} + a\big]. \tag{3.5.45}$$

We assume the intervals $I_{a,1}^{(b)}$ and $I_{a,2}^{(b)}$ are disjoint for all $a \in [0, 2\tilde{d}]$ and $b \in [0, m]$.[2] When

[2]The reason for this assumption will be seen below, in the material immediately following (3.5.79).

this is not true, the argument below is carried out analogously, but instead using a single connected interval. We describe the necessary modifications in Section 3.5.6.

Definition 3.5.6. For any particle configuration $\boldsymbol{\xi} \in \mathbb{N}^N$ with $\operatorname{supp} \boldsymbol{\xi} \subset I^{(b)}_{\widetilde{d}+\psi\ell}$, we further set

$$\chi_1^{(b)}(\boldsymbol{\xi}) = \sum_{i \in I^{(b)}_{\widetilde{d}+\psi\ell,1}} \xi_i, \qquad \chi_2^{(b)}(\boldsymbol{\xi}) = \sum_{i \in I^{(b)}_{\widetilde{d}+\psi\ell,2}} \xi_i, \tag{3.5.46}$$

which denote the number of particles in $\boldsymbol{\xi}$ in $I^{(b)}_{\widetilde{d}+\psi\ell,1}$ and $I^{(b)}_{\widetilde{d}+\psi\ell,2}$, respectively. For any integers $k_1, k_2, b \geq 0$ and $n \geq 1$, recall $A(\mathbf{q}, k)$ from (3.5.41) and set

$$\Omega^{(b)}(k_1, k_2) = \Big\{ \boldsymbol{\xi} \in \mathbb{N}^N : \operatorname{supp} \boldsymbol{\xi} \subset I^{(b)}_{\widetilde{d}+\psi\ell}, \chi_1^{(b)}(\boldsymbol{\xi}) = k_1, \chi_2^{(b)}(\boldsymbol{\xi}) = k_2 \Big\}; \tag{3.5.47}$$

$$\Omega^{(b)}(n) = \bigcup_{k_1+k_2=n} \Omega^{(b)}(k_1, k_2); \qquad A(k_1, k_2) = A(\mathbf{q}, i_1)^{k_1} A(\mathbf{q}, i_2)^{k_2}. \tag{3.5.48}$$

We also define the restricted intervals

$$\Phi^{(b)}(k_1, k_2) = \left\{ \boldsymbol{\xi} \in \mathbb{N}^N : \operatorname{supp} \boldsymbol{\xi} \subset I^{(b)}_{-\psi\ell}, \sum_{i \in I^{(b)}_{-\psi\ell,1}} \xi_i = k_1, \sum_{i \in I^{(b)}_{-\psi\ell,2}} \xi_i = k_2 \right\}; \tag{3.5.49}$$

$$\Phi^{(b)}(n) = \bigcup_{k_1+k_2=n} \Phi^{(b)}(k_1, k_2). \tag{3.5.50}$$

The following two definitions provide certain operators on the space of functions on particle configurations and an auxiliary flow. Similar definitions appeared in [44, Section 7.2].

Definition 3.5.7. Fix integers $k_1, k_2 \geq 0$. For integers $a, b \geq 0$, we define operators $\mathrm{Flat}_a^{(b)} = \mathrm{Flat}_{a;k_1,k_2}^{(b)}$ and $\mathrm{Av}^{(b)} = \mathrm{Av}^{(b)}_{k_1,k_2}$ on the space of functions $f \colon \mathbb{N}^N \to \mathbb{C}$ as follows. For each particle configuration $\boldsymbol{\xi} \in \mathbb{N}^N$ and function $f \colon \mathbb{N}^N \to \mathbb{C}$, set

$$\big(\mathrm{Flat}_a^{(b)}(f)\big)(\boldsymbol{\xi}) = \begin{cases} f(\boldsymbol{\xi}), & \text{if } \operatorname{supp} \boldsymbol{\xi} \subset I_a^{(b)}, \\ A(k_1, k_2), & \text{otherwise}; \end{cases} \tag{3.5.51}$$

$$\mathrm{Av}^{(b)}(f) = \widetilde{d}^{-1} \sum_{a=1}^{\widetilde{d}} \mathrm{Flat}_a^{(b)}(f). \tag{3.5.52}$$

Definition 3.5.8. Adopting the notation of Definition 3.5.7, we define the flow $g_s(\boldsymbol{\xi}) = g_s^{(b)}(\boldsymbol{\xi}) = g_s^{(b)}(\boldsymbol{\xi}; k_1, k_2)$ for $s \geq t_0$ by

$$\partial_s g_s = \mathcal{S}(s) g_s, \quad \text{with initial data} \quad g_{t_0}(\boldsymbol{\xi}) = (\mathrm{Av}^{(b)}\, h_{t_0})(\boldsymbol{\xi}). \tag{3.5.53}$$

For each $s \geq t_0$, let $\widetilde{\boldsymbol{\xi}} = \widetilde{\boldsymbol{\xi}}(s) = \widetilde{\boldsymbol{\xi}}(s; k_1, k_2) = \widetilde{\boldsymbol{\xi}}^{(b)}(s; k_1, k_2) \in \mathbb{N}^N$ denote a maximizing particle configuration for $g_s^{(b)}$:

$$g_s^{(b)}(\widetilde{\boldsymbol{\xi}}) = \max_{\boldsymbol{\xi} \in \Omega^{(b)}(k_1, k_2)} g_s^{(b)}(\boldsymbol{\xi}; k_1, k_2). \tag{3.5.54}$$

When there are multiple maximizers, we pick one arbitrarily (in a way such that $\widetilde{\boldsymbol{\xi}}(s)$ remains piecewise constant in s).

3.5.4 Proof of Proposition 3.5.5

To establish Proposition 3.5.5, we begin with the following lemma providing bounds on $A(\mathbf{q}, \boldsymbol{\xi})$ (recall (3.5.41)).

Lemma 3.5.9. *For any integer $m \geq 0$, there exists a constant $C = C(m) > 0$ such that the following holds with overwhelming probability with respect to $\mathcal{M}$ for sufficiently small $\mathfrak{c} = \mathfrak{c}(m) > 0$. First, for any particle configuration $\boldsymbol{\xi} \in \mathbb{N}^N$ with $m = \mathcal{N}(\boldsymbol{\xi})$ particles such* *that* $\operatorname{supp} \boldsymbol{\xi} \subset \big[(1/2 - c_1)N, (1/2 + c_1)N\big]$, *we have that*

$$\big|A(\mathbf{q}, \boldsymbol{\xi})\big| \le C\psi^m. \tag{3.5.55}$$

Second, for any integers $k_1, k_2, b \geq 0$ *with* $k_1 + k_2 \leq m$ *and* $b \leq m+1$, *we have that*

$$\max_{b \in [0, m+1]} \max_{\boldsymbol{\xi} \in \Omega^{(b)}(k_1, k_2)} \big|A(\mathbf{q}, \boldsymbol{\xi}) - A(k_1, k_2)\big| < CN^{-3\mathfrak{d}}. \tag{3.5.56}$$

Proof. Recalling the definition (3.5.41) of A, the denominator $\operatorname{Im} m_\alpha(\gamma_{i_j} + \mathrm{i}\eta)$ of each $A(\mathbf{q}, i_j)$ is bounded below by a uniform constant by (3.3.6). Moreover, the numerator of each $A(\mathbf{q}, i_j)$ is bounded above by $C\psi$ with overwhelming probability by the second part of Lemma 3.5.2. Together these estimates yield (3.5.55).

To establish (3.5.56), observe by (3.5.22), (3.2.11) and (3.2.2) that there exists a constant $c > 0$ such that, for any $\delta > 0$; $j \in \{1, 2\}$; and $k \in I^{(b)}_{\widetilde{d} + \psi\ell, j}$, we have with overwhelming probability that

$$\Big|\big\langle \mathbf{q}, \mathbf{R}(t, \widehat{\gamma}_k + \mathrm{i}\eta)\mathbf{q}\big\rangle - \big\langle \mathbf{q}, \mathbf{R}(t, \widehat{\gamma}_{i_j} + \mathrm{i}\eta)\mathbf{q}\big\rangle\Big| \lesssim_{m,\delta} \frac{\psi^4}{\sqrt{N\eta}} + N^\delta \Bigg(N^{-c} + \frac{\big|\widehat{\gamma}_k - \widehat{\gamma}_{i_j}\big| + \eta}{t - \tau}\Bigg). \tag{3.5.57}$$

To bound the right side of (3.5.57), we first note that by (3.3.24), there exists a constant $c' > 0$ such that for indices i, j with $|i - N/2| < c'N$ and $|j - N/2| < c'N$,

$$|\widehat{\gamma}_i - \widehat{\gamma}_j| \le |\widehat{\gamma}_i - \gamma_i(t)| + |\widehat{\gamma}_j - \gamma_j(t)| + |\gamma_i(t) - \gamma_j(t)| \le 2\eta + CN^{-1}|i - j|. \tag{3.5.58}$$

In the last inequality we used (3.3.24), the fact that $N^{\delta - 1/2} \leq \eta$ for $\delta \leq \mathfrak{c}$, and the fact that the density $\rho_{\mathrm{fc},t}(t, x)$ satisfies $c \leq \rho_{\mathrm{fc},t}(t, x) \leq C$ for $|x| \leq C^{-1}$. The latter fact is [78, Lemma 3.2], whose hypotheses are satisfied in this case by the first inequality in (3.3.11) and the first inequality in (3.3.6). We further observe using (3.5.58) that for $k \in I^{(b)}_{\widetilde{d}+\psi\ell,j}$,

$$\left|\widehat{\gamma}_{i_j} - \widehat{\gamma}_k\right| \lesssim N^{-1}|i_j - k| + \eta \lesssim 2(10b\widetilde{d} + 3\widetilde{d})N^{-1} + \eta \leq 30b\widetilde{d}N^{-1} + \eta \leq 31bN^{3\mathfrak{d}}\eta, \quad (3.5.59)$$

where in the last inequality we used the fact that $\widetilde{d} = \lfloor \ell\psi^{5m}N^{\mathfrak{d}} \rfloor$ (recall (3.5.43)), where $\ell = \psi^{4m+1}N^{1+\mathfrak{d}}\eta$ and $\mathfrak{d} = 50(m+1)\mathfrak{c}$ (recall (3.5.8)). Further using the facts that $\eta \gg N^{-1/2}$ and $t - \tau = N^{7\mathfrak{d}}\psi\eta$ (recall (3.5.9)), it follows from (3.5.57), after taking $\mathfrak{c} = \mathfrak{c}(m) > 0$ sufficiently small, that with overwhelming probability

$$\left|\left\langle \mathbf{q}, \mathbf{R}(t, \widehat{\gamma}_k + \mathrm{i}\eta)\mathbf{q}\right\rangle - \left\langle \mathbf{q}, \mathbf{R}(t, \widehat{\gamma}_{i_j} + \mathrm{i}\eta)\mathbf{q}\right\rangle\right| \lesssim_m N^{-4\mathfrak{d}}. \quad (3.5.60)$$

We now note that by (3.3.4), $\varrho_\alpha(x) > c$ for a constant $c > 0$ and all x in a neighborhood of zero that contains all $\gamma_k^{(\alpha)}$ such that $k \in I^{(b)}_{\widetilde{d}+\psi\ell,j}$. The definition (3.1.7) then implies that for $j, k \in I^{(b)}_{\widetilde{d}+\psi\ell,j}$, $\left|\gamma_j^{(\alpha)} - \gamma_k^{(\alpha)}\right| \leq CN^{-1}|j - k|$. Using this fact along with (3.3.5) implies that for $k \in I^{(b)}_{\widetilde{d}+\psi\ell,j}$,

$$\left|\operatorname{Im} m_\alpha(\gamma_{i_j} + \mathrm{i}\eta) - \operatorname{Im} m_\alpha(\gamma_k + \mathrm{i}\eta)\right| \leq C|i_j - k| + C\eta \leq 31bN^{3\mathfrak{d}}\eta \ll 1, \quad (3.5.61)$$

by the same calculation as in (3.5.59).

Thus, the bound (3.5.56) follows from (3.5.60), (3.5.55), and (3.5.61). □

The following lemma, which we will establish in Section 3.5.5 below, essentially states that the difference $g_s(\widetilde{\boldsymbol{\xi}}) - A(k_1, k_2)$ is either nearly negative or its derivative is bounded by a negative multiple of itself. Here, we recall the intervals $\Phi^{(b)}(k_1, k_2)$ from Definition 3.5.6.

Lemma 3.5.10. *Fix integers $b, n, k_1, k_2 \geq 0$ such that $n \leq m$, $b \leq m + 1$, and $k_1 + k_2 = n$. If $\mathfrak{c} = \mathfrak{c}(m) > 0$ (recall (3.2.15)) is chosen small enough, then there exist constants $C = C(b, n) > 0$ and $c = c(b, n) > 0$ such following holds with overwhelming probability with respect to $\mathcal{M}$. Fix a realization of $\mathbf{X}_0$ and an associated eigenvalue trajectory $\boldsymbol{\lambda} = \big(\boldsymbol{\lambda}(s)\big)_{t_0 \leq s \leq t}$. There exists a countable subset $\mathcal{C} = \mathcal{C}(\mathbf{X}_0, \boldsymbol{\lambda}) \subset [t_0, t]$ such that, for $s \in [t_0, t] \backslash \mathcal{C}$,* *the continuous function* $g_s^{(b)}\big(\widetilde{\boldsymbol{\xi}}(s;k_1,k_2);k_1,k_2\big)$ *is differentiable and satisfies either*

$$g_s^{(b)}(\widetilde{\boldsymbol{\xi}}) - A(k_1,k_2) \le N^{-1}, \tag{3.5.62}$$

or

$$\begin{aligned} \partial_s\big(g_s^{(b)}(\widetilde{\boldsymbol{\xi}}) - A(k_1,k_2)\big) \le \frac{C}{\eta}\bigg(\psi \max_{\boldsymbol{\xi}} \Big| h_s(\boldsymbol{\xi}) - A\big(\chi_1^{(b+1)}(\boldsymbol{\xi}), \chi_2^{(b+1)}(\boldsymbol{\xi})\big)\Big| + N^{-\mathfrak{d}}\bigg) \\ - \frac{c}{\eta}\big(g_s^{(b)}(\widetilde{\boldsymbol{\xi}}) - A(k_1,k_2)\big), \end{aligned} \tag{3.5.63}$$

where the maximum in (3.5.63) *is taken over all* $\boldsymbol{\xi} \in \Phi^{(b+1)}(k_1-1,k_2) \cup \Phi^{(b+1)}(k_1,k_2-1)$.

Given Lemma 3.5.10, we can now establish Proposition 3.5.5.

Proof of Proposition 3.5.5. First observe that, for any $n \le m$ and $\boldsymbol{\xi} \in \Phi^{(m-n+1)}(n)$, we have

$$\big|h_t(\boldsymbol{\xi}) - g_t^{(m-n+1)}(\boldsymbol{\xi})\big| \le \Big|\big(\mathcal{U}_{\mathcal{B}}(t_0,t)h_{t_0} - \mathcal{U}_{\mathcal{S}}(t_0,t)h_{t_0}\big)(\boldsymbol{\xi})\Big| + \big|\mathcal{U}_{\mathcal{S}}(t_0,t)(h_{t_0} - \mathrm{Av}^{(m-n+1)}\, h_{t_0})(\boldsymbol{\xi})\big| \tag{3.5.64}$$

$$\lesssim_m \psi^{4m} N \ell^{-1}(t-t_0) + e^{-c\psi}. \tag{3.5.65}$$

Here, the first term from (3.5.65) follows from Lemma 3.5.3, where the containment $\operatorname{supp}\boldsymbol{\xi} \subset \big[(1/2-c)N, (1/2+c)N\big]$ holds since $i_1, i_2 \in \big[(1/2-c_1)N, (1/2+c_1)N\big]$. The second term in (3.5.65) follows from (3.5.17), which applies due to the facts that $\big(h_{t_0} - \mathrm{Av}^{(m-n+1)}\, h_{t_0}\big)(\boldsymbol{\xi}') = 0$ whenever $\operatorname{supp}\boldsymbol{\xi}' \subseteq I_0^{(m-n+1)}$ and that $\widetilde{d}_n(\boldsymbol{\xi},\boldsymbol{\xi}') > \psi\ell$ for any $\boldsymbol{\xi} \in \Phi^{(m-n+1)}(n)$ and $\operatorname{supp}\boldsymbol{\xi}' \not\subseteq I_0^{(m-n+1)}$.

We now define a discretization of the interval $[t_0,t]$ by

$$t_k = t_0 + km^{-1}\psi\eta, \quad \text{for} \quad 0 \le k \le m, \tag{3.5.66}$$

and we will show for each integer $n \in [0, m]$ that, with overwhelming probability,

$$\sup_{s \in [t_n, t]} \max_{\boldsymbol{\xi} \in \Phi^{(m-n+1)}(n)} \Big| h_s(\boldsymbol{\xi}) - A\big(\chi_1^{(m-n+1)}(\boldsymbol{\xi}), \chi_2^{(m-n+1)}(\boldsymbol{\xi})\big) \Big| \lesssim_n \frac{\psi^n}{N^{\mathfrak{d}}}. \tag{3.5.67}$$

Given the $n = m$ case of (3.5.67) and using the facts that $\boldsymbol{\zeta} \in \Phi^{(1)}(m)$ and $t_m = t$, we obtain with overwhelming probability that

$$\Big| h_t(\boldsymbol{\zeta}) - A\big(\chi_1^{(1)}(\boldsymbol{\zeta}), \chi_2^{(1)}(\boldsymbol{\zeta})\big) \Big| \lesssim_m \frac{\psi^m}{N^{\mathfrak{d}}}. \tag{3.5.68}$$

Our conclusion, (3.5.42), then follows from the facts that $\chi_1^{(1)}(\boldsymbol{\zeta}) = j_1 = \zeta_{i_1}$ and $\chi_2^{(1)}(\boldsymbol{\zeta}) = j_2 = \zeta_{i_2}$; and our choices $\psi = N^{\mathfrak{c}}$ and $\mathfrak{d} = 50(m+1)\mathfrak{c}$.

So, it suffices to prove (3.5.67) and therefore the two estimates

$$\sup_{s \in [t_n, t]} \max_{\boldsymbol{\xi} \in \Phi^{(m-n+1)}(n)} \Big(h_s(\boldsymbol{\xi}) - A\big(\chi_1^{(m-n+1)}(\boldsymbol{\xi}), \chi_2^{(m-n+1)}(\boldsymbol{\xi})\big) \Big) \lesssim_n \frac{\psi^n}{N^{\mathfrak{d}}}; \tag{3.5.69}$$

$$\sup_{s \in [t_n, t]} \max_{\boldsymbol{\xi} \in \Phi^{(m-n+1)}(n)} \Big(A\big(\chi_1^{(m-n+1)}(\boldsymbol{\xi}), \chi_2^{(m-n+1)}(\boldsymbol{\xi})\big) - h_s(\boldsymbol{\xi}) \Big) \lesssim_n \frac{\psi^n}{N^{\mathfrak{d}}}. \tag{3.5.70}$$

To do this, we induct on $n \in [0, m]$. The base case $n = 0$ is trivial, since $\boldsymbol{\xi} \in \Phi^{(m+1)}(0)$ implies that $h_s(\boldsymbol{\xi}) = 1 = A\big(\chi_1^{(m+1)}(\boldsymbol{\xi}), \chi_2^{(m+1)}(\boldsymbol{\xi})\big)$. For the induction step, we assume the induction hypothesis (3.5.67) holds for $n-1$ and prove (3.5.69) and (3.5.70) for n.

We will in fact only establish (3.5.69), as the proof of (3.5.70) is entirely analogous (by in what follows replacing the maximizer $\widetilde{\boldsymbol{\xi}}$ of $g^{(b)}$ with the minimizer). To that end, for any two fixed integers $k_1, k_2 \geq 0$ with $k_1 + k_2 = n$, it suffices to show that

$$\sup_{s \in [t_n, t]} \big(g_s^{(m-n+1)}(\widetilde{\boldsymbol{\xi}}; k_1, k_2) - A(k_1, k_2) \big) \lesssim_n \frac{\psi^n}{N^{\mathfrak{d}}} \tag{3.5.71}$$

holds with overwhelming probability, where we have abbreviated $\widetilde{\boldsymbol{\xi}} = \widetilde{\boldsymbol{\xi}}^{(m-n+1)}(s; k_1, k_2)$. Indeed, given (3.5.71), (3.5.69) follows upon letting (k_1, k_2) range over all pairs of integers summing to n; the fact that $\widetilde{\boldsymbol{\xi}}$ maximizes $g^{(m-n+1)}$ over $\Omega^{(m-n+1)}(k_1, k_2)$; the fact that

$\Phi^{(m-n+1)}(k_1,k_2) \subseteq \Omega^{(m-n+1)}(k_1,k_2)$; and (3.5.65).

To establish (3.5.71), we first apply the $b = m-n+1$ case of Lemma 3.5.10. Since (3.5.62) implies (3.5.71), we may assume that (3.5.63) holds. Then the induction hypothesis (3.5.67) (whose n is equal to $n-1$ here); the fact that $\boldsymbol{\xi} \in \Phi^{(m-n+2)}(k_1-1,k_2) \cup \Phi^{(m-n+2)}(k_1,k_2-1)$ implies $\boldsymbol{\xi} \in \Omega^{(m-n+1)}(n-1)$ (since $10\widetilde{d} > 2\psi\ell$); and (3.5.63) together yield that the bound

$$\partial_s\big(g_s^{(m-n+1)}(\widetilde{\boldsymbol{\xi}};k_1,k_2) - A(k_1,k_2)\big) \le C(m,n)\frac{\psi^n}{N^{\mathfrak{d}}\eta} - \frac{c(m,n)}{\eta}\big(g_s^{(m-n+1)}(\widetilde{\boldsymbol{\xi}};k_1,k_2) - A(k_1,k_2)\big), \tag{3.5.72}$$

holds for all $s \in [t_{n-1},t] \setminus \mathcal{C}$ (for some countable subset $\mathcal{C}$) with overwhelming probability.

