

## ON TWO THEOREMS REGARDING EXPONENTIAL STABILITY

*Pham Viet Hai*

There are two remarkable results in the theory of stability of dynamical systems which were obtained by E. A. BARBASHIN in 1967 and R. DATKO in 1970. After the seminal researches of DATKO and BARBASHIN, there has been a great number of works devoted to this results. For the case of discrete-time systems analogous results have been obtained. This paper will give the new versions which unify the discrete-time versions of BARBASHIN's theorem and DATKO's theorem.

### 1. INTRODUCTION

The theory of semigroups of linear operators has been developed extensively and the results have found many applications. In recent years an increasing interest in the study of evolution equations has been developed. The interest arises from a need to extend classical results on ordinary differential equations. In this paper, we consider the exponential stability of linear skew-evolution semiflows, generated by a new class of evolution equations. The used method is the integral condition. We associate a linear skew-evolution semiflow to an integral condition and then study the exponential stability in the neighborhood of this integral inequality. Since the present paper is closely related to previous articles we briefly recall some results proved in references.

One of the most famous results was given by R. DATKO. DATKO's theorem states that an evolution family  $\left\{U(t, s)\right\}_{t \geq s \geq 0}$  is uniformly exponentially stable

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if and only if there exists  $b \in \mathcal{N}$  (the set of all continuous, non-decreasing functions  $c : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  with  $c(0) = 0$  and  $c(t) > 0$  for each  $t > 0$ ) such that  $\sup_{s \geq 0} \int_s^\infty b(\|U(\tau, s)x\|) \, d\tau < \infty$  for every  $x \in \mathbb{X}$ . We have a weaker result as follows.

**Theorem 1.** *An evolution family  $\left\{U(t, s)\right\}_{t \geq s \geq 0}$  is uniformly exponentially stable if and only if there exists  $b \in \mathcal{N}$  such that  $\sup_{s \geq 0} \int_s^\infty b(\|U(\tau, s)\|) \, d\tau < \infty$ .*

Theorem 1 can be interpreted as the uniform DATKO’s condition. Analogous results for discrete-time case were first obtained by K. M. PRZYLUSKI, S. ROLEWICZ and Z. ZABCZYK in [12], [15]. An interesting idea has been presented by NEERVEN in [9], where he introduced the theory of Banach function spaces. He proved that a  $C_0$ -semigroup is exponentially stable if and only if all its orbits lie in a certain Banach function space over  $\mathbb{R}_+$ . Another approach was given in [11] where P. PREDA, A. POGAN and C. PREDA characterized the uniform exponential stability of evolution families in terms of existence of some functionals on sequence (function) spaces. In fact, they are merely generalizations of Banach sequence (function) spaces in [8]. Using functionals on sequence spaces, the authors generalized discrete-time versions due to K. M. PRZYLUSKI, S. ROLEWICZ and Z. ZABCZYK.

**Theorem 2.** *An evolution family  $\left\{U(t, s)\right\}_{t \geq s \geq 0}$  is uniformly exponentially stable if and only if there exists  $F \in \mathcal{F}$  such that  $A_F := \left\{x \in \mathbb{X} : \sup_{t_0 \geq 0} F(\varphi(x, t_0, \cdot)) < \infty\right\}$  is a set of the second Baire category, where  $\mathcal{F}$  is the set of all functionals  $F : \mathcal{S} \rightarrow [0, \infty]$  with the properties*

1.  $F(s_1) \leq F(s_2)$ , for every  $s_1 \leq s_2$ .
2. There exists a  $c > 0$  such that  $F(\alpha \mathcal{X}_{\{n\}}) \geq c\alpha$ , for every  $(\alpha, n) \in \mathbb{R}_+^* \times \mathbb{N}$ .
3.  $\lim_{n \rightarrow \infty} F(\alpha \mathcal{X}_{\{0, \dots, n\}}) = \infty$ , for all  $\alpha \in \mathbb{R}_+^*$ .

Here  $\mathcal{X}_A$  denotes the characteristic function of a set  $A$ .

Recently, Theorem 2 has been improved in [5] for a more general case.

**Theorem 3.** *The linear skew-product semiflow  $\pi_0 = (\Phi_0, \sigma_0)$  is uniformly exponentially stable if and only if there exists  $F \in \mathcal{F}$  and a non-decreasing sequence  $(t_n) \subset \mathbb{R}_+$  such that:*

1.  $\delta := \sup(t_{n+1} - t_n) < \infty$ .
2. The set  $A_F := \left\{x \in \mathbb{X} : \sup_{\theta, n} F(\varphi(x, \theta, n, \cdot)) < \infty\right\}$  is of the second Baire category, where

$$\varphi(x, \theta, n, j) = \begin{cases} \|\Phi_0(\theta, t_j)x\|, & \text{for } j \in \{0, \dots, n\}, \\ 0, & \text{for } j \notin \{0, \dots, n\}. \end{cases}$$

It is worth mentioning that in (uniform) DATKO's condition, the integrand is the first parameter of the evolution family. Integral characterizations with the second parameter as integrand were first introduced by BARBASHIN. BARBASHIN's theorem states that an evolution family  $\{U(t, s)\}_{t \geq s \geq 0}$  is uniformly exponentially stable if and only if  $\sup_{t \geq 0} \int_0^t \|U(t, \tau)\| d\tau < \infty$ . In fact, he formulated it for non-autonomous differential equations in the frame work of finite dimensional spaces:

$$(1) \quad U'(t) = A(t)U(t), U(0) = I, \text{ for } t \geq 0,$$

here  $A(\cdot)$  and  $U(\cdot)$  are quadratic matrices.  $I$  is the identity matrix of the same order as  $A(t)$  and the mapping  $t \rightarrow A(t)$  is continuous. This theorem was the starting point for the article [7], where the discrete-time version of BARBASHIN theorem was given.

**Theorem 4.** *The linear skew-evolution semiflow  $\pi$  is uniformly exponentially stable if and only if there exist  $F \in \mathcal{H}(\mathbb{N})$ ,  $\mathcal{K} > 0$ ,  $b \in \mathcal{N}$  and a non-decreasing sequence  $(t_n) \subset \mathbb{R}_+$  such that  $\sup_{n, m, \in \mathbb{N}, \theta \in \ominus} F(\varphi_b(\theta, m, n, \cdot)) \leq \mathcal{K}$ , where*

$$\varphi_b(\theta, m, n, j) = \begin{cases} b(\|\Phi(m + t_n, m + t_j, \sigma(m + t_j, m, \theta))\|), & \text{for } j \in \{0, \dots, n\}, \\ 0, & \text{for } j \notin \{0, \dots, n\}, \end{cases}$$

and  $\mathcal{H}(\mathbb{N})$  is the set of all functionals  $F : \mathcal{S} \rightarrow [0, \infty]$  with the following properties

1.  $F(s_1) \leq F(s_2)$ , for every  $s_1 \leq s_2$ .
2.  $\lim_{n \rightarrow \infty} \inf_{\alpha > 0} \frac{F(\alpha \mathcal{X}_{\{0, \dots, n\}})}{\alpha} = \infty$ .

The goal of this paper is to present the most general approach for both discrete-time versions of BARBASHIN's condition and uniform DATKO's condition. Thus, we extend the results in [7], [12], [15].

