Applicable Analysis and Discrete Mathematics
available online at http://pefmath.etf.rs

Appl. Anal. Discrete Math. **4** (2010), 66–80. doi:10.2298/AADM100217016Z

CONVERGENCE PROPERTIES OF THE q-DEFORMED BINOMIAL DISTRIBUTION

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We consider the q-deformed binomial distribution introduced by S. C. Jing: *The q-deformed binomial distribution and its asymptotic behaviour,* J. Phys. A **27** (2) (1994), 493–499 and W. S. Chung et al: *q-deformed probability and binomial distribution,* Internat. J. Theoret. Phys. **34** (11) (1995), 2165–2170 and establish several convergence results involving the Euler and the exponential distribution; some of them are q-analogues of classical results.

1. INTRODUCTION

The q-deformed binomial distribution $QD(n,\tau,q)$ was introduced by Jing [**10**] in connection with the q-deformed boson oscillator and by Chung et al. [**5**]. Its probabilities are given by

$$\text{(1)}\qquad \mathbb{P}(X_{QD}=x)=\begin{bmatrix} n\\ x\end{bmatrix}_q \tau^x(\tau;q)_{n-x}, \qquad 0\le x\le n,\ 0\le\tau\le 1,\ 0<q<1,$$

where

$$\begin{bmatrix} n\\ x\end{bmatrix}_q=\frac{(q;q)_n}{(q;q)_x(q;q)_{n-x}} \qquad \text{and} \qquad (z;q)_n=\prod_{i=0}^{n-1}(1-zq^i)$$

are the q-binomial coefficient and the q-shifted Pochhammer symbol; an introduction to the q-calculus and basic hypergeometric series can be found in Gasper and Rahman [**6**]. This distribution was studied by many authors and has applications in physics as well as in approximation theory due to the q-Bernstein polynomials and the q-Bernstein operator (see Section 2 for details).

2000 Mathematics Subject Classification. Primary. 33D90; Secondary. 05A30, 60F05.
Keywords and Phrases. q-deformed binomial distribution, q-Poisson distribution, limit theorems, Euler-distribution, exponential distribution.

It is well known that for $n \to \infty$ (and fixed τ) the q-deformed binomial distribution converges to an Euler distribution. This paper is devoted to the study of sequences of q-deformed binomially distributed random variables $X_n \sim QD(n, \tau_n, q)$ with parameter sequence (τ_n) depending on n (a similar analysis for Kemp's q-binomial distribution has been done by Gerhold and Zeiner [**7**]).

The present paper is organised as follows. In Section 2 we give all definitions of q-calculus and q-distributions we need in the following and we sum up some important properties of the q-deformed binomial distribution. Section 3 deals with parameter sequences τ_n where τ_n tends to a limit $c \in [0, 1)$, in particular with the case of constant mean. The pertinent limit law in this case is the Heine distribution and we establish a q-analogue of the convergence of the classical binomial distribution with constant mean to the Poisson distribution. In Section 4 we investigate parameter sequences with limit 1. Depending on the growth rate of the parameter sequence we obtain a degenerate, a truncated-exponential like or an exponential limit law. Remarkably all these limits are independent of q.

2. NOTATION AND DEFINITIONS

Throughout the paper we use the notation of Gasper and Rahman [**6**]. Besides the definitions of the q-binomial coefficient and the q-shifted Pochhammer symbol we need the q-number $[x]_q$ of x defined by

$$[x]_q := \frac{1 - q^x}{1 - q};$$

for $q \to 1$ we have $[x]_q \to x$. Moreover, we will need two q-analogues of the exponential function:

$$e_q(z) = \frac{1}{(z; q)_\infty}, \qquad z \in \mathbb{C} \setminus \{q^{-i}, i = 0, 1, 2, \ldots\}, \quad |q| < 1,$$

and $E_q(z) = (-z; q)_\infty$. Here the limit relations $e_q((1-q)z) \to e^z$ and $E_q((1-q)z) \to e^z$ hold, as $q \to 1$.

The Euler distribution $E(\lambda, q)$ with parameter λ is defined by

$$\mathbb{P}(X_E = x) = \frac{\lambda^x}{(q; q)_x} (\lambda; q)_\infty = \frac{\lambda^x}{(q; q)_x} E_q(-\lambda).$$

This is a q-analogue of the Poisson distribution since $E((1-q)\lambda, q) \to P(\lambda)$ for $q \to 1$. For properties and applications of this distribution we refer to Johnson, Kemp and Kotz [**12**], Benkherouf and Bather [**1**], Biedenharn [**2**], Kemp [**13**, **14**, **16**], Charalambides and Papadatos [**4**] and Ostrovska [**19**, **20**].