In particular, if we define $F\colon [t_0,t] \to \mathbb{R}$ by

$$F(s) = F_{m,n;k_1,k_2}(s) = g_s^{(m-n+1)}(\widetilde{\boldsymbol{\xi}};k_1,k_2) - A(k_1,k_2), \tag{3.5.73}$$

then there exist constants $c = c(m,n) > 0$ and $C = C(m,n) > 0$ such that

$$\partial_s\left(F(s) - C\frac{\psi^n}{N^{\mathfrak{d}}}\right) \le -\frac{c}{\eta}\left(F(s) - C\frac{\psi^n}{N^{\mathfrak{d}}}\right), \qquad \text{for each } s \in [t_{n-1},t] \setminus \mathcal{C}. \tag{3.5.74}$$

Thus, integration and the fact that $t_n - t_{n-1} = \frac{\psi\eta}{m}$ together yield for $s \in [t_n,t]$ that

$$F(s) \le \exp\Big(-\frac{c}{\eta}(s-t_{n-1})\Big)\left(F(t_{n-1}) - C\frac{\psi^n}{N^{\mathfrak{d}}}\right) + C\frac{\psi^n}{N^{\mathfrak{d}}} \le \exp\Big(-\frac{c\psi}{m}\Big)F(t_{n-1}) + C\frac{\psi^n}{N^{\mathfrak{d}}}. \tag{3.5.75}$$

To bound $\big|F(t_{n-1})\big|$, observe that

$$\big|F(t_{n-1})\big| \le \big\|g_{t_{n-1}}^{(m-n+1)}\big\|_\infty + \big|A(k_1,k_2)\big| \le \big\|g_{t_0}^{(m-n+1)}\big\|_\infty + C(m)\psi^m \le C^m N^{m/2} + C(m)\psi^m. \tag{3.5.76}$$

Here, to deduce the first inequality, we used the definition (3.5.73) of F. To deduce the second, we used the fact that $\big\|g_s^{(b)}\big\|_\infty \le \big\|g_{s'}^{(b)}\big\|_\infty$ whenever $s' \le s$ (since $\mathcal{S}$ is the generator of a Markov process) and (3.5.55). To deduce the third inequality, we used (3.5.53) and (3.5.3).

Then (3.5.73), (3.5.75), and (3.5.76) together imply (3.5.71), from which we deduce the proposition. $\square$

3.5.5 Proof of Lemma 3.5.10

We first establish Lemma 3.5.10 assuming (3.5.92) below; the latter will be proven as Lemma 3.5.11. Throughout this section, for $s \in [t_0, t]$ we occasionally abbreviate $\{\lambda_j\}_{1 \le j \le N} = \{\lambda_j(s)\}_{1 \le j \le N}$.

Proof of Lemma 3.5.10. The differentiability of $g_s(\widetilde{\boldsymbol{\xi}}) = g_s^{(b)}(\widetilde{\boldsymbol{\xi}}; k_1, k_2)$ follows from the general fact that the maximum of finitely many differentiable functions on an interval I is itself differentiable, away from a countable set $\mathcal{C}$. Thus, for any fixed $s \in [t_0, t] \setminus \mathcal{C}$, it remains to upper bound $g_s(\widetilde{\boldsymbol{\xi}}) - A(k_1, k_2)$ and its derivative. To that end, we may assume that

$$g_s(\widetilde{\boldsymbol{\xi}}) - A(k_1, k_2) > N^{-1}, \tag{3.5.77}$$

for otherwise (3.5.62) would hold. In this case, we set $\widetilde{\boldsymbol{\xi}} = (\widetilde{\xi}_1, \widetilde{\xi}_2, \ldots, \widetilde{\xi}_N)$ and use (3.5.11) to write

$$\partial_s\big(g_s(\widetilde{\boldsymbol{\xi}}) - A(k_1, k_2)\big) = \mathcal{S}(s) g_s(\widetilde{\boldsymbol{\xi}}) = \sum_{0 < |j-k| \le \ell} c_{jk}(t) 2\widetilde{\xi}_j (1 + 2\widetilde{\xi}_k)\big(g_s(\widetilde{\boldsymbol{\xi}}^{jk}) - g_s(\widetilde{\boldsymbol{\xi}})\big). \tag{3.5.78}$$

Now let $\operatorname{supp} \widetilde{\boldsymbol{\xi}} = \{j_1, j_2, \ldots, j_h\}$. We claim that

$$g_s(\widetilde{\boldsymbol{\xi}}^{j_p k}) \le g_s(\widetilde{\boldsymbol{\xi}}), \quad \text{for any integers } p \in [1, h] \text{ and } k \in [j_p - \ell, j_p + \ell]. \tag{3.5.79}$$

To see this first observe that, since $\widetilde{\boldsymbol{\xi}}$ maximizes g_s over $\Omega(k_1, k_2) = \Omega^{(b)}(k_1, k_2)$, (3.5.79) holds if $\widetilde{\boldsymbol{\xi}}^{j_p k} \in \Omega(k_1, k_2)$. So, let us assume instead that $\widetilde{\boldsymbol{\xi}}^{j_p k} \notin \Omega(k_1, k_2)$, meaning that there exists some $v \in \{1, 2\}$ such a particle originally at site $j_p \in I^{(b)}_{\widetilde{d} + \psi\ell, v}$ in $\widetilde{\boldsymbol{\xi}}$ jumped out of the interval $I^{(b)}_{\tilde{d}+\psi\ell,v}$. This implies that $k \notin I^{(b)}_{\tilde{d}+\psi\ell}$, since the particle can jump at most ℓ sites by (3.5.79), but the disjoint intervals $I^{(b)}_{\tilde{d}+\psi\ell,1}$ and $I^{(b)}_{\tilde{d}+\psi\ell,2}$ are at least $\tilde{d} \gg \ell$ sites apart by hypothesis.

Then, (3.5.51) and (3.5.52) together yield $(\operatorname{Av} h_{t_0})(\boldsymbol{\xi}) - A(k_1,k_2) = 0$ unless $\operatorname{supp}\boldsymbol{\xi} \subseteq I^{(b)}_{\tilde{d}}$. Thus, any particle configuration $\boldsymbol{\xi} \in \mathbb{N}^N$ in the support of $\operatorname{Av} h_{t_0} - A(k_1,k_2)$ must satisfy $\widetilde{d}_n\big(\boldsymbol{\xi}, \widetilde{\boldsymbol{\xi}}^{j_p k}\big) \geq \psi\ell$. Hence, the finite speed of propagation estimate (3.5.17) yields

$$\big|g_s(\widetilde{\boldsymbol{\xi}}^{j_p k}) - A(k_1,k_2)\big| = \Big|\mathcal{U}_{\mathcal{S}}(t_0,s)\big((\operatorname{Av} h_{t_0}) - A(k_1,k_2)\big)(\widetilde{\boldsymbol{\xi}}^{j_p k})\Big| \lesssim_n \exp\Big(-\frac{c\psi}{2}\Big) < \frac{1}{N}, \tag{3.5.80}$$

which contradicts (3.5.77).

We now set $z_{j_p} = \lambda_{j_p} + \mathrm{i}\eta$ and use (3.5.79), the definition (3.5.4) of the c_{ij}, and the fact that $\widetilde{\xi}_j(2\widetilde{\xi}_k + 1) \geq 1$ when $\widetilde{\boldsymbol{\xi}}^{j_p k} \neq \widetilde{\boldsymbol{\xi}}$ to bound the sum in (3.5.78) over j by

$$(3.5.78) \leq \sum_{p=1}^{h} \sum_{0<|k-j_p|\leq\ell} \frac{g_s(\widetilde{\boldsymbol{\xi}}^{j_p k}) - g_s(\widetilde{\boldsymbol{\xi}})}{N(\lambda_{j_p} - \lambda_k)^2} \leq \frac{1}{N} \sum_{p=1}^{h} \sum_{0<|k-j_p|\leq\ell} \frac{g_s(\widetilde{\boldsymbol{\xi}}^{j_p k}) - g_s(\widetilde{\boldsymbol{\xi}})}{(\lambda_{j_p} - \lambda_k)^2 + \eta^2} \tag{3.5.81}$$

$$= \frac{1}{N\eta} \sum_{p=1}^{h} \sum_{0<|k-j_p|\leq\ell} \bigg(\operatorname{Im} \frac{g_s(\widetilde{\boldsymbol{\xi}}^{j_p k})}{z_{j_p} - \lambda_k} - \operatorname{Im} \frac{A(k_1,k_2)}{z_{j_p} - \lambda_k} \bigg) \tag{3.5.82}$$

$$- \frac{1}{N\eta}\big(g_s(\widetilde{\boldsymbol{\xi}}) - A(k_1,k_2)\big) \sum_{p=1}^{h} \sum_{0<|k-j_p|\leq\ell} \operatorname{Im} \frac{1}{z_{j_p} - \lambda_k}. \tag{3.5.83}$$

Since the first bound in (3.5.18) and the fact that $N^{-1}\ell \gg \eta$ yields

$$\sum_{p=1}^{h} \sum_{0<|k-j_p|\leq\ell} \operatorname{Im} \frac{1}{z_{j_p} - \lambda_k} = \sum_{p=1}^{h} \sum_{0<|k-j_p|\leq\ell} \frac{\eta}{(\lambda_{j_p} - \lambda_k)^2 + \eta^2} \geq \sum_{p=1}^{h} \sum_{k:|\lambda_k - \lambda_{j_p}|\leq\eta} \frac{\eta}{2\eta^2} \geq cN, \tag{3.5.84}$$

we have that

$$(3.5.83) \leq -\frac{c}{\eta}\big(g_s(\widetilde{\boldsymbol{\xi}}) - A(k_1,k_2)\big). \tag{3.5.85}$$

To bound (3.5.82), we fix $p \in [1,h]$, recall $\mathcal{B}$ from (3.5.5), and employ the decomposition

$$\frac{1}{N} \sum_{0<|k-j_p|\leq \ell} \left(\operatorname{Im} \frac{g_s(\widetilde{\boldsymbol{\xi}}^{j_p k})}{z_{j_p} - \lambda_k} - \operatorname{Im} \frac{A(k_1,k_2)}{z_{j_p} - \lambda_k} \right) \tag{3.5.86}$$

$$= \frac{1}{N} \operatorname{Im} \sum_{0<|k-j_p|\leq \ell} \frac{\big(\mathcal{U}_{\mathcal{S}}(t_0,s) \operatorname{Av}^{(b)} h_{t_0}\big)(\widetilde{\boldsymbol{\xi}}^{j_p k}) - \big(\operatorname{Av}^{(b)} \mathcal{U}_{\mathcal{S}}(t_0,s) h_{t_0}\big)(\widetilde{\boldsymbol{\xi}}^{j_p k})}{z_{j_p} - \lambda_k} \tag{3.5.87}$$

$$+ \frac{1}{N} \operatorname{Im} \sum_{0<|k-j_p|\leq \ell} \frac{\big(\operatorname{Av}^{(b)} \mathcal{U}_{\mathcal{S}}(t_0,s) h_{t_0}\big)(\widetilde{\boldsymbol{\xi}}^{j_p k}) - \big(\operatorname{Av}^{(b)} \mathcal{U}_{\mathcal{B}}(t_0,s) h_{t_0}\big)(\widetilde{\boldsymbol{\xi}}^{j_p k})}{z_{j_p} - \lambda_k} \tag{3.5.88}$$

$$+ \frac{1}{N} \operatorname{Im} \sum_{0<|k-j_p|\leq \ell} \left(\frac{\big(\operatorname{Av}^{(b)} \mathcal{U}_{\mathcal{B}}(t_0,s) h_{t_0}\big)(\widetilde{\boldsymbol{\xi}}^{j_p k})}{z_{j_p} - \lambda_k} - \operatorname{Im} \frac{A(k_1,k_2)}{z_{j_p} - \lambda_k} \right). \tag{3.5.89}$$

The terms (3.5.87) and (3.5.88) may be bounded as in the content following [42, (3.64)]:[3]

$$(3.5.87) \lesssim_n \frac{\psi^{n+1}\ell}{\widetilde{d}}, \quad (3.5.88) \lesssim_n \frac{\psi^n N(t-t_0)}{\ell}. \tag{3.5.90}$$

For brevity we only prove here the second inequality in (3.5.90) and refer the reader to [42] for details on the first. Using Lemma 3.5.3 (which applies as $\operatorname{supp} \widetilde{\boldsymbol{\xi}}^{j_p k} \subseteq \big[(1/2-c)N, (1/2+c)N\big]$, since $\operatorname{supp} \widetilde{\boldsymbol{\xi}} \subseteq \big[(1/2-c_1)N, (1/2+c_1)N\big]$ and $|k-j_p| \leq \ell$), we find

$$\begin{aligned} &\Big| \big(\operatorname{Av}^{(b)} \mathcal{U}_{\mathcal{S}}(t_0,s) h_{t_0}\big)(\widetilde{\boldsymbol{\xi}}^{j_p k}) - \big(\operatorname{Av}^{(b)} \mathcal{U}_{\mathcal{B}}(t_0,s) h_{t_0}\big)(\widetilde{\boldsymbol{\xi}}^{j_p k}) \Big| \\ &\qquad \leq \Big| \big(\mathcal{U}_{\mathcal{S}}(t_0,s) h_{t_0} - \mathcal{U}_{\mathcal{B}}(t_0,s) h_{t_0}\big)\big(\widetilde{\boldsymbol{\xi}}^{j_p k}\big) \Big| \lesssim_n \frac{\psi^n N(t-t_0)}{\ell}, \end{aligned} \tag{3.5.91}$$

which implies the second bound in (3.5.90).

Next, as Lemma 3.5.11 below, we show that, for any fixed $p \in [1,h]$ and $s \in [t_0,t] \setminus \mathcal{C}$,

$$(3.5.89) \lesssim_n \psi \Big| h_s(\widetilde{\boldsymbol{\xi}} \setminus j_p) - A\big(\chi_1^{(b+1)}(\widetilde{\boldsymbol{\xi}} \setminus j_p), \chi_2^{(b+1)}(\widetilde{\boldsymbol{\xi}} \setminus j_p)\big) \Big| + N^{-\mathfrak{d}}. \tag{3.5.92}$$

Combining these bounds and using the choices of t_0 from (3.5.9); ℓ from (3.5.8); and $\widetilde{d}$ from

[3] When bounding (3.5.87), $I_a^{(b)}$ plays the role of the interval $[b_1 - a, b_2 + a]$ in [42].

(3.5.43), we obtain for fixed $p \in [1, h]$ and $s \in [t_0, t] \setminus \mathcal{C}$ that (3.5.86) satisfies

$$\frac{1}{N} \sum_{0 < |k - j_p| \le \ell} \left(\operatorname{Im} \frac{g_s(\widetilde{\boldsymbol{\xi}}^{j_p k})}{z_{j_p} - \lambda_k} - \operatorname{Im} \frac{A(k_1, k_2)}{z_{j_p} - \lambda_k} \right) \tag{3.5.93}$$

$$\lesssim_n \max_{\boldsymbol{\xi} \in \Omega^{(b+1)}(k_1 - 1, k_2) \cup \Omega^{(b+1)}(k_1, k_2 - 1)} \psi \Big| h_s(\boldsymbol{\xi}) - A\big(\chi_1^{(b+1)}(\boldsymbol{\xi}), \chi_2^{(b+1)}(\boldsymbol{\xi})\big) \Big| + N^{-\mathfrak{d}}. \tag{3.5.94}$$

Summing over $p \in [1, h]$, inserting this into (3.5.82), and using (3.5.85) then completes the proof. $\square$

We conclude this section by establishing (3.5.92).

Lemma 3.5.11. *Retain the hypotheses and notation of the proof of Theorem 3.2.8. Then equation* (3.5.92) *holds.*

Proof. From complete delocalization (3.5.16), we have with overwhelming probability that

$$\max_{p \in [1, h]} h_s(\widetilde{\boldsymbol{\xi}}^{j_p k}) \lesssim_n \psi^n. \tag{3.5.95}$$

Now, for any particle configuration $\boldsymbol{\xi} \in \mathbb{N}^N$, define $a_{\boldsymbol{\xi}} = a_{\boldsymbol{\xi}; s} \in [0, 1]$ through the equation

$$\operatorname{Av}(f)(\boldsymbol{\xi}) = a_{\boldsymbol{\xi}} h_s(\boldsymbol{\xi}) + (1 - a_{\boldsymbol{\xi}}) A(k_1, k_2) \qquad \text{if } h_s(\boldsymbol{\xi}) \ne A(k_1, k_2), \tag{3.5.96}$$

and set $a_{\boldsymbol{\xi}} = 0$ if $h_s(\boldsymbol{\xi}) = A(k_1, k_2)$. Since $\mathcal{U}_{\mathcal{B}}(t_0, s) h_{t_0} = h_s$, the first term of (3.5.89) equals

$$\begin{aligned}
\frac{1}{N} \operatorname{Im} \sum_{0 < |k - j_p| \le \ell} & \frac{a_{\widetilde{\boldsymbol{\xi}}^{j_p k}} h_s(\widetilde{\boldsymbol{\xi}}^{j_p k}) + (1 - a_{\widetilde{\boldsymbol{\xi}}^{j_p k}}) A(k_1, k_2)}{z_{j_p} - \lambda_k} \\
&= \frac{1}{N} \operatorname{Im} \sum_{0 < |k - j_p| \le \ell} \frac{a_{\widetilde{\boldsymbol{\xi}}} h_s(\widetilde{\boldsymbol{\xi}}^{j_p k}) + (1 - a_{\widetilde{\boldsymbol{\xi}}}) A(k_1, k_2)}{z_{j_p} - \lambda_k} \\
&\quad + \frac{1}{N} \operatorname{Im} \sum_{0 < |k - j_p| \le \ell} \frac{(a_{\widetilde{\boldsymbol{\xi}}^{j_p k}} - a_{\widetilde{\boldsymbol{\xi}}}) \big(h_s(\widetilde{\boldsymbol{\xi}}^{j_p k}) - A(k_1, k_2) \big)}{z_{j_p} - \lambda_k} \\
&= \frac{1}{N} \operatorname{Im} \sum_{0 < |k - j_p| \le \ell} \frac{a_{\widetilde{\boldsymbol{\xi}}} h_s(\widetilde{\boldsymbol{\xi}}^{j_p k}) + (1 - a_{\widetilde{\boldsymbol{\xi}}}) A(k_1, k_2)}{z_{j_p} - \lambda_k} + O_n \left(\frac{\ell \psi^n}{\widetilde{d}} \right).
\end{aligned} \tag{3.5.97}$$

In the last line we used (3.5.55); (3.5.95); the fact that $\operatorname{Im} \sum_{0<|k-j_p|\le \ell} (z_{j_p} - \lambda_k)^{-1} \le \operatorname{Im} m_N(s, z_{j_p}) \le C$, which follows from the first part of Lemma 3.5.2, (3.3.13), and (3.3.6); and the bound

$$\left| a_{\widetilde{\boldsymbol{\xi}}} - a_{\widetilde{\boldsymbol{\xi}}^{j_p k}} \right| \le \frac{d(\widetilde{\boldsymbol{\xi}}, \widetilde{\boldsymbol{\xi}}^{j_p k})}{\widetilde{d}} \le \frac{\ell}{\widetilde{d}}, \tag{3.5.98}$$

which follows from the definition of $a_{\widetilde{\boldsymbol{\xi}}}$ and the definition of the Av operator, since the sums defining $\operatorname{Av}(f)(\widetilde{\boldsymbol{\xi}})$ and $\operatorname{Av}(f)(\widetilde{\boldsymbol{\xi}}^{j_p k})$ can differ in at most ℓ terms. The equation (3.5.97) implies

$$(3.5.89) = \frac{a_{\widetilde{\boldsymbol{\xi}}}}{N} \sum_{0<|k-j_p|<\ell} \left(\frac{\eta h_s(\widetilde{\boldsymbol{\xi}}^{j_p k})}{(\lambda_{j_p} - \lambda_k)^2 + \eta^2} - \frac{\eta A(k_1, k_2)}{(\lambda_{j_p} - \lambda_k)^2 + \eta^2} \right) + O_n\left(\frac{\ell \psi^n}{\widetilde{d}} \right). \tag{3.5.99}$$

Through (3.5.18) and a dyadic decomposition analogous to the one used in the proof of Lemma 3.4.7 (see also the proof of [42, Lemma 3.5] for more details), one has with overwhelming probability that

$$\frac{1}{N} \sum_{|k-j_p|>\ell} \frac{\eta}{(\lambda_{j_p} - \lambda_k)^2 + \eta^2} \le \frac{CN\eta}{\ell}, \tag{3.5.100}$$

which by (3.5.95) implies with overwhelming probability

$$\left| \frac{1}{N} \sum_{|k-j_p|>\ell} \frac{\eta h_s(\widetilde{\boldsymbol{\xi}}^{j_p k})}{(\lambda_{j_p} - \lambda_k)^2 + \eta^2} \right| \le \frac{CN\eta\psi^n}{\ell}. \tag{3.5.101}$$

Also,

$$\frac{1}{N} \sum_{k=1}^{N} \frac{\eta}{(\lambda_{j_p} - \lambda_k)^2 + \eta^2} = \operatorname{Im} \sum_{k=1}^{N} \frac{1}{z_{j_p} - \lambda_k} = \operatorname{Im} m_N(s, z_{j_p}). \tag{3.5.102}$$

We conclude from (3.5.99), (3.5.100), (3.5.95), (3.5.101), and (3.5.102) that with overwhelming probability

$$(3.5.99) \le \frac{a_{\widetilde{\boldsymbol{\xi}}}}{N} \sum_{k=1}^{N} \frac{\eta h_s(\widetilde{\boldsymbol{\xi}}^{j_p k})}{(\lambda_{j_p} - \lambda_k)^2 + \eta^2} - a_{\widetilde{\boldsymbol{\xi}}} A(k_1, k_2) \operatorname{Im} m_N(s, z_{j_p}) + O_n\left(\frac{\psi^n}{N\eta} + \frac{\psi^n N\eta}{\ell} + \frac{\ell\psi^n}{\widetilde{d}} \right), \tag{3.5.103}$$

where we also used (3.5.55) and (3.5.95) to restore the term with index $k = j_p$ in the sum, which accrues an error of size $O\left(\psi^n/N\eta\right)$.