## 2. FUNCTIONALS ON SEQUENCE (FUNCTION) SPACES

In this section, the notations  $\mathcal{S}$  and  $\mathcal{M}$  stand respectively for the set of all positive sequences and functions. We write  $s_1 \leq s_2$  if  $s_1(j) \leq s_2(j)$  for every  $j \in \mathbb{K}$ . Here

$$\mathbb{K} = \begin{cases} \mathbb{N}, & \text{if } s_1, s_2 \in \mathcal{S}, \\ \mathbb{R}_+, & \text{if } s_1, s_2 \in \mathcal{M}. \end{cases}$$

This section is devoted to the basic material on functionals on sequence (function) spaces and their basic properties needed in the sequel. In each definition, illustrative examples will be discussed.

**Definition 1.**  $\mathbb{H}_{k, q}(\mathbb{N})$  is the set of all functionals  $F : \mathcal{S} \rightarrow [0, \infty]$  with the following properties

1.  $F(s_1) \leq F(s_2)$  for  $s_1 \leq s_2$ ,
2. there exists  $c > 0$  such that

$$(2) \quad F(\alpha \mathcal{X}_{\{n+k, \dots, n+k+q\}}) \geq c\alpha, \text{ for every } \alpha \geq 0,$$

3.

$$(3) \quad \lim_{n \rightarrow \infty} F(\alpha \mathcal{X}_{\{0, \dots, n\}}) = \infty, \text{ for every } \alpha > 0.$$

Among the important properties of the family  $H_{k, q}(\mathbb{N})$  is the following.

**Proposition 2.** For each  $L > 0$  and  $F \in \mathbb{H}_{k, q}(\mathbb{N})$ , we have

$$\lim_{n \rightarrow \infty} \inf_{\tau \in (0, L]} \frac{F(\tau \mathcal{X}_{\{0, \dots, n\}})}{\tau^2} = \infty.$$

**Proof.** Let  $a_n := \inf_{\tau \in (0, L]} \frac{F(\tau \mathcal{X}_{\{0, \dots, n\}})}{\tau^2}$ . Then  $(a_n)$  is a non-decreasing sequence. We need to prove that

$$(4) \quad \lim_{n \rightarrow \infty} a_n = \infty.$$

Assume that (4) does not hold. It follows that there exists an  $a_* \in (0, \infty)$  satisfying  $a_* = \lim_{n \rightarrow \infty} a_n = \sup_{n \in \mathbb{N}} a_n < \infty$ . For each  $n \in \mathbb{N}$ , by the definition of infimum,

there exists  $\beta_n \in (0, L]$  such that  $\frac{F(\beta_n \mathcal{X}_{\{0, \dots, n\}})}{\beta_n^2} \leq a_n + \frac{1}{n+1}$ . Now let  $n \geq k+q$ .

Using (2), we obtain

$$a_n + \frac{1}{n+1} \geq \frac{F(\beta_n \mathcal{X}_{\{0, \dots, n\}})}{\beta_n^2} \geq \frac{F(\beta_n \mathcal{X}_{\{n-q, \dots, n\}})}{\beta_n^2} \geq \frac{c\beta_n}{\beta_n^2} = \frac{c}{\beta_n},$$

which implies

$$(5) \quad \inf_{n \geq k+q} \beta_n > 0.$$

On the other hand,

$$(6) \quad \inf_{n \leq k+q} \beta_n = \min_{n \leq k+q} \beta_n > 0.$$

Combining (5) with (6), there exists  $\beta_* := \inf_{n \in \mathbb{N}} \beta_n > 0$ . Using this, we have

$$\begin{aligned} F(\beta_* \mathcal{X}_{\{0, \dots, n\}}) &\leq F(\beta_n \mathcal{X}_{\{0, \dots, n\}}) \leq L^2 \frac{F(\beta_n \mathcal{X}_{\{0, \dots, n\}})}{\beta_n^2} \\ &\leq L^2 \left( a_n + \frac{1}{n+1} \right) \leq L^2(a_* + 1). \end{aligned}$$

Taking the limit of both sides of the above inequality as  $n \rightarrow \infty$  we obtain a contradiction to the condition (3). This completes the proof.  $\square$

**Definition 3.**  $\mathbb{H}_{k, q}(\mathbb{R}_+)$  is the set of all functionals  $G : \mathcal{M} \rightarrow [0, \infty]$  with the following properties

1.  $G(f_1) \leq G(f_2)$  for  $f_1 \leq f_2$ ,
2. there exists  $c > 0$  such that

$$(7) \quad G(\alpha \mathcal{X}_{[n+k, n+k+q]}) \geq c\alpha, \text{ for every } \alpha \geq 0,$$

- 3.

$$(8) \quad \lim_{n \rightarrow \infty} G(\alpha \mathcal{X}_{[0, n]}) = \infty, \text{ for every } \alpha > 0.$$

**Definition 4.**  $\mathbf{H}(\mathbb{N})$  is the set of all functionals  $F : \mathcal{S} \rightarrow [0, \infty]$  with the properties:

1.  $F(s_1) \leq F(s_2)$ , for every  $s_1 \leq s_2$ .
2.  $\lim_{m \rightarrow \infty} \inf_{n \in \mathbb{N}, \alpha > 0} \frac{F(\alpha \mathcal{X}_{\{n, \dots, n+m\}})}{\alpha} = \infty$ .

Similarly, we define the family  $\mathbf{H}(\mathbb{R}_+)$  as follows

**Definition 5.**  $\mathbf{H}(\mathbb{R}_+)$  is the set of all functionals  $G : \mathcal{M} \rightarrow [0, \infty]$  with the properties:

1.  $G(f_1) \leq G(f_2)$ , for every  $f_1 \leq f_2$ .
2.  $\lim_{m \rightarrow \infty} \inf_{n \in \mathbb{N}, \alpha > 0} \frac{G(\alpha \mathcal{X}_{[n, n+m]})}{\alpha} = \infty$ .

EXAMPLE 6. A trivial example of a functional on sequence space which lies in  $\mathbb{H}_{k, q}(\mathbb{N})$  and  $\mathbf{H}(\mathbb{N})$ , is the series  $F_1(s) := \sum_{j=0}^{\infty} s(j)$ .

EXAMPLE 7. An example of functionals on sequence space which lies in  $\mathbb{H}_{k, q}(\mathbb{N})$  (but not in  $\mathcal{H}(\mathbb{N})$ ), is given by  $F_2 : \mathcal{S} \rightarrow [0, \infty]$  and

$$F_2(s) := \left( \prod_{j=0}^k s(j) \right) \left( \prod_{j=k+1}^{\infty} (1 + s(j)) \right).$$

One can easily check that  $F_2$  lies in  $\mathbb{H}_{k+1, q}(\mathbb{N})$  but does not lie in  $\mathcal{H}(\mathbb{N})$ .

EXAMPLE 8. An example of functional on sequence space which lies in  $\mathcal{H}(\mathbb{N})$  (but not in  $\mathbb{H}_{k, q}(\mathbb{N})$ ), is given by  $F_3 : \mathcal{S} \rightarrow [0, \infty]$  and

$$F_3(s) := \sum_{j=0}^{\infty} s((j+1)(k+q+1)).$$

One can easily check that  $F_3$  lies in  $\mathcal{H}(\mathbb{N})$  but does not lie in  $\mathbb{H}_{k, q}(\mathbb{N})$ .

EXAMPLE 9. Our last example is a functional  $G : \mathcal{M} \rightarrow [0, \infty]$  which lies in  $\mathbb{H}_{k, q}(\mathbb{R}_+)$  and  $\mathbf{H}(\mathbb{R}_+)$ . It is given by  $G(f) := \int_0^{\infty} f(\tau) d\tau$ .