Our main object of interest is the q-deformed binomial distribution $QD(n, \tau, q)$ defined in (1). This distribution is a q-analogue of the classical binomial distribution, since in the limit $q \to 1$ the q-deformed binomial distribution with parameter

(n, τ, q) reduces to the binomial distribution with parameters (n, τ). The limit $n \to \infty$ of random variables $X_n \sim QD(n, \tau, q)$ leads to an Euler distribution with parameter $\lambda = \tau$. If we denote the probabilities (1) by $p_n(x, \tau)$, then the following recurrence relation holds (see VIDENSKII [**21**, Section 3]):

$$p_n(x, \tau) = \tau p_{n-1}(x - 1, \tau) + (1 - \tau)p_{n-1}(x, q\tau). \tag{2}$$

For details and further properties we refer to JING [**10**], JING and FAN [**11**], KEMP [**15**, **16**], the encyclopedic book JOHNSON, KEMP and KOTZ [**12**], and to CHARALAMBIDES [**3**]. CHUNG et al. [**5**], KUPERSHMIDT [**17**] and IL'INSKI [**8**] gave representations of the q-deformed binomial distribution as a sum of dependent and not identically distributed random variables.

As mentioned above the q-deformed binomial distribution and the Euler distribution appear in particular both in physics ([**2**, **5**, **10**, **11**]) and in approximation theory. The q-Bernstein polynomials of order n are defined by

$$B_n(f(t), q; x) = \sum_{r=0}^{n} f\left(\frac{[r]_q}{[n]_q}\right) \begin{bmatrix} n \\ r \end{bmatrix}_q x^r (x; q)_{n-r},$$

where f is a continuous function on the interval $[0, 1]$. There exists a vast literature on these polynomials, closely related to the distributions under consideration are e.g. [**3**, **9**, **18**, **19**, **20**, **21**].

3. PARAMETER SEQUENCES WITH LIMIT < 1

In the present section we study sequences of random variables X_n which are $QD(n, \tau_n, q)$-distributed, where the parameters τ_n converge to a limit $c \in [0, 1)$. In particular we prove a q-analogue of the convergence of the classical binomial distribution with constant mean to a Poisson distribution.

As noted above the sequence converges in the case of constant parameters $\tau_n = \tau$ to an Euler distribution with parameter τ. The following proposition is a mild generalisation of the convergence to an Euler distribution mentioned in the previous section and shows that the Euler distribution is the limit distribution for every convergent parameter sequence τ_n with limit in $[0, 1)$.

Proposition 3.1. *Let $X_n \sim QD(n, \tau_n, q)$. Then, for $n \to \infty$,*

$$X_n \to E(\tau, q)$$

if $\tau_n \to \tau$ and $0 \le \tau < 1$.

Proof. Note that

$$\mathbb{P}(X_n = x) = \begin{bmatrix} n \\ x \end{bmatrix}_q \tau_n^x \prod_{i=0}^{n-x} (1 - \tau_n q^i).$$

The q-binomial coefficient tends to $1/(q;q)_x$. For the product apply the dominated convergence theorem to its logarithm to see that it converges to $E_q(-\tau)$. $\square$

We are now interested in special choices of the parameters τ_n such that the limit $X(q)$ of the sequence $X_n(q)$ converges to a Poisson distribution for $q \to 1$. From the previous theorem we deduce immediately the following corollary.

Corollary 3.2. *Let $X_n \sim QD(n, \tau_n(q), q)$ with $\tau_n(q) \to \dfrac{\lambda}{n}$ for $q \to 1$ and $\tau_n(q) \to \tau(q)$ for $n \to \infty$ with the additional property $\dfrac{\tau(q)}{1-q} \to \lambda$ in the limit $q \to 1$ (recall that we assume $\tau(q) < 1$ in this section). Then the following diagram is commutative:*

$$\begin{array}{ccc} QD(n,\tau_n,q) & \xrightarrow{n\to\infty} & E(\tau(q),q) \\ {\scriptstyle q\to 1}\big\downarrow & & \big\downarrow{\scriptstyle q\to 1} \\ B\left(n,\dfrac{\lambda}{n}\right) & \xrightarrow[n\to\infty]{} & P(\lambda) \end{array}$$

One very natural way to choose the parameters is to set $\tau_n = \dfrac{\lambda}{[n]_q}$.

Our next goal is to establish a convergence result, which is analogous to the convergence of the classical binomial distribution with constant mean to a Poisson distribution and reduces in the limit $q \to 1$ to that theorem. For this purpose we start with an elementary fact.