By (3.3.12) and (3.3.13), there exists a constant $c > 0$ such that, with overwhelming probability,

$$\left|\operatorname{Im} m_N(s, z_{j_p}) - \operatorname{Im} m_\alpha(\gamma_{j_p} + \mathrm{i}\eta)\right| \tag{3.5.104}$$

$$\leq \left|\operatorname{Im} m_N\big(s, z_{j_p}\big) - \operatorname{Im} m_\alpha(z_{j_p})\right| + \left|\operatorname{Im} m_\alpha(z_{j_p}) - \operatorname{Im} m_\alpha(\gamma_{j_p} + \mathrm{i}\eta)\right| \tag{3.5.105}$$

$$= \left|\operatorname{Im} m_\alpha(z_{j_p}) - \operatorname{Im} m_\alpha(\gamma_{j_p} + \mathrm{i}\eta)\right| + O\left(N^{-c}\right). \tag{3.5.106}$$

Moreover, by rigidity estimate from from the first part of Lemma 3.5.2, the combination of (3.3.23) and (3.3.24), and (3.3.5), we have that

$$\left|\operatorname{Im} m_\alpha\big(z_{j_p}\big) - \operatorname{Im} m_\alpha\big(\gamma_{j_p} + \mathrm{i}\eta\big)\right| \leq C\big|\lambda_{j_p}(s) - \gamma_{j_p}\big| + C\eta \leq C\left(\frac{\psi}{N} + N^{-c} + \eta\right) \leq CN^{-c}, \tag{3.5.107}$$

Thus, (3.5.106) and (3.5.107) yield

$$\left|\operatorname{Im} m_N\big(s, z_{j_p}\big) - \operatorname{Im} m_\alpha(\gamma_{j_p} + \mathrm{i}\eta)\right| \leq CN^{-c}, \tag{3.5.108}$$

which, together with (3.5.55), yields

$$\left|A(k_1, k_2) \operatorname{Im} m_N\big(s, z_{j_p}\big) - A(k_1, k_2) \operatorname{Im} m_\alpha(\gamma_{j_p} + \mathrm{i}\eta)\right| \lesssim_n \psi^n N^{-c} \tag{3.5.109}$$

with overwhelming probability. Therefore (3.5.103) implies

$$(3.5.89) \leq \frac{a_{\widetilde{\boldsymbol{\xi}}}}{N} \sum_{k=1}^{N} \frac{\eta h_s(\widetilde{\boldsymbol{\xi}}^{j_p k})}{\big(\lambda_{j_p}(s) - \lambda_k\big)^2 + \eta^2} - a_{\widetilde{\boldsymbol{\xi}}} A(k_1, k_2) \operatorname{Im} m_\alpha(\gamma_{j_p} + \mathrm{i}\eta) \tag{3.5.110}$$

$$+ O_n\left(\frac{\psi^n}{N\eta} + \frac{\psi^n N\eta}{\ell} + \frac{\ell \psi^n}{\widetilde{d}} + N^{-\mathfrak{d}} \right) \tag{3.5.111}$$

holds with overwhelming probability, where we used that $\psi^n N^{-c} = O(N^{-\mathfrak{d}})$ if $\mathfrak{c}$ is chosen sufficiently small (depending only on the value m from Lemma 3.5.10). Using the definition of h_s from (3.5.13) (and recalling from (3.2.13) that $a(2j) = (2j - i)!!$), we find that the first term on the right side of (3.5.110) is equal to

$$a_{\widetilde{\boldsymbol{\xi}}} \sum_{k=1}^{N} \mathbb{E}\left[\left(\prod_{1 \leq q \leq h} \frac{\Big(N\big\langle \mathbf{q}, \mathbf{u}_{j_q}(s)\big\rangle^2\Big)^{\widetilde{\xi}_q - \mathbf{1}_{p=q}}}{a\big(2(\widetilde{\xi}_q - \mathbf{1}_{p=q})\big)} \right) \left(\frac{\eta \big\langle \mathbf{q}, \mathbf{u}_k(s)\big\rangle^2}{(\lambda_{j_p} - \lambda_k)^2 + \eta^2} \right) \frac{a\big(2(\widetilde{\xi}_k)\big)}{a\big(2(\widetilde{\xi}_k + 1)\big)} \,\middle|\, \boldsymbol{\lambda}, \mathcal{F}_{t_0} \right] \tag{3.5.112}$$

$$\leq a_{\widetilde{\boldsymbol{\xi}}} \mathbb{E}\left[\left(\prod_{1 \leq q \leq h} \frac{\Big(N\big\langle \mathbf{q}, \mathbf{u}_{j_q}(s)\big\rangle^2\Big)^{\widetilde{\xi}_q - \mathbf{1}_{p=q}}}{a\big(2(\widetilde{\xi}_q - \mathbf{1}_{p=q})\big)} \right) \left(\sum_{k=1}^{N} \frac{\eta \big\langle \mathbf{q}, \mathbf{u}_k(s)\big\rangle^2}{(\lambda_{j_p} - \lambda_k)^2 + \eta^2} \right) \,\middle|\, \boldsymbol{\lambda}, \mathcal{F}_{t_0} \right]. \tag{3.5.113}$$

We now have with overwhelming probability that

$$\sum_{k=1}^{N} \frac{\eta \big\langle \mathbf{q}, \mathbf{u}_k(s)\big\rangle^2}{(\lambda_{j_p} - \lambda_k)^2 + \eta^2} = \operatorname{Im} \Big\langle \mathbf{q}, \mathbf{R}\big(s, \lambda_{j_p}(s) + \mathrm{i}\eta\big) \mathbf{q} \Big\rangle \tag{3.5.114}$$

$$= \operatorname{Im} \big\langle \mathbf{q}, \mathbf{R}(t_0, \widehat{\gamma}_{j_p}(s) + \mathrm{i}\eta) \mathbf{q} \big\rangle + O_{n,\mathfrak{c}}\left(\frac{\psi^4}{\sqrt{N\eta}} + N^{\mathfrak{c}}\left(\frac{t - t_0}{t_0 - \tau} + N^{-c} \right) \right. \tag{3.5.115}$$

$$\left. + N^{\mathfrak{c}}\left(N^{-c} + \frac{\psi N^{-1} + 2\eta}{t_0 - \tau} \right) \right). \tag{3.5.116}$$

In the last equality, we used (3.5.21), (3.5.22), $N\eta \gg N^{1/2}$, $s \in [t_0, t]$, and the overwhelming probability estimate

$$\left|\lambda_{j_p}(s)-\widehat{\gamma}_{j_p}(s)\right| \le \left|\lambda_{j_p}(s)-\gamma_{j_p}(s)\right| + \left|\gamma_{j_p}(s)-\widehat{\gamma}_{j_p}(s)\right| \le \frac{\psi}{N}+\eta, \tag{3.5.117}$$

which follows from the rigidity estimate in the first item in Lemma 3.5.2 and (3.3.24) (with $\eta \gg N^{-1/2}$). By (3.5.16) and the fact that $\psi = N^{\mathfrak{c}}$, this yields

$$(3.5.113) \le a_{\widetilde{\boldsymbol{\xi}}}\mathbb{E}\left[\left(\prod_{1\le q\le h}\frac{\left(N\big\langle \mathbf{q},\mathbf{u}_{j_q}(s)\big\rangle^2\right)^{\widetilde{\xi}_q-\mathbf{1}_{p=q}}}{a\big(2(\widetilde{\xi}_q-\mathbf{1}_{p=q})\big)}\right)\operatorname{Im}\Big\langle \mathbf{q},\mathbf{R}\big(t_0,\widehat{\gamma}_{j_p}(s)+\mathrm{i}\eta\big)\mathbf{q}\Big\rangle\,\Bigg|\,\boldsymbol{\lambda},\mathcal{F}_{t_0}\right] \tag{3.5.118}$$

$$+O_{n,\mathfrak{c}}\left(\frac{\psi^{n+4}}{\sqrt{N\eta}}+\psi^{n+1}\left(\frac{t-t_0}{t_0-\tau}+N^{-c}+\frac{\psi N^{-1}+2\eta}{t_0-\tau}\right)\right). \tag{3.5.119}$$

We now recognize that the second factor inside the expectation on the right side of (3.5.118) is measurable with respect to $\mathcal{F}_{t_0}$. We may therefore factor it out of the expectation and rewrite the previous bound as

$$(3.5.113) \le a_{\widetilde{\boldsymbol{\xi}}}h_s(\widetilde{\boldsymbol{\xi}}\setminus j_p)\operatorname{Im}\Big\langle \mathbf{q},\mathbf{R}\big(t_0,\widehat{\gamma}_{j_p}(s)+\mathrm{i}\eta\big)\mathbf{q}\Big\rangle \tag{3.5.120}$$

$$+O_{n,\mathfrak{c}}\left(\frac{\psi^{n+4}}{\sqrt{N\eta}}+\psi^{n+1}\left(\frac{t-t_0}{t_0-\tau}+N^{-c}\right)+\frac{\psi N^{-1}+2\eta}{t_0-\tau}\right)\Bigg). \tag{3.5.121}$$

Using again the computation (3.5.115) with $s=t$, and (3.5.95) yields

$$(3.5.113) \le a_{\widetilde{\boldsymbol{\xi}}}h_s(\widetilde{\boldsymbol{\xi}}\setminus j_p)\operatorname{Im}\Big\langle \mathbf{q},\mathbf{R}\big(t,\widehat{\gamma}_{j_p}(s)+\mathrm{i}\eta\big)\mathbf{q}\Big\rangle \tag{3.5.122}$$

$$+O_{n,\mathfrak{c}}\left(\frac{\psi^{n+4}}{\sqrt{N\eta}}+\psi^{n+1}\left(\frac{t-t_0}{t_0-\tau}+N^{-c}+\frac{\psi N^{-1}+2\eta}{t_0-\tau}\right)\right). \tag{3.5.123}$$

This, together with the definition (3.5.41) of $A(\mathbf{q}, j_p)$ and (3.5.110), gives

$$\begin{aligned}(3.5.89) \leq a_{\widetilde{\boldsymbol{\xi}}} \operatorname{Im} m_\alpha(\gamma_{j_p} + \mathrm{i}\eta) \Big(A(\mathbf{q}, j_p) h_s(\widetilde{\boldsymbol{\xi}} \setminus j_p) - A(k_1, k_2) \Big) \qquad (3.5.124)\\ + O_{n,\mathfrak{c}} \left(\frac{\psi^{n+4}}{\sqrt{N\eta}} + \psi^{n+1} \left(\frac{t - t_0}{t_0 - \tau} + N^{-c} + \frac{\psi N^{-1} + 2\eta}{t_0 - \tau} \right) \right) \qquad (3.5.125)\\ + O_{n,\mathfrak{c}} \left(\frac{\psi^4}{N\eta} + \frac{\psi^n N\eta}{\ell} + \frac{\ell \psi^n}{\widetilde{d}} + N^{-\mathfrak{d}} \right). \qquad (3.5.126)\end{aligned}$$

Recalling that in (3.2.15), (3.5.8), (3.5.9), and (3.5.43), we fixed small $\mathfrak{d}(m) > 0$ such that $\mathfrak{d} = 50(1+m)\mathfrak{c}$ (recall $\psi = N^{\mathfrak{c}}$) and chose parameters so that $N^{-1/2} \ll \eta \ll \tau \leq t_0 \leq t$:

$$\eta = N^{-\mathfrak{a}}\psi, \quad \ell = \psi^{5m+1} N^{1+\mathfrak{d}} \eta, \quad t_0 = t - \psi\eta, \quad \tau = t - N^{7\mathfrak{d}}\psi\eta, \quad \widetilde{d} = \ell \psi^{5m} N^{\mathfrak{d}}. \quad (3.5.127)$$

Then choosing $\mathfrak{c}$ sufficiently small, we deduce from (3.5.124) that

$$(3.5.89) \leq a_{\widetilde{\boldsymbol{\xi}}} \operatorname{Im} m_\alpha(\gamma_{j_p} + \mathrm{i}\eta) \Big(A(\mathbf{q}, j_p) h_s(\widetilde{\boldsymbol{\xi}} \setminus j_p) - A(k_1, k_2) \Big) + O_n \left(N^{-\mathfrak{d}} \right). \qquad (3.5.128)$$

To complete the argument, it suffices to show that

$$\begin{aligned}&a_{\widetilde{\boldsymbol{\xi}}} \operatorname{Im} m_\alpha(\gamma_{j_p} + \mathrm{i}\eta) \Big(A_t(\mathbf{q}, j_p) h_s(\widetilde{\boldsymbol{\xi}} \setminus j_p) - A_t(k_1, k_2) \Big)\\ &\qquad \leq C\psi \left| h_s(\widetilde{\boldsymbol{\xi}} \setminus j_p) - A_t\big(\chi_1^{(b+1)}(\widetilde{\boldsymbol{\xi}} \setminus j_p), \chi_2^{(b+1)}(\widetilde{\boldsymbol{\xi}} \setminus j_p)\big) \right| + O_n(N^{-\mathfrak{d}}). \qquad (3.5.129)\end{aligned}$$

We recall that, for $j_p \in \big[(1/2 - c)N, (1/2 + c)N\big]$, there exists $C > 0$ such that $\big|\operatorname{Im} m_\alpha(\gamma_{j_p} + \mathrm{i}\eta)\big| < C$, which holds by (3.3.6). This, together with (3.5.56), and the definition (3.5.41) of

$A(\mathbf{q}, \boldsymbol{\xi})$, we obtain

$$\operatorname{Im} m_\alpha(\gamma_{j_p} + \mathrm{i}\eta) \left(A(\mathbf{q}, j_p) h_s(\widetilde{\boldsymbol{\xi}} \setminus j_p) - A(k_1, k_2) \right) \tag{3.5.130}$$

$$= \operatorname{Im} m_\alpha(\gamma_{j_p} + \mathrm{i}\eta) \left(A(\mathbf{q}, j_p) h_s(\widetilde{\boldsymbol{\xi}} \setminus j_p) - A\big(\mathbf{q}, \widetilde{\boldsymbol{\xi}}\big) \right) + O_n\left(N^{-\mathfrak{d}}\right) \tag{3.5.131}$$

$$= \operatorname{Im} m_\alpha(\gamma_{j_p} + \mathrm{i}\eta) A(\mathbf{q}, j_p) \left(h_s(\widetilde{\boldsymbol{\xi}} \setminus j_p) - A(\mathbf{q}, \widetilde{\boldsymbol{\xi}} \setminus j_p) \right) + O_n\left(N^{-\mathfrak{d}}\right) \tag{3.5.132}$$

$$= \operatorname{Im} m_\alpha(\gamma_{j_p} + \mathrm{i}\eta) A(\mathbf{q}, j_p) \left(h_s(\widetilde{\boldsymbol{\xi}} \setminus j_p) - A\big(\chi_1^{(b+1)}(\widetilde{\boldsymbol{\xi}} \setminus j_p), \chi_2^{(b+1)}(\widetilde{\boldsymbol{\xi}} \setminus j_p)\big) + O_n\left(N^{-2\mathfrak{d}}\right) \right) \tag{3.5.133}$$

$$+ O_n\left(N^{-\mathfrak{d}}\right). \tag{3.5.134}$$

Combining the last line with (3.5.55) and using the bound $\left| \operatorname{Im} m_\alpha(\gamma_{j_p} + \mathrm{i}\eta) \right| < C$ again, we see

$$\begin{aligned} &\operatorname{Im} m_\alpha(\gamma_{j_p} + \mathrm{i}\eta) \left(A(\mathbf{q}, j_p) h_s(\widetilde{\boldsymbol{\xi}} \setminus j_p) - A(k_1, k_2) \right) \\ &\qquad \le C\psi \left| h_s(\widetilde{\boldsymbol{\xi}} \setminus j_p) - A\big(\chi_1^{(b+1)}(\widetilde{\boldsymbol{\xi}} \setminus j_p), \chi_2^{(b+1)}(\widetilde{\boldsymbol{\xi}} \setminus j_p)\big) \right| + O_n(N^{-\mathfrak{d}}). \end{aligned} \tag{3.5.135}$$

Then (3.5.129) follows because $|a_{\widetilde{\boldsymbol{\xi}}}| \le 1$. □

3.5.6 Outline of the proof of Theorem 3.2.8 in the general case

Previously, we assumed when defining the interval I_a in (3.5.44) that $|\operatorname{supp} \boldsymbol{\zeta}| = 2$. Consider now the general case where $|\operatorname{supp} \boldsymbol{\zeta}| = n'$ for $n' \ge 1$. Set $\operatorname{supp} \boldsymbol{\zeta} = \{i_1, i_2, \dots, i_{n'}\}$, with $i_1 < i_2 < \cdots < i_{n'}$, recall $m = \mathcal{N}(\boldsymbol{\zeta})$, and define

$$I_a^{(b)}(\boldsymbol{\zeta}) = \bigcup_{j=1}^{n'} I_{a,j}^{(b)}, \quad \text{where for all } 1 \le j \le n', \quad I_{a,j}^{(b)} = [i_j - 10b\tilde{d} - a, i_j + 10b\tilde{d} + a] \tag{3.5.136}$$

under the assumption that all the intervals $I_{2\tilde{d},j}^{(m)}$ are disjoint; we will describe the appropriate definition when this is not the case below. We further set, for each $j \in [1, n']$ and particle configuration $\boldsymbol{\xi} \in \mathbb{N}^N$,

$$\chi_j^{(b)}(\boldsymbol{\xi}) = \sum_{i \in I^{(b)}_{\widetilde{d}+\psi\ell,j}} \xi_i. \tag{3.5.137}$$

For any integer $n' \geq 1$ and n'-tuple $\mathbf{k} = (k_1, k_2, \ldots, k_{n'})$ of nonnegative integers, we define

$$\Omega^{(b)}(\mathbf{k}) = \Big\{ \boldsymbol{\xi} \in \mathbb{N}^N : \operatorname{supp} \boldsymbol{\xi} \subset I^{(b)}_{\widetilde{d}+\psi\ell}, \chi_j^{(b)}(\boldsymbol{\xi}) = k_j \text{ for } j \in [1, n'] \Big\}; \tag{3.5.138}$$

$$\Omega^{(b)}(n) = \bigcup_{|\mathbf{k}|=n} \Omega(\mathbf{k}); \qquad A(\mathbf{k}) = \prod_{j=1}^{n'} A(\mathbf{q}, i_j)^{k_j}, \tag{3.5.139}$$

where $|\mathbf{k}| = \sum_{j=1}^{n'} k_j$. We also define the restricted intervals

$$\Phi^{(b)}(\mathbf{k}) = \Big\{ \boldsymbol{\xi} \in \mathbb{N}^N : \operatorname{supp} \boldsymbol{\xi} \subset I^{(b)}_{-\psi\ell}, \sum_{i \in I^{(b)}_{-\psi\ell,j}} \xi_i = k_j \text{ for } j \in [1, n'] \Big\}; \tag{3.5.140}$$

$$\Phi^{(b)}(n) = \bigcup_{|\mathbf{k}|=n} \Phi^{(b)}(\mathbf{k}). \tag{3.5.141}$$

The operator $\mathrm{Flat}_{a;\mathbf{k}}$ is then defined as in (3.5.51), except with $A(k_1, k_2)$ there replaced by $A(\mathbf{k})$ here. Given this change, Av is defined as in (3.5.52). Additionally define the flow $g_s(\boldsymbol{\xi}) = g_s^{(b)}(\boldsymbol{\xi}) = g_s^{(b)}(\boldsymbol{\xi}; \mathbf{k})$ as in (3.5.53), and also the maximizer $\widetilde{\boldsymbol{\xi}} = \widetilde{\boldsymbol{\xi}}(s) = \widetilde{\boldsymbol{\xi}}(s, \mathbf{k}) = \widetilde{\boldsymbol{\xi}}^{(b)}(s, \mathbf{k}) \in \mathbb{N}^N$ by

$$g_s(\widetilde{\boldsymbol{\xi}}) = \max_{\boldsymbol{\xi} \in \Omega(\mathbf{k})} g_s^{(b)}(\boldsymbol{\xi}; \mathbf{k}). \tag{3.5.142}$$

Now the argument then proceeds as before. Specifically, the dichotomy in Lemma 3.5.10 becomes that $g_s^{(b)}(\widetilde{\boldsymbol{\xi}})$ satisfies either

$$g_s^{(b)}(\widetilde{\boldsymbol{\xi}}) - A(\mathbf{k}) \leq \frac{1}{N}, \tag{3.5.143}$$

or

$$\partial_s\big(g_s^{(b)}(\widetilde{\boldsymbol{\xi}})-A(\mathbf{k})\big)\leq\frac{C}{\eta}\bigg(\psi\max_{\boldsymbol{\xi}}\Big|h_s(\boldsymbol{\xi})-A\big(\chi_1^{(b+1)}(\boldsymbol{\xi}),\ldots,\chi_{n'}^{(b+1)}(\boldsymbol{\xi})\big)\Big|+N^{-\mathfrak{d}}\bigg)\tag{3.5.144}$$

$$-\frac{c}{\eta}\big(g_s^{(b)}(\widetilde{\boldsymbol{\xi}})-A(\mathbf{k})\big),\tag{3.5.145}$$

where the maximum in (3.5.144) is taken over $\boldsymbol{\xi}\in\bigcup_{1\leq j\leq n'}\Phi^{(b+1)}(k_1,\ldots,k_j-1,\ldots,k_{n'})$. The proof of this claim is the same as the one in Section 3.5.5, and the proof of the main result given this claim is the same as in Section 3.5.4.