REMARK 5. It is clear that  $\mathbf{H}(\mathbb{N}) \subset \mathcal{H}(\mathbb{N})$ ,  $\mathbf{H}(\mathbb{R}_+) \subset \mathcal{H}(\mathbb{R}_+)$  and  $\mathbb{H}_{0,0}(\mathbb{N}) = \mathcal{F}$ ,  $\mathbb{H}_{0,0}(\mathbb{R}_+) = \mathcal{G}$ .

Throughout this paper, the notation  $\mathcal{S}_\delta$  stands for the set of all non-decreasing sequence  $(t_n) \in \mathcal{S}$  provided that  $\delta := \sup_{n \in \mathbb{N}}(t_{n+1} - t_n) < \infty$ .

### 3. LINEAR SKEW-EVOLUTION SEMIFLOWS

As mentioned above, we prove the main results for linear skew-evolution semiflows. So we next come to the basic notations of linear skew-evolution semiflows. Let  $\mathbb{X}$  be a Banach space,  $\mathcal{L}(\mathbb{X})$ -the space of all bounded linear operators of  $\mathbb{X}$  into itself and  $T := \{(t, s), t \geq s \geq 0\}$ . We denote the norm of vectors on  $\mathbb{X}$  and operators on  $\mathcal{L}(\mathbb{X})$  by  $\|\cdot\|$ , and  $(\ominus, d)$  is a metric space.

**Definition 10.** An evolution semiflow on  $\ominus$  is a continuous mapping  $\sigma : T \times \ominus \rightarrow \ominus$  with the properties

1.  $\sigma(t, t, \theta) = \theta$ , for all  $\theta, t$ .
2.  $\sigma(t, s, \sigma(s, r, \theta)) = \sigma(t, r, \theta)$ , for all  $t \geq s \geq r \geq 0, \theta \in \ominus$ .

EXAMPLE 11.

1. Given a semiflow  $\sigma_0 : \ominus \times \mathbb{R}_+ \rightarrow \ominus$ , we can define an evolution semiflow  $\sigma : T \times \ominus \rightarrow \ominus$  as  $\sigma(t, s, \theta) := \sigma_0(\theta, t - s)$ .
2. The mapping  $\sigma(t, s, \theta) := \frac{t+1}{s+1} \theta$  is also an evolution semiflow, where  $\ominus = \mathbb{R}$ .

**Definition 12.** Suppose that  $\sigma$  is an evolution semiflow on  $\ominus$ . An evolution cocycle over evolution semiflow  $\sigma$  is an operator-valued function  $\Phi : T \times \ominus \rightarrow \mathcal{L}(\mathbb{X})$  with the following properties:

1.  $\Phi(t, t, \theta) = I$ , the identity operator on  $\mathbb{X}$ , for all  $(t, \theta) \in \mathbb{R}_+ \times \ominus$ .
2.  $\Phi(t, r, \theta) = \Phi(t, s, \sigma(s, r, \theta))\Phi(s, r, \theta)$ , for all  $t \geq s \geq r \geq 0$  and  $\theta \in \ominus$ .
3. In addition, there are  $M, \omega$  such that  $\|\Phi(t + s, s, \theta)x\| \leq Me^{\omega t} \|x\|$  for all  $(t, s, \theta, x) \in \mathbb{R}_+^2 \times \ominus \times \mathbb{X}$ .

**Definition 13.** The linear skew-evolution semiflow, associated with the above evolution cocycle, is the dynamical system  $\pi = (\sigma, \Phi)$  on  $\mathcal{E} := \mathbb{X} \times \ominus$  defined by  $\pi : \mathbb{X} \times \ominus \times T \rightarrow \mathbb{X} \times \ominus$ ,

$$\pi(x, \theta, t, s) := (\Phi(t, s, \theta)x, \sigma(t, s, \theta)).$$

Throughout the paper, we denote the constants  $M, \omega$  defined in Definition 12.

EXAMPLE 14. One can easily verify that  $C_0$ -semigroups, evolution families and linear skew-product semiflows are particular cases of linear skew-evolution semiflows.

EXAMPLE 15. The classical example of an evolution cocycle arises as the solution operator for a variational equation. Suppose that  $\sigma$  is an evolution semiflow on the compact metric space  $\ominus$  and  $\{A(\theta) : \theta \in \ominus\}$  be a family of linear operators on  $\mathbb{X}$ . If an evolution cocycle  $\Phi(\cdot, \cdot, \theta)x$  solves the variational equation

$$u'(t) = A(\sigma(t, t_0, \theta))u(t), \quad t \geq t_0$$

then the pair  $\pi = (\Phi, \sigma)$  is a linear skew-evolution semiflow.

EXAMPLE 16 (See [7]). Let  $\pi = (\Phi, \sigma)$  to be a linear skew-evolution semiflow on  $\mathcal{E}$ . We can define some linear skew-evolution semiflows in the following ways.

1. The pair  $\pi_\beta = (\Phi_\beta, \sigma)$ , where  $\Phi_\beta(t, s, \theta) = e^{-\beta(t-s)}\Phi(t, s, \theta)$ , is also a linear skew-evolution semiflow on  $\mathcal{E}$ .
2. If  $P : \ominus \rightarrow \mathcal{L}(\mathbb{X})$  is a strongly continuous bounded mapping, there is an unique linear skew-evolution semiflow  $\pi_P = (\Phi_P, \sigma_P)$  on  $\mathcal{E}$  such that

$$\Phi_P(t, s, \theta)x = \Phi(t, s, \theta)x + \int_s^t \Phi(t, \tau, \sigma(\tau, s, \theta))P(\sigma(\tau, s, \theta))\Phi_P(\tau, s, \theta)x \, d\tau,$$

for  $t \geq s \geq 0$  and  $(\theta, x) \in \ominus \times \mathbb{X}$ .

EXAMPLE 17. Let  $\{U(t, s)\}_{t \geq s \geq 0}$  be an evolution family and  $\{P(\theta)\}_{\theta \in \ominus}$  a bounded strongly continuous family of idempotent operators with the property that

$$P(\theta)U(t, s) = U(t, s)P(\theta), \quad t \geq s \geq 0, \quad \theta \in \ominus$$

then the pair  $\pi = (\Phi, \sigma)$  defined by

$$\sigma(t, s, \theta) = \theta, \quad \Phi(t, s, \theta) = P(\theta)U(t, s), \quad t \geq s \geq 0, \quad \theta \in \ominus$$

is a linear skew-evolution semiflow.

**Definition 18.** *The linear skew-evolution semiflow  $\pi = (\Phi, \sigma)$  is said to be uniformly exponentially stable if there exist  $K, \nu > 0$  such that  $\|\Phi(t + s, s, \theta)x\| \leq Ke^{-\nu t} \|x\|$  for all  $(t, s, \theta, x) \in \mathbb{R}_+^2 \times \ominus \times \mathbb{X}$ .*

The next two lemma contain key arguments for what follows.

**Lemma 6.** *If there exist two constants  $p > 0$  and  $c \in (0, 1)$  such that  $\|\Phi(p + m, m, \theta)\| \leq c$  for all  $(\theta, m) \in \ominus \times \mathbb{N}$  then the linear skew-evolution semiflow  $\pi = (\Phi, \sigma)$  is uniformly exponentially stable.*

**Proof.** See [4], [6]. □

In the following, instead of  $\sup_{n, m \in \mathbb{N}, \theta \in \ominus}$  we write  $\sup$ .