Lemma 3.3. *Let $f_n(x)$, $n \in \mathbb{N}$, be a sequence of continuous functions which converges pointwise to a continuous limit $f(x)$. Assume that for each n the function $f_n(x)$ has a single root $\hat{x}_n$, and $f(x)$ has a single root $\hat{x}$, and that $f(y)f(z) < 0$ for $y < \hat{x}$ and $z > \hat{x}$. Then $\hat{x}_n \to \hat{x}$.*

Proof. W.l.o.g. we may assume that $f(z) > 0$ for $z > \hat{x}$. For given $\varepsilon > 0$ choose a $\delta(\varepsilon) < \min(f(\hat{x}+\varepsilon), -f(\hat{x}-\varepsilon))$. Then there exists an $N = N(\delta(\varepsilon))$ such that for all $n \geq N$ we have $|f_n(\hat{x}+\varepsilon) - f(\hat{x}+\varepsilon)| < \delta(\varepsilon)$. Therefore $f_n(\hat{x}+\varepsilon) > 0$. Moreover there exists an $M = M(\delta(\varepsilon))$ such that for all $n \geq M$ we have $|f_n(\hat{x}-\varepsilon) - f(\hat{x}-\varepsilon)| < \delta(\varepsilon)$. Therefore $f_n(\hat{x} - \varepsilon) < 0$. Hence, by continuity, for all $n \geq \max(N, M)$ we have $|\hat{x} - \hat{x}_n| < 2\varepsilon$. $\square$

The essential key to apply this lemma is the following representation of the means $\mu_n(\tau, q)$, which allows us to extract important properties of the means easily.

Lemma 3.4. *The means $\mu_n(\tau, q)$ have the representation*

$$\mu_n(\tau, q) = \sum_{j=1}^{n} (q;q)_{j-1} \begin{bmatrix} n \\ j \end{bmatrix}_q \tau^j.$$

Proof. We proceed by induction. For $n = 1$ this is obviously true. Now suppose that the statement is true for $n - 1$. In order to calculate $\mu_n(\tau, q)$ we use the recurrence relation (2). Hence we have

$$\mu_n(\tau,q)=\sum_{x=1}^{n} xp_n(x,\tau)=\tau\sum_{x=1}^{n} xp_{n-1}(x-1,\tau)+(1-\tau)\sum_{x=1}^{n-1} xp_{n-1}(x,q\tau).$$

Shifting the summation index in the first sum, splitting this sum and using the induction hypothesis yields

$$\begin{aligned}\mu_n(\tau,q)=\tau\sum_{j=1}^{n-1}(q;q)_{j-1}\begin{bmatrix}n-1\\ j\end{bmatrix}_q\tau^j+\sum_{x=1}^{n}\tau p_{n-1}(x-1,\tau)\\ +(1-\tau)\sum_{j=1}^{n-1}(q;q)_{j-1}\begin{bmatrix}n-1\\ j\end{bmatrix}_q\tau^j q^j.\end{aligned}$$

The second sum reduces to τ. Collecting powers of τ gives

$$\begin{aligned}\mu_n(\tau,q)=\tau\left(1+\begin{bmatrix}n-1\\ 1\end{bmatrix}_q\right)\\ +\sum_{j=2}^{n}\left((q;q)_{j-1}\begin{bmatrix}n-1\\ j\end{bmatrix}_q+(q;q)_{j-2}\begin{bmatrix}n-1\\ j-1\end{bmatrix}_q(1-q^{j-1})\right)\tau^j.\end{aligned}$$

Consequently the desired result follows by the recurrence relation for the q-binomial coefficients (see e.g. [**6**, (I.45)]). □

Remark 3.5. An alternative way to prove this lemma is to use Kemp's [**15**, p. 300] representation of the probability generating function, to differentiate and to manipulate the sum.

Using the monotonicity of the q-binomial coefficients in n we immediately get the following proposition.

Proposition 3.6. *The means $\mu_n(\tau,q)$ are strictly increasing in n (for $\tau>0$) and τ.*

Now we turn to the convergence result:

Theorem 3.7. *Fix $\mu>0$ and choose the parameter $\tau_n=\tau_n(q,\mu)$ of the q-deformed binomial distribution such that $\mu_n=\mu$. Then we have*

(i) *The sequence $QD(n,\tau_n,q)$ converges for $n\to\infty$ to an Euler distribution $E(\tau,q)$, where $\tau=\lim_{n\to\infty}\tau_n$.*

(ii) *For fixed n, $QD(n,\tau_n,q)$ tends to a binomial distribution $B\left(n,\frac{\mu}{n}\right)$ in the limit $q\to 1$.*

(iii) *For $q\to 1$, the Euler distribution $E(\tau,q)$ converges to a Poisson distribution with parameter μ.*

So we obtain the following commutative diagram:

$$\begin{array}{ccc} QD(n,\tau_n(q),q) & \xrightarrow{n\to\infty} & E(\tau(q),q) \\ {\scriptstyle q\to 1}\Big\downarrow & & \Big\downarrow{\scriptstyle q\to 1} \\ B\left(n,\dfrac{\mu}{n}\right) & \xrightarrow[n\to\infty]{} & P(\mu) \end{array}$$