When the $I_{2\tilde{d},j}^{(m)}$ are not disjoint, we instead partition $\bigcup_{j=1}^{n'}I_{2\tilde{d},j}^{(m)}$ into a union of disjoint intervals $\widehat{I}_l^{(m)}$ as follows. There exist an integer $v\in[1,n']$ and indices $1\leq j_1<j_2<\ldots<j_v\leq n'$ such that the intervals

$$\widehat{I}_u^{(m)}=\bigcup_{j=j_u}^{j_{u+1}-1}I_{2\tilde{d},j}^{(m)}\tag{3.5.146}$$

are mutually disjoint over all $u\in[1,v]$ (where we set $j_{v+1}=n'+1$), but such that $I_{2\tilde{d},j}^{(m)}\cap I_{2\tilde{d},j+1}^{(m)}$ is nonempty for each $j\in[j_u,j_{u+1}-2]$. We then can make the above definitions using instead the intervals

$$J_{a,l}^{(b)}=[i_{j_l}-10b\tilde{d}-a,i_{j_{l+1}-1}+10b\tilde{d}+a],\tag{3.5.147}$$

which are disjoint for all $a\in[0,2\tilde{d}]$ (since the $\widehat{I}_u^{(m)}$ are). For instance, we set $\chi_j^{(b)}(\boldsymbol{\xi})=\sum_{i\in J_{\tilde{d}+\psi\ell,j}^{(b)}}\xi_i$, and

$$\Omega^{(b)}(\mathbf{k})=\Big\{\boldsymbol{\xi}\in\mathbb{N}^N:\operatorname{supp}\boldsymbol{\xi}\subset J_{\tilde{d}+\psi\ell}^{(b)},\chi_j^{(b)}(\boldsymbol{\xi})=k_j\text{ for }j\in[1,v]\Big\},\tag{3.5.148}$$

and similarly for the other intervals and quantities.

Let us motivate this procedure by very briefly considering the case $|\operatorname{supp}\boldsymbol{\zeta}|=2$, with $\operatorname{supp}\boldsymbol{\zeta}=\{i_1,i_2\}$, as in the material after (3.5.44). However, we now suppose that $|i_2-i_1|\leq 20m\widetilde{d}+2\widetilde{d}$, so that the intervals defined in (3.5.45) are not disjoint. Then, according to the above, we instead work on the single connected interval $J_{a,1}^{(b)}=[i_1-10b\tilde{d}-a,i_2+10b\tilde{d}+a]$.

Then Av is defined as in (3.5.52), and we observe that Lemma 3.5.9 holds with (3.5.56) replaced by the inequality

$$\max_{b\in[0,m+1]} \max_{\boldsymbol{\xi}\in\Omega^{(b)}(k)} \big|A(\mathbf{q},\boldsymbol{\xi}) - A(k)\big| < CN^{-3\mathfrak{d}}, \tag{3.5.149}$$

so that $A(\mathbf{q},\boldsymbol{\xi})$, up to a small error, does not depend on $\boldsymbol{\xi}$ for $\boldsymbol{\xi}\in\Omega^{(m+1)}(k)$. Additionally, it is permissible to apply the finite speed of propagation estimate as in the material immediately following (3.5.79), which would not be the case if we retained two disjoint but nearby or overlapping intervals and attempted the original argument (since then a particle could jump from one interval to the other). The same reasoning underlies the argument in the general case.

3.6 Scaling limit

In Section 3.5 we identified the moments of $\mathbf{X}_t$ through entries of the resolvent $\mathbf{R}(t,z)$. Here, we determine the scaling limit of these entries, as N tends to infinity. In Section 3.6.1, we recall some preliminary material from previous works. In Section 3.6.2 we compute the scaling limits of the moments $\mathbb{E}\big[\operatorname{Im} R_\star(E+\mathrm{i}\eta)^p\big]$ for $p\in\mathbb{N}$, as η tends to 0, and establish Proposition 3.1.4 as a consequence. In Section 3.6.3 we compute the scaling limits of the moments $\mathbb{E}\big[\operatorname{Im} R_{ii}(E+\mathrm{i}\eta)^p\big]$, as N tends to ∞, and prove Theorem 3.2.9. Throughout this section, we recall t from (3.2.10).

3.6.1 Order parameter for $\mathbf{X}_t$

In this section we recall several results on the diagonal resolvent entries of $\mathbf{X}_t$. From Definition 3.1.3 and the content following it, the scaling limits of these entries as N tends to ∞ and $z\in\mathbb{H}$ is fixed are given by the random variable $R_\star(z)$.

A key property of $R_\star(z)$, shown in [32], is that it satisfies a "recursive distributional equation," which may be considered as a limiting analogue of the usual Schur complement formula.

Lemma 3.6.1 ([32, Theorem 4.1])**.** *Denote by $\{\xi_k\}_{k\geq 1}$ a Poisson process on $\mathbb{R}_+$ with intensity measure $\left(\frac{\alpha}{2}\right)x^{-\alpha/2-1}\,dx$. For any $z \in \mathbb{H}$, the random variable $R_\star(z)\colon \mathbb{H} \to \mathbb{H}$ satisfies the equality in law*

$$R_\star(z) \overset{d}{=} -\left(z + \sum_{k=1}^{\infty} \xi_k R_k(z)\right)^{-1}, \tag{3.6.1}$$

where $\big(R_k(z)\big)_{k\geq 1}$ is an i.i.d. sequence with distribution $R_\star(z)$ independent from the process $\{\xi_k\}_{k\geq 1}$.

Next we discuss a certain order parameter, which is essentially given by the $\frac{\alpha}{2}$-th moment of (linear combinations of the imaginary and real parts of) $R_\star$. In what follows, for any $u, h \in \mathbb{C}$, we recall from [33, Section 5.1] the inner product

$$h.u = (\operatorname{Re} u)h + (\operatorname{Im} u)\bar{h}. \tag{3.6.2}$$

Definition 3.6.2. For any $z \in \mathbb{H}$ and $u \in \mathbb{C}$, we define $\gamma_z^\star(u)\colon \mathbb{C} \to \mathbb{C}$ by

$$\gamma_z^\star(u) = \Gamma\left(1 - \frac{\alpha}{2}\right)\mathbb{E}\left[\big(-\mathrm{i}R_\star(z).u\big)^{\alpha/2}\right]. \tag{3.6.3}$$

The following lemma establishes a lower bound on $\operatorname{Re}\gamma_z^\star$ and the existence of a limit for $\gamma_z^\star$ as $\operatorname{Im} z$ tends to 0. It will be proved in Section 3.7 below.

Lemma 3.6.3. *There exists a constant $c > 0$ such that the following two statements hold. First, we have the uniform lower bound*

$$\inf_{\substack{z\in\mathbb{H}\\ |z|\leq c}} \inf_{u\in\mathbb{S}^1_+} \operatorname{Re}\gamma_z^\star(u) > c. \tag{3.6.4}$$

Second, for every real number $E \in [-c, c]$, there exists a function $\gamma_E^\star\colon \mathbb{S}^1_+ \to \mathbb{C}$ such that the following holds. Let $\{E_j\}_{j\geq 1}$ and $\{\eta_j\}_{j\geq 1}$ denote sequences of real numbers such that

$\lim_{N\to\infty} E_N = E$ *and* $\lim_{N\to\infty} \eta_N = 0$. *Then, denoting* $z_N = E_N + \mathrm{i}\eta_N$, *we have*

$$\lim_{N\to\infty} \sup_{u\in\mathbb{S}^1_+} \left| \gamma^\star_{z_N}(u) - \gamma^\star_E(u) \right| = 0. \tag{3.6.5}$$

We next recall from [6, (7.37)] notation for a particular analog of $\gamma^\star_z$ for finite N that will be useful for us. For any real number $s \geq 0$ and index set $\mathcal{I} \subset \mathbb{N} \cap [1, N]$, let $\mathbf{R}^{(\mathcal{I})}(s, z) = \big(\mathbf{X}_s - z\big)^{-1} = \big\{ R_{ij}(s, z) \big\}$ denote the resolvent of $\mathbf{X}^{(\mathcal{I})}_s$, which we define as the matrix $\mathbf{X}_s$ but whose rows and columns with indices in $\mathcal{I}$ set to zero.

Definition 3.6.4. Fix a real number $s \geq 0$. For any index set $\mathcal{I} \subset \mathbb{N} \cap [1, N]$ and complex number $z \in \mathbb{H}$, the function $\gamma^{(\mathcal{I})}_z(u) \colon \mathbb{K}^+ \to \mathbb{C}$ is defined by

$$\gamma^{(\mathcal{I})}_z(u) = \gamma^{(\mathcal{I})}_{z;s}(u) = \Gamma\left(1 - \frac{\alpha}{2}\right) \frac{1}{N - |\mathcal{I}|} \sum_{\substack{1\leq k\leq N \\ k\notin\mathcal{I}}} \big(- \mathrm{i} R^{(\mathcal{I})}_{kk}(s, z).u \big)^{\alpha/2} \frac{|g_k|^\alpha}{\mathbb{E}\big[|g_k|^\alpha\big]}, \tag{3.6.6}$$

where $\mathbf{g} = (g_1, g_2, \ldots, g_N)$ is a vector of i.i.d. standard Gaussian random variables[4] independent from $\mathbf{X}_s$. If $\mathcal{I} = \varnothing$, we abbreviate $\gamma_z = \gamma^{(\mathcal{I})}_z$.

We next have the following local law stating that $\mathbb{E}\big[\gamma_z(u)\big] \approx \gamma^\star_z(u)$. It is a consequence of [6, Theorem 7.6], where the Ω_z there is equal to $\gamma^\star_z$ here by [34, Lemma 4.4].

Lemma 3.6.5 ([6, Theorem 7.6],[34, Lemma 4.4])**.** *There exist constants* $K > 0$ *and* $C = C(\delta) > 0$ *such that the following holds. Fix a real number* $\delta > 0$ *with* $\delta < \max\big\{ \frac{(b-1/\alpha)(2-\alpha)}{20}, \frac{1}{2} \big\}$, *and abbreviate* $\widetilde{\mathcal{D}} = \widetilde{\mathcal{D}}_{K,\delta}$ *(recall* (3.3.8)*). Then, for any* $s \in [0, t]$, *we have*

$$\sup_{z\in\widetilde{\mathcal{D}}} \sup_{u\in\mathbb{S}^1_+} \Big| \mathbb{E}\big[\gamma_z(u)\big] - \gamma^\star_z(u) \Big| \leq C N^{-\alpha\delta/8}, \tag{3.6.7}$$

where expectation is taken with respect to both $\mathbf{X}_s$ *and the Gaussian variables* g_k.

Like Lemma 3.3.2, [6, Theorem 7.6] was only stated in the case $s = 0$ in [6], but it is

[4] These Gaussian variables will be useful in (3.6.41) below, when applying a Hubbard–Stratonovich type transform.

quickly verified that the same proof applies for arbitrary $s \in [0, t]$, especially since $\mathbf{H} + s^{1/2}\mathbf{W}$ satisfies the conditions in Definition 3.1.1 for $s \in [0, t]$ if $\mathbf{H}$ does.

We next have the following lemma, which can be viewed as an analog of Lemma 3.6.1 for finite N. In what follows, we recall from Definition 3.1.1 that there exist random variables $\{Z_{ij}\}_{1 \le i,j \le}$ that are mutually independent (up to the symmetry condition $Z_{ij} = Z_{ji}$) and have the following properties. First, each Z_{ij} has law $N^{-1/\alpha} Z$, where Z is α-stable; second, each $N^{1/\alpha}(H_{ij} - Z_{ij})$ is symmetric and has finite variance.

The first and second bounds in (3.6.9) below are consequences of [6, Proposition 7.11] and [6, Proposition 7.9], respectively.

Lemma 3.6.6 ([6]). *Define the $\{Z_{ij}\}$ and, for each integer $j \in [1, N]$, set*

$$S_{jj} = -\left(z - \sum_{k \ne j} Z_{jk}^2 R_{kk}^{(j)} \right)^{-1}. \tag{3.6.8}$$

Then, with overwhelming probability we have the bounds

$$\max_{1 \le j \le N} \mathbb{E}\big[|S_{jj} - R_{jj}| \big] \le \frac{C(\log N)^C}{(N\eta^2)^{\alpha/8}}, \qquad \max_{1 \le j \le N} \big\{ |R_{jj}|, |S_{jj}| \big\} \le C(\log N)^C. \tag{3.6.9}$$

We conclude this section with the following concentration estimate, which is essentially [6, Proposition 7.17]. Although it was only stated in [6] for the case when $\mathcal{I}$ is a single index, the fact that it can be extended to all $\mathcal{I}$ of uniformly bounded size is a quick consequence [6, Lemma 5.6].

Lemma 3.6.7 ([6, Proposition 7.17]). *There exists a constant $K > 0$ such that the following holds. Fix a real number $\delta > 0$, and abbreviate $\widetilde{\mathcal{D}} = \widetilde{\mathcal{D}}_{K,\delta}$ from (3.3.8). For every index set $\mathcal{I} \subset [1, N]$, there exists a constant $C = C\big(s, |\mathcal{I}|\big) > 0$ such that, with overwhelming probability, we have*

$$\sup_{z \in \widetilde{\mathcal{D}}} \sup_{u \in \mathbb{S}_+^1} \Big| \gamma_z^{(\mathcal{I})}(u) - \mathbb{E}\big[\gamma_z(u)\big] \Big| \le \frac{C(\log N)^C}{N^{s/2}\eta^{\alpha/2}}. \tag{3.6.10}$$

Here, the expectation is taken with respect to both $\mathbf{X}_s$ *and the Gaussian variables* g_k*.*

3.6.2 Tightness

The following lemma computes the scaling limits of moments of $\operatorname{Im} R_\star(E+\mathrm{i}\eta)$, as η tends to 0. We recall $\gamma_E^\star$ from Lemma 3.6.3.

Lemma 3.6.8. *There exists a constant* $c > 0$ *such that the following holds. Fix a real number* $E \in [-c, c]$*, and let* $\{E_N\}_{N \geq 1}$ *and* $\{\eta_N\}_{N \geq 1}$ *be sequences of real numbers such that* $\lim_{N\to\infty} E_N = E$ *and* $\lim_{N\to\infty} \eta_N = 0$*. Then, for each* $p \in \mathbb{N}$*, we have that*

$$\lim_{N\to\infty} \mathbb{E}\Big[\big(\operatorname{Im} R_\star(E_N + \mathrm{i}\eta_N) \big)^p \Big] = 2^{-p} \left(\mathfrak{X} + \overline{\mathfrak{X}} + \sum_{a=1}^{p-1} \binom{p}{a} \mathfrak{Y}(a) \right), \tag{3.6.11}$$

where $\mathfrak{X} = \mathfrak{X}_p$ *and* $\mathfrak{Y}(a) = \mathfrak{Y}_p(a)$ *are defined by*

$$\mathfrak{X} = \frac{1}{\Gamma(p)} \int_{\mathbb{R}_+} t^{p-1} \exp\big(\mathrm{i}Et - t^{\alpha/2} \gamma_E^\star(1) \big)\, dt, \tag{3.6.12}$$

and

$$\begin{aligned} \mathfrak{Y}(a) = \frac{1}{\Gamma(a)\Gamma(p-a)} \int_{\mathbb{R}_+^2} & t^{a-1} s^{p-a-1} \exp\big(\mathrm{i}E(t-s) \big) \\ & \times \exp\left(- (t^2+s^2)^{\alpha/4} \gamma_E^\star \left(\frac{t+\mathrm{i}s}{\sqrt{t^2+s^2}} \right) \right) dt\, ds. \end{aligned} \tag{3.6.13}$$

Thus, the left side of (3.6.11) *exists, depends only on* E *and* p*, and is uniformly continuous in* E*.*

Proof. For brevity, we set $R_\star = R_\star(E_N + \mathrm{i}\eta_N)$. We first express moments of $\operatorname{Im} R_\star$ in terms of $\gamma_z^\star$ (recall Definition 3.6.2). To that end, we fix $p \in \mathbb{N}$ and use the identity $2\mathrm{i} \operatorname{Im} R_\star = R_\star - \overline{R_\star}$ to write

$$(\operatorname{Im} R_\star)^p = (2\mathrm{i})^{-p} (R_\star - \overline{R_\star})^p = (2\mathrm{i})^{-p} \sum_{a=0}^{p} \binom{p}{a} R_\star^a (-\overline{R_\star})^{p-a}. \tag{3.6.14}$$

So, to establish (3.6.11), it suffices to show for each integer $a \in [1, p]$ that

$$\lim_{N \to \infty} \mathbb{E}\big[(-\mathrm{i}R_\star)^p\big] = \mathfrak{X}; \qquad \lim_{N \to \infty} \mathbb{E}\big[(-\mathrm{i}R_\star)^a (\mathrm{i}\overline{R_\star})^{p-a}\big] = \mathfrak{Y}(a). \tag{3.6.15}$$

We only establish the second equality in (3.6.15), as the proof of the former is entirely analogous.

To that end, let $R_1, R_2, \ldots$ denote i.i.d. complex random variables whose laws are given by $R_\star$, and let $\{\xi_k\}_{k \geq 1}$ denote a Poisson point process with intensity measure $\big(\frac{\alpha}{2}\big) x^{-\alpha/2-1}\, dx$ (independent from the $\{R_k\}$), as in Lemma 3.6.1. Then, Lemma 3.6.1 implies

$$\mathbb{E}\big[(-\mathrm{i}R_\star)^a (\mathrm{i}\overline{R_\star})^b\big] = \mathbb{E}\Bigg[\bigg(-\mathrm{i}\sum_{k=1}^{\infty} \xi_k R_k - \mathrm{i}z\bigg)^{-a} \bigg(\mathrm{i}\sum_{k=1}^{\infty} \xi_k \overline{R_k} + \mathrm{i}\overline{z}\bigg)^{-b}\Bigg], \tag{3.6.16}$$

for any $a, b \geq 0$. Next, recall the integral formula

$$w^{-\beta} = \frac{1}{\Gamma(\beta)} \int_{\mathbb{R}_+} t^{\beta-1} \exp(-wt)\, dt, \qquad \text{for } \operatorname{Re} w > 0 \text{ and } \beta > 0. \tag{3.6.17}$$

For brevity, set $A = \sum_{k=1}^{\infty} \xi_k R_k(z)$. Abbreviating $z = z_N = E_N + \mathrm{i}\eta_N$, (3.6.17) implies for $a, b > 0$ that

$$(-\mathrm{i}A - \mathrm{i}z)^{-a} (\mathrm{i}\overline{A} + \mathrm{i}\overline{z})^{-b} \stackrel{d}{=} \frac{1}{\Gamma(a)\Gamma(b)} \int_{\mathbb{R}_+^2} t^{a-1} s^{b-1} \exp\big(\mathrm{i}t(z + A) - \mathrm{i}s(\bar{z} + \overline{A})\big)\, dt\, ds \tag{3.6.18}$$

$$= \frac{1}{\Gamma(a)\Gamma(b)} \int_{\mathbb{R}_+^2} t^{a-1} s^{b-1} \exp(\mathrm{i}tA - \mathrm{i}s\overline{A}) \exp\left(\mathrm{i}tz - \mathrm{i}s\bar{z}\right)\, dt\, ds. \tag{3.6.19}$$

We recall the Lévy–Khintchine formula (see [32, (4.5)]): for any i.i.d. complex random variables $\{w_k\}_{k \geq 1}$ such that $\operatorname{Re} w_k \geq 0$ holds almost surely, we have

$$\mathbb{E}\Bigg[\exp\bigg(-\sum_{k=1}^{\infty} \xi_k w_k\bigg)\Bigg] = \exp\bigg(-\Gamma\Big(1 - \frac{\alpha}{2}\Big)\mathbb{E}\big[w_1^{\alpha/2}\big]\bigg). \tag{3.6.20}$$

Since $\operatorname{Im} R_\star \geq 0$, (3.6.16), (3.6.19), and (3.6.20) together imply

$$\mathbb{E}\big[(-\mathrm{i}R_\star)^a (\mathrm{i}\overline{R_\star})^b\big] = \frac{1}{\Gamma(a)\Gamma(b)} \int_{\mathbb{R}_+^2} t^{a-1} s^{b-1} \exp\bigg(-\Gamma\Big(1 - \frac{\alpha}{2}\Big) \mathbb{E}\big[(\mathrm{i}s\overline{R_\star} - \mathrm{i}tR_\star)^{\alpha/2}\big]\bigg) \tag{3.6.21}$$

$$\times \exp(\mathrm{i}tz - \mathrm{i}s\bar{z})\, dt\, ds. \tag{3.6.22}$$

Recalling $\gamma_z^\star$ from (3.6.2), it follows that

$$\begin{aligned}\mathbb{E}\big[(-\mathrm{i}R_\star)^a (\mathrm{i}\overline{R_\star})^b\big] = \frac{1}{\Gamma(a)\Gamma(b)} \int_{\mathbb{R}_+^2} t^{a-1} s^{b-1} \exp\bigg(-(t^2+s^2)^{\alpha/4} \gamma_z^\star\Big(\frac{t+\mathrm{i}s}{\sqrt{t^2+s^2}}\Big)\bigg) \\ \times \exp(\mathrm{i}tz - \mathrm{i}s\bar{z})\, dt\, ds.\end{aligned} \tag{3.6.23}$$

Next, observe by (3.6.4)and (3.6.5), there exists a constant $c > 0$ such that

$$\sup_{\substack{z \in \mathbb{H} \\ |z| < c}} \inf_{u \in \mathbb{S}_+^1} \operatorname{Re} \gamma_z^\star(u) > c; \qquad \lim_{N \to \infty} \sup_{u \in \mathbb{S}_+^1} \big| \gamma_{z_N}^\star(u) - \gamma_E^\star(u) \big| = 0. \tag{3.6.24}$$

Therefore, (3.6.23); the dominated convergence theorem; and the fact that

$$\int_{\mathbb{R}_+^2} s^{a-1} t^{b-1} \exp\big(-c(s^2+t^2)^{\alpha/4}\big)\, ds\, dt < \infty, \tag{3.6.25}$$

together imply for $a \in [1, p-1]$ that $\lim_{N \to \infty} \mathbb{E}\big[(-\mathrm{i}R_\star)^a (\mathrm{i}\overline{R_\star})^{p-a}\big] = \mathfrak{Y}(a)$; this establishes the second statement in (3.6.15). The proof of the first is entirely analogous and is therefore omitted. Now (3.6.11) follows from (3.6.14) and (3.6.15).