**Lemma 7.** *If there exist  $\delta > 0$  and  $(c_n) \in \mathcal{S}_\delta$  satisfying the hypotheses*

1.  $\lim_{n \rightarrow \infty} c_n = \infty$ ,
2.  $\ell := \sup \|\Phi(c_n + m, m, \theta)\| < \infty$ ,

then the linear skew-evolution semiflow  $\pi = (\Phi, \sigma)$  is uniformly bounded.

**Proof.** First, we prove that  $\sup_{t \in \mathbb{R}^+} \sup_{m \in \mathbb{N}} \|\Phi(t + m, m, \theta)\| < \infty$ . Indeed, for each  $t \in \mathbb{R}^+$ , we consider the two following cases. The first case is  $t \leq c_0$ . In this case, we always have  $\|\Phi(t + m, m, \theta)\| \leq Me^{\omega c_0}$ . The second case is  $t \geq c_0$ . Since  $\lim_{n \rightarrow \infty} c_n = \infty$  we can find an  $n \in \mathbb{N}$  such that  $t \in [c_n, c_{n+1}]$ . Thus we have

$$\|\Phi(t + m, m, \theta)\| \leq Me^{\omega(t - c_n)} \|\Phi(c_n + m, m, \theta)\| \leq Mle^{\omega(c_{n+1} - c_n)} \leq Mle^{\omega\delta}.$$

In the next step we prove that  $\sup_{t \in \mathbb{R}^+} \sup_{s \in \mathbb{R}^+} \|\Phi(t + s, s, \theta)\|$ . For each pair  $(t, s) \in \mathbb{R}_+^2$ , we also consider two cases. The first case is  $t \leq 1$ . In this case, we always have  $\|\Phi(t + s, s, \theta)\| \leq Me^{\omega t} \leq Me^{\omega}$ . The second case is  $t \geq 1$ . Then  $t + s \geq 1 + [s] > s$ . We have

$$\Phi(t + s, s, \theta) = \Phi(t + s, 1 + [s], \sigma(1 + [s], s, \theta))\Phi(1 + [s], s, \theta).$$

So,

$$\begin{aligned} \|\Phi(t + s, s, \theta)\| &\leq \|\Phi(t + s, 1 + [s], \sigma(1 + [s], s, \theta))\| \|\Phi(1 + [s], s, \theta)\| \\ &\leq Me^{\omega(1 + [s] - s)} \|\Phi(t + s, 1 + [s], \sigma(1 + [s], s, \theta))\| \\ &\leq Me^{\omega} \|\Phi(t + s, 1 + [s], \sigma(1 + [s], s, \theta))\| \\ &\leq Me^{\omega} \max\{Me^{\omega c_0}, Mle^{\omega\delta}\}. \end{aligned}$$

This implies the desired result. □

#### 4. MAIN RESULTS

In this section, some necessary and sufficient conditions for uniform exponential stability are given in terms of the existence of some functionals on sequence spaces. We present how discrete methods can be used in order to characterize the asymptotic properties. We start with a characterization regarding the family  $\mathbf{H}(\mathbb{N})$ .

**Theorem 8.** *The linear skew-evolution semiflow  $\pi = (\Phi, \sigma)$  is uniformly exponentially stable if and only if there exist  $\alpha, \beta, \mathcal{K}, L_1, L_2 > 0$ ;  $(a_n) \in \mathcal{S}_\alpha$ ,  $(b_n) \in \mathcal{S}_\beta$ ;  $b \in \mathcal{N}$  and  $F \in \mathbf{H}(\mathbb{N})$  such that*

1.  $\sum_{j=0}^{\infty} (a_j + b_j) > 0$ ,
2.  $\sup F(\varphi_b(\theta, m, n, \cdot)) \leq \mathcal{K} < \infty$ ,
3.  $\sup \|\Phi(a_n + b_n + m, a_n + b_j + m, \sigma(a_n + b_j + m, m, \theta))\| \leq L_1$ ,
4.  $\sup \|\Phi(a_n + m, m, \theta)\| \leq L_2$ ,

where

$$\varphi_b(\theta, m, n, j) = \begin{cases} b(\|\Phi(a_n + b_j + m, a_j + m, \sigma(a_j + m, m, \theta))\|), & \text{for } j \in \{0, \dots, n\}, \\ 0, & \text{for } j \notin \{0, \dots, n\}. \end{cases}$$

**Proof.** Necessity. Suppose that  $\pi = (\Phi, \sigma)$  is uniformly exponentially stable and let  $K, \nu$  be two constants given in Definition 18. We have

$$\begin{aligned} \sum_{j=0}^n \|\Phi(n+j+m, j+m, \sigma(j+m, m, \theta))\| &\leq \sum_{j=0}^m Ke^{-\nu n} = \frac{K(n+1)}{e^{\nu n}} \\ &\leq K\frac{1}{\nu} + K\left|1 - \frac{1}{\nu}\right|. \end{aligned}$$

Thus we only take  $a_j = b_j = j$ ;  $\alpha = \beta = 1$ ;  $\mathcal{K} = K\frac{1}{\nu} + K\left|1 - \frac{1}{\nu}\right|$ ;  $L_1 = L_2 = K$ ;  $F(s) = \sum_{n=0}^{\infty} s(n)$ ;  $k = q = 0$ ;  $b(t) = t$ .

Sufficiency. Since  $\sum_{j=0}^{\infty} (a_j + b_j) > 0$  there exists  $\ell \in \mathbb{N}$  such that  $a_\ell + b_\ell > 0$ . Using the hypothesis (2) of Definition 4, we can fix the natural number  $k_1 \geq \ell$  such that

$$(9) \quad F(\alpha \mathcal{X}_{\{n, \dots, n+k_1\}}) \geq \frac{\mathcal{K}\alpha}{b\left(\frac{1}{2L_1L_2}\right)},$$

for every  $n \in \mathbb{N}$  and  $\alpha \in \mathbb{R}_+$ . For all arbitrary  $j \in \{0, \dots, k_1\}$ , we have

$$\begin{aligned} \Phi(a_{k_1} + b_{k_1} + m, m, \theta) &= \Phi(a_{k_1} + b_{k_1} + m, a_{k_1} + b_j + m, \sigma(a_{k_1} + b_j + m, m, \theta)) \\ &\quad \Phi(a_{k_1} + b_j + m, a_j + m, \sigma(a_j + m, m, \theta))\Phi(a_j + m, m, \theta). \end{aligned}$$

Thus,

$$\|\Phi(a_{k_1} + b_{k_1} + m, m, \theta)\| \leq L_1L_2 \|\Phi(a_{k_1} + b_j + m, a_j + m, \sigma(a_j + m, m, \theta))\|.$$

This implies that

$$\varphi_b(\theta, m, k_1, \cdot) \geq b\left(\frac{\|\Phi(a_{k_1} + b_{k_1} + m, m, \theta)\|}{L_1L_2}\right) \mathcal{X}_{\{0, \dots, k_1\}}.$$

Hence we have

$$\begin{aligned} \mathcal{K} \geq F(\varphi_b(\theta, m, k_1, \cdot)) &\geq F\left(b\left(\frac{\|\Phi(a_{k_1} + b_{k_1} + m, m, \theta)\|}{L_1L_2}\right) \mathcal{X}_{\{0, \dots, k_1\}}\right) \\ &\geq b\left(\frac{\|\Phi(a_{k_1} + b_{k_1} + m, m, \theta)\|}{L_1L_2}\right) \frac{\mathcal{K}}{b\left(\frac{1}{2L_1L_2}\right)}. \end{aligned}$$

This leads us to the following estimate:

$$(10) \quad \|\Phi(a_{k_1} + b_{k_1} + m, m, \theta)\| \leq \frac{1}{2},$$

for every  $(\theta, m) \in \ominus \times \mathbb{N}$ , taking into account that  $a_{k_1} + b_{k_1}$  does not depend on  $\theta, m$ . On the other hand, we have  $a_{k_1} + b_{k_1} \geq a_\ell + b_\ell > 0$ . Using Lemma 6, we have that  $\pi = (\Phi, \sigma)$  is uniformly exponentially stable.  $\square$

Using Theorem 8, we can prove the following result.