Proof. First we check that for given μ, q and large n there exists a unique τ_n with $\mu_n(\tau_n,q)=\mu$. The function $\mu_n(\tau,q)$ is continuous and strictly increasing in n and τ by the previous theorem. Moreover, we have $\lim\limits_{\tau\to 0}\mu_n(\tau,q)=0$. If we choose τ_n such that $\tau_n\to 1$ then $\mu_n(\tau_n,q)$ becomes arbitrarily large. Consequently there is a unique solution of $\mu_n(\tau,q)=\mu$. By Lemma 3.3 the sequence τ_n converges to a limit τ where τ is the unique solution of $\mu_E(\tau,q)=\mu$, where $\mu_E(\tau,q)$ is the mean of an Euler-distribution with parameters τ and q. This mean can be written as

$$\mu_E(\tau,q)=\sum_{i=0}^{\infty}\frac{q^i\tau}{1-q^i\tau},$$

see [**13**] or take the limit $n\to\infty$ (using the dominated convergence theorem) in Lemma 3.4 and manipulate the sum (i.e. expand the denominator as a geometric series and change the order of summation).

Again by Lemma 3.3 we get that $\tau_n\to\mu/n$. It remains to check that $\tau/(1-q)$ converges to μ in the limit $q\to 1$. But this is again a consequence of Lemma 3.3 since $\tau/(1-q)$ is the unique solution of $\mu_E((1-q)\tau,q)=\mu$ and $\mu_E((1-q)\tau,q)$ tends to τ for $q\to 1$. $\square$

4. PARAMETER SEQUENCES WITH LIMIT 1

In this section we investigate sequences X_n of random variables, where X_n is $QD(n,\tau_n,q)$-distributed and the parameters τ_n converge to 1. The behaviour of the sequences X_n depends on the growth rate of τ_n. Therefore we will distinguish three cases: Firstly we examine the case $\tau_n^n\to 1$, where it will turn out that the limit distribution is degenerate. Then we study the case $\tau_n^n\to c$ with $0<c<1$. Here the limit law depends only on c and is a truncated exponential distribution. Finally we turn to the case $\tau_n^{f(n)}\to c$ where $0<c<1$ and $f(n)=o(n)$; this will lead to an exponential distribution.

Consider sequences of random variables $X_n\sim QD(n,\tau_n,q)$ with $\tau_n\to 1$ and additionally $\tau_n^n\to 1$ first. Then we have the following theorem:

Theorem 4.1. *Let $X_n\sim QD(n,\tau_n,q)$ with $\tau_n\to 1$ and $\tau_n^n\to 1$. Then $n-X_n$ converges to the point measure at* 0.

Proof. The probability that $Y_n=n-X_n$ is equal to 0 is given by

$$\mathbb{P}(Y_n=0)=\tau_n^n$$

which converges to 1 by assumption. □

Now let us investigate sequences $X_n \sim QD(n, \tau_n, q)$, where $\tau_n \to 1$ and $\tau_n^n \to c$ for a $c \in (0, 1)$. Before we can establish the distribution of the limit of such a sequence, we start with several lemmas, which allow us to compute the asymptotic behaviour of certain sums of probabilities of $QD(n, \tau_n, q)$-distributed random variables and their means and variances.

The first lemma is an analogue to Lemma 3.4 and gives an alternative representation of the variance:

Lemma 4.2. *The second moment of $X_n(\tau, q)$ can be written as*

$$\sum_{x=1}^{n} x^2 \begin{bmatrix} n \\ x \end{bmatrix}_q \tau^x (\tau; q)_{n-x} = \sum_{j=1}^{n} {}_n a_j \tau^j$$

with

$${}_n a_j = \begin{bmatrix} n \\ j \end{bmatrix}_q (q; q)_{j-1} \left(1 + 2 \sum_{i=1}^{j-1} \frac{1}{1-q^i} \right).$$

Proof. We prove this by induction. The case $n = 1$ is obvious. To compute $\mathbb{E}(X_n^2)$ we use the recurrence (2) again and shift the summation index. This gives

$$V_n := \sum_{x=1}^{n} x^2 p_n(x, \tau) = \tau \sum_{x=0}^{n-1} (x^2 + 2x + 1) p_{n-1}(x, \tau) + (1 - \tau) \sum_{x=1}^{n} x^2 p_{n-1}(x, q\tau).$$

By splitting sums and by using Lemma 3.4 and the induction hypothesis we find

$$V_n = \tau \sum_{j=1}^{n-1} {}_{n-1}a_j \tau^j + 2\tau \sum_{j=1}^{n-1} (q; q)_{j-1} \begin{bmatrix} n-1 \\ j \end{bmatrix}_q \tau^j + \tau + (1 - \tau) \sum_{j=1}^{n-1} {}_{n-1}a_j q^j \tau^j.$$

Collecting powers of τ yields

$$V_n = \tau \left(1 + \begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q \right) + \sum_{j=2}^{n} \left({}_{n-1}a_{j-1}(1 - q^{j-1}) + 2 \begin{bmatrix} n-1 \\ j-1 \end{bmatrix}_q (q; q)_{j-2} + {}_{n-1}a_j q^j \right) \tau^j.$$