That the left side of (3.6.11) depends only on E and p holds since the same is true for $\mathfrak{X}$ and $\mathfrak{Y}$. Similarly, to verify the uniform continuity of the left side of (3.6.11) in E, it suffices to do the same for $\mathfrak{X}$ and $\mathfrak{Y}$. The latter follows from the continuity in E for the integrands on the right sides of (3.6.12) and (3.6.13), the first bound in (3.6.24), (3.6.25), and the dominated convergence theorem. $\square$

Remark 3.6.9. The proof of Lemma 3.6.8 implies that $\lim_{N\to\infty} \mathbb{E}\big[(-\mathrm{i}R_\star)^a(\mathrm{i}\overline{R_\star})^{p-a}\big]$ is equal to $\overline{\mathfrak{X}}$ if $a = 0$, to $\mathfrak{Y}(a)$ if $a \in [1, p-1]$, and to $\mathfrak{X}$ if $a = p$.

Now we can quickly establish Proposition 3.1.4.

Proof of Proposition 3.1.4. Since Lemma 3.6.8 implies that $\mathbb{E}\Big[\big(\operatorname{Im} R_\star(E + \mathrm{i}\eta)\big)^2\Big]$ is uniformly bounded in $\eta > 0$, the sequence $\big\{\operatorname{Im} R_\star(E+\mathrm{i}\eta)\big\}_{\eta>0}$ of random variables is tight. This establishes the first claim of the proposition. The second is a direct consequence of Lemma 3.6.8. $\square$

3.6.3 Scaling limit of $A(\mathbf{q}, \boldsymbol{\xi})$

We begin with the limit of the numerator of $A(\mathbf{q}, \boldsymbol{\xi})$. To compute the scaling limits of the moments of $A(\mathbf{q}, \boldsymbol{\xi})$, we first show that the off-diagonal resolvent entries in the numerator of $A(\mathbf{q}, \boldsymbol{\xi})$ are negligible. Here, we recall the $\widehat{\gamma}_i = \widehat{\gamma}_i(t)$ from (3.2.18).

Lemma 3.6.10. *For all real numbers $\delta > 0$; integers $m, n > 0$; and unit vectors $\mathbf{q} = (q_1, q_2, \ldots, q_N) \in \mathbb{R}^N$ with $|\operatorname{supp} \mathbf{q}| = m$, there exist constants $c > 0$ (independent of δ, m, and n) and $C = C(\delta, m, n) > 0$ such that the following holds. Let $\{k_1, k_2, \ldots, k_n\} \subset [1, N]$ denote an index sequence such that $\max_{1\le j\le n} |k_j - N/2| < cN$; let $\operatorname{supp} \mathbf{q} = \{j_1, j_2, \ldots, j_m\}$; and let t be as in* (3.2.10)*. Then, for $\eta \geq N^{\delta-1/2}$,*

$$\left| \mathbb{E}\bigg[\prod_{i=1}^{n} \operatorname{Im}\Big\langle \mathbf{q}, \mathbf{R}\big(t, \widehat{\gamma}_{k_i} + \mathrm{i}\eta\big)\mathbf{q}\Big\rangle\bigg] - \mathbb{E}\bigg[\prod_{i=1}^{n} \operatorname{Im} \sum_{h=1}^{m} q_{j_h}^2 R_{j_h j_h}\big(t, \widehat{\gamma}_{k_i} + \mathrm{i}\eta\big)\bigg] \right| \leq \frac{CN^\delta}{\sqrt{N\eta}}. \quad (3.6.26)$$

Proof. First observe that

$$\mathbb{E}\bigg[\prod_{i=1}^{n} \operatorname{Im}\Big\langle \mathbf{q}, \mathbf{R}\big(t, \widehat{\gamma}_{k_i} + \mathrm{i}\eta\big)\mathbf{q}\Big\rangle\bigg] = \mathbb{E}\bigg[\prod_{i=1}^{n}\sum_{a=1}^{m}\sum_{b=1}^{m} q_{j_a} q_{j_b} \operatorname{Im} R_{j_a j_b}\big(t, \widehat{\gamma}_{k_i} + \mathrm{i}\eta\big)\bigg] \quad (3.6.27)$$

$$= \sum_{\mathbf{a},\mathbf{b}} \mathbb{E}\bigg[\prod_{i=1}^{n} q_{j_{a(i)}} q_{j_{b(i)}} \operatorname{Im} R_{j_{a(i)} j_{b(i)}}\big(t, \widehat{\gamma}_{k_i} + \mathrm{i}\eta\big)\bigg], \quad (3.6.28)$$

where in the right side of (3.6.28), $\mathbf{a} = \big(a(1), a(2), \ldots, a(n)\big)$ and $\mathbf{b} = \big(b(1), b(2), \ldots, b(n)\big)$ are summed over all sequences of $\{1, 2, \ldots, m\}^n$.

It suffices to bound by $CN^{\delta}(N\eta)^{-1/2}$ any summand on the right side of (3.6.28) for which there exists some $i' \in [1, n]$ such that $a(i') \neq b(i')$. To that end, observe that the second bound in (3.3.11) (to bound $\big|R_{j_{a(i)} j_{b(i)}}\big| \leq N^{\delta/2n}$ with overwhelming probability for $i \neq i'$); (3.3.2); the fact that $q_j \leq 1$ for each $j \in [1, N]$; and the exchangeability of the matrix entries of $\mathbf{X}_t$ together imply that any such term is bounded by

$$N^{\delta/2} \mathbb{E}\Big[\Big|R_{12}\big(t, \widehat{\gamma}_{k_{i'}} + \mathrm{i}\eta\big)\Big|\Big]. \tag{3.6.29}$$

To estimate this quantity, abbreviate $R_{ij} = R_{ij}(t, \widehat{\gamma}_{k_{i'}} + \mathrm{i}\eta)$, and observe that the Ward identity (3.3.3) and the exchangeability of $\mathbf{X}_t$ together imply that

$$\mathbb{E}\big[|R_{12}|\big] \leq \Big(\mathbb{E}\big[|R_{12}|^2\big]\Big)^{1/2} \leq \left(\mathbb{E}\left[\frac{1}{N-1}\sum_{j=1}^{N} |R_{1j}|^2\right]\right)^{1/2} \leq \frac{2\mathbb{E}[\operatorname{Im} R_{11}]^{1/2}}{\sqrt{N\eta}}. \tag{3.6.30}$$

Using the second bound in (3.3.11), and the deterministic bound (3.3.2) on the exceptional set where the former estimate does not apply, yields $\mathbb{E}[\operatorname{Im} R_{11}] \leq C(\log N)^C$, and so

$$\mathbb{E}\big[|R_{12}|\big] \leq \frac{CN^{\delta/2}}{\sqrt{N\eta}}. \tag{3.6.31}$$

Together with (3.6.28) and (3.6.29), this implies (3.6.26). □

In [32, Theorem 2.8] it was shown for fixed $z \in \mathbb{H}$ that the diagonal resolvent elements $G_{ii}(z)$ of the matrix $\mathbf{H}$ are asymptotically independent. The next lemma is a version of this result (for the perturbed model $\mathbf{X}_t$) when $\eta = \operatorname{Im} z$ is simultaneously tending to 0. Theorem 3.2.9 is then deduced quickly as a consequence. Below, we recall $\mathcal{R}_\star(E)$ from Definition 3.1.5.

Proposition 3.6.11. *There exists a constant $c > 0$ such that the following holds. Fix* *integers $m, n > 0$ and a unit vector $\mathbf{q} = (q_1, q_2, \ldots, q_N) \in \mathbb{R}^N$ with $|\operatorname{supp} \mathbf{q}| = m$. Let $\operatorname{supp} \mathbf{q} = \{j_1, j_2, \ldots, j_m\}$, and let $t \in \mathbb{R}_{>0}$ be as in (3.2.10). Fix a real number $E \in [-c, c]$, and let $\{\eta_N\}_{N \geq 1}$ and $\big\{E_N^{(i)}\big\}_{N \geq 1}$ for each integer $i \in [1, n]$ be sequences of real numbers such that*

$$\lim_{N \to \infty} \eta_N = 0; \qquad \eta_N \gg N^{-1/2}; \qquad \lim_{N \to \infty} E_N^{(1)} = E; \qquad \max_{1 \leq i \leq n} \big| E_N^{(i)} - E_N^{(1)} \big| \ll \eta_N. \quad (3.6.32)$$

Then, letting $\big\{\mathcal{R}_{j_i}(E)\big\}_{i \in [1,m]}$ be i.i.d. random variables each with law $\mathcal{R}_\star(E)$, we have that

$$\lim_{N \to \infty} \mathbb{E}\left[\prod_{i=1}^n \operatorname{Im} \sum_{k=1}^m q_{j_k}^2 R_{j_k j_k}\big(t, E_N^{(i)} + \mathrm{i}\eta_N\big)\right] = \mathbb{E}\left[\bigg(\sum_{k=1}^m q_{j_k}^2 \mathcal{R}_{j_k}(E)\bigg)^n\right]. \qquad (3.6.33)$$

Proof. It suffices to show that, for any sequences of nonnegative integers $\mathbf{n}^{(i)} = \big(n_1^{(i)}, n_2^{(i)}, \ldots, n_m^{(i)}\big)$ for $1 \leq i \leq n$ with $N_k = \sum_{i=1}^n n_k^{(i)}$, we have

$$\lim_{N \to \infty} \mathbb{E}\left[\prod_{i=1}^n \prod_{k=1}^m \Big(\operatorname{Im} R_{j_k j_k}\big(t, E_N^{(i)} + \mathrm{i}\eta_N\big)\Big)^{n_k^{(i)}}\right] = \prod_{k=1}^m \mathbb{E}\big[\mathcal{R}_\star(E)^{N_k}\big]. \qquad (3.6.34)$$

To ease notation, we detail the proof of (3.6.34) when $(m, n) = (1, 1)$ and outline it when $(m, n) = (1, 2)$ and $(m, n) = (2, 2)$, which are largely analogous. We omit the proofs in the remaining cases, since they are very similar to those of the $(m, n) \in \big\{(1, 2), (2, 2)\big\}$ cases.

To that end, first assume $(m, n) = (1, 1)$; abbreviate $j_1 = j$, $z = E_N^{(1)} + \mathrm{i}\eta_N$, and $R_{ik} = R_{ik}(t, z)$; and set $p = n_1^{(1)}$. We compute $\lim_{N \to \infty} \mathbb{E}\big[(\operatorname{Im} R_{jj})^p\big]$. As in the proof of Lemma 3.6.8, we use the identity $2\mathrm{i} \operatorname{Im} R_{jj} = R_{jj} - \overline{R}_{jj}$ to write

$$\mathbb{E}\big[(\operatorname{Im} R_{jj})^p\big] = (2\mathrm{i})^{-p} \mathbb{E}\big[(R_{jj} - \overline{R}_{jj})^p\big] = (2\mathrm{i})^{-p} \sum_{a=0}^p (-1)^{p-a} \binom{p}{a} \mathbb{E}\big[R_{jj}^a \overline{R}_{jj}^{p-a}\big]. \qquad (3.6.35)$$

Now, recall from Definition 3.1.1 that each entry of $\mathbf{H}$ has law $N^{-1/\alpha}(Z + J)$, where Z is α-stable and J has finite variance. For each $1 \leq i \leq j \leq N$, let $\{Z_{ij}\}$ denote mutually independent random variables with law $N^{-1/\alpha} Z$ such that $N^{1/\alpha} |H_{ij} - Z_{ij}|$ has uniformly bounded variance. Following (3.6.8), for any subset $\mathcal{I} \subset [1, N]$ and index $j \in [1, N] \setminus \mathcal{I}$, set $\mathcal{J} = \mathcal{I} \cup \{j\}$ and define

$$S_{jj}^{(\mathcal{I})} = S_{jj}^{(\mathcal{I})}(z) = -\left(z + \sum_{k \notin \mathcal{J}} Z_{kj}^2 R_{kk}^{(\mathcal{J})} \right)^{-1}. \tag{3.6.36}$$

If $\mathcal{I}$ is empty, then we abbreviate $S_{jj} = S_{jj}^{(\mathcal{I})}$.

Then (3.6.9) and the deterministic bounds given by (3.3.2) and $|S_{jj}| \le \eta^{-1}$ together imply that there exists a constant $C = C(p) > 0$ such that

$$\mathbb{E}\big[|R_{jj}^a - S_{jj}^a| \big] + \mathbb{E}\big[|\overline{R}_{jj}^a - \overline{S}_{jj}^a| \big] \le \frac{C (\log N)^C}{(N\eta^2)^{\alpha/8}}, \tag{3.6.37}$$

for any $a \in [0, p]$. Then, (3.6.37) and (3.6.9) together imply for any $a, b \in [0, p]$ that

$$\mathbb{E}\big[|R_{jj}^a \overline{R}_{jj}^b - S_{jj}^a \overline{S}_{jj}^b| \big] \le \mathbb{E}\big[|R_{jj}|^a |\overline{R}_{jj}^b - \overline{S}_{jj}^b| \big] + \mathbb{E}\big[|S_{jj}|^b |R_{jj}^a - S_{jj}^a| \big] \le \frac{C (\log N)^C}{(N\eta^2)^{\alpha/8}}. \tag{3.6.38}$$

So, by (3.6.35), the definition (3.1.3) of $R_\star$, and Proposition 3.1.4, it suffices to show for each integer $a \in [0, p]$ that

$$\lim_{N \to \infty} \mathbb{E}\big[S_{jj}^a \overline{S}_{jj}^{p-a} \big] = \lim_{\eta \to 0} \mathbb{E}\big[R_\star(E + \mathrm{i}\eta)^a \overline{R_\star}(E + \mathrm{i}\eta)^{p-a} \big]. \tag{3.6.39}$$

Recalling (3.6.12), (3.6.13), and Remark 3.6.9, the right side of is equal to $\mathrm{i}^{-p}\overline{\mathfrak{X}}$ if $a = 0$, to $\mathrm{i}^{-p}(-1)^a \mathfrak{Y}(a)$ if $p \in [1, a-1]$, and to $\mathrm{i}^p \mathfrak{X}$ if $a = p$. Let us only show (3.6.39) in the case $a \in [1, p-1]$, as the cases $a \in \{0, p\}$ are entirely analogous.

To that end, we proceed similarly to as in the proof of Lemma 3.6.8. More specifically, by (3.6.36) and (3.6.17), we deduce that

$$(-\mathrm{i}S_{jj})^a (\mathrm{i}\overline{S}_{jj})^b = \frac{1}{\Gamma(a)\Gamma(b)} \int_{\mathbb{R}_+^2} t^{a-1} s^{b-1} \exp\left(\mathrm{i}tz - \mathrm{i}s\bar{z} + \mathrm{i} \sum_{k \ne i} Z_{ik}^2 \big(tR_{kk}^{(j)} - s\overline{R}_{kk}^{(j)} \big) \right) dt\, ds. \tag{3.6.40}$$

To analyze the right side of (3.6.40), observe by [33, Corollary B.2] (whose proof proceeds by first applying a type of Hubbard–Stratonovich transform to linearize the exponential in the $\{Z_{jk}\}$, and then using (3.6.20) to evaluate the expectation)

$$\mathbb{E}\Bigg[\exp\bigg(\mathrm{i}\sum_{k\neq j} Z_{jk}^2\big(tR_{kk}^{(j)} - s\overline{R}_{kk}^{(j)}\big)\bigg)\Bigg] = \mathbb{E}\Bigg[\exp\bigg(-\frac{(-2\mathrm{i})^{\alpha/2}\sigma^\alpha}{N}\sum_{k\neq j}\big(tR_{kk}^{(j)} - s\overline{R}_{kk}^{(j)}\big)^{\alpha/2}|g_k|^\alpha\bigg)\Bigg], \tag{3.6.41}$$

where $\sigma > 0$ is as in (3.1.2), the g_k are i.i.d. standard Gaussian random variables, and the expectation is taken with respect to the Z_{jk} on the left side and the g_k on the right.

Therefore, by the definition (3.6.6) of $\gamma_z^{(j)}$, we find

$$\mathbb{E}\Bigg[\exp\bigg(-\frac{(-2\mathrm{i})^{\alpha/2}\sigma^\alpha}{N}\sum_{k\neq j}\big(tR_{kk}^{(j)} - s\overline{R}_{kk}^{(j)}\big)^{\alpha/2}|g_k|^\alpha\bigg)\Bigg] = \mathbb{E}\left[\exp\left(-\frac{N-1}{N}\gamma_z^{(j)}(t+\mathrm{i}s)\right)\right], \tag{3.6.42}$$

where we have used the fact (see [6, (7.39)]) that $\mathbb{E}\big[|g_k|^\alpha\big] = 2^{-\alpha/2}\sigma^{-\alpha}\Gamma\big(1-\frac{\alpha}{2}\big)$. Thus,

$$(3.6.42) = \mathbb{E}\Bigg[\exp\bigg(-\frac{N-1}{N}(t^2+s^2)^{\alpha/4}\gamma_z^{(j)}\Big(\frac{t+\mathrm{i}s}{\sqrt{t^2+s^2}}\Big)\bigg)\Bigg]. \tag{3.6.43}$$

Using (3.6.10) and (3.6.7), the fact that $\operatorname{Im} z \gg N^{-1/2}$, and the deterministic estimate $\operatorname{Re}\gamma_z^{(j)}(u) \geq 0$ on the exceptional event where (3.6.10) and (3.6.7) do not hold, we obtain

$$(3.6.43) = \exp\bigg(-\frac{N-1}{N}(t^2+s^2)^{\alpha/4}\gamma_z^{\star}\Big(\frac{t+\mathrm{i}s}{\sqrt{t^2+s^2}}\Big) + O(N^{-c})\bigg) + O(N^{-10}). \tag{3.6.44}$$

Combining (3.6.40), (3.6.41), (3.6.42), (3.6.43), and (3.6.44) yields

$$\mathbb{E}\left[(-\mathrm{i}S_{jj})^a(\mathrm{i}\overline{S}_{jj})^b\right] \tag{3.6.45}$$

$$= \frac{1}{\Gamma(a)\Gamma(b)} \int_{\mathbb{R}_+^2} t^{a-1}s^{b-1}\exp(\mathrm{i}tz - \mathrm{i}s\bar{z}) \tag{3.6.46}$$

$$\times \Bigg(\exp\bigg(- \frac{N-1}{N}(t^2+s^2)^{\alpha/4}\gamma_z^\star\Big(\frac{t+\mathrm{i}s}{\sqrt{t^2+s^2}}\Big) + O(N^{-c})\bigg) + O(N^{-10})\Bigg)\, dt\, ds. \tag{3.6.47}$$

By (3.6.24) (with the z_N there equal to z here), (3.6.25), and the fact that $\operatorname{Im} z \gg N^{-1/2}$, we deduce from (3.6.47) and the dominated convergence theorem that

$$\lim_{N\to\infty} \mathbb{E}\left[(-\mathrm{i}S_{jj})^a(\mathrm{i}\overline{S}_{jj})^b\right] \tag{3.6.48}$$

$$= \frac{1}{\Gamma(a)\Gamma(b)} \int_{\mathbb{R}_+^2} t^{a-1}s^{b-1}\exp\left(\mathrm{i}E(t-s) - (t^2+s^2)^{\alpha/4}\gamma_E^\star\Big(\frac{t+\mathrm{i}s}{\sqrt{t^2+s^2}}\Big)\right) dt\, ds. \tag{3.6.49}$$

By Remark 3.6.9 and (3.6.13), this yields (3.6.39) when $a \in [1, p-1]$. The cases when $a \in \{0, p\}$ are handled analogously and therefore omitted. This therefore establishes (3.6.34) in the case $m = 1 = n$.