**Theorem 9.** *The linear skew-evolution semiflow  $\pi = (\Phi, \sigma)$  is uniformly exponentially stable if and only if there exist  $\alpha, \beta, \mathcal{K} > 0$ ;  $(a_n) \in \mathcal{S}_\alpha$ ,  $(b_n) \in \mathcal{S}_\beta$ ;  $b \in \mathcal{N}$  and  $F \in \mathbf{H}(\mathbb{N})$  such that*

1.  $\sum_{j=0}^{\infty} (a_j + b_j) > 0$ .
2.  $\sup F(\varphi_b(\theta, m, n, \cdot)) \leq \mathcal{K} < \infty$ ,

where

$$\varphi_b(\theta, m, n, j) = \begin{cases} b(\|\Phi(a_n + b_j + m, a_j + m, \sigma(a_j + m, m, \theta))\|), & \text{for } j \in \{0, \dots, n\}, \\ 0, & \text{for } j \notin \{0, \dots, n\}. \end{cases}$$

**Proof.** Necessity. This part of the proof is similar to that of Theorem 8 and hence is omitted.

Sufficiency. Since  $\sum_{j=0}^{\infty} (a_j + b_j) > 0$  there is  $\ell \in \mathbb{N}$  such that  $a_\ell + b_\ell > 0$ . From the hypothesis (2) of Definition 4, we can fix  $k \geq \ell$  such that

$$(11) \quad F(\alpha \mathcal{X}_{\{n, \dots, n+k\}}) \geq \frac{\mathcal{K}\alpha}{b(1)},$$

for every  $n \in \mathbb{N}$  and  $\alpha \in \mathbb{R}_+$ . We divide the proof into three cases depending on the convergence of sequences  $(a_n)$ ,  $(b_n)$ .

**Case 1.**  $\sup_{n \in \mathbb{N}} a_n = \infty$ .

In this case, we show that  $\|\Phi(a_n + b_k + m, m, \theta)\|$  is uniformly bounded. Indeed, for each  $n \in \mathbb{N}$ , there are two subcases which may occur. The first subcase is  $n \leq k$ . It is clear that

$$(12) \quad \|\Phi(a_n + b_k + m, m, \theta)\| \leq M e^{\omega(a_k + b_k)}.$$

The second subcase is  $n \geq k$ . For all arbitrary  $j \in \{0, \dots, k\}$ , we have

$$\begin{aligned} \Phi(a_n + b_k + m, m, \theta) &= \Phi(a_n + b_k + m, a_n + b_j + m, \sigma(a_n + b_j + m, m, \theta)) \\ &\quad \Phi(a_n + b_j + m, a_j + m, \sigma(a_j + m, m, \theta))\Phi(a_j + m, m, \theta), \end{aligned}$$

which implies

$$\begin{aligned} &\|\Phi(a_n + b_k + m, m, \theta)\| \\ &\leq M e^{\omega(b_k - b_j)} \|\Phi(a_n + b_j + m, a_j + m, \sigma(a_j + m, m, \theta))\| M e^{\omega a_j} \\ &\leq M^2 e^{\omega(a_k + b_k)} \|\Phi(a_n + b_j + m, a_j + m, \sigma(a_j + m, m, \theta))\|. \end{aligned}$$

We can rewrite

$$\varphi_b(\theta, m, n, j) \geq b \left( \frac{\|\Phi(a_n + b_k + m, m, \theta)\|}{M^2 e^{\omega(a_k + b_k)}} \right),$$

or

$$\varphi_b(\theta, m, n, \cdot) \geq b \left( \frac{\|\Phi(a_n + b_k + m, m, \theta)\|}{M^2 e^{\omega(a_k + b_k)}} \right) \mathcal{X}_{\{0, \dots, k\}}.$$

From (11), we can follow

$$\begin{aligned} \mathcal{K} \geq F(\varphi_b(\theta, m, n, \cdot)) &\geq F \left( b \left( \frac{\|\Phi(a_n + b_k + m, m, \theta)\|}{M^2 e^{\omega(a_k + b_k)}} \right) \mathcal{X}_{\{0, \dots, k\}} \right) \\ &\geq b \left( \frac{\|\Phi(a_n + b_k + m, m, \theta)\|}{M^2 e^{\omega(a_k + b_k)}} \right) \frac{\mathcal{K}}{b(1)}. \end{aligned}$$

This yields

$$(13) \quad \|\Phi(a_n + b_k + m, m, \theta)\| \leq M^2 e^{\omega(a_k + b_k)}.$$

(12) and (13) lead us to use Lemma 7 for  $c_n := a_n + b_k$ . Thus, there exists  $L > 0$  such that  $\sup \|\Phi(t + s, s, \theta)\| \leq L < \infty$ . Using Theorem 8,  $\pi$  is uniformly exponentially stable. We now consider the next case.

**Case 2.**  $\mathcal{A} := \sup_{n \in \mathbb{N}} a_n < \infty$  and  $\sup b_n = \infty$ .

For every  $n \in \mathbb{N}$ , we also consider two subcases to show the uniform boundedness of  $\|\Phi(a_n + b_n + m, m, \theta)\|$ . The first subcase is  $n \leq k$ . It is clear that

$$(14) \quad \|\Phi(a_n + b_n + m, m, \theta)\| \leq M^2 e^{\omega(a_k + b_k)}.$$

The second subcase is  $n \geq k$ . For all arbitrary  $j \in \{n - k, \dots, n\}$ , we have

$$\begin{aligned} \Phi(a_n + b_n + m, m, \theta) &= \Phi(a_n + b_n + m, a_n + b_j + m, \sigma(a_n + b_j + m, m, \theta)) \\ &\quad \Phi(a_n + b_j + m, a_j + m, \sigma(a_j + m, m, \theta)) \Phi(a_j + m, m, \theta). \end{aligned}$$

So

$$\begin{aligned} &\|\Phi(a_n + b_n + m, m, \theta)\| \\ &\leq M e^{\omega(b_n - b_j)} \|\Phi(a_n + b_j + m, a_j + m, \sigma(a_j + m, m, \theta))\| M e^{\omega a_j} \\ &\leq M^2 e^{\omega(\mathcal{A} + \beta(n-j))} \|\Phi(a_n + b_j + m, a_j + m, \sigma(a_j + m, m, \theta))\| \\ &\leq M^2 e^{\omega(\mathcal{A} + \beta k)} \|\Phi(a_n + b_j + m, a_j + m, \sigma(a_j + m, m, \theta))\|. \end{aligned}$$

It is easy to conclude that

$$\varphi_b(\theta, m, n, \cdot) \geq b \left( \frac{\|\Phi(a_n + b_n + m, m, \theta)\|}{M^2 e^{\omega(\mathcal{A} + \beta k)}} \right) \mathcal{X}_{\{n-k, \dots, n\}}.$$

Thus we obtain

$$\begin{aligned} \mathcal{K} \geq F(\varphi_b(\theta, m, n, \cdot)) &\geq F \left( b \left( \frac{\|\Phi(a_n + b_n + m, m, \theta)\|}{M^2 e^{\omega(\mathcal{A} + \beta k)}} \right) \mathcal{X}_{\{n-k, \dots, n\}} \right) \\ &\geq b \left( \frac{\|\Phi(a_n + b_n + m, m, \theta)\|}{M^2 e^{\omega(\mathcal{A} + \beta k)}} \right) \frac{\mathcal{K}}{b(1)}. \end{aligned}$$

The last estimate is equivalent to

$$(15) \quad \|\Phi(a_n + b_n + m, m, \theta)\| \leq M^2 e^{\omega(A+\beta k)}.$$

According to Lemma 7, there exists  $L > 0$  such that  $\sup \|\Phi(t + s, s, \theta)\| \leq L$ . Using Theorem 8,  $\pi$  is uniformly exponentially stable. We now consider the final case.