The first term gives $\begin{bmatrix} n \\ 1 \end{bmatrix}_q \tau$ and the coefficient of τ^j in the sum equals

$$\begin{bmatrix} n-1 \\ j-1 \end{bmatrix}_q (q; q)_{j-2} \left(1 + 2 \sum_{i=1}^{j-2} \frac{1}{1-q^i} \right) [1 - q^{j-1}] + 2 \begin{bmatrix} n-1 \\ j-1 \end{bmatrix}_q (q; q)_{j-2} + \begin{bmatrix} n-1 \\ j \end{bmatrix}_q (q; q)_{j-1} \left(1 + 2 \sum_{i=1}^{j-1} \frac{1}{1-q^i} \right) q^j,$$

which implies the statement by using the recurrence relation of the q-binomial coefficients again. □

The next three lemmas are devoted to the asymptotic behaviour of sums of powers of θ_n, where $0 < \theta_n < 1$ and $\theta_n \to 1$.

Lemma 4.3. *If $f(n) \to \infty$ for $n \to \infty$ and $\theta_n \le 1$ such that $\theta_n^{f(n)} \to c$ with $0 < c < 1$, then*

$$\sum_{i=0}^{\infty} \theta_n^i \sim \frac{-f(n)}{\log c}, \qquad n \to \infty.$$

Proof. Since $c < 1$ almost all θ_n must be smaller than 1. Thus we assume w.l.o.g. that $\theta_n < 1$ and obtain

$$\sum_{i=0}^{\infty} \theta_n^i = \frac{1}{1-\theta_n} \sim -\frac{1}{\log \theta_n}$$

using the substitution $\theta_n = 1 + x_n$ in the elementary equivalence

$$\log(1+x) \sim x, \quad x \to 0. \tag{3}$$

Since $f(n) \log \theta_n \sim \log c$, the statement follows. □

Lemma 4.4. *For $\theta_n \le 1$ and $\theta_n \to 1$, $\theta_n^{f(n)} \to c$ with $c \in (0,1)$ and $g(n)/f(n) \sim \beta$, $g(n) \le n$ we have*

$$\sum_{i=0}^{\lfloor g(n) \rfloor} \theta_n^i \sim \frac{c^\beta - 1}{\log c} f(n)$$

and

$$\sum_{i=0}^{\lfloor g(n) \rfloor} \begin{bmatrix} n \\ i \end{bmatrix}_q \theta_n^i \sim e_q(q) \frac{c^\beta - 1}{\log c} f(n)$$

as $n \to \infty$.

Proof. We rewrite the first sum as

$$\sum_{i=0}^{\lfloor g(n) \rfloor} \theta_n^i = \frac{1 - \theta_n^{\lfloor g(n) \rfloor + 1}}{1 - \theta_n}.$$

The growth of the denominator is given in Lemma 4.3, and the numerator tends to $1 - c^\beta$, since $\theta_n^{\lfloor g(n) \rfloor} = \theta_n^{g(n) - \{g(n)\}} \to c^\beta$ because of $\theta_n \to 1$.

To get the asymptotic of the second sum we write

$$\sum_{i=0}^{\lfloor g(n) \rfloor} \begin{bmatrix} n \\ i \end{bmatrix}_q \theta_n^i = \sum_{i=0}^{\lfloor \sqrt{g(n)} \rfloor} \begin{bmatrix} n \\ i \end{bmatrix}_q \theta_n^i + \sum_{\lfloor \sqrt{g(n)} \rfloor + 1}^{\lfloor g(n) - \sqrt{g(n)} \rfloor - 1} \begin{bmatrix} n \\ i \end{bmatrix}_q \theta_n^i + \sum_{\lfloor g(n) - \sqrt{g(n)} \rfloor}^{\lfloor g(n) \rfloor} \begin{bmatrix} n \\ i \end{bmatrix}_q \theta_n^i.$$

The first and the third sum on the right-hand side are $\mathcal{O}(\sqrt{g(n)})$ and therefore asymptotically negligible. The second sum is bounded by

$$\frac{(q;q)_n}{(q;q)_{\lfloor\sqrt{g(n)}\rfloor+1}(q;q)_{n-\lfloor\sqrt{g(n)}\rfloor-1}} \sum_{\lfloor\sqrt{g(n)}\rfloor+1}^{\lfloor g(n)-\sqrt{g(n)}\rfloor-1} \theta_n^i \leq \sum_{\lfloor\sqrt{g(n)}\rfloor+1}^{\lfloor g(n)-\sqrt{g(n)}\rfloor-1} \begin{bmatrix} n \\ i \end{bmatrix}_q \theta_n^i$$
$$\leq \frac{(q;q)_n}{(q;q)_{\lfloor n/2\rfloor}^2} \sum_{\lfloor\sqrt{g(n)}\rfloor+1}^{\lfloor g(n)-\sqrt{g(n)}\rfloor-1} \theta_n^i.$$

By the first part of this lemma the lower and the upper bound has the asserted asymptotic. □

Lemma 4.5. *If $\theta_n \leq 1$ and $\theta_n \to 1$ with $\theta_n^n \to c$ for $0 < c < 1$, then*

$$\sum_{i=0}^{n} i\theta_n^i \sim \frac{1-c+c\log c}{\log^2 c} n^2$$

and

$$\sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix}_q i\theta_n^i \sim e_q(q)\frac{1-c+c\log c}{\log^2 c} n^2$$

as $n \to \infty$.