Next let us outline how to establish (3.6.34) in the case $(m,n) = (1,2)$. We abbreviate $z_1 = E_N^{(1)} + \mathrm{i}\eta_N$, $z_2 = E_N^{(2)} + \mathrm{i}\eta_N$, and $j = j_1$. Then following (3.6.35), it suffices to show for any integers $a, b, c, d \geq 0$ that

$$\lim_{N\to\infty} \mathbb{E}\big[R_{jj}(z_1)^a\overline{R}_{jj}(z_1)^bR_{jj}(z_2)^c\overline{R}_{jj}(z_2)^d\big] = \lim_{\eta\to 0}\mathbb{E}\big[R_\star(E+\mathrm{i}\eta)^{a+c}\overline{R_\star}(E+\mathrm{i}\eta)^{b+d}\big]. \tag{3.6.50}$$

We write

$$\mathbb{E}\big[R_{jj}(z_1)^a\overline{R}_{jj}(z_1)^b R_{jj}(z_2)^c\overline{R}_{jj}(z_2)^d\big] = \big[R_{jj}(z_1)^{a+c}\overline{R}_{jj}(z_1)^{b+d}\big] \tag{3.6.51}$$

$$+\,\mathbb{E}\Big[R_{jj}(z_1)^a\overline{R}_{jj}(z_1)^{b+d}\big(R_{jj}(z_2)^c - R_{jj}(z_1)^c\big)\Big] \tag{3.6.52}$$

$$+\,\mathbb{E}\Big[R_{jj}(z_1)^a\overline{R}_{jj}(z_1)^b R_{jj}(z_2)^c\big(\overline{R}_{jj}(z_2)^d - \overline{R}_{jj}(z_1)^d\big)\Big]. \tag{3.6.53}$$

The first term is the main one, and it was shown in the preceding case, as (3.6.38) and (3.6.39), that

$$\lim_{N\to\infty}\mathbb{E}\big[R_{jj}(z_1)^{a+c}\overline{R}_{jj}(z_1)^{b+d}\big] = \lim_{\eta\to 0}\mathbb{E}\big[R_\star(E+\mathrm{i}\eta)^{a+c}\overline{R_\star}(E+\mathrm{i}\eta)^{b+d}\big]. \tag{3.6.54}$$

The latter two terms are error terms, and they tend to zero asymptotically. Let us show this for (3.6.53), as the other term is similar.

Since $R_{jj}(z) - R_{jj}(w) = (w-z)\sum_{a=1}^N R_{ja}(z)R_{aj}(w)$ by (3.3.1), we have that

$$\big|\partial_z R_{jj}(z)\big| \le \left|\sum_{a=1}^N R_{ja}(z)R_{aj}(z)\right| \le \sum_{a=1}^N \big|R_{ja}(z)\big|^2 = \frac{\operatorname{Im} R_{jj}(z)}{\eta} \le (\log N)^C\eta^{-1} \tag{3.6.55}$$

with overwhelming probability, where in the equality we used (3.3.3), and in the last bound we used (3.3.11). Integrating $\partial_z R_{jj}(z)^d = dR_{jj}^{d-1}\partial_z R_{jj}$ from z_1 to z_2 yields

$$\big|R_{jj}(z_1)^d - R_{jj}(z_2)^d\big| \le d|z_1 - z_2|(\log N)^{dC}\eta^{-1} \tag{3.6.56}$$

with overwhelming probability. Therefore, with overwhelming probability, we have

$$\left|R_{jj}(z_1)^a\overline{R}_{jj}(z_1)^b R_{jj}(z_2)^c\big(\overline{R}_{jj}(z_2)^d-\overline{R}_{jj}(z_1)^d\big)\right| \tag{3.6.57}$$

$$\ll C|z_1-z_2|(\log N)^{dC}\eta^{-1}\left|R_{jj}(z_1)^a\overline{R}_{jj}(z_1)^b R_{jj}(z_2)^c\right| \tag{3.6.58}$$

$$\ll |z_1-z_2|\eta^{-1}(\log N)^{(a+b+c+d)C}\ll 1, \tag{3.6.59}$$

where we used (3.3.11) in the last estimate, as well as the hypothesis (3.6.32) that $|E_N^{(1)}-E_N^{(2)}|\ll\eta$ to bound $|z_1-z_2|$. On the complementary event, we use the trivial bound (3.3.2). Together, these show that

$$\lim_{N\to\infty}\mathbb{E}\Big[R_{jj}(z_1)^a\overline{R}_{jj}(z_1)^b R_{jj}(z_2)^c\big(\overline{R}_{jj}(z_2)^d-\overline{R}_{jj}(z_1)^d\big)\Big]=0, \tag{3.6.60}$$

as desired. We have therefore established that (3.6.53) is small; since the same holds for (3.6.52), (3.6.54) implies (3.6.50).

Now let us outline how to establish (3.6.34) in the case $(m,n)=(2,2)$. We abbreviate $z_1=E_N^{(1)}+\mathrm{i}\eta_N$ and $z_2=E_N^{(2)}+\mathrm{i}\eta_N$, and we assume for notational convenience that $j_1=1$ and $j_2=2$. As in (3.6.35), it suffices to show for any integers $a,b,c,d\geq 0$ that

$$\lim_{N\to\infty}\mathbb{E}\left[R_{11}(z_1)^a\overline{R}_{11}(z_1)^b R_{22}(z_2)^c\overline{R}_{22}(z_2)^d\right] \tag{3.6.61}$$

$$=\lim_{\eta\to 0}\mathbb{E}\big[R_\star(E+\mathrm{i}\eta)^a\overline{R_\star}(E+\mathrm{i}\eta)^b\big]\mathbb{E}\big[R_\star(E+\mathrm{i}\eta)^c\overline{R_\star}(E+\mathrm{i}\eta)^d\big]. \tag{3.6.62}$$

In what follows, we assume that $a,b,c,d>0$ for notational simplicity. Note that [34, Lemma 5.5] implies for each $i\neq j$ that

$$\mathbb{E}\Big[\big|R_{jj}-R_{jj}^{(i)}\big|\Big]\leq\frac{C}{N\eta}. \tag{3.6.63}$$

It quickly follows from (3.6.63), (3.6.9), the deterministic bound (3.3.2) that

$$\lim_{N\to\infty}\mathbb{E}\big[R_{11}(z_1)^a\overline{R}_{11}(z_1)^b R_{22}(z_2)^c\overline{R}_{22}(z_2)^d\big] = \lim_{N\to\infty}\mathbb{E}\Big[R_{11}^{(2)}(z_1)^a\overline{R}_{11}^{(2)}(z_1)^b R_{22}^{(1)}(z_2)^c\overline{R}_{22}^{(1)}(z_2)^d\Big], \tag{3.6.64}$$

as in (3.6.38). As before, (3.6.9) and (3.3.2) together imply that

$$\lim_{N\to\infty}\mathbb{E}\Big[R_{11}^{(2)}(z_1)^a\overline{R}_{11}^{(2)}(z_1)^b R_{22}^{(1)}(z_2)^c\overline{R}_{22}^{(1)}(z_2)^d\Big] = \lim_{N\to\infty}\mathbb{E}\Big[S_{11}^{(2)}(z_1)^a\overline{S}_{11}^{(2)}(z_1)^b S_{22}^{(1)}(z_2)^c\overline{S}_{22}^{(1)}(z_2)^d\Big], \tag{3.6.65}$$

and so it suffices to show that

$$\lim_{N\to\infty}\mathbb{E}\Big[S_{11}^{(2)}(z_1)^a\overline{S}_{11}^{(2)}(z_1)^b S_{22}^{(1)}(z_2)^c\overline{S}_{22}^{(1)}(z_2)^d\Big] \tag{3.6.66}$$

$$= \lim_{\eta\to 0}\mathbb{E}\big[R_\star(E+\mathrm{i}\eta)^a\overline{R_\star}(E+\mathrm{i}\eta)^b\big]\mathbb{E}\big[R_\star(E+\mathrm{i}\eta)^c\overline{R_\star}(E+\mathrm{i}\eta)^d\big]. \tag{3.6.67}$$

Once again using (3.6.36) and (3.6.17), we find

$$\mathbb{E}\Big[\big(-\mathrm{i}S_{11}^{(2)}(z_1)\big)^a\big(\mathrm{i}\overline{S}_{11}^{(2)}(z_1)\big)^b\big(-\mathrm{i}S_{22}^{(1)}(z_2)\big)^c\big(\mathrm{i}\overline{S}_{22}^{(1)}(z_2)\big)^d\Big] \tag{3.6.68}$$

$$= \frac{1}{\Gamma(a)\Gamma(b)\Gamma(c)\Gamma(d)}\int_{\mathbb{R}_+^4} t^{a-1}s^{b-1}x^{c-1}y^{d-1}\exp\left(\mathrm{i}tz_1 + \mathrm{i}xz_2 - \mathrm{i}s\overline{z_1} - \mathrm{i}y\overline{z_2}\right) \tag{3.6.69}$$

$$\times\,\mathbb{E}\left[\exp\left(\mathrm{i}\sum_{k\notin\{1,2\}} Z_{1k}^2\big(tR_{kk}^{(12)} - s\overline{R}_{kk}^{(12)}\big)\right)\exp\left(\mathrm{i}\sum_{k\notin\{1,2\}} Z_{2k}^2\big(xR_{kk}^{(12)} - y\overline{R}_{kk}^{(12)}\big)\right)\right] dt\,ds\,dx\,dy. \tag{3.6.70}$$

We now condition on $\{h_{ij}\}_{i,j\notin\{1,2\}}$, which makes the two exponential terms in the previous line conditionally independent. Then by following (3.6.41), (3.6.42), (3.6.43), (3.6.44), and

(3.6.47), we obtain

$$\mathbb{E}\Big[\big(-\mathrm{i}S_{11}^{(2)}(z_1)\big)^a\big(\mathrm{i}\overline{S}_{11}^{(2)}(z_1)\big)^b\big(-\mathrm{i}S_{22}^{(1)}(z_2)\big)^c\big(\mathrm{i}\overline{S}_{22}^{(1)}(z_2)\big)^d\Big] \tag{3.6.71}$$

$$= \frac{1}{\Gamma(a)\Gamma(b)\Gamma(c)\Gamma(d)}\int_{\mathbb{R}_+^4} t^{a-1}s^{b-1}x^{c-1}y^{d-1}\exp\left(\mathrm{i}tz_1+\mathrm{i}xz_2-\mathrm{i}s\overline{z_1}-\mathrm{i}y\overline{z_2}\right) \tag{3.6.72}$$

$$\times\Bigg(\exp\Bigg(-\frac{N-2}{N}\Bigg((t^2+s^2)^{\alpha/4}\gamma_{z_1}^{\star}\Big(\frac{t+\mathrm{i}s}{\sqrt{t^2+s^2}}\Big)+(x^2+y^2)^{\alpha/4}\gamma_{z_2}^{\star}\Big(\frac{x+\mathrm{i}y}{\sqrt{x^2+y^2}}\Big)+O(N^{-c})\Bigg)\Bigg) \tag{3.6.73}$$

$$+O(N^{-10})\Bigg)\,dt\,ds\,dx\,dy. \tag{3.6.74}$$

Thus (3.6.24) (with the z_N there equal to z_1 and z_2 here), (3.6.25), the dominated convergence theorem, (3.6.13), and Remark 3.6.9 together give

$$\lim_{N\to\infty}\mathbb{E}\Big[\big(-\mathrm{i}S_{11}^{(2)}(z_1)\big)^a\big(\mathrm{i}\overline{S}_{11}^{(2)}(z_1)\big)^b\big(-\mathrm{i}S_{22}^{(1)}(z_2)\big)^c\big(\mathrm{i}\overline{S}_{22}^{(1)}(z_2)\big)^d\Big] \tag{3.6.75}$$

$$= \frac{1}{\Gamma(a)\Gamma(b)\Gamma(c)\Gamma(d)}\int_{\mathbb{R}_+^4} t^{a-1}s^{b-1}x^{c-1}y^{d-1}\exp\big(\mathrm{i}E(t-s+x-y)\big) \tag{3.6.76}$$

$$\times\exp\Bigg(-(t^2+s^2)^{\alpha/4}\gamma_E^{\star}\Big(\frac{t+\mathrm{i}s}{\sqrt{t^2+s^2}}\Big)-(x^2+y^2)^{\alpha/4}\gamma_E^{\star}\Big(\frac{x+\mathrm{i}y}{\sqrt{x^2+y^2}}\Big)\Bigg)\,dt\,ds\,dx\,dy \tag{3.6.77}$$

$$= \mathfrak{Y}_{a+b}(a)\mathfrak{Y}_{c+d}(c) = \lim_{\eta\to 0}\mathbb{E}\big[R_\star(E+\mathrm{i}\eta)^a\overline{R_\star}(E+\mathrm{i}\eta)^b\big]\mathbb{E}\big[R_\star(E+\mathrm{i}\eta)^c\overline{R_\star}(E+\mathrm{i}\eta)^d\big], \tag{3.6.78}$$

from which we deduce (3.6.66). □

Proof of Theorem 3.2.9. We will apply Proposition 3.6.11, with the η_N there equal to the $\eta = N^{\mathfrak{c}-\mathfrak{a}}$ here (recall (3.2.2)) and the $E_N^{(j)}$ there equal to the $\widehat{\gamma}_{k_j}$ here. To that end, we must verify the assumptions (3.6.32) of that proposition. The first and second statements there follow from the fact that $\eta = N^{\mathfrak{c}-\mathfrak{a}}$, that $\mathfrak{c}$ is sufficiently small, and the fact that $\mathfrak{a} < \frac{1}{2}$ (by (3.2.2)). The third follows from the fact that $\lim_{N\to\infty}\gamma_{k_1} = E$ and (3.3.23).

To verify the fourth, we must show that $|\widehat{\gamma}_{k_1}-\widehat{\gamma}_{k_j}| \ll \eta$. To that end, observe since

$|k_1 - k_j| \leq N^{1/2}$, we have for any fixed $\delta > 0$ and $j \in [1, n]$ that

$$|\widehat{\gamma}_{k_1} - \widehat{\gamma}_{k_j}| \leq \big|\widehat{\gamma}_{k_1} - \gamma_{k_j}(t)\big| + \big|\widehat{\gamma}_{k_j} - \gamma_{k_j}(t)\big| + \big|\gamma_{k_1}(t) - \gamma_{k_j}(t)\big| \tag{3.6.79}$$

$$\lesssim N^{\delta - 1/2} + N^{1+4\mathfrak{c}}|k_1 - k_j| \lesssim N^{\delta + 4\mathfrak{c} - 1/2} \tag{3.6.80}$$

for sufficiently large N, where we used (3.3.24) and (3.5.18). Then, since $\eta \gg N^{-1/2}$, we may choose δ and $\mathfrak{c}$ small enough that the last bound in (3.6.32) of Proposition 3.6.11 is also satisfied.

Now the theorem follows from Lemma 3.6.10; Proposition 3.6.11; the facts that

$$\lim_{N \to \infty} \operatorname{Im} m_\alpha(\gamma_k + \mathrm{i}\eta) = \pi \rho_\alpha(E)$$

(as $\lim_{N \to \infty} \gamma_k = E$) and $\mathcal{U}_\star(E) = \big(\pi \varrho_\alpha(E)\big)^{-1} \mathcal{R}_\star(E)$ (see Definition 3.1.5); and (3.3.6). $\square$

3.7 Proofs of results from Section 3.3 and Section 3.6

In this section, we prove results from Section 3.3 and Section 3.6 which are used in the rest of the paper. We begin with the proof of Lemma 3.6.3, since facts derived in the course of that proof will be useful for proving the statements from Section 3.3.

For any $w \in \mathbb{C}$, we let $\mathcal{H}_w$ denote the space of $\mathcal{C}^1$ functions $g \colon \mathbb{K}^+ \to \mathbb{C}$ such that $g(\lambda u) = \lambda^w g(u)$ for each $\lambda \in \mathbb{R}_+$. Following [34, (10)], we define for any $r \in [0, 1)$ a norm on $\mathcal{H}_w$ by

$$\|g\|_r = \|g\|_\infty + \sup_{u \in \mathbb{S}^1_+} \sqrt{\big|(\mathrm{i}.u)^r \partial_1 g(u)\big|^2 + \big|(\mathrm{i}.u)^r \partial_2 g(u)\big|^2}, \tag{3.7.1}$$

where $\partial_1 g(x + \mathrm{i}y) = \partial_x g(x + \mathrm{i}y)$ and $\partial_2 g(x + \mathrm{i}y) = \partial_y g(x + \mathrm{i}y)$, and we recall $\|g\|_\infty = \sup_{u \in \mathbb{S}^1_+} \big|g(u)\big|$. We let $\mathcal{H}_{w,r}$ denote the closure of $\mathcal{H}_w$ in $\| \cdot \|_r$, which is a Banach space.

Following [34, (11), (12), (13)], we define for any complex numbers $u \in \mathbb{S}^1_+$ and $h \in \overline{\mathbb{K}}$, and any function $g \in \mathcal{H}_{\alpha/2}$, the function

$$F_h(g)(u) = \int_0^{\pi/2} \left(\int_{\mathbb{R}_+^2} \left(\left(e^{-r^{\alpha/2} g(e^{\mathrm{i}\theta}) - (rh.e^{\mathrm{i}\theta})} - e^{-r^{\alpha/2} g(e^{\mathrm{i}\theta} + uy) - (yrh.u) - (rh.e^{\mathrm{i}\theta})} \right) \right. \right. \tag{3.7.2}$$

$$\left. \left. \times\, r^{\alpha/2-1} y^{-\alpha/2-1} drdy \right) \right) (\sin 2\theta)^{\alpha/2-1} d\theta. \tag{3.7.3}$$

Further, for any $z \in \mathbb{H}$, the map $G_z(f) \colon \mathbb{S}_+^1 \to \mathbb{C}$ is given by

$$G_z(f)(u) = \frac{\alpha}{2^{\alpha/2}\Gamma(\alpha/2)^2} F_{-\mathrm{i}z}(f)(\mathrm{i}\bar{u}). \tag{3.7.4}$$

The following lemma from [34] indicates that the function $\gamma_z^\star$ from Definition 3.6.2 is a fixed point of G_z.

Lemma 3.7.1 ([34, Lemma 4.4])**.** *For any $z \in \mathbb{H}$ and $u \in \mathbb{S}_+^1$, we have $\gamma_z^\star(u) = G_z(\gamma_z^\star)(u)$.*

Proof of Lemma 3.6.3. For the first statement, we use [34, Proposition 3.3], which shows that there exists $c > 0$ such that, uniformly in $|z| < c$,

$$\frac{\gamma_z^\star(e^{\mathrm{i}\pi/4})}{\Gamma\left(1 - \frac{\alpha}{2}\right)} = 2^{\alpha/4} \mathbb{E}\left[\left(\operatorname{Im} R_\star(z) \right)^{\alpha/2} \right] > c. \tag{3.7.5}$$

We now compute, for any $u \in \mathbb{S}_+^1$,

$$\frac{\operatorname{Re} \gamma_z^\star(u)}{\Gamma\left(1 - \frac{\alpha}{2}\right)} = \mathbb{E}\left[\operatorname{Re} \left(-\mathrm{i} R_\star(z).u \right)^{\alpha/2} \right] \geq \mathbb{E}\left[\left(\operatorname{Re} \left(-\mathrm{i} R_\star(z).u \right) \right)^{\alpha/2} \right] \geq \mathbb{E}\left[\left(\operatorname{Im} R_\star(z) \right)^{\alpha/2} \right] \geq c. \tag{3.7.6}$$

In the first inequality, we used the fact that $\operatorname{Re} a^r \geq (\operatorname{Re} a)^r$ for any $a \in \mathbb{K}$ and $r \in (0,1)$ (see [34, Lemma 5.10]). The second inequality follows from $\operatorname{Re}(a.u) \geq \operatorname{Re} a$ for any $u \in \mathbb{S}_1^+$ and $a \in \mathbb{K}^+$. The final inequality follows from (3.7.5). This completes the proof of the first claim.

Set $z = E + \mathrm{i}\eta$. We now establish convergence of the order parameter $\gamma_z^\star$ as $\eta \to 0$. We note that for any $\tau > 0$, there exists $c = c(\tau) > 0$ such that

$$\|\gamma_z^\star - \gamma_w^\star\|_r \le \|\gamma_z^\star - \gamma_0^\star\|_r + \|\gamma_w^\star - \gamma_0^\star\|_r \le \tau \tag{3.7.7}$$

if $z, w \in \mathbb{H}$ satisfy $|z| < c$ and $|w| < c$. The final inequality follows from the first displayed equation in the proof of [34, Proposition 3.3].

Then (3.7.7) and [34, Proposition 3.4] together imply there exist constants $C, c > 0$ such that

$$\|\gamma_w^\star - \gamma_z^\star\|_r \le C\|\gamma_w^\star - G_z(\gamma_w^\star)\|_r = C\big\|G_w(\gamma_w^\star) - G_z(\gamma_w^\star)\big\|_r \tag{3.7.8}$$

for $z, w \in \mathbb{H}$ such that $|z| < c$ and $|w| < c$. In the equality, we used that $\gamma_w^\star$ is a fixed point for G_w, as stated in Lemma 3.7.1.

We now claim

$$\big\|G_w(\gamma_w^\star) - G_z(\gamma_w^\star)\big\|_r \le C|w - z|. \tag{3.7.9}$$

In the proof of [34, Lemma 4.2] (in the final lines), it was shown that the partial (Fréchet) derivative of $F_h(g)$ in either the real or imaginary part of h has finite $\|\cdot\|_r$ norm, and the exact derivative was calculated. Further, the derivative may be bounded in the $\|\cdot\|_r$ norm by a constant C using [34, (20)] when $g = \gamma_w^\star$, which is uniform in z, w with $z, w \in \mathbb{H}$ and $|z|, |w| \le c$.[5] Also, it is continuous by the computation following [34, (21)]. By definition (3.7.4), the same is true for $G_h(g)$, and we obtain (3.7.9) by integration using the fundamental theorem of calculus. Combining this with (3.7.8) we obtain the Lipschitz estimate

$$\|\gamma_w^\star - \gamma_z^\star\|_r \le c|w - z| \tag{3.7.10}$$

for any $r \in [0, 1)$. This estimate implies that $\lim_{\eta\to 0} \gamma_{E+\mathrm{i}\eta}^\star$ exists as a function in $\mathcal{H}_{\alpha/2,r}$, which we denote by $\gamma_E^\star$. □

[5]We remark that although the constant C in the bound [34, (20)] depends on $\operatorname{Re} g$ and degenerates as $\operatorname{Re} g$ goes to zero, here $g = \gamma_w^\star$ and $\inf_{u\in\mathbb{S}^+} \operatorname{Re} g(u) > c$ for some $c > 0$, from (3.6.4). So we obtain the claimed bound.