**Case 3.**  $\mathcal{A} := \sup a_n < \infty$  and  $\mathcal{B} := \sup b_n < \infty$ .

We have

$$(16) \quad \|\Phi(a_n + b_n + m, a_n + b_j + m, \sigma(a_n + b_j + m, m, \theta))\| \leq M e^{\omega(b_n - b_j)} \leq M e^{\omega \mathcal{B}},$$

$$(17) \quad \|\Phi(a_n + m, m, \theta)\| \leq M e^{\omega a_n} \leq M e^{\omega \mathcal{A}}.$$

(16) and (17) show that we can use Theorem 8 for the case  $L_1 := M e^{\omega \mathcal{B}}$  and  $L_2 := M e^{\omega \mathcal{A}}$ . This completes the proof.  $\square$

The following corollary is immediate.

**Corollary 19.** *The linear skew-evolution semiflow  $\pi = (\Phi, \sigma)$  is uniformly exponentially stable if and only if there exist  $\alpha, \beta, \mathcal{K} > 0$ ;  $(a_n) \in \mathcal{S}_\alpha$ ,  $(b_n) \in \mathcal{S}_\beta$ ;  $b \in \mathcal{N}$  such that*

$$(18) \quad \sup \sum_{j=0}^n b (\|\Phi(a_n + b_j + m, a_j + m, \sigma(a_j + m, m, \theta))\|) \leq \mathcal{K} < \infty.$$

REMARK 10. From Corollary 19, we have two important remarks as follows:

1. If  $b_j = 0$  then Corollary 19 is BARBASHIN's condition.
2. If  $a_j = 0$  then Corollary 19 is uniform DATKO's condition.

Next we replace the family  $\mathbf{H}(\mathbb{N})$  with  $\mathbb{H}_{k, q}(\mathbb{N})$  to obtain similar results as Theorem 8 and Theorem 9.

**Theorem 11.** *The linear skew-evolution semiflow  $\pi = (\Phi, \sigma)$  is uniformly exponentially stable if and only if there exist  $\alpha, \beta, \mathcal{K}, L_1, L_2 > 0$ ;  $(a_n) \in \mathcal{S}_\alpha$ ,  $(b_n) \in \mathcal{S}_\beta$  and  $k, q \in \mathbb{N}$ ;  $F \in \mathbb{H}_{k, q}(\mathbb{N})$  such that*

1.  $\sum_{j=0}^{\infty} (a_j + b_j) > 0$ ,
2.  $\sup F(\varphi(\theta, m, n, \cdot)) \leq \mathcal{K} < \infty$ ,
3.  $\sup \|\Phi(a_n + b_n + m, a_n + b_j + m, \sigma(a_n + b_j + m, m, \theta))\| \leq L_1$ ,
4.  $\sup \|\Phi(a_n + m, m, \theta)\| \leq L_2$ ,
5.  $\sup \|\Phi(a_n + b_n + m, m, \theta)\| \leq L_3$ ,

where

$$\varphi(\theta, m, n, j) = \begin{cases} \|\Phi(a_n + b_j + m, a_j + m, \sigma(a_j + m, m, \theta))\|, & \text{for } j \in \{0, \dots, n\}, \\ 0, & \text{for } j \notin \{0, \dots, n\}. \end{cases}$$

**Proof.** Necessity. Suppose that  $\pi = (\Phi, \sigma)$  is uniformly exponentially stable and let  $K, \nu$  be two constants given by Definition 18. We have that

$$\begin{aligned} \sum_{j=0}^n \|\Phi(n+j+m, j+m, \sigma(j+m, m, \theta))\| &\leq \sum_{j=0}^m Ke^{-\nu n} = \frac{K(n+1)}{e^{\nu n}} \\ &\leq K\frac{1}{\nu} + K\left|1 - \frac{1}{\nu}\right|. \end{aligned}$$

Thus we only take  $a_j = b_j = j; \alpha = \beta = 1; \mathcal{K} = K\frac{1}{\nu} + K\left|1 - \frac{1}{\nu}\right|; L_1 = L_2 = L_3 = K;$

$$F(s) = \sum_{n=0}^{\infty} s(n); k = q = 0.$$

Sufficiency. Since  $\sum_{j=0}^{\infty} (a_j + b_j) > 0$  there exists  $\ell \in \mathbb{N}$  such that  $a_\ell + b_\ell > 0$ .

Using Proposition 2, we can choose  $n_0 > \ell$  such that

$$(19) \quad \frac{F(\tau \mathcal{X}_{\{0, \dots, n_0\}})}{\tau^2} \geq 4\mathcal{K}L_1^2L_2^2,$$

for every  $\tau \in \left(0, \frac{L_3}{L_1L_2}\right]$ . For all arbitrary  $j \in \{0, \dots, n_0\}$ , we have

$$\begin{aligned} &\Phi(a_{n_0} + b_{n_0} + m, m, \theta) \\ &= \Phi(a_{n_0} + b_{n_0} + m, a_{n_0} + b_j + m, \sigma(a_{n_0} + b_j + m, m, \theta)) \\ &\Phi(a_{n_0} + b_j + m, a_j + m, \sigma(a_j + m, m, \theta))\Phi(a_j + m, m, \theta). \end{aligned}$$

So

$$\|\Phi(a_{n_0} + b_{n_0} + m, m, \theta)\| \leq L_1L_2 \|\Phi(a_{n_0} + b_j + m, a_j + m, \sigma(a_j + m, m, \theta))\|.$$

This leads to

$$(20) \quad \varphi(\theta, m, n_0, \cdot) \geq \left(\frac{\|\Phi(a_{n_0} + b_{n_0} + m, m, \theta)\|}{L_1L_2}\right) \mathcal{X}_{\{0, \dots, n_0\}}.$$

Applying  $F$  on both sides of (20), we have

$$\begin{aligned} \mathcal{K} &\geq F\left(\left(\frac{\|\Phi(a_{n_0} + b_{n_0} + m, m, \theta)\|}{L_1L_2}\right) \mathcal{X}_{\{0, \dots, n_0\}}\right) \\ &\geq \left(\frac{\|\Phi(a_{n_0} + b_{n_0} + m, m, \theta)\|}{L_1L_2}\right)^2 4\mathcal{K}L_1^2L_2^2, \text{ using (19).} \end{aligned}$$

One can easily conclude that  $\|\Phi(a_{n_0} + b_{n_0} + m, m, \theta)\| \leq \frac{1}{2}$ . On the other hand,  $a_{n_0} + b_{n_0} \geq a_\ell + b_\ell > 0$ . This means that all conditions of Lemma 6 work. Thus  $\pi$  is uniformly exponentially stable.  $\square$