Proof. To estimate this sum we use Lemma 4.3 again and the identity

$$\sum_{i=0}^{n} it^i = \frac{t(1-t^n-nt^n(1-t))}{(1-t)^2}.$$

Hence, setting $t = \theta_n$,

$$\sum_{i=0}^{n} i\theta_n^i \sim (1-c-n\theta_n^n(1-\theta_n))\frac{n^2}{\log^2 c} \sim (1-c+c\log c)\frac{n^2}{\log^2 c}.$$

Here we used that under the assumption $\theta_n^n \to c$ we have $(1-\theta_n)n \to -\log c$. This can easily be seen from the equivalence (3). The asymptotic for the sum with the q-binomial coefficient is obtained as in Lemma 4.4. □

Now we are ready to establish the essential key in proving the convergence result: we give the asymptotic behaviour of sums of probabilities and the means and variances of $QD(n, \tau_n, q)$-distributed random variables.

Lemma 4.6. *Let X_n be $QD(n, \tau_n, q)$-distributed and denote by $\mu_n(\tau_n, q)$ and $\sigma_n^2(\tau_n, q)$ the corresponding mean and variance. If $\tau_n \to 1$ and $\tau_n^n \to c$ with*

$0 < c < 1$ *and* $f(n) \sim \beta n$, $f(n) < n$, *then*

$$\sum_{x=0}^{\lfloor f(n) \rfloor} \tau_n^x \begin{bmatrix} n \\ x \end{bmatrix}_q (\tau_n; q)_{n-x} \sim 1 - c^\beta,$$

$$\mu_n(\tau_n, q) \sim \frac{c-1}{\log c} n,$$

$$\sigma_n^2(\tau_n, q) \sim \frac{1 + 2c \log c - c^2}{(\log c)^2} n^2,$$

as $n \to \infty$.

Proof. We start with the first assertion. Since $f(n) < n$ we can write

$$S_n := \sum_{x=0}^{\lfloor f(n) \rfloor} \tau_n^x \begin{bmatrix} n \\ x \end{bmatrix}_q (\tau_n; q)_{n-x} = (1 - \tau_n) \sum_{x=0}^{\lfloor f(n) \rfloor} \tau_n^x \begin{bmatrix} n \\ x \end{bmatrix}_q \prod_{i=1}^{n-x-1} (1 - \tau_n q^i).$$

The summands are bounded by $e_q(q)^2$, hence

$$S_n \sim (1 - \tau_n) \sum_{x=\lfloor \sqrt{n} \rfloor}^{\lfloor f(n) \rfloor - \lfloor \sqrt{n} \rfloor} \tau_n^x \begin{bmatrix} n \\ x \end{bmatrix}_q \prod_{i=1}^{n-x-1} (1 - \tau_n q^i) =: \hat{S}_n.$$

Estimating the product and using again the boundedness of the summands yields

$$\begin{aligned} \hat{S}_n &\leq (1 - \tau_n)(\tau_n; q)_{n - \lfloor f(n) \rfloor + \lfloor \sqrt{n} \rfloor - 1} \sum_{x=\lfloor \sqrt{n} \rfloor}^{\lfloor f(n) \rfloor - \lfloor \sqrt{n} \rfloor} \tau_n^x \begin{bmatrix} n \\ x \end{bmatrix}_q \\ &\sim (1 - \tau_n)(\tau_n; q)_{n - \lfloor f(n) \rfloor + \lfloor \sqrt{n} \rfloor - 1} \sum_{1}^{n} \tau_n^x \begin{bmatrix} n \\ x \end{bmatrix}_q =: \hat{\hat{S}}_n. \end{aligned}$$

As in the proof of Proposition 3.1 and with use of Lemma 4.4 (with $g(n) := f(n)$ and $f(n) := n$) we obtain

$$\hat{\hat{S}}_n \sim (1 - \tau_n) \frac{1}{e_q(q)} e_q(q) \frac{c^\beta - 1}{\log c} n \sim 1 - c^\beta.$$

In an analogous way we find a lower bound of $\hat{S}_n$ that is asymptotically equivalent to $1 - c^\beta$.