Proof of Lemma 3.3.1. We begin with the first estimate of (3.3.4). By [32, (3.4)] and [32, Theorem 4.1], we have $\operatorname{Im} m_\alpha(z) = \mathbb{E}\big[\operatorname{Im} R_\star(z)\big]$ for $z \in \mathbb{H}$. Recalling $\lim_{\eta\to 0} \operatorname{Im} m_\alpha(E+\mathrm{i}\eta) = \pi\varrho_\alpha(E)$, it then suffices to show that $\lim_{\eta\to 0} \mathbb{E}\big[\operatorname{Im} R_\star(E+\mathrm{i}\eta)\big]$ is Lipschitz in E. For c small enough, by (3.6.11), this limit is given by $(\mathfrak{X} + \overline{\mathfrak{X}})/2$, where

$$\mathfrak{X}(E) = \int_{\mathbb{R}_+} \exp\big(\mathrm{i}Et - t^{\alpha/2}\gamma_E^\star(1)\big)\, dt. \tag{3.7.11}$$

Further, if c is small enough, by Lemma 3.6.3 we have the uniform lower bound

$$\inf_{|E|\le c} \inf_{u\in\mathbb{S}^1_+} \operatorname{Re}\gamma_E^\star(u) > c'. \tag{3.7.12}$$

Define

$$F(x, y, w) = \int_{\mathbb{R}_+} \exp\big(\mathrm{i}xt - t^{\alpha/2}(y + \mathrm{i}w)\big)\, dt. \tag{3.7.13}$$

By (3.7.10), $\big|\gamma_{E_1}^\star(1) - \gamma_{E_2}^\star(1)\big| \le C|E_1 - E_2|$ for some constant C and $E_1, E_2 \in [-c, c]$, if c is small enough. Using this inequality, to show $\mathfrak{X}(E)$ is Lipschitz in E, it suffices by the fundamental theorem of calculus to show the partial derivatives $\partial_x F(x,y,w)$, $\partial_y F(x,y,w)$, and $\partial_w F(x,y,w)$ are uniformly bounded by a constant when $|x| \le c$ and $y > c'$. This follows straightforwardly after differentiating under the integral sign (which is permissible as the integrand is dominated in absolute value by $\exp(-t^{\alpha/2}y) \le \exp(-t^{\alpha/2}c')$).

We have shown that the density $\varrho_\alpha(x)$ is continuous in a neighborhood of zero. By [32, Theorem 1.6(ii)], $\varrho_\alpha(0)$ is positive and bounded. Therefore the second claim in (3.3.4) follows from the first after possibly decreasing c.

For (3.3.5), we first suppose $|E_1 - E_2| \le c/10$ and $|E_1| \le c/2$, where c is the constant from the previous part of this proof. We use the definition of the Stieltjes transform to write

$$\operatorname{Im} m_\alpha\big(E_1 + \mathrm{i}\eta\big) = \operatorname{Im} \int_{\mathbb{R}} \frac{\varrho_\alpha(x)\, dx}{x - E_1 - \mathrm{i}\eta}, \quad \operatorname{Im} m_\alpha\big(E_2 + \mathrm{i}\eta\big) = \operatorname{Im} \int_{\mathbb{R}} \frac{\varrho_\alpha\big(x + E_2 - E_1\big)\, dx}{x - E_1 - \mathrm{i}\eta}, \tag{3.7.14}$$

where in the second equality we used the change of variables $x \mapsto x + E_2 - E_1$. Computing the imaginary part of the integrand directly, we have

$$\big|\operatorname{Im} m_\alpha\big(E_1 + \mathrm{i}\eta\big) - \operatorname{Im} m_\alpha\big(E_2 + \mathrm{i}\eta\big)\big| \le \eta \int_{\mathbb{R}} \frac{\big|\varrho_\alpha(x) - \varrho_\alpha(x + E_2 - E_1)\big|}{(x - E_1)^2 + \eta^2}\, dx \tag{3.7.15}$$

$$= \eta \int_{[-c/2, c/2]} \frac{\big|\varrho_\alpha(x) - \varrho_\alpha(x + E_2 - E_1)\big|}{(x - E_1)^2 + \eta^2}\, dx \tag{3.7.16}$$

$$+ \eta \int_{[-c/2, c/2]^c} \frac{\big|\varrho_\alpha(x) - \varrho_\alpha\big(x + E_2 - E_1\big)\big|}{(x - E_1)^2 + \eta^2}\, dx. \tag{3.7.17}$$

Using (3.3.4), $|x| \le c/2$, and $|E_1 - E_2| \le c/10$, we find that

$$(3.7.16) \le C|E_1 - E_2| \int_{\mathbb{R}} \frac{\eta}{(x - E_1)^2 + \eta^2}\, dx \le C|E_1 - E_2|. \tag{3.7.18}$$

By [32, Theorem 1.6], the density $\varrho_\alpha(x)$ is uniformly bounded. Therefore

$$(3.7.17) \le C \int_{[-c/2, c/2]^c} \frac{\eta}{(x - E_1)^2 + \eta^2}\, dx = C\eta \int_{[-c/2, c/2]^c} \frac{1}{(x - E_1)^2}\, dx \tag{3.7.19}$$

$$\le Cc^{-1}\eta. \tag{3.7.20}$$

In the last line, we used the hypothesis that $|E| \le c/2$ to show the integral is uniformly bounded and adjusted the value of C. This completes the proof of (3.3.5) after decreasing c if necessary.

We now address the first claim of (3.3.6). Again using the uniform boundedness of $\varrho_\alpha(x)$

over all $x \in \mathbb{R}$, we deduce

$$\operatorname{Im} m_\alpha(E + \mathrm{i}\eta) = \int_{\mathbb{R}} \frac{\eta \varrho_\alpha(x)\, dx}{\eta^2 + (E - x)^2} \le C. \tag{3.7.21}$$

We also note, again using (3.3.4), that

$$\operatorname{Im} m_\alpha(E + \mathrm{i}\eta) \ge \int_{[-c,c]} \frac{\eta \varrho_\alpha(x)\, dx}{\eta^2 + (E - x)^2} \ge c \int_{[-c,c]} \frac{\eta\, dx}{\eta^2 + (E - x)^2}, \tag{3.7.22}$$

and the latter quantity is uniformly bounded below η tends to zero because $E \in (-c, c)$. Thus, after decreasing c if necessary, we have $\operatorname{Im} m_\alpha(E + \mathrm{i}\eta) > c$ when $\eta, |E| < c$.

The second claim of (3.3.6) follows after noting (using the bounds on the density ϱ_α given in (3.3.4)) that for any c, there exists $c' > 0$ such that $\left|\gamma_i^{(\alpha)}\right| < c$ for all $i \in [(1/2 - c')N, (1/2 + c')N]$. □

Proof of Lemma 3.3.7. We prove only the first claim in detail. The proof of the second is analogous, and the third follows from the second by (3.3.14).

By a standard stochastic continuity argument, it suffices to prove the desired bound holds with overwhelming probability at fixed γ; all bounds below will be independent of γ. Let $C > 0$ be a parameter. A straightforward calculation shows that for any $1 \le i \le N$, the law of the sum of the absolute value of the entries of the i-th row and column of the matrix $\mathbf{H}^\gamma$ has a power law tail with parameter α. This implies, by Hoeffding's inequality, that with overwhelming probability there are at most $2C^{-\alpha}N$ such i whose corresponding sum is greater than C. When this holds, after removing at most $2C^{-\alpha}N$ rows and columns, the largest eigenvalue is at most C in absolute value, since the largest absolute value of a row of a matrix bounds the magnitude of its largest eigenvalue. Then eigenvalue interlacing [52, Lemma 7.4] implies that there are at most $4C^{-\alpha}N$ eigenvalues of $\mathbf{H}^\gamma$ outside of the interval $[-C, C]$.

By [18, Theorem 1.1] (or [32, Theorem 1.2]), for any fixed compact interval $I \subset \mathbb{R}$,

$$\mathbb{E}\left[\mu_N(I)\right] \to \mu_\alpha(I) \tag{3.7.23}$$

as N tends to ∞. Here $\mu_N = \mu_N^{(\gamma)}$ denotes the empirical spectral distribution of $\mathbf{H}^\gamma$.

For any $C > 0$, let $I_C = [c_1/2, C]$. Then (3.7.23) and the concentration estimate [33, Lemma C.1] imply that for any choice of C there exists $N(C)$ such that

$$\mu_N(I_C) \leq C^{-1} + \mu_\alpha(I_C) \tag{3.7.24}$$

holds for any $N > N(C)$ with overwhelming probability. By the symmetry of μ_α and the second estimate in (3.3.4), we have $\mu_\alpha(I_C) \leq (1/2 - \delta)$ for some $\delta = \delta(c_1) > 0$ such that $\lim_{c_1 \to 0} \delta(c_1) = 0$. Combining (3.7.24) with the estimate for eigenvalues lying outside $[-C, C]$, we find

$$\mu_N\big([c_1, \infty)\big) \leq C^{-1} + (1/2 - \delta) + 4C^{-1/\alpha} < 1/2 - \delta/2 \tag{3.7.25}$$

for large enough C, with overwhelming probability. Then the $(1/2 - \delta/2)N$-th eigenvalue is less than c_1 with overwhelming probability. A similar argument shows that the $(1/2+\delta/2)N$-th eigenvalue is greater than c_1 with overwhelming probability. This completes the proof. $\square$

Before proceeding to the proof of Lemma 3.3.8, we require the following preliminary lemma. We recall $m_N(s,z)$ and its expectation $\widehat{m}_N(s,z) = \mathbb{E}\big[\widehat{m}_N(s,z)\big]$ from (3.2.16), and μ_s from (3.2.17). Let $\widehat{\mu}_s = \mathbb{E}[\mu_s]$, which is symmetric about the origin. We further define the counting functions

$$n_s(E) = \frac{1}{N}\Big|\big\{i : \lambda_i(s) \leq E\big\}\Big| = \mu_s\big((-\infty, E]\big), \quad \widehat{n}_s(E) = \widehat{\mu}_s\big((-\infty, E]\big). \tag{3.7.26}$$

Lemma 3.7.2. *Retain the notation of Lemma 3.3.8. There exists a constant $c_0 > 0$ such* *that, for any $E \in [-c_0, c_0]$, we have with overwhelming probability that*

$$\sup_{s \in [N^{-1/2+\delta}, N^{-\delta}]} \big| n_s(E) - \widehat{n}_s(E) \big| \leq N^{\delta - 1/2}. \tag{3.7.27}$$

Proof. This proof will largely follow the calculations of [78, Section 7.3], with some modifications to account for the fact that the spectral distributions we consider are not compactly supported.

To that end, we begin with a tail bound on the smallest eigenvalue, $\lambda_1 = \lambda_1(s)$, of $\mathbf{X}_s$. A straightforward calculation shows that, for any $1 \leq i \leq N$, the law of the sum of the absolute value of the entries of the i-th row and column of the matrix $\mathbf{X}_s$ has a power law tail with parameter α. Thus, since the largest such sum bounds $|\lambda_1|$ above, we deduce for any $t > 1$ that

$$\mathbb{P}\big(\lambda_1 < -t\big) \leq CNt^{-\alpha}. \tag{3.7.28}$$

Now, we must show that (3.7.27) holds on an event of probability at least $1 - N^{-D}$ for any fixed $D > 0$ and sufficiently large N. Throughout the remainder of this proof, set $B = \alpha^{-1}(D+3)$ so, by (3.7.28), $\mathbb{P}(\lambda_1 > -N^B) \geq 1 - N^{-D-2}$. Thus, we may work on the event on which $\lambda_1 > -N^B$.

By the Helffer–Sjöstrand formula (see, for example, [52, Chapter 11]), for any smooth and compactly supported function $f \colon \mathbb{R} \to \mathbb{R}$,

$$f(u) = \frac{1}{\pi} \int_{\mathbb{R}^2} \frac{\mathrm{i} y f''(x) g(y) + \mathrm{i}\big(f(x) + \mathrm{i} y f'(x)\big) g'(y)}{u - x - \mathrm{i} y} \, dx \, dy, \tag{3.7.29}$$

where g is any smooth, compactly supported function that is 1 in a neighborhood of 0. Set $E_1 = -N^{4B}$ and fix some $E_2 \in [-(2K)^{-1}, (2K)^{-1}]$, where K is the constant from Lemma 3.3.3. Let $\eta = N^{-1/2+\delta}$, and let f be a smooth function satisfying $f(E) = 0$ for $E \notin [E_1 - 1, E_2 + \eta]$ and $f(E) = 1$ for $E \in [E_1, E_2]$. We can select f such that $\big|f(x)\big| \leq 1$ for all $x \in \mathbb{R}$; $\big|f'(x)\big| \leq C$ and $\big|f''(x)\big| \leq C$ for $x \in [E_1 - 1, E_1]$; and $\big|f'(x)\big| \leq C\eta^{-1}$ and

$\left|f''(x)\right| \le C\eta^{-2}$ for $x \in [E_2, E_2+\eta]$. We also let $g(y)$ be a smooth function satisfying $g(y) = 1$ for $|y| \le N^{10B}$; $g(y) = 0$ for $|y| > N^{10B} + 1$; we may select g such that $0 \le g(y) < 1$ and $\left|g'(y)\right| < C$ for all $y \in \mathbb{R}$.

Write $\mu_\Delta = \mu_s - \widehat{\mu}_s$ and let $m_\Delta(z) = m_N(s,z) - \widehat{m}_N(s,z)$ be the Stieltjes transform of μ_Δ. Our first goal is to prove that

$$\left|\int_{\mathbb{R}} f(E)\, d\mu_\Delta(E)\right| \le CN^{\delta/2-1/2} \tag{3.7.30}$$

with probability at at least $1 - N^{-D-1}$, for large enough N. Using(3.7.29), we find

$$\begin{aligned}\left|\int_{\mathbb{R}} f(E)\, d\mu_\Delta(E)\right| &\le C\left|\int_{\mathbb{R}^2} yf''(x)g(y)\operatorname{Im} m_\Delta(x+\mathrm{i}y)\,dx\,dy\right| &(3.7.31)\\ &\quad + C\int_{\mathbb{R}^2}\left|f(x)g'(y)\right|\left|\operatorname{Im} m_\Delta(x+\mathrm{i}y)\right|dx\,dy &(3.7.32)\\ &\quad + C\int_{\mathbb{R}^2}\left|yf'(x)g'(y)\right|\left|\operatorname{Re} m_\Delta(x+\mathrm{i}y)\right|dx\,dy. &(3.7.33)\end{aligned}$$

Now, since [6, (5.13)] states

$$\mathbb{P}\left[\left|m_N(s,z) - \widehat{m}_N(s,z)\right| > \frac{4\log N}{N^{1/2}\operatorname{Im} z}\right] \le 2\exp\big(-(\log N)^2\big), \tag{3.7.34}$$

we have (by a standard stochastic continuity argument) with overwhelming probability that

$$\sup_{|y|\le N^{20B}}\left|ym_\Delta(x+\mathrm{i}y)\right| \le 5N^{-1/2}\log N. \tag{3.7.35}$$

Now let us bound the quantities (3.7.31), (3.7.32), and (3.7.33). We begin with the latter. To that end, observe that since $\operatorname{supp} f' \subseteq [E_1-1, E_1]\cup[E_2, E_2+\eta]$; since $\operatorname{supp} g' \subseteq [-N^{10B}-1, -N^{10B}]\cup[N^{10B}, N^{10B}+1]$; since $\left|f'(x)\right| \le C$ for $x \in [E_1-1, E_1]$; since $\left|f'(x)\right| \le C\eta^{-1}$ for $x \in [E_2, E_2+\eta]$; and since $\left|g'(y)\right| \le C$, we have by (3.7.35) that (3.7.33) $\le C(\log N)N^{-1/2}$ with overwhelming probability.

Similarly, to bound (3.7.32), observe that $\left|f(x)\right| \le 1$; that $\left|g'(x)\right| \le C$; that $\operatorname{supp} f$ is contained in the interval $[-N^{4B}, N^{4B}]$ of length at most $2N^{4B}$; and that $\operatorname{supp} g' \subset [-N^{10B} - 1, -N^{10B}] \cup [N^{10B}, N^{10B}+1]$, on which we have $\big|m_\Delta(z)\big| \le N^{-10B}$ (due to the deterministic bound $\big|m_\Delta(z)\big| < |\operatorname{Im} z|^{-1}$). Together, these yield the deterministic estimate (3.7.32) $\le N^{-1}$.

It therefore suffices to bound (3.7.31), to which end we write

$$
\begin{aligned}
(3.7.31) \le \bigg| \int_{|x-E_1|\le 2, |y|\le 10} & y f''(x) g(y) \operatorname{Im} m_\Delta(x+\mathrm{i}y)\, dx\, dy \bigg| && (3.7.36)\\
& + \bigg| \int_{|x-E_1|\le 2, |y|> 10} y f''(x) g(y) \operatorname{Im} m_\Delta(x+\mathrm{i}y)\, dx\, dy \bigg| && (3.7.37)\\
& + \bigg| \int_{|x-E_2|\le 2\eta, |y|\le \eta} y f''(x) g(y) \operatorname{Im} m_\Delta(x+\mathrm{i}y)\, dx\, dy \bigg| && (3.7.38)\\
& + \bigg| \int_{|x-E_2|\le 2\eta, |y|> \eta} y f''(x) g(y) \operatorname{Im} m_\Delta(x+\mathrm{i}y)\, dx\, dy \bigg|. && (3.7.39)
\end{aligned}
$$

We must bound the terms (3.7.36), (3.7.37), (3.7.38), and (3.7.39); we begin with the former. Since we restricted to the event of probability $1-N^{-D-2}$ on which $\lambda_1 > -N^B$, and since $E_1 = N^{4B}$, the definition of $m_N(s,z)$ shows that $\big|m_N(s, x+\mathrm{i}y)\big| \le CN^{-4B}$ for $x \in [E_1-2, E_1+2]$. Using the trivial bound (3.3.2) on the complementary event and then taking expectation shows that $|\widehat{m}_N(s, x+\mathrm{i}y)| \le CN^{-4B} + N^{-D-2}y^{-1}$ for $x \in [E_1-1, E_1+1]$, if D is sufficiently large. Hence, with probability $1-N^{-D-2}$, we have $\big|m_\Delta(x+\mathrm{i}y)\big| \le N^{-4B} + N^{-D}y^{-1}$ for $x \in [E_1-2, E_1+2]$. Combining this with the fact that $\big|f''(x)\big|$ for $x \in [E_1-1, E_1+1]$, we obtain (3.7.36) $\le N^{-1}$.

To estimate (3.7.37), we first integrate by parts in x, using the identity $\partial_x \operatorname{Im} m_\Delta = -\partial_y \operatorname{Re} m_\Delta$ and the fact that $\operatorname{supp} f' \subseteq [E_1-1, E_1+1]$ to deduce

$$
(3.7.37) = \bigg| \int_{|x-E_1|\le 2, |y|>10} y f'(x) g(y) \partial_y \big(\operatorname{Re} m_\Delta(x+\mathrm{i}y) \big)\, dx\, dy \bigg|. \qquad (3.7.40)
$$

Integrating by parts in y and using the fact that $\partial_y\big(yg(y)\big) = g(y) + yg'(y)$ then gives

$$(3.7.37) \le \left|\int_{|x-E_1|\le 2, |y|>10} f'(x)\big(g(y)+yg'(y)\big)\operatorname{Re} m_\Delta(x+\mathrm{i}y)\,dx\,dy\right| \tag{3.7.41}$$

$$+\left|\int_{|x-E_1|\le 2} f'(x)10g(10)\operatorname{Re} m_\Delta(x+10\mathrm{i})\,dx\right|. \tag{3.7.42}$$

To bound (3.7.42), we use (3.7.35) and the fact that $\big|f'(x)\big| \le C$ for $x \in [E_1-2, E_1+2]$ to deduce that (3.7.42) $\le C(\log N)N^{-1/2}$ with overwhelming probability. To estimate (3.7.41), we again use the the facts that $\big|f'(x)\big| \le C$ for $x \in [E_1-2, E_1+2]$; that $\operatorname{supp} g \subseteq [-N^{10B}-1, N^{10B}+1]$; that $0 \le g(y) \le 1$; that $\operatorname{supp} g' \subseteq [N^{10B}, N^{10B}+1]$; $g'(y) \le C$; and (3.7.35) to deduce

$$(3.7.41) \le CN^{-1/2}\log N\left|\int_{|y|>10}|y|^{-1}\big(g(y)+yg'(y)\big)\,dy\right| \tag{3.7.43}$$

$$\le CN^{-1/2}\log N\left(\left|\int_{10}^{N^{11B}}|y|^{-1}dy\right| + \left|\int_{|y|>10} g'(y)\,dy\right|\right) \le CN^{-1/2}(\log N)^3, \tag{3.7.44}$$

with overwhelming probability and for sufficiently large N. Hence, (3.7.37) $\le CN^{-1/2}(\log N)^3$.