**Theorem 12.** *The linear skew-evolution semiflow  $\pi = (\Phi, \sigma)$  is uniformly exponentially stable if and only if there exist  $\alpha, \beta, \mathcal{K} > 0$ ;  $(a_n) \in \mathcal{S}_\alpha$ ,  $(b_n) \in \mathcal{S}_\beta$  and  $k, q \in \mathbb{N}$ ;  $F \in \mathbb{H}_{k, q}(\mathbb{N})$  such that*

1.  $\sum_{j=0}^{\infty} (a_j + b_j) > 0$ ,
2.  $\sup F(\varphi(\theta, m, n, \cdot)) \leq \mathcal{K} < \infty$ ,

where

$$\varphi(\theta, m, n, j) = \begin{cases} \|\Phi(a_n + b_j + m, a_j + m, \sigma(a_j + m, m, \theta))\|, & \text{for } j \in \{0, \dots, n\}, \\ 0, & \text{for } j \notin \{0, \dots, n\}. \end{cases}$$

**Proof.** The techniques used to prove the necessity are similar to those used in the necessity part of Theorem 11. Thus it remains to prove the sufficiency. We consider the following three cases.

**Case 1.**  $\sup a_n = \infty$ . For each  $n \in \mathbb{N}$ , there are two subcases. The first subcase is  $n \leq k + q$ . Then

$$\|\Phi(a_n + b_{k+q} + m, m, \theta)\| \leq M e^{\omega(a_{k+q} + b_{k+q})}.$$

The second subcase is  $n \geq k + q$ . For all arbitrary  $j \in \{k, \dots, k + q\}$  we have

$$\begin{aligned} & \Phi(a_n + b_{k+q} + m, a_k + m, \sigma(a_k + m, m, \theta)) \\ &= \Phi(a_n + b_{k+q} + m, a_n + b_j + m, \sigma(a_n + b_j + m, m, \theta)) \\ & \Phi(a_n + b_j + m, a_j + m, \sigma(a_j + m, m, \theta)) \Phi(a_j + m, a_k + m, \sigma(a_k + m, m, \theta)). \end{aligned}$$

This yields

$$\begin{aligned} & \|\Phi(a_n + b_{k+q} + m, a_k + m, \sigma(a_k + m, m, \theta))\| \\ & \leq M e^{\omega b_{k+q}} \|\Phi(a_n + b_j + m, a_j + m, \sigma(a_j + m, m, \theta))\| M e^{\omega a_j} \\ & \leq M^2 e^{\omega(a_{k+q} + b_{k+q})} \|\Phi(a_n + b_j + m, a_j + m, \sigma(a_j + m, m, \theta))\|, \end{aligned}$$

It follows that

$$\begin{aligned} \varphi(\theta, m, n, \cdot) & \geq \varphi(\theta, m, n, \cdot) \mathcal{X}_{\{k, \dots, k+q\}} \\ & \geq \left( \frac{\|\Phi(a_n + b_{k+q} + m, a_k + m, \sigma(a_k + m, m, \theta))\|}{M^2 e^{\omega(a_{k+q} + b_{k+q})}} \right) \mathcal{X}_{\{k, \dots, k+q\}}. \end{aligned}$$

We obtain

$$\begin{aligned} \mathcal{K} & \geq F(\varphi(\theta, m, n, \cdot)) \\ & \geq F\left(\left(\frac{\|\Phi(a_n + b_{k+q} + m, a_k + m, \sigma(a_k + m, m, \theta))\|}{M^2 e^{\omega(a_{k+q} + b_{k+q})}}\right) \mathcal{X}_{\{k, \dots, k+q\}}\right) \\ & \geq \left(\frac{c}{M^2 e^{\omega(a_{k+q} + b_{k+q})}}\right) \|\Phi(a_n + b_{k+q} + m, a_k + m, \sigma(a_k + m, m, \theta))\|. \end{aligned}$$

Thus,

$$\|\Phi(a_n + b_{k+q} + m, a_k + m, \sigma(a_k + m, m, \theta))\| \leq \frac{\mathcal{K}M^2 e^{\omega(a_{k+q} + b_{k+q})}}{c}.$$

From

$$\Phi(a_n + b_{k+q} + m, m, \theta) = \Phi(a_n + b_{k+q} + m, a_k + m, \sigma(a_k + m, m, \theta))\Phi(a_k + m, m, \theta),$$

we have

$$\begin{aligned} \|\Phi(a_n + b_{k+q} + m, m, \theta)\| &\leq \|\Phi(a_n + b_{k+q} + m, a_k + m, \sigma(a_k + m, m, \theta))\| \\ &\quad \times \|\Phi(a_k + m, m, \theta)\| \\ &\leq \left( \frac{\mathcal{K}M^2 e^{\omega(a_{k+q} + b_{k+q})}}{c} \right) M e^{\omega a_k}. \end{aligned}$$

Therefore,

$$\|\Phi(a_n + b_{k+q} + m, m, \theta)\| \leq \max \left\{ \left( \frac{\mathcal{K}M^2 e^{\omega(a_{k+q} + b_{k+q})}}{c} \right) M e^{\omega a_k}, M e^{\omega(a_{k+q} + b_{k+q})} \right\}.$$

Using Lemma 7 for  $c_n := a_n + b_{k+q}$ , we get the uniform boundedness of  $\Phi(t+s, s, \theta)$ . Assume that

$$\sup \|\Phi(t+s, s, \theta)\| \leq L.$$

We now use Theorem 11 for  $L_1 := L$  and  $L_2 := L$ .

**Case 2.**  $\mathcal{A} := \sup a_n < \infty$  and  $\sup b_n = \infty$ .

Now we show that  $\|\Phi(a_n + b_n + m, m, \theta)\|$  is uniformly bounded. For each  $n \in \mathbb{N}$ , there are two subcases which may occur. The first subcase is  $n \leq k + q$ . Then

$$(21) \quad \|\Phi(a_n + b_n + m, m, \theta)\| \leq M e^{\omega(a_{k+q} + b_{k+q})}.$$

The second subcase is  $n \geq k + q$ . For all arbitrary  $j \in \{n - q, \dots, n\}$ , we have

$$\begin{aligned} &\Phi(a_n + b_n + m, m, \theta) \\ &= \Phi(a_n + b_n + m, a_n + b_j + m, \sigma(a_n + b_j + m, m, \theta)) \\ &\quad \Phi(a_n + b_j + m, a_j + m, \sigma(a_j + m, m, \theta))\Phi(a_j + m, m, \theta). \end{aligned}$$

From

$$\begin{aligned} &\|\Phi(a_n + b_n + m, m, \theta)\| \\ &\leq M e^{\omega(b_n - b_j)} \|\Phi(a_n + b_j + m, a_j + m, \sigma(a_j + m, m, \theta))\| M e^{\omega a_j}, \\ &\leq M^2 e^{\omega \mathcal{A}} e^{\omega \beta(n-j)} \|\Phi(a_n + b_j + m, a_j + m, \sigma(a_j + m, m, \theta))\| \\ &\leq M^2 e^{\omega(\mathcal{A} + \beta q)} \|\Phi(a_n + b_j + m, a_j + m, \sigma(a_j + m, m, \theta))\|, \end{aligned}$$

we have

$$\varphi(\theta, m, n, \cdot) \geq \varphi(\theta, m, n, \cdot) \mathcal{X}_{\{n-q, \dots, n\}} \geq \left( \frac{\|\Phi(a_n + b_n + m, m, \theta)\|}{M^2 e^{\omega(\mathcal{A} + \beta q)}} \right) \mathcal{X}_{\{n-q, \dots, n\}}.$$