Now we prove the second proposition of the lemma: Use Lemma 3.4, easy estimates of the q-Pochhammer symbol and the asymptotics given in Lemma 4.4 to obtain

$$\mu_n(\tau_n, q) \leq \sum_{j=1}^{\lfloor \sqrt{n} \rfloor} \frac{(q; q)_n}{(q; q)_{\lceil n/2 \rceil}^2} + (q; q)_{\lfloor \sqrt{n} \rfloor} \sum_{j=\lfloor \sqrt{n} \rfloor}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q \tau_n^j \sim \frac{1}{e_q(q)} e_q(q) \frac{c-1}{\log c} n$$

and

$$\mu_n(\tau_n, q) \geq (q;q)_n \sum_{j=1}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q \tau_n^j \sim \frac{c-1}{\log c} n.$$

Similarly we proceed for the second moments of $X_n(\tau_n, q)$ and estimate with use of Lemma 4.5

$$\begin{aligned} \mathbb{E}(X_n^2) &\geq \sum_{j=1}^{n} (q;q)_{j-1} \left(1 + 2(j-1)\right) \begin{bmatrix} n \\ j \end{bmatrix}_q \tau_n^j \\ &\geq 2(q;q)_n \sum_{j=1}^{n} (j-1) \begin{bmatrix} n \\ j \end{bmatrix}_q \tau_n^j \sim 2\frac{1-c+c\log c}{(\log c)^2} n^2. \end{aligned}$$

To bound the second moment from above we split the sum into two parts

$$\begin{aligned} \mathbb{E}(X_n^2) \leq \sum_{j=1}^{\lfloor\sqrt{n}\rfloor} \frac{(q;q)_n}{(q;q)_{\lceil n/2\rceil}^2} \left(1 + \frac{2n}{1-q}\right) \\ + \sum_{j=\lfloor\sqrt{n}\rfloor}^{n} (q;q)_{j-1} \left(1 + 2\sum_{i=1}^{j-1} \frac{1}{1-q^{j-i}}\right) \begin{bmatrix} n \\ j \end{bmatrix}_q \tau_n^j. \end{aligned}$$

The first sum is $\mathrm{o}(n^2)$, and splitting the inner sum in the second term we obtain

$$\begin{aligned} \mathbb{E}(X_n^2) = o(n^2) &+ \sum_{j=\lfloor\sqrt{n}\rfloor}^{n} (q;q)_{j-1} \left(1 + 2\sum_{i=\lfloor\sqrt{j}\rfloor}^{j-1} \frac{1}{1-q^{j-i}}\right) \begin{bmatrix} n \\ j \end{bmatrix}_q \tau_n^j \\ &+ \sum_{j=\lfloor\sqrt{n}\rfloor}^{n} (q;q)_{j-1} \left(1 + 2\sum_{i=1}^{\lfloor\sqrt{j}\rfloor} \frac{1}{1-q^{j-i}}\right) \begin{bmatrix} n \\ j \end{bmatrix}_q \tau_n^j. \end{aligned}$$

Here the first sum is $\mathrm{o}(n^2)$ again and easy estimates of the second term yield

$$\begin{aligned} \mathbb{E}(X_n^2) &\leq o(n^2) + 2(q;q)_{\lfloor\sqrt{n}\rfloor} \sum_{j=\lfloor\sqrt{n}\rfloor}^{n} j \frac{1}{1-q^{j-\lfloor\sqrt{j}\rfloor-1}} \begin{bmatrix} n \\ j \end{bmatrix}_q \tau_n^j \\ &\leq o(n^2) + 2(q;q)_{\lfloor\sqrt{n}\rfloor} \frac{1}{1-q^{n-\sqrt{n}-1}} \sum_{j=1}^{n} j \begin{bmatrix} n \\ j \end{bmatrix}_q \tau_n^j \\ &\sim 2\frac{1-c+c\log c}{(\log c)^2} n^2. \end{aligned}$$

Thus

$$\mathbb{E}(X_n^2(\tau_n, q)) \sim 2\frac{1-c+c\log c}{(\log c)^2} n^2.$$

Hence

$$\sigma_n^2(\tau_n, q) = \mathbb{E}(X_n^2(\tau_n, q)) - \mu_n(\tau, q)^2 \sim \left(2\frac{1 - c + c\log c}{(\log c)^2} - \left(\frac{c-1}{\log c}\right)^2\right) n^2$$
$$\sim \frac{1 + 2c\log c - c^2}{(\log c)^2} n^2,$$

which completes the proof. □

After this analysis of the means and variances it is now easy to obtain the limiting distribution of the sequence X_n.