To bound (3.7.38), first recall that the function $y\operatorname{Im} m(x+\mathrm{i}y)$ is increasing in y, for any Stieltjes transform m of a positive measure. Therefore, (3.7.35) implies with overwhelming probability that

$$\sup_{y\le\eta} y\operatorname{Im} m_\Delta(x+\mathrm{i}y) \le \eta\operatorname{Im} m_\Delta(x+\mathrm{i}\eta) \le CN^{-1/2}\log N \tag{3.7.45}$$

Putting this estimate into (3.7.38) and using that $\big|f''(x)\big|$ vanishes except on $[E_2, E_2+\eta]$, where it is at most $C\eta^{-2}$, and that $\big|g(y)\big|$ is 1 for $|y| \le \eta$, we deduce that (3.7.38) $\le C(\log N)N^{-1/2}$.

For the term (3.7.39), we integrate by parts as we did for (3.7.37) to obtain

$$(3.7.39) \leq \left| \int_{|x-E_2|\leq 2\eta, |y|>\eta} f'(x)\partial_y\big(g(y) + yg'(y)\big) \operatorname{Re} m_\Delta(x+iy)\, dx\, dy \right| \tag{3.7.46}$$

$$+ \left| \int_{|x-E_2|\leq 2\eta} f'(x)\eta g(\eta) \operatorname{Re} m_\Delta(x + \mathrm{i}\eta)\, dx \right|. \tag{3.7.47}$$

We use (3.7.35) to estimate (3.7.46) $\leq C(\log N)^3 N^{-1/2}$ and (3.7.47) $\leq C(\log N)N^{-1/2}$, in the same way we bounded (3.7.41) (in (3.7.44)) and (3.7.42), except now we note that $f'(x)$ vanishes off of $x \in [E_2, E_2+\eta]$, where it is at most $C\eta^{-1}$. This shows (3.7.39) $\leq C(\log N)^3 N^{-1/2}$, and so (3.7.31) $\leq C(\log N)^3 N^{-1/2}$, with overwhelming probability. Combining our estimates on (3.7.31), (3.7.32), and (3.7.33), we deduce (3.7.30) holds with probability at least $1 - N^{-D-1}$.

We now use (3.7.30) to estimate the difference between the eigenvalue counting functions for the measures μ_s and $\widehat{\mu}_s$. We recall that we are working on the set of probability at least $1 - N^{-D-2}$ where $\lambda_1 > -N^B$. Therefore, recalling the definition of $f(x)$, we see $n_s(E_2) \leq \int_{-\infty}^{\infty} f(x)\, d\mu_s(x) \leq n_s(E_2 + \eta)$ for any $E_2 \in [-(2K)^{-1}, (2K)^{-1}]$ on this event. The case of of $\widehat{n}_s(E)$ is slightly more delicate, since we must estimate the contribution to the mass of $\widehat{\mu}_s$ from eigenvalues in the interval $(-\infty, -N^B]$. With probability at least $1 - N^{-D-2}$, there are no eigenvalues in this interval. On the complementary event, trivially have $\int_{-\infty}^{-N^B} d\mu_s(x) \leq 1$, since μ_s is a probability measure. Therefore, $\int_{-\infty}^{-N^B} d\widehat{\mu}_s(x) \leq N^{-2}$ by taking expectation and using the previous two observations.

Using (3.7.30) and the overwhelming probability estimate $\big|n_s(E_2 + \eta) - n_s(E_2)\big| \leq C\eta$ (which follows from the second inequality of (3.3.21)), we deduce that, with probability at least $1 - CN^{-D-1}$,

$$\begin{aligned}\widehat{n}_s(E_2) - n_s(E_2) &\le \widehat{n}_s(E_2) - n_s(E_2 + \eta) + C\eta &(3.7.48)\\ &\le \int_{\mathbb{R}} f(x) d\widehat{\mu}_s(x) + CN^{-2} - \int_{\mathbb{R}} f(x) d\mu_s(x) + C\eta &(3.7.49)\\ &\le CN^{\delta/2-1/2} + C\eta \le CN^{\delta/2-1/2}. &(3.7.50)\end{aligned}$$

Similarly, now using the bound $\big|\widehat{n}_s(E_2 + \eta) - \widehat{n}_s(E_2)\big| \le C\eta$ (which follows from the overwhelming probability estimate $\big|n_s(E_2 + \eta) - n_s(E_2)\big| \le C\eta$ after taking expectation and using the trivial bound $\big|n_s(E_2 + \eta) - n_s(E_2)\big| \le 1$ on the set where this estimate does not hold), we obtain

$$\begin{aligned}n_s(E_2) - \widehat{n}_s(E_2) &\le n_s(E_2 + \eta) - \widehat{n}_s(E_2 + \eta) + C\eta &(3.7.51)\\ &\le \int_{\mathbb{R}} f(x) d\mu_s(x) + CN^{-2} - \int_{\mathbb{R}} f(x) d\widehat{\mu}_s(x) + C\eta &(3.7.52)\\ &\le CN^{\delta/2-1/2} + C\eta \le CN^{\delta/2-1/2}, &(3.7.53)\end{aligned}$$

with probability at least $1 - CN^{-D-1}$. Hence, with probability at least $1 - N^{-D}$, we conclude $\big|n_s(E) - \widehat{n}_s(E)\big| \le N^{\delta-1/2}$, for $|E| \le (2K)^{-1}$. $\square$

We are now ready for the proof of Lemma 3.3.8.

Proof of Lemma 3.3.8. We start with the first claim. By (3.3.12) and (3.3.13), we have with overwhelming probability that

$$\sup_{s \in [N^{-1/2+\delta}, N^{-\delta}]} \sup_{z \in \widetilde{\mathcal{D}}} \big|m_\alpha(z) - m_N(s, z)\big| < N^{-\alpha\delta/16}. \tag{3.7.54}$$

By using (3.3.2) on the set where this does not hold and taking expectation, we deduce the deterministic estimate

$$\sup_{s\in[N^{-1/2+\delta},N^{-\delta}]}\sup_{z\in\widetilde{\mathcal{D}}}\left|m_\alpha(z)-\widehat{m}_N(s,z)\right|<CN^{-\alpha\delta/16}. \tag{3.7.55}$$

Next, we define

$$n_\alpha(E)=\int_{-\infty}^{E}\varrho_\alpha(x)\,dx. \tag{3.7.56}$$

The distribution functions $n_\alpha(E)-n_\alpha(0)=\int_0^E\varrho_\alpha(x)\,dx$ and $\widehat{n}_s(E)-\widehat{n}_s(0)=\int_0^E d\widehat{\mu}_s(x)$ may be compared using [52, (11.3)] (see also the proof of [52, Lemma 11.3]), which by (3.7.55) gives

$$\left|n_\alpha(E)-\widehat{n}_s(E)-n_\alpha(0)+\widehat{n}_s(0)\right|\le CN^{-c} \tag{3.7.57}$$

for some $c=c(\delta)>0$ and $E\in[-c_0,c_0]$, where $c_0>0$ is sufficiently small.

There are two cases: N is even and N is odd. When N is even, by the symmetry of the measures $\widehat{\mu}_s$ and μ_α, we see that $\widehat{\gamma}_{N/2}(s)=\gamma_{N/2}^{(\alpha)}=0$. We consider this case first.

We will show that for any $c_1>0$ sufficiently small, there exists $c_2>0$ such that, if $|i-N/2|\le c_2N$, then $\left|\widehat{\gamma}_i(s)\right|<c_1$. To that end, observe that (3.5.18) implies, with overwhelming probability,

$$\int_0^v d\mu_s\ge cv, \tag{3.7.58}$$

for any $v\in[N^{\delta-1},c_0]$ (after decreasing c_0, if necessary). Taking expectation and using that μ_s is a nonnegative measure, we find

$$\int_0^v d\widehat{\mu}_s\ge\frac{cv}{2},\qquad\text{for any } v\in[N^{\delta-1},c_0]. \tag{3.7.59}$$

We may suppose by symmetry that $i\ge N/2$, so that $\widehat{\gamma}_i(s)\ge 0$ by the symmetry of $\widehat{\mu}_s(x)$. By definition,

$$\int_0^{\widehat{\gamma}_i(s)}d\widehat{\mu}_s=\frac{i-N/2}{N}\le c_2. \tag{3.7.60}$$

If we choose $c_2<c_1c/4$, then $\widehat{\gamma}_i(s)>c_1$ produces a contradiction with (3.7.59). By (3.3.4), we also have that for any $c_1 > 0$ sufficiently small, there exists $c_2 > 0$ such that, if $|i - N/2| \leq c_2 N$, then $|\gamma_i| < c_1$. We take c_2 small enough so that $|i - N/2| \leq c_2 N$ implies $\gamma_i, \widehat{\gamma}_i(s) \in [-c_0, c_0]$, and consider just this set of indices in what follows.

We observe that, for $s > 0$, $\widehat{\mu}_s$ is absolutely continuous with respect to Lebesgue measure, since each entry of $\mathbf{X}_s$ is. Then, by the definition of γ_i and $\widehat{\gamma}_i(s)$,

$$\int_0^{\widehat{\gamma}_i(s)} \varrho_\alpha(x)\,dx + \int_{\widehat{\gamma}_i(s)}^{\gamma_i} \varrho_\alpha(x)\,dx = \int_0^{\gamma_i} \varrho_\alpha(x)dx = \frac{i - N/2}{N} = \int_0^{\widehat{\gamma}_i(s)} d\widehat{\mu}_s. \tag{3.7.61}$$

Using (3.7.57) and the fact that $\varrho_\alpha(x)$ is bounded below on $[-K, K]$ by some constant c' by (3.3.4), we see

$$N^{-c} \geq \left| \int_{\gamma_i(s)}^{\gamma_i} \varrho_\alpha(x)\,dx \right| \geq c' \big| \gamma_i - \widehat{\gamma}_i(s) \big| \tag{3.7.62}$$

deterministically, which completes the proof when N is even.

When N is odd, we have $\widehat{\gamma}_{\lfloor N/2 \rfloor}(s) = -\widehat{\gamma}_{\lceil N/2 \rceil}(s)$ by symmetry. If $\widehat{\gamma}_{\lceil N/2 \rceil}(s) > N^{-1+\delta}$, then setting $v = \big|\widehat{\gamma}_{\lceil N/2 \rceil}(s)\big|$ in (3.7.59) yields and also using the fact that

$$N^{-1} = \int_{\widehat{\gamma}_{\lfloor N/2 \rfloor}(s)}^{\widehat{\gamma}_{\lceil N/2 \rceil}(s)} d\widehat{\mu}_s = 2 \int_0^{\widehat{\gamma}_{\lceil N/2 \rceil}(s)} d\widehat{\mu}_s \geq \frac{cN^{\delta - 1}}{2}, \tag{3.7.63}$$

which is a contradiction. Thus, $\big|\widehat{\gamma}_{\lfloor N/2 \rfloor}(s)\big| = \big|\widehat{\gamma}_{\lceil N/2 \rceil}(s)\big| \leq CN^{-1+\delta}$. We write

$$\int_{\widehat{\gamma}_{\lceil N/2 \rceil}(s)}^{\widehat{\gamma}_i(s)} d\widehat{\mu}_s = \int_{\gamma_{\lceil N/2 \rceil}(s)}^{\widehat{\gamma}_i(s)} \varrho_\alpha(x)\,dx + \int_{\widehat{\gamma}_i(s)}^{\gamma_i} \varrho_\alpha(x)\,dx. \tag{3.7.64}$$

Since $\big|\widehat{\gamma}_{\lceil N/2 \rceil}(s)\big| \leq CN^{-1+\delta}$, and by (3.3.4), $\big|\gamma_{\lceil N/2 \rceil}(s)\big| \leq CN^{-1+\delta}$, we have

$$\big|\widehat{\gamma}_{\lceil N/2 \rceil}(s) - \gamma_{\lceil N/2 \rceil}(s)\big| \leq CN^{-1+\delta}. \tag{3.7.65}$$

In conjunction with (3.3.4), this shows

$$\int_{\widehat{\gamma}_{\lceil N/2 \rceil}(s)}^{\widehat{\gamma}_i(s)} d\widehat{\mu}_s = \int_{\widehat{\gamma}_{\lceil N/2 \rceil}(s)}^{\widehat{\gamma}_i(s)} \varrho_\alpha(x)\,dx + \int_{\widehat{\gamma}_i(s)}^{\gamma_i} \varrho_\alpha(x)\,dx + O(N^{-1+\delta}). \tag{3.7.66}$$

We may then proceed using (3.7.57) as before in (3.7.62) to complete the proof.

The proof of the second claim, (3.3.24), uses (3.7.27) and proceeds similarly, except there is no need to treat the cases of even and odd N separately. By (3.3.7) and the discussion following (3.7.60), there exists $c_2 > 0$ such that $|i - N/2| \leq c_2 N$ implies $\gamma_i, \lambda_i(s) \in [-c_0, c_0]$ with overwhelming probability. We then write

$$\int_{-\infty}^{\widehat{\gamma}_i(s)} d\widehat{\mu}_s = \int_{-\infty}^{\widehat{\gamma}_i(s)} d\mu_s + \int_{\widehat{\gamma}_i(s)}^{\lambda_i(s)} d\mu_s. \tag{3.7.67}$$

Then using (3.7.27), with overwhelming probability we have

$$N^{\delta/2-1/2} \geq \left| \int_{\widehat{\gamma}_i(s)}^{\lambda_i(s)} d\mu_s \right|. \tag{3.7.68}$$

Assuming to the contrary that $\left|\lambda_i(s) - \widehat{\gamma}_i(s)\right| \geq N^{\delta-1}$, we use (3.5.18) again to show that, with overwhelming probability,

$$CN^{\delta/2-1/2} \geq \left| \int_{\widehat{\gamma}_i(s)}^{\lambda_i(s)} d\mu_s \right| \geq c' \left|\lambda_i(s) - \widehat{\gamma}_i(s)\right|. \tag{3.7.69}$$

Thus, $\left|\lambda_i(s) - \widehat{\gamma}_i(s)\right| \leq CN^{\delta/2-1/2}$, which is a contradiction and so $\left|\lambda_i(s) - \widehat{\gamma}_i(s)\right| \geq N^{\delta-1}$. Finally, we estimate

$$\left|\gamma_i(s) - \widehat{\gamma}_i(s)\right| \leq \left|\gamma_i(s) - \lambda_i(s)\right| + \left|\lambda_i(s) - \widehat{\gamma}_i(s)\right| \leq N^{-1+\delta} + CN^{-1/2+\delta} \lesssim N^{-1/2+\delta} \tag{3.7.70}$$

using (3.3.14) to bound $\left|\gamma_i(s) - \lambda_i(s)\right|$, which proves (3.3.24) □

3.8 Convergence in distribution

Proposition 3.8.1. *For $\alpha \in (2/3, 2) \setminus \mathcal{A}$, there is a unique limit point $\mathcal{R}_\star(E)$ of the sequence of random variables $\left\{ \operatorname{Im} R_\star(E + \mathrm{i}\eta) \right\}_{\eta>0}$ in the weak topology. For $\alpha \in (1, 2) \setminus \mathcal{A}$,* *the conclusions of Theorem 3.1.8, Theorem 3.1.7, and Corollary 3.1.9 hold in the sense of convergence in distribution.*

Proof. We begin by showing that there exist constants $C > 1 > c > 0$ such that, for $z \in \mathbb{H}$ with $|z| < c$, the random variable $R_\star(z)$ satisfies the tail bound

$$\mathbb{P}\big(\operatorname{Im} R_\star(z) > s\big) \leq \exp\left(-\frac{s^{\alpha/(2-\alpha)}}{C}\right). \tag{3.8.1}$$

To that end, let $R_1(z), R_2(z), \dots$ denote mutually independent random variables each with law $R_\star(z)$. By the Lévy–Khintchine formula (3.6.20),

$$\mathbb{E}\left[\exp\left(-t \operatorname{Im}\sum_{k=1}^{\infty} \xi_k R_k(z)\right)\right] = \exp\left(-t^{\alpha/2}\Gamma\left(1-\frac{\alpha}{2}\right)\mathbb{E}\left[\big(\operatorname{Im} R_\star(z)\big)^{\alpha/2}\right]\right) \leq \exp\left(-\frac{2t^{\alpha/2}}{C}\right), \tag{3.8.2}$$

for some $C > 0$, where we used that $\mathbb{E}\big[\big(\operatorname{Im} R_\star(z)\big)^{\alpha/2}\big] > c'$ for z in a neighborhood of 0 (see (3.7.5)).

We now compute, using (3.6.1),

$$\mathbb{P}\big(\operatorname{Im} R_\star(z) > Ct^{1-\alpha/2}\big) \leq \mathbb{P}\left(\operatorname{Im}\sum_{k=1}^{\infty}\xi_k R_k(z) < \frac{t^{\alpha/2-1}}{C}\right) \tag{3.8.3}$$

$$= \mathbb{P}\left(\exp\left(-t\operatorname{Im}\sum_{k=1}^{\infty}\xi_k R_k(z)\right) > \exp\left(-\frac{t^{\alpha/2}}{C}\right)\right). \tag{3.8.4}$$

In the first equality, we used $\operatorname{Im} R_k(z) > 0$. Now applying Markov's inequality to (3.8.2) and (3.8.3), we obtain

$$\mathbb{P}\big(\operatorname{Im} R_\star(z) > Ct^{1-\alpha/2}\big) \leq \exp\left(-\frac{t^{\alpha/2}}{C}\right). \tag{3.8.5}$$

Setting $s = Ct^{1-\alpha/2}$, we obtain (3.8.1) for a new value of C.

Let $\mathcal{R}_\star(E)$ be the limit point in the weak topology of $\{\operatorname{Im} R_\star(E + i\eta)\}_{\eta>0}$ from Definition 3.1.5. Note that since the tail bound (3.8.1) holds for $\operatorname{Im} R_\star(z)$ uniformly in z, it also holds for $\mathcal{R}_\star(E)$.

Using the bound (3.8.1) for $\mathcal{R}_\star(E)$, we see that for all k, the k-th moment of $\mathcal{R}_\star(E)$ is bounded by $(Ck)^{k(2-\alpha)/\alpha}$ for some $C > 0$. Therefore, the series $\sum_{k\geq 1} \mathbb{E}\big[\mathcal{R}_\star(E)^k\big]^{-1/2k}$ diverges when $\alpha \in (2/3, 2)$. By Carleman's condition for positive random variables (the Stieltjes moment problem) [89, p. 21], this implies that $\mathcal{R}_\star(E)$ is determined by its moments when $\alpha \in (2/3, 2)$. By Proposition 3.1.4, these moments are the same for any subsequential limit of $\{\operatorname{Im} R_\star(z)\}_{\eta>0}$, so $\mathcal{R}_\star(E)$ is only possible subsequential limit. Therefore the sequence converges in distribution to $\mathcal{R}_\star(E)$.

Similar reasoning may be applied to the quantities $\mathcal{N}^2 \cdot \mathcal{R}_\star(E)$ appearing in Theorem 3.1.8, Theorem 3.1.7, and Corollary 3.1.9. By Stirling's formula, the k-th moment of $\mathcal{N}^2$ is bounded by $(Ck)^k$ for some $C > 0$, so the moments of $\mathcal{N}^2 \cdot \mathcal{R}_\star(E)$ are bounded by $(Ck)^{k(1+(2-\alpha)/\alpha)}$. For $\alpha \in (1, 2)$, $1 + (2-\alpha)/\alpha < 2$ and Carleman's condition applies. This completes the proof. $\square$

3.9 Quantum unique ergodicity of eigenvectors

For any $a_N \colon [1, N] \cap \mathbb{N} \to [-1, 1]$ we denote by $|a_N| = \big|1 \leq i \leq N : a_N(i) \neq 0\big|$ the cardinality of the integer support of a_N. We define $\langle \mathbf{u}_k, a_N \mathbf{u}_k\rangle = \sum_{i=1}^N |\mathbf{u}_k(i)|^2 a_N(i)$.

Corollary 3.9.1. *For all $\alpha \in (0, 2) \setminus \mathcal{A}$, there exists $c = c(\alpha) > 0$ such that the following holds. Fix any index sequence $k = k(N)$ such that $\lim_{N\to\infty} \gamma_k = E$ for some $E \in \mathbb{R}$ satisfying $|E| < c$. Then for every $\delta > 0$, for any $a_N \colon [1, N] \cap \mathbb{N} \to [-1, 1]$ such that $\sum_{i=1}^\infty a_N(i) = 0$ and $|a_N| \to \infty$,*

$$\mathbb{P}\left(\left| \frac{N}{|a_N|} \langle \mathbf{u}_k, a_N \mathbf{u}_k \rangle > \delta \right| \right) \to 0. \tag{3.9.1}$$

Proof. Letting $m_2 = \mathbb{E}\big[\mathcal{U}_\star(E)\big]$, we compute

$$\mathbb{E}\Big[\Big(\frac{N}{|a_N|}\langle \mathbf{u}_k, a_N \mathbf{u}_k\rangle\Big)^2\Big] = \frac{1}{|a_N|^2}\mathbb{E}\left[\left(\sum_{i=1}^{N} a_N(i)\big(N|\mathbf{u}_k(i)|^2 - m_2\big)\right)^2\right]$$
$$\leq \max_{\substack{i_1,i_2\in[1,N]\\ i_1\neq i_2}} \mathbb{E}\Big[\big(N|\mathbf{u}_k(i_1)|^2 - m_2\big)\big(N|\mathbf{u}_k(i_2)|^2 - m_2\big)\Big] + \frac{1}{|a_N|}\max_{i\in[1,N]} \mathbb{E}\left[\big(N|\mathbf{u}_k(i)|^2 - m_2\big)^2\right]. \tag{3.9.2}$$

The conclusion applies after applying Markov's inequality to the second moment computed in (3.9.2) and applying Theorem 3.1.7. The hypothesis that $|a_N| \to \infty$ ensures the second term in the second moment computation tends to zero. □

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