We obtain

$$\begin{aligned} \mathcal{K} &\geq F \left( \left( \frac{\|\Phi(a_n + b_n + m, m, \theta)\|}{M^2 e^{\omega(\mathcal{A} + \beta q)}} \right) \mathcal{X}_{\{n-q, \dots, n\}} \right) \\ &\geq \left( \frac{c}{M^2 e^{\omega(\mathcal{A} + \beta q)}} \right) \|\Phi(a_n + b_n + m, m, \theta)\|. \end{aligned}$$

Thus,

$$(22) \quad \|\Phi(a_n + b_n + m, m, \theta)\| \leq \frac{\mathcal{K} M^2 e^{\omega(\mathcal{A} + \beta q)}}{c}.$$

Since (22) and (21), we have

$$\|\Phi(a_n + b_n + m, m, \theta)\| \leq \max \left\{ \frac{\mathcal{K} M^2 e^{\omega(\mathcal{A} + \beta q)}}{c}, M e^{\omega(a_k + q + b_{k+q})} \right\}.$$

Once again, using Lemma 7, we obtain the uniform boundedness of  $\Phi(t + s, s, \theta)$ . From Theorem 11, we obtain the uniform exponential stability of  $\pi$ .

**Case 3.**  $\mathcal{A} := \sup a_n < \infty$  and  $\mathcal{B} := \sup b_n < \infty$ .

With the conditions above, we have

$$(23) \quad \|\Phi(a_n + b_n + m, a_n + b_j + m, \sigma(a_n + b_j + m, m, \theta))\| \leq M e^{\omega(b_n - b_j)} \leq M e^{\omega \mathcal{B}},$$

$$(24) \quad \|\Phi(a_n + m, m, \theta)\| \leq M e^{\omega a_n} \leq M e^{\omega \mathcal{A}},$$

and

$$(25) \quad \|\Phi(a_n + b_n + m, m, \theta)\| \leq M e^{\omega(a_n + b_n)} \leq M e^{\omega(\mathcal{A} + \mathcal{B})}.$$

We now use Theorem 11 for  $L_1 := M e^{\omega \mathcal{B}}$ ;  $L_2 := M e^{\omega \mathcal{A}}$  and  $L_3 := M e^{\omega(\mathcal{A} + \mathcal{B})}$ . This completes the proof.  $\square$

The following theorems establish a connection between discrete-time versions and continuous-time versions.

**Theorem 13.** *The linear skew-evolution semiflow  $\pi = (\Phi, \sigma)$  is uniformly exponentially stable if and only if there exist  $\mathcal{K} > 0$ ;  $b \in \mathcal{N}$  and  $G \in \mathbf{H}(\mathbb{R}_+)$  such that  $\sup G(\eta_b(\theta, m, n, \cdot)) \leq \mathcal{K} < \infty$ , where*

$$\eta_b(\theta, m, n, \tau) = \begin{cases} b(\|\Phi(n + \tau + m, \tau + m, \sigma(\tau + m, m, \theta))\|), & \text{for } \tau \in [0, n], \\ 0, & \text{for } \tau \notin [0, n]. \end{cases}$$

**Proof.** Necessity. Suppose that  $\pi = (\Phi, \sigma)$  is uniformly exponentially stable and let  $K, \nu > 0$  be two constants given by Definition 18. We have that

$$\int_0^n \|\Phi(n + \tau + m, \tau + m, \sigma(\tau + m, m, \theta))\| \, d\tau \leq \int_0^n K e^{-\nu m} \, d\tau = \frac{Kn}{e^{\nu n}} \leq \frac{K}{\nu}.$$

Thus we only take  $b(t) = t$ ,  $G(f) := \int_0^\infty f(\tau) \, d\tau$  and  $\mathcal{K} := \frac{K}{\nu}$ .

Sufficiency. Let  $(a_j), (b_j)$  be two sequences given by  $a_j := j$ ,  $b_j := j + 1$ . It is not difficult to show that  $(a_j), (b_j) \in \mathcal{S}_1$ . We consider the function  $a : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  given by  $a(t) := b\left(\frac{t}{M^2 e^{2\omega}}\right)$ . It is clear that  $a \in \mathcal{N}$ . For each  $s \in \mathcal{S}$ , we define two maps  $f(s, \cdot) : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ ,  $F_G : \mathcal{S} \rightarrow [0, \infty]$  given by

$$f(s, \tau) := s([\tau]), \quad F_G(s) := G(f(s, \cdot)).$$

Because of the hypothesis  $G \in \mathbf{H}(\mathbb{R}_+)$ , we deduce that  $F_G \in \mathbf{H}(\mathbb{N})$ . From the equality

$$\begin{aligned} & \Phi(n + 1 + [\tau] + m, [\tau] + m, \sigma([\tau] + m, m, \theta)) \\ &= \Phi(n + 1 + [\tau] + m, n + \tau + m, \sigma(n + \tau + m, m, \theta)) \\ & \Phi(n + \tau + m, \tau + m, \sigma(\tau + m, m, \theta)) \Phi(\tau + m, [\tau] + m, \sigma([\tau] + m, m, \theta)), \end{aligned}$$

we have

$$\begin{aligned} & \frac{\|\Phi(n + 1 + [\tau] + m, [\tau] + m, \sigma([\tau] + m, m, \theta))\|}{M^2 e^{2\omega}} \\ & \leq \|\Phi(n + \tau + m, \tau + m, \sigma(\tau + m, m, \theta))\|. \end{aligned}$$

It is not difficult to show that

$$(26) \quad \mathcal{K} \geq G(\eta_b(\theta, m, n, \cdot)) \geq G(f(\varphi_a, \cdot)) \geq F_G(\varphi_a(\theta, m, n, \cdot)).$$

Using Theorem 9, we obtain that  $\pi$  is uniformly exponentially stable. □

In the following we give a continuous-time version of Theorem 12.

**Theorem 14.** *The linear skew-evolution semiflow  $\pi = (\Phi, \sigma)$  is uniformly exponentially stable if and only if there exist  $\mathcal{K} > 0$  and  $k, q \in \mathbb{N}$ ,  $G \in \mathbb{H}_{k, q}(\mathbb{R}_+)$  such that  $\sup G(\eta(\theta, m, n, \cdot)) \leq \mathcal{K} < \infty$ , where*

$$\eta(\theta, m, n, \tau) = \begin{cases} \|\Phi(n + \tau + m, \tau + m, \sigma(\tau + m, m, \theta))\|, & \text{for } \tau \in [0, n], \\ 0, & \text{for } \tau \notin [0, n]. \end{cases}$$

**Proof.** The rest of the proof is similar to that of Theorem 13 and is omitted. □

We can rewrite Theorem 13 as follows.

**Corollary 20.** *The linear skew-evolution semiflow  $\pi = (\Phi, \sigma)$  is uniformly exponentially stable if and only if there exist  $\mathcal{K} > 0$  and  $b \in \mathcal{N}$  such that*

$$\sup_0^n \int_0^n b(\|\Phi(n + \tau + m, \tau + m, \sigma(\tau + m, m, \theta))\|) d\tau \leq \mathcal{K} < \infty.$$

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Faculty of Mathematics, Mechanics and Informatics,  
College of Science,  
Viet Nam National University, Ha Noi,  
334, Nguyen Trai, Thanh Xuan, Ha Noi,  
Viet Nam  
E-mail: phamviethai86@gmail.com

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