Theorem 4.7. *Let $Y_n \sim QD(n, q, \tau_n)$ with $\tau_n \to 1$ and $\tau_n^n \to c$ with $0 < c < 1$. Then the sequence of the normalised random variables $X_n = (Y_n - \mu_n)/\sigma_n$ converges to a limit X with*

$$\mathbb{P}(X \le x) = 1 - e^{c-1} e^{-\sqrt{1+2c\log c - c^2}\,x}$$

for

$$x \in \left[-\frac{1-c}{\sqrt{1 + 2c\log c - c^2}}, \frac{c - \log c - 1}{\sqrt{1 + 2c\log c - c^2}}\right)$$

and

$$\mathbb{P}(X \le x) = 1 \qquad \text{for} \qquad x = \frac{c - \log c - 1}{\sqrt{1 + 2c\log c - c^2}}.$$

Proof. The support of X is given by

$$\left[\lim_{n\to\infty} -\frac{\mu_n(\tau_n, q)}{\sigma_n(\tau_n, q)}, \lim_{n\to\infty} \frac{n - \mu_n(\tau_n, q)}{\sigma_n(\tau_n, q)}\right].$$

Using Lemma 4.6 the stated support follows immediately.

Computing the distribution function of X yields with use of Lemma 4.6

$$\mathbb{P}(X_n \le x) = \sum_{0 \le y \le \sigma_n x + \mu_n} \tau_n^y \begin{bmatrix} n \\ y \end{bmatrix}_q (\tau_n; q)_{n-y} \sim 1 - c^\alpha$$

with

$$\alpha = \frac{\sqrt{1 + 2c\log c - c^2}}{-\log c} x + \frac{c-1}{\log c}$$

for

$$x < \frac{c - \log c - 1}{\sqrt{1 + 2c\log c - c^2}}.$$

Simplifying c^α yields the theorem. □

Now we turn to the third case, which treats sequences of random variables $X_n \sim QD(n, \tau_n, q)$ where $\tau_n \to 1$ and $\tau_n^{f(n)} \to c$ for a $c \in (0,1)$ and $f(n) = o(n)$.

This case is very similar to the previous one, and so we start with an analogue of Lemma 4.5

Lemma 4.8. *Let $f(n) \to \infty$, $f(n) = o(n)$, $\theta_n^{f(n)} \to c$ with $0 < c < 1$. Then*

$$\sum_{i=0}^{n} i\theta_n^i \sim \frac{f(n)^2}{\log^2 c} \qquad \text{and} \qquad \sum_{i=0}^{n} i \begin{bmatrix} n \\ i \end{bmatrix}_q \theta_n^i \sim e_q(q) \frac{f(n)^2}{\log^2 c}$$

as $n \to \infty$.

Proof. Follow the proof of Lemma 4.5 and observe that $n\theta_n^n(1-\theta_n)$ tends to zero. $\square$

Following the proof of Lemma 4.6 and using Lemma 4.8 instead of Lemma 4.5 we obtain

Lemma 4.9. *If $\tau_n \to 1$ and $\tau_n^{f(n)} \to c$ with $0 < c < 1$ and $f(n) = o(n)$, $g(n) \sim \beta f(n)$, then*

$$\sum_{x=0}^{\lfloor g(n) \rfloor} \tau_n^x \begin{bmatrix} n \\ x \end{bmatrix}_q (\tau_n; q)_{n-x} \sim 1 - c^{\beta},$$

$$\mu_n(\tau_n, q) \sim \frac{-f(n)}{\log c},$$

$$\sigma_n^2(\tau_n, q) \sim \frac{f(n)^2}{(\log c)^2},$$

as $n \to \infty$.

As an immediate consequence we get the distribution of the limit of X_n, which is an exponential distribution and is again independent of q.

Theorem 4.10. *Let $Y_n \sim QD(n, q, \tau_n)$ with $\tau_n \to 1$ and $\tau_n^{f(n)} \to c$ with $0 < c < 1$ and $f(n) = o(n)$. Then the sequence of the normalised random variables $X_n = (Y_n - \mu_n)/\sigma_n$ converges to a normalised exponential distribution with parameter 1, i.e.*

$$\mathbb{P}(X \le x) = 1 - e^{-x-1}, \qquad x \ge -1.$$

Proof. Lemma 4.9 yields immediately that the support of the limit distribution is $[-1, \infty)$. Computing the distribution function gives

$$\mathbb{P}(X \le x) = \sum_{0 \le y \le \sigma_n x + \mu_n} \tau_n^y \begin{bmatrix} n \\ y \end{bmatrix}_q (\tau_n; q)_{n-y} \sim 1 - c^{\frac{x+1}{-\log c}} = 1 - e^{-x-1}. \qquad \square$$

Comparing this result with Theorem 3.7 we see that this corresponds to taking the limit $c \to 0$.

Acknowledgement. I thank the anonymous referees for all the remarks that improved the presentation, in particular for shorter proofs of Lemma 3.4, Lemma

4.2, and Lemma 4.3 and and for directing my attention to the connections with q-Bernstein polynomials.

M. Zeiner was supported by the NAWI-Graz project and the Austrian Science Fund project S9611 of the National Research Network S9600 Analytic Combinatorics and Probabilistic Number Theory.

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(Received November 12, 2009)
(Revised February 17, 2010)